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Relativistické stochastické procesy, jejich popis a užití ve finančních trzích

Relativistic stochastic processes, their description and use in financial markets

Diplomová práce / Master's thesis

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Název práce:
Relativistické stochastické procesy, jejich popis a užití ve finančních trzích

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Abstrakt: Teória stochastických procesov sa za ostatné desaťročia stala dôležitým nástrojom pri popise vývoja cien finančných aktív. Black-Scholes-Merton oceňovací model, pôvodne publikovaný v roku 1973, poskytol účastníkom na trhu matematický nástroj na určenie férových cien európskych opcií, pričom tento model bol bez väčších problémov používaný až do tzv. čierneho pondelka v roku 1987, kedy boli pozorované výrazné odchýlky medzi reálnymi dátami z trhu a predpoveďami pochádzajúcimi z teórie. Dnes sa ukazuje, že relativistická fyzika nám umožňuje za istých okolností tieto problémy odstrániť a upraviť pôvodný model tak, aby fungoval za nových podmienok. V tejto práci sa snažíme priblížiť úlohu relativistickej fyziky v tejto oblasti finančnej matematiky a vysloviť zopár pozorovaní a nápadov, ktoré by mohli dopomôcť tejto oblasti v jej ďalšom rozvoji.

Klíčová slova: dráhový integrál, hamiltonián, kvantová mechanika, oceňovanie opcií, risk-neutrálna miera, rozdelenie pravdepodobnosti, špeciálna relativita, stochastický proces

Title:
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Abstract: During the several past decades, stochastic processes have become important tool in description of financial assets prices evolution. Black-Scholes-Merton formula, originally published in 1973, provided financial practitioners with mathematics-based way, how to determine fair prices of european style options, which was succesfully used until so called Black Monday in 1987, when crucial dicsrepancies between real market data and theory were observed. As actual research shows, relativistic physics is able to get rid of these problems, obviously, at some expense. In this work, we try to elucidate the role of relativistic physics in the field of financial mathematics and come up with some observations and ideas, which could help this field in its further development.

Key words: hamiltonian, option valuation, path integral, probability distribution, quantum mechanics, risk-neutral measure, special relativity, stochastic process

Contents

Introduction

In the literature, one can find many sources dealing with problem of relativistic generalizations of stochastic processes. The reason to motivate people for making such relativistic generalizations is obvious. If the theory of classical mechanics was not Lorentz covariant, no one tried to doubt about the Maxwell's theory of electricity and magnetism formulated in the Maxwell's equations, which seemed to be Lorentz covariant, however not covariant with respect to Galilean transformations. It was classical Newtonian mechanics, which was reformulated to the form, which already possess the property of Lorentz covariance. This treatment of mechanics is known as Special Relativity. The same process was then applied to the theory of random motion. The original papers of Einstein and Smoluchowski began with the mathematically rigorous treatment of this problem and came to so called *diffusion equation*, which is the partial differential equation for probability density of particle being in some coordinate $\boldsymbol{x}$ at time t. The solution of such an equation with appropiate initial conditions is nothing but Wiener process. They began solely with describing particles motion in the configuration space of positions, or to be more correct, involving the time into the space, the Minkowski space-time. There has been also another approach. One of them was to begin with the differential equation for particle's momentum, which was nothing but adding the random term to the original differential equation representing Newton's second law of motion. If the mometum is calculated, one can calculate the position from the coupled equation additionally. This approach is called *phase-space approach* and the solution for momentum is called *Ornstein-Uhlenbeck process*. In this thesis, we will revise both of this approaches and present its relativistic generalizations. The relativistic generalization of the diffusion equation in one dimension also called *telegrapher's equation* is then used to relativistically generalize the Black-Scholes-Merton valuation model. This model is stochastic model, hence equivalent to quantum theory via extended Wick's rotation. However, as it will be obvious later, this correspondence can be ambiguous. It allows us to work directly with the option price as well as with probability distribution of underlying security. Both approaches are presented and the Dirac equation in 1+1 dimensions equivalence to telegrapher's equation is then applied to the derivation of relativistic generalization of pricing Hamiltonian. Some ideas and results concerning older Klein-Gordon equation equivalence are extended, too. As the Schrodinger approach to treat quantum mechanics is equivalent to path integral interpretation, we present the path integral equivalence to Feynman-Kac formula and its use to calculate the price of the option in relativistic case and consequences of this approach. At the very end of the thesis, we offer and revise some ideas, which could possibly remove the undesirable consequences arising from the relativistic quantum mechanics approach.

Chapter 1

Phase space relativistic stochastic processes

1.1 Nonrelativistic Brownian motion

Firstly mentioned by Jan Ingenhousz in 1784 (as irregular motion of coal dust particles on the alcohol surface), motion of mass particle immersed in the fluid was the subject of interest for many people. Starting with botanist Robert Brown in 1827, who used microscope to observe pollen particles performing random motion in the liquid, it took several decades to successfully describe this motion using standard means of mathematics. Mainly, for our purposes, it is very important to remind French mathematician Louis Bachelier and his 1900 PhD. thesis *"Theory of speculation"* [59] where he proposed that logarithms of stock prices evolution are concrete realizations of the Wiener process.
First physical explanations of Brownian motion brought physicists Sutherland, Einstein [1] and Smoluchowski [60].
In [1] Albert Einstein presumed, that the length of the step performed by mass particle during its random motion in the liquid is the itself random variable. From this and the fact, that the steps are sufficiently small for the use of Taylor expansion but sufficienty big to be able consider myriad of collisions still independent, he obtained the equation:

$$\frac{\partial\rho(x,t)}{\partial t} = \mathcal{D}\frac{\partial^2\rho(x,t)}{\partial x^2} \tag{1.1}$$

with $\rho(x,t)$ being the probability of finding the particle at the position x at time t (Einstein confined himself to the case of the probability being dependent only on position and time). But this was well-known diffusion equation with diffusion coefficient D and solution

$$\rho(x,t) = \frac{1}{\sqrt{4\pi\mathcal{D}t}}e^{-\frac{x^2}{4\mathcal{D}t}} \tag{1.2}$$

So the position of the particle randomly moving in the liquid was given by Wiener process with $\mathbb{E}(X_t) = 0$ and $Var(X_t) = \mathbb{E}(X_t^2) = 2Dt$. In other words, the position of the particle spreads out from the origin of the reference frame linearly in time.
Albert Einstein has also drawn the way how to calculate the relationship between temperature of the liquid and the diffusion coefficient. It is based on the fact, that after some relaxation time the dynamic equilibrium between the particle and the the enviroment establishes. This leads to the so called *Einstein relation*:

$$\mathcal{D} = \frac{k_B T}{6\pi\eta r} \tag{1.3}$$

where the T represents equlibirium temperature, η the dynamic viscosity of the fluid and r is the radius of the particle immersed in the fluid (the spherical shape of the particles was presumed).

1.2 Ornstein-Uhlenbeck process

The Einstein's method for describing the Brownian motion through probability distribution of particle's position is not the only one. Famous French physicist Paul Langevin [3],[4] and Dutch physicists Ornstein and Uhlenbeck in [2] unveiled another approach on how to describe the Brownian motion. They proposed phenomenological equation expressing both drag force as well as random fluctuations in the generalized form of classical Newton's second law of mechanics [2, 3, 4, 5, 6, 7, 9, 10, 11]

$$m\frac{dv(t)}{dt} = -\gamma m v(t) + \sqrt{2D}.\xi(t) \tag{1.4}$$

where m is the mass of the particle randomly moving in the fluid, v is its velocity, while ξ represents so called stochastic driving function usually considered to be Gaussian white noise; γ and D stand for friction coefficient and noise parameter, respectively. In the literature, authors work with different definitions of the noise parameter, hence the equation 1.4 usually gets formally different forms(compare e.g. [10] and [7] or [12]). To compare previous work of Einstein, one must also be able to calculate the position process. If one calculates the process of particle's velocity, then it is possible to obtain position from

$$\frac{dx(t)}{dt} = v(t). \tag{1.5}$$

Gaussian white noise is characterised by

$$\mathbb{E}[\xi(t)] = 0, \tag{1.6}$$

$$\mathbb{E}[\xi(t)\xi(s)] = \delta(t-s). \tag{1.7}$$

The first condition expresses the fact that in average, the motion of the massive particle should be determined by classical equation of motion with drag force only while the second one says that collisions of the particle with the particles of the fluid should be independent on each other.

1.3 Relativistic Ornstein-Uhlenbeck process

Up to now, from the beginning of the 20th century and times of Einstein and Smoluchowski, in the literature one finds several attempts on how to make the theory of Brownian motion consistent with the framework of special relativity. We just remind the approach of Schay, Dudley and Hakim [13], followed by Ben Ya'acov and Boyer. The more recent publications include Debbash and al., approach of Dunkel and Hänggi or Mukopadhyay et al. [14], who consider Brownian particle even in the presence of electromagnetic field in 1+1 dimension. In this section we briefly review the most recent contributions to this topic. For the purposes of derivative pricing, it is useful to be interested in 1+1 dimensional case only.

1.3.1 Dunkel-Hänggi approach

Dunkel and Hänggi in [7] began with the system of uncoupled stochastic differential equations

$$\frac{dx(t)}{dt} = v(t), \tag{1.8}$$

$$m_0\frac{dv(t)}{dt} = -\nu m_0 v(t) + L(t). \tag{1.9}$$

Comparing this system with the one from previous section, we identify ν with friction coefficient and Langevin force $L(t)$ with $(2D)^{1/2}\xi(t)$. Therefore we find

$$\mathbb{E}[L(t)] = 0, \tag{1.10}$$

$$\mathbb{E}[L(t)L(s)] = 2D\delta(t-s). \tag{1.11}$$

Physically, the above equations have clear interpretation as the equations of motion, however, mathematically, it is more tricky. The problem is, that one can not guarantee, whether the solution of the equation 1.9 obtained by formal integration exists, or if exists, on which domain it exists and whether it is sufficient to have only initial velocity condition to obtain unique solution. It can be shown [15], that it is possible to be written in the form

$$d[m_0 v(t)] = -\nu m_0 v(t)dt + dW(t) \tag{1.12}$$

where $W(t)$ is one-dimensional Wiener process multiplied with constant $\sqrt{2D}$ hence having the variance of $\sigma^2 = 2Dt$ with $dW(t) = W(t+dt) - W(t)$ being its increments. This formulation is already more understandable, as the notion of stochastic integral is already known.
The original equations are then made Lorentz covariant. This is reached by the observation, that in the frame of coordinates temporarily connected with the test particle moving in the fluid (exactly at time t), the Lorentz covariant equations generally reduce to its nonrelativistic counterparts. [17] Firstly, the deterministic part of equation 1.9 is treated (i.e. the thermal effects of the heat bath are not considered). Then the equation 1.9 simplifies to the form

$$m\frac{dv(t)}{dt} = -\nu m_0 v(t) \tag{1.13}$$

Exploiting the classical Galilean coordinate transformation, this equation alongside the expression for kinetic energy of the particle can be written in the comoving frame in laboratory frame at time t (all quantities in this frame are marked with *).[1]

[1] From our point of view, it is important to express the fact, that Hänggi and Dunkel did not explicitly explained the reader their flow of ideas neccessary to derive this key equations. Firstly, it is good to realise, that heat bath frame moves with constant average velocity V with respect to some thought absolute frame connected with the static borders of the heat bath. In this concrete reference frame, using the asumption, that $V << c$ (and hence Galilean transformation), the original equation of motion gets the form

$$m_0\frac{dv'}{dt'} = -\nu m_0(v'(t') \pm V) \tag{1.14}$$

Then, in arbitrary reference frame moving relatively either to borders of heat bath or heat bath itself , the heat bath liquid will have constant average velocity $\tilde{V}$. The equation of motion then reads

$$m_0\frac{d\tilde{v}}{d\tilde{t}} = -\nu m_0(\tilde{v}(\tilde{t}) \pm \tilde{V}) \tag{1.15}$$

$$m_0 \frac{dv_*}{dt_*}(t) = -\nu m_0 (v_*(t) - V_*) \tag{1.19}$$
$$\frac{dE_{k*}}{dt_*} = -\nu m_0 v_*^2(t_*(t)) - \nu m_0 v_*(t_*(t)) V_*$$

The above equations are the equations for motion and change of kinetic energy of any particle moving in liquid without considering random effects of the enviroment in arbitrary reference frame which is in relative motion to heath bath frame. In concrete time t, when according to assumptions used by Hänggi and Dunkel, $v_*(t_*(t)) = v_*(t) = 0$, the original equations become

$$m_0 \frac{dv_*}{dt_*}(t) = \nu m_0 (V_*) \tag{1.20}$$
$$\frac{dE_{k*}}{dt_*} = 0$$

In order to prepare these equations in the form suitable for relativistic generalization (i.e. "(1+1)-vector" formulation), the standard relativistic notation is used (e.g. [17]) and the above equations are written in the form

$$\frac{d{p_*}^\alpha}{d\tau} = {f_*}^\alpha \tag{1.21}$$

where

$${F_*}^\alpha = -m_0 \nu \begin{pmatrix} 0 \\ v_* - V_* \end{pmatrix} \tag{1.22}$$

and $d\tau = dt_* \left(\sqrt{1 - \frac{v_*}{c}} \right)$ is proper time of immersed particle. Correctness of this "ansatz" stands out from the fact, that in non-relativistic limit, when gamma factor $\gamma = \frac{1}{(\sqrt{1-\frac{v_*}{c}})}$ is in very vicinity of one, the previous equation acquire the form of the original equations 1.20. In this generalized version, the $p_*^\alpha = (\frac{E_*}{c}, p_*) = (\frac{E_*}{c}, mv_*)$ with E_* being total energy of the particle and $m = \frac{m_0}{\sqrt{1-\frac{v_*}{c}}}$. Then it is simple to

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At this moment, one can choose the concrete reference frame, which is comoving with the Brownian particle at heath bath laboratory time t. This frame is signed with $*$. So finally we get the equation

$$m_0 \frac{dv_*}{dt_*}(t) = -\nu m_0 (v_*(t_*(t)) \pm V_*), \tag{1.16}$$

where t expresses the fact, that this equation is the special case of relativistic equations of motion in this time, when the frame is comoving with the particle.
The derivation of the first time derivative of kinetic energy exploits its not very well known and non-trivial Galilean transformation rule in the form [18]

$$\frac{d\tilde{E}_k}{d\tilde{t}} = \frac{dE_k}{dt} + R\boldsymbol{F(t)}.\boldsymbol{u} \tag{1.17}$$

where R represents orthogonal square matrix with dimension correspoding to those of the system configuration space (rotations), $\boldsymbol{F(t)}$ is the deterministic drag force acting on the immersed particle and finally the velocity vector $\boldsymbol{u}$ characterizes the uniform motion of the one reference frame with respect to another. In the notation used above, we make the identification $\boldsymbol{F(t)} = -\nu m_0 v(t)$, $\boldsymbol{u} = V$, $R = 1$. Therefore, the derivative of kinetic energy in $*$ frame has the form

$$\frac{dE_{k*}}{dt_*}(t) = -\nu m_0 v_*^2(t_*(t)) - \nu m_0 v_*(t_*(t)) V_* \tag{1.18}$$

observe, that in limit $v_* \to 0$, the latter equations really become former ones 1.20.
To obtain manifestly covariant formulation using the 2-vector properties of 2-velocity, the correction, so called friction tensor ${\nu_*}^{\alpha}{}_{\beta} = \begin{pmatrix} 0 & 0 \\ 0 & \nu \end{pmatrix}$, is introduced. With the help of this tensor, the equations in $*$ frame have the form

$$\frac{d{p_*}^{\alpha}}{d\tau} = -m{\nu_*}^{\alpha}{}_{\beta}(u_*^{\beta} - U_*^{\beta}). \tag{1.23}$$

where one finds only quantities which are already manifestly covariant with respect to application of the Lorentz group. At this point, it is not difficult to apply particular Lorentz transformation on the last equations and express these in corresponding reference frame.
The relativistic covariant equations we obtained have given us generalization of deterministic part so far. To be able to use relativistic generalization in the field of econophysics, one must involve the stochastic part too. One would expect, that the relativistic distribution will be different from the standard distribution of Wiener process used in the classical case. In [7], the same procedure as with deterministic part generalization is used, e.g. the fact, that in coordinate frames temporarily comoving with Brownian particle, relativistic equations reduce to the nonrelativistic counterparts. In the following parts of the text, we will try to elucidate their flow of ideas and briefly summarize these results. To be consistent with deterministic part generalization formulated in terms of 2-vectors, one must also think of stochastic part as a 2-vector

$$dW^{\alpha} = (0, dW). \tag{1.24}$$

In the reference frame, connected with the moving particle in heath bath laboratory time t, one may assume, that the 0'th component of the generalized random part will be

$$dW_*^0(t) = 0. \tag{1.25}$$

At the first sight, this is not obvious. In the deterministic part, where the energetic term is introduced, one explicitly uses the stochastic differential equation comprising stochastic term, the increment of Wiener process. The reason, why one needs not to work with the stochastic term is, that in the first step of the whole process, only the deterministic part of relativistic generalization is treated and so Wiener process increment is intentionally not imported to equations. In this situation however, it is possible for stochastic term to occur in the zero'th component and therefore it is not clear, why one should completely eliminate stochastic part of this, let's say, energy component. The reason is, that the zeroth component will be equal to zero as the total energy of the particle is constant [17]. Probability distribution density increment for nonrelativistic stochastic part, i.e. Wiener process increment in comoving frame reads:

$$f_*(dW_*(t)) = \frac{1}{\sqrt{4\pi m_0 \nu k T dt_*}} e^{-\frac{dW_*(t)^2}{4m_0\nu kT dt_*}}. \tag{1.26}$$

where we used the fact that $D = m_0 \nu kT$. From this formula, one has to properly modify it to obtain joint probability distribution for (1+1) stochastic force vector on the one hand and ensure relativistic covariance on the other. The joint probability distribution in comoving frame has form

$$f_*(dW_*^{\alpha}(t)) = \frac{1}{\sqrt{4\pi D dt_*}} e^{-\frac{D_{*\alpha\beta} dW_*^{\alpha}(t) dW_*^{\beta}(t)}{2}} \delta(dW_*^0(t)). \tag{1.27}$$

where $D_{*\alpha\beta}$ is correlation tensor

$$D_{*\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2Ddt_*} \end{pmatrix} \tag{1.28}$$

The Dirac delta function expresses the condition of zeroness on the zero'th component of the stochastic force. After integration one obtains

$$f_*(dW_*^0(t)) = \delta(dW_*^0(t)). \tag{1.29}$$

which coresponds with the assumption that zero'th component of the stochastic force (1+1) vector should be zero; the probability is fully concentrated at one point. The last probability density can be equivalently written as

$$f_*(dW_*^\alpha(t)) = \frac{c}{\sqrt{4\pi Ddt_*}} e^{-\frac{D_{*\alpha\beta} dW_*^\alpha(t) dW_*^\beta(t)}{2}} \delta(u_{*\alpha}(t) dW_*^\alpha(t)) \tag{1.30}$$

This joint probability denstity is now changed by imposing Lorentz transformation on it and so made valid for arbitrary Lorentz frame. This general form is written as

$$f(dW^\alpha(\tau)) = \frac{c}{\sqrt{4\pi Dd\tau}} e^{-\frac{D_{\alpha\beta} dW^\alpha(\tau) dW^\beta(\tau)}{2}} \delta(u_\alpha(\tau) dW^\alpha(\tau)) \tag{1.31}$$

where $D_{\alpha\beta}$ has now the general form

$$D_{\alpha\beta} = \frac{1}{2D\tau}(\eta_{\alpha\beta} + \frac{u_\alpha u_\beta}{c^2}). \tag{1.32}$$

Expressions for energy derivative, particle position and momentum evolution in the original classical Ornstein-Uhlenbeck interpretation, however already relativistic covariant are then

$$dx^\alpha(\tau) = \frac{p^\alpha(\tau)}{m_0} d\tau \tag{1.33}$$

$$dp^\alpha(\tau) = \nu_\beta^\alpha (p^\beta(\tau) - m_0 U^\beta) d\tau + dW^\alpha(\tau) \tag{1.34}$$

with general form of friction tensor

$$\nu_\beta^\alpha = \nu(\eta_\beta^\alpha + \frac{u^\alpha u_\beta}{c^2}). \tag{1.35}$$

Hänggi and Dunkel have in [7], [8]therefore given us very interesting way how to make nonrelativistic equations of motion relativistic covariant. By simple Lorentz transformations can one then obtain momentum distribution from view of the observer in every inertial reference frame. Because the underlying assets in Black-Scholes model are originally distributed according to log-normal distribution, we will try to derive relativistic distribution for particle position. Our aim is to obtain stochastic differential equation equivalently represented by Fokker-Planck equation which should provide us with generalized normal Gaussian distribution through some reasonable analytical solution of above mentioned equations. Despite all observations we presented until now and effort to express the stochastic differential equations in the formalism of 2-vectors one can immediately verify, even without the arguments with relativistic limit of laws of motion for particle in the co-moving frame, that the whole theory is in fact relativistic [6]. If one writes the Langevin equations for 2-vector of momentum

$$p^\mu = \begin{pmatrix} \frac{E}{c} \\ p \end{pmatrix} \tag{1.36}$$

then if one assumes, that the future description of relativistic particle will allow the momentum $p^1 = p$ to acquire values $p \in \mathbb{R}$, then for the velocity of the particle we get

$$\frac{|p|}{p_0} = \frac{|m_0\gamma(v)v|}{\frac{E}{c}} = \frac{m_0\gamma(v)|v|}{\frac{m_0\gamma(v)c^2}{c}} = \frac{|v|}{c} \tag{1.37}$$

and finally

$$\frac{|v|}{c} = \frac{|p|}{\frac{E}{c}} = \frac{|p|}{\sqrt{p^2 + m_0^2c^2}} < 1. \tag{1.38}$$

for all admissible values of momentum $p \in \mathbb{R}$, which implies, that the whole theory based on the only assumption with momentum values is already implicitly relativistic.

1.3.2 Generalized position distribution - approximation method approach

First way how one can try to resolve this task is to exploit equivalence between partial differential equations of Fokker-Planck type and stochastic Langevin equations. In accordance with [7], [8], we write the first component of 1.33 in laboratory frame of heath bath

$$dp = -\nu p dt + dW(t), \tag{1.39}$$

where the distribution of random increments dW is now not normal. This distribution can be derived from 1.31 and exact calculation is attached in Appendix D. We exploit the fact that random variable

$$Y(t) = \frac{dW}{\sqrt{\gamma}}, \tag{1.40}$$

where γ stands for Lorentz factor has normal distribution known from non-relativistic model. Hence it is possible to rewrite the last equation as

$$dp = -\nu p dt + \sqrt{\gamma} L(t) dt \tag{1.41}$$

where we apart from the last remark used the symbolic relation between Wiener process increments and stochastic force $L(t)$ from previous subsection. In this moment one can rewrite this equation by formally dividing it time increment dt and get

$$\frac{dp}{dt} = -\nu p + \sqrt{\gamma} L(t). \tag{1.42}$$

This procedure was done solely in formal manner and in the same way as one works with differentials in non-relativistic theory. The reason why one rewrite the mathematical *correct* interpretation into this more physical and heuristic form is the known correspondence between Fokker-Planck equation and stochastic differential equations [11], [16]. Hence it is natural, aiming to obtain stochastic differential equation for position in the form

$$\dot{x} = h(x,t) + g(x,t)\Gamma(t) \tag{1.43}$$

where $L(t) = \Gamma(t)\sqrt{m_0 \nu k T} = \Gamma(t)\sqrt{D}$. The corresponding Fokker-Planck equation is

$$\dot{f}(x,t) = \Big(-\frac{\partial}{\partial x} D^{(1)}(x,t) + \frac{\partial^2}{\partial x^2} D^{(2)}(x,t)\Big) f(x,t) \tag{1.44}$$

where drift and diffusion coefficients $D^{(1)}(x,t)$ and $D^{(2)}(x,t)$ are dependent of used stochastic calculus discretization rule. If we were able to obtain relativistic generalized equation for position of particle, then we would get coresponding F-P equation. The last task would be then to solve (analytically or numerically) that equation and obtain position distribution for particle, which would be relativistic invariant. So we continue with conveniently rewriting equation 1.42 as

$$\frac{d(\gamma m_0 v)}{dt} = -\nu\gamma m_0 v + \sqrt{\gamma}L(t). \tag{1.45}$$

After explicit calculation of the derivative (although with use of Stratonovich derivative rule) on the left side one obtains

$$m_0\Big[(\frac{\gamma^3}{c^2}\dot{x}^2 + \gamma)\ddot{x}\Big] = -\nu\gamma m_0\dot{x} + \sqrt{\gamma}L(t). \tag{1.46}$$

Already in this stage the potential obstacle may be seen immediately. Not only because this equation is not in the form which it can be seen in [11] or [16] and therefore easily converted into corresponding F-P equation, but also due to the fact, that this equation is most probably neither not explicitly solvable (see for instance Chapter 4 in[19]). To see this, one can use the assumption aiming to remove the second derivatives. If we assume that in the equilibrium, the inertial force of the particle is much weaker than the frictional one, i.e. all expressions containing $m_0\ddot{x}$ disappear as they are very small, then in these part of motion the equation of motion reduces to

$$\dot{x} = \frac{L(t)}{\nu m_0\sqrt{\gamma}}. \tag{1.47}$$

At first sight, this form seems to be much easier to manipulate with, but the opposite is true, as the Lorentz factor implicitly depends on position derivative variable (i.e. the velocity of the particle). One could try to solve this equation using known numerical methods, the problem is, that this solution would not be exact. This could incorporate systematic error in the generalized option valuation model from the very beginning. As we mentioned, the calculation of the last derivative was done with use of Stratonovich calculus rules. However, in the world of finance, the Itô calculus is used more frequently. This follows from the known fact, that so called discretization rule exploited by Itô calculus is better unterstandable when dealing with prices of stocks (as the prices of stock is known at most in the present moment and can not be determined exactly for future times). Therefore, it is natural to try apply Itô distretization rule also in this case and seek for diferential equation which ideally won't be difficult to solve. One then gets following expression

$$m_0\Big[(\gamma + \frac{\dot{x}^2\gamma^3}{c^2})\ddot{x} + \frac{1}{2}\frac{\partial}{\partial\dot{x}}(\gamma + \frac{\dot{x}^2\gamma^3}{c^2})G^2\Big] = -\nu\gamma m_0\dot{x} + \sqrt{\gamma}L(t). \tag{1.48}$$

where we used Itô lemma (see Appendix), in which we naturally assigned $Y(X(t),t) \equiv u(X(t),t) = m_0 v\gamma$ with $X(t) \equiv \dot{x}$.

However, as the exact stochastic differential for velocity is not known, the coefficient G and hence its square can not be calculated exactly. But what can already be seen, one of two cases will occur.

Firstly, the coefficient G will not comprise second derivative of velocity. Together with fact, that the derivative before it certainly does not comprise expression $m_0\ddot{x}$ it implies that the result will be much more complicated than the expression with use of classic chain rule from Stratonovich calculus, therefore it does not solve our problem. On the contrary, if the square G^2 will comprise the "acceleration" of particle multiplied by its rest mass, then with the assumption $m_0\ddot{x} \approx 0$ in the equilibrium state the left side of the latter equation 1.48 will be the same as the former 1.46, i.e. equal to zero.

All these observations simply say that generalized Langevin equation for position must be obtained in different way.

1.3.3 Generalized position distribution - substitution method approach

In the quest for generalized Langevin equations in laboratory frame connected with the heat bath one may try to directly substitute for the momentum. If we write the second equation in 1.33 in heat bath frame, we get [[7], eq. 38a]

$$dp(t) = -\nu p dt + dW(t) \tag{1.49}$$

so we begin with the same differential as in the previous subsection. In the next step we use the formal solution to this SDE [[7], eq.45]

$$p(t) = p_0 e^{-\nu t} + e^{-\nu t}\int_0^t e^{\nu s} dW(s) \tag{1.50}$$

with $p_0 \equiv p(0)$ being initial condition for momentum. Another ingredient is the first of two coupled equations 1.33 although now in the heat bath frame

$$dx(t) = \frac{p(t)dt}{m_0\gamma} \tag{1.51}$$

After inserting the formal solution into this form one gets

$$dx(t) = \frac{[p_0 e^{-\nu t} + e^{-\nu t}\int_0^t e^{\nu s} dW(s)]dt}{m_0\gamma} \tag{1.52}$$

and finally after relocation of coefficients

$$dx(t) = \frac{p_0 e^{-\nu t}}{m_0\gamma(p)}dt + \frac{e^{-\nu t}dt\int_0^t e^{\nu s} dW(s)}{m_0\gamma(p)} \tag{1.53}$$

where we emphasized Lorentz's factor dependence on momentum.
It is clear, that ideally one would want to compute the stochastic integral in the last equation and proceed with further relocations to obtain well known form of Langevin equation, although now with generalized friction and drift coefficients. But the computation of stochastic integral is possible only in the case when the generalized (originally Gaussian) distribution is semimartingale. For proving this, one can exploit the fact, that every Lévy process is semimartingale [29] and try to prove, that the process is of Lévy kind. For this sake, we remind definition of Lévy process.

Definition 1.3.1. *Let $X = (X(t), t \geqslant 0)$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then this process is called the process with independent increments if for each $n \in \mathbb{N}$ and each $0 \leqslant t_1 < t_2 \leqslant ... < t_{n+1} < +\infty$ the radnom variables in the form $X(t_{j+1} - X_j), 1 \leqslant j \leqslant n)$ are independent.*

Definition 1.3.2. *Let's have the stochastic process $X = (X(t), t \geqslant 0)$ defined on probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The one say that process $X(t)$ has stationary increments, if $X(t_{j+1}) - X(t_j) \overset{d}{=} X(t_{j+1} - t_j) - X(0)$ for each fixed $n \in \mathbb{N}$ and $1 \geqslant j \geqslant n$.*

After this revision of basic notions known from probability theory (full introduction to theory of stochastic processes can be found e.g. in [26], [38], [25]), one can define the proper Lévy stochastic process

Definition 1.3.3. *Stochastic process $X = X(t)$ is called a Lévy process if the following conditions are held*

- $X(0) = 0$ *almost surely,*
- X *independent and stationary increments,*
- X *is so called stochastically continous, i.e. for all* $a > 0$ *and for all* $s \leqslant 0$

$$\lim_{t\to s} \mathbb{P}(|X(t) - X(s)| > a) = 0.$$

As the stationarity is presumed, this immediately implies, that the last condition may be modified to the equivalent form

$$\lim_{t\to 0} \mathbb{P}(|X(t)| > a) = 0.$$

So one can proceed in this way, but as one inspects the whole theory more carefully once more, then exploiting the properties of the random increment $dW(t)$ from above the last equation can be written as

$$dx(t) = \frac{p_0 e^{-\nu t}}{m_0\gamma(p)}dt + \frac{e^{-\nu t}dt \int_0^t e^{\nu s}\sqrt{\gamma(p)}dW(s)}{m_0\gamma(p)} \tag{1.54}$$

where $dW(t)$ already is the normal Gaussian increment. So it means, that we can work with the framework of Itô integral as revised in App.B. But even if we utilize the Itô lemma from there, already without computing one sees, that the whole result have to be multiplied by time increment dt and therefore adjustment of the latter equation to the form

$$dx^1(t) = h^1(\boldsymbol{x})dt + g^1_s(\boldsymbol{x}) \star dW^s(t) \tag{1.55}$$

is not possible as the first term with time increment dt would comprise some function of random variable $W(t)$.

1.3.4 Generalized position distribution - multiple variables approach

As one could see, the biggest problems arisen in the process of obtaining generalized position distribution is either the fact, that the sum of generalized Wiener increments with their parameters implicitly dependent on time is not generally the same kind of generalized distribution with some specific coefficients calculated from the known ones in original generalized distributions, or the standard approach when trying to calculate mean values together with other moments originally needed as sufficient information to determine exact form of Gaussian distribution will not work as coefficients in front of random variables are not constants anymore. Instead, they are random variables, too.
In order to partially circumvent this problem, one can exploit theory of Fokker-Planck equation in multiple dimensions. Specifically, the Klein-Kramers equation for Brownian motion in one dimension [11]

$$\frac{\partial f(x,v,t)}{\partial t} = \Big[-\frac{\partial}{\partial x}v + \frac{\partial}{\partial v}[\nu v + f'(x)] + \frac{\nu kT}{m}\frac{\partial^2}{\partial v^2}\Big] f(x,v,t) \tag{1.56}$$

may be generalized.
If we use generalized Langevin equations in the laboratory heat bath frame, i.e.

$$\frac{dx(t)}{dt} = \frac{p(t)}{m_0\gamma} \tag{1.57}$$

$$\frac{dp(t)}{dt} = -\nu p(t) + \sqrt{\gamma}L(t) \tag{1.58}$$

then directly exploiting multi-variable theory for FPE (see Appendix and [11]) and realizing the relation $p(t) = m_0\gamma v(t)$ one obtains generalized PDE for joint probability distribution in phase space for position and velocity

$$\frac{\partial f(x,p,t)}{\partial t} = \Big[-\frac{\partial}{\partial x}\Big(\frac{p(t)}{m_0\gamma(p)}\Big) + \frac{\partial}{\partial p}\Big(\nu p(t)\Big) + \frac{\partial^2}{\partial p^2}\Big(\gamma(p)D\Big)\Big]f(x,p,t) \tag{1.59}$$

The first thing that has to be verified is whether in non-relativistic limit one recovers the non-relativistic original equation. At this point, relations $p = m_0\gamma v$ and $\gamma = \frac{1}{\sqrt{(1-\frac{v^2}{c^2})}} = \sqrt{1+\frac{p^2}{m_0^2c^2}}$ have to be reminded. Non-relativistic limit means that $v << c$, hence $\gamma \to 1$ and finally $p \to m_0 v$. After adjusting all terms in the last equation, one recovers the original non-relativistic equation as written above. One now want tries to resolve the generalized Klein-Kramers equation. Stationary solution was already presented in [6]. If one has general Langevin-type equation

$$dp(t) = -\alpha(p)pdt + [2D(p)]^{1/2} \bullet dW(t) \tag{1.60}$$

where $dW(t)$ has the same distrbution as that in App. C and $\bullet$ sign to represent Hänggi-Climontovich approach to stochastic integration, together with coupled differential for particle position

$$dx^{\alpha}(t) = \frac{p^{\alpha}(t)}{m_0\gamma}dt \tag{1.61}$$

for $\alpha \in \{0,1\}$ for 1-dimensional problem one gets with respect to post-point (i.e. Hänggi-Klimontovich) discretization rule following Fokker-Planck equation

$$\frac{\partial f}{\partial t}(t,x,p) + \frac{p}{E}\frac{\partial f}{\partial x}(t,x,p) = \frac{\partial}{\partial p}\Big[\alpha(p)pf(t,x,p) + D(p)\frac{\partial f}{\partial p}(t,x,p)\Big] \tag{1.62}$$

where $E = \sqrt{p^2c^2 + m_0^2c^4}$ and represents the total energy of particle. If one now wants to transform this equation to the world of Itô discretization, then one has to use following transformation relation

$$\alpha_{Hanggi-Klimontovich}(p)p = \alpha(p)_{Ito}p + \frac{\partial D(p)}{\partial p}. \tag{1.63}$$

After substituting this into previous partial differential equation, one gets exact the same generalized equation as that written above. So up to now we verified the consistency of theory from two different sources and proved, that the generalized equation 1.59 has correct non-relativistic limit behaviour. The reason, why this was important is the fact, that if one has the equation in the form 1.62, then one can immediately write stationary solution for it

$$f(x,p) = \mathfrak{N}\exp\Big[-\int_{p_0}^{p}\frac{\alpha(p')p'}{D(p')}dp'\Big] \tag{1.64}$$

where $\mathfrak{N}$ stands for normalization constant and p_0 is boundary condition. Substituting for original Stratonovich coefficient one gets

$$f(x,p) = \Re \exp\Big[-\int_{p_0}^{p} \frac{\alpha_{Ito}(p')p' + \frac{\partial D(p')}{\partial p'}}{D(p')} dp'\Big]. \tag{1.65}$$

Now substituting for coefficient $\alpha_{Ito}(p') = \nu p'$ and $D(p') = \gamma(p')D$ one gets

$$f(x,p) = \Re \exp\Big[-\int_{p_0}^{p} \Big(\frac{\nu p'^2}{\gamma(p')D} + \frac{1}{\gamma(p')D}\frac{\partial}{\partial p'}(\gamma(p')D)\Big)dp'\Big] \tag{1.66}$$

and because the diffusion coefficient D is constant, after cancelling it one gets

$$f(x,p) = \Re \exp\Big[-\int_{p_0}^{p} \Big(\frac{\nu p'^2}{\gamma(p')D} + \frac{1}{\gamma(p')}\frac{\partial}{\partial p'}(\gamma(p'))\Big)dp'\Big] \tag{1.67}$$

Generally, the last equation expresses relativistic generalized stationary joint probability distribution for both position and momentum of relativistic particle randomly moving in liquid. However, this would be valuable for us only in the case, when coefficient $\alpha(t,x,p)$ and $D(t,x,p)$ would be explicitly dependent on position of the particle which is not our case, as both the coefficients were explicitply dependent on momentum only. To be more clear, in our case

$$\alpha(t,x,p) = \nu \tag{1.68}$$

together with second coefficient

$$D(t,x,p) = \gamma(p)D. \tag{1.69}$$

with $D = k_B T m_0 \nu$ being the noise parameter known from relativistic model in [7] and [8] and mentioned above. Now it can be immediately seen, that not only both the coefficients are not position dependent, but one of them is constant. Therefore, stationary solution presented for instance in [6], [7] [8] and mentioned above are not useful for finding explicit position distribution for relativistic particle. We remind that all these calculations were made in laboratory heat bath frame.

When it comes to analysis of all techniques or attempts from obtaining generalize distribution above, one can conlude, that even if we weren't able to derive the analytic expression for distribution function for particle's position, we are able to say, that the resulting process would certainly not be Markovian. This was already mentioned in [31], [32], [6], [56]. In the first two works there is proven that if one tries to formulate stochastic problem in the space-time of particle which would be covariant under application of Lorentz group elements, this is impossible unless the process itself is non-Markovian. In the language of diffusion equations, the generalized diffusion equation for probability density of such a process would certainly contain derivatives order higher than one. However, this could be problem in generalized valuation process derivation, as the original Black-Scholes model assumes that stock prices are driven by Markov process and obviously, this has it's logic. Markovian assumption incorporates the notion of unpredictability of prices evolution in the model. In other words, trying to use hypotetically obtained process from above mentioned ways would give as a non-Markovian process, which couldn't be used in the valutation model, as this would mean, that the price of the stock in the following trading day could be potentially influenced by some price occured in the past. However, we will see, that second order partial differential equation are important in the world of valuation but in little different way.

1.4 Covariant formulation of diffusion equation

Aiming to obtain generalized position ditribution of immersed particle, we worked with covariant formulation of stochastic differential equation according to [7], [8] up to now and immediately after deriving it we transited ourselves into laboratory frame connected with heath bath moving with velocity V. Although we argued that the transition from phase space to space-time would not give us generalized Markovian process and thus analytical computation would bring no added value for asset valuation, numerical analysis is still possible and one could possibly utilize method of numerical analysis to get some interesting results. To make the theory of relativistic stochastic processes on phase space complete, one should also present covariant form corresponding Fokker-Planck equation in 1+1 dimension and show that whole theory is fully consistent. We will similarly to [6], [39] start from Langevin equations in the laboratory frame of heath bath. In the first subsection, we only demonstrate reparametrisation with respect to proper time of moving particle.

1.4.1 Relativistic Fokker-Planck equation in proper time

According to App. C we may write them with respect to Itô discretization rule

$$dx(t) = h^1(x, p)dt \tag{1.70}$$

$$dp(t) = h^2(x, p)dt + g^2(x, p) \star dW(t) \tag{1.71}$$

where we used the exact notation as in App. C and expressed all variables in their physical names (i.e. position and momentum). Substituting for the functions h, g with respect to the notation already used above one gets

$$dx(t) = \frac{p(t)}{m_0\gamma(p)}dt \tag{1.72}$$

and for the second coupled equation for momentum

$$dp(t) = -\nu p(t)dt + \sqrt{2D\gamma(p)} \star dW(t) \tag{1.73}$$

again with diffusion constant $D = m_0\nu kT$. Now we can use the corresponding Fokker-Planck equation and get

$$\frac{\partial f(t, x, p)}{\partial t} = \frac{\partial}{\partial x}\Big(- \frac{p(t)}{m_0\gamma(p)}f(t, x, p)\Big) + \frac{\partial}{\partial p}\Big(\nu p(t)f(t, x, p) + \frac{\partial}{\partial p}(\gamma(p)Df(t, x, p))\Big) \tag{1.74}$$

So we obtained special form of corresponding diffusion equation for the set of coupled Langevin stochastic differential equations in laboratory frame (i.e. heath bath frame). The task in this subsection is to reparametrise the FPE in laboratory frame in terms of proper time of moving particle. The reason for doing it is that the whole theory could in the future possibly be extended to the framework of general relativity.
As it can be seen in App.A, we have the following relationship between proper time τ and laboratory time t.

$$\Delta\tau = \gamma(v)\Delta t \tag{1.75}$$

where velocity v stands for the velocity of particle seen by observer in the heat bath laboratory frame. In the infinitesimal limit together with explicitly substituting for Lorentz factor one obtains

$$d\tau(t) = \sqrt{1 - \frac{v^2}{c^2}}dt \tag{1.76}$$

One has to realize that the differential for proper time is now stochastic and not deterministic, as the velocity of the particle is basically different in each instant of time and no one can know the instant velocity of particle in the following time. Before one starts with proper time formulation of FPE, it is convenient to rewrite the space-time Langevin equation in laboratory frame utilizing 2-vector notation

$$dx^{\alpha}(t) = c\frac{p^{\alpha}}{p^{0}}dt. \tag{1.77}$$

where $\alpha \in \{0, 1\}$ and $p^0 = \frac{E}{c}$. Using the 2-vector of momentum one is able to rewrite the relation between proper time and laboratory time.

$$d\tau(t) = \frac{dt}{\gamma c} \tag{1.78}$$

Now one wants to substitute for position as well as momentum. So one clearly has $x^{\alpha}(\tau) = x^{\alpha}(t(\tau))$ and $p^{\alpha}(\tau) = p^{\alpha}(t(\tau))$. Substituting for laboratory time in the 2-vector formulation of position one gets

$$dx^{\alpha}(\tau) = c\frac{p^{\alpha}}{p^{0}}\gamma c d\tau. \tag{1.79}$$

and exploiting the definition of relativistic energy $p^0 = m_0\gamma c$ one immediately gets

$$dx^{\alpha}(\tau) = c\frac{p^{\alpha}}{m_0}d\tau. \tag{1.80}$$

So the position part is already reparametrized. The last part of derivation is to transform the left side of momentum stochastic differential equation. So up to now we were able to obtain

$$dp(\tau) = -\nu p(\tau)\gamma c\tau + \sqrt{2D\gamma(p)} \star dW(\tau) \tag{1.81}$$

which we can again rewrite in the form

$$dp(\tau) = -\nu p(\tau)\frac{p^{0}}{m_0}d\tau + \sqrt{2D\gamma(p)} \star dW(\tau). \tag{1.82}$$

and therefore the only thing we don't know is the Wiener process increment expresses in terms of proper time. To be able to derive relationship between Wiener process increment with respect to proper time and laboratory time one has to proceed according to [29]

Definition 1.4.1 ("Cadlag" process)**.** *A stochastic process X is said to be cadlag if it almost surely has sample paths which are right continuous, with left limits.*

The next ingredient is the definition of stopping time

Definition 1.4.2 (Stopping time)**.** *Let's have a complete probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with given filtration* $\mathbb{F} = (\mathcal{F}_t)_{0 \leqslant t \leqslant +\infty}$. *Then a random variable* $T : \Omega \to [0, +\infty]$ *is a stopping time if the event* $\{T \leqslant t\} \in \mathcal{F}_t$ *for every choice of time* $t \in [0, +\infty]$.

So the stopping time is time, when the arbitrary random game stops after some previously given rules. The condition on the stopping time is just expressing the fact, that the moment of stopping the random process is dependent only on the information known up to that stopping time.

Definition 1.4.3 (Stopped process)**.** *Let be* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *be a complete filtered probability space which fulfills following conditions*

- *$\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$,*
- *$\mathcal{F}_t = \cap_{u>t}\mathcal{F}_u$ for all t, so in other words, the filtration $\mathbb{F}$ is right continuous.*

If one has defined stochastic process X on this probability space and stopping time T, then one can introduce the notion of stopped process in the form

$$X_t^T = X_{t\wedge T} = X_t \mathbb{1}_{t<T} + X_T \mathbb{1}_{t>T}.$$

Definition 1.4.4 (Local martingale)**.** *An adapted, cadlag process X is a local martingale if there exists a sequence of increasing stopping times T_n with $\lim_{n\to+\infty} T_n = +\infty$ almost surely such that $X_{t\wedge T_n}\mathbb{1}_{T_n>0}$ is a uniformly integrable martingale for each n. Such a sequence (T_n) of stopping times is then called a fundamental sequence.*

After reminding some neccessary definitions we can finally proceed to the theorem useful for finding the relationship between the Wiener process increment with respect to proper and laboratory time.

Theorem 1.4.1 (Lévy's theorem)**.** *Let $\mathbf{X} = (X_1, ..., X_n)$ be continuous local martingale such that*

$$[X^i, X^j]_t = \begin{cases} t & \text{if } i = j \\ 0 & \text{if} i \neq j \end{cases}$$

where $[\cdot,\cdot]_t$ represents quadratic covariation of stochastic process. Then $\mathbf{X}$ is a standard n-dimensional Wiener process.

So the task is to utilize the last theorem to prove that if one has time increment in the primed frame of reference $dt' = H(t, p^\alpha)dt$, where $H(t, p^\alpha)$ is is strictly positive, continuous adapted process with respect to time with additional conditions $\mathbb{P}\Big(\int_0^t H(s)ds < +\infty \forall t\Big) = 1$ together with $\mathbb{P}\Big(\int_0^{+\infty} H(s)sds = +\infty\Big) = 1$, then Wiener process increment fulfills the relation

$$dW'(t') = \sqrt{H(t, p^\alpha)}dW(t). \tag{1.83}$$

If one will work with 1-dimensional version of the last equation only and chooses the process $X(t) = \int_0^t \sqrt{H(s, p^\alpha)}dW(s)$, it is again proven in [39] that the condition of quadratic covariance is fulfilled and hence the relationship between Wiener process increments after substituting for concrete H now is

$$dW(t) = \sqrt{\gamma c}dW(\tau) = \sqrt{\frac{p^0}{m_0}}dW(\tau) \tag{1.84}$$

So finally we get reprametrized stochastic differential equation for momentum in the form

$$dp(\tau) = -\nu p(\tau)\frac{p^0}{m_0}d\tau + \sqrt{\frac{2D\gamma(p)p^0}{m_0}} \star dW(\tau). \tag{1.85}$$

Now one is completely prepared for writing the corresponding Fokker-Planck equation reparametrized in proper time

$$\Big(\frac{\partial}{\partial\tau} + c\frac{p^\alpha}{m_0}\frac{\partial}{\partial x^\alpha}\Big)f(\tau, x^\alpha, p^\alpha) = \frac{\partial}{\partial p}\Big[\nu p(\tau)\frac{p^0}{m_0}f(\tau, x^\alpha, p^\alpha) + \frac{\partial}{\partial p}\Big(\frac{D\gamma(p)p^0}{m_0}f(\tau, x^\alpha, p^\alpha)\Big)\Big]. \tag{1.86}$$

1.4.2 Relativistic Fokker-Planck equation in moving frame

The same procedure is usable in situations when one has to write corresponding diffusion equation in moving reference frame. One assumes, that the inertial reference frame is moving with the constant velocity V with respect to laboratory heath bath frame. With the well-known relations from special relativity one can immediately write

$$x'^{\nu}(t) = \Lambda^{\nu}_{\mu} x^{\mu}(t) \tag{1.87}$$

and the 4-momentum fulfills exact the same relation

$$p'^{\nu} = \Lambda^{\nu}_{\mu} p^{\mu}(t) \tag{1.88}$$

where the Lorentz matrix has now dimension 2×2 and the form

$$\Lambda^{\mu}_{\nu} = \begin{bmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{bmatrix} \tag{1.89}$$

The transformation relation for times in both reference frames has the form

$$d(t') = c \frac{\Lambda^{0}_{\mu} p^{\mu}}{p^0} dt \tag{1.90}$$

because in every inertial frame of reference one has

$$dx^{\nu} = c \frac{p^{\nu}}{p^0} dt \tag{1.91}$$

and using the equation 1.88 one gets

$$dt' = c \frac{p'^0}{(\Lambda^{-1})^{0}_{\mu} p'^{\mu}(t')} dt \tag{1.92}$$

with Λ^{-1} represents the matrix of inverse Lorentz transformation. So this is the first ingredient into the transformation. The next one is according to the previous section the increment of Wiener process

$$dW(t') = \sqrt{\Big[\frac{(\Lambda^{-1})^{0}_{\mu} p'^{\mu}}{p'^0}\Big]} dW(t) \tag{1.93}$$

and for drift and diffusion coefficients in momentum part of stochastic differential equation

$$h(t', x', p') = \Big[\frac{(\Lambda^{-1})^{0}_{\mu} p'^{\mu}}{p'^0}\Big] \Lambda^{1}_{\nu} h^{\nu}\Big((\Lambda^{-1})^{0}_{\mu} x'^{\mu}, (\Lambda^{-1})^{1}_{\mu} x'^{\mu}, (\Lambda^{-1})^{1}_{\mu} p'^{\mu}\Big) \tag{1.94}$$

$$g(t', x', p') = \sqrt{\Big[\frac{(\Lambda^{-1})^{0}_{\mu} p'^{\mu}}{p'^0}\Big]} \Lambda^{1}_{\nu} g^{\nu}\Big((\Lambda^{-1})^{0}_{\mu} x'^{\mu}, (\Lambda^{-1})^{1}_{\mu} x'^{\mu}, (\Lambda^{-1})^{1}_{\mu} p'^{\mu}\Big) \tag{1.95}$$

where the inverse Lorentz transformations in the arguments are just expressing the fact that we are evaluating the coordinates in original frame of reference (i.e. without primes) to the moving one. The zero'th components h^0 and g^0 are expressed as

$$h^0(t, x, p) = \frac{h(t, x, p) p}{p^0} + \frac{g(t, x, p)^2}{2} \Big(\frac{1}{p^0} - \frac{p^2}{(p^0)^3}\Big) \tag{1.96}$$

$$g^0(t,x,p) = \frac{pg(t,x,p)}{p^0}. \tag{1.97}$$

To obtain transformed FPE one just has to insert this formulas to the well known standard set up

$$dx'^{\alpha} = \Big(\frac{p'^{\alpha}}{p'^{0}}\Big)dt' \tag{1.98}$$

$$dp'(t') = h(t',x',p')dt' + g(t',x',p') \star dW'(t') \tag{1.99}$$

and after that use the equivalence formulas presented in the App.C. This is only an encoded compact version of the Lorentz boost aplied to Langevin or Fokker-Planck equation in the laboratory frame and one is able to get the same result when evaluating derivatives in the original Fokker-Planck equation with respect to primed coordinates in the moving frame of reference. So one can start from Fokker-Planck equation obtained from Langevin equation with respect to Itô discretization rule

$$\frac{\partial f(t,x,p)}{\partial t} = \frac{\partial}{\partial x}\Big(-\frac{p(t)}{m_0\gamma(p)}f(t,x,p)\Big) + \frac{\partial}{\partial p}\Big(\nu p(t) f(t,x,p) + \frac{\partial}{\partial p}(\gamma(p)Df(t,x,p)\Big) \tag{1.100}$$

It is convenient to rewrite this equation to Hänggi-Klimontovich post-point discretization world for the sake of to be able to explicitly see the coefficient $g(t',x',p')$ in the moving frame. We get

$$\frac{\partial f(t,x,p)}{\partial t} = \frac{\partial}{\partial x}\Big(-\frac{p(t)}{m_0\gamma(p)}f(t,x,p)\Big) + \frac{\partial}{\partial p}\Big(\nu p(t) f(t,x,p) + \gamma(p)D\frac{\partial}{\partial p}(f(t,x,p)\Big) \tag{1.101}$$

At this moment to transform this equation to moving frame of reference one has only to express all the variables in the primed ones and use the fact, that probability density $f(t,x,p)$ is Lorentz scalar [6], [16] . So firstly one conveniently rearranges the terms and gets

$$\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\frac{p(t)}{m_0\gamma(p)}\Big)f(t,x,p) = \frac{\partial}{\partial p}\Big(\nu p(t) f(t,x,p) + \gamma(p)D\frac{\partial}{\partial p}(f(t,x,p)\Big) \tag{1.102}$$

from which its left side can be rewritten once more as

$$\Big(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\frac{p(t)}{p^0}\Big)f(t,x,p) \tag{1.103}$$

Dividing the last expression by c and multiplying by p^0 we get

$$\Big(p^0\frac{\partial}{\partial(ct)} + p(t)\frac{\partial}{\partial x}\Big)f(t,x,p) \tag{1.104}$$

so we have $p^{\alpha}\partial_{\alpha}f(t,x,p)$ and exploitng the fact that $p^{\alpha}\partial_{\alpha}f(t,x,p) = p'^{\alpha}\partial'_{\alpha}f'(t',x',p')$ one immediately obtains the form of the left hand side as seen by observer in moving frame of reference (i.e. primed one) with very interesting additional result, that the left side of the Fokker-Planck equation automatically becomes manifestly Lorentz covariant. When it comes to the right side, one gets

$$\frac{\partial p'}{\partial p}\frac{\partial}{\partial p'}\Big(\nu p(p')f(t(t'),x(x'),p(p')) + \gamma(p(p'))D\frac{\partial p'}{\partial p}\frac{\partial}{\partial p'}f(t(t'),x(x'),p(p')) \tag{1.105}$$

and again exploiting the Lorentz transformation matrix this time applied to 2-vector of momentum one gets

$$\frac{p'^0}{p^0}\frac{\partial}{\partial p'}\Big(\nu p(p')f(t(t'),x(x'),p(p'))+\gamma(p(p'))D\frac{\partial p'}{\partial p}\frac{\partial}{\partial p'}f(t(t'),x(x'),p(p'))\qquad(1.106)$$

Substituting for derivatives, $p(p')$, multiplying by p^0 and finally dividing by c we obtain

$$\frac{p'^0}{c}\frac{\partial}{\partial p'}\Big(\nu\gamma(V)\Big(\frac{V}{c}p'^0+p'^1\Big)f'+\gamma(p')D\frac{p'^0}{p^0}\frac{\partial}{\partial p'}(f')\Big)\qquad(1.107)$$

to modify the last expression into the nicer form, one can assume that $c = 1$ (geometrical units) and get

$$\Big(\frac{\partial}{\partial t}+\frac{p'}{p'^0}\frac{\partial}{\partial x'}\Big)f'(t',x',p')=\frac{\partial}{\partial p'}\Big(\nu\gamma(V)\Big(\frac{V}{c}p'^0+p'^1\Big)f'(t',x',p')+\gamma(p')D\frac{p'^0}{p^0}\frac{\partial}{\partial p'}f'(t',x',p')\Big)\qquad(1.108)$$

So one is able to explicitly transform the Fokker-Planck equation from inertial laboratory frame to arbitrary frame moving with constant velocity V with respect to the heath bath laboratory frame. Hänggi-Klimontovich discretization was very convenient because one may immediately observe transformed coefficients h' and g' and write corresponding stochastic differential equations in moving frame of reference

$$dx'(t')=\frac{p'}{p'^0}dt'\qquad(1.109)$$

$$dp'(t')=h(t',x',p',V)dt'+g(t',x',p',V)dW(t')\qquad(1.110)$$

where

$$h(t',x',p',V)=-\nu\gamma(V)\Big(\frac{V}{c}p'^0+p'\Big)\qquad(1.111)$$

and

$$g(t',x',p',V)=\sqrt{2\frac{\gamma(V)Dp'^0}{\gamma(V)(p'^0+Vp')}}\qquad(1.112)$$

1.4.3 Manifestly covariant form of relativistic Fokker-Planck equation

To obtain the covariant form of the diffusion Fokker-Planck equation one has to realize that the the FPE in moving frame of reference is the most general form of expressing the diffusion equation, as either the frame of reference of observer is laboratory heath bath frame or the observer is moving with respect to heath bath laboratory frame with velocity V. So one can drop the primes from all variables as the derived form of FPE in moving frame is valid in all inertial frames moving with any finite velocity. As one could see above, the left hand side is already manifestly covariant, as it consists of product of Lorentz invariant $p^\alpha\partial_\alpha$ with Lorentz scalar $f(t,x,p)$. On the right hand side one is able to derive

$$\frac{p^0}{\partial p}g(p^0(p),p)=\epsilon^{\alpha\beta}p_\alpha\frac{\partial}{\partial p^\beta}g(p^0,p)\qquad(1.113)$$

where $p_\alpha = g_{\alpha\beta}p^\beta$ with notation from App.A and $\epsilon^{\alpha\beta}$ being analogy of Levi-Civita tensor from 3 dimensions (i.e. $\epsilon^{01} = -1$, $\epsilon^{10} = -\epsilon^{01} = 1$ and $\epsilon^{00} = \epsilon^{11} = 0$). Next one has to realize that

$$p^0 = \gamma(V)(p'^0 + \frac{V}{c}p') \tag{1.114}$$

and using the 2-velocity of the heath bath observed from arbitrary frame of reference moving with respect to that heath bath frame of reference with realative velocity V in the form

$$U^\alpha = \gamma(V)(1, -V) \tag{1.115}$$

one can write

$$U^\alpha p'_\alpha = \gamma(V)(p'^0 + \frac{V}{c}p') \tag{1.116}$$

but as we said that dropping the primes will not affect generality or correctness of the result, one can immediately write

$$p^\alpha \partial_\alpha f = \epsilon^{\alpha\beta} p_\alpha \frac{\partial}{\partial p^\beta}\Big[-\nu U^\rho p_\rho f + \frac{\gamma(p)D}{U^\sigma p_\sigma} \epsilon^{\omega\zeta} p_\omega \frac{\partial}{\partial p^\zeta} f \Big] \tag{1.117}$$

which is manifestly covariant form of the relativistic Fokker-Planck equation as D, ν are constants and $\gamma(p) = \gamma(p')$.

Chapter 2

Black-Scholes-Merton valuation model

In this chapter we briefly revise thougths, ideas and basic sketches of derivations of original Black-Scholes-Merton valuation model. One could say, that determining the right price for some financial security is not a problem that can be resolved only by employing the apparatus of financial mathematics which stands on the rich theory of probability, which was in the form as one knows nowadays firstly formulated by Soviet physicist A.N.Kolmogorov in 1933, because random behaviour of asset's price evolution should automatically imply, that this randomness must be undoubtedly present in equations trying to give us an answer to the original question, what is the actual right price for tradeable financial derivative as i.e. option. And basically, he would be right. Obviously, on the market there is a huge number of participants which are risk-averse and therefore the prices of equities, derivatives and other tradeable assets are usually overvalued. When trying to describe some laws describing the market, and prices of derivatives should be of course be a part of them, then neccessarily one has to count with the risk-aversion on the market as intristic property of these laws. This would immediately imply, that prices calculated from this presumptions would not be only random variables dependent on spot prices of underlying securities, but they must have been influenced by random risk-aversions of market participants and therefore the right price of derivative could not be simply determined. However, in early 1970's, Fischer Black, Myron Scholes and Robert Merton invented a method which can determine the price as deterministic specified number independent of randomness hidden in the model coming from either underlying security or various levels of risk-aversion of investors. This method stands on the fact, that if one is holder of the derivative as well as some share of underlying security, then one can completely eliminate market risk and therefore determine the price of the derivative precisely without stochasticity. We tackled with problem of financial derivative valuation in our bachelor thesis already, but here we present the whole problem in more extended way. Firstly, the traditional method of deriving Black-Scholes-Merton equation is presented, after that the risk-neutral valuation method is introduced, as this method is regarded as one of the cornerstone of derivative valuation. In the very end of this chapter, the mathematical treatment of the whole problem is reviewed, as the reader may be interested in the more rigorous approach to the whole problem. From this point of view, it will be also easier to present and explain the analogies coming from the world of stochastic processes and quantum mechanics in the last chapter. The main sources for this chapter are [40], [25], [10] and most of the information in the literature comes from the original works [41], [42].

2.1 Binomial tree valuation model

The binomial tree compactly illustrates the problem on the market with derivatives and respective underlying securities. It is even possible to show, that beginning with this simple model, one can in the

limit case come directly to Black-Scholes model.
We want to determine the right unbiased value of option traded on the market at initial time t_0 with corresponding underlying asset spot price (t_0). To do this, one presumes Δ shares of asset at long position and one short position option to be our portfolio. We are firstly interested in the portofolio value in time $t_1 > t_0 = 0$. If the stock price rises to the new value $S(t_1) = S(t_0)u$, then one has

$$S(t_0)u\Delta - C(t_1)_u \tag{2.1}$$

where expression $[(u-1).100]$ represents the percentual rise of original spot price. On the contrary, if the asset price plummets down, than one has

$$S(t_0)d\Delta - C_(t_1)_d \tag{2.2}$$

where similarly $[(1-d).100]$ stands for percentual decrease in spot price. To make this portfolio risk-less, as one wants, both expressions has to equal, so

$$S(t_0)u\Delta - C(t_1)_u = S(t_0)d\Delta - C(t_1)_d \tag{2.3}$$

And one gets the expression for number of shares to hold to make the portfolio riskless

$$\Delta = \frac{C(t_1)_u - C(t_1)_d}{S(t_0)u - S(t_0)d} \tag{2.4}$$

So the portfolio is riskless and the value of it in the initial time t_0 can be expressed by

$$\Pi(t_0) = \exp^{-r(t_1-t_0)}(S(t_0)u\Delta - C(t_1)_u) \tag{2.5}$$

or equivalently for the case of price decrease

$$\Pi(t_0) = \exp^{-r(t_1-t_0)}(S(t_0)d\Delta - C(t_1)_d) \tag{2.6}$$

where we used non-standard notation with $\Pi(t_0)$ being portfolio value at initial time t_0. We used this notation, despite in [10] one finds $W(t_0)$,to avoid confusion, as in the previous chapter we used $W(t)$ to represent Wiener process. As the whole model is "up-down" symmetric, one can arbitrarily choose one of the previous two relations to determine the value of portfolio in t_0 in terms of t_0 quantities. In [40] one finds the "up"-version, thus we will change this and continue with "down" value

$$S(t_0)\Delta - C(t_0) = \exp^{-r(t_1-t_0)}(S(t_0)d\Delta - C(t_1)_d) \tag{2.7}$$

Now substituting for Δ and utilizing basic algebra one obtains

$$C(t_0) = \exp^{-r(t_1-t_0)}[pC(t_1)_u - (1-p)C(t_1)_d] \tag{2.8}$$

with

$$p = \frac{\exp^{r(t_1-t_0)} - d}{u - d} \tag{2.9}$$

So can see very interesting, important and general result. We derived this formula with assumption of risk-less portfolio and implicitly non-arbitrage assumption (so called *free lunch*), because we had always one price for underlying security in each time instance. It was not possible to transit to another market with different stock prices. What's more, the bracket also reminds of the discrete version of expected value of random variable well-known from probability theory. And the p really stands for probability. This is possible to find out from

$$\mathbb{E}[(S(t_1)] = pS(t_0)u + (1 - p)S(t_0)d. \tag{2.10}$$

Is one substitutes for p from above, one comes to the relation

$$\mathbb{E}[S(t_1)] = S(t_0)\exp^{r(t_1 - t_0)} \tag{2.11}$$

So if p is interpreted as probability, then one gets the correct expression for expected value of stock price at risk-free interest rate. One has to emphasize the fact, that the probability p represents so called *risk-neutral* probability in the risk-neutral world. This is obviously not the real world, nevertheless one can see, that the portfolio can be adjusted so that the risk is completely eliminated and hence the derivative price at initial time t_0 can be expressed as discounted exptected value of derivative prices at time t_1. One step binomial tree is not useful when trying to determine the price of the derivative, as the spot price of the underlying is changing rapidly, literally in each instance, so the model would be automatically only approximation and would comprise the systematic error. If one want to work with arbitrary time difference $t_1 - t_0$, then one divides this time interval into n parts and can establish multi-step binomial model according to previous theory. So one obtains

$$\max(S(t_0)u^j d^{n-j} - K, 0) \tag{2.12}$$

for the pay-off of option at time t_1, because $S(t_0)u^j d^{n-j}$ represents the price of the underlying at time t_1 if one presumes, that in the n-step tree there was n movements "up" and $n - j$ movements "down". We know, that probability that from n time steps, the price will rise in j cases and decrease in $n - j$ times is given by

$$p(n, j) = \frac{n!}{(n - j)!j!} p^j (1 - p)^{n-j}. \tag{2.13}$$

Therefore, using the same logic as in the on-step binomial tree one immediately obtains the option price at time t_0 $C(t_0)$ using the risk-neutral world measure

$$C(t_0) = \sum_{j=0}^{n} p^j (1 - p)^{n-j} \max(S(t_0)u^j d^{n-j} - K, 0) \tag{2.14}$$

In [40] it is proven, that this n-step binomial model converges to Black-Scholes valuation model in the limit of $n \to +\infty$.

2.2 Black-Scholes differential equation

The binomial tree model enables to the reader to understand the ideas behind the complicated mathematical facade, and this mathematical facade may, at the first sight, seem very unpleasant to work with. In this section we summarize all the presumptions made by F.Black, M.Scholes and independently R.C.Merton.

- On the market there is no credit risk, only the market risk. This stands for the fact that the prices of the drivatives are not driven by reliability of the counter party, but only stochastic fluctuations of the underlying security.
- The market with derivatives is completely liquid. This means that if the arbitrary market participant wants to sell the derivative, there is always way to do it, i.e. there is some demand for this asset on the market, independently of the actual market price of it. One also assumes, that on the market are no transactional costs and all the information is instantly available to all market participants.

- There is a possibility to trade continuosly in every instant of time.
- Underlying asset price is completely random and in each instant of time is driven by the log-normal probability distribution. This can be expressed by stochastic differential equation in the form

$$dS(t) = \mu(t)S(t)dt + \sigma S(t)dW(t) \tag{2.15}$$

- The underlying asset pays no dividends to holder of some shares of it.
- The volatility parameter σ is presumed to be constant, this in general may not be true about the drift coefficient $\mu(t)$, which in general can be changed, as the expected value of final profit can differ from each other at different instances of time and as we already said, this can be caused by unpredictabilities on the market, fluctuating risk-aversion of the other participants od the market, thus influencing the stock price. The price of the option can be determined exactly because of existence of risk-neutral measure, as drew in the previous section.
- The absence of arbitrage opportunities (i.e. free lunch). Investor is not allowed to be cummulating capital for free in the risk-less manner.

As we have seen in the previous chapter, the binomial tree valuation model exploits the existence of the risk-free probability measure. In the limit, one recovers Black-Scholes model. So the assumptions are fullfilled even here, at starting point of deriving standard Black-Scholes-Merton partial differential equation.
At this point, it is important to remind the general laws for financial markets, to see that up to now, both binomial valuation and assumptions in Black-Scholes-Merton are consistent. These laws will come from the mathematical theory of financial markets and later it will be interesting to observe how the theory bonds and restrict itself with conditions to be fulfilled to make Black-Scholes theory consistent. It will also responsible for the fact, that if one wants to generalize the original, then every change in the assumptions has to be done very carefully. The main source of the next lines is [25].

Definition 2.2.1 (Local martingale)**.** *An $\mathcal{F}_t$-adapted stochastic process $\{X(t\}$ is called a local martingale with respect to the given filtration, if there exists an increasing sequence of $\mathcal{F}_t$- stopping times τ_k such that*

$$\tau_k \to +\infty \text{ almost surely as index } k \to \infty.$$

and

$$X(\max(t, \tau + k)) \text{ is an } \mathcal{F}_t\text{-martingale for all } k.$$

Now the main definition from the mathematical formulation of Black-Scholes model comes

Definition 2.2.2 (Market)**.** *A market is an $\mathcal{F}_t^m$-adapted $(n+1)$-dimensional Itô process $X(t) = (X_0(t), ..., X_n(t))$ for $t \in [0, T]$ which one assumes to have the form*

$$dX_0(t) = \rho(t, \omega)X_0(t)dt \tag{2.16}$$

with $X_0(0) = 1$, and

$$dX_i(t) = \mu_i(t,\omega)dt + \sum_{j=1}^{m} \sigma_{ij}(t,\omega)dW_j(t) \tag{2.17}$$

$$= \mu_i(t,\omega)dt + \sigma_i(t,\omega)dW(t) \tag{2.18}$$

again with $X_i(0) = x_i$ with obvious assumption of general multidimensional Wiener process driving the stochasticity hidden in the model.

In the Black-Scholes model one work with $n = 2$ and $m = 1$. The components of the market vector are then created from some non-risky asset together with some risky asset. The frequently used notion of normalized market is remined (i.e. the mathematical formulation of so called *numeráire*). This can be seen, if one divides the whole market components by the first component $X_0(t)$ and claims, that the normalizator used will be called numeráire, i.e. the unit of "value" used on the market. The resulting process is also called a normalization.

Definition 2.2.3. *The market $\{X(t)\}$ is called normalized if $X_0(t) \equiv 1$.*

Definition 2.2.4. *A portfolio in the market $\{X(t)\}$ is an $(n + 1$ dimensional (t,ω)-measurable and $\mathcal{F}_t^m$-adapted stochastic process*

$$\Theta(t,\omega) = \Big(\Theta_0(t,\omega), ..., \theta_n(t,\omega)\Big). \tag{2.19}$$

So the portfolio simply expresses the number of units of each market asset. Then the value of the portfolio is defined by the expression

$$V^{\Theta}(t,\omega) = \theta(t).X(t) = \sum_{i=0}^{n} \Theta_i(t)X_i(t) \tag{2.20}$$

with dot signalizing the inner product on the space $\mathbb{R}^{n+1}$. The last part of the series of essential definition is a mathematically formulated notion of self-financing portfolio. This arises in the theory of Black-Scholes-Merton, when one tries to create a portfolio from risky and non-risky assets to ensure, that the portfolio will be completely riskless. This is reached by permanent buying and selling the respective parts of the portfolio.

Definition 2.2.5. *The portfolio $\Theta(t)$ is called risk-less, if*

$$\int_0^T \Big\{\Big|\Theta_0(s)\rho(s)X_0(s) + \sum_{i=0}^{n} \Theta_i(s)\mu_i(s)\Big| + \sum_{j=1}^{m}\Big[\sum_{i=1}^{n} \theta_i(s)\sigma_{ij}(s)\Big]^2\Big\}ds < +\infty \tag{2.21}$$

together with

$$dV(t) = \Theta(t).dX(t) \tag{2.22}$$

The value of such a portfolio can be expressed as

$$V(t) = V(0) + \int_0^t \Theta(s).dX(s) \tag{2.23}$$

for all $t \in [0,T]$.

The first condition ensures, that the integral in the expression for value of self-financing portfolio exists. The value expression then stands for the fact, that in the self-financing portfolio, no additional sources of assets or cash from outside are added to change the portfolio value within investigated time interval. This reminds the notion of closed system from statistical physics. The market is in equilibrium, if there are not possibilites available to some market participants, which could change the value of their portofolios in their favour. Now we define the important notion tightly bonded with the notion of self-financing portfolio.

Definition 2.2.6. *A portoflio $\Theta(t)$ which is self-financing is called admissible, if the corresponding value process $V^{\Theta}(t)$ is (t, ω) almost surely lower bounded, i.e. there exists $K = K(\theta) < +\infty$ such that*

$$V^{\Theta}(t, \omega) \geqslant -K \tag{2.24}$$

for almost all $(t, \omega) \in [0, T] \times \Omega$.

In other words, if the market is admissible, then there exists some restriction given from the external enviroment, which plays the role of maximal debt, which one can hold when being on the market.

Definition 2.2.7. *An admissible portfolio $\Theta(t)$ is called an arbitrage in the market $\{X(t)\}$, if the corresponding value process $V^{\theta}(t)$ satisfies $V^{\theta}(0) = 0$ and*

$$V^{\Theta}(t) \geqslant 0 \text{ almost surely and } \mathbb{P}[V^{Theta}(T) > 0] > 0.$$

So here one can see another analogy with thermodynamics and statistical physics. The concept of srbitrage simply expresses the fact, that in the market there exists an equilibrium. To generalize valuation techniques woth assumptions of arbitrage to be present on the market, one must have to work with non-equilibrium thermodynamics theory analogies. Now we will present one of the most imporant result of this general mathematical theory of valuation.

Definition 2.2.8. *A probability measure $\mu \sim \mathbb{P}$, such that the normalized process $\{\bar{X}(t)\}$ is a (local) martingale with respect to μ is called an equivalent (local) martingale measure.*

Lemma 2.2.1. *Suppose, there exists a measure μ on ${\mathcal{F}_T}^m$ such that $\mathbb{P} \sim \mu$, for $\mathbb{P}$ being the original measure and such that the normalized market price process $\{\bar{X}(t)\}$ is a local martingale with respect to μ. Then the market $\{X(t)\}$has no arbitrage.*

In the last section of this chapter, e will present so called Girsanov theorem, which will ensure, that under certain conditions, there will exist equivalent measure, which ensures the stock prices to be described by martingale process, hence enabling to use risk-neutral valuation when trying to determine the fair option price. Therefore, if one did not assume absence of arbitrage, then the risk-neutral valuation could obviously not be used. Now one can see, that every assumption made by Black-Scholes and Merton are of tremendous importance and even one assumption absence could destroy the validity of the model consistence.

The last notion which we decided to present here is a notion of market completeness. This market property comprises the assumption of negligible transaction costs, guarantees the abundance of perfect information and possibility to determine the price of every asset in every time.

Theorem 2.2.2. *A market $\{X(t)\}$ is complete if and only if there exists one and only one equivalent martingale measure for the normalized market $\{\bar{X}(t)\}$.*

This consequence of more complicated mathematics behind it toghether with results about arbitrage presented above cover all market laws which are behind the assumptions given by Black-Scholes and Merton, today taught as *fundamental theorems of asset pricing*.

After we picked the essentials from the mathematical formulation of Black-Scholes model assumptions, now one want to derive the original Black-Scholes-Merton partial differential equation and present it's solution. We presumed that the stock prices are modeled by geometrical Brownian motion. The option price should be some function of spot price $S(t)$ and time t, mathematically $C(S(t), t)$. Using Itô's lemma one immediately obtains

$$d(C(S(t),t)) = \Big(\frac{\partial C(S(t),t)}{\partial S}\mu(t)S(t) + \frac{\partial C(S(t),t)}{\partial t} + \frac{1}{2}\frac{\partial^2 C(S(t),t)}{\partial S^2}\sigma^2 S(t)^2\Big)dt + \frac{\partial C(S(t),t)}{\partial S}\sigma S(t)dW(t) \tag{2.25}$$

The randomness in the latter equation is hidden in the last term containing $W(t)$. This randomness can be eliminated through the special choice of assets in one's portfolio. If one chooses exactly $\frac{\partial C(S(t),t)}{\partial S}$ shares of underlying and 1 short position on derivative at the actual price $C(S(t), t)$, then the the whole portfolio can be completely hedged and hence the risk eliminated. The value of this kind of portfolio is in accordance with the previous ideas equal to

$$\Pi(t) = \frac{\partial C(S(t),t)}{\partial S}S - C(S(t),t). \tag{2.26}$$

So the finite difference in the portfolio value is

$$\Delta\Pi = \frac{\partial C(S(t),t)}{\partial S}\Delta S - \Delta C(S(t),t) \tag{2.27}$$

and substituting for $\Delta C(S(t), t)$ and $\Delta S(t)$ yields

$$\Delta\Pi = \Big(-\frac{\partial C(S(t),t)}{\partial t} - \frac{1}{2}\frac{\partial^2 C(S(t),t)}{\partial S^2}\sigma^2 S^2\Big)\Delta t. \tag{2.28}$$

Now one can already see, that the change in above choosed portfolio value is completely riskless. This was besides the right choice of portfolio reached by the assumption made about underlying price evolution (i.e. log-normal distribution). So this together with the absence of arbitrage opportunities gives

$$\Delta\Pi = r\Pi\Delta t \tag{2.29}$$

and comparing the portfolio value difference and rearranging the terms gives us the Black-Scholes-Merton partial differential equation

$$\frac{\partial C(S(t),t)}{\partial t} + rS\frac{\partial C(S(t),t)}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C(S(t),t)}{\partial S^2} = rC(S(t),t). \tag{2.30}$$

The solution can be obtained by setting up boundary conditions, for call option one has

- $C(S(t_1), t_1) = \max(S(t_1) - K), 0)$
- $C(0, t) = 0$
- $C(S(t), t) \sim S$ in the case when $S \to +\infty$

where t_1 stands for maturity time of option in accordance with previous notation. For put option one has

- $P(S(t_1), t_1) = \max(K - S(t_1), 0)$
- $P(0, t) = K\exp[-r(t_1 - t_0)]$
- $P(S, t) = P(S(t), t) \to 0$ in the case when $S \to +\infty$.

In the previous boundary conditions as well as in the following, K respresents the strike price. Aiming to obtain solution to this problem one first employs the variable transformations in the form

$$C = Kf(x, \tau) \tag{2.31}$$
$$S = Ke^x \tag{2.32}$$
$$t_0 = t_1 - \frac{\tau}{(\sigma^2/2)} \tag{2.33}$$

which leads to modified equation of the form

$$\frac{\partial f}{\partial \tau} = \frac{\partial^2}{\partial x^2} + (\kappa - 1)\frac{\partial f}{\partial x} - \kappa f \tag{2.34}$$

with $\kappa = \frac{2r}{\sigma^2}$ and modified boundary conditions

$$f(x, 0) = \max(e^x - 1, 0) \tag{2.35}$$
$$f(x, \tau) \to 0 \text{in the case when} x \to -\infty \tag{2.36}$$
$$f(x, \tau) \sim e^x \text{when} x \to +\infty \tag{2.37}$$

One now observes that the latter equation resembles the diffusion equation and hence apply last transformation in the form

$$f(x, \tau) = e^{ax+b\tau} g(x, \tau) \tag{2.38}$$

and finally denoting

$$a \equiv -\frac{1}{2}(\kappa - 1) \tag{2.39}$$

and

$$b \equiv a^2 + (\kappa - 1)a - \kappa = -\frac{1}{4}(\kappa + 1)^2 \tag{2.40}$$

one obtains the well-known diffusion equation

$$\frac{\partial g}{\partial \tau} = \frac{\partial^2 g}{\partial x^2} \tag{2.41}$$

with boundary conditions

$$g(x, 0) = \max(e^{(\kappa+1)x/2} - e^{(\kappa-1)x/2}, 0) \tag{2.42}$$
$$g(x, \tau) \overset{|x| \to +\infty}{\to} e^{(\kappa+1)x/2 - (\kappa+1)^2\tau/4} \text{for} \tau > 0 \tag{2.43}$$

The diffusion equation is then resolved with the well-known method of Green's function. The most important semi-result which one gets in the process of using that method of solution and is worth to be emphasized is the fact, that

$$g(x, \tau) = \int_{-\infty}^{+\infty} p(x, \tau | y, 0) g(y, 0) dy \tag{2.44}$$

where

$$p(x, \tau | y, 0) = \frac{1}{\sqrt{4\pi\tau}} \exp^{-\frac{(x-y)^2}{4\tau}} . \tag{2.45}$$

So one simply see, that trying to resolve the original equation leads firstly to classic diffusion equation and the solution to this mathematical problem may be expressed in similar manner than that from the first section of this chapter which dealt with binomial trees valuation. The transformed function of the call option price reminds the expected value of future option pay-offs with respect to risk-neutral probability measure, although in this case the simple probability distribution is switched to transitional probability function. This is another example of the fact, that binomial tree is just tightly connected with the Black-Scholes valuation model. What's more, this expression is very similar to the equation known from quantum mechanics, to be concrete, when one wants to express the probability amplitude $\psi(t_1)$ for quantum particle at time t_1 in terms of the probability amplitude $\psi(t_0)$ at time t_0, one is integrating the latter amplitude with integral kernel called *propagator*. So there is direct analogy between propagator in quantum mechanics and transitional probability in the world of stochastic processes. The difference is, that in the first case one works with amplitudes, while the latter case deals with probabilities. This analogy will be exploited in the last chapter of this thesis.
After restoring the original variables, one obtains the final result in the form

$$C(S(t), t) = S N(d_1) - K e^{-r(t_1 - t_0)} N(d_2) \tag{2.46}$$

with

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\Big(-\frac{z^2}{2}\Big) \tag{2.47}$$

and

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(t_1 - t_0)}{\sigma\sqrt{t_1 - t_0}} \tag{2.48}$$

$$d_2 = d_1 - \sigma\sqrt{t_1 - t_0} \tag{2.49}$$

From the calculated price for European call option one can immediately calculate the price for it's put counterpart exploiting the so called *put-call parity* [10]

$$P(S(t), t) = -S\Big[1 - N(d_1)\Big] + K e^{-r(t_1 - t_0)}\Big[1 - N(d_2)\Big] \tag{2.50}$$

The fact, that the function of option price may be expressed as the the expected value is nothing but the the consequence of rich theory of stochastic processes which will be briefly presented in the next section.

2.3 Diffusions properties and Feynman-Kac formula

In this section we briefly revise the essential results coming from the rigorous theory of Itô diffusions. The Black-Scholes theory presented from the more practical point of view, as presented in the most of available literature (see i.e. [26], [40], [10]) is in general valuable for understanding the basic notions of the valuation theory from the more heuristic point of view. The reader is usually not acquainted with the hard mathematical rigor hidden somewhat behind the valuation "scenes". What's more, the researchers and students interested in the problematic of derivatives valuation and stochastic processes stand in front of the dillema, i.e. they can start studying the literature either very easy and not sufficient to completely understand everything neccessary or too technically difficult and hiding the objective of the problem. To see the real connection between both approaches used frequently to derive the well-known Black-Scholes formula (i.e. risk-neutral valuation and solving Black-Scholes PDE) and still be able to express the thoughts in rigorous way, we assume, that [25] is ideal. So this automatically imply, that is was also the main source of information needed for this section.
Firstly, one needs to understand the basic notions of special kind of stochastic processes called "Itô diffusions".

Definition 2.3.1. *A time-homonegenous Itô diffusion is a stochastic process $X_t(\omega) = X(t,\omega) : [0,+\infty] \times \Omega \to \mathbb{R}^n$, which satisfies a stochastic differential equation in the form*

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dW(t) \tag{2.51}$$

$t \leqslant s$, $X(s) = x$, where $W(t)$ is m-dimensional Wiener process and functions $\mu : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ satisfy the condition

$$|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \leqslant D|x - y|$$

for $x, y \in \mathbb{R}^n$ and $|\sigma| = \sqrt{\sum_{i,j} |\sigma_{ij}|^2}$, D being non-negative constant.

The time homogenity of the resulting stochastic process $X(t)$ means that the random variable

$$X^{s,x}_{s+h} = x + \int_s^{s+h} \mu(X^{s,x}_u)du + \int_s^{s+h} \sigma(S^{s,x}_u)dW(u) \tag{2.52}$$

which is after the variable translation $u = s + v$ equal to

$$X^{s,x}_{s+h} = x + \int_0^{h} \mu(X^{s,x}_{s+v})dv + \int_0^{h} \sigma(S^{s,x}_{s+v})d\tilde{W}(v) \tag{2.53}$$

where $\tilde{W}(v) = W(s + v) - W(s)$ for arbitrary $v \geqslant 0$ (about which can be proven, that is again Wiener process) and

$$X^{0,x}_{h} = x + \int_0^{h} \mu(X^{0,x}_{v})dv + \int_0^{h} \sigma(X^{0,x}_{v})dW(v) \tag{2.54}$$

have the same probability distribution laws. The notation $X^{s,x}_{s+h}$ means, that this is random variable coming from investigated stochastic process X in the time $s+h$ under additional condition, that $X(s) = x$. According to this notation aiming to generalise the traditional assumption that $X(0) = 0$ and start building theory with the most general presumptions one introduces the new probability measure Q^x for fixed $x \in \mathbb{R}^n$. This measure will be giving the same result as the original measure (in [25] written as P^0), with respect to which the process $W(t)$ is Wiener process. As usual, one defines the $\sigma-$ algebra generated by random variables X_t^y for fixed $x \in \mathbb{R}^n$. On this subsets of sample space Ω, one defines

$$Q^x\left[X_{t_1} \in E_1, ..., X(t_k) \in E_k\right] = P^0\left[X_{t_1}^x \in E_1, ..., X_{t_k}^x \in E_k\right] \tag{2.55}$$

with sets E_i coming from the σ- algebra given on the set of possible outcomes (which is usually Borel algebra). With this notation one is able to define the Markov property of Itô diffusion in more sophisticated and even compact way.

Theorem 2.3.1 (Markov property)**.** *Let f be a bounded Borel function $f : \mathbb{R}^n \to \mathbb{R}^n$ and X Itô diffusion. Then for arbitrary combinations of $t, h \geqslant 0$*

$$\mathbb{E}^x\left[f(X_{t+h})|\mathcal{F}_t\right] = \mathbb{E}^{X_t}\left[f(X_h)\right]$$

where $\mathcal{F}_t$ stands for filtration generated by generally m- dimensional Brownian motion. The theorem expresses the fact, that all Itô diffusions fulfill the Markov property, i.e. that the future behaviour of random stochastic process from some time $t + h$ further is not influenced by the history of the process (i.e. the values of the process at times smaller than time $t + h$. This property importance arises when one realizes, that stochastic process given by the stochastic differential equation

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t) \tag{2.56}$$

i.e. geometric Wiener process which drives the stock prices, fulfills the conditions from the definition of Itô diffusion and hence it is Markov process. So one can see, that until now, everything is consistent. With using the definition of stopping time τ given in the definition 1.4.2, one can generalize the last theorem even for this time of random variables, so

$$\mathbb{E}^x\left[f(X_{\tau+h})|\mathcal{F}_\tau\right] = \mathbb{E}^{X_\tau}\left[f(X_h)\right] \tag{2.57}$$

for all $h \geqslant 0$.

On our way to the final result, which will be the so called Feynman-Kac formula linking the general mathematical form of risk-neutral valuation presented with partial differential equations in the next section, one has to define the important notion of the generator of an Itô diffusion.

Definition 2.3.2. *Let $X = \{X_t\}$ be a time homogenous Itô diffusion in $\mathbb{R}^n$. The infinitesimal generator G of X_t is operator defined by*

$$Af(x) = \lim_{t\to 0} \frac{\mathbb{E}^x[f(X_t)] - f(x)}{t}.$$

The set of all functions f such that the limit exists at some concrete $x \in \mathbb{R}^n$ is denoted by $\mathcal{D}_A(x)$ and $\mathcal{D}_A$ is then the set of functions for which the limit exists for all x.

From the definition of Itô diffusion process it is obvious, that it is also well-known classical Itô process. To be able to calculate the explicit formula for Itô diffusion, one utilizes very useful lemma, which is valid not only for the Itô diffusions, but even for the standard Itô processes

Lemma 2.3.2 (key to partial differential operators)**.** *Let's have X be an Itô process in $\mathbb{R}^n$ of the form*

$$X_t^x = x + \int_0^t u(s)ds + \int_0^t v(s)dW(s)$$

with $\{W\}$ being m-dimensional Wiener process. Let $f \in C_0^2(\mathbb{R}^n)$ (i.e. f is presumed to be the continuosly differentiable up to second order with compact support). Let be τ be a stopping time with respect to filtration $\{\mathcal{F}_t\}$ generated by Brownian motion and assume, that $\mathbb{E}^x[\tau] < +\infty$. Assume that $u(t,\omega)$ and $v(t,\omega)$ are bounded on the set of variables (t,ω), such that n-dimensional value of the process $X(t,\omega)$ belongs to the support of f. Then

$$\mathbb{E}^x\Big[f(X_\tau)\Big] = f(x) + \mathbb{E}^x\Big[\int_0^\tau \Big(\sum_{i=1}^n u_i(s)\frac{\partial f}{\partial x_i}(X_s) + \frac{1}{2}\sum_{i,j=1}^n (vv^T)_{i,j}(s)\frac{\partial^2}{\partial x_i \partial x_j}(X_s)\Big)ds\Big]$$

with probability law $\mathbb{E}^x$ denoted in the same manner as before.

Now, if one applies the last lemma to the stochastic process in the general form of stochastic differential equation

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dW(t) \tag{2.58}$$

then for $f(X(t))$ continuously differentiable up to second order with compact supports one immediately has

$$Af(x) = \sum_{i=1}^n \mu_i(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^n (\sigma\sigma^T)_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j} \tag{2.59}$$

when one chooses special value for stopping time $\tau = t$ and the definition of the generator. In the definition, one can use Fubini 's theorem for switching the order of integration (the expected value $\mathbb{E}^x$ respresents the first integration, the integral $\int_0^t$ the second one) and the basic definition of integral as the function of the upper bound (fundamental theorem of calculus). Now analyze the following important case for future investigations. If the function $C(S(T-t), T-t)$ with special choice $t = 0$ (i.e. $C(S(T), T) := f(S(T)) = f(X(T))$ for the maturity option price as known from Black-Scholes valuation model under the obvious assumption is chosen, then it must be zero for $S(0) < 0$ and $S(0) \leqslant K$ in the case of call option. The above mentioned theory would give us the infinitezimal generator for partial differential equation, which is simply the operator acting on the function $C(S(T), T) \equiv C(S(T))$ according to the last lemma and remark. Substituting to the 1-dimensional process gives

$$dX(T) = \mu S(T)dT + \sigma S(T)dW(T) \tag{2.60}$$

which can be simply seen as the stochastic differential equation driving the stochastic process of maturity spot prices distributions $S(T)$, one gets the infinitesimal generator

$$A[C(S(T)] = \mu S(t)\frac{\partial C(S(T))}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C(S(T))}{\partial S^2} \tag{2.61}$$

which already really resembles the part of Black-Scholes-Merton valuation equation. This result gives us the motivation, that we are moving forward in right direction. This special case is crucial to analyze, as the extensions of it play the main role in Black-Scholes risk-neutral valuation model through Feynman-Kac formula. If one focus on the assumption made on the function $f(S(T)) = C(S(T), T)$ (i.e. compact support and differentiability), one can see, that the support of it $S(T) = [K, +\infty)$ is simply not compact interval. Therefore, if one wants to use the Feynman-Kac approach and connect the world of quantum mechanics through path integral with stochastic processes, the assumptions on the funcion f in the previous text must be loosened. Then one must to prove, that the whole theory is still valid. In [44] it is shown, that the whole presented theory is valid, if the following assumption on $f(x)$ are fulfilled

$$|f(x)| \leqslant L(1 + \|x\|^{2\lambda}) \tag{2.62}$$

with $\lambda \geqslant 1$, $L \geqslant 0$, the $f(x)$ is continuous function and finally

$$f(x) \geqslant 0 \tag{2.63}$$

for $\forall x \in \mathbb{R}^n$. This assumptions are already fulfilled despite the fact, that the support of the function was not compact and hence one can proceed further.

On our way to desired big general result in the form of Feynman-Kac formula, the stochastic equivalent for the Feynman path integral from the world of quantum mechanics, one has to remind the useful Dynkin's lemma (formula), used in proof of the the so called Kolmogorov theorem. The Kolmogorov theorem is then only one step from the Feynman-Kac theorem.

Lemma 2.3.3 (Dynkin)**.** *Let $f \in C_0^2(\mathbb{R}^n)$ and let τ be a stopping time, $\mathbb{E}^x[\tau] < +\infty$. Then*

$$\mathbb{E}^x\Big[f(X_\tau)\Big] = f(x) + \mathbb{E}^x\Big[\int_0^\tau Af(X_s)ds\Big].$$

where X stands for Itô diffusion

The Dynkin formula is nothing but the lemma which we called the key to partial differential equations above with the sums in the second expected value expressed in terms of the infinitesimal generator. If one denotes the $\mathbb{E}^x\Big[f(X_t)\Big] := u(t,x)$ and $\frac{\partial u}{\partial t} := \mathbb{E}^x\Big[Af(X_t)\Big]$, then the next theorem simply says, that the right side of this derivative can be expressed in terms of the function $u(t,x)$.

Theorem 2.3.4 (Kolmogorov's backward)**.** *Let $f \in C_0^2(\mathbb{R}^n)$ as usual. If one defines*

$$u(t,x) = \mathbb{E}^x\Big[f(X_t)\Big]$$

,

then $u(t,\cdot) \in \mathcal{D}_A$ for each time t and

$$\frac{\partial u}{\partial t} = Au \tag{2.64}$$

$$u(0,x) = f(x) \tag{2.65}$$

for $t > 0$ and every $x \in \mathbb{R}^n$.

and finally the generalization of it

Theorem 2.3.5 (Feynman (1948), Kac (1949))**.** *Let $f \in C_0^2(\mathbb{R}^n)$ (also valid for the loosened conditions in [44]) and $q \in C(\mathbb{R}^n)$. Assume that q is lower bounded.*

- *Put*

$$v(t,x) = \mathbb{E}^n\Big[\exp\Big(-\int_0^t q(X_s)ds\Big)f(X_t)\Big]. \tag{2.66}$$

Then

$$\frac{\partial v}{\partial t} = Av - qv \tag{2.67}$$

$$v(0,x) = f(x) \tag{2.68}$$

for $t > 0$ and $x \in \mathbb{R}^n$.

- *Moreover, if $w(t,x) \in C^{1,2}(\mathbb{R}\times\mathbb{R}^n)$ is bounded on $K\times\mathbb{R}^n$ for each compact $K\subset\mathbb{R}$ and w solves the latter initial problem, then $w(t,x) = v(t,x)$, with v given with the expected value 2.66.*

Now if one chooses $f(x(t)|_{t=0}) = C(S(T-0)) = C(S(T)) = \max(S(T)-K,0)$ as already done above, $q=\rho$ for some constant $\rho > 0$ and writes the initial problem 2.67, 2.68 which is presumed to be fulfilled, then according to the Feynman-Kac formula one obtains

$$u(t,x(t)) = e^{-\rho t}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi t}}e^{-\frac{y^2}{2t}}\Big(x\exp\big(\alpha-\frac{1}{2}\beta^2)t+\beta y\big)-K\Big)^{+}dy \tag{2.69}$$

If one identifies $\alpha,\rho = r$ and $\beta=\sigma$, one immediately sees, that initial value problem 2.67, 2.68 becomes Black-Scholes partial differential equation for the price of the option $C(t,S) = u(t,x)$ and in the last expression then recovers the risk-neutral valuation formula for the price of the derivative. This confirms the "feeling", that because of binomial tree valuation model convergence to Black-Scholes in the limit of infinite steps, the risk-neutral expected value valuation method agrees with the result got from the Black-Scholes partial differential equation.
At this point, at the very end of this section, we emphasize that in the latter equation, the coefficients α,ρ had to be replaced by risk-free interest rate r. If we substituted $\alpha,\rho=\mu$, i.e. drigt coefficient from the evolution of underlying security price, the result could not be interpreted as the price of the option at some time $T-t$. The reason is, that if one calculated the expected value of such a price formula, it would not satisfy the martingale condition, which one desires from the option price process. There is no reason to presume, that the option price process, from which we take expected value should be not martingale. What's more, one would like to see the option price process to be martingale. But if one presumes (in accordance with assumptions given by Black, Scholes and Merton) that

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \tag{2.70}$$

then the price of the option, for which is reasonable to presume that it will be naturally expressed as

$$C(S(t),t) \sim e^{-r(T-t)}f(S(t)) \tag{2.71}$$

can not be martingale. The reason is , that the geometric Wiener process is not martingale itself. One would guess, that the dream of expressing the prices in very elegant way calculating the discounted expected value of stock prices only is not possible. But as we could see in the binomial tree, it is possible. But how one knows, that this is possible generally? What if it was a coincidence? In 1958, Soviet mathematician I.V.Girsanov gave us the answer to this complication [45]. in this case we will use the version as presented in [10].

Theorem 2.3.6 (Girsanov)**.** *Let $\{\gamma(t)\}$ for $t\in[0,T]$ be a measurable process which satisifies Novikov's conidition in the form*

$$\mathbb{E}\Big[\exp\Big(\frac{1}{2}\int_0^T\gamma^2(t)dt\Big)\Big] < +\infty \tag{2.72}$$

Furthermore, define two processes $\{L(t)\}$ and $\{W^{\mu}(t)\}$ by expressions

$$L(t) = \exp\Big(-\int_0^t\gamma(t')dW(t') - \frac{1}{2}\int_0^t\gamma(t')^2dt'\Big) \tag{2.73}$$

and

$$W^{\mu}(t) = W(t) + \int_0^t\gamma(t')dt' \tag{2.74}$$

where $\{W(t)\}$ is Wiener process with respect to the measure ν. Then, $\{L(t)\}$ is a martingale and $\{W^{\mu}(t)\}$ is also a Wiener process under the equivalent probaility measure μ with Radon-Nikodým derivative

$$\frac{d\mu}{d\nu} = L(T). \tag{2.75}$$

Without this theorem, one could guess, that because

$$e^{-r(T-t)}\mathbb{E}^{\nu}\Big[S(T)|\mathcal{F}_t\Big] = S(t)e^{(\mu-r)(T-t)} \tag{2.76}$$

it is sufficient to put $\mu = r$. This comes from the fact, that if one writes

$$\tilde{S}(t) = S(t)e^{-rt} \tag{2.77}$$

which is nothing but discounted stock prices process with SDE

$$d\tilde{S}(t) = (\mu - r)\tilde{S}(t)dt + \sigma dW(t) \tag{2.78}$$

one gets the expected value as in 2.76. But this presumption would not be true. According to Girsanov theorem, inserting the expression for $dW(t)$ expresses in terms of new $dW^{\mu}(t)$ gives

$$d\tilde{S}(t) = \Big(\mu - r - \sigma\gamma(t)\Big)\tilde{S}(t)dt + \sigma\tilde{S}(t)dW^{\mu}(t) \tag{2.79}$$

and to make this process a martingale, one has to choose

$$\gamma = \frac{\mu - r}{\sigma} \tag{2.80}$$

and get

$$d\tilde{S}(t) = \tilde{S}(t)\sigma dW^{\mu}(t). \tag{2.81}$$

So one got the martingale representation of the presumed stock prices stochastic process and Girsanov theorem guaranteed, that the random driving term $dW^{\mu}(t)$ represents the Wiener process increments, however, under new measure, which is connected with the former through Radon-Nikodým derivative. The option price expressed as the expectation value of function of stock prices, can be therefore obtained in this form, but with respect to so called *risk-neutral* measure, in above text signed by μ. This is the measure used in the binomial tree model in the previous chapter.

2.4 Risk-neutral valuation

In the last section of the chapter focused on revising the most essential knowledge from the standard Black-Scholes-Merton theory of derivatives valuation we assume to be very important to present the brief derivation of the fact, that risk-neutral valuation approach, which accompanied us from the very beginning of this chapter already in the binomial tree model, gives one exactly the same results as partial differential equation solution approach. This is neccessary, if one wants to conclude the whole chapter with the at least heuristic proof, that the very elegant and smooth risk neutral valuation is fully equivalent with the quite technical and not so direct method of resolving complicated Black-Scholes partial differential equation. We already know, that mathematically, these two methods are connected through the Feynman-Kac formula. The main source to this section is [40].
As we presume in accordance with the previous theory, the right price $C(S,t)$ of the European Call option can be expressed as

$$C(S(t), t) = e^{-r(t_1 - t_0)}\mathbb{E}^P\Big[\max(S(t_1) - K, 0)\Big] \tag{2.82}$$

where the notation $\mathbb{E}^P$ stands for the expected value of the random variable in the parentheses calculated with respect to the risk-neutral probability measure P. To express this relation in terms of exact, non-random quantities, one has to evaluate the expected value. Within the assupmtions enabling to derive the Black-Scholes model, one presumes that

$$dS(t) = \mu(t)S(t)dt + \sigma S(t)dW(t). \tag{2.83}$$

This means, that $S(t)$ is log-normally distributed random variable with mean value $\mathbb{E}^P = S(t_0)e^{rT}$ and the standard deviation $\sigma\sqrt{T}$.Here we assumed that $t_0 = 0$ and $t_1 = T$ as usually seen in the literature. Hence, one can utilize the next lemma

Lemma 2.4.1. *Let X be the log-normally distributed random variable. Let $\tilde{\sigma}$ be the standard deviation of* $\ln X$*. Then*

$$\mathbb{E}\Big[\max(X - K, 0)\Big] = \mathbb{E}(X)N(d_1) - KN(d_2)$$

where for the coefficients d_1 and d_2 following equalities hold

$$d_1 = \frac{\ln[\mathbb{E}(X)/K] + \tilde{\sigma}^2/2}{\tilde{\sigma}}$$

$$d_2 = \frac{\ln[\mathbb{E}(X)/K] - \tilde{\sigma}^2/2}{\tilde{\sigma}}$$

where the function $N(\cdot)$ is the same as in the 2.47.

to exactly calculate the desired expected value. Due to the fact, that this lemma is the key to showing the equivalence between risk-neutral valuation principle and solving Black-Scholes partial differential equation with appropiate boundary conditions, it deserves to be presented together with it's proof.

Proof. Let $f(X)$ be the probability density function of random variable X. From this one can immediately write

$$\mathbb{E}[\max(X - K, 0)] = \int_K^{+\infty} (X - K)f(X)dX \tag{2.84}$$

for the expectation value of the $\max[(X - K), 0)]$. Because the expected value of the log-normal variable equals $\exp(\mu + \frac{\sigma^2}{2})$, the variable $\ln(X)$ has the mean $m = \ln[\mathbb{E}(X)] - \frac{\tilde{\sigma}^2}{2}$. Now one defines new random variable in the form

$$\tilde{X} = \frac{\ln(X) - m}{\tilde{\sigma}} \tag{2.85}$$

Then exploiting the basic properties of normal distribution known from basic probability theory course, one gets that $\tilde{X}$ is normally distributed with the mean value $\mathbb{E}(\tilde{X}) = 0$ and standard deviation $\sigma_{\tilde{X}} = 1$. So in the language of probability distributions, random variable $\tilde{X}$ has the distribution

$$f(\tilde{X}) = \frac{1}{\sqrt{2\pi}}e^{-\tilde{x}^2/2} \tag{2.86}$$

With this variable transformation, the original expected value may be rewritten as

$$\mathbb{E}[\max(X - K, 0)] = \int_{(lnK-m)/\tilde{\sigma}}^{+\infty} e^{\tilde{X}\tilde{\sigma}+m} f(\tilde{X})dX - K \int_{(lnK-m)/\tilde{\sigma}}^{+\infty} f(\tilde{X})dX \tag{2.87}$$

where the term in the first integral $e^{\tilde{X}\tilde{\sigma}+m} f(\tilde{X})$ can be rewritten (using uprava na stvorec) into the form

$$e^{m+\frac{\tilde{\sigma}^2}{2}} f(\tilde{X} - \tilde{\sigma}) \tag{2.88}$$

and therefore one rewrites the original sum of two integrals in 2.87 as

$$\mathbb{E}[\max(X - K, 0)] = e^{m+\frac{\tilde{\sigma}^2}{2}} \int_{(\ln K-m)/\tilde{\sigma}}^{+\infty} f(\tilde{X} - \tilde{\sigma})d\tilde{X} - K \int_{(\ln K-m)/\tilde{\sigma}}^{+\infty} f(\tilde{X})d\tilde{X}. \tag{2.89}$$

At this point, one only has to realise, that if the function $N(\cdot)$ represents the probability distribution of the random variable with mean value of zero and a standard deviation of 1, then the the first integral gives one the probability, that that random variable will be greater than the value

$$\frac{(\ln K - m)}{\tilde{\sigma}} - \tilde{\sigma}. \tag{2.90}$$

i.e. in the language of cumulative distribution functions one gets, that the first integral is equal to

$$1 - N\Big[\frac{(\ln K - m)}{\tilde{\sigma}} - \tilde{\sigma}\Big]. \tag{2.91}$$

The similar logic stands behind expressing the second integral. If one even returns to the original variables (i.e. one substitutes for m), the expression becomes

$$\mathbb{E}\Big[\max(X - K, 0)\Big] = \mathbb{E}(X)N(d_1) - KN(d_2) \tag{2.92}$$

which proves the statement of the lemma. □

So mixing the statement of the previous lemma together with the spot prices evolution assumption and the risk-neutral valuation formula, one gets

$$C(S(t), t) = e^{-rT}\Big[S(0)e^{rT}N(d_1) - KN(d_2)\Big] \tag{2.93}$$

hence after multiplying the exponentials

$$C(S(t), t) = S(0)N(d_1) - Ke^{-rT}N(d_2) \tag{2.94}$$

with d_1 and d_2 being the same as in the 2.48 and 2.49. So the risk-neutral valuation approach is really equivalent to the solving the differential equaiton approachw, hence respecting the mathematical background and of course giving the same result in the form Black-Scholes formula.

Chapter 3

Quantum finance

3.1 BSM model problems

3.1.1 Volatility modifications

It is well-known, that financial markets, in our case the markets with financial derivatives came to the point, where the original Black-Scholes model is not valid and fair prices for the options are not given by it anymore. This has meant the big problem for financial practitioners and market participants, as they were used to work with that model, back then, the only one consistent model giving the fair prices of the option on the market. The way, how they were able to observe inconsistence between reality and Black-Scholes-Merton theory was the method of so called *implied volatility*. In each moment of trading one picks the actual price of the option on the market and using remaining parameters in the Black-Scholes model is able to calculate the desired parameters. When they let the volatility σ constant as in the Black-Scholes assumptions, they got the prices of the underlying which were different from original Black-Scholes presumptions. On the contrary, if one let the prices of the underlying at the state, which is directly observed on the market, then the volatility is not constant. In fact, this observed volatility changes $\sigma(S(t), K)$ are obviously functions of stock prices and also strike price coming from the option contract. Depending on data, one can get etiher the smile-shape of the graph curve, i.e. so called *volatility smile*, or frown like shape, which does not a name until now, it could be called *volatility frown*. There are several ways, how to tackle with these problems. First is completely change the underlying asset price evaluation driving process. The original geometric Brownian motion in the form

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \tag{3.1}$$

with $\mu \in \mathbb{R}$ and $\sigma > 0$

as presented in the above chapter are then changed to have fat tails. But before 1987 market crash, the original was working well [43] and the in spite of the fact, that the fat tails distributions were not incorporated theory yet. This posed a problem, as one used the observed prices on the market, even those prices have not behaved with respect to geometric Brownian motion and the volatility was still assumed to be constant. However, the option prices were not located far away from their actual prices. At this moment, financial priactitioners introduced so called *local volatility* as function $\sigma(S(t), t)$. This volatility is changing not with respect to strike price, but is dependent on the actual spot price.As the volatility is the part of the probability distribution function for spot price, the model is made up so that this volatility changes itself to modify the distribution function to make the original one exhibiting fat tails in accordance with observation. But here another problem arises. As the local volatility does not depend on strike price, the real dynamics can not be captured properly. This led to models with *stochastic volatility* [40], [46], [47], [48], [12]. The general presumption of these models is

$$d\sigma(t) = \alpha(\sigma(t), t)dt + \beta(\sigma(t), t)d\tilde{W}(t) \tag{3.2}$$

with $\tilde{W}(t)$ being some Wiener process, generally different form the original one. The general form presented in [12] refering other models such as [48] and many more is of the form

$$\frac{d\sigma^2(t)}{dt} = \lambda + \kappa\sigma^2(t) + \omega\sigma^{2\alpha}\xi(t) \tag{3.3}$$

where $\lambda, \kappa, \omega, \alpha \in \mathbb{R}$ are constants and λ, κ has to be chosen such that σ^2 is positive and finally $\xi(t)$ is a white noise as used at the very beginning of the first chapter. The whole valuation model expressed in terms of Langevin equations, now two coupled stochastic differential equations is then expresses as

$$dS(t) = \mu S(t)dt + S(t)\sqrt{\sigma^2}\xi_1(t) \tag{3.4}$$

$$\frac{d\sigma^2(t)}{dt} = \lambda + \kappa\sigma^2(t) + \omega\sigma^{2\alpha}\xi_2(t) \tag{3.5}$$

with relations

$$< \xi_1(t)\xi_1(t') >=< \xi_2(t)\xi_2(t') >= \frac{1}{\rho} < \xi_1(t)\xi(t') >= \delta(t - t') \tag{3.6}$$

and correlation $\rho \in [-1, 1]$. Using the similar logic as when one derives the Black-Scholes model partial differential equation, but now already for the function $C(S(t), \sigma^2(t), t)$ dependent on two stochastic variables, one get the generalized Black-Scholes stochastic volatility equation in the form

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + (\lambda + \kappa(\sigma^2))\frac{\partial C}{\partial(\sigma^2)} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} + \rho\omega\sigma^{1+2\alpha}S\frac{\partial^2 C}{\partial S\,\partial(\sigma^2)} + \omega^2\sigma^{4\alpha}\frac{\partial^C}{\partial(\sigma^2)^2} = rC \tag{3.7}$$

where $C = C(S(t), \sigma^2(t), t)$. This BSM equation equivalent is called *Merton-Garman equation*.
The main difference in the derivaiton of the latter equation is the fact, that volatility is not explicitly tradeable on the markets, therefore the idea of getting rid of the risk by constant selling and buying the assets on the market can not be used in this case. However, one can involve the risk non-zero parameter into some of the constants (e.g. in [12] redefining coefficient λ) to get to the same point, as in the deriving BSM partial differential equation.
But as explained in [43], this approach leads to scientific incosistency, simply, the model is then non-falsifiable. Hence above mentioned methods are just blind paths and are rather practical and temporary tools than scientific approach despite using robust and rigorous theory behind. But despite all these problems connected with the stochastic volatility, we remind those models from one important reason. Like the original BSM model, stochastic volatility models are able to be expressed in the form similar to that used in the world of quantum mechanics, which is convenient for future generalizations connected with another problems of original BSM valuation, we will present in the next subsection.

3.1.2 Relativistic effects on the market

The traditional and well-known drawback of original Black-Scholes model is the the fact, that volatility when compared to the real market data is not constant quantity in time. However, this is not the only problem connected with the original model. The second one is tigthly bonded with the rapid development of computers and modern technologies. Nowadays, one can observe so called *high-speed trading* on the market. The investors are simply exploiting the information tehcnology advance to be one or more steps forward comparing their rivals. But it is not only high-frequency trading, which motivates us to modificate and generalize the original valuation BSM model [49], [50], [51]. It is also the effect of special relativity, to be concrete the finiteness of speed of light and its consequences on the arbitrage on the market. It is already well-known, that there are intraplanetary places, which can provide the investor with arbitrage advantage comparing other investors, especially those investing from other places. In [55] there is presented this effect eith use of Vašíček model [54]

$$dr(t) = a(b - r(t))dt + \sigma dW(t) \tag{3.8}$$

where $r(t)$ stands for linear combination of two cointegrated stochastic processes representing some equity fulfilling this assumption, b represents long term mean level around which future trajectories of portfolio combination $r(t)$ will evolve, $dW(t) = \xi(t)dt$ is already known Wiener increment given by random Langevin force present in the market and a representing speed of reversion, characterizing the rate of portfolio fluctuations or regrouping around the long term mena level. One can find many asset combinations where the condition of cointegration holds. As [55] says, cointegrated asset behavior was observed in pairs of higly correlated stocks, futures and corresponding spot prices, large portfolios of stocks and even foreign currency exchange market. So one has

$$du(x,t) = a\Big[\frac{u(x+h, t-\frac{h}{c}) + u(x-h, t-\frac{h}{c})}{2} - u(x,t)\Big]dt \tag{3.9}$$

from the Vašíček model, because the random white noise was turned off at every instant and the portfolio price $u(x,t)$ is evolving around the mean value of portfolio prices in the neighbouring trading nodes (sites) although with time retardation $t - \frac{h}{c}$. The nodes are in mutual distances of h and the last presumption was, that the speed of information propagating is $c = 3.10^8 m.s^{-1}$. The value h is very small and hence one can utilize Taylor expansion for both time and position. Firstly, one rewrites the latter expression as

$$\frac{\partial u(x,t)}{\partial t} = \frac{a}{2}\Big[u(x+h, t-\frac{h}{c}) - u(x,t) + u(x-h, t-\frac{h}{c}) - u(x,t)\Big] \tag{3.10}$$

and after expanding up to second order in position and first in time one obtains

$$\frac{\partial u(x,t)}{\partial t} \approx a\Big[\frac{h^2}{2}\frac{\partial^2 u(x, t-\frac{h}{2c})}{\partial x^2} - \frac{h}{c}\frac{\partial u(x, t-\frac{h}{2c})}{\partial t}\Big]. \tag{3.11}$$

Now, one is told to shift time forward by $\frac{h}{2c}$ and after that expand also the left side which will give us

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{ahc}\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{2(\frac{h}{c} + \frac{1}{a})}{h^2}\frac{\partial u(x,t)}{\partial t} = 0. \tag{3.12}$$

which is partial differential equation usually called telegrapher's equation usually presented as relativistic generalization of non-relativistic diffusion. [6], [56], [33].
In the market the relativistic effect can be observed also according to [43]. These effect comes from the fact, that the evolution of market prices is random and in general influenced by the market friction. On the market there always someone, who is not willing to pay the required amount of money and therefore, the speed of price evolution is considerably retarded.

3.2 Valuation and Quantum Mechanics analogies

3.2.1 Underlying price equivalence

In e.g. [12], [6], [56] one can find the observed equivalences between the world of quantum mechanics and the world of derivatives valuation. These observarions are base solely on suitable variable transformations and. The original Black-Scholes valuation model can be paired with the the world of non-relativistic quantum mechanics in two ways. the first is, that partial differential equations with diffusion character playing the important role in valuation theory are equivalent to diffusion equations from theory of stochastic processes, which is equivalent to the non-relativistic quantum theory. So lets' have

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \tag{3.13}$$

which is the stochastic differential equation for spot prices. According to theory of multidimensional Fokker-Planck equation (see e.g. App.C) one can immediately write corresponding diffusion equation for this stochastic evolution in the form

$$\frac{\partial f(S(t),t)}{\partial t} = -\mu\frac{\partial}{\partial S}f(S(t),t) + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial S^2}S^2 f(S(t),t) \tag{3.14}$$

This is Fokker-Planck equation for geometric Wiener process which logarithm behaves like Wiener process with drift μ. But one is able to make the equation for that drift Wiener process. If one uses Itô lemma as presented in App.B, one immediately gets

$$d(\ln(S(t)) = \frac{\partial \ln(S(t))}{\partial t}dt + \frac{\partial \ln(S(t))}{\partial S}\Big(\mu S(t)dt + \sigma S(t)dW(t)\Big) + frac12\frac{\partial^2 \ln(S(t))}{\partial S^2}\sigma^2 S^2(t)dt \tag{3.15}$$

and rearranging the terms one gets

$$d(\ln(S(t))) = \Big(\mu - \frac{1}{2}\sigma^2\Big)dt + \sigma dW(t). \tag{3.16}$$

so we simply got the stochastic differential equation for Wiener process with drift μ. For this process, again from the Fokker-Planck-Langevin equations equivalence, we get

$$\frac{\partial f(x,t)}{\partial t} + \Big(\mu - \frac{1}{2}\sigma^2\Big)\frac{\partial f(x,t)}{\partial x} = \frac{1}{2}\sigma^2\frac{\partial^2 f(x,t)}{\partial x^2} \tag{3.17}$$

where $x \equiv \ln(S(t))$. At this moment, one may transform the variable

$$\xi(t) = x - \delta t \tag{3.18}$$

where we denoted $\delta = \Big(\mu - \frac{1}{2}\sigma^2\Big)$ and in the following $\gamma = \frac{\sigma^2}{2}$. Applying this transformation one gets

$$\frac{\partial f(\xi,t)}{\partial t} + \delta\frac{\partial f(\xi,t)}{\partial \xi} = \gamma\frac{\partial^2 f(\xi,t)}{\partial \xi^2} \tag{3.19}$$

and applying additional time transformation with

$$\tau = \gamma t \tag{3.20}$$

one obtains

$$\frac{\partial f(\xi,\tau)}{\partial \tau} = \frac{\partial^2 f(\xi,\tau)}{\partial \xi^2} \tag{3.21}$$

However, this is well-known diffusion equation equivalent to heat equation. This problem has it's fundamental solution in the form

$$f_F(\xi(\tau),\tau) = \frac{1}{\sqrt{4\pi\tau}} \exp\Big(-\frac{\xi(\tau)^2}{4\tau}\Big) \tag{3.22}$$

for initial condition

$$f(\xi(0),0) = \delta(\xi - 0) \tag{3.23}$$

where we emphasized the initial coniditon with position value $\xi(0) = 0$. The general solution for arbitrary initial condition

$$f(\xi(\tau_0),\tau_0) = \delta(\xi - \xi_0) \tag{3.24}$$

for the shifted time $t_0 = 0 \to t_0 \neq 0$ is

$$f(\xi,\tau) = \iint \frac{1}{\sqrt{4\pi\tilde{\tau}}} \exp\Big(-\frac{\tilde{\xi}(\tilde{\tau})^2}{4\tilde{\tau}}\Big)\delta(\tilde{\xi} - (\xi - \xi_0))\delta(\tilde{\tau} - (\tau - \tau_0))\, d\tilde{\xi}\, d\tilde{\tau} \tag{3.25}$$

which only means that

$$f(\xi,\tau) = \frac{1}{\sqrt{4\pi(\tau-\tau_0)}} \exp\Big(-\frac{(\xi-\xi_0)^2}{4(\tau-\tau_0)}\Big) \tag{3.26}$$

which is nothing but Gaussian probability density for Wiener process. After restoring variables, one obtains final result in the form

$$f(S(t),t) = \frac{1}{S(t)\sqrt{2\pi\sigma^2(t-t_0)}} \exp\Big[-\frac{\Big(\ln(\frac{S(t)}{S(0)}) - (\mu - (\frac{\sigma^2}{2}))(t-t_0)\Big)^2}{2\sigma^2(t-t_0)}\Big] \tag{3.27}$$

which is nothing but the probability distribution function for geometric Wiener process created by log-normally distributed random variables at each time. The same result we would get if we presumed from the beginning, that drift coefficient is zero $\mu = 0$. So overall one can see, that from classical Wiener process it is very easy to come to process with log-normal random variables, i.e. geometric Wiener process. The price evolution of underlying security $S(t)$ is therefore equivalently described the Wiener process with drift and due to variable transformations also classical Wiener process with standard Gaussian distribution. Therefore it is usual to start from the diffusion equation in the form 3.21 and in the process of generalization work with it, not with the original geometric Brownian motion, for which the corresponding diffusion equation is more complicated. So now, to finally show the correspondence between the underlying security price process and quantum mechanics, one has to remind the non-relativistic particle Schrodinger equation in one dimension

$$-i\hbar\frac{\partial\psi(x,t)}{\partial t} = \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) \tag{3.28}$$

and rearranging the constants toghether with denoting $c = \frac{i\hbar}{2m}$ and $\psi(x,t) = f(x,t)$ one immediately gets

$$\frac{\partial f}{\partial t} = c\frac{\partial^2 f}{\partial x^2} \tag{3.29}$$

which is the diffusion equation in the form as we have seen in 3.21, with the additional constant in front of the second derivative on the r.h.s. So one can see, that non-relativistic quantum mechanics theory in Schrondinger wave function approach is equivalent to stochastic underlying asset evolution up to some constant c. The mathematical difference stems from the fact, that to make quantum mechanics work, one must assume, that wave function $\psi(x,t) \in L^2(\mathbb{R})$ while in the world of stochastic processes, we have weaker condition $f(x,t) \in L^1(\mathbb{R})$ for some chosen time t. So one can build the relativistic generalization of derivative valuation on this type of equivalence between stochastic processes and quantum mechanics. If one has this equivalence, then exploiting the relativistic quantum mechanics theory and doing similar variable transformations would lead to relativistic generalization of stochastic process described by the generalized probability density given as solution of that generalized equation. This approach can be seen in [43], [57] and will be revised later.

3.2.2 Derivative price equivalence

The second way, how to show the equivalence between quantum mechanics and stochastic processes, concretely option valuation theory is to start directly from the the expressions describing the price of the option. This idea is widely presented in the [12] and its generalizations can be seen in [52], [53]. The classic BSM partial differential equation written in the form

$$\frac{\partial C(S(t),t)}{\partial t} = -\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S(t),t)}{\partial S^2} - rS\frac{\partial C(S(t),t)}{\partial S} + rC(S(t),t) \tag{3.30}$$

can be rewritten in the form

$$\frac{\partial C(x,\tilde{t})}{\partial \tilde{t}} = \Big[-\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \Big(\frac{1}{2}\sigma^2 - r\Big)\frac{\partial}{\partial x} + r\Big]C(x,\tilde{t}) \tag{3.31}$$

which is reached by already known variable transform $x(\tilde{t}) = \ln(S(\tilde{t}))$ or equivalently $S(t) = e^{x(t)}$. This equation is equivalent to the Schrodinger equation, if one presumes, that

$$\hbar = 1 \tag{3.32}$$

and *Wick's rotation*

$$\tilde{t} = it \tag{3.33}$$

($\tilde{t}$ will simply be denoted as t always when one deals with valuation equations) with new Black-Scholes option pricing Hamiltonian operator

$$H_{BS} = \Big[-\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \Big(\frac{1}{2}\sigma^2 - r\Big)\frac{\partial}{\partial x} + r\Big] \tag{3.34}$$

which is however not hermitian as the Hamiltonians in the quantum theory

$$H_{BS}{}^{\dagger} = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} - \Big(\frac{1}{2}\sigma^2 - r\Big)\frac{\partial}{\partial x} + r \neq H_{BS} \tag{3.35}$$

so the interpretation of eigenvalues as possible values obtained during measurement of observables known from quantum mechanics can not be true at least when dealing with financial counterpart of hamiltonian operator H_{BS} - some kind of "financial energy". To illustrate that the quantum notation encoding the option price is not possible solely for quite simple Black-Scholes valuation model, one can do the same for stochastic volatility Merton-Garman equation

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + (\lambda + \kappa(\sigma^2))\frac{\partial C}{\partial(\sigma^2)} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} + \rho\omega\sigma^{1+2\alpha}S\frac{\partial^2 C}{\partial S\,\partial(\sigma^2)} + \omega^2\sigma^{4\alpha}\frac{\partial^C}{\partial(\sigma^2)^2} = rC \quad (3.36)$$

with extended variable transformation in the form

$$S(t) = e^x \quad (3.37)$$

together with

$$\sigma^2 = e^y \quad (3.38)$$

for square of stochastic volatility. The Merton-Garman equation becomes

$$\frac{\partial C}{\partial t} + \left(r - \frac{e^y}{2}\right)\frac{\partial C}{\partial x} + \left(\lambda e^{-y} + \kappa - \frac{\omega^2}{2}e^{2y(\alpha-1)}\right)\frac{\partial C}{\partial y} + \frac{e^y}{2}\frac{\partial^2 C}{\partial x^2} + \rho\omega e^{y(\alpha-1/2)}\frac{\partial^2 C}{\partial x\partial y} + \omega^2 e^{2y(\alpha-1)}\frac{\partial^C}{\partial y^2} = rC \quad (3.39)$$

and one can rewrite this equation into the same form as Black-Scholes equation

$$\frac{\partial C}{\partial t} = H_{MG}C \quad (3.40)$$

with Merton-Garman Hamiltonian

$$H_{MG} = -\left(r - \frac{e^y}{2}\right)\frac{\partial}{\partial x} - \left(\lambda e^{-y} + \kappa - \frac{\omega^2}{2}e^{2y(\alpha-1)}\right)\frac{\partial}{\partial y} - \frac{e^y}{2}\frac{\partial^2}{\partial x^2} - \rho\omega e^{y(\alpha-1/2)}\frac{\partial^2}{\partial x\partial y} - \frac{\omega^2 e^{2y(\alpha-1)}}{2}\frac{\partial}{\partial y^2} + r \quad (3.41)$$

In the previous chapter we presented Feynman-Kac formula. This formula says, that function solving special form of partial differential equation can be expressed as expected value of this function in terms of given Itô process with respect to risk-neutral measure equivalent to the original probability measure. And here comes another analogy of quantum mechanics comparing it with option valuation. Because Black-Scholes option price is the solution of the same type of partial differential equation as that in the Feynman-Kac theorem, it can be expressed as

$$C(S(t), t) = \int_0^{+\infty} f(S(T), T|S(t), t)g(S(T))\frac{dS(T)}{S(T)} \quad (3.42)$$

where $f(S(T), T|S(t), t)$ stands for transition probability for spot price of underlying security being at time t in state $S(t)$ and at some later (i.e. maturity) time T in state $S(T)$ and $g(S(T))$ represents pay-off function (i.e. value of option at maturity time T). In terms of $x(t) = \ln(S(t))$ the last equation has the same form

$$C(x(t), t) = \int_{-\infty}^{+\infty} f(x(T), T|x(t), t)g(x(T))dx(T) \quad (3.43)$$

In the case of stochastic volatility model, the equation would be the same with additional variable σ^2 representing stochastic volatilty and can be found in [12]. At this moment, one can see another analogy with quantum mechanics, the desired fair option price can be represented as integral form comprising "stochastic propagator" (i.e. transitional probability $f(S(T), T|S(t), t)$) multiplied by the fair price of option given by it's pay-off, which is nothing but $g(S(T)) = \max(S(T) - K, 0)$ for call option. The only difference between classical quantum mechanics and stochastic valuation is the fact, that quantum propagator gives the evolution of wave function forward in time, while here it is evolution backward from T to t. So if one could calculate the propagator for the quantum system connected through the already mentioned stochastic-quantum equivalence, this would simply mean, that he would automatically get the wanted fair option price at time t. The integral kernel, or propagator is determined by the Hamiltonian of the system. In non-relativistic quantum mechanics, this is generally expressed by

$$H(x, \frac{\partial^2}{\partial x^2}, t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t) \tag{3.44}$$

for the mass particle moving in the enviroment under some potential force acting on it. Generally, in the theory of quantum mechanics, this propagator can be calculated directly from

$$f(x(t_f), t_f|x(t_0), t_0) =< x(t_f)|\hat{U}(t_f, t_0)|x(t_0) > \tag{3.45}$$

where the unitary operator $\hat{U}(t_f, t_0)$ called *evolution operator* can be written in terms of above mentioned Hamiltonian in the cases, where the potential part is not explicitly dependent on time variable. We can then write

$$\hat{U}(t_f, t_0) = \exp\Big[-\frac{i}{\hbar}\hat{H}(x, \frac{\partial^2}{\partial x^2})(t_f - t_0)\Big]. \tag{3.46}$$

So the original expression for the option price can be rewritten in the form

$$C(x(t), t) = \int_{-\infty}^{+\infty} < x(t)|\exp\Big[-\hat{H}(x, \frac{\partial^2}{\partial x^2})(T - t)\Big]|x(T) > g(x(T))dx(T) \tag{3.47}$$

where we similarly to previous sections used the notation $t = t_0$ and $t_f = T$ and of course all this analogy occurs in Wicked rotated imaginary time together with presumption $\hbar = 1$. The difference with the quantum mechanics is immediately seen, as the pricing integral kernel is evolving the system backwards the time, in QM we see forward evolution. One has to be aware of the fact, that all of this theory presented here is analogically as in the quantum mechanics usable only in the situations, in which the Hamiltonian operator is not explicitly dependent on time coordinate, in other words, when it does not comprise potential term or this potential term is not explicitly dependent on time variable. In the [12] it can be seen, that "Financial Hamiltonians" with general potential term can be seen as tool for calculating the *barrier options*, which we will not be further interested in.
The standard Black-Scholes-Merton matrix element has the form

$$f_{BSM}(x(T), T|x(t), t) =< x(t)|\exp\Big[-\hat{H}_{BS}(x, \frac{\partial^2}{\partial x^2})(T - t)\Big]|x(T) > \tag{3.48}$$

If one tries to compute the matrix element, one uses the resolution of indentity

$$\int_{-\infty}^{+\infty} dp|p >< p| = \mathbb{1} \tag{3.49}$$

and inserts it into the previous relation

$$f_{BSM}(x(T), T|x(t), t) = \int_{-\infty}^{+\infty} < x(t)|\exp\Big[-\hat{H}_{BS}(x, \frac{\partial^2}{\partial x^2})(T - t)\Big]|p(t) >< p(t)|x(T) > dp(t). \tag{3.50}$$

This is convenient, because

$$< p|x >= \frac{1}{\sqrt{2\pi}}e^{-ixp} \tag{3.51}$$

$$< x|p >= \frac{1}{\sqrt{2\pi}} e^{+ixp} \tag{3.52}$$

where one has to remember, that in the world of finance $\hbar = 1$ and one also has

$$< x|H_{BS}|p >\equiv H_{BS} < x|p >= H_{BS} \frac{1}{\sqrt{2\pi}} e^{+ixp}. \tag{3.53}$$

These relations then help with evaluating the transitional probability kernel and it becomes

$$f_{BSM}(x(T), T|x(t), t) = e^{-r(T-t)} \int_{-\infty}^{+\infty} \frac{dp(t)}{\sqrt{2\pi}} \exp\Big[-\frac{1}{2}(T-t)\sigma^2 p(t)^2\Big] \exp\Big[ip(t)(x(t)-x(T))+(T-t)(r-\frac{\sigma^2}{2})\Big] \tag{3.54}$$

and using well-known Gaussian integrals one gets

$$f_{BSM}(x(T), T|x(t), t) = e^{-r(T-t)} \frac{1}{\sqrt{2\pi(T-t)\sigma^2}} \exp\Big[-\frac{1}{2(T-t)\sigma^2}\Big(x(t) - x(T) + (T-t)(r - \frac{\sigma^2}{2})\Big)^2\Big] \tag{3.55}$$

but this is exatly the same pricing kernel (or "financial propagator") as in the case 3.27. So this quantum mechanics-financial equivalence gives the same results as traditional methods as risk-neutral valuation or solving the Black-Scholes-Merton partial differential equation. The same results are obtained when working with e.g. stochastic volatility Merton-Garman model. Therefore one can work, besides the previous observed equivalence between quantum wave functions and probability distributions for underlying prices, with this approach when trying to generalize Black-Scholes-Merton valuation model, too, to be concrete, for instance utilizing the methods of relativistic quantum mechanics.

3.3 Telegrapher's equation

It could be very surprising, that besides the framework of quantum mechanics, the pieces from the theory of electromagnetism can be also exploited when trying to generalize the valuation processes. This stems from the effort of physical community to be able to describe relativistic generalizations of original Einstein's-Smoluchowski model of Brownian motion comprising the heat equation-like diffusion equation [1] [60]. Wiener process seemed to be very useful to describe various physicial phenomena and from the times, when Louis Bachelier used the Wiener process for modelling stock prices on the market in 1900, the diffusion processes already describing phenomena in physics, biology etc. naturally soaked in the world of mathematical finance. The relativistic generalization of original Brownian motion theory was suggested in 1926 by V.A.Fock [58] and the original diffusion equation became so called *telegraph-damped wave equation*. Another apporoach leading to the same result was suggested by Goldstein in 1950 [36]. However, the telegraph equation first occured in the works of W.Thomson (and yes, that Lord Kelvin) from 1854 [61] to describe propagation of electric signal going through transatlantic cable. Because the fact, that the framework of the Telegrapher's equation and the properties of it will be crucial in the following, it is very important to remind the main results. The main source for this section is [33]. Counting process can be defined as

$$N(t) = \max\{n : \tau_n \leqslant t\} \tag{3.56}$$

where τ_n respresents the time, when the observed phenomenon in th system occured n-th time. The random variable τ_i is then nothing but stopping time for i-th observed occurence. With these konwledge, one can define

Definition 3.3.1 (Poisson process). *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be given probability space. Then the adapted counting stochastic process $N = \{N(t)\}$ on this space is called Poisson process , if it has properties*

- *The number of phenomenon occurences in the arbitrary time intervals of $(t, t+\tau)$ depends only on chosen τ, not on t, i.e. stationarity of increments,*
- *The distribution of phenomenon occurences in arbitray time interval $(t, t+\tau$ does not depend on history of the process (i.e. on number of occurences in times $\tilde{t} < t$ = Markov property),*
- $\mathbb{P}\Big[N(t+\Delta t) \neq N(t) = \lambda\Delta t + O(\Delta t)\Big]$

for small time intervals Δt where $\lambda > 0$ is called process intensity.

From this definition it is possible to prove, that

$$\mathbb{P}\big[N(t) - N(s) = k\big] = \frac{[\lambda(t-s)]^k}{k!} e^{-\lambda(t-s)}. \tag{3.57}$$

The sequence of stopping times (t_i) with condition $\tau_0 = 0$ is then distributed like

$$\mathbb{P}\Big[\tau_k - \tau_{k-1}\Big] = e^{-\lambda t} \tag{3.58}$$

where obviously $t \geqslant 0$ together with *Erlang distribution*

$$f(\tau_i, t) = \frac{(\lambda t)^{i-1}}{(i-1)!} \lambda e^{-\lambda t}. \tag{3.59}$$

These properties for counting Process stochastic process will be used to define Goldstein-Kac telegraph process. One first define *direction process* $\{D(t)\}$ in the form

$$D(t) = D(0)(-1)^{N(t)} \tag{3.60}$$

which is simply process encoding the actual direction of particle moving on line and randomly switching its direction throughout the time. For forward (right) direction, $D(t) = +1$, for backward (left) direction $D(t) = -1$. Additional assumption for direction process is

$$\mathbb{P}\Big[D(0) = +1\Big] = \mathbb{P}\Big[D(0) = -1\Big] = \frac{1}{2}. \tag{3.61}$$

If the particle is moving on the line with constant velocity, let denote it v, then the particle position on the line is given by

$$X(t) = v \int_0^t D(s) ds = vD(0) \int_0^t (-1)^N (s) ds \tag{3.62}$$

The stochastic process created by the random variables $X(t)$ is called *Goldstein-Kac telegrapher process*. If the particle started with forward direction, then one would have

$$X^+(t) = +v \int_0^t (-1)^{N(s)} ds \tag{3.63}$$

and analogously for initial backward direction

$$X^-(t) = -v \int_0^t (-1)^{N(s)} ds. \tag{3.64}$$

One can define the probability densities to the telegrapher process and also to the processes $X^+(t)$, $X^-(t)$. For this sake, one has to realize, that

$$\mathbb{P}\Big[X^+(t) = +vt\Big] = \mathbb{P}\Big[X^-(t) = -vt\Big] = e^{-\lambda t}. \tag{3.65}$$

This is simmple the mathematical expression of the fact, that the chance of particle being at the point of the maximal reach, so $+vt$ for the process $X^+(t)$ and $-vt$ for $X^-(t)$ are given by the probability distribution of driving Poisson process. To be concrete, the particle will come to the maximal reach point if and only if the $N(t) = 0$ and the probability for such counting process value is given by

$$\mathbb{P}[N(t) = 0\Big] = e^{-\lambda t} \tag{3.66}$$

And realizing that telegraph process $X(t)$ is just compound of the above described partial processes $X^+(t)$, $X^-(t)$ one gets

$$\mathbb{P}\Big[X(t) = +vt\Big] = \mathbb{P}\Big[X(t) = -vt\Big] = \frac{1}{2}e^{-\lambda t}. \tag{3.67}$$

In both cases of initial condition (i.e. $D(0) = +1$ or $D(0) = -1$) the particle moving for the fixed time t can not run over distance bigger than $|x| = vt$, the points in bigger distance are simply unreachable, in the terminology of special relativity, they are "space-like". All the knowledge from above will now enable us to write the probability densities for processes $X(t)$, $X^+(t)$ and $X^-(t)$. So one can write

$$p_+(x, t) = e^{-\lambda t}\delta(x - vt) + P_+(x, t)\mathbb{1}_{|x|<vt} \tag{3.68}$$

similarly

$$p_-(x, t) = e^{-\lambda t}\delta(x + vt) + P_-(x, t)\mathbb{1}_{|x|<vt} \tag{3.69}$$

and finally for telegraph process

$$p_-(x, t) = \frac{1}{2}e^{-\lambda t}\Big[\delta(x + vt) + \delta(x - vt)\Big] + P(x, t)\mathbb{1}_{|x|<vt}. \tag{3.70}$$

where $P(x, t) = \frac{P_+(x,t)+P_-(x,t)}{2}$.

At this moment, one would like to express the latter probability densities in terms of their counterparts. Because

$$X(t) = D(0)vt\mathbb{1}_{\tau>t} + \Big[D(0)vt + X_r(t - \tau)\Big]\mathbb{1}_{\tau<t} \tag{3.71}$$

where $D(0)$ denotes the already mentioned initial direction of particle's motion and X_r represents the motion starting at the opposite direction $-D(0)$ and is independent of the original motion $X(t)$. Form this fact it follows that

$$p_+(x, t) = e^{-\lambda t}\delta(x - vt) + \int_0^t p_-(x - vs, t - s)\lambda e^{-\lambda s}ds \tag{3.72}$$

and

$$p_-(x, t) = e^{-\lambda t}\delta(x + vt) + \int_0^t p_-(x + vs, t - s)\lambda e^{-\lambda s}ds \tag{3.73}$$

Similar relations can be observed for probability densities, that in time t the particle is at position $x(t)$ and it's direction is either $D(t) = +1$, or $D(t) = -1$. For the first case one has

$$f(x,t) = \frac{1}{2}e^{-\lambda t}\delta(x - vt) + \int_0^t b(x - v(t-s), s)\lambda e^{-\lambda(t-s)}ds \tag{3.74}$$

and similarly for the case $D(t) = -1$

$$b(x,t) = \frac{1}{2}e^{-\lambda t}\delta(x + vt) + \int_0^t f(x + v(t-s), s)\lambda e^{-\lambda(t-s)}ds \tag{3.75}$$

The relations between the above mentioned probability densities fucntion are as following

$$p(x,t) = \frac{1}{2}\Big(p_+(x,t) + p_-(x,t)\Big) \tag{3.76}$$

and

$$p(x,t) = f(x,t) + b(x,t). \tag{3.77}$$

The main reason, why one wants to obtain explicit formulas for probability densities connected with the telegrapher process $X(t)$ is, that they fulfill two kinds of well-known equations. Now is the moment to present the first result from the theory of telegrapher's process which is usually seen in the literature (i.e. [34])

Theorem 3.3.1 (Kolmogorov equations)**.** *Probability densities $p_+(x,t)$, $p_-(x,t)$, $f(x,t)$, $b(x,t)$ fulfill the following system of partial differential equations (so called Cattaneo system)*

$$\frac{\partial p_+(x,t)}{\partial t} + v\frac{\partial p_+(x,t)}{\partial x} + \lambda p_+(x,t) - \lambda p_-(x,t) = 0 \tag{3.78}$$

$$-\lambda p_+(x,t) + \frac{\partial p_-(x,t)}{\partial t} - v\frac{\partial p_-(x,t)}{\partial x} + \lambda p_-(x,t) = 0 \tag{3.79}$$

$$\frac{\partial f(x,t)}{\partial t} + v\frac{\partial f(x,t)}{\partial x} + \lambda f(x,t) - \lambda b(x,t) = 0 \tag{3.80}$$

$$-\lambda f(x,t) + \frac{\partial b(x,t)}{\partial t} - v\frac{\partial b(x,t)}{\partial x} + \lambda b(x,t) = 0 \tag{3.81}$$

for $\forall x \in [-vt, +vt]$ with initial conditions

$$p_+(x,0) = p_-(x,0) = \delta(x) \tag{3.82}$$

$$f(x,0) = b(x,0) = \frac{1}{2}\delta(x) \tag{3.83}$$

Moreover, for $\forall x \in \mathbb{R}/\{[-vt, +vt]\}$

$$p_+(x,t) = p_-(x,t) = f(x,t) = b(x,t) = 0. \tag{3.84}$$

It is possible to show, that these coupled differential equations are equivalent to the set of second order partial differential equations valid for each component separately usually called *telegrapher's equation* or *damped-wave equations*. To be precise, the following theorem holds

Theorem 3.3.2 (Telegrapher's equation)**.** *Let $\xi(x, t)$ be arbitrary function from the set of probability densities $p_+(x, t)$, $p_-(x, t)$, $f(x, t)$ and $b(x, t)$. Then $\xi(x, t)$ is the solution of the so called telegrapher's partial differential equation in the form*

$$\frac{\partial^2 \xi(x,t)}{\partial t^2} + 2\lambda \frac{\partial \xi(x,t)}{\partial t} = v^2 \frac{\partial^2 \xi(x,t)}{\partial x^2} \tag{3.85}$$

where $t > 0$, $x \in \mathbb{R}$ with initial conditions

$$p_\pm|_{t=0} = \delta(x) \qquad \frac{\partial p_\pm}{\partial t}|_{t=0} = \mp v\delta'(x) \tag{3.86}$$

$$f|_{t=0} = b|_{t=0} = \frac{1}{2}\delta(x) \qquad \frac{\partial f}{\partial t}|_{t=0} = -\frac{\partial b}{\partial t}|_{t=0} = -\frac{v}{2}\delta'(x) \tag{3.87}$$

It is clear that the last equation is also valid for the linear combination probability density $p(x, t)$ with slightly modified coefficients. Solving this equation for the probaiblity density $p(x, t)$ gives

$$p(x,t) = \frac{e^{-\lambda t}}{2v}\Big[\delta(vt + x) + \delta(vt - x)\Big] + \frac{e^{-\lambda t}}{2v}\Big[\lambda I_0\Big(\frac{\lambda}{v}\sqrt{v^2t^2 - x^2}\Big) + \frac{\partial}{\partial t} I_0\Big(\frac{\lambda}{v}\sqrt{v^2t^2 - x^2}\Big)\Big]\mathbb{1}_{|x|<vt} \tag{3.88}$$

for any $t > 0$, $x \in \mathbb{R}$ where $I_0(x)$ denotes modified Bessel function in the form

$$I_0(x) = \sum_{m=0}^{+\infty} \frac{1}{m!\Gamma(m+1)}\Big(\frac{x}{2}\Big)^{2m} = \sum_{m=0}^{+\infty} \frac{1}{(m!)^2}\Big(\frac{x}{2}\Big)^{2m} \tag{3.89}$$

The most important property of the probability distribution function $p(x, t)$ crucial for the use of the theory of Goldstein-Kac telegrapher process in relativistic generalization of Black-Scholes valuation theory is the fact, that under certain conditions it converges to the probability distribution function of Wiener process.

Theorem 3.3.3 (Telegrapher's process convergence)**.** *Let $\{X(t)\}$, $t < 0$ be the Goldstein-Kac telegrapher process comprising parameters of velocity v and intensity λ. Then if*

$$v \to +\infty \qquad \lambda \to +\infty \tag{3.90}$$

together with

$$\frac{v^2}{\lambda} \to \sigma^2, \tag{3.91}$$

then

$$X(t) \xrightarrow{d} W(t) \tag{3.92}$$

where $W(t)$ is the Wiener process which starts from $W(0) = 0$ with zero mean value $\mu = 0$ and variance σ^2.

the rigorous proof may be again found in [33]. Heuristically, the convergence to the diffusion process is not so surprising, as the infinity particle's velocity limit should provide us with some kind of distribution, for which the probability of particle being at "space-like" unreachable point at every time is non-zero. In mathematical terms, the compact support of the probability $p(x, t)$ $[-vt, vt]$ should be widening together with rising particle's velocity v.
For this moment, this is everything neccessary for further utilizing the theory of telegrapher process in the next section.

3.4 Relativistic Quantum Mechanics approach

After revising all important topics neccessarry for Black-Scholes-Merton valuation model generalization, one is able to understandably indroduce the brief summary of already known results and try to investigate and possibly brings new ideas and extensions of them. As we alredy said in the previous sections, the reasons for relativistic generalization of original Black-Scholes theory are in general two. The first is fact, that one has to deal with consequences of high-frequency trading in the market and therefore incorporate the principles the special relativity into the theory [55], [50], [49], [43]. Second reason stems from the fact, that the assumption with constantness of volatility parameter σ in the BSM theory turned out to be wrong, especially after Black Monday in 1987. There are relativistic generalizations of the BSM model, which can flatten that volatility smile or volatility frown [53], [52], [57]. In the following, except for canonical quantum mechanics interpretation analogy with the valuation theory, one can observe the efforts for using the framework of path integral when determining the option price [57] [12]. To begin with analysis of the first reason which led to generalization of Black-Scholes valuation model, we will briefly revise the approach of [43].
According to this idea, there exists the concept of so called "market speed of light" c_m arising from the fact, that if the underlying security has the spot price $S(t)$ at time t, then the spot price value change to $S(t + \Delta t)$ at later time can not be instant, i.e. the expression

$$v_m \equiv \lim_{\Delta t \to 0} \frac{S(t + \Delta t) - S(t)}{\Delta t} \tag{3.93}$$

is not only always smaller than $+\infty$, so $v_m < +\infty$, but is also upper bounded. This upper bound for such a velocity is then denoted c_m (very nice comparision is done, when one can imagine the market as some conductive material and effective spot price velocity has then an analogy with electrons moving throughout of that material with effective velocity, which is much smaller than maximal possible speed that electron can experience). This value is given by the speed of light in vacuum c, because the signal about concrete change of spot price transmits with the speed of light in the vacuum and also from desire of investors for buying or selling the underlying security. From the real market data [43] it turns out ,that

$$\left|\frac{dx(t)}{dt}\right| = \frac{\left|\frac{dS(t)}{dt}\right|}{S(t)} < \frac{c_m}{S(t)} < 1 \tag{3.94}$$

So the primary aim is to find the probability distribution $p_r(x, t)$, which will serve as the integration kernel (or as said above financial propagator) in the risk-neutral Feynman-Kac valuation formula, i.e.

$$C(x(t), t) = e^{-r(T-t)} \int_{-\infty}^{+\infty} f(x(T), T|x(t), t) g(x(T)) dx(T) \tag{3.95}$$

or in the variables of spot price itself $S(t) = e^x$ as

$$C(S(t), t) = e^{-r(T-t)} \int_{0}^{+\infty} f(S(T), T|S(t), t) g(S(T)) \frac{dS(T)}{S(T)} \tag{3.96}$$

This is then rewritten with notation

$$C(S(t), t) = e^{-r(T-t)} \int_{0}^{+\infty} f(x(S(T)), T - t) g(S(T)) \frac{dS(T)}{S(T)} \tag{3.97}$$

where

$$x(S(T)) = \ln \frac{S(t)}{S(T)} - +\Big(r - \frac{1}{2}\sigma^2\Big)(T - t). \tag{3.98}$$

And now is the moment, when one needs the main idea of this approach. The question is, how to find out the function $f(X(S(T)), T-t)$? After getting acquainted with the previous section on telegrapher's equation, the answer is quite simple. One will use the fact already known from the previous text, to be concrete, that Schrodinger equation is equivalent to non-relativistic diffusion equation through Wick rotation. The one of several relativistic generalizations of Schrodinger equation is Dirac equation in 1 dimension for position and 1 dimension for time. In the following we show, that this (1+1) equation is equivalent to Kolmogorov equations from the previous section and hence equivalent to telegrapher's equation.
The Dirac equation in (1+1) dimension has the form [35],[23]

$$\Big(i\hbar\gamma^{\mu}\partial_{\mu} - m_0 c\Big)\psi(x,t) = 0 \tag{3.99}$$

with γ^{μ} being gamma matrices, however in our case the greek-indices represents the 2-dimensional analogy to the standard 4-dimensional Minkowski space case, i.e. $\mu \in \{0, 1\}$. Breaking down the notation one obtains

$$\Big(i\hbar\gamma^{0}\partial_{0} + i\hbar\gamma^{1}\partial_{1} - m_0 c\Big)\psi(x,t) = 0 \tag{3.100}$$

multiplying by the speed of light in vacuum c

$$\Big(ic\hbar\gamma^{0}\partial_{0} + ic\hbar\gamma^{1}\partial_{1} - m_0 c^2\Big)\psi(x,t) = 0 \tag{3.101}$$

and finally breaking down the notation of 4-gradient (App.A) one obtains

$$\Big(i\hbar\gamma^{0}\frac{\partial}{\partial t} + ic\hbar\gamma^{1}\frac{\partial}{\partial x} - m_0 c^2\Big)\psi(x,t) = 0 \tag{3.102}$$

now in order to get this form of the 1+1 Dirac equation in form similar to that used in i.e.[34], [38] one has to multiply the whole equation with the matrix $\gamma^{0^{-1}}$ from the left side, hence get

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} + i\hbar c\Big(\gamma^{0^{-1}}\gamma^{1}\Big)\frac{\partial\psi(x,t)}{\partial x} - m_0 c^2\gamma^{0^{-1}}\psi(x,t) = 0 \tag{3.103}$$

and rearranging all terms into

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = m_0 c^2\gamma^{0^{-1}}\psi(x,t) - i\hbar c\Big(\gamma^{0^{-1}}\gamma^{1}\Big)\frac{\partial\psi(x,t)}{\partial x} \tag{3.104}$$

which is the same form of 1+1 Dirac equation as could be seen in above mentioned works. This derivations is crucial, as it is not explicitly done ine both of that sources. The last but certainly not least step is to establish following correspondence

$$\gamma^{0^{-1}} = \sigma_1 \tag{3.105}$$

$$\gamma^{0^{-1}}\gamma^{1} = \sigma_3 \tag{3.106}$$

to obtain the equation in exactly the same form as can be seen there, where σ_1, σ_3 denote Pauli matrices in the form

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.107}$$

and

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.108}$$

So one has the new form of 1+1 Dirac equation

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = \Big(- ic\hbar\sigma_3 \frac{\partial}{\partial x} + \sigma_1 m_0 c^2 \Big)\psi(x,t) \tag{3.109}$$

for Dirac spinor

$$\psi(x,t) = \begin{pmatrix} \psi_+(x,t) \\ \psi_-(x,t) \end{pmatrix}. \tag{3.110}$$

At the very end of this derivation, one has to verify, whether the Clifford algebra relations hold, i.e.

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_{2\times 2} \tag{3.111}$$

but this can be immediately seen, if one exploits relations from 3.105 and 3.106. Simple matrix multiplications gives us

$$\gamma^0 = \sigma_1 \tag{3.112}$$

and

$$\gamma^1 = \sigma_1\sigma_3. \tag{3.113}$$

Utilizing the well-known rules for Pauli matrices already gives the desired result 3.111. Now when the Dirac equation in (1+1) dimension is derived, one can change the variables to obtain already well-known telegrapher's equation. Using

$$u(x,t) = e^{\frac{im_0 c^2 t}{\hbar}} \psi(x,t) \tag{3.114}$$

or equivalently expressed as

$$\psi(x,t) = e^{\frac{-im_0 c^2 t}{\hbar}} u(x,t). \tag{3.115}$$

Inserting this expression to the (1+1) Dirac equation coupled system for each Dirac spinor component in the form

$$i\hbar \frac{\partial \psi_+(x,t)}{\partial t} = -ic\hbar \frac{\partial \psi_+(x,t)}{\partial x} + m_0 c^2 \psi_-(x,t) \tag{3.116}$$

$$i\hbar \frac{\partial \psi_-(x,t)}{\partial t} = +ic\hbar \frac{\partial \psi_-(x,t)}{\partial x} + m_0 c^2 \psi_+(x,t) \tag{3.117}$$

gives

$$\frac{u_+(x,t)}{\partial t} = -c\frac{\partial u_+(x,t)}{\partial x} + \frac{im_0 c^2}{\hbar}\Big(u_+(x,t) - u_-(x,t)\Big) \tag{3.118}$$

$$\frac{u_-(x,t)}{\partial t} = +c\frac{\partial u_-(x,t)}{\partial x} + \frac{im_0 c^2}{\hbar}\Big(u_-(x,t) - u_+(x,t)\Big) \tag{3.119}$$

which is nothing but Kolmogorov equations from the theory of Goldstein-Kac telegrapher process section with imaginary "intensity" coefficient

$$\lambda = -\frac{im_0c^2}{\hbar}. \tag{3.120}$$

and particle's velocity denoted as c. The problem is, that intensity of stochastic process schould be real, not imaginary number. This problem is resolved, when one introduces additional transformation, this is the transformation of time and it is nothing but old well-known Wick's rotation $\tilde{t} = -it$ together with introducing Euclidean speed of light $\tilde{c} = -ic$ and transiting into Euclidean world. If one applies this rotation, he gets

$$\frac{u_+(x,\tilde{t})}{\partial\tilde{t}} = -\tilde{c}\frac{\partial u_+(x,\tilde{t})}{\partial x} + \frac{m_0\tilde{c}^2}{\hbar}\Big(u_+(x,\tilde{t}) - u_-(x,\tilde{t})\Big) \tag{3.121}$$

$$\frac{u_-(x,\tilde{t})}{\partial\tilde{t}} = +\tilde{c}\frac{\partial u_-(x,\tilde{t})}{\partial x} + \frac{m_0\tilde{c}^2}{\hbar}\Big(u_-(x,\tilde{t}) - u_+(x,\tilde{t})\Big). \tag{3.122}$$

Now one can see the similar equivalence between (1+1) Dirac equation and Kolmogorov equations hence telegrapher's equations as there was equivalence between Schrodinger equation and classical diffusion equation. To be complete, we write the telepgrapher's equation for each component separately

$$\frac{\partial^2 u_+(x,t)}{\partial t^2} - c^2\frac{\partial^2 u_+(x,t)}{\partial x^2} = +2\frac{im_0c^2}{\hbar}\frac{\partial u_+(x,t)}{\partial t} \tag{3.123}$$

$$\frac{\partial^2 u_-(x,t)}{\partial t^2} - c^2\frac{\partial^2 u_-(x,t)}{\partial x^2} = +2\frac{im_0c^2}{\hbar}\frac{\partial u_-(x,t)}{\partial t}. \tag{3.124}$$

In the classical limit $c \to +\infty$ one recovers the classical diffusion equation equivalent to the Schrodinger equation, for example for the component $u_+(x,t)$ one gets

$$-\frac{\partial^2 u_+(x,t)}{\partial x^2} = +2\frac{im_0}{\hbar}\frac{\partial u_+(x,t)}{\partial t} \tag{3.125}$$

and finally dividing the last equation with $2m_0$ together with multiplying with $\hbar^2$ gives

$$-\frac{\hbar^2}{2m_0}\frac{\partial^2 u_+(x,t)}{\partial x^2} = +i\hbar\frac{\partial u_+(x,t)}{\partial t} \tag{3.126}$$

which is nothing but non-relativistic diffusion equation with rearranged coefficients (again with imaginary diffusion coefficient, for getting rid of this, one can simply make Wick's rotation) and again incorporating wave equation - probability density equivalence $u_+(x,t) \equiv \psi_+(x,t)$ gets

$$-\frac{\hbar^2}{2m_0}\frac{\partial^2 \psi_+(x,t)}{\partial x^2} = +i\hbar\frac{\partial \psi_+(x,t)}{\partial t} \tag{3.127}$$

which is nothing but Schrodinger equation from non-relativistic quantum mechanics.
If one use the above described equivalence between the (1+1) dimensional Dirac equation for particle with spin 0 and telegrapher's equation through already known Kolmogorov equations one will get that the original normal Gaussian distribution of Wiener process as the solution of classical diffusion equation which drives the evolutions of underlying asset price in the original BSM model has to be substituted by

$$p_r(x,t) = \frac{e^{-\lambda t}}{2c_m}\Big[\delta(c_m t + x) + \delta(c_m t - x)\Big] + \frac{e^{-\lambda t}}{2c_m}\Big[\lambda I_0\Big(\frac{\lambda}{c_m}\sqrt{{c_m}^2 t^2 - x^2}\Big) + \frac{\partial}{\partial t} I_0\Big(\frac{\lambda}{c_m}\sqrt{{c_m}^2 t^2 - x^2}\Big)\Big]\mathbb{1}_{|x|<c_m t} \tag{3.128}$$

This equation is already relativistic, because in accordance with given theory of telegrapher's equation it does not allow the particle to be at positions $x > c_m t$, the c_m is again denoting the market velocity of light. Using the variable transformation in the limit $\lambda = \frac{{c_m}^2}{\sigma^2}$ and asymptotic expansion one comes to the following representation of $p(x, T-t)$

$$p_r(x, T-t) \approx \frac{e^{-\frac{x^2}{2\sigma^2(T-t)}}}{\sqrt{2\pi\sigma^2(T-t)}}\Big(1 + \frac{1}{{c_m}^2} f_r(x, T-t)\Big) \tag{3.129}$$

with

$$f_r(x, T-t) = -\frac{\sigma^2}{8(T-t)} + \frac{x^2}{2(T-t)^2} - \frac{x^4}{8\sigma^2(T-t)^3} \tag{3.130}$$

The easiest and heuristic way to see, that the latter expansion is valid follows from the fact, that

$$I_0(a) \approx \frac{e^a}{\sqrt{2\pi a}}\Big(1 + \frac{1}{8a}\Big) \tag{3.131}$$

holds for $a >> 1$. The argument $\frac{\lambda}{c_m}\sqrt{{c_m}^2 t^2 - x^2} = \lambda t\sqrt{1 - \frac{x^2}{t{c_m}^2}}$ is in the investigated limit $c_m \to +\infty$, $\lambda \to +\infty$ certainly greater than 1 and using the Taylor expansion for it gives

$$\frac{\lambda}{c_m}\sqrt{{c_m}^2 t^2 - x^2} \approx \lambda t - \frac{\lambda x^2}{2{c_m}^2 t} - \frac{\lambda x^4}{8{c_m}^4 t^3}. \tag{3.132}$$

The way how one can use the Taylor expansion to obtain the last expression is to consider function

$$f(a) = \sqrt{1-a}. \tag{3.133}$$

The Taylor expansion up to second order for this function in the variable a around the point $a = 0$ then reads

$$f(a) \approx 1 - \frac{a}{2} - \frac{a^2}{8} \tag{3.134}$$

and substituting $a = \frac{x^2}{t{c_m}^2}$ gives the desired result. The reason, why one takes the expansion coefficients put to the second order is the fact, that the higher order terms does include the market speed of light c_m only in denominator and hence they are not important. Then the generalized Black-Scholes formula with the latter expansion of telegrapher's probability distribution has the form [43]

$$C(S(t), t) = \frac{Ke^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}}\int_0^{y_{max}}\Big((e^y - 1)e^{-\frac{(x-y)^2}{2\sigma^2(T-t)}}\Big)\Big(1 - \frac{1}{{c_m}^2} f(y, T-t)\Big)dy \tag{3.135}$$

with

$$x(S(t)) = \ln\frac{S(t)}{K} + \Big(r - \frac{1}{2}\sigma^2\Big)(T-t) \tag{3.136}$$

and

$$y_{max} = \sqrt{2\sigma^2(T-t) + \sigma(T-t)\sqrt{3\sigma^2 + 8{c_m}^2(T-t)}} \tag{3.137}$$

The generalized relativistic stochastic differential equation for underlying security stock price has the form

$$dS(t) = \mu S(t)dt + c_m S(t)(-1)^{N(t)}dt \tag{3.138}$$

which means, that the use of Itô formula to derive the generalized relativistic partial differential equation can not be used, as the Poisson process $N(t)$ term simply not fulfill the assumptions from the Itô formula as given in App.B and one has to exploit the theory of jump-processes and generalized Itô calculus, which is not the purpose of this thesis and can be found in [37], [29].
So as we could see, the latter approach is focused on relativistic generalization of the probability distribution of underlying security. We will exploit the same framework of telegrapher's equation with Dirac equation equivalence later, but in somewhat different manner. For get acquainted with this kind of different approach focused on the price of the option itself (not underlying), we will revise the main idea and results [52], [53]. In the approach of relativistic generalizing the equations driving directly the option price utilizing the tools of quantum mechanics one observe direct correspondence between Black-Scholes-Merton partial differential equation

$$\frac{\partial C(S,t)}{\partial t} + rS\frac{\partial C(S,t)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S,t)}{\partial^2 S} = rC(S,t) \tag{3.139}$$

can be mapped on Schrodinger equation itself through similar transformation as the Wick's rotation in the previous approach. To be concrete, if one applies the variable transformation on the Schrodinger equation 3.127 in the form

$$t = -i\tilde{t} \qquad \hbar = 1 \tag{3.140}$$

$$m = \frac{1}{\sigma^2} \qquad x = \ln S \tag{3.141}$$

$$\psi(x,t) = e^{-(\alpha x + \beta t)} C(S(x), \tilde{t}(t)) \tag{3.142}$$

where

$$\alpha = \frac{1}{\sigma^2}\left(\frac{\sigma^2}{2} - r\right) \tag{3.143}$$

then one obtains the original BSM partial differential equation. So in this case, the connecting transformation is a little bit more difficult and it seems that it is just an extension of the former Wick's rotation. This transformation is then naturally used in the case of Klein-Gordon relativistic equation

$$-\frac{\hbar^2}{c^2}\frac{\partial^2\psi(x,t)}{\partial t^2} + \hbar^2\frac{\partial^2\psi(x,t)}{\partial x^2} = {m_0}^2 c^2 \psi(x,t) \tag{3.144}$$

with slight change

$$\psi_{KG}(x,t) = e^{-(\alpha x + (\beta - \nu)t)} C(S(x), \tilde{t}(t)). \tag{3.145}$$

This adjustment in the transformation is due to the non-relativistic limit of Klein-Gordon equation, to be precise

$$\psi_{KG}(x,t) \stackrel{!}{=} e^{-\frac{im_0 c^2 t}{\hbar}} \psi_S(x,t) \tag{3.146}$$

because the expression

$$\psi_{KG}(x,t) e^{\frac{im_0 c^2 t}{\hbar}} \to \psi_S(x,t) \tag{3.147}$$

as $c \to +\infty$ [23] and where $\psi_S(x,t)$ represents the wave function being the solution of Schrodinger equation, whereas $\psi_{KG}(x,t)$ is wave function from Klein-Gordon equation. After aplying this transformation to Klein-Gordon equation, one obtains

$$\frac{1}{2\nu}\frac{\partial^2 C(S,t)}{\partial t^2} + \Big(1 - \frac{\beta}{\nu}\Big)\frac{\partial C(S,t)}{\partial t} + rS\frac{\partial C(S,t)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S,t)}{\partial S^2} = (r - \frac{\beta^2}{2\nu})C(S,t) \tag{3.148}$$

where

$$\beta = \frac{1}{2\sigma^2}\Big(\frac{\sigma^2}{2} + r\Big) \tag{3.149}$$

and

$$\nu = \frac{c^2}{\sigma^2} \tag{3.150}$$

and in the limit $\nu \to +\infty$ one recovers the original Black-Scholes-Merton partial differential equation. In the case of [53], to obtain the generalized price of the option is calculated directly through solving the generalized Black-Scholes-Merton equation. However, they chose the equivalent possibility to solve the Klein-Gordon equation with $\hbar = 1$, i.e. they have

$$-\frac{\partial^2 \psi(x,t)}{\partial t^2} + c^2 \frac{\partial^2 \psi(x,t)}{\partial x^2} = \nu^2 \psi \tag{3.151}$$

with terminal condition

$$\psi(x(T),T) = e^{-\Big(\alpha x + (\beta - \nu)T\Big)} \max\Big((e^x - K), 0\Big) \tag{3.152}$$

To make the latter terminal condition to be the initial condition one apply transformation

$$\phi(x,t) = \psi(cx, T-t) \tag{3.153}$$

After that they solve the differential equation with method of Fourier transform. The detailed derivation can be found in [53]. What is important for us, that the exact analytical solution is possible to obtain for $\alpha \neq 0$ only. In the case of the situation when risk-free interest rate r is equal to the half of the square of volatility σ one has to utilize numerical methods to approximate the fair price of the option. In the case of $\alpha \neq 0$ one obtains

$$C_r = e^{-\beta T}(S e^{\nu - \sqrt{\nu^2 - c^2(1-\alpha)^2}T} - K e^{\nu - \sqrt{\nu^2 - c^2\alpha^2}} T \mathbb{1}_{\alpha<0} + \frac{1}{2\pi}\frac{S^\alpha}{K^{1-\alpha}} \int_{-\infty}^{+\infty} \frac{(S/K)^{iy} e^{-(\sqrt{c^2 y^2 + \nu^2} - \nu)T}}{(\alpha + iy)(\alpha + iy - 1)} \Big) dy \tag{3.154}$$

for call option and the put option can be obtained through generalized put-call parity relation

$$C_r - P_r = e^{-\beta T}\Big(S e^{(\nu - \sqrt{\nu^2 - c^2(1-\alpha)^2})T} - K e^{(\nu - \sqrt{\nu^2 - c^2\alpha^2} T}\Big). \tag{3.155}$$

The volatility smile will be also flattened, as demonstrated in the mentioned article, which supports the importance of this way of generalization. When it comes to the integration kernel within the Feynman-Kac formula (for the starting time $t = 0$), i.e. the distribution for underlying prices, one comes to the probability density in the form [62]

$$f_r = e^{rT}\frac{\partial^2 C_r}{\partial K^2} = e^{(r-\beta)T}\frac{1}{2\pi}\frac{S^\alpha}{K^{(1+\alpha)}}\int_{-\infty}^{+\infty}\Big(\frac{S}{K}\Big)^{iy}e^{-(\sqrt{c^2y^2+\nu^2}-\nu)T}dy \tag{3.156}$$

In the [52] the solutions to the equation 3.148 is obtained in different manner. The original equation is simplified through the observation, that if one make another transformation in the form

$$z = \ln S + i\sqrt{c^2}t \tag{3.157}$$

and the equation becomes

$$4\frac{\partial^2 C(z)}{\partial z\partial\bar{z}} + 2\Big(\bar{A}\frac{\partial C(z)}{\partial z} + A\frac{\partial C(z)}{\partial\bar{z}}\Big) + \Big(A\bar{A} - \nu^2\Big)C(z) = 0 \tag{3.158}$$

with notation

$$A = -\frac{1}{\sigma^2}\Big[\Big(\frac{\sigma^2}{2} - r\Big) - \frac{i}{2\sqrt{c^2}}\Big(\big(\frac{\sigma^2}{2} + r\big)^2 - 2c^2\Big)\Big] \tag{3.159}$$

then if $\nu^2 << \bar{A}A$, the equation 3.158 becomes

$$4\frac{\partial^2 C(z)}{\partial z\partial\bar{z}} + 2\Big(\bar{A}\frac{\partial C(z)}{\partial z} + A\frac{\partial C(z)}{\partial\bar{z}}\Big) + \Big(A\bar{A}\Big)C(z) = 0. \tag{3.160}$$

Solving this equation and restore original variables x and t gives

$$C(x(t), t) = \int_{-\infty}^{+\infty}\frac{\exp\Big(\beta t - \nu t + \alpha(x(t) - x(0)\Big)}{\pi}\frac{\sqrt{c^2}t}{(x(t) - x(0))^2 + (ct)^2}C(x(0), 0)dx(0) \tag{3.161}$$

where the integral kernel represents the probability density function of stochastic Cauchy distribution.
So the main disadvantage of the latter approach stems from the fact, that the generalized Black-Scholes-Merton equation is approximated and hence one must count with the systematic error even before he start solving it. On the other hand, the approach of [53] has to tackle woth problems when evaluating integrals for coefficient $\alpha = 0$.
To remedy these disadvantages, we tried to exploit the approach of [12] as presented above in this chapter to find out the price of the call option with use of the same Klein-Gordon relativistic equation. The advantage of this approach will be the fact, that one can circumvent the problem of finding solutions for generalized BSM partial differential equation. The generalized BSM equation 3.148 can be rewritten to the form

$$\frac{\partial C(x,t)}{\partial t} = \Big[-c_1\frac{\partial^2}{\partial t^2} - c_2\frac{\partial}{\partial x} - c_3\Big(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\Big) + c_4\Big]C(x,t) \tag{3.162}$$

with constants

$$c_1 = \frac{\frac{1}{2\nu}}{1-\frac{\beta}{\nu}} \qquad c_2 = \frac{r}{1-\frac{\beta}{\nu}} \tag{3.163}$$

$$c_3 = \frac{\frac{\sigma^2}{2}}{1-\frac{\beta}{\nu}} \qquad c_4 = \frac{r-\frac{\beta^2}{2\nu}}{1-\frac{\beta}{\nu}} \tag{3.164}$$

and similar as in the original Black-Scholes-Merton treatment, introduce Klein-Gordon financial Hamiltonian

$$H_{KG} = \Big[-c_1\frac{\partial^2}{\partial t^2} - c_2\frac{\partial}{\partial x} - c_3\Big(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\Big) + c_4\Big] \tag{3.165}$$

and calculate the matrix element for integral kernel of the form

$$f_{KG}(x(T), T|x(t), t) =< x(t)|\exp\Big((T-t)H_{KG}\Big)|x(T)> \tag{3.166}$$

Substituting for Klein-Gordon Hamiltonian and inserting resolution of identity as in the previous sections gives us

$$f_{KG}(x(T), T|x(t), t) = \int_{-\infty}^{+\infty} < x(t)|\exp\Big((T-t)(c_1\hat{E}+ic_2\hat{p}+c_3(\hat{p}^2-i\hat{p})+c_4)\Big)|p>< p|x(T)> dp \tag{3.167}$$

with

$$\hat{E} = i\frac{\partial}{\partial t} \tag{3.168}$$

and

$$\hat{p} = -i\frac{\partial}{\partial x} \tag{3.169}$$

being operators of energy and momentum coming from quantum mechanics, however now, not surprisingly with $\hbar = 1$. Using the relation

$$< x|\hat{E} = E < x| \tag{3.170}$$

and properties of acting momentum operator $\hat{p}$ on bra co-vector $< x|$ gives

$$f_{KG}(x(T), T|x(t), t) = \int_{-\infty}^{+\infty} \frac{1}{2\pi} e^{-ipx(T)} e^{+ipx(t)} e^{-(T-t)(c_1E^2+ic_2p+c_3(p^2-ip)+c_4)}dp. \tag{3.171}$$

The last thing in the deriving the pricing integral kernel one has to utilize Gaussian integration, concretely well-known formula

$$\int_{-\infty}^{+\infty} e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}} \tag{3.172}$$

which gives us

$$f_{KG}(x(T), T|x(t), t) = e^{-(T-t)(c_4+c_1E^2)}\frac{1}{2\sqrt{(T-t)c_3\pi}}\exp\Big(-\frac{(x(t)-x(T)+(T-t)c_2-c_3)^2}{4(T-t)c_3}\Big). \tag{3.173}$$

As one can see from the our definition of coefficients, the latter expression converges to classical Black-Scholes-Merton pricing integral kernel as $\nu \to +\infty$. So the generalized formula for call option price in original observed quantities reads

$$C(S(t),t) = \tag{3.174}$$

$$= e^{-(T-t)(c_4+c_1E^2)}\frac{1}{2\sqrt{(T-t)c_3\pi}}\int_0^{+\infty}\exp\Big(-\frac{(x(t)-x(T)+(T-t)c_2-c_3)^2}{4(T-t)c_3}\Big)\max\Big(S(T)-K,0\Big)\frac{dS(T)}{S(T)}. \tag{3.175}$$

which is surpsisingly the Gaussian distribution, however with different relativistic risk-neutral measure $r_r = c_4 + c_1E^2$. The total energy of particle in physics therefore represents the analogical quantity for financial world quantity interpreted as additional term creating the new risk-neutral interest rate r_r. One can immediately see problems with this probability density. For underlying price $x > ct$ it has non-zero probabilities and hence it seems that the underlying price despite using the framework of quantum mechanics does not work. Mapping directly through equations describing the option price therefore seems not so reliable method, how to generalize the classical models of valuation. It is true, that this should follow from the calculating generalized Klein-Gordon Hamiltonian matrix element with the same method as used in original BSM model. The new Hamiltonian comprises additional term representing the square of the operator of energy $\hat{E}$ and in the original basis the new Hamiltonian is simply likely not diagonal. However, as we already seen, and surprisingly, the non-relativistic limit for coefficients is still preserved.

One however may not trying to find the "right" basis to properly modify Klein-Gordon equation as to obtain more appropiate results, one can utilize the tools coming from the already mentioned barrier options valuation. This theory works with special case of options, which value is immediately zero when the the certain value for spot price is exceeded.[12] What's more, one does not have to use the relativistic QM. The price is then again obtained as

$$C(x(t),t) = \int_{-\infty}^{+\infty} f_{Barrier}(x(T),T|x(t),t)g(x(T))dx(T) \tag{3.176}$$

with integral kernel

$$f_{Barrier}(x(T),T|x(t),t) = f_{BSM}(x(T),T|x(t),t) - \Big(\frac{e^x}{e^B}\Big)f_{BSM}(x(T),T|2B-x(t),t) \tag{3.177}$$

for $x(t), x(T) < B$, where $f_{BSM}(x(T),T|x(t),t)$ represents the original Black-Scholes integral kernel and B is already mentioned barrier with $B \equiv c_m t$, where c_m is maximal market speed of lighta as presented above. For $x(t), x(T) > B$ or $x(t) > B, x(T) < B$ the density is zero. One can derive this integral kernel with extended Black-Scholes-Merton Hamiltonian

$$H_{Barrier} = H_{BSM} + V(x) \tag{3.178}$$

with potential in the form

$$V(x) = \begin{cases} +\infty, & \text{if } x \geqslant c_m t \\ r, & \text{if } x < c_m t \end{cases}. \tag{3.179}$$

One can immediately see, that not only the condition on relativistic restriction is automatically inserted into the model, what's more, we can observe, that the price is always smaller in the areas with

non-zero probability.
The latter idea useful for circumventing the problem of relativistic approach can be used universally whenever one has the difficulties with calculating the pricing integral kernel. Now we will derive the relativistic financial Hamiltonian for the (1+1) Dirac equation. We will help ourselves with the Kolmogorov equations for the functions $u_+(x,t)$ and $u_-(x,t)$ which are equivalent to the telegrapher's equation, to remind

$$\frac{\partial^2 u_+(x,t)}{\partial t^2} - c^2\frac{\partial^2 u_+(x,t)}{\partial x^2} = +2\frac{im_0c^2}{\hbar}\frac{\partial u_+(x,t)}{\partial t} \tag{3.180}$$

$$\frac{\partial^2 u_-(x,t)}{\partial t^2} - c^2\frac{\partial^2 u_-(x,t)}{\partial x^2} = +2\frac{im_0c^2}{\hbar}\frac{\partial u_-(x,t)}{\partial t} \tag{3.181}$$

substituting for

$$u_+(x,t) = \psi_+(x,t)e^{\frac{im_0c^2t}{\hbar}} \tag{3.182}$$

gives us

$$2\frac{\partial^2\psi_+}{\partial t^2} + \Big(\frac{4im_0c^2}{\hbar} - 2\Big)\frac{\partial\psi_+(x,t)}{\partial t} + \Big[2\Big(\frac{im_0c^2}{\hbar}\Big)^2 - 2\Big(\frac{im_0c^2}{\hbar}\Big)\Big]\psi_+(x,t) = 0 \tag{3.183}$$

It is not difficult to see, that because of the same form of telegrapher's equation for the Dirac spinor component $\psi_-(x,t)$, the latter result will be exactly the same also for this component. From this we also try to answer the question from [43], what the each single components from the Dirc spinor $\psi(x,t)$ mean in the context of finances. As our approach is valid for each single component and the telegrapher's equation for each single component has the same form, the financial meaning of these component is not something affecting the pricing integral kernel directly, hence the components has no visible meaning in the world of finance. To continue, one similarly to Klein-Gordon equation, uses following and not surprising transform

$$t = i\tilde{t}, \qquad \hbar = 1 \tag{3.184}$$

$$m = \frac{1}{\sigma^2}, \qquad x = \ln S \tag{3.185}$$

$$\psi_+(x,t) = e^{-(\alpha x+\beta t)}C(S(x), t(\tilde{t})) \tag{3.186}$$

which after substituting gives us

$$-2\Big(\beta^2C - \beta\frac{\partial C}{\partial t} - \beta\frac{\partial C}{\partial t} + \frac{\partial^2 C}{\partial t^2}\Big) + \Big(4v^2+2i\Big)\Big(\beta C + \frac{\partial C}{\partial t}\Big) + 2\Big(-v^4 - iv^2\Big)C + i\sigma^2\Big(\alpha^2 C - \alpha\frac{\partial C}{\partial x} - \alpha\frac{\partial C}{\partial x} + \frac{\partial^2 C}{\partial x^2}\Big) = 0 \tag{3.187}$$

and after terms rearrangement one gets

$$\frac{\partial C}{\partial t}\Big(4\beta+4v^2+2i\Big) - 2\frac{\partial^2 C}{\partial t^2} - 2i\sigma^2\alpha\frac{\partial C}{\partial x} + i\sigma^2\frac{\partial^2 C}{\partial x^2} + C\Big(-2\beta^2 - \beta(2i+4v^2) - 2v^2(2v^2+1) + i\sigma^2\alpha^2\Big) = 0 \tag{3.188}$$

In comparision with equation 3.148 one immediately sees the difference in quite complicated coefficients, which also comprise imaginary units. However, we still obtained nothing but generalized

Black-Scholes-Merton partial differential equation for the price of European call option. Writing the latter equation in the Schrodinger form gives

$$\frac{\partial C}{\partial t} = \Big[\Big(-\frac{2}{\gamma}\Big)\frac{\partial^2 C}{\partial t^2} + \Big(\frac{i\sigma^2}{\gamma}\Big)\frac{\partial^2}{\partial x^2} - \Big(\frac{2i\sigma^2\alpha}{\gamma}\Big)\frac{\partial}{\partial x}\Big]C = 0. \tag{3.189}$$

with

$$\gamma = 4\beta + 4\nu^2 + 2i. \tag{3.190}$$

one can now immediately identify

$$H_{D+} = H_{D-} = \Big(-\frac{2}{\gamma}\Big)\frac{\partial^2 C}{\partial t^2} + \Big(\frac{i\sigma^2}{\gamma}\Big)\frac{\partial^2}{\partial x^2} - \Big(\frac{2i\sigma^2\alpha}{\gamma}\Big)\frac{\partial}{\partial x} \tag{3.191}$$

as the new generalized BSM Hamiltonian obtained through the equivalence between Dirac equation in (1+1) dimension and telegrapher's equation for both Dirac spinor components $\psi_+(x,t)$, $\psi_-(x,t)$. Recalling the approach of calculating the pricing integral kernel one can even without calculations see, that this Hamiltonian will incorporate imaginary units in the denominator into the expected value integral. This is strange and needs to be better understood in further research.
At the end of this chapter, we briefly revise path integral treatment of the problem of option valuation and propose some ideas for possible further investigations. The main source will be [57].
If one wants to calculate the propagator for non-relativistic particle moving with one degrees of freedom, one can use the advantage of Feynman's path integral formulation [63], [64], [65], [66] and write

$$f(x_{t_f}, t_f | x_{t_0}, t_0) = \int_{x_{t_0}}^{x_{t_f}} \mathcal{D}x \exp\Big(\frac{i}{\hbar}S\Big) \tag{3.192}$$

where

$$S = \int_{t_0}^{t_f} dt L\Big(x, \frac{dx}{dt}, t\Big) \tag{3.193}$$

is the functional of action with general Lagrangian

$$L\Big(x, \frac{dx}{dt}, t\Big) = \frac{1}{2}m\dot{x}^2 - V(x,t) \tag{3.194}$$

and integral "measure" $\mathcal{D}x$ given by relation

$$\int_{x_{t_0}}^{x_{t_f}} \mathcal{D}x \exp\Big(\frac{i}{\hbar}S\Big) = \lim_{n\to+\infty}\prod_{i=1}^{n}\sqrt{\frac{m}{2\pi i\hbar\Delta t_i}}\int\prod_{i=1}^{n-1} dx_i \exp\Big(\frac{iS_n}{\hbar}\Big) \tag{3.195}$$

with partial action S_n in the form

$$S_n = \sum_{i=1}^{n} \Delta t_i \Big[\frac{m(x_i - x_{i-1})^2}{2(\Delta t_i)^2} - V(x_{i-1}, t_{i-1})\Big] \tag{3.196}$$

where x_i represents dividing discretization points of the arbitrary path connecting points x_{t_0} and x_{t_f} with t_i being the times, in which the particle passes through the discretization points x_i, $i \in \hat{n}$. The similar results one obtains in the case of already well-known Wick's rotation $t = -it$ for time coordinate (so called Euclidean time). The propagator then becomes

$$f_E(x_{t_f}, t_f|x_{t_0}, t_0) = \int_{x_{t_0}}^{x_{t_f}} \mathcal{D}x \exp\Big(-\frac{1}{\hbar}S_E\Big) \tag{3.197}$$

where

$$S_E = \int_{t_0}^{t_f} dt L_E\Big(x, \frac{dx}{dt}, t\Big) \tag{3.198}$$

and

$$L_E\Big(x, \frac{dx}{dt}, t\Big) = \frac{1}{2}m\dot{x}^2 + V(x,t). \tag{3.199}$$

The Euclidean integral measure is the same with the exception of argument of exponential

$$\int_{x_{t_0}}^{x_{t_f}} \mathcal{D}x \exp\Big(-\frac{1}{\hbar}S\Big) = \lim_{n\to+\infty} \prod_{i=1}^{n} \sqrt{\frac{m}{2\pi i\hbar\Delta t_i}} \int \prod_{i=1}^{n-1} dx_i \exp\Big(-\frac{S_n}{\hbar}\Big) \tag{3.200}$$

And as one expects from the above analyzed transformations, introducing additional transformation $\hbar = 1$, $m = \frac{1}{\sigma^2}$ for the Euclidean expressions allows to express the expected value of some random variable $g(x(T))$ with $x(T)$ being random variable picked from the adapted normal Gaussian distributed stochatic process $\{X(t)\}$ with respect to some filtration (e.g. generated by the process itself), in other words Wiener process.

$$\mathbb{E}^{\mathbb{Q}}\Big(g(x(T)\Big) = \lim_{n\to+\infty} \prod_{i=1}^{n-1} \int dx_i \prod_{i=1}^{n} \sqrt{\frac{1}{2\pi\sigma^2\Delta t_i}} \exp\Big(-\frac{(\Delta x_i)^2}{2\sigma^2\Delta t_i}\Big) g(x(T)) \tag{3.201}$$

where $\mathbb{Q}$ is the probability measure, with respect to which the Wiener process $X(t)$ is martingale. Now we already know, that this measure is so called risk-neutral measure as introduced in the Chapter 2 and the function $g(x(T))$ is nothing but the pay-off function of call or alternatively put option. Multiplying it with term

$$\exp\Big(-r(T-t)\Big) \tag{3.202}$$

would give us the fair value of the option $C(x(t), t)$ known from Feynmann-Kac formula and risk-neutral valuation. So one can again see the equivalence between the worlds of quantum mehanics and stochastic process which is already unsurprisingly visible through extended Wick's rotation used before, with the difference, that now one uses the path integral formulation of quantum mechanics. To make the theory relativistic, in [57] is used generalized relativistic Langrangian

$$L = -m_0c^2\sqrt{1-\frac{v^2}{c^2}}. \tag{3.203}$$

Firstly, this gives the non-relativistic limit in the form,

$$L \approx \frac{1}{2}m_0\dot{x} - m_0c^2 \tag{3.204}$$

which is not the same as the non-relativistic Lagrangian for free particle. However, the equations of motion are not changed due to the fact, that the second term is only constant rest energy of it and hence it does not affect the equations of motion. Then one wants exploit the extended Wick's rotation to tranfer from QM world to stochastic world. The "rotated" action of "rotated" Lagrangian then becomes

$$S_E = m_0 c^2 \int dt \sqrt{1 + \frac{v^2}{c^2}} \tag{3.205}$$

At this point, we find very important to emphasize the biggest difference between the relativistic approach of [52], [53]. Even though in every relativistic generalization, or even in the non-relativistic correspondence between quantum mechanics and stochastic processes, one uses the tools of Wick's rotation or it's extended version, in the path integral approach of [57], the financial analogy of "particle" is able to exceed the speed of light on the market. The action S_E is still real and hence mathematically treatable. There is no reason to change the rules well-known from physics to work with such an action. This is quite surprising to us, as the relativistic corresposndence can be perceived or if one wants seen in different manner without formally doing anything. The same transformations may be seen differently. Calculation of the relativistic pricing integral kernel gives

$$f_E(x_{t_f}, t_f | x_{t_0}, t_0) = \mathcal{N} \int_0^{+\infty} ds \frac{m_0}{2\pi\hbar s} \exp\Big(- \frac{m_0[(x(t_f) - x(t_0))^2 + c^2(t_f - t_0)^2]}{2\hbar s} - \frac{m_0 c^2 s}{2\hbar} \Big) \tag{3.206}$$

The normalization constant is determined from the normalization condition

$$\int_{-\infty}^{+\infty} f_E(x_{t_f}, t_f | x_{t_0}, t_0) dx_{t_f} = 1. \tag{3.207}$$

which is nothing but the mathematical interpretation of the fact, that the particle must go somewhere. The Fourier transform is utilized to transform the last transition density and come to the normalization coefficient

$$\mathcal{N} = c \exp\Big(\frac{m_0 c^2}{\hbar}(t_f - t_0)\Big) \tag{3.208}$$

with the exponential with $m_0 c^2$ standing for particle's rest energy implicitly present in the model through the Euclidean relativistic action S_E. However, this was identified as problem, because the model is no more invariant under rotations in Euclidean space $SO(2)$. So this is one of the first disadvantages if the model. However, in the finance, the symmetries do not have their real meaning, hence one can just forget this fact. Inserting the normalization coefficient and integrating in the transitional probability one obtains

$$f_E(x_{t_f}, t_f | x_{t_0}, t_0) = \frac{m_0 c}{\pi\hbar} \exp\Big(\frac{m_0 c^2}{\hbar}(t_f - t_0)\Big) K_0\Big(\frac{m_0 c}{\hbar} \sqrt{(x(t_f) - x(t_0))^2 + c^2(t_f - t_0)^2}\Big) \tag{3.209}$$

where

$$K_\alpha(x) = \int_0^{+\infty} d\xi \exp\Big(- x \cosh(\xi)\Big) \cosh(\alpha\xi) \tag{3.210}$$

However, there is still problem with this transition probability when one tries to construct martingale stochastic process which is neccessary for valuation process. Finally, the transition probability density is using transformation $t_f - t_0 \to t$, $x(t_f) - x_(t_0) \to x$ and properties of functions K_α expressed as

$$f_E(x, t) = \frac{c}{\pi\sigma^2} \frac{ct}{\sqrt{x^2 + c^2 t^2}} K_1\Big(\frac{c}{\sigma^2} \sqrt{x^2 + c^2 t^2}\Big) \exp\Big(\frac{c^2 t}{\sigma^2}\Big). \tag{3.211}$$

The whole treatment of the valuation problem with method of path integral is quite complicated, as one needs to solve too many problems on his way for obtaining the generalized formula to option price. The first idea to resolve the problem of difficulty and too many obstacles within the relativistic framework would then be to use another form of Lagrangian. One could simply utilize the framework of quantum field theory and Lagrangians in the form

$$\mathcal{L}_{KGF}(\psi, \partial_\mu \psi, t) = \frac{1}{2}\partial_\mu \psi \partial^\mu \psi - \frac{1}{2}m^2\psi^2 \tag{3.212}$$

for Klein-Gordon field Lagrangian density or

$$\mathcal{L}_{DF}(\psi, \partial_\mu \psi, t) = \psi^\dagger \gamma^0 \Big(i\hbar c \gamma^\mu \partial_\mu - m_0 c^2\Big)\psi \tag{3.213}$$

for Dirac Lagrangian density. One then obtains the Langrangian function from the relation

$$L(\psi, \partial_\mu \psi, t) = \int dx \mathcal{L}(\psi, \partial_\mu \psi, t) \tag{3.214}$$

This is just an idea and regarding it's difficulty it is just let it to further research in this area. Equivalent approach would be also utilize the tools from [12] and directly compute the pricing relativistic integral kernel

$$\hat{H}_{DF} \tag{3.215}$$

where indices of $\hat{H}_{DF}$ represents the Dirac Hamiltonian operator coming not from relativistic quantum mechanics, but from quantum field theory and this is then exploited in the theory of path integral in quantum field theory. For doing this, firstly, one would have to compute the (1+1) Dirac field Langrangian denstity in terms of above mentioned Weyl representation

$$\mathcal{L}_{DF}(\psi, \partial_\mu \psi) = \psi^\dagger \sigma^1 \Big(i\hbar \sigma_1 \partial_0 + (\sigma_1 \sigma_3)\partial_1 - m_0 c^2\Big)\psi \tag{3.216}$$

where we used the identification of gamma matrices as they were already calculated above. Breaking down the notation and utilizing quantum-stochastic transformation (i.e. extended Wick's rotation) of variables as used in [53] and which we used several times above in the form

$$t \rightarrow it, \qquad m = \frac{1}{\sigma^2} \tag{3.217}$$

$$\psi(x,t) = \exp\Big(-(\alpha x + (\beta - \nu)t)\Big)C(x,t) \tag{3.218}$$

with the same interpretation of coefficients α, β, ν as before one gets transformed Lagrangian in the form

$$\mathcal{L}_{DFF}(C, \partial_\mu C) = e^{-(\alpha x(\beta-\nu)t)} C^\dagger \sigma_1 \Big(\sigma_1 \frac{\partial}{\partial t} + \sigma_1 \sigma_3 \frac{\partial}{\partial x} - \nu\Big) e^{-(\alpha x + (\beta - \nu)t)} C \tag{3.219}$$

where the second F letter in the index emphasizes the stochastic financial world. At this moment, the option price would be spinor with two components similarly to Dirac spinor in (1+1) dimensions, which is something new comparing the previous techniques, as in that techniques one could utilize the equivalence of coupled Kolmogorov equations and telegrapher's equation and therefore work with each component separately and as we could see there, there was no difference in pricing kernel Hamiltonian H_{D+} or H_{D-}. This situation will be most probably different. The financial Dirac Hamiltonian will be then given by

$$H_{DFF} = \int dx \mathcal{L}_{DFF}. \tag{3.220}$$

What follows can be, as we said before, subject of further investigations.

Appendix A

Special Relativity

In this part we briefly revise basic knowledge coming from theory of special relativity. The main sources for this part are [17], [22], [21], [23]
Theory of Special Relativity is technically a consequence of the fact, that laws of electromagnetism, which are sleeky, conveniently and elegantly encoded in the Maxwell equations published by Maxwell in 1873 within work named *A Treaticity on Electricity and Magnetism* are not covariant under standard Galilean tranformations used in the classical mechanics. It is based on two postulates:

1. (Principle of relativity) Laws of nature are invariant in all inertial frame of reference. There exist no privileged frame of reference.
2. (Constant velocity principle) The speed of light in the vacuum is finite and equal in all inertial frames of reference.

These two postulates are basically obvious to presume. The requirement on invariance of natural laws in all inertial frames of reference comes from the observation that in all such inertial frames, all known experiments aiming to verify these laws have the same result. The second postulate is logical as well, as the results of 1887's *Michelson - Morley experiment* refused the theory of light aether. Several approaches for derivation of STR are known, we pick one of them.
From all inertial frames of reference we pick one named S and then other frame named S', which will be moving with respect to S' with constant velocity $\boldsymbol{v}$. The position vector $\boldsymbol{r}$ of particle can be decomposed into perpendicular and parallel part

$$\boldsymbol{r} = r_{\perp} + r_{\parallel} \tag{A.1}$$

with respect to the direction of the velocity vector $\boldsymbol{v}$. Transformed position vector then reads

$$\boldsymbol{r'} = \boldsymbol{r}_{\perp} + \gamma(\boldsymbol{r}_{\parallel} - \boldsymbol{v}t) \tag{A.2}$$

as the S' is moving in the direction of parralel component of radiusvector, i.e. in direction of velocity. Coefficient γ now serves as mathematical tool for explicitly distinguish Galileo and Lorentz transformation. If now one subsitutes for both radiusvector components in S, then the transformation relation will be immediately obtained. Using

$$|\boldsymbol{r}_{\parallel}| = \frac{\boldsymbol{r}_{\parallel}.\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{\boldsymbol{r}.\boldsymbol{v}}{|\boldsymbol{v}|} \tag{A.3}$$

one gets final transformation of the position coordinates in the form

$$\boldsymbol{r'} = -\gamma \boldsymbol{v} t + \boldsymbol{r} + (\gamma - 1)\frac{\boldsymbol{r}.\boldsymbol{v}\boldsymbol{v}}{|\boldsymbol{v}|^2}. \tag{A.4}$$

However, this general formula is not neccesary in real world, not even to derive unknown parameter γ. Reason is obvious, in the cases when frame S' is moving in arbitrary direction with respect to S, then basis in the frame S will be immediately adjusted so that S' is moving in 1 direction only. This direction could be then chosen parallel with, for instance, x-axis. Therefore, from this moment it is possible to work with simplified form of the last vector equality for movement along x-asis:

$$x' = \gamma(x - vt) \tag{A.5}$$

$$x = \gamma(x' + vt') \tag{A.6}$$

Now for deriving still unknown coefficient γ, second postulate has to be used. In the moment, when the origins of S and S' will be on the same place, torch is turned on. Wavefronts of light are then propagated in each of this frame of reference separately and can be localized in places $x = ct$, $x' = ct'$. respectively. Subsituting this and then multiplying these equations will give us well-known relation

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{A.7}$$

Now using this coefficient and inserting for position x' into relation for position x gives us transformation of time

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{A.8}$$

and in the case when the frame axis would be not adjusted according to the direction of velocity

$$t' = \frac{t - \frac{\boldsymbol{v}\boldsymbol{r}}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{A.9}$$

All these result can be expressed in the form of matrix multiplication as

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{bmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1 + (\gamma-1)\frac{\beta_x^2}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} \\ \gamma\beta_y & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & 1 + (\gamma-1)\frac{\beta_y^2}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} \\ \gamma\beta_z & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} & 1 + (\gamma-1)\frac{\beta_z^2}{\beta^2} \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}. \tag{A.10}$$

It turns out, that the matrix multiplication may be perceived as representation of element of abstract Lorentz group acting on vector picked from Minkowski space, which is vector space consisting of elements - called 4-vectors, trasforming according to Lorentz transformation, i.e. some group element of Lorentz group (every transformation which does not move with origin of frame of reference). As Minkowski space is a vector space, then one can define contravariant components of 4-vector.

$$x^\alpha = (x^0, x^1, x^2, x^3) = (ct, x, y, z) \tag{A.11}$$

is contravariant version of vector called *event*. The covariant version can be obtained by

$$x_\alpha = g_{\alpha\beta}x^\beta \tag{A.12}$$

where tensor $g_{\alpha\beta}$ represents the metric tensor in Minkowski space

$$g_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1). \tag{A.13}$$

From relation

$$x^\alpha = g^{\alpha\beta}x_\beta = g^{\alpha\beta}g_{\beta\gamma}x^\gamma \tag{A.14}$$

one immediately writes

$$g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha_\gamma. \tag{A.15}$$

where δ^α_γ is Kronecker delta symbol. With this notation, one is able to write the inner product of two elements (i.e. 4-vectors) of Minkowski space

$$a^\alpha b_\alpha = a^\alpha g_{\alpha\beta}b^\beta = +a^0b^0 - a^1b^1 - a^2b^2 - a^3b^3 \tag{A.16}$$

With this introduced notation, the general Lorentz transformation (in this case called *boost*) A.10 can be written as

$$x'^\alpha = \Lambda^\alpha_\beta x^\beta \tag{A.17}$$

where Λ is usual notation representing Lorentz transformation. Now imagine, that one has clocks in the origin of frame of reference S as well as in S'. Both origins are at same place in time $t = t' = 0$. Once the signal is given, frame S' starts moving with constant velocity v in given direction. S axis may be again adjusted such that for simplicity x-direction is the same as direction of S' movement. If now the time elapsed from the beginning of such translation movement is measured in S as well as S', one gets relation

$$\Delta t = \frac{\Delta t' + \frac{v\Delta x'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{A.18}$$

As the clocks placed in S' are not moving (i.e. $\Delta x' = 0$) the last equation gives

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{A.19}$$

and the *proper time* $\Delta\tau = \Delta t'$ is defined. It is invariant quantity and always represents the shortest time comparing set of all possible times in other inertial frames of reference. The special meaning in special relativity has also quantity called *space-time interval* Δs

$$(\Delta s)^2 = (c\Delta t)^2 - \Delta^2 x - \Delta^2 y - \Delta^2 z \tag{A.20}$$

which represents space-time distance of two events in the Minkowski space. It is invariant of Lorentz transformations. This means, that cause and effect of that cause have the same order in all inertial frames of reference and frame of reference with effect occuring earlier that cause of it does not exist. Hence if we calculate this interval in the inertial frame connected with particle we get

$$(\Delta s)^2 = (c\Delta\tau)^2 \tag{A.21}$$

and speed of light in vacuum c is constant. This gives us another useful formulation of proper time and also proof that proper time is invariant, too ($\Delta\tau' = \Delta\tau$). Spacetime in every inertial frame of reference is then divided into three regions

- $(\Delta s)^2 = 0$: so called lightlike interval. Two separate events connected by this type of interval must have been connected by signal travelling with speed of light in vacuum if causal connected.
- $(\Delta s)^2 > 0$: so called timelike interval. Two separate events connected by timelike interval represents events occured to arbitrary mass particle able to move with speeds smaller than c.
- $(\Delta s)^2 < 0$: so called spacelike interval. Events connected with this type of interval cannot be causally connected. One may not be effect of another.

Using greek-index notation and the presented fact, that spacetime interval is invariant under Lorentz transformations, one obtains restriction conditions on elements of Lorentz matrix

$$\Lambda^\alpha_\beta \Lambda^\gamma_\delta g_{\alpha\gamma} = g_{\beta\delta} \tag{A.22}$$

which implies that $\det^2\Lambda = 1$. This gives us two possibilities, either $\det\Lambda = 1$ (so called *proper* Lorentz transformations, fo instance boosts or rotations), or $\det\Lambda = -1$ (i.e. *improper* Lorentz transformations, for instance parity inverse or time reversal).
Choosing $\beta = \gamma = 0$ in the last relation gives another condition

$$(\Lambda^0{}_0)^2 = 1 + \sum_{i=1}^{3} (\Lambda^i_0)^2 \tag{A.23}$$

from which is clear, that we have either $\Lambda^0{}_0 \leqslant -1$ (i.e. *non-ortochronous, reversing direction of time*, or $\Lambda^0{}_0 \geqslant +1$ (i.e. *ortochronous*, preserving direction of time). So overall, Lorentz group is divided into 4 disjoint disconnected components.
With above mentioned definitions remaining important 4-vectors living in the Minkowski space are defined.

$$u^\alpha := \frac{dx^\alpha}{d\tau} = c\frac{dx^\alpha}{ds} \tag{A.24}$$

is 4-velocity of particle, where ds is infinitesimal version of original finite spacetime interval Δs. Second derivative with respect to proper time gives

$$a^\alpha := \frac{du^\alpha}{d\tau} = c\frac{du^\alpha}{ds} \tag{A.25}$$

The significant importance, particularly in theory of Brownian motion and Ornstein-Uhlenbeck process has 4-momentum

$$p^\alpha := m_0 u^\alpha = m_0(\gamma c, \gamma \boldsymbol{v}). \tag{A.26}$$

and using the well-known relativistic energy expression for mass particle

$$E = mc^2 = m_0\gamma c^2 \tag{A.27}$$

one gets another used formulation

$$p^\alpha = (\frac{E}{c}, \boldsymbol{p}). \tag{A.28}$$

Finally, calculating norm of the 4-momentum one obtains

$$p^\alpha p_\alpha = \frac{E^2}{c^2} - \boldsymbol{p}^2 = m_0^2 u^\alpha u_\alpha = m_0^2 c^2. \tag{A.29}$$

where we used the fact, that norm of 4-velocity is equal to c^2. Hence we got famous energy-momentum relation

$$E^2 = p^2 c^2 + m_0^2 c^4. \tag{A.30}$$

In addition to previous most important 4-vectors we add last two frequently used, namely 4-gradient

$$\partial^\alpha = \Big(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}) = (\frac{\partial}{\partial ct}, -\frac{\partial}{\partial \boldsymbol{x}}\Big) \tag{A.31}$$

and 4-current density

$$J^\alpha = (\rho c, j^1, j^2, j^3) = (\rho c, \boldsymbol{j}) \tag{A.32}$$

where ρ is charge density and $\boldsymbol{j}$ classical current density.

Appendix B

Itô stochastic calculus, Stochastic differential equations

Next lines will introduce the theory of stochastic integral. After that, concept of stochastic differential equation will be presented, as these two topics are tightly-bonded together. Main sources for this appendix are [24], [26], [25],[27].
In the world of econophysics, many problems are based on solving integral looking like

$$I(f,t,\omega) := \int_0^t f(s,\omega)dg(s,\omega) \tag{B.1}$$

where objects $f(s,\omega)$ and $g(s,\omega)$ represents basically some stochastic processes (or more accurately sample path of that stochastic process), especially if talking about above mentioned form of integral, symbol $f(s,\omega)$ can possibly stand for deterministic function, while $g(s,\omega)$ comes always as stochastic process. At the first sight, no difficulties may be seen, but after further investigations concerning most usual examples of integral, one see, that traditional approaches such as Riemann-Stieltjes integral or Lebesgue-Stieljes integral break down.
Neverheless, standard way of defining stochastic integral is possible under strictly given conditions:

- the functions f and g (or sample paths of stochastic process) must not have any discontinuities at the same point (i.e. at some certain time $s \in [0,t]$)
- sample path f must have bounded p-variation and the g has bounded bounded q-variation for some $p > 0$ and $q > 0$ such that $p^{-1} + q^{-1} > 1$.

For example, in the cases, when function f is differentiable with bounded derivation, then B.1 is perceived exactly same as classical integral known from deterministic calculus, although practically, the calculation in of it in terms of Brownian motion path is not always easy. Stochastic integral of the type

$$I(f,t,\omega) := \int_0^t W(s,\omega)dW(s,\omega) \tag{B.2}$$

with $W(s,\omega)$ being Brownian motion sample path is demostrative example, that Riemann-Stieltjes integral approach may not be used. It can be also shown, that if integral of type B.1 exist in traditional Riemann-Stieltjes sense, then the path $g(s,\omega)$ must have bounded variation. [26][29] But we already know, that sample paths of Brownian motion are good example of paths not having bounded variation.
Naturally, we get to the point, where one could try resolve this problem by

$$I(f,t,\omega) := \int_0^t f(s,\omega)g'(s)ds \tag{B.3}$$

but as every sample path of Brownian motion is non-differentiable, the time derivative can not be correctly defined and therefore this treatment of the stochastic integral won't work as well.

The stochastic integral definition will continue firstly with so called simply processes and their sample paths:

Definition B.0.1. *Stochastic process Φ is called simple (elementary) process if its sample path has the form*

$$\Phi(a,b,\omega) = \textstyle\sum_{i=0}^{n} c_i \chi_{[t_{i-1},t_i)}$$

for arbitrary partition of the interval $[a,b] \subset \mathbb{R}$: $a = t_0 < t_1 < ... < t_n = b$ and c_i being random variable.

Then the *Itô stochastic integral* of simple process is defined as

$$I(f,a,b,\omega) \equiv \int_a^b \Phi(t,\omega)dW(t,\omega) := \sum_{i=0}^{n} c_i[W(t_i,\omega) - W(t_{i-1},\omega)] \tag{B.4}$$

Set of integrands, which Itô stochastic integral was originally defined on, is

Definition B.0.2. *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be probability space. Let $W(t,\omega)$, $t \in [0,+\infty]$, be Wiener process defined on such a probability space. Let $\mathcal{F} = \{\mathcal{F}_t\}_{t\in[0,+\infty]}$ be natural filtration generated by Wiener process. Let $\mathfrak{M}^2[0,+\infty]$ be the set of functions (or sample paths of stochastic processes defined on above probability space with real values $((\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$ which fulfill following criteria*

- *map $(t,\omega) \to f(t,\omega) \in \mathfrak{M}^2[0,+\infty]$ is $\mathcal{B}\times\mathcal{F}$ measurable ($\mathcal{B}$ represents Borel σ-algebra on $[0,+\infty]$,*
- *$f(t,\omega)$ is $\mathcal{F}_t$ - adapted,*
- *$\|f(t,\omega)\|_2^2 = \mathbb{E}(\int_a^b f^2(t,\omega)dt) < +\infty$ for arbitrary $a,b \in [0,+\infty]$*

The space of previously mentioned simple (elementary) processes is subset of the $\mathfrak{M}^2[0,+\infty]$. Integral is already defined there, so the last part is to define integral on its complement. To be more precise, the set $\mathfrak{M}^2[a,b]$ (notation signs for the fact, that we are now interested in closed interval $[a,b] \subset [0,+\infty]$, i.e. subset of original set of sample paths) is then closure of the subset of simple processes, or in other words, the set od simple processes is dense in $\mathfrak{M}^2[a,b]$. The class of simple processes on the arbitrary interval $[a,b]$ will be denoted as $\mathfrak{M}_0^2[a,b]$
However, to be able to present integral for sample paths from $\mathfrak{M}^2[a,b]$, one has to remind one of several properties of Itô integral for sample paths of simple processes:

Lemma B.0.1 (Isometry property)**.** *Let $\Phi(t,\omega)$ be sample path of simple proces defined above. Then*

$$\mathbb{E}\Big[\Big(\int_a^b \Phi(t,\omega)dW(t,\omega)\Big)^2\Big] = \mathbb{E}\Big[\int_a^b \Phi^2(t,\omega)dt\Big].$$

If we now imagine the set of all simple processes on arbitrary interval $[a, b]$ as complete vector space with defined inner product

$$\langle f(t,\omega); g(t,\omega)\rangle := \mathbb{E}\Big[\int_a^b f(t,\omega)g(t,\omega)dt\Big] \tag{B.5}$$

where $f(t,\omega) \wedge g(t,\omega) \in \mathfrak{M}_0^2[a,b]$, then the expression on the right side has the meaning of the norm generated by the last equation. This is not some weird kind of coincidence as the original paper [24] automatically presumes, that the class of sample paths of stochastic process has the form of $L^2([a,b]\times\Omega)$ space. The above stressed isometry property also accounts for the fact, that properties of original space will be inherited by the space of possible outcomes. The next lemma stands for the fact, that space $\mathfrak{M}^2[a,b]$ is indeed dense closure of $\mathfrak{M}_0^2[a,b]$

Lemma B.0.2. *Let $f(t,\omega) \in \mathfrak{M}^2[a,b]$. Then there exists a sequence $\{\Phi_n(t,\omega)\} \in \mathfrak{M}_0^2[a,b]$ so that*

$$\lim_{n\to\infty} \mathbb{E}\Big[\int_a^b |f(t,\omega) - \Phi_n(t,\omega)|^2 dt\Big] = 0$$

At this point it is important to emphasize, that the sequence of simple processes $\Phi_n(t,\omega)$ may be chosen as $\Phi_n(t,\omega) = \sum_{i=0}^n f(t_i,\omega)\chi_{[t_i,t_{i+1}]}(t)$. This choice of sequence is even used to prove the last lemma, see e.g.[25]. The reason to stressing this is that if one works with this choice of converging sequence of simple processes, then one obtains the frequently used expression of definition of Itô integral in the form

$$\int_a^b f(t,\omega)dW(t,\omega) = L^2 - \lim_{n\to+\infty} \sum_{\pi_n} f(t_i,\omega)(W(t_{i+1} - W_{t_i})(\omega) \tag{B.6}$$

where π_n expresses the sequence of partitions of finite interval $[a,b]$ with it's mesh converging to zero.
Using all information from above one is able to define the Itô integral for whole set $\mathfrak{M}^2[a,b]$.

Definition B.0.3. *Let $f(t,\omega) \in \mathfrak{M}^2[a,b]$. Then the Itô integral of f on interval $[a,b]$ is defined as*

$$I(f,a,b,\omega) \equiv \int_a^b f(t,\omega)dW(t,\omega) := L^2 - \lim_{n\to\infty} \int_a^b \Phi_n(t,\omega)dW(t,\omega)$$

where $\{\Phi_n\} \in \mathfrak{M}_0^2[a,b]$ is sequence such that

$$\lim_{n\to\infty} \mathbb{E}\Big[\int_a^b |f(t,\omega) - \Phi_n(t,\omega)|^2 dt\Big] = 0$$

The limit from integral definition above is well-defined as the image of $\mathfrak{M}_0^2[a,b]$ under Itô integration is complete space and

$$\lim_{m,n\to\infty} \mathbb{E}\Big[\int_a^b |\Phi_m(t,\omega) - \Phi_n(t,\omega)|^2 dW(t,\omega)\Big] = 0 \tag{B.7}$$

which came from the relation

$$\lim_{m,n\to\infty} \mathbb{E}\Big[\int_a^b |\Phi_m(t,\omega) - \Phi_n(t,\omega)|^2 dW(t,\omega)\Big] = \lim_{m,n\to\infty} \mathbb{E}\Big[\int_a^b |\Phi_m(t,\omega) - \Phi_n(t,\omega)|^2 dt\Big] \tag{B.8}$$

and then realizing that

$$\lim_{m,n\to\infty} \mathbb{E}\Big[\int_a^b |\Phi_m(t,\omega)-\Phi_n(t,\omega)|^2 dt\Big] = \|\Phi_m(t,\omega)-\Phi_n(t,\omega)\|^2. \tag{B.9}$$

So it is sufficient to prove, that last norm converges to zero. But this will follow from the fact that

$$\|\Phi_m(t,\omega)-\Phi_n(t,\omega)\| = \|\Phi_m(t,\omega)-f(t,\omega)+f(t,\omega)-\Phi_n(t,\omega)\| \leqslant \|\Phi_m(t,\omega)-f(t,\omega)\|+\|f(t,\omega)-\Phi_n(t,\omega)\|. \tag{B.10}$$

The right side of the last equation is zero. This follows from the the last equation from B.0.3. This proves that approximating sequence $\{\Phi_n\} \in \mathfrak{M}_0^2[a,b]$ is Cauchy. However, in complete spaces each Cauchy sequence converges and therefore the integral limit is well-defined. It is also neccesary to prove, that the value of integral is not dependent on the choice of converging sequence of simple processes. As this can be found in almost every book dedicated to theory of stochastic integral, we will not do so here. Next we will remind some important properties of Itô stochastic integral

Theorem B.0.3. *Let $f, g \in \mathfrak{M}^2[a,b]$. Then for almost all $\omega \in \Omega$*

- $\int_a^b f dW(t) = \int_a^s f dW(t) + \int_s^b f dW(t)$*, where* $0 \leqslant a < s < b$*,*
- $\int_a^b (cf+g) dW(t) = c\int_a^b f dW(t) + \int_a^b g dW(t)$ *where c is constant,*
- $\mathbb{E}[\int_a^b f dW(t)] = 0$*,*
- $\int_0^t f dW(s)$ *is* $\mathcal{F}_t$ *measurable,*
- $\mathbb{E}[(\int_a^b f dW(t))^2] = \mathbb{E}[(\int_a^b f^2 dt]$.

Apart from all these properties, one property is of special importance, as other types of stochastic integrals (Stratonovich, Hanggi-Climontovich etc.) defined through different choice of discretization point in simple process definition generally do not carry it is the fact that

Theorem B.0.4. *Let $f(t,\omega) \in \mathfrak{M}^2[0,t]$. Then*

$$I(f,t,\omega) = \int_0^t f(s,\omega) dW(s)$$

is martingale with respect to $\mathcal{F}_t$.

This property does not come for free. We have to count for the fact, that despite we got result, which is martingale, the theory will be slightly tricky in other corner of it. And it is the chain rule, which is not the same as in the ordinary deterministic calculus. Before we present this chain rule modification, conditions on integrands will be slightly relaxed

Definition B.0.4. *Let $\tilde{\mathfrak{M}}^2[0,+\infty]$ be a set of stochastic processes (sample paths of stochastic processes) fulfilling following conditions*

- *map* $(t,\omega) \to f(t,\omega) \in \tilde{\mathfrak{M}}^2[0,+\infty]$ *is* $\mathcal{B}\times\mathcal{F}$ *measurable (*$\mathcal{B}$ *represents Borel* σ*-algebra on* $[0,+\infty]$*,*
- *there exists an increasing family of* σ*-algebras* $\mathcal{H}_t$*,* $t \geq 0$ *such that*

  1. *$W(t)$ is a martingle with respect to $\mathcal{H}_t$*
  2. *$f(t,\omega)$ is $\mathcal{H}_t$-adapted*

- $\mathbb{P}\Big[\int_a^b f^2(t,\omega)dt < +\infty\Big] = 1$ *for arbitrary* $a, b \in [0, +\infty]$

It may be proven, that with above assumptions on integrands the generalized Itô integral can be defined, although the mean square convergence will be replaced by weaker convergence in probability. At this moment one can proceed in defining Itô process

Definition B.0.5. *Let* $W(t)$ *be 1-dimensional Wiener process on probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with* $t \in [0, +\infty]$. *Then Itô process* $X(t)$ *is a stochastic process on the same probability space given by condition*

$$X(t) = X(0) + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dW(s)$$

where $v \in \tilde{\mathfrak{M}}^2[0, +\infty]$ *and* u *is* $\mathcal{H}_t$*-adapted and fulfilling weaker condition comparing* v

$$\mathbb{P}\Big[\int_0^t |u(s,\omega)|ds < +\infty\Big] = 1$$

Now one is fully prepared for stating theorem generalizing deterministic chain rule known from calculus

Theorem B.0.5 (1-D Itô lemma (formula))**.** *Let* $L(t)$ *be an Itô process given by*

$$dX(t) = udt + vdW(t)$$

Let $g(t,x) \in C^2([0, +\infty] \times \mathbb{R})$. *Then*

$$Y(t) = g(t, X(t))$$

is Itô process and such function of stochastic process has the differential of the form

$$dY(t, X(t)) = \frac{\partial g}{\partial t}(t, X(t))dt + \frac{\partial g}{\partial x}(t, X(t))\Big(udt + vdW(t)\Big) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X(t))v^2 dt$$

Last theorem from 1-D theory of Itô stochastic calculus will be useful rule similar to that from deterministic calculus

Theorem B.0.6 (Integration by parts)**.** *Let* $f(t)$ *be continuous function of bounded variation in* $[0, t]$. *Then*

$$\int_0^t f(s)dW(s) = f(t)W(t) - \int_0^t W(s)df(s)$$

In the last part of this appendix we will briefly remind the basics from the theory of stochastic differential equations. We will be focused on the equation in the form

$$\frac{dX(t)}{dt} = b(t, X(t)) + \sigma(t, X(t))\Gamma(t) \tag{B.11}$$

with $\Gamma(t)$ being some random term. This equation is obviously simply the generalization of standard deterministic differential equation. If we multiply this equation by dt we get

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t) \tag{B.12}$$

where $dW(t)$ stands for infinitesimal increment of Wiener process. So it is clear now, that white noise represents the derivative of Wiener process increment. This notation is more or less formal as it is known, that Wiener process sample paths are non-differentiable at each point. However, modified representation of stochastic problem described by the last equation is mathematically treatable. In fact, it is Itô lemma, which is able to tackle with similar problems. Integrating B.12 gives us

$$X(t) = X(0) + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s). \tag{B.13}$$

Now using stochastic integral theory presented in this appendix earlier one can see that the last interpretation of original equation is already mathematically treatable. The remaining question is whether this equation has solution and if it has, then how many. For the sake of generality, we will present here the multidimensional version of the problem.

Theorem B.0.7. *Let $b(\cdot,\cdot) : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma(\cdot,\cdot) : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be measurable functions satisfying*

$$|b(t,x)| + |\sigma(t,x)| \leqslant c(1 + |x|)$$

for $\forall x \in \mathbb{R}^n$, $t \in [0,T]$ and $c \in \mathbb{R}$ being some constant (the absolute value is understood as $|\sigma|^2 = \sum |\sigma_{ij}|^2$) such that

$$|b(t,x) - b(t,y)| + |(\sigma(t,x) - \sigma(t,y)| \leqslant d|x - y|$$

for other constant $d \in \mathbb{R}$ and $y \in \mathbb{R}^n$. Let Z be a random variable which is independent of the σ-algebra generated by Wiener process $W(t)$ for all possible times with additional condition

$$\mathbb{E}[|Z|^2] < +\infty.$$

Then the stochastic differential equation

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t)$$

where $t \in [0,T]$ with initial condition $X(0) = Z$ has a unique t-continuous solution $X(t,\omega)$. For this solution also holds following

- *$X(t,\omega)$ is adapted to filtration $\mathcal{F}_t^Z$ generated by initial conidition Z and Wiener process up to time t,*
- $\mathbb{E}[\int_0^T |X(t)|^2 dt] < +\infty$

Practically, above mentioned theorem is to be used in situations, where the new equation is derived and its solution has to be found. The reason is that in this cases, it is not so easy to get the solution immediately from observation and one has to tackle with anwering the fundamental question of solution existence first. One can find intuitive explanations of additional condition inequalities in the theorem in the literature (e.g.[25]). For example, the very first inequality guarantees, that the solution will not behave wildly at finite times on which the solution process is defined.

The second inequality presumption (i.e. Lipshitz conidition) is then important for uniqueness of the solution.

However, the uniqueness and existence theorem as written above is covering the solutions of stochastic differential equation on given probability space only, as concrete version of Wiener process is already given in advance. Even initial conidition is presumed to depend on history of sample paths of Brownian motion. This type of solution is then called *strong solution*. There also exists a theory of *weak solutions*. The basic difference is, that in that case, the version of Brownian motion is not given and one condition which has to be fulfilled, in comparision with quite many additional assumptions in the theorem on existence and uniqueness of strong solution, is that after substituting the weak solution into the equation, it has to be formally fulfilled then. From this is clear, that every strong solution is also weak solution, the reversed implication is however not true. [25] [30].

Theory of Itô stochastic calculus as we showed here in this appendix is not the most general one. Up to this moment, we were assuming that integrator is Wiener process. But Wiener process is by far not the most general type of stochastic processes. In the process of generalizing physical phenomenons it is natural to expect that original distributions being not Lorentz covariant will change to be Lorentz covariant or be part of equations of motion modified to be Lorentz covariant.

Appendix C

Fokker-Planck equation

Next appendix is dedicated to the theory of special type of master equation, so called Fokker-Planck equation. As every master equation, Fokker-Planck equation will encode time change of probability at some place at given time. Unlike Chapman-Kolmogorov equation, Fokker-Planck equation is a partial differential equation. Firstly, Fokker-Planck equation in one dimension with its coefficients is derived, after that Pawula theorem and its consequences are reminded and finally Langevin Fokker-Planck equivalence will be discussed. The main sources for this appendix are [11], [16], [10], [6].
Basic assupmtion when deriving Fokker-Planck equation is the general validity of equation

$$f(x,t+\tau) = \int f(x,t+\tau|x',t)f(x',t) \tag{C.1}$$

which is a basic relation between transition probability density known from theory of continous stochastic processes and corresponding probability density. Next we think of time τ as very small time increment. Besides these two presumptions, we have one additional, that we already know all the moments

$$M(n,x',t,\tau) = \mathbb{E}\Big[(x-x')^n\Big] = \int (x-x')^n f(x,t+\tau|x',t)dx. \tag{C.2}$$

for the random variable $x(t+\tau)$. There exist several ways of derivation. We chose one where the use of characteristic funtion is not neccessary. The transition probability density in C.1 may be expressed as

$$f(x,t+\tau|x',t) = \int \delta(y-x)f(y,t+\tau|x',t)dy \tag{C.3}$$

which is nothing but exploiting the properties of Dirac-delta generalized function. Now the Dirac delta function is expressed with use of following modification in argument

$$\delta(x-y) = \delta(x'-x+y-x') \tag{C.4}$$

where x' stands for the position of particle at time t as above. Now the right side of the last equation may be formally expanded into Taylor series

$$\delta(x'-x+y-x') = \sum_{n=0}^{+\infty}(\frac{\partial^n}{\partial x'^n})\delta(x'-x)\frac{(y-x')^n}{n!} \tag{C.5}$$

and exploiting the properties of generalized derivative of Dirac delta fucntion we may next write that

$$\delta(y-x) = \sum_{n=0}^{+\infty}((-1)^n \frac{\partial^n}{\partial x^n})\delta(x'-x)\frac{(y-x')^n}{n!}. \tag{C.6}$$

With this expansion, one can substitute to equation C.3 and get

$$f(x,t+\tau|x',t) = \sum_{n=0}^{+\infty}\frac{1}{n!}(-1)^n(\frac{\partial^n}{\partial x^n})\int(y-x')^n f(y,t+\tau|x',t)dy\delta(x'-x) \tag{C.7}$$

which can be modified to

$$f(x,t+\tau|x',t) = \Big[1+\sum_{n=1}^{+\infty}\frac{1}{n!}(-1)^n\frac{\partial^n}{\partial x^n}M(n,x,t,\tau)\Big]\delta(x-x') \tag{C.8}$$

where one used the fact, that $\delta(x-x')f(x') = \delta(x'-x)f(x')$ together with normalization condition for transition probability $\int f(y,t+\tau|x',t)dy = 1$. If one now inserts this expression into original assupmtion C.1, one gets

$$f(x,t+\tau) = \int\Big[1+\sum_{n=1}^{+\infty}\frac{1}{n!}(-1)^n\frac{\partial^n}{\partial x^n}M(n,x,t,\tau)\Big]\delta(x-x')f(x',t)dx' \tag{C.9}$$

which after calculating the first term gives

$$f(x,t+\tau) = f(x,t)+\sum_{n=1}^{+\infty}(-1)^n\frac{\partial^n}{\partial x^n}\Big[\frac{M(n,x,t,\tau)}{n!}\Big]f(x,t) \tag{C.10}$$

After rearranging terms one gets

$$f(x,t+\tau)-f(x,t) = \sum_{n=1}^{+\infty}(-1)^n\frac{\partial^n}{\partial x^n}\Big[\frac{M(n,x,t,\tau)}{n!}\Big]f(x,t) \tag{C.11}$$

However, one of the first assumption was, that τ stands for very small time increment. Hence one can use Taylor expansion up to second term in time variable around time t to find

$$f(x,t+\tau)-f(x,t) = \frac{\partial}{\partial t}f(x,t)\tau + O(\tau^2). \tag{C.12}$$

Neglecting terms with time powers of order higher than one and substituting to the last equation one gets

$$\frac{\partial}{\partial t}f(x,t)\tau = \sum_{n=1}^{+\infty}(-1)^n\frac{\partial^n}{\partial x^n}\Big[\frac{M(n,x,t,\tau)}{n!}\Big]f(x,t) \tag{C.13}$$

To get rid of τ in front of derivative on the left side, one assumes, that n-th moment is expandable into Taylor series in τ around $\tau = 0$, so

$$M(n,x,t,\tau) = \lim_{\tau\to 0}\frac{1}{\tau}\mathbb{E}\Big[(x-x')^n\Big]\tau + O(\tau^2) \tag{C.14}$$

where the 0-th derivative term is equal to zero as the transitional probability distribution in the expected value calculation will be $f(x,t|x',t) = \delta(x-x')$. The first term in this expansion is usually written as $D^{(n)}(x,t)\tau$. With this notation one comes to the final equation for derivative of probability density in the form

$$\frac{\partial}{\partial t} f(x,t) = \sum_{n=1}^{+\infty} (-1)^n \frac{\partial^n}{\partial x^n} \Big[\frac{D^{(n)}(x,t)}{n!}\Big] f(x,t) \tag{C.15}$$

The right side of the equation is known as *Kramers-Moyal forward expansion*. If one write the right side of it explicitly up to the term with $n = 2$ neglecting all higher terms, then the Kramers-Moyal expansion becomes so called *Fokker-Planck forward equation*. This equation is also valid for the transitional probability distribution $f(x,t|x',t')$, as it is nothing but the standard probability distribution $f(x,t)$ with special initial condition $f(x,t') = \delta(x - x')$. One therefore gets

$$\frac{\partial f(x,t|x',t')}{\partial t} = \sum_{n=1}^{+\infty} (-1)^n \frac{\partial^n}{\partial x^n} \Big[\frac{D^{(n)}(x,t)}{n!}\Big] f(x,t|x',t') \tag{C.16}$$

with rewritten initial condition in the form

$$f(x,t'|x',t') = \delta(x - x') \tag{C.17}$$

In similar way as we derived Kramers-Moyal forward expansion, one can derive Kramers-Moyal backward expansion in the form

$$\frac{\partial f(x,t|x',t')}{\partial t'} = -\sum_{n=1}^{+\infty} D^{(n)}(x',t') \frac{\partial^n}{\partial x'^n} f(x,t|x',t') \tag{C.18}$$

It is possible to show that both solutions, i.e. solutions obtained from Kramers-Moyal forward equation and backward equation are equivalent. Therefore it does not depend on which one decide to solve. The advantage of the latter is the fact, that derivative operator on the right side of expansion affect only the transition probability density, not additional coefficient as it is in forward equation. The difference is also the fact, that while in the forward expansion one derive with respect to time t, which is later than the starting time $t' < t$, backward equation sees the transition probability as evolving back in time, hence the intuitive perception of the backward is not so natural as the motion is happening in the reverse time flow. This is the price one has to pay for not differentiating the coefficient $D^{(n)}(x',t')$ in front of the transitional probability density.
The question what will happen in the case when one cuts the infinite series at some point (i.e. chosen n) was investigated by Pawula.

Theorem C.0.1 (Pawula)**.** *Let's have differential expansion of the type C.16 for some stochastic process $X(t)$. If we assume, that transition probability density $f(x,t|x',t')$ can acquire positive values only, then the expansion stops either after the first term or after the second term.*

The Pawula theorem is usually used to reason, why one usually meets truncated right sides of the Kramers-Moyal equation. It can be also shown, that the Langevin equations encoding continous stochastic process given together with standard "delta" conditions are equivalently fully represented by truncated Kramers-Moyal expansion at $n = 2$, hence being Fokker-Planck equations. However, there are examples, where the approximation of probability distribution function of observed stochastic process is better approximated by Kramers-Moyal expansions truncated at higher terms. Of course, for the price, that on the tails will be som negative probability regions. The generalization of Fokker-Planck equation to n dimensions has the form

$$\frac{\partial f(\boldsymbol{x},t|\boldsymbol{x}',t')}{\partial t} = \sum_{\nu=1}^{+\infty} (-1)^\nu \frac{\partial^\nu}{\partial x_{j_1}...\partial x_{j_\nu}} D^\nu j_1,...,j_\nu(\boldsymbol{x},t) f(\boldsymbol{x},t|\boldsymbol{x}',t) \tag{C.19}$$

where $\boldsymbol{x} = (x_1, ..., x_n)$ and indeces $j_1, ..., j_\nu \in \hat{n}$. The n-dimensional initial condition is now

$$f(\boldsymbol{x}, t'|\boldsymbol{x}', t') = \delta(\boldsymbol{x} - \boldsymbol{x}'). \tag{C.20}$$

In the previous notation, one uses Einstein summation convention through indeces $j_1, ..., j_\nu \in \hat{n}$. If one multiplies the last equation by $f(\boldsymbol{x}', t')$ and after that integrates the whole equation with respect to $\boldsymbol{x}'$, then one gets the Kramers-Moyal expansion encoding the probability density $f(\boldsymbol{x}, t)$

$$\frac{\partial f(\boldsymbol{x}, t)}{\partial t} = \sum_{\nu=1}^{+\infty} (-1)^\nu \frac{\partial^\nu}{\partial x_{j_1} ... \partial x_{j_\nu}} D^\nu j_1, ..., j_\nu(\boldsymbol{x}, t) f(\boldsymbol{x}, t) \tag{C.21}$$

As it was written at the very beginning of this chapter, in the last part of this appendix we will summarize equivalence between Langevin equation and Fokker-Planck equation with respect to various discretization points. To be more exact, Langevin equation, which is only another name for stochastic differential equation, is not fully equivalent to Fokker-Planck equation or Kramers-Moyal equation. The point is, that the derivation of coefficients $D^n(\boldsymbol{x}, t)$ is dependent on the choice of discretization points, which in result leads to different expressions. One can obtain various differential equation representations of the same Langevin equation.
The set of general (even coupled) stochastic differential equations (i.e. Langevin equations) in the form

$$dx^i(t) = h^i(\boldsymbol{x}) dt + g_s^i(\boldsymbol{x}) \star dW^s(t) \tag{C.22}$$

where $i \in \hat{n}$ represents the actual number of concrete stochastic differential equation in the set, $\boldsymbol{x} = (x^1, x^2, ..., x^n)$ and index $s \in \hat{m}$ stands for the the general occurence of several terms with random behaviour, independent of each other. To be concrete, random increments $dW^j(t)$ here has Gaussian distribution

$$f(dW^j(t)) = \frac{1}{\sqrt{2\pi dt}} \exp\Big(\frac{-dW^s(t)^2}{2dt}\Big). \tag{C.23}$$

The star sign signalizes, that after integrating respective equations, one has to use the Itô stochastic calculus when evaluating it. Corresponding partial differential equation (i.e. Fokker-Planck equation) reads

$$\frac{\partial f(\boldsymbol{x}, t)}{\partial t} = \frac{\partial}{\partial x^i}\Big[- h^i(\boldsymbol{x}) f(\boldsymbol{x}, t) + \frac{1}{2}\frac{\partial}{\partial x^j}(g_s^i(\boldsymbol{x}) g_s^j(\boldsymbol{x}) f(\boldsymbol{x}, t))\Big] \tag{C.24}$$

As we were dealing with the stationary solution of Fokker-Planck equation also in *post-point dicretization rule* (i.e. transport form, kinetic form or Hänggi-Klimontovich approach) signed with the symbol $\bullet$, we also remind Fokker-Planck equation for the Langevin equation with this type of discretization. We start with the same stochastic differential equation

$$dx^i(t) = h^i(\boldsymbol{x}) dt + g_s^i(\boldsymbol{x}) \bullet dW^s(t) \tag{C.25}$$

with the same gaussian probability distribution for random increments $dW^j(t)$. Although now during derivating the corresponding partial differential equation one has to consider different discretization point (i.e. original t_i for intervals $[t_i, t_{i+1})$ and stochatic integral evaluation are switched to point t_{i+1}). From the same derivation process then one obtains

$$\frac{\partial f(\boldsymbol{x}, t)}{\partial t} = \frac{\partial}{\partial x^i}\Big[- h^i(\boldsymbol{x}) f(\boldsymbol{x}, t) + \frac{1}{2} g_s^i(\boldsymbol{x}) g_s^j(\boldsymbol{x}) \frac{\partial}{\partial x^j} f(\boldsymbol{x}, t)\Big] \tag{C.26}$$

One can see, that the main difference is ordering of coefficients and neighboring derivative operators. So if one chooses the post-point discretization in evaluating stochastic integral arisen in the integrating of original equation, then one is supposed to differentiate the probability distribution only. Solving this kind fo equation is therefore more straightforward in the sense, that solving the equation will immediately give the solution for probability distribution only, not multiplied by other coefficients.
Except for above mentioned discretization points, even one more is often used and is called *Stratonovich* or *mid-point rule*. It means nothing but the discretization point when evaluating integral will be chosen as $\frac{t_{i+1}-t_i}{2}$ for intervals $[t_i, t_{i+1})$. Here one has

$$dx^i(t) = h^i(\boldsymbol{x})dt + g^i_s(\boldsymbol{x}) \circ dW^s(t) \tag{C.27}$$

while now one gets the differential equation of the type

$$\frac{\partial f(\boldsymbol{x}, t)}{\partial t} = \frac{\partial}{\partial x^i}\Big[- h^i(\boldsymbol{x}) f(\boldsymbol{x}, t) + \frac{1}{2} g^i_s(\boldsymbol{x}) \frac{\partial}{\partial x^j} (g^j_s(\boldsymbol{x}) f(\boldsymbol{x}, t))\Big]. \tag{C.28}$$

Appendix D

Derivation of non-Gaussian increment distribution

We explicitly derive the probability distribution for relativistic generalization of original Wiener process increment denoted $dW(t)$. As mentioned in the Chapter 1, one starts from the joint probability distribution

$$f(dW^{\alpha}(\tau)) = \frac{c}{\sqrt{4\pi D d\tau}} e^{-\frac{D_{\alpha\beta} dW^{\alpha}(\tau) dW^{\beta}(\tau)}{2}} \delta(u_{\alpha}(\tau) dW^{\alpha}(\tau)) \tag{D.1}$$

for practical reasons of simplifying notation, in this Appendix we assign $dW^{\alpha}(\tau) \equiv w^{\alpha}(\tau)$ in accordance with [7].
So one can write

$$w^{1}(\tau) = \int d(w^{0}(\tau)) f(w^{\alpha}(\tau)) \tag{D.2}$$

Rewriting the right side of the last equation one gets

$$\int d(w^{0}(\tau)) \frac{c}{\sqrt{4\pi D d\tau}} \exp\Big(-\frac{w_{\alpha}(\tau) w^{\alpha}(\tau)}{4D d\tau}\Big) \delta(u_{\alpha} w^{\alpha}(\tau)) \tag{D.3}$$

using the same metric convention as in [7], one can break down the inner products of 2-vectors and write

$$\frac{c}{\sqrt{4\pi D d\tau}} \int d(w^{0}(\tau)) \exp\Big(-\frac{w^{1}(\tau)^{2} - w^{0}(\tau)^{2}}{4D d\tau}\Big) \delta\Big(c\gamma w^{0}(\tau) - \gamma v w^{1}(\tau)\Big) \tag{D.4}$$

extracting the term which is not subject of integration one gets

$$\frac{c}{\sqrt{4\pi D d\tau}} \exp\Big(-\frac{w^{1}(\tau)^{2}}{4D d\tau}\Big) \int d(w^{0}(\tau)) \exp\Big(-\frac{w^{0}(\tau)^{2}}{4D d\tau}\Big) \delta\Big(c\gamma w^{0}(\tau) - \gamma v w^{1}(\tau)\Big) \tag{D.5}$$

To treat the integral with Dirac function properly, one makes formal transform of variable $w^{0}(\tau)$ in the form

$$\tilde{w}^{0}(\tau) \equiv c\gamma w^{0}(\tau) \tag{D.6}$$

and the last expression becomes

$$\frac{1}{\sqrt{4\pi D d\tau}} \exp\Big(-\frac{w^1(\tau)^2}{4Dd\tau}\Big) \int \frac{d\tilde{w}^0(\tau)}{\gamma} \exp\Big(-\frac{\tilde{w}^0(\tau)^2}{4D(c\gamma)^2 d\tau}\Big)\delta\Big(\tilde{w}^0(\tau) - \gamma v w^1(\tau)\Big) \tag{D.7}$$

and one is able to calculate the integral according to well-known properties of Dirac function. After doing it, one can write

$$\frac{1}{\sqrt{4\pi D d\tau\gamma^2}} \exp\Big(-\frac{w^1(\tau)^2}{4Dd\tau}\Big) \exp\Big(\frac{(\gamma v w^1(\tau))^2}{(c\gamma)^2 4Dd\tau}\Big) \tag{D.8}$$

and finally

$$\frac{1}{\sqrt{4\pi D d\tau\gamma^2}} \exp\Big(-\frac{w^1(\tau)^2}{4\pi Dd\tau}\Big(1-\frac{v^2}{c^2}\Big)\Big). \tag{D.9}$$

The very last step to obtain desired expression for probability density is to utilize definition of Lorentz factor

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{D.10}$$

which gives

$$f(dW^1(\tau)) = f(w^1(\tau)) = \frac{1}{\sqrt{4\pi D d\tau\gamma^2}} \exp\Big(-\frac{w^1(\tau)^2}{4\pi D\gamma^2 d\tau}\Big) \tag{D.11}$$

To write the probabilty density in laboratory heath bath frame, one has to exploit the relationship between proper time τ and laboratory time t, i.e.

$$d\tau = \frac{dt}{\gamma}. \tag{D.12}$$

Inserting this relation into the probability density leads to

$$f(dW^1(t)) = \frac{1}{\sqrt{4\pi D dt\gamma}} \exp\Big(-\frac{w^1(\tau)^2}{4\pi D\gamma dt}\Big) \tag{D.13}$$

which is exactly the expression, which can be found in i.e.[7].

Conclusion

In this thesis we tried to shed a light on the theory of stochastic processes and its relativistic generalizations, at first in the phase space manner with use of coupled stochastic differential equations describing so called Ornstein-Uhlenbeck process and after that continued with space-time approach described by diffusion equation and its relativistic generalization, which is in one space dimension nothing but the telegrapher's equation equivalent to the coupled partial differential equations called Kolmogorov equations. We tried to exploit the phase space description in order to obtain transitional probability serving in the option valuation model as the integral pricing kernel within framework of risk-neutral valuation, which is equivalent to Feynman-Kac formula in the theory of stochastic processes. However, as one could see, all three approaches have not lead to satisfying result. In the phase space, one is at most able to express the stationary distribution of the position in accordance with works [6], [7], [8], [5]. With the coefficients occuring in the relativistic generalizations of the Ornstein-Uhlenbeck process, the resulting stationary distribution is neither dependent on position. One may not even calculate the corresponding integral and see, that the resulting stationary distribution would be position independent. What's more, if one could derive the non-stationary probability distribution function, there is certainity, that resulting distribution would be non-Markovian [32], [13], [56]. If one wanted use this kind of probabilty, it is not consistent with the market description without assuming some unexpected extreme events in it (despite it already was used in several cases, e.g. [43]). But what one could do, was to utilize the mutual correspondence between Dirac equation in (1+1) dimension and telegrapher's equation, which is nothing but the relativistic generalization [56], [6], [5], [33] of diffusion equation toghether with this exploit the theory of financial analogy to quantum mechanics. Generally, there are two ways how to exploit this theory. First, one can still be interested in relativistic generalization of underlying security price evolution process and this price incorporation into the original Black-Scholes-Merton valuation model or work directly with the option price. The first option was investigated in [43] and so we exploited the theory of [12], [52], [53] to directly find the financial Hamiltonian for each component of Dirac spinor in (1+1) dimensions. We could see, that the imaginary units occured there and hence giving us the suggestion, that within this framework, the generalized option price could be imaginary. But we did not explicitly calculated the matrix elements, so this is just our guess. Klein-Gordon financial Hamiltonian was also presented and if the matrix elements were calculated correctly, one could observe Gaussian distribution with different coefficients with energy of the particle serving as generalization of risk-free interest rate. This could be part of the further investigations to precisely compute respective matrix elements and see what one gets. We also reminded, how the barrier options valuation theory [12] may be utilized to describe the market speed of light [43] finiteness. What's more, we presented the fact, that each component of Dirac spinor in (1+1) dimensions is equivalent to each other and no differences are observed. This comes from the equivalence with telegrapher's equation for each component. In the very end of the thesis we briefly presented the theory of path integral and it's use in the valuation of options [57]. We revised the problems, which arise when one want to use it and presented some ideas how one can extend the theory and possibly get rid of that problems. In the future, one can also exploit superstatistical approach to path integral for relativistic particle bonding the theory of path integral and stochastic mass of the particle [67]. However, these topics are too broad to be part of this thesis and are let to be the subject of other future analysis. Overall, from various approaches presented in this thesis, it is very difficult to decide, which model is the best relativistic generalization of the original Black-Scholes valiuation model. Each of them has its advantages, disadvantages and even areas, which are not sufficiently explored.

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