Nesamosdružené relativistické bodové interakce a jejich aproximace pomocí nelokálních potenciálů

VÝZKUMNÝ ÚKOL

Bc. Lukáš Heriban **2021**

Czech technical university in Prague
Faculty of Nuclear Sciences and Physical Engineering

Non-self-adjoint relativistic point interactions and their approximations by non-local potentials

Research project

Author: **Bc. Lukáš Heriban**
Supervisor: **Ing. Matěj Tušek, Ph.D.**
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I declare that I have prepared my thesis independently and that I used only the sources listed in the Bibliography.

In Prague on 30th August 2021 Bc. Lukáš Heriban

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Author: Bc. Lukáš Heriban

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Supervisor: Ing. Matěj Tušek, Ph.D., Faculty of Nuclear Sciences and Physical Engineering, CTU

Abstract: The research project deals with the Dirac operator with non–local potential given by the projection on a fixed scaled function from $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ multiplied by complex matrix $\mathbb{A}$. The norm–resolvent limit of this not necessarily self–adjoint operator is discussed in this thesis. Furthermore, the rigorous expression for the norm resolvent limit is compared to the formal limit of the Dirac operator with non–local potential. This formal limit corresponds to the norm–resolvent limit. In other words, renormalization of the coupling constant does not occur. This property will lead to generalization of the definition of the Dirac operator with relativistic point interaction. Moreover, the spectrum of this newly defined operator is discussed. Remarkable spectral transitions in special cases will be presented. Finally, this spectral transition will be explained by examining ε–pseudospectrum of the operator.

Key words: Dirac operator, local potentials, non–self adjoint operators, non–local potentials, point interactions, unbounded operators

Název práce:

Nesamosdružené relativistické bodové interakce a jejich aproximace pomocí nelokálních potenciálů

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Abstrakt: Tato práce se zabývá volným Diracovým operátorem s nelokálním potenciálem v podobě projekce v $L^2(\mathbb{R})$ na pevně zvolenou škálovanou funkci z $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ tenzorově vynásobené komplexní 2×2 maticí $\mathbb{A}$. Následně je zde řešen problém uniformní konvergence v resolventě, takto definovaného, ne nutně samosdruženého operátoru. Poté je v této práci provedeno porovnání této operátorové limity s formální limitou daného operátoru. Ukážeme, že tato formální limita se skutečně shoduje s operátorovou limitou. Jinými slovy nedochází k takzvané renormalizaci vazebných konstant. Této vlastnosti je dále využito ke zobecnění definice operátoru relativistické bodové interakce i na nesamosdružené případy. V práci je také diskutována problematika spektrální analýzy tohoto nově definovaného operátoru. V práci jsou představené pozoruhodné spektrální přechody pro speciální případy matice $\mathbb{A}$. K vysvětlení těchto spektrálních přechodů vedlo prozkoumání chování ε – pseudospektra v kritických případech.

Klíčová slova: bodové interakce, Diracův operátor, lokální potenciály, nelokální potenciály, neomezené operátory, nesamosdružené operátory

Contents

1 Introduction

The one–dimensional Dirac operator H_m acting like

$$H_m\psi(x) = -i\frac{\mathrm{d}}{\mathrm{d}x} \otimes \sigma_1\psi(x) + m \otimes \sigma_3\psi(x)$$

$$\psi \in \mathrm{Dom}\, H_m = W^{1,2}(\mathbb{R}) \otimes \mathbb{C}^2,\ m \geq 0$$

perturbed at one point is an important exactly solvable model of quantum mechanics. Approximations of this mathematical model were rigorously discussed for the first time by Šeba in [5] where he studied exclusively the electrostatic and Lorentz scalar point interactions. More general definition of self–adjoint relativistic point interaction H^Λ was discussed by Benvegnu and Dabrowski in [7], where H^Λ acts like the free Dirac operator H_m on functions from the Sobolev space with transmission condition at the point of interaction

$$\psi(0+) = \Lambda\psi(0-)$$

$$\Lambda = \omega \begin{pmatrix} \alpha & i\beta \\ -i\gamma & \delta \end{pmatrix},$$

$$\omega = \mathrm{e}^{i\varphi}, \alpha\delta - \beta\gamma = 1, \varphi, \alpha, \beta, \delta, \gamma \in \mathbb{R}.$$

Non–self–adjoint models extending quantum mechanics have been studied since the beginning of 21st century and there are only few papers discussing non–self–adjoint non–relativistic point interaction for example [1]. However, as far as I know, non–self–adjoint case of relativistic point interaction has not been studied yet.

This work is focused on using not necessary self–adjoint non–local potential in the form of the projection $1/\varepsilon^2\, |v(x/\varepsilon)\rangle\langle v(x/\varepsilon)|$ on a fixed scaled function v in $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ multiplied by a complex matrix $\mathbb{A} \in \mathbb{C}^{2,2}$ with the differential Dirac operator as an approximation to some unbounded operator.

$$H_\varepsilon^{\mathbb{A}} = -i\frac{\mathrm{d}}{\mathrm{d}x} \otimes \sigma_1 + m \otimes \sigma_3 + \frac{1}{\varepsilon^2}|v(x/\varepsilon)\rangle\langle v(x/\varepsilon)| \otimes \mathbb{A} \tag{1}$$

We already showed [2] that in the self–adjoint case the norm resolvent limit of (1) corresponds to the relativistic point interaction discussed in [7]. We will show that also for non–self–adjoint matrices $\mathbb{A}$ the norm–resolvent limit exists and we will call the limit the non–self–adjoint relativistic point interaction.

Idea of using non–local potential to study approximation of relativistic point interaction comes from the paper [5], where two matrices $\mathbb{A}$ are studied explicitly. We will generalize this result to any complex matrix $\mathbb{A}$.

In [5] Šeba also discussed comparison of the formal limit and the operator limit. He showed that by starting with the Dirac operator with local potential $1/\varepsilon\, h(x/\varepsilon)$ the formal limit will not correspond to the proper operator limit. This phenomena is called renormalization of the coupling constant. More general case of the local potential and its limit was already discussed by Hughes [8],[9] and by Tušek [6]. Šeba also showed that in two special cases of self–adjoint matrix $\mathbb{A}$ using non–local potential will not lead to the renormalization. We already showed that this property is preserved in the most general self–adjoint case of non–local potential [2]. We will show that this property is also preserved in non–self–adjoint case. Moreover, this approach extends naturally the definition of relativistic point interactions also to non–self–adjoint case.

Furthermore, an expression for finding the spectrum of newly defined general relativistic point interaction will be derived. We will discussed remarkable spectral transition in the non–self–adjoint case. We will also find an implicit equation for eigenvalues and eigenfunctions of non–local approximations of the relativistic point interactions.

2 Notation

We denote by $L^p(U;\mathcal{H})$ the Banach space of integrable functions in the pth power on the domain U with values in a Hilbert space $\mathcal{H}$. We denote the norm in $L^p(U;\mathcal{H})$ by $\|.\|_p$. If $L^p(U;\mathbb{C})$ we will simply write $L^p(U)$ and if $U = \mathbb{R}$ we will write L^p. For the scalar product on the Hilbert space L^2 we will use the symbol $\langle\cdot|\cdot\rangle$. We will also identify $L^2(U)\otimes\mathbb{C}^2$ with $L^2(U;\mathbb{C}^2)$. Abusing notation we will denote

$$\langle f|\psi\rangle = \begin{pmatrix}\langle f|\psi_1\rangle \\ \langle f|\psi_2\rangle\end{pmatrix}, \langle f|\mathbb{B}\rangle = \begin{pmatrix}\langle f|\mathbb{B}_{11}\rangle & \langle f|\mathbb{B}_{12}\rangle \\ \langle f|\mathbb{B}_{21}\rangle & \langle f|\mathbb{B}_{22}\rangle\end{pmatrix}$$

for $f \in L^2(\mathbb{R})$, $\psi \in L^2(\mathbb{R};\mathbb{C}^2)$ and $\mathbb{B} \in L^2(\mathbb{R};\mathbb{C}^{2,2})$. For an integral operator K we will write its integral kernel as $K(x,y)$. We denote $f * g$ the convolution of two functions f and g. For bounded operators on the Hilbert space $\mathcal{H}$ we will use the symbol $\mathcal{B}(\mathcal{H})$. For a matrix $\mathbb{B} \in \mathbb{C}^{n,m}$ we will sometimes denote $|\mathbb{B}|$ for the Frobenius norm of the matrix $\mathbb{B}$

$$|\mathbb{B}|^2 = \|\mathbb{B}\|_2^2 = \sum_{i,j=1}^{n,m} |\mathbb{B}_{ij}|^2.$$

Note that the Frobenius norm is submultiplicative which can be proved using the Cauchy–Schwarz inequality

$$|\mathbb{A}\mathbb{B}|^2 = \sum_{i=1}^{n}\sum_{j=1}^{k}\langle\vec{a}_i|\vec{b}_j\rangle^2 \le \sum_{i=1}^{n}\sum_{j=1}^{k}\|\vec{a}_i\|_2^2\|\vec{b}_j\|_2^2 = |\mathbb{A}|^2|\mathbb{B}|^2.$$

3 Relativistic point interactions

Let us start with the definition of our non–local potential as a projection on a scaled vector v from $L^2(\mathbb{R};\mathbb{R}) \cap L^1(\mathbb{R};\mathbb{R})$ multiplied by a 2×2 complex matrix. Using the bra–ket notation we can write the Dirac operator with the non–local potential in the following way

$$\begin{aligned} H_\varepsilon^{\mathbb{A}} &= H_m + W_\varepsilon(x) \otimes \mathbb{A} \\ W_\varepsilon &= \frac{1}{\varepsilon^2}|v(x/\varepsilon)\rangle\langle v(x/\varepsilon)| =: |v_\varepsilon(x)\rangle\langle v_\varepsilon(x)|, \end{aligned} \tag{2}$$

where H_m is the free Dirac operator defined as

$$\begin{aligned} H_m &= -i\frac{\mathrm{d}}{\mathrm{d}x} \otimes \sigma_1 + m \otimes \sigma_3 \\ \mathrm{Dom}(H_m) &= W^{1,2}(\mathbb{R}) \otimes \mathbb{C}^2. \end{aligned} \tag{3}$$

Here $W^{1,2}$ stands for the Sobolev space, σ_i are the Pauli matrices and m is a non–negative constant. Its resolvent is the integral operator with the integral kernel given by

$$R_z(x,y) = \frac{i}{2}(\mathbb{Z}(z) + \mathrm{sgn}(x-y)\sigma_1)\mathrm{e}^{ik(z)|x-y|}, \tag{4}$$

where

$$\mathbb{Z}(z) = \begin{pmatrix} \zeta(z) & 0 \\ 0 & \zeta^{-1}(z) \end{pmatrix},$$

$$\zeta(z) = \frac{z+m}{k(z)} \text{ and } k(z) = \sqrt{z^2 - m^2}, \mathrm{Im}k(z) \geq 0.$$

We already found out [2] that for a self–adjoint matrix $\mathbb{A}$ the Dirac operator with the non–local potential converge in the norm–resolvent sense to the operator of the self–adjoint relativistic point interaction $H^{\mathbb{A}}$ [7] acting like

$$\begin{aligned} (H^{\mathbb{A}}\psi)(x) &= (H_m\psi)(x), \ \forall x \in \mathbb{R}\setminus\{0\} \\ \psi \in \mathrm{Dom}(H^{\mathbb{A}}) &= \{\varphi \in W^{1,2}(\mathbb{R}\setminus\{0\}) \otimes \mathbb{C}^2 \mid (2i - \sigma_1\mathbb{A})\varphi(0+) = (2i + \sigma_1\mathbb{A})\varphi(0-)\}. \end{aligned} \tag{5}$$

Recall that if we take the Dirac operator with a scaled potential V_ε which goes to the delta potential

$$H_m + V_\varepsilon \otimes \mathbb{A}, \tag{6}$$

one can find a formal limit of this operator as $H_m + \mathbb{A}\delta$. This formal limit can be rewritten as the free Dirac operator with the transmission condition at the point of interaction. We will begin with the expression

$$(H_m + \mathbb{A}\delta)\psi. \tag{7}$$

If we define

$$\psi(0) := \frac{\psi(0+) + \psi(0-)}{2}$$

then to cancel singular parts in (7) the following condition must be true

$$(2i\sigma_1 + \mathbb{A})\psi(0-) = (2i\sigma_1 - \mathbb{A})\psi(0+).$$

Because of that we can see that the formal limit of (6) correspond the the operator (5).

This formal limit would seem as a good guess for the operator limit. However, it is now well known [5],[6],[8] that for the special case of the potential V_ε which is the local potential

$$V_\varepsilon = \frac{1}{\varepsilon} h\left(\frac{x}{\varepsilon}\right)$$

the operator limit does not correspond to the formal limit. This phenomena is known as the renormalization of the coupling constant.

We have already shown in the self–adjoint case [2], as one can see above, that for the non–local potential (2) the renormalization of coupling constant does not occur. In other words, the formal limit for the non–local potential is the same as the norm–resolvent limit of the operator.

We can extend these results also for the non–self–adjoint case of the matrix $\mathbb{A}$. We will show that the resolvent of the operator also converge to some bounded operator. We will show that the limit is the resolvent of a unbounded operator also acting like H_m with certain boundary condition at the point of interaction and we will call this operator the non–self–adjoint relativistic point interaction.

3.1 General relativistic point interaction and its non–local approximation

Let us firstly state one of the main results of this paper which is the existence of norm–resolvent limit of the Dirac operator with scaled non–local potential multiplied by any complex matrix.

Theorem 3.1.1. *Let the matrix $\mathbb{A}$ in the definition of the Dirac operator with the non–local potential (2) be any complex matrix and $z \in \mathbb{C} \setminus \{(-\infty, -m] \cup [m, +\infty)\}$ such that the matrix*

$$(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))$$

is invertible. Then the resolvent of the non–local potential converges in the operator norm to the bounded integral operator

$$R_z^{\mathbb{A}}(x, y) = R_z(x, y) - R_z(x, 0)(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}R_z(0, y) \tag{8}$$

as $\varepsilon \to 0$.

First, we recall the Minkowski integral inequality which plays the main role in the proof of the theorem above.

Proposition 3.1.1 (Minkowski integral inequality)**.** *Let (U_1, μ_1) and (U_2, μ_2) be σ–finite measure spaces and $g : U_1 \times U_2 \to \mathbb{R}$ is measurable, non–negative function. Let $p \geq 1$ then*

$$\left(\int_{U_2}\left|\int_{U_1} g(x, y)\,\mathrm{d}\mu_1(x)\right|^p \mathrm{d}\mu_2(y)\right)^{\frac{1}{p}} \leq \int_{U_1}\left(\int_{U_2} |g(x, y)|^p\,\mathrm{d}\mu_2(y)\right)^{\frac{1}{p}} \mathrm{d}\mu_1(x) \tag{9}$$

Proof. Denote $F(y) = \int_{U_1} g(x, y)\mu_1(x)$ then using the Fubini theorem and the Hölder inequality we obtain

$$\|F\|^p_{L^p(U_2,\mathrm{d}\mu_2)} = \int_{U_2}\left|\int_{U_1} g(x, y)\,\mathrm{d}\mu_1(x)\right| |F^{p-1}(y)|\,\mathrm{d}\mu_2(y) \leq \int_{U_1}\int_{U_2} |g(x, y)||F^{p-1}(y)|\,\mathrm{d}\mu_2(y)\,\mathrm{d}\mu_1(x) \leq$$

$$\leq \int_{U_1}\left(\int_{U_2} |g(x, y)|^p\,\mathrm{d}\mu_2(y)\right)^{\frac{1}{p}} \mathrm{d}\mu_1(x)\|F\|^{p-1}_{L^p(U_2,\mathrm{d}\mu_2)}$$

If $\|F\|_{L^p(U_2,\mathrm{d}\mu_2)} < +\infty$ we get the wanted inequality. If $\|F\|_{L^p(U_2,\mathrm{d}\mu_2)} = +\infty$ we can choose monotone sequences $V_n^1 \subset U_1, V_k^2 \subset U_2$ such that $\forall k, n \in \mathbb{N}, \mu_1(V_n^1), \mu_2(V_k^2) < +\infty$ and $V_n^1 \to U_1$, $V_k^2 \to U_2$. Then for every pair of V_n^1 and V_k^2 inequality holds and by letting $k, n \to +\infty$ we obtain the result. □

Furthermore we will need following lemmas.

Lemma 3.1.1. $(I + \langle v_\varepsilon | \mathbb{A} R_z v_\varepsilon \rangle)^{-1} \overset{u}{\to} (I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}$

Proof. By [[12], theorem IV 1.16]

$$(I + \langle v_\varepsilon | \mathbb{A} R_z v_\varepsilon \rangle)^{-1} \overset{u}{\to} (I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1} \text{ if and only if}$$

$$\langle v_\varepsilon | \mathbb{A} R_z v_\varepsilon \rangle \overset{u}{\to} \frac{i}{2}\mathbb{A}\mathbb{Z}(z)$$

$$|\langle v_\varepsilon | \mathbb{A} R_z v_\varepsilon \rangle - \frac{i}{2}\mathbb{A}\mathbb{Z}(z)|^2 = \left\| \begin{matrix} a\zeta(z)(E_\varepsilon - \frac{i}{2}) & b\zeta^{-1}(z)(E_\varepsilon - \frac{i}{2}) \\ c\zeta(z)(E_\varepsilon - \frac{i}{2}) & d\zeta^{-1}(z)(E_\varepsilon - \frac{i}{2}) \end{matrix} \right\|_2^2 \leq$$
$$\leq \|\mathbb{A}\|_2^2 (|\zeta(z)|^2 + |\zeta^{-1}(z)|^2)|E_\varepsilon - \frac{i}{2}|^2 \overset{\varepsilon \to 0^+}{\to} 0$$

□

Lemma 3.1.2. $\sup_{y\in\mathbb{R}} \int_\mathbb{R} |R_z(x,y)|^2 \,\mathrm{d}x < +\infty$

Proof.

$$|R_z(x,y)| = \|\frac{i}{2}(\mathbb{Z}(z) + \operatorname{sgn}(x-y)\sigma_1)\mathrm{e}^{ik(z)|x-y|}\|_2 \leq \frac{1}{2}(\|\mathbb{Z}(z)\|_2 + \|\sigma_1\|_2)|\mathrm{e}^{ik(z)|x-y|}| \leq$$
$$\leq \frac{1}{2}(\|\mathbb{Z}(z)\|_2 + \|\sigma_1\|_2)\mathrm{e}^{-\mathrm{Im}k(z)|x-y|}$$

This implies

$$\int_\mathbb{R} |R_z(x,y)|^2 \,\mathrm{d}x \leq \frac{1}{4}(\|\mathbb{Z}(z)\|_2 + \|\sigma_1\|_2)^2 \int_\mathbb{R} \mathrm{e}^{-2\mathrm{Im}k(z)|x-y|}\,\mathrm{d}x =$$
$$= \frac{1}{4}(\|\mathbb{Z}(z)\|_2 + \|\sigma_1\|_2)^2 \int_\mathbb{R} \mathrm{e}^{-2\mathrm{Im}k(z)|x|}\,\mathrm{d}x < +\infty.$$

We can see that the estimate does not depend on $y \in \mathbb{R}$. □

Lemma 3.1.3. $\lim_{\varepsilon\to 0} \int_\mathbb{R} \left(\int_\mathbb{R} |R_z(x,\varepsilon s) - R_z(x,0)||v(s)| \,\mathrm{d}s \right)^2 \,\mathrm{d}x = 0$

Proof. Using the Minkowski integral inequality from Proposition 3.1.1 we get

$$\int_\mathbb{R} \left(\int_\mathbb{R} |R_z(x,\varepsilon s) - R_z(x,0)||v(s)| \,\mathrm{d}s \right)^2 \,\mathrm{d}x \leq$$
$$\leq \left(\int_\mathbb{R} |v(s)| \left(\underbrace{\int_\mathbb{R} |R_z(x,\varepsilon s) - R_z(x,0)|^2 \,\mathrm{d}x}_{f_\varepsilon(s)} \right)^{\frac{1}{2}} \mathrm{d}s \right)^2.$$

Using Lemma 3.1.2 one can see that $f_\varepsilon(s)$ is uniformly bounded by some constant $C \geq 0$.

$$\forall \varepsilon > 0, \forall s \in \mathbb{R}, |f_\varepsilon(s)| < C \in \mathbb{R}$$

Because of that we can drag the limit into the outer integral.

Next, we will deal with the inner integral. Using Young´s inequality we get the following estimate

$$|R_z(x,\varepsilon s) - R_z(x,0)|^2 \leq 2|R_z(x,\varepsilon s)|^2 + 2|R_z(x,0)|^2$$

Then for a fixed $s \in \mathbb{R}$, a fixed constant δ and a small ε such that $|\varepsilon s| < \delta$ we get

$$|R_z(x,\varepsilon s)|^2 \leq \frac{1}{4}(\|\mathbb{Z}(z)\|_2 + \|\sigma_1\|_2)^2 \mathrm{e}^{-2\mathrm{Im}k(z)|x-\varepsilon s|} \leq \frac{1}{4}(\|\mathbb{Z}(z)\|_2 + \|\sigma_1\|_2)^2 m(x),$$

where $m(x)$ is the dominating integrable function defined as

$$\forall x \in \mathbb{R},\ m(x) = \mathrm{e}^{-2\mathrm{Im}k(z)(|x|-\delta)}.$$

We can see that

$$2|R_z(x,\varepsilon s)|^2 + 2|R_z(x,0)|^2 \leq \frac{1}{2}(\|\mathbb{Z}(z)\|_2 + \|\sigma_1\|_2)^2 m(x) + 2|R_z(x,0)|^2 \in L^1(\mathbb{R}).$$

Then by using the Lebesgue dominant convergent theorem two times we obtain the desired result. □

Now we can prove Theorem 3.1.1.

Proof. We will mimic the proof of the limit of the resolvent of the non–local potential from [2] but now using any complex matrix $\mathbb{A}$. We will assume that the potential is normalized to one.

From the resolvent formula and the form of the $H_\varepsilon^{\mathbb{A}}$ we get

$$R_{z,\varepsilon}^{\mathbb{A}} = (H_\varepsilon^{\mathbb{A}} - z)^{-1} = R_z(I + (|v_\varepsilon\rangle\langle v_\varepsilon| \otimes \mathbb{A})R_z)^{-1},$$

where R_z is the resolvent of the free Dirac operator.

1) We will find the inverse of the operator $(I + (|v_\varepsilon\rangle\langle v_\varepsilon| \otimes \mathbb{A})R_z)$

$$\psi + \underbrace{\langle v_\varepsilon|\mathbb{A}R_z\psi\rangle}_{\vec{k}} v_\varepsilon = g \Rightarrow \vec{k} + \langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon \otimes \vec{k}\rangle = \langle v_\varepsilon|\mathbb{A}R_z g\rangle$$

$$\vec{k} + \langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle\vec{k} = \langle v_\varepsilon|\mathbb{A}R_z g\rangle$$

$$\vec{k} = (I + \langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle)^{-1}\langle v_\varepsilon|\mathbb{A}R_z g\rangle$$

If we substitute for the vector $\vec{k}$ and we assume that inverse of the matrix $I + \langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle$ exists we get

$$\psi = g - (I + \langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle)^{-1}\langle v_\varepsilon|\mathbb{A}R_z g\rangle v_\varepsilon.$$

Existence of the inverse of the matrix $I + \langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle$ will be proved further in the text by explicitly inverting the matrix.

This yields

$$(I + (|v_\varepsilon\rangle\langle v_\varepsilon| \otimes \mathbb{A})R_z)^{-1} = I - (I + \langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle)^{-1}(|v_\varepsilon\rangle\langle v_\varepsilon| \otimes \mathbb{A})R_z.$$

Which means that we get the resolvent of the operator $H_\varepsilon^{\mathbb{A}}$ in the following form

$$R_{z,\varepsilon}^{\mathbb{A}} = R_z - R_z \underbrace{(I + \langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle)^{-1}}_{2)}(|v_\varepsilon\rangle\langle v_\varepsilon|\mathbb{A})R_z.$$

2) Now we need to find the inverse of $(I + \langle v_\varepsilon | \mathbb{A} R_z v_\varepsilon \rangle)$. If we set

$$\mathbb{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \in \mathbb{C}$$

then we get

$$\mathbb{A} R_z v_\varepsilon = \frac{i}{2} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \zeta(z) \int_{\mathbb{R}} \mathrm{e}^{ik(z)|x-y|} v_\varepsilon(y)\,\mathrm{d}y & \int_{\mathbb{R}} \operatorname{sgn}(x-y) \mathrm{e}^{ik(z)|x-y|} v_\varepsilon(y)\,\mathrm{d}y \\ \int_{\mathbb{R}} \operatorname{sgn}(x-y) \mathrm{e}^{ik(z)|x-y|} v_\varepsilon(y)\,\mathrm{d}y & \zeta^{-1}(z) \int_{\mathbb{R}} \mathrm{e}^{ik(z)|x-y|} v_\varepsilon(y)\,\mathrm{d}y \end{pmatrix} =$$

$$= \begin{pmatrix} bS + a\zeta(z)E & aS + b\zeta^{-1}(z)E \\ dS + c\zeta(z)E & cS + d\zeta^{-1}(z)E \end{pmatrix}, \text{ where}$$

$$E = \frac{i}{2} \int_{\mathbb{R}} \mathrm{e}^{ik(z)|x-y|} v_\varepsilon(y)\,\mathrm{d}y$$

$$S = \frac{i}{2} \int_{\mathbb{R}} \operatorname{sgn}(x-y) v_\varepsilon(y) \mathrm{e}^{ik(z)|x-y|}\,\mathrm{d}y$$

$$\langle v_\varepsilon | \mathbb{A} R_z v_\varepsilon \rangle = \begin{pmatrix} bS_\varepsilon + a\zeta(z)E_\varepsilon & aS_\varepsilon + b\zeta^{-1}(z)E_\varepsilon \\ dS_\varepsilon + c\zeta(z)E_\varepsilon & cS_\varepsilon + d\zeta^{-1}(z)E_\varepsilon \end{pmatrix}$$

$$E_\varepsilon = \frac{i}{2} \int_{\mathbb{R}^2} v_\varepsilon(x) \mathrm{e}^{ik(z)|x-y|} v_\varepsilon(y)\,\mathrm{d}y\,\mathrm{d}x$$

$$S_\varepsilon = \frac{i}{2} \int_{\mathbb{R}^2} \operatorname{sgn}(x-y) v_\varepsilon(x) \mathrm{e}^{ik(z)|x-y|} v_\varepsilon(y)\,\mathrm{d}y\,\mathrm{d}x$$

We can see that the integral S_ε is equal to zero because its integrand is an antisymmetric function. So the inverse of the matrix $(I + \langle v_\varepsilon | \mathbb{A} R_z v_\varepsilon \rangle) = (I + E_\varepsilon \mathbb{A}\mathbb{Z}(z))$ is

$$(I + \langle v_\varepsilon | \mathbb{A} R_z v_\varepsilon \rangle)^{-1} = \frac{1}{(1 + d\zeta^{-1}(z)E_\varepsilon)(1 + a\zeta(z)E_\varepsilon) - bcE_\varepsilon^2} \begin{pmatrix} 1 + d\zeta^{-1}(z)E_\varepsilon & -b\zeta^{-1}(z)E_\varepsilon \\ -c\zeta(z)E_\varepsilon & 1 + a\zeta(z)E_\varepsilon \end{pmatrix}.$$

Employing the dominated convergence theorem we can see that the pointwise limit of E_ε is

$$E_\varepsilon \overset{\varepsilon\to 0+}{\to} \frac{i}{2} \left(\int_{\mathbb{R}} v \right)^2 = \frac{i}{2},$$

which implies that the pointwise limit of the matrix $(I + \langle v_\varepsilon | \mathbb{A} R_z v_\varepsilon \rangle)^{-1}$ is

$$(I + \langle v_\varepsilon | \mathbb{A} R_z v_\varepsilon \rangle)^{-1} \overset{\varepsilon\to 0+}{\to} (I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}. \tag{10}$$

If we denote matrices above like

$$M_\varepsilon = (I + \langle v_\varepsilon | \mathbb{A} R_z v_\varepsilon \rangle)^{-1}$$

$$M = (I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}$$

we can rewrite kernels of the resolvent of the non–local potential and its pointwise limit as follows

$$R^{\mathbb{A}}_{z,\varepsilon}(x,y) = R_z(x,y) - \int_{\mathbb{R}^2} R_z(x, \varepsilon s) v(s) M_\varepsilon \mathbb{A} v(t) R_z(\varepsilon t, y)\,\mathrm{d}s\,\mathrm{d}t$$

$$R_z^{\mathbb{A}}(x,y) = R_z(x,y) - \int_{\mathbb{R}^2} R_z(x,0)v(s)M\mathbb{A}v(t)R_z(0,y)\,\mathrm{d}s\,\mathrm{d}t$$

We will study a convergence of the operator $R_\varepsilon^{\mathbb{A}}$ to the operator $R^{\mathbb{A}}$ in the Hilbert–Schmidt norm which will imply a convergence in the operator norm. Now we will tried to find the estimate for the Hilbert–Schmidt norm of the difference of the operator $R_\varepsilon^{\mathbb{A}}$ and $R^{\mathbb{A}}$. We will use the inequality $(a+b)^2 \le 2a^2+2b^2$ several times in the following inequalities.

$$\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}^2} R_z(x,\varepsilon s)v(s)M_\varepsilon\mathbb{A}v(t)R_z(\varepsilon t,y) - R_z(x,0)v(s)M\mathbb{A}v(t)R_z(0,y)\,\mathrm{d}s\,\mathrm{d}t\right|^2\,\mathrm{d}x\,\mathrm{d}y =$$

$$= \int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2}(R_z(x,\varepsilon s) - R_z(x,0))v(s)M_\varepsilon\mathbb{A}v(t)R_z(\varepsilon t,y) +$$

$$+ R_z(x,0)v(s)(M_\varepsilon\mathbb{A}v(t)R_z(\varepsilon t,y) - M\mathbb{A}v(t)R_z(0,y))\,\mathrm{d}s\,\mathrm{d}t\Big|^2\,\mathrm{d}x\,\mathrm{d}y \le$$

$$\le \int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^2}|(R_z(x,\varepsilon s) - R_z(x,0))v(s)M_\varepsilon\mathbb{A}v(t)R_z(\varepsilon t,y)| +$$

$$+ |R_z(x,0)v(s)(M_\varepsilon\mathbb{A}v(t)R_z(\varepsilon t,y) - M\mathbb{A}v(t)R_z(0,y))|\,\mathrm{d}s\,\mathrm{d}t\Big)^2\,\mathrm{d}x\,\mathrm{d}y$$

$$\le 2\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}|(R_z(x,\varepsilon s) - R_z(x,0))v(s)M_\varepsilon\mathbb{A}v(t)R_z(\varepsilon t,y)|\,\mathrm{d}s\,\mathrm{d}t\right)^2\,\mathrm{d}x\,\mathrm{d}y +$$

$$+2\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}|R_z(x,0)v(s)(M_\varepsilon\mathbb{A}v(t)R_z(\varepsilon t,y) - M\mathbb{A}v(t)R_z(0,y))|\,\mathrm{d}s\,\mathrm{d}t\right)^2\,\mathrm{d}x\,\mathrm{d}y \le$$

$$\le 2\underbrace{\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}|(R_z(x,\varepsilon s) - R_z(x,0))v(s)M_\varepsilon\mathbb{A}v(t)R_z(\varepsilon t,y)|\,\mathrm{d}s\,\mathrm{d}t\right)^2\,\mathrm{d}x\,\mathrm{d}y}_{a)} +$$

$$+4\underbrace{\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}|R_z(x,0)v(s)(M_\varepsilon - M)\mathbb{A}v(t)R_z(\varepsilon t,y)|\,\mathrm{d}s\,\mathrm{d}t\right)^2\,\mathrm{d}x\,\mathrm{d}y}_{b)} +$$

$$+4\underbrace{\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}|R_z(x,0)v(s)M\mathbb{A}v(t)(R_z(\varepsilon t,y) - R_z(0,y))|\,\mathrm{d}s\,\mathrm{d}t\right)^2\,\mathrm{d}x\,\mathrm{d}y}_{c)}$$

We will estimate each of the terms a),b) and c) separately.

a)

$$a) \le \underbrace{\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|R_z(x,\varepsilon s) - R_z(x,0)||v(s)|\,\mathrm{d}s\right)^2\mathrm{d}x}_{\text{By lemma 3.1.3 }\to 0}\underbrace{\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|M_\varepsilon\mathbb{A}||v(t)||R_z(\varepsilon t,y)|\,\mathrm{d}t\right)^2\mathrm{d}y}_{=K_\varepsilon}$$

Since matrix M_ε converge to the matrix M by Lemma 3.1.1 it is uniformly bounded by some constant $C \ge 0$. Then by using this, Lemma 3.1.2 and the Minkowski integral inequality 3.1.1 we get following

$$K_\varepsilon \le C \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |v(t)||R_z(\varepsilon t, y)| \,\mathrm{d}t \right)^2 \mathrm{d}y \le C \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |v(t)|^2 |R_z(\varepsilon t, y)|^2 \,\mathrm{d}y \right)^{\frac{1}{2}} \mathrm{d}t \right)^2 =$$

$$= C \left(\int_{\mathbb{R}} |v(t)| \left(\int_{\mathbb{R}} |R_z(\varepsilon t, y)|^2 \,\mathrm{d}y \right)^{\frac{1}{2}} \mathrm{d}t \right)^2 \le \tilde{C}$$

That yields a) $\to 0$ as $\varepsilon \to 0$.

b) Using Lemmas 3.1.1 and 3.1.2 and the Minkowski integral inequality we get

$$\text{b)} \le \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |R_z(x,0) v(s)(M_\varepsilon - M)\mathbb{A} v(t) R_z(\varepsilon t, y)| \,\mathrm{d}s\,\mathrm{d}t \right)^2 \mathrm{d}x\,\mathrm{d}y \le$$

$$\le \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |R_z(x,0)||v(s)| \,\mathrm{d}s \right)^2 \mathrm{d}x |M_\varepsilon - M||\mathbb{A}| \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |R_z(\varepsilon t, y)||v(t)| \,\mathrm{d}t \right)^2 \mathrm{d}y \le$$

$$\le \left(\int_{\mathbb{R}} |v(s)| \,\mathrm{d}s \right)^2 \int_{\mathbb{R}} |R_z(x,0)|^2 \,\mathrm{d}x |M_\varepsilon - M||\mathbb{A}| \left(\int_{\mathbb{R}} |v(t)| \left(\int_{\mathbb{R}} |R_z(\varepsilon t, y)|^2 \,\mathrm{d}y \right)^{\frac{1}{2}} \mathrm{d}t \right)^2 \le$$

$$\le \underbrace{\left(\int_{\mathbb{R}} |v(s)| \,\mathrm{d}s \right)^2 \int_{\mathbb{R}} |R_z(x,0)|^2 \,\mathrm{d}x}_{<+\infty} \underbrace{|M_\varepsilon - M|}_{\to 0} |\mathbb{A}| \underbrace{\left(\int_{\mathbb{R}} |v(t)| \,\mathrm{d}t \right)^2 \sup_{\iota \in \mathbb{R}} \int_{\mathbb{R}} |R_z(\iota, y)|^2 \,\mathrm{d}y}_{<+\infty} \to 0.$$

c) Similarly to a).

This means that we get the convergence in the Hilbert–Schmidt norm which implies $R^{\mathbb{A}}_{z,\varepsilon} \xrightarrow{u} R^{\mathbb{A}}_z$. □

Thus we get the limit in the operator norm for the resolvent of the Dirac operator with non–local potential but we do not know if it is a resolvent of some operator. In the self–adjoint case we had got a candidate in a form of self–adjoint relativistic point interaction [7] which was also proved as norm–resolvent limit of the operator. As we already mentioned the pointwise limit of the self–adjoint operator $H^{\mathbb{A}}_\varepsilon$ with a hermitian matrix $\mathbb{A}$ coincide with its norm resolvent limit. We can assume that this property will be preserved in the general case of any complex matrix $\mathbb{A}$.

For clarity purposes we will rewrite pointwise limit of $H^{\mathbb{A}}_\varepsilon$ with any complex matrix $\mathbb{A}$ as the Dirac operator with a transmission condition at the point of interaction. We can easily see that the pointwise limit of $H^{\mathbb{A}}_\varepsilon$ can be formally written as $H_m + \mathbb{A}\delta$. Now we will find maximal domain of the latter operator which, in particular, means that we need $(H_m + \mathbb{A}\delta)\psi \in L^2(\mathbb{R}; \mathbb{C}^2)$. In other words we need singular parts to be cancelled out [2] which yields the following condition

$$-i\sigma_1(\psi(0+) - \psi(0-)) + \mathbb{A}\frac{\psi(0+) + \psi(0-)}{2} = 0$$
$$(2i\sigma_1 + \mathbb{A})\psi(0-) = (2i\sigma_1 - \mathbb{A})\psi(0+).$$

Therefore, our pointwise limit corresponds to the operator $H^{\mathbb{A}}$ acting as the Dirac operator with the transmission condition at the point of interaction.

$$(H^{\mathbb{A}}\psi)(x) = (H_m\psi)(x), \ x \in \mathbb{R} \setminus \{0\}$$
$$\psi \in \operatorname{Dom} H^{\mathbb{A}} = \{\psi \in W^{1,2}(\mathbb{R} \setminus \{0\}) \otimes \mathbb{C}^2 \mid (2i\sigma_1 + \mathbb{A})\psi(0-) = (2i\sigma_1 - \mathbb{A})\psi(0+)\}. \tag{11}$$

Definition 3.1.1. *Let matrix $\mathbb{A}$ be any 2×2 complex matrix. Then we will call the operator $H^{\mathbb{A}}$ given by (11) the Hamiltonian of the relativistic point interaction.*

Note that if matrix $\mathbb{A}$ is self–adjoint and $(2i\sigma_1 + \mathbb{A})$ is regular then this definition 3.1.1 coincides with the definition of the relativistic point interaction introduced in [7] as the self–adjoint extension of the Dirac operator perturbated in the point of interaction. Our definition includes all self–adjoint extensions and of course also non–self–adjoint cases.

One can easily check that $H^{\mathbb{A}}$ is a self–adjoint operator if and only if matrix $\mathbb{A}$ is self–adjoint. We can try to find out if $H^{\mathbb{A}}$ is at least closed operator in the most general case. For this recall the trace theorem.

Theorem 3.1.2 (Trace theorem)**.** *Let U be bounded open subset of $\mathbb{R}^n$ with C^1–boundary and $p \in [1, +\infty)$. Then there exists operator $\operatorname{Tr} \in \mathcal{B}(W^{1,p}(U), L^p(\partial U))$ such that $\forall \psi \in W^{1,p}(U) \cap C(\overline{U})$, $\operatorname{Tr} \psi = \psi|_{\partial U}$.*

Proof. Can be found for example in [11]. □

Theorem 3.1.3. *Let matrix $\mathbb{A}$ be any complex matrix. Then the operator of relativistic point interaction $H^{\mathbb{A}}$ (11) is densely defined closed operator.*

Proof. We can see that $H^{\mathbb{A}}$ is densely defined operator in $L^2(\mathbb{R}; \mathbb{C}^2)$ because $W^{1,2}(\mathbb{R} \setminus \{0\})$ is a dense subset of $L^2(\mathbb{R})$.

If we decompose a function from $\operatorname{Dom} H^{\mathbb{A}}$ into a sum of functions on a positive and negative half–line of $\mathbb{R}$ respectively, and use the Trace theorem 3.1.2 the domain of $H^{\mathbb{A}}$ can be written as follows

$$\operatorname{Dom} H^{\mathbb{A}} = \{\varphi = \varphi_- \oplus \varphi_+ \in W^{1,2}(\mathbb{R}^-; \mathbb{C}^2) \oplus W^{1,2}(\mathbb{R}^+; \mathbb{C}^2) \mid (2i - \sigma_1\mathbb{A}) \operatorname{Tr} \varphi_+ = (2i + \sigma_1\mathbb{A}) \operatorname{Tr} \varphi_-\}. \tag{12}$$

By Trace theorem we get a bounded linear operator $\operatorname{Tr} \in \mathcal{B}(W^{1,2}(\mathbb{R}^\pm; \mathbb{C}^2), \mathbb{C}^2)$ and certain constant $C \geq 0$ such that

$$|\operatorname{Tr} \varphi_\pm| \leq C \|\varphi_\pm\|_{W^{1,2}(\mathbb{R}^\pm; \mathbb{C}^2)}. \tag{13}$$

To prove that $H^{\mathbb{A}}$ is closed we choose a convergent sequence from $\operatorname{Dom} H^{\mathbb{A}}$ with a convergent sequence of its images.

$$\begin{aligned} \varphi_n &= \varphi_{n,-} \oplus \varphi_{n,+} \in \operatorname{Dom} H^{\mathbb{A}} \\ \varphi_n &\to \varphi = \varphi_- \oplus \varphi_+ \in L^2(\mathbb{R}; \mathbb{C}^2) \\ H^{\mathbb{A}}\varphi_n &\to \psi = \psi_- \oplus \psi_+ \in L^2(\mathbb{R}; \mathbb{C}^2) \end{aligned} \tag{14}$$

Now we need to prove that $\varphi \in \operatorname{Dom} H^{\mathbb{A}}$ and $\psi = H^{\mathbb{A}}\varphi$. Firstly, note that

$$H^{\mathbb{A}}\varphi_n = (-i\sigma_1\varphi'_{n,-} + m\sigma_3\varphi_{n,-}) \oplus (-i\sigma_1\varphi'_{n,+} + m\sigma_3\varphi_{n,+}).$$

Then (14) gives us convergence of functions $\varphi_{n,\pm}$ in $W^{1,2}(\mathbb{R}^\pm; \mathbb{C}^2)$. Spaces $W^{1,2}(\mathbb{R}^\pm; \mathbb{C}^2)$ are complete spaces, that implies

$$\varphi_\pm \in W^{1,2}(\mathbb{R}^\pm; \mathbb{C}^2) \text{ and}$$

$$-i\sigma_1\varphi'_\pm + m\sigma_3\varphi_\pm = \psi_\pm.$$

Now we need to determine if $\varphi_\pm$ fulfils the transmission condition of $\operatorname{Dom} H^{\mathbb{A}}$ (12). Since functions $\varphi_\pm$ converge in $W^{1,2}(\mathbb{R}^\pm;\mathbb{C}^2)$, (13) gives us convergence of their traces in $\mathbb{C}^2$ space. Because $\varphi_n \in \operatorname{Dom} H^{\mathbb{A}}$ we get

$$\forall n \in \mathbb{N}, (2i - \sigma_1\mathbb{A}) \operatorname{Tr} \varphi_{n,+} = (2i + \sigma_1\mathbb{A}) \operatorname{Tr} \varphi_{n,-}.$$

Letting $n \to +\infty$ we obtain

$$(2i - \sigma_1\mathbb{A}) \operatorname{Tr} \varphi_+ = (2i + \sigma_1\mathbb{A}) \operatorname{Tr} \varphi_-.$$

We conclude that $\varphi \in \operatorname{Dom} H^{\mathbb{A}}$ and $H^{\mathbb{A}}\varphi = \psi$. This means that the operator $H^{\mathbb{A}}$ is a densely defined closed operator. □

Theorem 3.1.1 gives us the norm resolvent limit of the Dirac operator with a non–local potential. Since we assume that the renormalization will not happen in the general case and the operator $H^{\mathbb{A}}_\varepsilon$ most likely converges to the relativistic point interaction $H^{\mathbb{A}}$, it is reasonable to state the following theorem.

Theorem 3.1.4. *Let the matrix $\mathbb{A}$ be any complex matrix and $z \in \mathbb{C} \setminus \{(-\infty, -m] \cup [m, +\infty)\}$ be such that*

$$(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))$$

is a regular matrix. Then the operator

$$R^{\mathbb{A}}_z = R_z - R_z(x,0)(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}R_z(0,y) \tag{15}$$

is the resolvent of the operator $H^{\mathbb{A}}$ given in (11).

Proof. Since the operator $R^{\mathbb{A}}_z$ is Hilbert–Schmidt, and thus it is a bounded operator, it is sufficient to check following two statements

1. $$\operatorname{Ran} R^{\mathbb{A}}_z \subset \operatorname{Dom} H^{\mathbb{A}} \text{ and } \forall\psi \in \operatorname{Dom} R^{\mathbb{A}}_z, (H^{\mathbb{A}} - z)R^{\mathbb{A}}_z\psi = \psi,$$

2. $$\forall\psi \in \operatorname{Dom} H^{\mathbb{A}}, R^{\mathbb{A}}_z(H^{\mathbb{A}} - z)\psi = \psi.$$

1. First let $\varphi \in \operatorname{Ran} R^{\mathbb{A}}_z$ then we will check if $\varphi \in \operatorname{Dom} H^{\mathbb{A}}$. Since $\varphi \in \operatorname{Ran} R^{\mathbb{A}}_z$ then there exists $\psi \in \operatorname{Dom} R^{\mathbb{A}}_z$ such that

$$\varphi(x) = R^{\mathbb{A}}_z\psi(x)$$

$$\varphi(x) = \int_{\mathbb{R}} R_z(x,y)\psi(y)\,dy - R_z(x,0)(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\int_{\mathbb{R}} R_z(0,y)\psi(y)\,dy.$$

That yields

$$\varphi(0+) = \int_{\mathbb{R}} R_z(0,y)\psi(y)\,dy - \frac{i}{2}(\mathbb{Z}(z) + \sigma_1)(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\int_{\mathbb{R}} R_z(0,y)\psi(y)\,dy$$
$$\varphi(0-) = \int_{\mathbb{R}} R_z(0,y)\psi(y)\,dy - \frac{i}{2}(\mathbb{Z}(z) - \sigma_1)(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\int_{\mathbb{R}} R_z(0,y)\psi(y)\,dy$$

We can see that the transmission condition

$$(2i - \sigma_1\mathbb{A})\varphi(0+) = (2i + \sigma_1\mathbb{A})\varphi(0-)$$

holds if and only if

$$(2i-\sigma_1\mathbb{A})(I-\frac{i}{2}(\mathbb{Z}(z)+\sigma_1)(I+\frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}) = (2i+\sigma_1\mathbb{A})(I-\frac{i}{2}(\mathbb{Z}(z)-\sigma_1)(I+\frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A})$$
$$2i-\sigma_1\mathbb{A}+(2i-\sigma_1\mathbb{A})(\mathbb{Z}(z)+\sigma_1)(2i-\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A} = 2i+\sigma_1\mathbb{A}+(2i+\sigma_1\mathbb{A})(\mathbb{Z}(z)-\sigma_1)(2i-\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}$$

$$-\sigma_1\mathbb{A}+2i\sigma_1(2i-\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}-\sigma_1\mathbb{A}\mathbb{Z}(z)(2i-\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A} =$$
$$= \sigma_1\mathbb{A}-2i\sigma_1(2i-\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}+\sigma_1\mathbb{A}\mathbb{Z}(z)(2i-\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}$$

$$-\sigma_1\mathbb{A}+\sigma_1(2i-\mathbb{A}\mathbb{Z}(z))(2i-\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A} = \sigma_1\mathbb{A}-\sigma_1(2i-\mathbb{A}\mathbb{Z}(z))(2i-\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}$$
$$-\sigma_1\mathbb{A}+\sigma_1\mathbb{A} = \sigma_1\mathbb{A}-\sigma_1\mathbb{A}.$$

We conclude that for any complex matrix $\mathbb{A}$ and $z\in\mathbb{C}\setminus\{(-\infty,-m]\cup[m,+\infty)\}$ such that $(I+i/2\ \mathbb{A}\mathbb{Z}(z))^{-1}$ exists every $\varphi\in\operatorname{Ran}R_z^{\mathbb{A}}$ fulfils the transmission condition of the Hamiltonian of the relativistic point interaction.

Since $R_z(x,y)$ is the resolvent of the Dirac operator and $R_z(x,0)$ standing alone is in $W^{1,2}(\mathbb{R}\setminus\{0\};\mathbb{C}^{2,2})$ we finally get

$$\operatorname{Ran}R_z^{\mathbb{A}}\subset\operatorname{Dom}H^{\mathbb{A}}$$

Now let us check that $R^{\mathbb{A}}$ is the right inverse of $(H^{\mathbb{A}}-z)$.

$$(H^{\mathbb{A}}-z)R_z^{\mathbb{A}}\psi(x) = \Big(-i\frac{\mathrm{d}}{\mathrm{d}x}\sigma_1+m\sigma_3-z\Big)\Big(\int_{\mathbb{R}}\frac{i}{2}(\mathbb{Z}(z)+\operatorname{sgn}(x-y)\sigma_1)\mathrm{e}^{ik(z)|x-y|}\psi(y)\,\mathrm{d}y+$$
$$+\frac{1}{4}(\mathbb{Z}(z)+\sigma_1\operatorname{sgn}(x))\mathrm{e}^{ik(z)|x|}(I+\frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\int_{\mathbb{R}}(\mathbb{Z}(z)+\operatorname{sgn}(-y)\sigma_1)\mathrm{e}^{ik(z)|y|}\psi(y)\,\mathrm{d}y\Big)$$

For $x>0$ we get following

$$(H^{\mathbb{A}}-z)R_z^{\mathbb{A}}\psi(x) =$$

$$= -i\frac{\mathrm{d}}{\mathrm{d}x}\sigma_1\int_{\mathbb{R}}\frac{i}{2}(\mathbb{Z}(z)+\operatorname{sgn}(x-y)\sigma_1)\mathrm{e}^{ik(z)|x-y|}\psi(y)\,\mathrm{d}y- \tag{16}$$
$$-i\frac{\mathrm{d}}{\mathrm{d}x}\sigma_1\frac{1}{4}(\mathbb{Z}(z)+\sigma_1)(\mathrm{e}^{ik(z)|x|})(I+\frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\int_{\mathbb{R}}(\mathbb{Z}(z)+\operatorname{sgn}(-y)\sigma_1)\mathrm{e}^{ik(z)|y|}\psi(y)\,\mathrm{d}y- \tag{17}$$
$$-\begin{pmatrix} z-m & 0\\ 0 & z+m\end{pmatrix}\int_{\mathbb{R}}\frac{i}{2}(\mathbb{Z}(z)+\operatorname{sgn}(x-y)\sigma_1)\mathrm{e}^{ik(z)|x-y|}\psi(y)\,\mathrm{d}y- \tag{18}$$
$$-\begin{pmatrix} z-m & 0\\ 0 & z+m\end{pmatrix}\frac{1}{4}(\mathbb{Z}(z)+\sigma_1)\mathrm{e}^{ik(z)|x|}(I+\frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\int_{\mathbb{R}}(\mathbb{Z}(z)+\operatorname{sgn}(-y)\sigma_1)\mathrm{e}^{ik(z)|y|}\psi(y)\,\mathrm{d}y \tag{19}$$

Since

$$\zeta(z)k(z) = z+m \text{ and } \zeta^{-1}(z)k(z) = z-m,$$

we get the following

$$(16) = \frac{1}{2}\sigma_1 \frac{\mathrm{d}}{\mathrm{d}x}\int_x^{+\infty} (\mathbb{Z}(z) - \sigma_1)\mathrm{e}^{ik(z)|x-y|}\psi(y)\,\mathrm{d}y +$$

$$+ \frac{1}{2}\sigma_1 \frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{x} (\mathbb{Z}(z) + \sigma_1)\mathrm{e}^{ik(z)|x-y|}\psi(y)\,\mathrm{d}y =$$

$$= -\frac{1}{2}\sigma_1(\mathbb{Z}(z) - \sigma_1)\psi(x) + \frac{1}{2}\sigma_1 \int_x^{+\infty} (\mathbb{Z}(z) - \sigma_1)(-ik(z))\mathrm{e}^{ik(z)|x-y|}\psi(y)\,\mathrm{d}y +$$

$$+ \frac{1}{2}\sigma_1(\mathbb{Z}(z) + \sigma_1)\psi(x) + \frac{1}{2}\sigma_1 \int_{-\infty}^{x} (\mathbb{Z}(z) + \sigma_1)(ik(z))\mathrm{e}^{ik(z)|x-y|}\psi(y)\,\mathrm{d}y =$$

$$= \psi(x) + \begin{pmatrix} z-m & 0 \\ 0 & z+m \end{pmatrix} \int_{\mathbb{R}} \frac{i}{2}(\mathbb{Z}(z) + \mathrm{sgn}(x-y)\sigma_1)\mathrm{e}^{ik(z)|x-y|}\psi(y)\,\mathrm{d}y$$

$$(17) = -\frac{1}{2}ik(z)(\mathbb{Z}(z) - \sigma_1)\mathrm{e}^{ik(z)|x|}(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\int_{\mathbb{R}} R_z(0,y)\psi(y)\,\mathrm{d}y$$

$$(19) = \frac{1}{2}ik(z)(\mathbb{Z}(z) - \sigma_1)\mathrm{e}^{ik(z)|x|}(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\int_{\mathbb{R}} R_z(0,y)\psi(y)\,\mathrm{d}y.$$

We finally get

$$(H^{\mathbb{A}} - z)R_z^{\mathbb{A}}\psi = (16) + (17) + (18) + (19) = \psi(x),$$

and similarly for $x < 0$ we get the same result.

2. Now we need to check if $R_z^{\mathbb{A}}$ is also a left inverse of $(H^{\mathbb{A}} - z)$.

$$R_z^{\mathbb{A}}(H^{\mathbb{A}} - z)\psi(x) = (\int_{\mathbb{R}} \frac{i}{2}(\mathbb{Z}(z) + \mathrm{sgn}(x-y)\sigma_1)\mathrm{e}^{ik(z)|x-y|}(-i\frac{\mathrm{d}}{\mathrm{d}y}\sigma_1 + m\sigma_3 - z)\psi(y)\,\mathrm{d}y +$$

$$+\frac{1}{4}(\mathbb{Z}(z)+\mathrm{sgn}(x)\sigma_1)\mathrm{e}^{ik(z)|x|}(I+\frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\int_{\mathbb{R}}(\mathbb{Z}(z)+\mathrm{sgn}(-y)\sigma_1)\mathrm{e}^{ik(z)|y|}(-i\frac{\mathrm{d}}{\mathrm{d}y}\sigma_1 + m\sigma_3 - z)\psi(y)\,\mathrm{d}y) =$$

$$= \int_{\mathbb{R}} \frac{1}{2}(\mathbb{Z}(z) + \mathrm{sgn}(x-y)\sigma_1)\mathrm{e}^{ik(z)|x-y|}\frac{\mathrm{d}}{\mathrm{d}y}\sigma_1\psi(y)\,\mathrm{d}y - \tag{20}$$

$$- \frac{i}{4}(\mathbb{Z}(z) + \mathrm{sgn}(x)\sigma_1)\mathrm{e}^{ik(z)|x|}(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\int_{\mathbb{R}}(\mathbb{Z}(z) + \mathrm{sgn}(-y)\sigma_1)\mathrm{e}^{ik(z)|y|}\frac{\mathrm{d}}{\mathrm{d}y}\sigma_1\psi(y)\,\mathrm{d}y - \tag{21}$$

$$- \begin{pmatrix} z-m & 0 \\ 0 & z+m \end{pmatrix}\int_{\mathbb{R}} \frac{i}{2}(\mathbb{Z}(z) + \mathrm{sgn}(x-y)\sigma_1)\mathrm{e}^{ik(z)|x-y|}\psi(y)\,\mathrm{d}y - \tag{22}$$

$$- \begin{pmatrix} z-m & 0 \\ 0 & z+m \end{pmatrix}\frac{1}{4}(\mathbb{Z}(z) + \mathrm{sgn}(x)\sigma_1)\mathrm{e}^{ik(z)|x|}(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\int_{\mathbb{R}}(\mathbb{Z}(z) + \mathrm{sgn}(-y)\sigma_1)\mathrm{e}^{ik(z)|y|}\psi(y)\,\mathrm{d}y \tag{23}$$

Firstly, we consider $x > 0$. Using integration by parts we get the following

$$(20) = \int_x^{+\infty} \frac{1}{2}(\mathbb{Z}(z) - \sigma_1)\mathrm{e}^{i\zeta(z)z|x-y|}\frac{\mathrm{d}}{\mathrm{d}y}\sigma_1\psi(y)\,\mathrm{d}y +$$

$$+ \int_0^x \frac{1}{2}(\mathbb{Z}(z) + \sigma_1)\mathrm{e}^{ik(z)|x-y|}\frac{\mathrm{d}}{\mathrm{d}y}\sigma_1\psi(y)\,\mathrm{d}y +$$

$$+ \int_{-\infty}^{0} \frac{1}{2}(\mathbb{Z}(z) + \sigma_1)\mathrm{e}^{ik(z)|x-y|}\frac{\mathrm{d}}{\mathrm{d}y}\sigma_1\psi(y)\,\mathrm{d}y =$$

$$= \left[\frac{1}{2}(\mathbb{Z}(z) - \sigma_1)\mathrm{e}^{ik(z)|x-y|})\sigma_1\psi(y)\right]_{y\to x}^{y\to+\infty} + \left[\frac{1}{2}(\mathbb{Z}(z) + \sigma_1)\mathrm{e}^{ik(z)|x-y|}\sigma_1\psi(y)\right]_{y\to 0+}^{y\to x} +$$

$$+ \left[\frac{1}{2}(\mathbb{Z}(z) + \sigma_1)\mathrm{e}^{ik(z)|x-y|}\sigma_1\psi(y)\right]_{y\to-\infty}^{y\to 0-} -$$

$$- \int_x^{+\infty} \frac{1}{2}(\mathbb{Z}(z) - \sigma_1)(ik(z))\mathrm{e}^{ik(z)|x-y|}\sigma_1\psi(y)\,\mathrm{d}y -$$

$$- \int_0^x \frac{1}{2}(\mathbb{Z}(z) + \sigma_1)(ik(z))\mathrm{e}^{ik(z)|x-y|}\sigma_1\psi(y)\,\mathrm{d}y -$$

$$- \int_{-\infty}^x \frac{1}{2}(\mathbb{Z}(z) + \sigma_1)(-ik(z))\mathrm{e}^{ik(z)|x-y|}\sigma_1\psi(y)\,\mathrm{d}y =$$

$$= \psi(x) + \begin{pmatrix} z-m & 0 \\ 0 & z+m \end{pmatrix} \int_{\mathbb{R}} \frac{i}{2}(\mathbb{Z}(z) + \operatorname{sgn}(x-y)\sigma_1)\mathrm{e}^{ik(z)|x-y|}\psi(y)\,\mathrm{d}y +$$

$$+ \frac{1}{2}(\mathbb{Z}(z) + \sigma_1)\mathrm{e}^{ik(z)|x|}\sigma_1(\psi(0-) - \psi(0+))$$

Similarly for $x < 0$ we get

$$(20) = \psi(x) + \begin{pmatrix} z-m & 0 \\ 0 & z+m \end{pmatrix} \int_{\mathbb{R}} \frac{i}{2}(\mathbb{Z}(z) + \operatorname{sgn}(x-y)\sigma_1)\mathrm{e}^{ik(z)|x-y|}\psi(y)\,\mathrm{d}y +$$

$$+ \frac{1}{2}(\mathbb{Z}(z) - \sigma_1)\mathrm{e}^{ik(z)|x|}\sigma_1(\psi(0-) - \psi(0+))$$

In other words for $\forall x \in \mathbb{R} \setminus \{0\}$

$$(20) = \psi(x) + \begin{pmatrix} z-m & 0 \\ 0 & z+m \end{pmatrix} \int_{\mathbb{R}} \frac{i}{2}(\mathbb{Z}(z) + \operatorname{sgn}(x-y)\sigma_1)\mathrm{e}^{ik(z)|x-y|}\psi(y)\,\mathrm{d}y +$$

$$+ \frac{1}{2}(\mathbb{Z}(z) + \operatorname{sgn}(x)\sigma_1)\mathrm{e}^{ik(z)|x|}\sigma_1(\psi(0-) - \psi(0+))$$

$$(21) = -\frac{i}{4}(\mathbb{Z}(z) + \operatorname{sgn}(x)\sigma_1)\mathrm{e}^{ik(z)|x|}(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\left[(\mathbb{Z}(z) - \sigma_1)\mathrm{e}^{ik(z)|y|}\sigma_1\psi(y)\right]_0^{+\infty} -$$

$$- \frac{i}{4}(\mathbb{Z}(z) + \operatorname{sgn}(x)\sigma_1)\mathrm{e}^{ik(z)|x|}(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\left[(\mathbb{Z}(z) + \sigma_1)\mathrm{e}^{ik(z)|y|}\sigma_1\psi(y)\right]_{-\infty}^{0} +$$

$$+ \frac{i}{4}(\mathbb{Z}(z) + \operatorname{sgn}(x)\sigma_1)\mathrm{e}^{ik(z)|x|}(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\int_0^{+\infty}(\mathbb{Z}(z) - \sigma_1)(ik(z))\mathrm{e}^{ik(z)|y|}\sigma_1\psi(y)\,\mathrm{d}y +$$

$$+ \frac{i}{4}(\mathbb{Z}(z) + \operatorname{sgn}(x)\sigma_1)\mathrm{e}^{ik(z)|x|}(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\int_{-\infty}^{0}(\mathbb{Z}(z) + \sigma_1)(-ik(z))\mathrm{e}^{ik(z)|y|}\sigma_1\psi(y)\,\mathrm{d}y =$$

$$= -\frac{i}{4}(\mathbb{Z}(z) + \operatorname{sgn}(x)\sigma_1)\mathrm{e}^{ik(z)|x|}(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\,((\mathbb{Z}(z) + \sigma_1)\sigma_1\psi(0-) - (\mathbb{Z}(z) - \sigma_1)\sigma_1\psi(0+)) +$$

$$+ \begin{pmatrix} z-m & 0 \\ 0 & z+m \end{pmatrix} \frac{1}{4}(\mathbb{Z}(z) + \operatorname{sgn}(x)\sigma_1)\mathrm{e}^{ik(z)|x|}(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\int_{\mathbb{R}}(\mathbb{Z}(z) + \operatorname{sgn}(-y)\sigma_1)\mathrm{e}^{ik(z)|y|}\psi(y)\,\mathrm{d}y$$

Results above imply

$$\begin{aligned}
&(20) + (21) + (22) + (23) = \\
&= \psi(x) - \frac{i}{4}(\mathbb{Z}(z) + \operatorname{sgn}(x)\sigma_1)\mathrm{e}^{ik(z)|x|}2i\sigma_1(\psi(0-) - \psi(0+)) - \\
&- \frac{i}{4}(\mathbb{Z}(z) + \operatorname{sgn}(x)\sigma_1)\mathrm{e}^{ik(z)|x|}(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\Big((\mathbb{Z}(z) + \sigma_1)\sigma_1\psi(0-) - (\mathbb{Z}(z) - \sigma_1)\sigma_1\psi(0+)\Big) = \\
&= \psi(x) - \frac{i}{4}(\mathbb{Z}(z) + \operatorname{sgn}(x)\sigma_1)\mathrm{e}^{ik(z)|x|}\underbrace{\Big(2i + (I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}(\mathbb{Z}(z) + \sigma_1)\Big)\sigma_1}_{\mathbb{B}_-}\psi(0-) - \\
&- \frac{i}{4}(\mathbb{Z}(z) + \operatorname{sgn}(x)\sigma_1)\mathrm{e}^{ik(z)|x|}\underbrace{\Big(2i - (I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}(\mathbb{Z}(z) - \sigma_1)\Big)\sigma_1}_{\mathbb{B}_+}\psi(0+).
\end{aligned}$$

Matrix $\mathbb{B}_-$ next to $\psi(0-)$ is

$$\begin{aligned}
\mathbb{B}_- = (2i + (I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}(\mathbb{Z}(z) + \sigma_1))\sigma_1 = (2i - 2i(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}(\frac{i}{2}\mathbb{A}\mathbb{Z}(z) + I - I + \frac{i}{2}\mathbb{A}\sigma_1))\sigma_1 = \\
= (I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}(2i\sigma_1 + \mathbb{A}).
\end{aligned}$$

Similarly matrix $\mathbb{B}_+$ next to $\psi(0+)$ is

$$(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}(2i\sigma_1 - \mathbb{A}).$$

If we take the transmission condition into account

$$(2i - \sigma_1\mathbb{A})\psi(0+) = (2i + \sigma_1\mathbb{A})\psi(0-),$$

we finally get that

$$\forall x \in \mathbb{R} \setminus \{0\},\ R_z^{\mathbb{A}}(H^{\mathbb{A}} - z)\psi(x) = \psi(x).$$

□

4 Spectral analysis

4.1 Spectrum of general relativistic point interactions

In this section we will study the spectrum of the operator $H^{\mathbb{A}}$. Since $H^{\mathbb{A}}$ is not necessarily self–adjoint it may happen that some points of its spectrum lie outside real numbers. We will find out if conditions under which we do not get the resolvent $R_z^{\mathbb{A}}$ from Theorem 3.1.4 are superfluous due to our procedure of finding the resolvent, or if they coincide with conditions for the spectral points.

Theorem 4.1.1. *Let $\mathbb{A}$ be any complex matrix. Then*

$$\sigma(H^{\mathbb{A}}) \setminus \{(-\infty, -m] \cup [m, +\infty)\} = \sigma_p(H^{\mathbb{A}}) \setminus \{(-\infty, -m] \cup [m, +\infty)\}$$

and $z \in \mathbb{C} \setminus \{(-\infty, -m] \cup [m, +\infty)\}$ is in the spectrum of the operator $H^{\mathbb{A}}$ give by (11) *if and only if z satisfies the following equation*

$$4 + 2i\,\mathrm{tr}(\mathbb{A}\mathbb{Z}(z)) - \det \mathbb{A} = 0. \tag{24}$$

Proof. Let us take $z \in \mathbb{C} \setminus \{(-\infty, -m] \cup [m, +\infty)\}$ then the eigenvalue equation is

$$\begin{gathered} -i\sigma_1 \frac{\mathrm{d}}{\mathrm{d}x}\psi + m\sigma_3\psi = z\psi, \psi \in \mathrm{Dom}\, H^{\mathbb{A}} \\ \frac{\mathrm{d}}{\mathrm{d}x}\psi = i\begin{pmatrix} 0 & z+m \\ z-m & 0 \end{pmatrix}\psi. \end{gathered} \tag{25}$$

Because the matrix on the right hand side of the equation is constant, it is easy to get its antiderivative. Using

$$\begin{pmatrix} 0 & z+m \\ z-m & 0 \end{pmatrix}^2 = (z^2 - m^2)I$$

we can compute the exponential of the antiderivative.

$$\begin{aligned} \exp\left(i\begin{pmatrix} 0 & z+m \\ z-m & 0 \end{pmatrix}x\right) &= \sum_{n=0}^{+\infty} \frac{i^n x^n}{n!}\begin{pmatrix} 0 & z+m \\ z-m & 0 \end{pmatrix}^n = \\ &= \left(\sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n}}{(2n)!}(z^2 - m^2)^n\right) I + i\left(\sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}(z^2 - m^2)^n\right)\begin{pmatrix} 0 & z+m \\ z-m & 0 \end{pmatrix} = \\ &= \cos(k(z)x)I + i\sin(k(z)x)\begin{pmatrix} 0 & \zeta(z) \\ \zeta^{-1}(z) & 0 \end{pmatrix}. \end{aligned}$$

This yields that the general solution to the equation (25) can be written in the following form.

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} = \begin{pmatrix} \cos(k(z)x) & i\zeta(z)\sin(k(z)x) \\ i\zeta^{-1}(z)\sin(k(z)x) & \cos(k(z)x) \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

Now we need to determine constants $C_1, C_2 \in \mathbb{C}$. We will find ψ on the intervals $(-\infty, 0)$ and $(0, +\infty)$ separately and then we will merge them via the transmission condition. Let us write $k(z) = \eta + i\gamma, \gamma \geq 0$ then

$$\psi_1 = (C_1 \cosh(\gamma x) - C_2\zeta(z)\sinh(\gamma x))\cos(\eta x) + i(C_2\zeta(z)\cosh(\gamma x) - C_1\sinh(\gamma x))\sin(\eta x)$$

$$\psi_2 = (C_2 \cosh(\gamma x) - C_1\zeta^{-1}(z)\sinh(\gamma x))\cos(\eta x) + i(C_1\zeta^{-1}(z)\cosh(\gamma x) - C_2\sinh(\gamma x))\sin(\eta x)$$

Therefore, to get a square-integrable solution we need $C_1 = \zeta(z)C_2$ on $(0, +\infty)$ and $C_1 = -\zeta(z)C_2$ on $(-\infty, 0)$. We conclude that

$$\psi_1(x) = \begin{cases} C_1 e^{ik(z)x}, & x \in (0, +\infty) \\ \tilde{C}_1 e^{-ik(z)x}, & x \in (-\infty, 0) \end{cases}$$

and

$$\psi_2(x) = \begin{cases} C_1\zeta^{-1} e^{ik(z)x}, & x \in (0, +\infty) \\ -\tilde{C}_1\zeta^{-1}(z) e^{-ik(z)x}, & x \in (-\infty, 0) \end{cases}$$

Recall that the transmission condition for $\psi \in \operatorname{Dom} H^{\mathbb{A}}$ reads as

$$(2i\sigma_1 - \mathbb{A})\psi(0+) = (2i\sigma_1 + \mathbb{A})\psi(0-),$$

where $\mathbb{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
This implies

$$\begin{aligned} (2i - a\zeta(z) - b)C_1 &= (-2i + a\zeta(z) - b)\tilde{C}_1 \\ (2i - c - d\zeta^{-1}(z))C_1 &= (2i + c - d\zeta^{-1}(z))\tilde{C}_1. \end{aligned} \tag{26}$$

A non–trivial solution $(C_1, \tilde{C}_1)$ of (26) exists if and only if

$$0 = \frac{1}{2}\begin{vmatrix} 2i - a\zeta(z) - b & 2i - a\zeta(z) + b \\ 2i - d\zeta^{-1}(z) - c & -2i + d\zeta^{-1} - c \end{vmatrix} = 4 + 2i\operatorname{tr}(\mathbb{A}\mathbb{Z}(z)) - \det\mathbb{A} \tag{27}$$

Recall that

$$z \in \sigma(H^{\mathbb{A}}) \text{ if and only if } R_z^{\mathbb{A}} = (H^{\mathbb{A}} - z)^{-1} \notin \mathcal{B}(\mathcal{H}).$$

So we can check under which condition Theorem 3.1.4 will not give us $R_z^{\mathbb{A}}$. By Theorem 3.1.4 only problem may occur if the matrix $(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))$ is not a regular matrix. This is true if and only if

$$0 = \det(2I + i\mathbb{A}\mathbb{Z}(z)) = 4 + 2i\operatorname{tr}(\mathbb{A}\mathbb{Z}(z)) - \det\mathbb{A}. \tag{28}$$

We can see that the condition under which $z \in \mathbb{C} \setminus \{(-\infty, -m] \cup [m, +\infty)\}$ is in the point spectrum of the operator $H^{\mathbb{A}}$ (27) is the same as the condition (28). This proves the theorem. □

Theorems 4.1.1 and 3.1.4 gives us a full picture of the spectral problem. The operator $H^{\mathbb{A}}$ has only point spectrum in $\mathbb{C} \setminus \{(-\infty, -m] \cup [m, +\infty)\}$ which can be found by examining (24). If we got the matrix $\mathbb{A}$ such that the condition (24) is not fulfilled for any $z \in \mathbb{C} \setminus \mathbb{R}$ then we get the operator $H^{\mathbb{A}}$ with purely real spectrum.

We can deal with the remaining set $(-\infty, -m] \cup [m, +\infty)$ by using [Theorem XIII.14 [10]]. Since $(-\infty, -m] \cup [m, +\infty)$ is equal to essential spectrum of the free Dirac operator this theorem implies that this set is is also equal to essential spectrum of the operator $H^{\mathbb{A}}$.

4.2 Spectral transitions

If the matrix $\mathbb{A}$ has a non–zero diagonal, the condition (24) is polynomial in z and that give us finite number of points in the spectrum in $\mathbb{C} \setminus \{(-\infty, -m] \cup [m, +\infty)\}$. Nevertheless we can see remarkable spectral transitions of our model if elements on the diagonal of the matrix $\mathbb{A}$ are equal to zero. Then (24) reduces to

$$0 = 4 - \det \mathbb{A}.$$

This yields that if matrix $\mathbb{A}$ has zeros on the diagonal and $\det \mathbb{A} \neq 4$ then by the theorem 4.1.1 we have no spectrum outside real numbers. However if $\mathbb{A}$ has zeros on the diagonal and its determinant is equal to 4, the condition (24) holds for every $z \in \mathbb{C} \setminus \{(-\infty, -m] \cup [m, +\infty)\}$. This implies that if the matrix

$$\mathbb{A} = \begin{pmatrix} 0 & 2b \\ -\frac{2}{b} & 0 \end{pmatrix}, \ b \in \mathbb{C} \setminus \{0\}$$

then the whole complex plane lies in the spectrum of the relativistic point interaction $H^{\mathbb{A}}$. Note that such $\mathbb{A}$ is never hermitian.

Also choosing $m = 0$ we can observe another interesting spectral transitions. In this case, matrix $\mathbb{Z}(z)$ takes a following form

$$\mathbb{Z}(z) = \operatorname{sgn}(\operatorname{Im}(z))I.$$

Then (24) looks like this

$$4 + \operatorname{sgn}(\operatorname{Im}(z))2i \operatorname{tr}(\mathbb{A}) - \det(\mathbb{A}) = 0. \tag{29}$$

We can further investigate (29) and find the exact expression for spectrum for the operator. Firstly discuss $\operatorname{tr} \mathbb{A} = 0$ then the condition will simplify into

$$4 - \det \mathbb{A} = 0.$$

This is a form of the condition we already discuss above. This means for matrices $\mathbb{A}$ such that $\operatorname{tr} \mathbb{A} = 0$ and $\det \mathbb{A} = 4$ the whole complex plane will belong to the spectrum of the operator.

Let us now discuss $\operatorname{tr} \mathbb{A} \neq 0$. Then we can (29) divide by $\operatorname{tr} \mathbb{A}$.

$$\operatorname{sgn}(\operatorname{Im}(z)) = \frac{\det \mathbb{A} - 4}{2i \operatorname{tr} \mathbb{A}}$$

This yields that if the following condition holds

$$\frac{\det \mathbb{A} - 4}{2i \operatorname{tr} \mathbb{A}} = \pm 1$$

then the whole upper respectively lower complex half–plane will be in the spectrum of the operator and the other one will not.

That gives us the following statement for the spectral transition of our model.

Theorem 4.2.1.

Let $m \neq 0$. If the matrix $\mathbb{A}$ fulfils that the elements of its diagonal are zeros and

$$\det \mathbb{A} = 4,$$

then $\sigma(H^{\mathbb{A}}) = \mathbb{C}$.

Let $m = 0$. If the matrix $\mathbb{A}$ fulfils following conditions

$$\det \mathbb{A} = 4 \textit{ and } \operatorname{tr} \mathbb{A} = 0$$

then $\sigma(H^{\mathbb{A}}) = \mathbb{C}$.

If

$$\operatorname{tr} \mathbb{A} \neq 0 \textit{ and } \frac{\det \mathbb{A} - 4}{2i \operatorname{tr} \mathbb{A}} = \pm 1$$

then the whole upper respectively lower complex half–plane belongs to the spectrum of the operator $H^{\mathbb{A}}$ and the latter does not.

4.3 Pseudospectrum of the relativistic point interaction

Now we would like to answer why this wild spectral transition, mentioned in section 4.2, appeared. It is clear that the spectrum of the point interaction 3.1.1 will not be denser while approaching critical transition condition because if we consider matrix

$$\mathbb{A}_\delta = \begin{pmatrix} 0 & (\delta + 2)b \\ \frac{\delta - 2}{b} & 0 \end{pmatrix} \tag{30}$$

then for arbitrarily small $\delta > 0$ the condition (28) does not hold for any $z \in \mathbb{C} \setminus \{(-\infty, -m] \cup [m, +\infty)\}$. That implies that this operator does not have any new points in its spectrum. On the other hand, if $\delta = 0$ then this condition holds for every z and this yields that the spectrum of this operator is a whole complex plane.

We will explain this remarkable spectral transitions with the pseudospectrum of the operator. We will show that for arbitrarily small ε by taking δ to zero the whole complex plane will eventually fall into ε–pseudospectrum of the operator 3.1.1 with the matrix $\mathbb{A}_\delta$ in its transmission condition define as

$$\mathbb{A}_\delta = \mathbb{A} + \delta\mathbb{B},$$

where the matrix $\mathbb{A}$ is the critical matrix

$$\mathbb{A} = \begin{pmatrix} 0 & 2b \\ \frac{-2}{b} & 0 \end{pmatrix}$$

and $\mathbb{B}$ is any fixed non–zero complex matrix. Note that for the fixed matrix $\mathbb{B}$ matrix $\mathbb{A} + \delta\mathbb{B}$ cannot be in the critical form.

Definition 4.3.1 (ε–pseudospectrum)**.** *Let A be a linear operator and $R_A(z)$ is its resolvent at $z \in \mathbb{C}$. Then we will call the set*

$$\sigma_\varepsilon(A) = \{z \in \mathbb{C} \mid \|R_A(z)\| > \varepsilon^{-1}\}$$

ε–pseudospectrum of the operator A. Here we use a convention that $\|R_A(z)\| = +\infty$ if $z \in \sigma(A)$.

Our main goal is to prove that the norm of the resolvent of the operator 3.1.1 with the matrix $\mathbb{A}_\delta$ will go to infinity as δ tends to zero. This will prove that for any ε and any $z \in \mathbb{C} \setminus \{(-\infty, -m] \cup [m, +\infty)\}$ this number z will eventually fall into ε–pseudospectrum.

Theorem 4.3.1. *For any $\varepsilon > 0$ and any number $z \in \mathbb{C} \setminus \sigma(H_m)$ there exists $\delta_0 > 0$ such that for every $0 < \delta < \delta_0$, $z \in \sigma_\varepsilon(H^{\mathbb{A}_\delta})$.*

Proof. Let us denote

$$\mathcal{K} = R_z(x,0)(I + \frac{i}{2}\mathbb{A}_\delta \mathbb{Z}(z))^{-1}\mathbb{A}_\delta R_z(0,y).$$

Then from a formula for the resolvent $R_z^{\mathbb{A}_\delta}$ of the operator $H^{\mathbb{A}_\delta}$ from Theorem 3.1.4 we get

$$R_z^{\mathbb{A}_\delta} = R_z - \mathcal{K}.$$

R_z is a bounded integral operator and because it is a resolvent of the free Dirac operator we explicitly know the norm of this operator

$$||R_z|| = \frac{1}{\mathrm{dist}(z, \sigma(H_m))}.$$

On the other hand, we expect that the norm of the operator $\mathcal{K}$ will go to infinity because the matrix $(I + \frac{i}{2}\mathbb{A}_\delta \mathbb{Z}(z))$ is going to a singular matrix as delta tends to zero.

$$\mathcal{K} = R_z(x,0)(I + \frac{i}{2}\mathbb{A}_\delta \mathbb{Z}(z))^{-1}\mathbb{A}_\delta R_z(0,y) = \frac{1}{\det(I + \frac{i}{2}\mathbb{A}_\delta \mathbb{Z}(z))} R_z(x,0)\mathbb{M}_\delta \mathbb{A}_\delta R_z(0,y),$$

where $\mathbb{M}_\delta = \det(I + \frac{i}{2}\mathbb{A}_\delta \mathbb{Z}(z))(I + \frac{i}{2}\mathbb{A}_\delta \mathbb{Z}(z))^{-1} \overset{\delta \to 0}{\to} \begin{pmatrix} 1 & -ib\zeta(z)^{-1} \\ \frac{i\zeta(z)}{b} & 1 \end{pmatrix}$.

Then we can finally write the norm of the $R_z^{\mathbb{A}_\delta}$ as

$$||R_z^{\mathbb{A}_\delta}|| = \frac{1}{|\det(I + \frac{i}{2}\mathbb{A}_\delta \mathbb{Z}(z))|} || \det(I + \frac{i}{2}\mathbb{A}_\delta \mathbb{Z}(z)) R_z - R_z(x,0)\mathbb{M}_\delta \mathbb{A}_\delta R_z(0,y)|| \geq$$

$$\geq \frac{1}{|\det(I + \frac{i}{2}\mathbb{A}_\delta \mathbb{Z}(z))|} ||R_z(x,0)\mathbb{M}_\delta \mathbb{A}_\delta R_z(0,y)|| - ||R_z||.$$

Because $\det(I + \frac{i}{2}\mathbb{A}_\delta \mathbb{Z}(z)) \overset{\delta \to 0}{\to} 0$ we conclude that

$$||R_z^{\mathbb{A}_\delta}|| \overset{\delta \to 0}{\to} +\infty$$

which proves the theorem. □

4.4 Eigenvalues and eigenfunctions of the Dirac operator with non–local potential

Question of a stability of the spectrum of the perturbation for self–adjoint operators is discussed for example in [Theorems VIII.23,VIII.24 [10]]. For a self–adjoint case if we have a norm–resolvent convergence at our disposal a spectrum of the limiting operator cannot expand nor contract rapidly. Even though a sudden contraction cannot happen also in a non–self–adjoint case, the same is not true for a sudden expansion as discussed in a paragraph after [Theorems VIII.23,VIII.24, [10]] and in [Section IV., §3, 2nd subsection, [12]]. In a following section we will try to demonstrate a convergence of the spectrum of the Dirac operator with not necessary self–adjoint non–local potential.

One can try to find a condition for eigenvalues of the operator H_m with the non–local potential (2) similar to the condition for the spectrum of the limiting operator (28) and see how this condition will behave while approaching its limit. Let us take $z \in \mathbb{C} \setminus \{(-\infty, -m] \cup [m, +\infty)\}$

$$H_\varepsilon^{\mathbb{A}}\psi = z\psi,\ \psi \in \mathrm{Dom}\, H_\varepsilon^{\mathbb{A}} = \mathrm{Dom}\, H_m = W^{1,2}(\mathbb{R}; \mathbb{C}^2)$$

$$-i\frac{\mathrm{d}}{\mathrm{d}x}\sigma_1\psi + m\sigma_3\psi + \mathbb{A}\langle v_\varepsilon|\psi\rangle v_\varepsilon = z\psi$$

$$(H_m - z)\psi = -\mathbb{A}\langle v_\varepsilon|\psi\rangle v_\varepsilon \in L^2(\mathbb{R};\mathbb{C}^2) \tag{31}$$

For $z \in \rho(H_m)$ the equation (31) is equivalent to the following

$$\exists\psi \in \mathrm{Dom}(H_m) \setminus \{0\}, \psi = -R_z\mathbb{A}\langle v_\varepsilon|\psi\rangle v_\varepsilon. \tag{32}$$

This implies that for a certain vector $\vec{a} \in \mathbb{C}^2$ a function ψ is in the following form

$$\psi = R_z\mathbb{A}\vec{a}v_\varepsilon. \tag{33}$$

If we substitute the form (33) into the equation (32) we get another equivalent expression of the (31)

$$(\exists\vec{a} \in \mathbb{C}^2)(\mathbb{A}\vec{a} \neq 0 \wedge R_z\mathbb{A}\vec{a}v_\varepsilon = -R_z\mathbb{A}\langle v_\varepsilon|R_z\mathbb{A}\vec{a}v_\varepsilon\rangle v_\varepsilon) \tag{34}$$

which we can rewrite as

$$(\exists\vec{a} \in \mathbb{C}^2)(\mathbb{A}\vec{a} \neq 0 \wedge \mathbb{A}\vec{a} = -\mathbb{A}\langle v_\varepsilon|R_z v_\varepsilon\rangle\mathbb{A}\vec{a}). \tag{35}$$

Finally, we get that (35) holds if and only if

$$(\exists\vec{a} \in \mathbb{C}^2)(\mathbb{A}\vec{a} \neq 0 \wedge (I + \mathbb{A}\langle v_\varepsilon|R_z v_\varepsilon\rangle)\mathbb{A}\vec{a} = 0) \tag{36}$$

which implies

$$\det(I + \langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle) = 0. \tag{37}$$

For a regular matrix $\mathbb{A}$ the equivalence between (36) and (37) is easily seen. For a singular matrix $\mathbb{A}$ a reverse implication still remains unproven.

Let us start with $\det(I + \langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle) = 0$. That implies

$$(\exists\vec{x} \in \mathbb{C}^2, \vec{x} \neq 0)((I + \langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle)\vec{x} = 0).$$

We need to prove that there exists some vector $\vec{a} \in \mathbb{C}^2$ such that $\mathbb{A}\vec{a} = \vec{x}$.

$$\begin{gathered}(I + \langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle)\vec{x} = 0\\ \vec{x} + \langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle\vec{x} = 0\\ \vec{x} + \mathbb{A}\langle v_\varepsilon|R_z v_\varepsilon\rangle\vec{x} = 0\\ \vec{x} = \mathbb{A}\underbrace{(-\langle v_\varepsilon|R_z v_\varepsilon\rangle\vec{x})}_{:=\vec{a}}\end{gathered}$$

which concludes that (36) and (37) are indeed in equivalence for any matrix $\mathbb{A}$.

For a clarity purpose only, if we consider $\det\mathbb{A} \neq 0$ then (34) can be simplified even more to the following form

$$(\exists\vec{a} \in \mathbb{C}^2, \vec{a} \neq 0)(\vec{a} = -\langle v_\varepsilon|R_z\mathbb{A}v_\varepsilon\rangle\vec{a}).$$

That is true if and only if

$$\det(I + \langle v_\varepsilon|R_z\mathbb{A}v_\varepsilon\rangle) = 0.$$

Fortunately, due to the parity in integrals and the form of R_z this is the exactly the same condition as (37).

Since determinant is a continuous function it is sufficient to check a limit of the matrix in the determinant. Using (4) we obtain

$$\langle v_\varepsilon | R_z \mathbb{A} v_\varepsilon \rangle = \frac{i}{2}\mathbb{Z}(z)\mathbb{A}\langle v_\varepsilon | \mathrm{e}^{ik(z)|x|} * v_\varepsilon \rangle + \frac{i}{2}\sigma_1 \mathbb{A}\langle v_\varepsilon | (\mathrm{sgn}(x)\mathrm{e}^{ik(z)|x|}) * v_\varepsilon \rangle \tag{38}$$

Since

$$\langle v_\varepsilon | (\mathrm{sgn}(x)\mathrm{e}^{ik(z)|x|}) * v_\varepsilon \rangle = \int_{\mathbb{R}^2} v_\varepsilon(y)\mathrm{sgn}(x-y)\mathrm{e}^{ik(z)|x-y|} v_\varepsilon(x)\,\mathrm{d}x\,\mathrm{d}y = 0$$

we can simplify (38) into the form

$$\langle v_\varepsilon | R_z \mathbb{A} v_\varepsilon \rangle = \frac{i}{2}\mathbb{Z}(z)\mathbb{A}\langle v_\varepsilon | \mathrm{e}^{ik(z)|x|} * v_\varepsilon \rangle. \tag{39}$$

Note that

$$\langle v_\varepsilon | \mathrm{e}^{ik(z)|x|} * v_\varepsilon \rangle = \int_{\mathbb{R}^2} v(y)v(x)\mathrm{e}^{ik(z)\varepsilon|x-y|}\,\mathrm{d}x\,\mathrm{d}y.$$

For the integrand we have an integrable majorant

$$|v(y)v(x)\mathrm{e}^{ik(z)\varepsilon|x-y|}| \leq |v(y)||v(x)|.$$

Then by the dominated convergence theorem we get

$$\langle v_\varepsilon | \mathrm{e}^{ik(z)|x|} * v_\varepsilon \rangle \overset{\varepsilon\to 0}{\to} 1$$

This yields that the matrix in the equation (39) converge to the following matrix

$$\frac{i}{2}\mathbb{Z}(z)\mathbb{A}\langle v_\varepsilon | \mathrm{e}^{ik(z)|x|} * v_\varepsilon \rangle \overset{\varepsilon\to 0}{\to} \frac{i}{2}\mathbb{Z}(z)\mathbb{A}.$$

Thus we can found the limit for the condition (37)

$$\det(I + \langle v_\varepsilon | R_z \mathbb{A} v_\varepsilon \rangle) \overset{\varepsilon\to 0}{\to} \det(I + \frac{i}{2}\mathbb{Z}(z)\mathbb{A}) \tag{40}$$

which is exactly the same result as we got in the previous section for the condition for points of spectrum of the operator $H^{\mathbb{A}}$. The condition $\det(I + \langle v_\varepsilon | R_z \mathbb{A} v_\varepsilon \rangle) = 0$ can be treated as an implicit relation for the implicit function $\varepsilon \to z(\varepsilon) \in \sigma(H^{\mathbb{A}}_\varepsilon)$.

We can further examine an eigenequation by explicitly choosing a function v and exactly calculate the matrix

$$(I + \langle v_\varepsilon | R_z \mathbb{A} v_\varepsilon \rangle)$$

to find a point spectrum of the operator $H^{\mathbb{A}}_\varepsilon$.

Let us take

$$v(x) = \begin{cases} \varepsilon^{-1}, & x \in (0, \varepsilon) \\ 0, & \text{everywhere else} \end{cases} \tag{41}$$

then

$$\langle v_\varepsilon | \mathrm{e}^{ik(z)|x|} * v_\varepsilon \rangle = \int_0^\varepsilon \int_0^\varepsilon \frac{1}{\varepsilon^2} \mathrm{e}^{ik(z)|x-y|} \,\mathrm{d}x\,\mathrm{d}y = \{x - y = t\} = \frac{1}{\varepsilon^2} \int_0^\varepsilon \int_{-y}^{\varepsilon - y} \mathrm{e}^{ik(z)|t|} \,\mathrm{d}t\,\mathrm{d}y =$$

$$= \frac{1}{\varepsilon^2} \int_0^\varepsilon \left(\int_0^{\varepsilon - y} \mathrm{e}^{ik(z)t} \,\mathrm{d}t + \int_{-y}^0 \mathrm{e}^{-ik(z)t} \,\mathrm{d}t \right) \mathrm{d}y = \frac{1}{\varepsilon^2} \int_0^\varepsilon \left(\left[\frac{\mathrm{e}^{ik(z)t}}{ik(z)} \right]_0^{\varepsilon - y} + \left[\frac{\mathrm{e}^{-ik(z)t}}{-ik(z)} \right]_{-y}^0 \right) \mathrm{d}y =$$

$$= \frac{1}{\varepsilon^2} \left(\frac{\mathrm{e}^{ik(z)(\varepsilon - y)}}{ik(z)} - \frac{2}{ik(z)} + \frac{\mathrm{e}^{ik(z)y}}{ik(z)} \right) \mathrm{d}y = \frac{2}{\varepsilon^2} \left(\frac{i\varepsilon k(z) + 1 - \mathrm{e}^{ik(z)\varepsilon}}{k^2(z)} \right)$$

This means that for the projection on a function (41) every $z \in \mathbb{C} \setminus \{(-\infty, -m] \cup [m, +\infty)\}$ that satisfies following equation is an eigenvalue of the operator $H_\varepsilon^{\mathbb{A}}$

$$\det \left(I + \frac{i}{\varepsilon^2} \left(\frac{i\varepsilon k(z) + 1 - \mathrm{e}^{ik(z)\varepsilon}}{k^2(z)} \right) \mathbb{Z}(z) \mathbb{A} \right) = 0$$

By denoting $f_\varepsilon(z) := \frac{i}{\varepsilon^2} \left(\frac{i\varepsilon k(z) + 1 - \mathrm{e}^{ik(z)\varepsilon}}{k^2(z)} \right)$ we can rewrite the condition as

$$\det(I + f_\varepsilon(z) \mathbb{Z}(z) \mathbb{A}) = 0.$$

However, finding solutions to this equation or even question of existence of a solution remains as the open problem.

5 Conclusion

In this work we found the norm–resolvent limit of the free Dirac operator H_m with not necessary self–adjoint non–local potential

$$H_\varepsilon^{\mathbb{A}} = H_m + \frac{1}{\varepsilon^2}|v(x/\varepsilon)\rangle\langle v(x/\varepsilon)| \otimes \mathbb{A}.$$

In the self–adjoint case the limit correspond to the relativistic point interaction described in [7]. As Šeba pointed out in [5] the norm–resolvent limit of the operator $H_\varepsilon^{\mathbb{A}}$ is the same as its formal limit. We concluded that renormalization of coupling constant does not occur for non–local potential.

Because of this property we defined general relativistic point interaction $H^{\mathbb{A}}$ as the limit of the operator $H_\varepsilon^{\mathbb{A}}$ for all complex 2×2 matrices $\mathbb{A}$.

$$(H^{\mathbb{A}}\psi)(x) = (H_m\psi)(x), \ \forall x \in \mathbb{R} \setminus \{0\} \text{ on domain}$$
$$\psi \in \operatorname{Dom} H^{\mathbb{A}} = \{\psi \in W^{1,2}(\mathbb{R}\backslash\{0\}) \otimes \mathbb{C}^2 \mid (2i\sigma_1 + \mathbb{A})\psi(0-) = (2i\sigma_1 - \mathbb{A})\psi(0+)\}$$

Furthermore, we discussed a spectrum of the operator $H^{\mathbb{A}}$ and found that $z \in \mathbb{C}\backslash\{(-\infty, -m] \cup [m, +\infty)\}$ is in the spectrum of the operator $H^{\mathbb{A}}$ if and only if following equation holds

$$0 = 4 + 2i \operatorname{tr}(\mathbb{A}\mathbb{Z}(z)) - \det \mathbb{A}.$$

We conclude that usage of a non–local potential to approximate the relativistic point interaction is more rewarding and natural in the case of the Dirac operator. We also see that the non–self–adjoint generalization of the relativistic point interaction behave similarly as in the self–adjoint case, except for the wild spectral transition from Section 4.2.

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