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NON–LINEAR ANALYSIS OF SLENDER MASONRY BEAM

Abstract

This paper deals with numerical analysis and design of slander prismatic masonry beams loaded predominantly by axial force and bending moment in plane of the principal moment of inertia. Because of the material non-linearity, classical mathematical theory of slender columns cannot be applied for masonry elements, therefore the proposed method uses iterative non–linear calculation considering both material and geometrical non–linearity.

Keywords

Masonry, material non–linearity, geometric non–linearity, second order analysis.

1 INTRODUCTION

Stability analysis is one of the most studied problems in the field of civil engineering. One reason is that the analysis is quite complex due to the many phenomena that need to be included. Recently proposed masonry structures are, considering their arrangement and resistance, usually designed as wall, which strongly emphasize the actual standards (especially [2]). If, however, the slender masonry element is to be designed, the method specified in [2] fails and leads to an underestimation of the impact of the stability problem.

The goal of the current research is an analysis of the real behaviour of slender masonry columns under compression and development of simplified methods for their design and verification in combination with the applicable technical regulations.

Accurate methods of modelling have to take into account nonlinear stress-strain diagram of masonry, changes in cross-section characteristics due to development of cracks and the geometric nonlinearity of the problem. At the early age only one non-linearity was consider in analysis, e.g. linear stress-strain diagram in compression with zero tensile strength. Nowadays the most common option is modelling using finite element method. Also the method of numerical integration - Runge-Kutta method - was used.

The object of this article is to develop an MATLAB® algorithm that takes into account the above-mentioned phenomena, i.e. the material and geometric nonlinearity, including changing the center of gravity position due to the cracks development. The prismatic rod loaded in the plane of the main inertia is considered in the analysis.

2 METHODS OF MODELLING

2.1 Stability analysis of theoretically straight column with linear behaviour

Euler's critical load for theoretically straight column:

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$$F_{cr}=\frac{\pi^2 EI}{L_{cr}^2} \tag{1}$$

where:

F_{cr} – is critical load, for which the column buckles [kN],

EI – is bending stiffness [kN.m^2] and

L_{cr} – is critical length [m].

Critical length depends on boundary conditions - see Fig. 1. For masonry column only full restraint or free end are applicable.

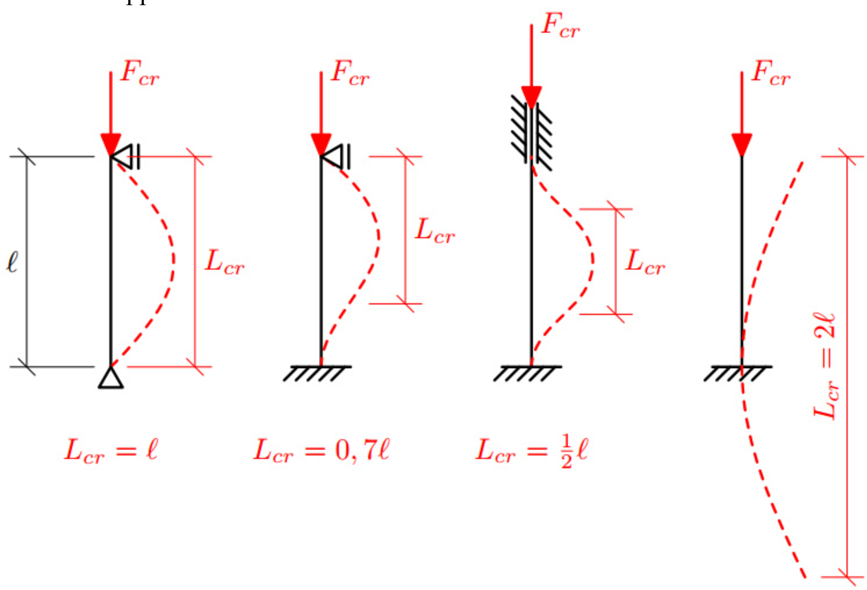


Fig. 1: Buckling shapes and critical lengths for various boundary conditions

In anlgorithm the buckling shapes according to Fig. 1 are used in a form of prescribed initial shape of column (called initial imperfection), see below.

2.2 Stability analysis of a beam with initial imperfections with linear behaviour

Model geometry is based on the real layout and dimensions of the analyzed structure. A crucial input data is the maximum estimated value of the initial imperfection of the structure, which is defined by the principles set out in [2] by:

$$e_0 = max(\frac{L}{450};\frac{b}{30};20\ mm) \tag{2}$$

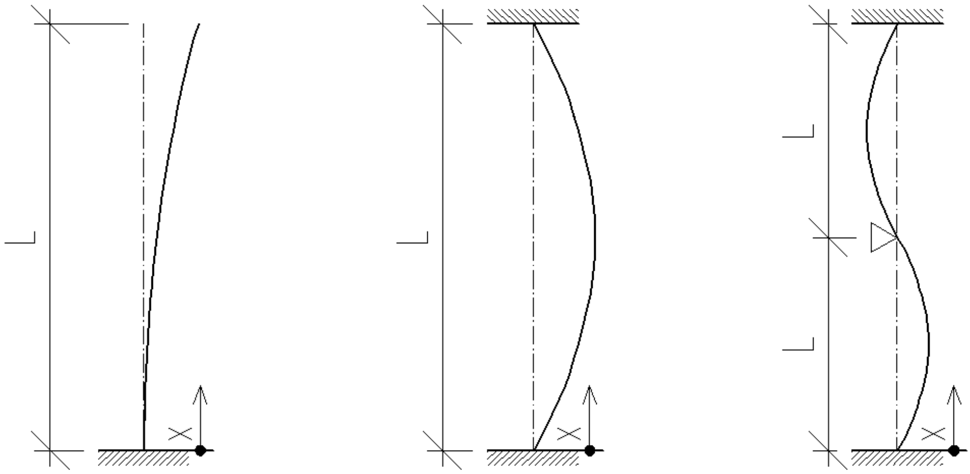
where:

e_0 – maximal initial displacement [mm],

b – is width of cross-section in the direction of buckling [mm] and

L – is effective length, for beam column it is distance of supports in analyzed direction [m].

The initial shape of the structure (imperfection) is determined depending on the expected buckling shape of the structure. Given the nature of masonry as a material with relatively small strength the buckling shape is considered to be sinusoidal, or part of sine function in any "span" of column, i.e. between the supports. The functions used for the approximation of the initial shape of the structure are shown on Fig. 2. In the case of unequal length of "fields" approximation functions have to be selected individually for each field.



$$e(x) = e_0 \cdot \left[\cos\left(\frac{\pi x}{2L}\right) - 1\right] \qquad e(x) = e_0 \cdot \sin\left(\frac{\pi x}{L}\right) \qquad e(x) = e_0 \cdot \sin\left(\frac{\pi x}{L}\right)$$

Fig. 2: Left: cantilever beam, middle: simply supported beam, right: the beam, which is supported on both sides by hinges and has a hinge support in the middle of height

In future, the number of initial imperfection shapes will be expanded so as to cover all the possibilities of structure geometry and design tolerances for the execution of building works according to the relevant technical regulations and standards.

Way of solving the problem:
Given the nature of the problem (above mentioned) solving is performed by method of inverse iterations. Solvee the system of equations of the form:

$$(K - KG).r = 0 \tag{3}$$

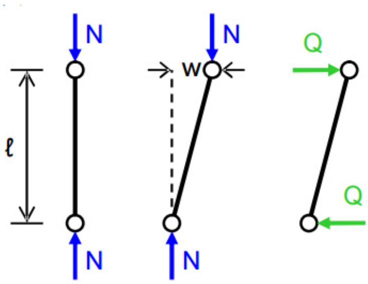
where:

K – is global stiffness matrix considering the boundary conditions (regular matrix) [kN/m],

KG – is a geometric stiffness matrix and

r – is a vector of displacement.

Geometric stiffness matrix for a beam hinged on both sides is derived from a simple equilibrium conditions:



$$Q \cdot l = N \cdot w \Rightarrow Q = \frac{N}{l} w$$

Fig. 3: Derivation of geometric stiffness matrix for a beam hinged on both sides

$$KG = \frac{N}{l}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4}$$

where:

N – is normal force in the beam [kN] and

l – is a beam length.

Geometric stiffness matrix for a beam fixed on both sides:

$$KG = \frac{N}{30l}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 36 & 3l & 0 & -36 & 3l \\ 0 & 3l & 4l\text{^}2 & 0 & -3l & -l\text{^}2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -36 & -3l & 0 & 36 & -3l \\ 0 & 3l & -l\text{^}2 & 0 & -3l & 4l\text{^}2 \end{bmatrix} \tag{5}$$

Equation (3) can be modified for the purpose of calculation to:

$$Kr=f+f_{ekv} \tag{6}$$

where:

f – is a load vector, and

f_{ekv} – is vector of equivalent load calculated as follows:

$$f_{ekv}=K_G r \tag{7}$$

In the (i+1)–th step of iteration we can calculate:

$$Kr^{i+1} = f + K_G^i r^i \tag{8}$$

Where:

i – is number of step. It is used in upper indices.

Analysis uses following algorithm:

$$Kr^0 = f \tag{9}$$

$$Kr^{i+1} = f + K_G^i r^i \tag{10}$$

$$abs(r^{i+1} - r^i) < \varepsilon \tag{11}$$

Equation (9) is solved first to obtain r^0, from which N^0, K_G^0 are calculatet. Then, from equation (10) we obtain r^{i+1} – vector of displacement of next step. This step is repeated until the criterion of equation (11) is fullfilled. In Matlab algorithm, error is chosen to be smaller than $\varepsilon = 10^{-8} m$.

3 MATERIAL PROPERTIES

3.1 In general

Masonry as a construction material is highly heterogeneous, consisting of separate masonry elements (bricks, blocks, etc.) and of joints filled with binder (mortar) usually of significantly lower strength and stiffness. Lower strength and stiffness of the mortar ensures uniform transfer of load between the masonry units, and even in places of local stress concentrations (i.e. where the asperity of the masonry elements occurs). High mortar strength and stiffness increase the strength of the entire composite material, but in areas of local asperities undesired crushing of masonry elements would occur, with consequent risk of failures.

3.2 Masonry material model

In practical terms, a discrete model of the structure, which takes into account the individual elements and their behaviour is unusable, particularly with regard to complicated determination of the input data, its own calculation and evaluation of the result. For the purpose of analysis and the structural design, the material is usually homogenized in an appropriate manner to preserve its properties in relation to the real behaviour of the structure modelled.

Consistent approach is chosen in this study. Masonry is considered as a homogeneous or homogenized material. It is assumed that the dimensions of masonry elements and the joints between them do not significantly affect the distribution of stress in a masonry member. Evaluation of the structure is then carried out only in the weakest cross-section, i.e. in the joints of brickwork. It is assumed that if the structure fulfils verification in the joints, it fulfils verification in all other cross-sections.

The real stress-strain diagram of masonry shows a non-linear behaviour:

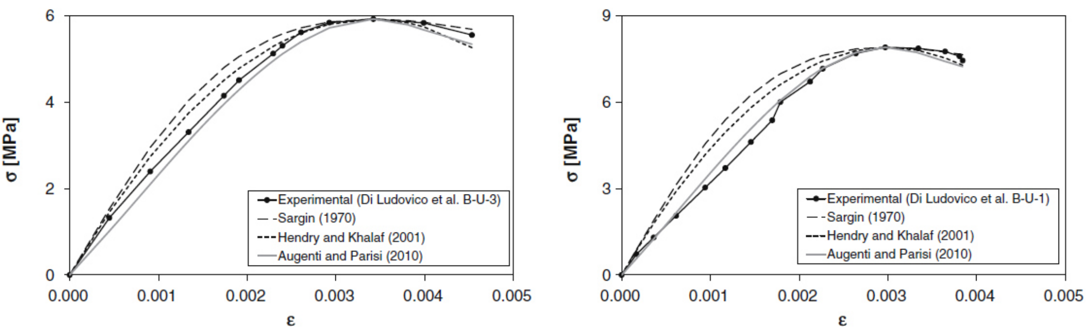


Fig. 4: Stress-strain diagram of masonry – experiments and curves, which try to fit it

The stress-strain diagrams were described for example Hendry and Khalaf [5], Augenti and Parisi [4] or Sargın [6].

In this work it is assumed that the material reacts only on the pressure and the occurrence of tensile stress leads to cracking and its opening. If subsequently (eg. in another load combination) tensile stress in cross-section vanishes, the cracks will close and cross-section acts as full again. Idealized stress-strain diagrams of such behaviour of the material for the individual limit states in accordance with [2] are shown on Fig. 5. Stress-strain diagrams are considered linear (SLS), or linearly-plastic (ULS).

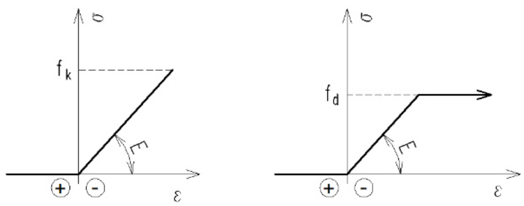


Fig. 5: Stress-strain diagrams of masonry used for analysis of structure. Left: Serviceability limit state, right: Ultimate limit state

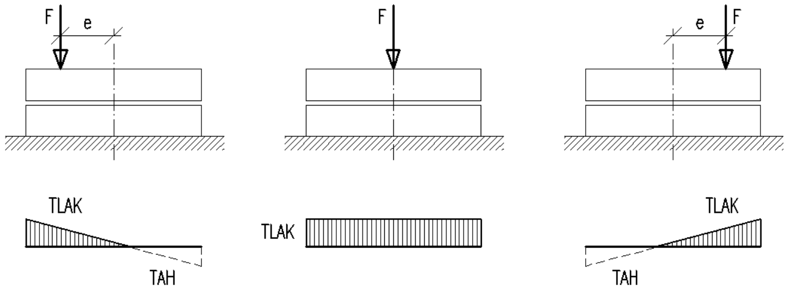


Fig. 6: Changes of cross-section properties in dependence of load. Upper: Load position, lower: Corresponding stress distribution

4 MECHANICAL MODELLING OF MASONRY COLUMNS

The introduction of the above-described material model causes significant changes in geometry and stiffness of the structure being modelled in dependence on the applied load. The widening of the cracks in the masonry together with the geometrical non-linearities leads to non-linear solution of the whole problem.

Principle of modelling of cross-sections and individual elements, changes in the stiffness and the geometry of the structure caused by the applied load are shown on Fig. 9. On the left picture may be seen, that tension in the part of cross-section makes this part excluded. The cross-section gets smaller and center of gravity moves - Δ. The move of centroid has a positive effect on the stresses (the bending moment gets smaller) but cross-section getting smaller has negative effect on stresses, which increase.

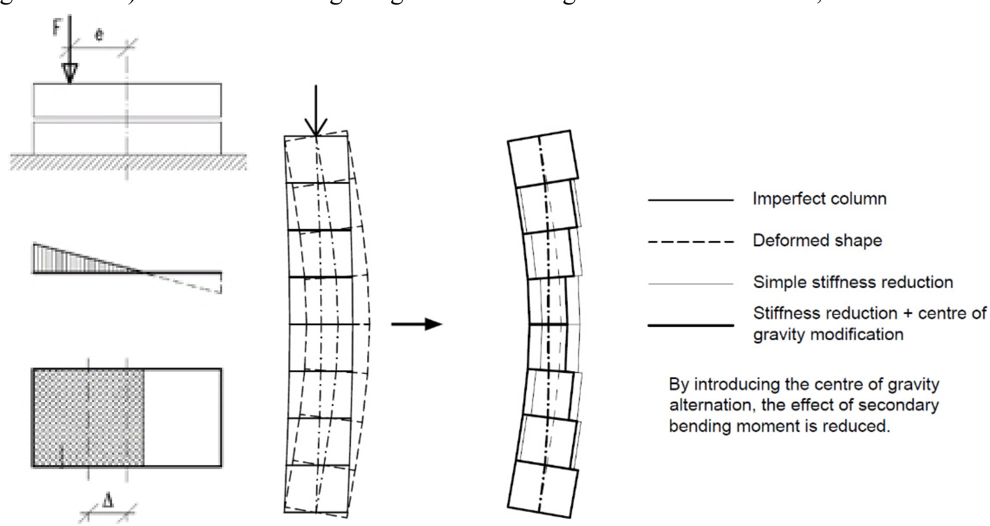


Fig. 7: Changes of cross-section and geometry of structure due to excluding the tensioned parts

The method of calculation (see also 2.2):

1) First step
    a. Dividing of the column to partial beam elements,
    b. Elastic analysis of vector r^0 with full cross-section – see equation (9),
    c. Calculation of stresses in the cross-section of each element and determining, whether the excluding of the tensioned part of cross-section executes, or the cross-section remains full,
    d. Filling of the matrix „prop“ – matrix of height, area and moment of inertia of cross-section in each step of calculation.
2) Second and every other step
    a. Calculation of Δ - move of cross-section of each element (from matrix „prop“ from last step),
    b. Calculation of matrix *K* with the new cross-section properties and changed geometry,
    c. Calculation of matrix *KG* – see chapter 2.2,
    d. Calculation of vector r with changed geometry and new matrix *K*,
    e. Calculation of stresses in the new cross-section and determining, whether the new cross-section gets even smaller or it grows bigger again,
    f. Filling of the matrix „prop“,
    g. Checking the condition of convergence – see equation (11).

5 EXAMPLE 1

Column hinged on both sides with square cross-section BxH=0.4x0.4 m with height 4.2 m was chosen. Its initial imperfect shape was chosen according to (2), initially inclined right. Axial load on the column was 770 kN and bending moments at both ends 39 kNm. These moments were chosen so, that it enlarge moment from imperfection. The final deflection and int. forces can be seen on Fig. 10.

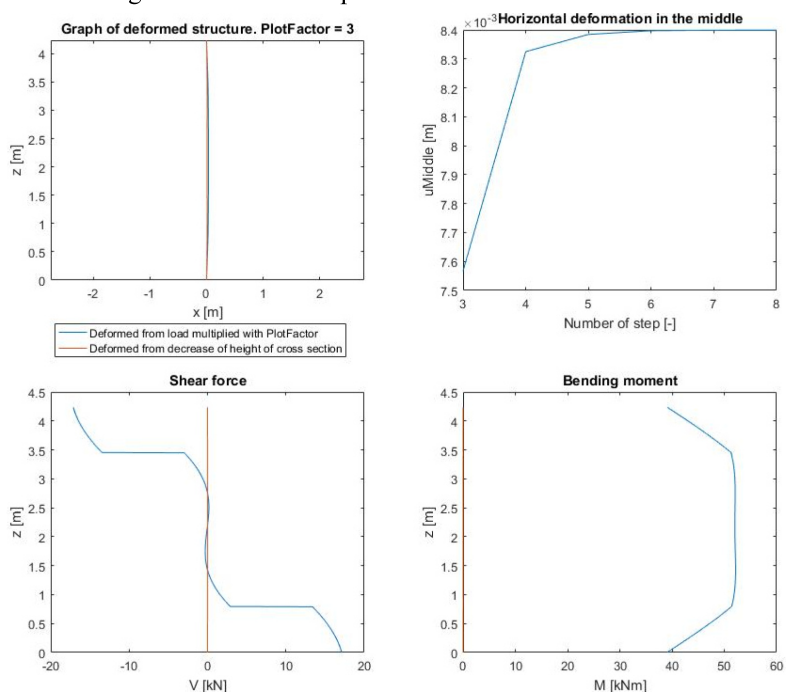


Fig. 8: Left top - final shape and lateral deflection (see also Fig. 11), right top – lateral deflection of the middle of the column in each step of calculation, left bottom – shear force in the final step,

right bottom – bending moment in the final step. All this is for the case, that cross-section reduces and the calculation converge to equilibrium

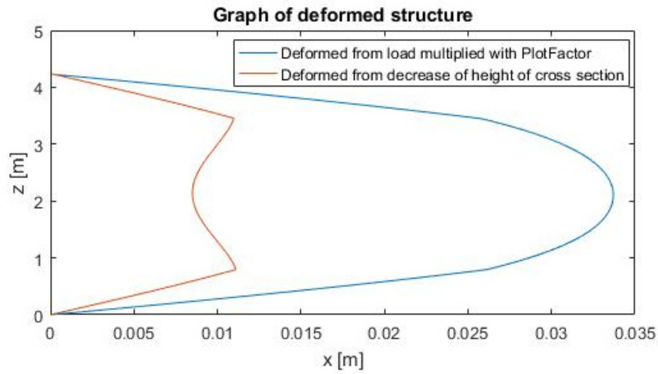


Fig. 9: Final shape of center of gravity and lateral deflection of this structure, non-proportional plot. It is for the case, that cross-section reduces and the calculation converge to equilibrium

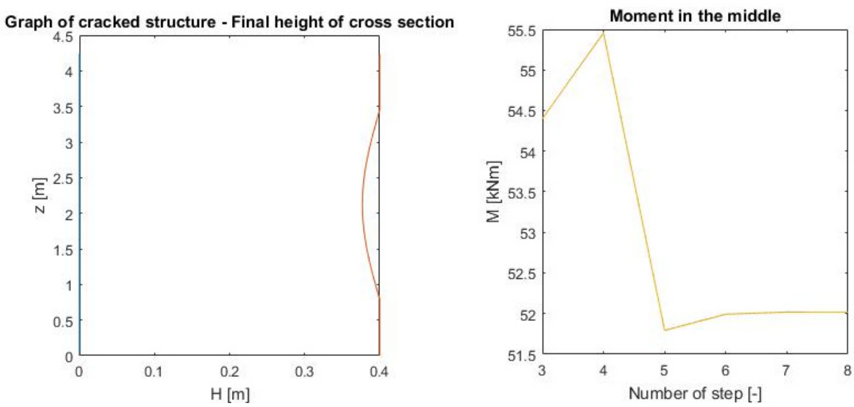


Fig. 10: Left – final height of cross-section at the final step, right – bending moment in the middle of height of the column in each step of calculation. It is for the case, that cross-section reduces and the calculation converge to equilibrium

The load in above example was chosen so that the material non-linearity will show up. If the load is higher than critical, the lateral deflection grows excessively large. In such a case, the calculation stops after few steps:

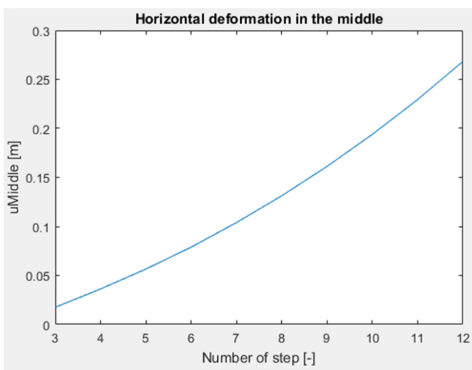


Fig. 11: Lateral deflection of the middle of the column in each step of calculation. It is for the case, that the cross-section reduces and the calculation didn't converge

The last possibility is, that the column cross-section doesn't reduce, because no tension occurred. So the calculation is carried out only with the assumption of linear behaviour of material.

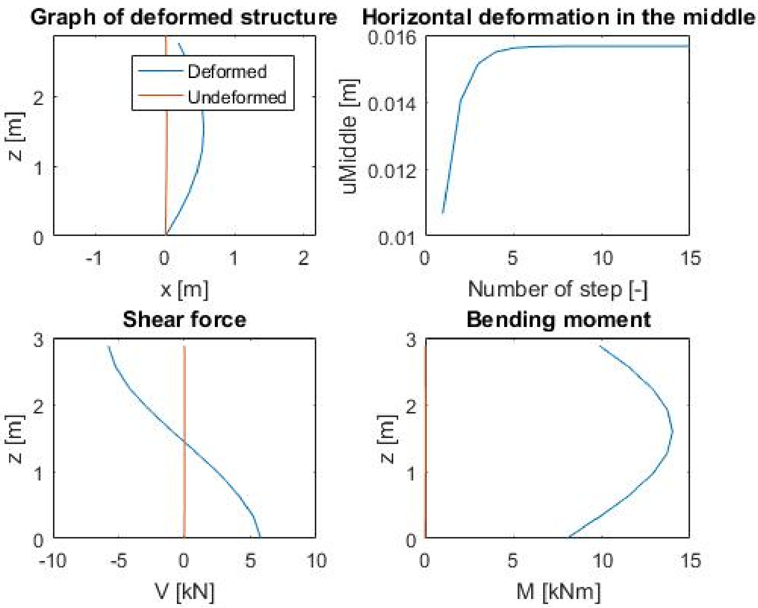


Fig. 12: Left top - final shape and lateral deflection, right top – lateral deflection of the middle of the column in each step of calculation, left bottom – shear force in the final step, right bottom – bending moment in the final step. All this is for the case, that cross-section doesn't reduce and the calculation converge to equilibrium

6 EXAMPLE 2

As an example, column, which rotations are fixed at both sides with square cross-section BxH=0.4x0.4 m with height 8.2 m was chosen. Its initial imperfect shape was chosen 0.18 m, initially inclined right. Axial load on the column was 770 kN. The final deformation and internal forces can be seen on Fig. 10.

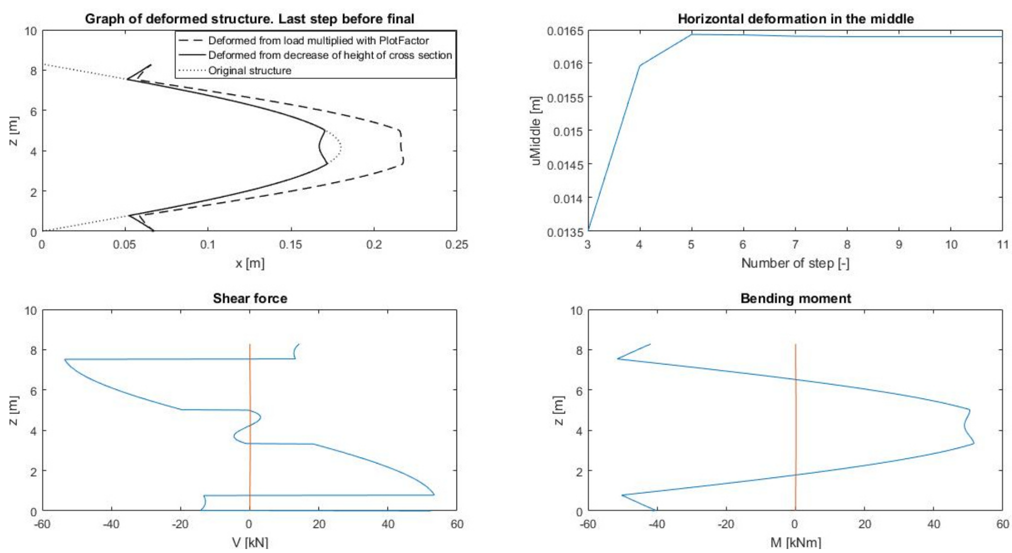


Fig. 15: Left top - final shape and lateral deflection, non-proportional plot, right top – lateral deflection of the middle of the column in each step of calculation, left bottom – shear force in the final step, right bottom – bending moment in the final step. All this is for the case, that cross-section reduces and the calculation converge to equilibrium

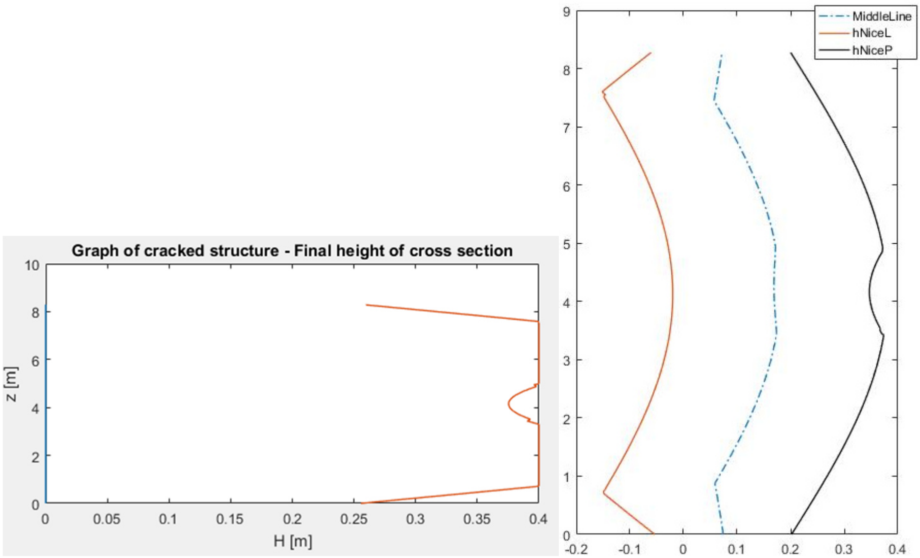


Fig. 16: Left - final height of cross-section at the final step, right - final shape of the column – without excluded tensioned parts of cross-section

7 CONCLUSIONS

An algorithm which can solve non-linear stability analysis of slender columns loaded in the plane of the main moment of inertia was implemented in the program MATLAB®. The program allows to analyze these structures till its failure. In the future extension of algorithm to columns loaded in general plane is assumed to be made. The authors also plan to extend the algorithm to general structures, not only columns under compression.

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