Czech Technical University in Prague
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Habilitation Thesis

Many-valued conjunctions

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Preface

This thesis is a collection of the following papers which have been published (or accepted for publication) in scientific journals and edited volumes.

[PV17] M. Petrík and Th. Vetterlein. “Rees coextensions of finite, negative tomonoids”. In: *J. Logic Comput.* 27.1 (2017), pp. 337–356.

[PV19] M. Petrík and Th. Vetterlein. “Rees coextensions of finite tomonoids and free pomonoids”. In: *Semigroup Forum* 99 (2019), pp. 345–367.

[PV14] M. Petrík and Th. Vetterlein. “Algorithm to generate the Archimedean, finite, negative tomonoids”. In: *Joint 7th International Conference on Soft Computing and Intelligent Systems and 15th International Symposium on Advanced Intelligent Systems* (Kitakyushu, Japan, Dec. 3–6, 2014). 2014, pp. 42–47.

[PV16] M. Petrík and Th. Vetterlein. “Algorithm to generate finite negative totally ordered monoids”. In: *IPMU 2016: 16th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems* (Eindhoven, Netherlands, June 20–24, 2016). Ed. by J. P. Carvalho et al. 2016.

[Pet15] M. Petrík. “New solutions to Mulholland inequality”. In: *Aequationes Math.* 89.4 (2015), pp. 1107–1122.

[Pet18] M. Petrík. “Dominance on strict triangular norms and Mulholland inequality”. In: *Fuzzy Sets and Systems* 335 (2018), pp. 3–17.

[Pet20] M. Petrík. “Dominance on continuous Archimedean triangular norms and generalized Mulholland inequality”. In: *Fuzzy Sets and Systems* (2020). In press, available online. DOI: `10.1016/j.fss.2020.01.012`.

The first four papers [PV19; PV17; PV14; PV16] deal with a rather practical question: How to obtain all possible many-valued conjunctions if a totally ordered finite set of truth degrees is given. A set of truth degrees together with a conjunction is described here by a finite, negative, totally ordered monoid and the procedure of obtaining all such monoids is performed step-wise: having a set of truth degrees of size $n \in \mathbb{N}$ endowed with a particular conjunction, the algorithm gives all conjunctions on a set of degrees of the size $n+1$ that can be seen as “extensions” of the given conjunction. More details are given in Chapter 2.

The next three papers [Pet20; Pet18; Pet15] deal with many-valued conjunctions from a different point of view. If the set of the truth values is represented by the real unit interval $[0,1]$ then the conjunction, which is now a mapping of the type $[0,1] \times [0,1] \to [0,1]$, is called a *triangular norm*. This notion refers to the origins of these operations which is the theory of probabilistic metric spaces. The attached papers deal with two problems related to this area which have been open for a rather long time:

- Does the Mulholland's condition characterize the whole set of solutions of Mulholland inequality?
- Is the dominance relation transitive on the set of triangular norms?

These problems, together with the achieved results, are described in Chapter 3.

A description of the motivations is given in Chapter 1 and the thesis is concluded by the attached papers.

Contents

Chapter 1

Introduction

1.1 Overview

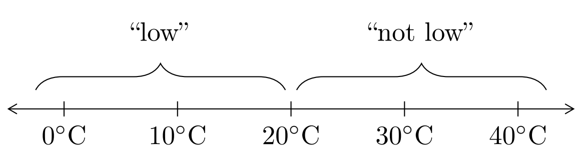
Many-valued logic (or fuzzy logic) is a concept which has been studied for a hundred years, already. Its origins can be dated back to 1913 to the works of Jan Łukasiewicz [Łuk13].

An intuitive way of introducing many-valued logic is to enrich the classical set of truth values, "false" and "true" (often denoted by 0 and 1, respectively), by additional truth values. Note that in many-valued logic it is often referred to the truth values as *truth degrees*. These truth degrees are typically lattice-ordered such that 1 is the highest degree while 0 is the lowest one. A *standard* choice is the real unit interval $[0, 1]$ or a set of n values equidistantly spread between 0 and 1 to form the set

$$\left\{0,\ \frac{1}{n-1},\ \frac{2}{n-1},\ \ldots,\ 1\right\}.$$

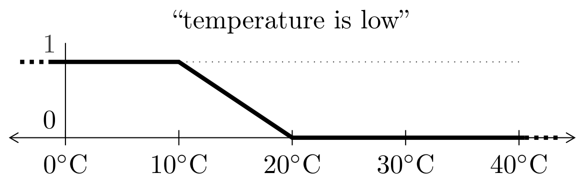
Notice that both these sets are totally (linearly) ordered which is generally not granted.

While 0 and 1 still represent "false" and "true", the additional degrees between them represent states of "partial truth". Let us demonstrate this on an example of a simple room temperature controller whose function could be described by the proposition "If the temperature is low then turn the heating on". In the classical two-valued case, in order to determine which values of the temperature are "low", we usually have no better choice than to establish a strict boundary on the scale of the possible temperature values, say 20°C:



Such strict distinguishing, however, brings only a little information and does not allow the controller to do more than to either turn the heating on or turn it off.

However, we may decide to determine the truth degree of the statement "the temperature is low" as a number from $[0,1]$, for example, in the following way:



Observe that the sharp boundary has disappeared. While the truth degree of the statement "10°C is a low temperature" is still "true" (1) and the truth degree of the statement "20°C is a low temperature" is "false" (0), the degrees for the temperatures in the range from 10°C to 20°C are somewhere in between. For example, the truth degree of the statement "11°C is a low temperature" is 0.9 which we can interpret as "almost true". As such approach gives more information, the controller can now set the heating low or high instead of only being able to turn it on or off.

When we extend the set of the truth values, we also need to adapt the logical calculus. Particularly, we need to give a suitable generalization of the logical operations (conjunction, implication, ...). This way we obtain what is usually called (the semantics of) many-valued, or fuzzy, logic [NPM99]. Remark that the notion "fuzzy" was introduced in 1965 by Zadeh when he presented his concept of fuzzy sets [Zad65].

Interestingly enough, the most prominent operation in many-valued logic is the conjunction; the reason is that the other important operations (namely the implication) can be uniquely determined by it (see Section 2.2). The conjunction is given axiomatically and it is not unique; for example, in the case of the standard interval of truth degrees $[0,1]$, a conjunction can be every binary operation on $[0,1]$ that is commutative, associative, monotone, and such that 1 is its unit element. Such an operation is often called a *triangular norm*[1] (or a *t-norm*, for short) [KMP00]. Analogously, we can define a conjunction on a finite totally ordered set of truth degrees. In such a case it is often called a *discrete triangular norm* (or a *discrete t-norm*) [BM03].

1.2 Construction of many-valued conjunctions

The thesis deals, as the title suggests, with many-valued conjunctions. Its first part is devoted to the question, how to obtain all possible conjunctions when a finite and totally ordered set of truth degrees is given. The question could be also formulated as "how to construct all possible discrete t-norms of size $n \in \mathbb{N}$" or,

[1] Triangular norms are described in detail in Section 3.1.

more algebraically, “how to determine all totally ordered MTL-algebras (MTL-chains) of size $n \in \mathbb{N}$”[2].

This question is significantly oriented towards possible applications. When implementing a many-valued reasoning system, it is reasonable to first establish the set of truth degrees. With respect to the possibilities of computer systems, this set will be finite and, most probably, totally ordered. Then, after establishing such set of truth degrees, we look for the representation of the logical operations. The conjunction comes as the first one since the rest of the operations can be (usually) uniquely determined by it.

Two-valued logic is the only case where the conjunction has a unique representation:

0	1	
0	1	1
0	0	0

In the case of three truth degrees, the conjunction can be implemented by two distinct discrete t-norms:

0	1/2	1	
0	1/2	1	1
0	0	1/2	1/2
0	0	0	0

0	1/2	1	
0	1/2	1	1
0	1/2	1/2	1/2
0	0	0	0

In the case of four truth degrees there are six discrete t-norms:

0	1/3	2/3	1	
0	1/3	2/3	1	1
0	0	0	2/3	2/3
0	0	0	1/3	1/3
0	0	0	0	0

0	1/3	2/3	1	
0	1/3	2/3	1	1
0	0	1/3	2/3	2/3
0	0	0	1/3	1/3
0	0	0	0	0

0	1/3	2/3	1	
0	1/3	2/3	1	1
0	1/3	1/3	2/3	2/3
0	1/3	1/3	1/3	1/3
0	0	0	0	0

0	1/3	2/3	1	
0	1/3	2/3	1	1
0	0	2/3	2/3	2/3
0	0	0	1/3	1/3
0	0	0	0	0

0	1/3	2/3	1	
0	1/3	2/3	1	1
0	1/3	2/3	2/3	2/3
0	0	1/3	1/3	1/3
0	0	0	0	0

0	1/3	2/3	1	
0	1/3	2/3	1	1
0	1/3	2/3	2/3	2/3
0	1/3	1/3	1/3	1/3
0	0	0	0	0

[2] MTL-algebras are described in Section 2.2.

In the case of five truth degrees there are 22 discrete t-norms, and so on; see Section 2.5 for higher numbers.

The number of possible representations of the conjunction grows rapidly (not slower than exponentially [PV14, Proposition 7.3]) with the number of truth degrees. The high amount of possible conjunctions is, however, not a drawback of many-valued logic since different conjunctions may have different properties and thus may be useful in different situations. In order to know which conjunction is suitable for our task, it may be handy to have the full list of them.

1.3 Dominance and Mulholland inequality

The second part of the thesis studies open problems related to the theory of probabilistic metric spaces [Men42; SS05] which is the origin of t-norms. A probabilistic metric space is a space where the distance between two points is expressed by a distribution function rather than by a positive real number and t-norms serve here to describe the triangular inequality of the metrics.

The dominance relation is a binary relation defined on the set of all t-norms and its importance emerges when one constructs Cartesian products of probabilistic metric spaces [Tar76]. It can be easily shown that the dominance relation is both reflexive and anti-symmetric. Therefore, it is natural to ask whether this relation is also transitive and hence an order.

Mulholland inequality is a functional inequality (its solution is a function) which was introduced in 1949 by Mulholland [Mul49] as a generalization of Minkowski inequality. Recall that Minkowski inequality is the triangular inequality of L^p-norms. In this inequality, Mulholland has replaced the power function $x \mapsto x^p$, $p > 0$, by an arbitrary increasing bijection of $[0, \infty]$ and he asked, under which conditions the inequality will be preserved. Of course, all power functions, as well as their positive multiples, naturally solve (satisfy) Mulholland inequality. However, there are more such solutions. Mulholland has provided a condition which is sufficient for a given function to solve the inequality but the question, whether this condition characterizes all the solutions (i.e., whether it is also a necessary condition), has remained open.

Mulholland inequality has brought attention of the researchers that were dealing with the dominance relation since it characterizes in a clear way the dominance on the set of strict t-norms [Tar84]; later it has been generalized to characterize even the whole set of continuous Archimedean t-norms[3] [SBM08].

The thesis brings two main results. It shows that the Mulholland's condition does not characterize all the solutions of Mulholland inequality and, based on this result, it proves that the dominance relation is transitive neither on the set of strict t-norms nor on the set of continuous Archimedean t-norms.

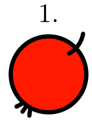
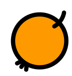
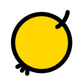
[3]Continuous Archimedean t-norms, as well as strict t-norms, form significant classes of these operations; see Section 3.1 for the definitions and for more details.

1.4 Example

The intention of this section is to provide a simple example in order to demonstrate a possible implementation of many-valued reasoning with a utilization of a conjunction.

1.4.1 Robot's task

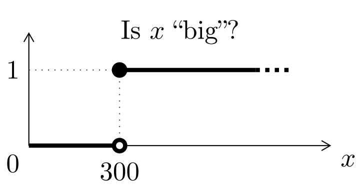
Let us consider a robot who's task is to collect three "ripe" apples from a tree in the garden. We define "ripe" apples as those that are "big" and "red" and we suppose that the robot is able to measure the size of an apple as a number in $]0,\infty[$ (its volume in cm^3) and its "non-redness" as a number in $\left[0,\sqrt{3}\right]$ (the Euclidean distance from the "most red" point with the coordinates $(1,0,0)$ in the red-green-blue unit cube). Suppose that the robot has found the following five apples:

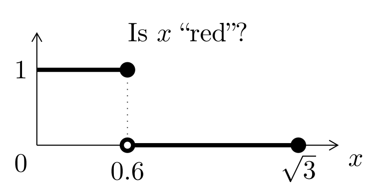
apple	1. 	2. 	3. 	4. 	5. 
size	400	240	280	300	180
non-redness	0.10	0.57	0.60	0.84	1.10

Clearly, the first apple is ripe while the fifth one is not. How to deal with the rest?

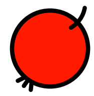
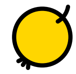
1.4.2 Two-valued reasoning

If the robot has two-valued reasoning implemented then, in order to distinguish between "big" and "not big" apples, we need to set a strict border in the range of the possible apple sizes; and we need to do the same for "red" and "not red". If we choose these borders as, for example, 300 and 0.6, respectively, then we will obtain the following answers to the questions "Is the size big?" and "Is the color red?":





The classification of the five apples is then the following:

apple	1. 	2. 	3. 	4. 	5. 
“big”	1	0	0	1	0
“red”	1	1	0	0	0
“big” and “red”	1	0	0	0	0

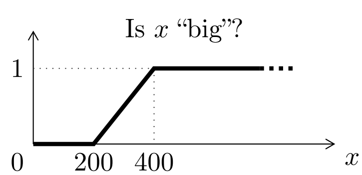
We can observe several drawbacks in this approach.

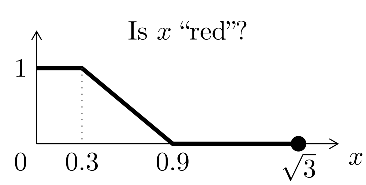
One drawback is that with such strict borders we may obtain different classifications for objects which actually have similar qualities. For example, despite the fact that the size of the third and the fourth apple is almost the same, the fourth one is “big” while the third one is not. Similarly, the second and the third apple have very similar color but they are classified differently, too.

Another drawback is that the classified objects lack an order. In our example, there is one positively classified object and four negatively classified objects. However, the robot needs to choose three of them and does not have any objective clue how to do it. Perhaps, in this particular case, we could benefit from the fact that the second and the fourth apple fulfill at least one of the required properties. Hence the robot would pick the first, the second, and the fourth apple. However, such a situation is not likely to happen every time and, surprisingly, the many-valued approach will give us different solution.

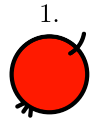
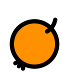
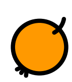
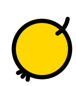
1.4.3 Many-valued reasoning

Let us see then how we could manage this task with a many-valued approach. For this example, we will choose the standard set of truth degrees which is the real unit interval $[0, 1]$. Instead of strict borders we will establish fuzzy borders, for example, in the following way:

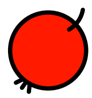
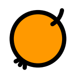
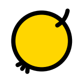




Thus, an answer to the question “Is the size big?” or “Is the color red?” is not strictly 0 or 1 but it can be any number from the interval $[0, 1]$. The classification of the five apples is then the following:

apple	1. 	2. 	3. 	4. 	5. 
“big”	1.00	0.20	0.40	0.50	0.00
“red”	1.00	0.55	0.50	0.10	0.00

Now we want to aggregate the truth degree of “big” and the truth degree of “red” into a single value representing the truth degree of “big and red”. We need a many-valued conjunction which, in this case, is represented by a t-norm. Let us use one of the most prototypical t-norms which is the product t-norm defined simply as the product of two real numbers. The degrees of “big and red” will be then the following:

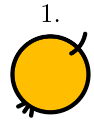
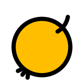
apple	1. 	2. 	3. 	4. 	5. 
“big and red”	1.00	0.11	0.20	0.05	0.00

The robot now has the possibility to sort the apples according to the truth degrees of “big and red”. Hence the robot picks the first, the second, and the third apple.

1.4.4 Remark on conjunctions

In this example, we have utilized the product t-norm as a conjunction. This is a particularly nice representation of the conjunction, it is well comprehensible, it is easy to compute, its definition is compact. However, it has one major drawback: it cannot be defined on finite sets! Which conjunction can we choose if the set is finite?

Instead of the product t-norm we can use, for example, the *minimum (Gödel) t-norm* which is evaluated as the smaller value of two numbers. A great benefit of this t-norm is that it can be defined on any set of truth degrees. However, it has a drawback, too. Consider the following two apples:

apple	1. 	2. 
“big”	1.00	0.50
“red”	0.25	0.25
“big and red”	0.25	0.25

The “ripeness” of both apples is evaluated by the minimum t-norm as equal although the first apple is apparently better. The reason is that the minimum t-norm does not reflect the higher one of its arguments.

Or, we could use the *Łukasiewicz t-norm* defined by:

$$x * y \quad = \quad \begin{cases} x + y - 1 & \text{if } x + y \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

This would be perhaps the best choice for our task. It has, however, some drawbacks, too. Its definition is rather unintuitive and it is also a little censorious: its evaluation is often equal to zero.

Of course, we can also use any of the other possible conjunctions. For example, in the case of 10 truth degrees there are 86417 of them (see Section 2.5).

Chapter 2

Rees coextensions of finite negative tomonoids

This part of the thesis deals with the question, how to describe all the one-element Rees coextensions of a given finite, negative, totally ordered monoid. With such a result, one is able to successively construct all the finite, negative, totally ordered monoids up to a given size. The constructed monoids naturally form a tree with the trivial monoid as the root while two monoids are in the relation if one can be obtained as a Rees quotient of the other. The intention of this chapter is to give an overview of the achieved results:

- We define a relational structure describing all the one-element Rees coextensions of a given finite, negative, totally ordered monoid.
- We give a system of rules to obtain these coextensions; this is done separately for the case of Archimedean finite, negative, totally ordered monoids and for the case of general (non-Archimedean) ones.
- These rules are also presented in the form of an algorithm. This algorithm has been implemented in the programming language Python and the implementation is available for download at `https://home.czu.cz/en/petrikm/coextensions`.
- We show that every one-element Rees coextension of a given finite, negative, totally ordered monoid corresponds to a downset that separates two particular elements of a specially created free partially ordered monoid.

2.1 Totally ordered monoids

We are going to represent a totally ordered set of truth values together with the conjunction by a totally ordered monoid.

Recall that a *monoid* is a set S with a binary operation $\odot \colon S \times S \to S$ such that:

(T1) $(a \odot b) \odot c = a \odot (b \odot c)$ for every $a, b, c \in S$,

(T2) there is an element $1 \in S$ such that $a \odot 1 = 1 \odot a = a$ for every $a \in S$.

A total (linear) order $\leq$ on a monoid S is called *compatible* if

(T3) $a \leq b$ implies both $a \odot c \leq b \odot c$ and $c \odot a \leq c \odot b$ for every $a, b, c \in S$.

In such a case, we speak about a *totally ordered monoid* [Eva+01], or a *tomonoid* for short. We call this tomonoid *commutative* if

(T4) $a \odot b = b \odot a$ for every $a, b \in S$,

negative[1] if 1 is the top element, and *finite* if S is a finite set. Here, we are interested exclusively in those tomonoids that are finite and negative, which we abbreviate as "*f. n. tomonoids*". Let us remark that, in contrast to the work of Evans et al. [Eva+01], we usually do not assume commutativity, although we deal with this case, too. A finite tomonoid always has a smallest element; in the case of f. n. tomonoids we denote it by 0.

The smallest tomonoid, called the *trivial tomonoid*, is the one that consists of the monoidal identity 1 alone. Tomonoids with at least two elements are called *non-trivial*. A negative tomonoid is called *Archimedean* if, for every $a, b \in S \setminus \{1\}$ such that $a \leq b$, there is an $n \in \mathbb{N}$ such that

$$b^n = \underbrace{b \odot b \odot \cdots \odot b}_{n\text{-times}} \leq a.$$

A f. n. tomonoid is Archimedean if and only if its only idempotent elements are 0 and 1. Recall that an element $a \in S$ is called *idempotent*, if $a \odot a = a$.

Throughout this thesis, f. n. tomonoids are often defined by slightly modified *Cayley tables*. An example of a non-commutative Archimedean f. n. tomonoid defined by its Cayley table can be seen in Figure 2.1.

2.2 Residuated lattices and MTL-algebras

Our motivation to study totally ordered monoids arise from residuated lattices and, particularly, MTL-algebras. This section intends to recall briefly these structures.

In many-valued logic, the set of the truth values is usually represented by a *residuated lattice* [Gal+07] which is an algebra $(S; \wedge, \vee, \odot, \backslash, /, 1)$ of the type $\langle 2, 2, 2, 2, 2, 0 \rangle$ where:

- $(S; \wedge, \vee)$ is a lattice,
- $(S; \odot, 1)$ is a monoid compatible with the lattice order,

[1] Remark that in the context of residuated lattices, usually the notion "integral" is used instead.

0	v	w	x	y	z	1	
0	v	w	x	y	z	1	1
0	0	v	v	w	x	z	z
0	0	0	0	v	v	y	y
0	0	0	0	v	v	x	x
0	0	0	0	0	v	w	w
0	0	0	0	0	0	v	v
0	0	0	0	0	0	0	0

Figure 2.1: Example of a Cayley table of a f. n. tomonoid. Note that, throughout the attached papers, the values between the bottom element 0 and the top element 1 are denoted by the alphabetically ordered lower-case letters such that z is the second greatest element, y is the third greatest element, etc. Remark that this f. n. tomonoid is non-commutative and Archimedean.

- $\backslash$ and $/$ are binary operations on S called the *left and the right residuum*, respectively, given by the axiom

$$a \odot b \leq c \quad \text{iff} \quad b \leq a\backslash c \quad \text{iff} \quad a \leq c/b \quad \text{for any} \quad a, b, c \in S, \tag{2.1}$$

  where $\leq$ denotes the lattice order.

Note that both residua are given by (2.1) uniquely and they can be determined also, for $a, b, c \in S$, by the formulas:

$$\begin{aligned} a\backslash c &= \sup \{b \in S \mid a \odot b \leq c\}, \\ c/b &= \sup \{a \in S \mid a \odot b \leq c\}. \end{aligned}$$

The monoidal operation $\odot$ is called a *multiplication* (also *strong conjunction*) and represents the logical conjunction (whereas the lattice operation $\wedge$, usually called *weak conjunction*, serves to implement the universal quantification).

The residua are understood as the left and the right division of the multiplication and they represent the left and the right logical implication, respectively. Remark that if we add the axiom of commutativity of $*$ then, of course, both the left and the right residuum become identical; in such a case, they are often denoted by the common symbol $\rightarrow$ which reflects their relation to the logical implication.

Examples of (some sub-varieties of) residuated lattices are the Boolean algebras, the Heyting algebras [Hey30], the MV-algebras [Cha58], the BL-algebras [Háj98a; Háj98b], or MTL-algebras [EG01]. These varieties form semantics of the classical logic, the intuitionistic logic [Hey30], the Łukasiewicz logic [Łuk30; ŁT30], the basic fuzzy logic (shortly BL) [Háj98a; Háj98b], and the monoidal t-norm based logic (shortly MTL) [EG01], respectively.

Note that BL and MTL can be considered as two prototypical many-valued logics. BL was introduced by Hájek [Háj98a; Háj98b] in 1998 and it covered the many-valued logics that were known at that time, namely, the Gödel-Dummet logic [Dum59; Göd32], the Łukasiewicz logic [Łuk30; ŁT30], and the product logic [HGE96]. The semantical counterpart of BL is represented by BL-algebras which play an analogous role as Boolean algebras for the classical logic. An example of a BL-algebra is the real unit interval $[0, 1]$ endowed with a continuous t-norm (see Definition 3.1.1) which represents a conjunction and the corresponding residuum which represents an implication; such a BL-algebra is called a *standard BL-algebra*. Hájek proved that BL is sound and complete with respect to the class of BL-algebras and Cignoli, Esteva, Godo, and Torrens proved that BL is sound and complete even with respect to the class of standard BL-algebras [Cig+00].

The t-norm in a standard BL-algebra is continuous, however, the existence of its residuum is conditioned by the left-continuity of the t-norm. This was an inspiration for Esteva and Godo to introduce MTL as the logic of left-continuous t-norms [EG01]. Analogously to BL, the semantical counterpart of MTL is represented by MTL-algebras and the real unit interval $[0, 1]$ endowed with a left-continuous t-norm and the corresponding residuum is the, so called, *standard MTL-algebra*. MTL is sound and complete with respect to the class of MTL-algebras, as well as to the class of standard MTL-algebras which was proven by Jenei and Montagna [JM02b].

Note that MTL-algebras are bounded, commutative, negative (integral) residuated lattices fulfilling the prelinearity axiom

$$(a \to b) \vee (b \to a) = 1 \quad \text{for every} \quad a, b \in S$$

and that BL-algebras are MTL-algebras fulfilling the divisibility axiom

$$a \odot (a \to b) = a \wedge b \quad \text{for every} \quad a, b \in S.$$

2.3 Summary of the attached papers

2.3.1 [PV17] Rees coextensions of finite, negative tomonoids

The paper deals with *one-element Rees coextension* of f. n. tomonoids which are defined the following way. Let $(S; \leq, \odot, 1)$ be a f. n. tomonoid and let $q \in S$. Define a relation $\approx_q$, for every $a, b \in S$, by

$$a \approx_q b \text{ if } a = b \text{ or } a, b \leq q.$$

It can be shown that $\approx_q$ is a tomonoid congruence. Actually, it is a *Rees congruence* [How76] if we see the tomonoid as a totally ordered semigroup. The

0	w	x	y	z	1	
0	w	x	y	z	1	1
0	0	w	w	x	z	z
0	0	0	w	w	y	y
0	0	0	0	w	x	x
0	0	0	0	0	w	w
0	0	0	0	0	0	0

0	x	y	z	1	
0	x	y	z	1	1
0	0	0	x	z	z
0	0	0	0	y	y
0	0	0	0	x	x
0	0	0	0	0	0

0	y	z	1	
0	y	z	1	1
0	0	0	z	z
0	0	0	y	y
0	0	0	0	0

0	z	1	
0	z	1	1
0	0	z	z
0	0	0	0

$\cdots$

Figure 2.2: Obtaining the Rees quotient $S/{\approx_\alpha}$ of a f. n. tomonoid can be accomplished by renaming all occurrences of the atom in the Cayley table to zero and, subsequently, by removing the lowest row and the lowest column in the table.

0	y	z	1	
0	y	z	1	1
0	0	y	z	z
0	0	0	y	y
0	0	0	0	0

0	x	y	z	1	
0	x	y	z	1	1
0	0	0	y	z	z
0	0	0	0	y	y
0	0	0	0	x	x
0	0	0	0	0	0

0	x	y	z	1	
0	x	y	z	1	1
0	0	x	y	z	z
0	0	0	x	y	y
0	0	0	0	x	x
0	0	0	0	0	0

0	x	y	z	1	
0	x	y	z	1	1
0	x	x	y	z	z
0	x	x	x	y	y
0	x	x	x	x	x
0	0	0	0	0	0

Figure 2.3: All the possible one-element coextensions of a f. n. tomonoid.

corresponding quotient $S/{\approx_q}$ is then called the *Rees quotient* of S by q. This process can be intuitively understood as "merging" all the truth values from 0 up to q into a single new element which will become the "new zero" in the resulting f. n. tomonoid. Reversing the process, S is called the *Rees coextension*[2] of $S/{\approx_q}$.

If q is the second lowest element (we call it the *atom* and denote it by α) then, creating the Rees quotient, we merge only the two lowest elements of the f. n. tomonoid. In such a case, S is called the *one-element Rees coextension* (or, shortly, *one-element coextension*) of $S/{\approx_q}$.

When a f. n. tomonoid is given, obtaining the Rees quotients is a rather simple process which is illustrated by Figure 2.2. The question that we ask here

[2]For the sake of consistency we have chosen the notion "coextension" since, according to the terminology of Howie [How76], the notion "extension" has a different meaning. (According to Howie, S is the "extension" of I by S where I is the ideal formed by the downset of q.)

is how to determine all one-element coextensions of a given f. n. tomonoid. Let us remark that while the Rees quotient of a f. n. tomonoid is given uniquely, Rees coextension is not. For example, the f. n. tomonoid depicted in Figure 2.3 has three possible one-element coextensions.

Although a Cayley table is suitable to define a single f. n. tomonoid, it is not suitable to describe all its possible one-element coextensions at once. For this purpose, we define a *tomonoid partition* [PV17, Section 3] which is an equivalence relation defined on the set of pairs of the tomonoid elements; two pairs are equivalent if and only if they are evaluated by the tomonoid operation to the same value; that is

$$(a, b) \sim (c, d) \quad \text{iff} \quad a \odot b = c \odot d,$$

where a, b, c, d are elements of the monoid. Such a structure shows to be practical for our purpose since the procedure determining all one-element coextensions does not determine for every pair of the resulting tomonoid to which value it has to be evaluated but rather states that sets of pairs must be evaluated in the same way without specifying the exact value.

The paper brings two results. The first one [PV17, Section 4] deals with Archimedean one-element coextensions of a given Archimedean f. n. tomonoid. It consists of a set of rules which serve to determine a tomonoid partition which describes all these coextensions.

The second result [PV17, Section 5] does the same but without the requirement of the Archimedean property. The first result is, actually, a special case of the second one. However, the result that concerns Archimedean f. n. tomonoids is stated explicitly since its set of rules is significantly simpler and, furthermore, it always yields a valid tomonoid partition, which is not the case of the more general result.

2.3.2 [PV19] Rees coextensions of finite tomonoids and free pomonoids

This paper, as well as the previous one, studies one-element coextensions of finite tomonoids. However, instead of describing them utilizing tomonoid partitions, it shows that each such coextension corresponds to a downset of a specially created free partially ordered monoid. Recall that a *partially ordered monoid* (a *pomonoid*) is a monoid endowed with a compatible partial order; tomonoids are special cases of pomonoids if the order is total. Let us summarize the main result.

Let $(G; \leq)$ be a partially ordered set. The *free pomonoid* $\mathcal{F}(G)$ on G is the monoid of words $a_1 \dots a_n$ where $n \geq 0$ and $a_1, \dots, a_n \in G$. The product is defined by concatenation of the words and the neutral element is given by the empty word ε. We define a partial order on $\mathcal{F}(G)$ by

$$a_1 \dots a_n \leq b_1 \dots b_n \text{ if } n = m \text{ and } a_1 \leq b_1, \dots, a_n \leq b_n.$$

Let $(S; \leq, \odot, 1)$ be a f. n. tomonoid of which we intend to describe the one-element coextensions. Let $\dot{0}$ be the smallest element of S and let ε_l and ε_r be its, not necessarily distinct, non-zero idempotent elements (recall that 1 is an idempotent element of any f. n. tomonoid). We define

$$\begin{aligned} S^\star &= S \setminus \{\dot{0}\}, \\ \bar{S} &= S^\star \mathbin{\dot{\cup}} \{0, \alpha\}, \end{aligned}$$

where 0 and α are new elements and we extend the total order on $S^\star$ to $\bar{S}$, letting $0 \leq \alpha \leq a$ for any $a \in S^\star$. Further, we define

$$\begin{aligned} \mathcal{N} \;=\; & \Big\{(a,b) \in \bar{S}^2 \mid a, b > \alpha \text{ and } ab = \dot{0} \text{ in } S\Big\} \\ & \cup\; \big\{0, \alpha\big\} \times \bar{S}^2 \;\cup\; \bar{S}^2 \times \big\{0, \alpha\big\}, \end{aligned}$$

i.e., $\mathcal{N}$ is the set of all the pairs for which the value of the monoid product (which is either 0 or α) needs to be defined. By $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$ we denote the free pomonoid over the chain $\bar{S}$, subject to the conditions [PV19, Definitions 3.3 and 4.1]:

(a) $a\,b = c$ for any $a, b, c \in S^\star$ fulfilling this equation in S,

(b) $\varepsilon = 1$,

(c) $a\,b \leq \alpha$ for any $(a, b) \in \mathcal{N}$,

(d) $0\,a = a\,0 = 0$ for any $a \in \bar{S}$,

(e) $a\,b\,c = b\,c$ for any $(b, c) \in \mathcal{N}$ and $a \geq \varepsilon_l$,

$a\,b\,c = a\,b$ for any $(a, b) \in \mathcal{N}$ and $c \geq \varepsilon_r$,

(f) $a\,b\,c = 0$ for any $(b, c) \in \mathcal{N}$ and $a < \varepsilon_l$,

$a\,b\,c = 0$ for any $(a, b) \in \mathcal{N}$ and $c < \varepsilon_r$.

We call $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$ the *free one-element Rees coextension of* S *w.r.t.* $(\varepsilon_l, \varepsilon_r)$, or simply the *free one-element* $(\varepsilon_l, \varepsilon_r)$*-coextension of* S.

It can be shown [PV19, Proposition 4.4] that $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$ is a finite structure which can be represented as the union of two intervals; one is a chain consisting of the elements of $S^\star$ and the second is a partially ordered set with the bottom element 0 and the top element α.

As the main result, every one-element coextension of S can be identified with a quotient of $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$ given by a downset that contains 0 but does not contain α [PV19, Propositions 3.4, 3.6, and 4.2].

2.3.3 [PV14] Algorithm to generate the Archimedean, finite, negative tomonoids

The paper gives a detailed description of the algorithm that produces all Archimedean one-element coextensions of a given Archimedean f. n. tomonoid. It has been published as the first one in the row and the terminology slightly differs from the rest.

2.3.4 [PV16] Algorithm to generate finite, negative totally ordered monoids

The paper, similarly to the previous one, gives a detailed description of the algorithm, however, dealing with general f. n. tomonoids. It contains also an example of the run of the algorithm and some suggestions how to implement the crucial methods.

2.4 Comparison with the results by other authors

Let us mention briefly the results that have been done by other authors and that have similar goals, that is, to find all finitely-valued conjunctions of a given size.

The algorithm by De Baets and Mesiar is the first one in the row and the authors have succeed to find all commutative f. n. tomonoids (that is, discrete t-norms) up to the size 14 [BM99, Table 1]. Unfortunately, the paper does not descrieb the algorithm.

The algorithm by Bartušek and Navara [Bar01; BN01a; BN01b; BN02] is based on one-element Rees coextensions, as well, although the authors do not use this notion. In this approach, all existing coextensions of a given commutative f. n. tomonoid are obtained by checking the associativity for every newly defined pair (x, y).

The algorithm by Bělohlávek and Vychodil [BV10] uses a recursive backtracking procedure to test every possible tomonoid multiplication table on associativity.

Since the mentioned algorithms are based on testing the associativity equation on triplets of values, we believe that our algorithm is more effective. Let us also stress that, thanks to the use of the tomonoid partition structure, the results are described in a significantly more compact way.

Finally, we refer to a very recent result by Bejines, Bruteničová, Chasco, Elorza, and Janiš [Bej+20] who gave upper bounds for the numbers of particular finite, negative, lattice-ordered monoids.

2.5 Table with numbers of finite, negative tomonoids

In order to verify the theoretical results, a program has been written in the programming language Python and run on a personal computer; it is downloadable at `https://home.czu.cz/en/petrikm/coextensions`. As one of the outputs, Table 2.1 presents the numbers of the generated f. n. tomonoids. Due to the modest computer resources, only the f. n. tomonoids up to the size 10 have been obtained. Results of such kind have been already presented by other authors [BN02, Table 2][BV10, Table 3][BM99, Table 1][KMP00, Table 7.1]. Our

size	all	Arch.	comm.	Arch. and comm.
1	1	1	1	1
2	1	1	1	1
3	2	1	2	1
4	8	2	6	2
5	44	8	22	6
6	308	44	94	22
7	2 641	333	451	95
8	27 120	3 543	2 386	471
9	332 507	54 954	13 775	2 670
10	5 035 455	1 297 705	86 417	17 387

Table 2.1: Numbers of finite, negative tomonoids according to their sizes. Here "Arch." stands for "Archimedean" and "comm." stands for "commutative".

contribution, in addition, shows also the numbers of non-commutative f. n. tomonoids.

Chapter 3

Dominance relation and Mulholland inequality

This part of the thesis deals with two problems which were open for a rather long time. The first one, dealing with Mulholland inequality, is known from the introduction of this inequality in 1949 [Mul49]. The second one, which deals with the dominance relation, was stated as an open problem in 1983 [SS05]. This chapter intends to give an overview of these problems and of the history of the related contributions.

3.1 Triangular norms

Definition 3.1.1. *A* triangular norm *[AFS06; KMP00] (or, shortly, a* t-norm*) is a binary operation* $*\colon [0,1]\times[0,1]\to[0,1]$ *such that, for every* $x,y,z\in[0,1]$*,*

1. $x*y=y*x$ *(commutativity),*
2. $(x*y)*z=x*(y*z)$ *(associativity),*
3. $x*1=1*x=x$ *(unit element),*
4. *if* $y\le z$ *then* $x*y\le x*z$ *and* $y*x\le z*x$ *(monotonicity).*

T-norms can be seen as special cases of the structures we were dealing with in Chapter 2. Indeed, the real unit interval $[0,1]$ endowed with a binary operation $*$ is a commutative, negative, totally ordered monoid if and only if the operation $*$ is a t-norm.

A t-norm is said to be *continuous* if it is continuous as a two-variable real function. A t-norm $*$ is said to be *Archimedean* if for every $x,y\in\,]0,1[$ such that $x<y$ there is $n\in\mathbb{N}$ such that

$$y_*^{(n)} \;=\; \underbrace{y*y*\ldots*y}_{n\text{-times}} \;<\; x.$$

Note that a continuous t-norm $*$ is Archimedean if and only if $x * x < x$ for every $x \in]0,1[$.

A t-norm is said to be *strict* if it is continuous and if its restriction to $]0,1] \times]0,1]$ is strictly increasing in each variable. A t-norm $*$ is said to be *nilpotent* if it is continuous and if for every $x \in]0,1[$ there is $n \in \mathbb{N}$ such that $x_*^{(n)} = 0$. A continuous Archimedean t-norm is either strict or nilpotent [KMP00, Corollary 3.30]. A t-norm $*$ is continuous and Archimedean if and only if there is a continuous decreasing injection $t\colon [0,1] \to [0,\infty]$ with $t(1) = 0$ such that, for every $x, y \in [0,1]$,

$$x * y \;=\; t^{(-1)}\Big(t(x) + t(y)\Big).$$

Here, $t^{(-1)}$ denotes the *pseudo-inverse* of t defined, in this particular case, by

$$t^{(-1)}(x) \;=\; \begin{cases} t^{-1}(x) & \text{if } x \leq t(0), \\ 0 & \text{otherwise.} \end{cases}$$

The mapping t is called the *additive generator* [KMP00, Section 3.2] of the continuous Archimedean t-norm $*$ and this generator is unique up to a multiplication by a positive real constant. For two t-norms $*_1$ and $*_2$ we write $*_1 \geq *_2$ if

$$x *_1 y \geq x *_2 y \quad \text{for every} \quad x, y \in [0,1]\,.$$

A prototypical example of a strict t-norm is the *product t-norm* $*_P$ given by

$$x *_P y \;=\; x \cdot y$$

for $x, y \in [0,1]$. It can be easily verified that one of its additive generators is the mapping

$$x \;\mapsto\; \begin{cases} -\log x & \text{if } x \in]0,1]\,, \\ \infty & \text{if } x = 0, \end{cases}$$

as well as any positive multiple of it.

A prototypical example of a nilpotent t-norm is the *Łukasiewicz t-norm* $*_L$ given, for $x, y \in [0,1]$, by

$$x *_L y \;=\; \begin{cases} x + y - 1 & \text{if } x + y \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

One of its additive generators is the mapping $x \mapsto 1 - x$.

An example of a t-norm which is continuous but not Archimedean is the *minimum t-norm* $*_M$ given, for $x, y \in [0,1]$, by

$$x *_M y \;=\; \min\{x, y\}.$$

An example of a non-continuous t-norm is the *drastic t-norm* $*_D$ given, for $x, y \in [0,1]$, by

$$x *_D y \;=\; \begin{cases} 0 & \text{if } x, y < 1, \\ \min\{x, y\} & \text{otherwise.} \end{cases}$$

3.2 Dominance relation

Dominance is a binary relation which was originally defined on the set of triangle functions by Tardiff [Tar76, Definition 3.4]. Later, it was naturally adapted for t-norms [Tar84, Definition 2] and even for more general operations [SMB02].

A t-norm $*_1$ is said to *dominate* a t-norm $*_2$ (we denote it by $*_1 \gg *_2$) if, for every $x, y, u, v \in [0,1]$,

$$(x *_2 y) *_1 (u *_2 v) \;\; \geq \;\; (x *_1 u) *_2 (y *_1 v). \tag{Dom}$$

The motivation to study the dominance relation comes from Tardiff who showed that this notion plays a crucial role when constructing Cartesian products of probabilistic metric spaces [Tar76, Theorem 3.5]. Later, this relation found its utilization also when working with t-norm based fuzzy equivalences and partitions [BM98, Theorem 2] and with their refinements [BM98, Theorem 5]. Let us summarize some of its known properties.

- For every two t-norms $*_1$ and $*_2$ we have that $*_1 \gg *_2$ implies $*_1 \geq *_2$. This can be proven simply by setting $y = u = 1$ in (Dom). Hence dominance is an anti-symmetric relation.
- It can be easily observed that $*_M \gg * \gg *_D$ for every t-norm $*$.
- For any t-norm $*$ we have $* \gg *$ which follows from the associativity and commutativity of $*$. Thus dominance is a reflexive relation.

Since the dominance relation on the set of t-norms is anti-symmetric and reflexive, it is natural to ask, whether it is also transitive and thus an order with the least element $*_D$ and the greatest element $*_M$. This question was stated in 1983 as an open problem for general associative binary operations defined on a partially ordered set:

Problem 3.2.1. *[SS05, Problem 12.11.3] Is the relation "dominates" always transitive? If not, under which conditions is it transitive?*

This question was answered by Sherwood who gave the following counterexample:

Example 3.2.2. *[AFS06, Example 4.2.3] Let* $S = \{0, 1, 2\}$ *be linearly ordered by* $0 < 1 < 2$ *and let* F, G, H *be the binary operations on* S *defined by the multiplication tables:*

2	2	2	2
1	1	0	2
0	0	1	2
F	0	1	2

2	2	2	2
1	1	1	2
0	0	1	2
G	0	1	2

2	2	2	2
1	1	2	2
0	0	1	2
H	0	1	2

Then F, G, H are commutative and associative operations on S, having 0 *as a common identity and such that $H \gg G$, $G \gg F$, but $H \not\gg F$, since*

$$\begin{array}{rrcccl} & H(F(1,1),F(1,0)) & = & H(0,1) & = & 1 \\ \textit{while} & F(H(1,1),H(1,0)) & = & F(2,1) & = & 2. \end{array}$$

Later, in 2003, the same problem was stated namely for t-norms:

Problem 3.2.3. *[AFS03, Problem 17] Is the dominance relation transitive, and hence a partial order, on the set of all t-norms? If not, for what subsets is this the case?*

It can be observed that Example 3.2.2 cannot be adapted for t-norms mainly because the operation F is not monotone.

3.2.1 Dominance on continuous t-norms

Concerning the question of the transitivity of the dominance relation, there were done many partial results with positive answers. These results mainly involved selected families of continuous Archimedean t-norms known from the literature [AFS06; KMP00]. There were also some results on non-Archimedean continuous t-norms; particularly, it was proven in 2005 by Saminger-Platz, De Baets, and De Meyer [SBM05] that the relation of dominance is transitive on the class of Mayor-Torrens t-norms and Dubois-Prade t-norms. However, the question whether the dominance relation is transitive on the whole set of t-norms remained open until the result by Sarkoci [Sar08] who gave a negative answer showing that the dominance relation is not transitive on the set of continuous t-norms. This result was revisited [SSB06; Sar14] providing more comprehensible and flexible tools to obtain the counter-examples needed for the proof.

Thus, it was shown that the dominance relation is not transitive on continuous t-norms. However, the mentioned counter-examples did not involve Archimedean t-norms. Therefore, the question whether, or not, the dominance relation is transitive on the set of continuous Archimedean t-norms remained open; this was also stated as a final remark in the paper by Sarkoci [Sar08, page 207].

3.2.2 Dominance on families of continuous Archimedean t-norms

In the literature [AFS06, Section 2.6][KMP00, Chapter 4], many examples of t-norms are known in the form of single-parametric families; see Table 3.1. T-norms that belong to a particular family are usually described by their additive generators given by expressions which involve a parameter taking its values from a real interval. Many of these families have emerged as solutions of functional equations and inequalities or answers to open questions. For example,

	name	$t_\lambda(x) =$	$\lambda \in$	type	$*_\alpha \gg *_\beta \Leftrightarrow$	transitive?
	Sugeno-Weber	$1 - \frac{\log(1+\lambda x)}{\log(1+\lambda)}$ $1 - x$ $-\log x$	$]-1, 0[\cup]0, \infty[$ $\{0\}$ $\{\infty\}$	nilpotent nilpotent strict	$\left(\alpha \leq \beta \text{ and } \alpha \leq \left(\frac{1-3\sqrt{\beta}}{3-\sqrt{\beta}}\right)^2\right)$ or $\alpha = \beta$	yes [KPS11]
1.	Schweizer-Sklar	$\frac{x^{-\lambda}-1}{\lambda}$ $-\log x$ $\frac{x^{-\lambda}-1}{\lambda}$	$]-\infty, 0[$ $\{0\}$ $]0, \infty[$	nilpotent strict strict	$\alpha \geq \beta$	yes [She84]
2.	Yager	$(1 - x)^\lambda$	$]0, \infty[$	nilpotent	$\alpha \geq \beta$	yes [KMP00]
3.	Hamacher	$\frac{1-x}{x}$ $\log \frac{\lambda+(1-\lambda)x}{x}$	$\{0\}$ $]0, \infty[$	strict strict	$\alpha = 0,\ \alpha = \beta,$ or $\beta = \infty$	yes [Sar05]
4.	Aczél-Alsina	$(-\log x)^\lambda$	$]0, \infty[$	strict	$\alpha \geq \beta$	yes [KMP00]
5.	Frank	$-\log x$ $-\log \frac{\lambda^x-1}{\lambda-1}$	$\{1\}$ $]0, 1[\cup]1, \infty[$	strict strict	$\alpha = 0,\ \alpha = \beta,$ or $\beta = \infty$	yes [Sar05]
6.		$-\log\left(1 - (1 - x)^\lambda\right)$	$]0, \infty[$	strict	?	?
7.		$(\lambda - 1)\log(\lambda + x - \lambda x)$	$[0, \infty[$	nilpotent	?	?
8.		$\frac{1-x}{1+x(\lambda-1)}$ $\frac{1-x}{x}$	$]0, \infty[$ $\{\infty\}$	nilpotent strict	$\alpha \geq \beta$	yes [Sam09]
9.		$\log(1 - \lambda \log x)$ $-\log x$	$]0, \infty[$ $\{\infty\}$	strict strict	$\alpha = \infty,\ \alpha = \beta$ or $\beta = 0$	yes [Sam09]
10.		$\log(2x^{-\lambda} - 1)$	$]0, \infty[$	strict	?	?
11.		$\log(2 - x^\lambda)$	$]0, \infty[$	nilpotent	?	?
12.	Dombi	$\left(\frac{1-x}{x}\right)^\lambda$	$]0, \infty[$	strict	$\alpha \geq \beta$	yes [KMP00]
13.		$\arccos x^\lambda$	$]0, \infty[$	nilpotent	?	?
14.		$\lambda \arctan(1 - x^\lambda)$	$\mathbb{R} \setminus \{0\}$	nilpotent	?	?
15.		$\log(1 - \log x)$ $(1 - \log x)^\lambda - 1$	$\{0\}$ $]0, \infty[$	strict strict	$\alpha \geq \beta$	yes [Sam09]
16.		$(x^{-\frac{1}{\lambda}} - 1)^\lambda$	$]0, \infty[$	strict	?	?
17.		$\left(1 - x^{\frac{1}{\lambda}}\right)^\lambda$	$]0, \infty[$	nilpotent	?	?
18.		$(\frac{\lambda}{x} + 1)(1 - x)$	$[0, \infty[$	strict	?	?
19.		$(-\log(1 - x))^{-\alpha}$	$]0, \infty[$	strict	$\alpha \geq \beta$	yes [Pet18]
20.		$\log \frac{(1+x)^{-\lambda}-1}{2^{-\lambda}-1}$	$\mathbb{R} \setminus \{0\}$	strict	?	?
21.		$e^{\frac{\lambda}{x-1}}$	$]0, \infty[$	nilpotent	$\alpha \geq \beta$	yes (not published)
22.		$\frac{1-x}{x}$ $e^{\frac{\lambda}{x}} - e^\lambda$	$\{0\}$ $]0, \infty[$	strict strict	$\alpha \geq \beta$	yes [Sam09]
23.		$-\log x$ $e^{x^{-\lambda}} - e$	$\{0\}$ $]0, \infty[$	strict strict	$\alpha \geq \beta$	yes [Sam09]
24.		$1 - \left(1 - (1 - x)^\lambda\right)^{\frac{1}{\lambda}}$	$]0, \infty[$	nilpotent	?	?
25.		$\arcsin(1 - x^\lambda)$	$]0, \infty[$	nilpotent	?	?

Table 3.1: **Parametric families of continuous Archimedean t-norms.** The numbering in the 1st column is according to Alsina, Frank, and Schweizer [AFS06, Table 2.6]; the names in the 2nd column are according to Klement, Mesiar, and Pap [KMP00, Chapter 4]; 3rd column: additive generators of the t-norms; 4th column: ranges of the parameter; 5th column describes whether the t-norms within the given parameter range are strict or nilpotent; 6th column: condition under which a t-norm of the family given by the parameter α dominates the t-norm given by the parameter β; 7th column: answer whether the dominance relation is transitive on the given family. (The table does not contain the limit cases that are not continuous Archimedean t-norms; see the cited literature.)

Hamacher t-norms are the only t-norms that are expressible as a quotient of two polynomials.

When we restrict our attention to the families of continuous Archimedean t-norms, the history of the results, that concerns with the transitivity of the dominance relation, is rather long and fruitful. Interestingly enough, all the answers are positive from which one may get a feeling that the relation shall remain transitive, if not on the whole class of continuous t-norms, then at least on the whole class of continuous Archimedean t-norms.

In 1984, Sherwood gave the historically first result proving that the dominance relation is transitive on Family 1 (Schweizer-Sklar t-norms) [She84].

In 2000, it was stated as a remark [KMP00, Example 6.17.v] that transitivity of the dominance can be proven easily in the case of the families where the additive generators do not differ up to a positive power given by the parameter. In such a case, if α and β are the parameters of two t-norms from a single family, $\alpha \geq \beta$ implies $*_\alpha \gg *_\beta$; this is the case of Family 2 (Yager t-norms), Family 4 (Aczél-Alsina t-norms), and Family 12 (Dombi t-norms). It follows from this fact that the dominance relation is transitive when we restrict to these families.

In 2005, Sarkoci showed that the dominance relation is transitive on Family 3 (Hamacher t-norms) and on Family 5 (Frank t-norms) [Sar05].

In 2009, Saminger-Platz proved transitivity of the dominance relation on Families 8, 9, 15, 22, and 23 [Sam09].

In 2011, Kauers, Pillwein, and Saminger-Platz have demonstrated that the dominance is transitive on the family of Sugeno-Weber t-norms [KPS11].

In 2018, a characterization of the dominance on Family 19 was given [Pet18, Proposition 7.1]; the transitivity follows from it.

On the other hand, we present here, as our result, that the dominance relation is transitive neither on the set of strict t-norms [Pet18] nor on the set of nilpotent t-norms [Pet20].

3.3 Mulholland inequality

Mulholland inequality was introduced by Mulholland [Mul49] as a generalization of Minkowski inequality which establishes the triangular inequality of L^p-norms and has the form

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{p}} \quad \leq \quad \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}$$

where $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ and $p \geq 1$. In his work, Mulholland replaced the power function by an arbitrary increasing bijection of $[0, \infty]$ and asked, under which conditions the inequality will be preserved. Hence an increasing bijection $f\colon [0, \infty] \to [0, \infty]$ is said to *solve Mulholland inequality* if

$$f^{-1}\left(\sum_{i=1}^{n} f(x_i + y_i)\right) \quad \leq \quad f^{-1}\left(\sum_{i=1}^{n} f(x_i)\right) + f^{-1}\left(\sum_{i=1}^{n} f(y_i)\right)$$

for every $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in [0, \infty]^n$.

Mulholland also provided a condition which assures that a given increasing bijection f of $[0, \infty]$ solves this inequality (for a proof, see also Kuczma [Kuc09, Theorem VIII.8.1]):

Theorem 3.3.1. *[Mul49, Theorem 1] Let $f\colon [0,\infty] \to [0,\infty]$ be an increasing bijection. If both f and $\log \circ f \circ \exp$ are convex then f solves Mulholland inequality.*

We refer to the assumptions of this theorem as the *Mulholland's condition* and, following the terminology of Matkowski [Mat92], we say that f is *geometrically convex* if $\log \circ f \circ \exp$ is convex.

The question, whether every function that solves Mulholland inequality also satisfies the Mulholland's condition (i.e., whether the Mulholland's condition characterizes the whole set of the solutions of the inequality), was open for a rather long time and the question of a structural characterization of the set of the solutions of Mulholland inequality has remained open until today. There were, however, attempts to find the answers.

In 1984, Tardiff introduced a different sufficient condition:

Theorem 3.3.2. *[Tar84, Theorem 1] Let $f\colon [0,\infty] \to [0,\infty]$ be a continuous, strictly increasing, everywhere differentiable convex function with $f(0) = 0$. If $\log \circ f' \circ \exp$ is a convex function, i.e. if f' is geometrically convex, then f solves Mulholland inequality.*

In 1993, Matkowski and Świątkowski have provided a necessary condition on functions to solve Mulholland inequality:

Theorem 3.3.3. *[MŚ93, Theorem 2] If an increasing bijection $f\colon [0,\infty] \to [0,\infty]$ solves Mulholland inequality then it is necessarily convex.*

In 1999, Sklar has posed a question whether the Tardiff's condition is equivalent to the Mulholland's one [Skl00].

This question has been answered in 2002 by Jarczyk and Matkowski who stated that the Tardiff's condition actually implies the Mulholland's one [JM02a]. This statement has been proven alternatively also by Baricz [Bar10].

Finally, it was shown that the answer to the question is negative and that there are functions that solve Mulholland inequality but do not satisfy the Mulholland's condition [Pet15].

3.4 Mulholland inequality and dominance

In 1984, Tardiff showed that Mulholland inequality is in a close correspondence with dominance of strict t-norms.

Theorem 3.4.1. *[Tar84, Theorem 3] Let $*_1$ and $*_2$ be two strict t-norms defined by their additive generators t_1 and t_2. Then $*_1$ dominates $*_2$ if and only if*

$f = t_1 \circ t_2^{-1}$ solves Mulholland inequality, that is, if and only if f satisfies:

$$\forall x, y, u, v \in [0, \infty] :$$
$$f^{-1}\Big(f(x+u) + f(y+v)\Big) \le f^{-1}\Big(f(x) + f(y)\Big) + f^{-1}\Big(f(u) + f(v)\Big).$$

This correspondence has been enlarged in 2008 by Saminger-Platz, De Baets, and De Meyer to the set of all continuous Archimedean t-norms introducing the notion of the *generalized Mulholland inequality* [SBM08]. The results of this thesis benefit greatly from these two achievements.

3.5 Summary of the attached papers

3.5.1 [Pet15] New solutions to Mulholland inequality

This paper brings the following two results on Mulholland inequality:

Definition 3.5.1. *[Pet15, Definition 3.4] An increasing bijection $f\colon [0, \infty] \to [0, \infty]$ is said to be k-*subscalable *for some given $k \in [0, \infty]$ if*

$$\forall a, b \in \,]0, \infty[\, , x \in [0, 1]\, , b - a \ge k\colon \quad \frac{f(ax)}{f(a)} \;\ge\; \frac{f(bx)}{f(b)}.$$

Theorem 3.5.2. *[Pet15, Theorem 4.5] Let $f\colon [0, \infty] \to [0, \infty]$ be an increasing bijection which is, for some $k \in [0, \infty]$,*

- *convex,*
- *k-subscalable,*
- *linear on $[0, k]$.*

Then f solves Mulholland inequality.

Theorem 3.5.3. *[Pet15, Theorem 6.3] The set of solutions to Mulholland inequality is not closed with respect to their compositions.*

The result described by Theorem 3.5.2 gives a sufficient condition in the same manner as the result of Mulholland (Theorem 3.3.1), however, it delimits a strictly larger set of solutions. The new condition covers all the solutions delimited by the Mulholland's condition and, moreover, there are functions that do not satisfy the Mulholland's condition while still solving Mulholland inequality [Pet15, Example 5.1].

The result described by Theorem 3.5.3 is important in the following paper [Pet18] where it serves to prove that the dominance relation is not transitive on the set of strict t-norms.

3.5.2 [Pet18] Dominance on strict triangular norms and Mulholland inequality

This paper deals with the set of strict t-norms and with the question whether the relation of dominance is transitive on this set. It brings an overview of the results that have been reached in this field, it broadens the set of counter-examples from the previous paper [Pet15] and, as the main results, it states:

Theorem 3.5.4. *[Pet18, Theorem 10.3] The dominance relation is in general not transitive on the set of strict triangular norms.*

3.5.3 [Pet20] Dominance on continuous Archimedean triangular norms and generalized Mulholland inequality

This paper deals with the generalization of Mulholland inequality introduced in 2008 by Saminger-Platz, De Baets, and De Meyer [SBM08] which can be formulated in the following way (compare with Theorem 3.4.1):

Theorem 3.5.5. *[SBM08, Theorem 1] Let $*_1$ and $*_2$ be two continuous Archimedean t-norms defined respectively by their additive generators t_1 and t_2, for every $x, y \in [0,1]$, as*

$$\begin{aligned} x *_1 y &= t_1^{(-1)}\big(t_1(x) + t_1(y)\big), \\ x *_2 y &= t_2^{(-1)}\big(t_2(x) + t_2(y)\big). \end{aligned}$$

Then $*_1$ dominates $*_2$ if and only if the functions $f = t_1 \circ t_2^{(-1)}$ and $f^{(-1)} = t_2 \circ t_1^{(-1)}$ satisfy

$$\begin{aligned} &\forall x, y, u, v \in [0, t_2(0)] : \\ &f^{(-1)}\Big(f(x+y) + f(u+v)\Big) \leq f^{(-1)}\Big(f(x) + f(u)\Big) + f^{(-1)}\Big(f(y) + f(v)\Big). \end{aligned}$$

If the latter inequality holds we say that f *solves the generalized Mulholland inequality*. Analogously to the case of strict t-norms and Mulholland inequality, the generalized Mulholland inequality gives a characterization of the dominance on all continuous Archimedean t-norms which involve both strict and nilpotent t-norms. The result presented by Theorem 3.4.1 can be seen as a special case of the result presented by Theorem 3.5.5.

The mentioned paper [SBM08] has also introduced the following result (compare with Theorem 3.3.1):

Theorem 3.5.6. *[SBM08, Theorem 6] Assume a function $f\colon [0,\infty] \to [0,\infty]$ and fixed values $d, e \in \,]0,\infty]$ such that:*

1. *$f(0) = 0$ and $f(d) = e$,*
2. *f is continuous and strictly increasing on $[0,d]$,*

3. $f(x) \geq e$ *for* $x \geq d$,

4. *convex on* $]0, d[$,

5. *geometrically convex on* $]0, d[$.

Assume, further, a function $f^{(-1)}\colon [0, \infty] \to [0, \infty]$ *defined by*

$$f^{(-1)}\colon \quad x \mapsto \begin{cases} f^{-1}(x) & \text{if } x \in [0, e], \\ d & \text{otherwise.} \end{cases}$$

Then f *and* $f^{(-1)}$ *solve the generalized Mulholland inequality.*

The present paper broadens the result of Theorem 3.5.6 analogously to Theorem 3.5.2:

Theorem 3.5.7. *[Pet20, Theorem 5.6] Assume a function* $f\colon [0, \infty] \to [0, \infty]$ *and fixed values* $d, e \in]0, \infty]$ *and* $k \in [0, d]$ *such that:*

1. $f(0) = 0$ *and* $f(d) = e$,

2. f *is continuous and strictly increasing on* $[0, d]$,

3. $f(x) \geq e$ *for* $x \geq d$,

4. *convex on* $[0, d]$,

5. *for every* $x \in [0, 1]$ *and for every* $a, b \in [0, d]$ *such that* $b - a \geq k$*:*

$$\frac{f(ax)}{f(a)} \geq \frac{f(bx)}{f(b)},$$

6. *linear on* $[0, k]$.

Assume, further, a function $f^{(-1)}\colon [0, \infty] \to [0, \infty]$ *defined by*

$$f^{(-1)}\colon \quad x \mapsto \begin{cases} f^{-1}(x) & \text{if } x \in [0, e], \\ d & \text{otherwise.} \end{cases}$$

Then f *and* f^{-1} *solve the generalized Mulholland inequality.*

The assumptions of Theorem 3.5.7 delimit a strictly larger set of solutions compared to Theorem 3.5.6. This is proven by introducing a parametric family of counter-examples [Pet20, Example 5.8] which, furthermore, help to prove that the set of solutions of the generalized Mulholland inequality is not closed with respect to their compositions [Pet20, Lemma 7.2]. The paper is concluded by the following result:

Theorem 3.5.8. *[Pet20, Theorem 8.1] The relation of dominance is not transitive on the set of nilpotent t-norms.*

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Appendix A

Attached papers

Rees coextensions of finite, negative tomonoids

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Abstract

A totally ordered monoid, or tomonoid for short, is a monoid endowed with a compatible total order. We deal in this article with tomonoids that are finite and negative, where negativity means that the monoidal identity is the top element. Examples can be found, for instance, in the context of finite-valued fuzzy logic. By a Rees coextension of a negative tomonoid S, we mean a negative tomonoid T such that a Rees quotient of T is isomorphic to S. We characterize the set of all those Rees coextensions of a finite, negative tomonoid that are by one element larger. We thereby define a method of generating all such tomonoids in a stepwise fashion. Our description relies on the level-set representation of tomonoids, which allows us to identify the structures in question with partitions of a certain type.

Keywords: Totally ordered monoids, tomonoid partition, Rees coextension.

1 Introduction

A totally ordered monoid, or tomonoid as we say shortly [7], is a monoid $(S;\odot,1)$ endowed with a total order $\leqslant$ that is compatible with the monoidal operation. The compatibility means that, for any $a,b,c \in S$, $a \leqslant b$ implies $a \odot c \leqslant b \odot c$ and $c \odot a \leqslant c \odot b$. Tomonoids occur in a number of different contexts. For instance, the term orders used in computational mathematics in connection with Gröbner bases can be identified with positive orders on $\mathbb{N}^n$ that are compatible with the addition. Hence term orders correspond to total orders making $\mathbb{N}^n$ into a tomonoid [6]. Numerous further examples can be found in the context of many-valued logics. In particular, fuzzy logics are based on an extended set of truth values and this set is usually totally ordered [10]. The conjunction in fuzzy logic is moreover commonly interpreted by an associative operation with which the total order is compatible. Hence we are naturally led to tomonoids.

The probably most familiar type of a tomonoid in fuzzy logic is the real unit interval endowed with the natural order, a left-continuous triangular norm, and the constant 1 [10, 18]. Note that this tomonoid is negative and commutative. Negativity means that the monoidal identity is the top element. This condition will be assumed throughout the present article as well. In contrast, our results are developed without the assumption of commutativity. We will see, however, that the adaptation to the commutative case does not cause difficulties.

A remarkable effort has been spent in the last decade on the problem of describing tomonoids in a systematic way, often under the assumption of negativity and mostly under the assumption

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of commutativity; we may, e.g., refer to [7, 11, 12]. In particular, the aforementioned triangular norms have been an intensive research field; see, e.g., [20, 24]. Although the examination of negative, commutative tomonoids has made from an algebraic perspective a considerable progress, a comprehensive classification of these structures has not yet been found.

Given the complexity of the problem, it seems to be reasonable to consider separately the finite case. This is what we do in this article. Immediate simplifications cannot be expected from this further restriction. Nothing seems to indicate that tomonoids are easier to describe under the finiteness assumption. Quite a few papers are devoted to finite tomonoids; see, for instance, [13, 25].

The starting point of the present article is the following simple observation. Let $(S; \leqslant, \odot, 1)$ be a finite negative tomonoid. Let 0 be the bottom element of S and let α be the atom of S, i.e., the smallest element apart from 0. Then the identification of α with the bottom element of S is a tomonoid congruence. With regard to the semigroup reduct, this is the Rees congruence by the ideal $\{0, \alpha\}$. The quotient is by one element smaller than S and forming the same kind of quotient repeatedly, we get a sequence of tomonoids eventually leading to the tomonoid that consists of the single element 1.

Seen from the other direction, each n-element negative tomonoid is the last entry in a sequence of n tomonoids the first of which is the one-element tomonoid and each other leads to its predecessor by the identification of its smallest two elements. Proceeding one step forward in this sequence means replacing a finite, negative tomonoid by a tomonoid that is by one element larger and whose Rees quotient by the ideal consisting of the bottom element and the atom is the original one. In this article, we specify all possibilities of enlarging the tomonoid in this way. Therefore, we propose a way of generating systematically all finite, negative tomonoids.

In accordance with the unordered case [9], we call a tomonoid whose quotient is a tomonoid S a *coextension* of S. We deal with Rees congruences, which are understood as usual but restricted to the case that the ideal is a downward closed set. Accordingly, we have chosen the notion of a *Rees coextension* to name our construction. Explicitly, a Rees coextension of a negative tomonoid S is a negative tomonoid T such that S is isomorphic to a Rees quotient of T. We focus on the case that the cardinality of the coextension is just by 1 larger and speak about *one-element* (Rees) coextensions then.

Coextensions of semigroups have been explored under various conditions, which are usually, however, in the present context quite special. For instance, coextensions of regular semigroups have been considered in [19]. In contrast, our present work is closely related to the theory of extensions of semigroups. A semigroup T is an *(ideal) extension* of a semigroup I by a semigroup S if I is an ideal of T and S is the Rees quotient of T by I. Semigroup extensions were first investigated in [3]; see also [4, Section 4.4] or [21, Chapter 3]. It is, moreover-straightforward to adapt the notion to the ordered case [15, 17]. What we study in this article are actually special ideal extensions. The latter terminology just reflects a different viewpoint: the extended and extending semigroups are denoted in the opposite way. That is, we study ideal extensions of a two-element semigroup by a finite negative tomonoid.

Ideal extensions of ordered semigroups were first studied by A. J. Hulin. In [15], Clifford's technique of constructing extensions by means of partial homomorphisms was adapted to the ordered case. However, if the extended semigroup does not possess an identity the method does not necessarily cover all possible extensions. A more general method, which is applicable to weakly reductive semigroups, is due to Clifford as well. The presumed condition, however, although found 'relatively mild' in [4], turns out to be quite special in the present context again. The ordered case was investigated along these lines by N. Kehayopulu and M. Tsingelis in [17].

Let us have a closer look at the method that we are going to discuss here. The construction requires the duplication of the bottom element; the latter is replaced a new bottom element and a new atom.

Then, the multiplication needs to be revised in all those cases that lead, in the original tomonoid, to the bottom element. Trying out some simple examples, we soon observe that this problem is more difficult than it looks. Because of the mutual interdependencies, to decide which pairs of elements multiply to the (new) bottom element and which pairs multiply to the atom is not straightforward.

To bring transparency into this problem, a framework in which the structures under consideration become manageable is desirable. The crucial property with which we have to cope is associativity. This property is fundamental in mathematics and numerous approaches exist to shed light on it. Let us enumerate some ideas that are applicable in our context.

- We can lead back the associativity of a monoid to the probably most common situation where this property arises: the addition of natural numbers. Naturally, this approach is limited to the commutative case. In fact, any commutative monoid, provided it is finitely generated, is a quotient of $\mathbb{N}^n$, where n is the number of generators. A description of tomonoids on this basis has been proposed, e.g. in [25].
- There is another situation in which associativity arises naturally: the composition of functions. In fact, we may represent any monoid as a monoid of mappings under composition. Namely, we may use the regular representation; see, e.g. [4]. In the presence of commutativity, we are led to a monoid of pairwise commuting, order-preserving mappings. The associativity is then accounted for by the fact that any two mappings commute. This idea is applied to tomonoids in [24].
- A third and once again totally different approach is inspired by the field of web geometry; see, e.g. [1, 2]. Here, a tomonoid is represented by its level sets. Associativity then corresponds to the so-called Reidemeister condition. This approach has been applied to triangular norms in [22, 23].

For our aims, any of these three approaches is worth being considered. Each of them has its benefits and drawbacks. The present article is devoted to the third approach.

We may represent any two-place function by means of its level sets. The idea is simple and means in our context the following. Let $(S;\leqslant)$ be a chain, i.e., a totally ordered set. Let $\odot\colon S\times S\to S$ be a binary operation on S, and consider the following equivalence relation on the set $S^2=S\times S$:

$$(a,b)\sim(c,d) \text{ if } a\odot b=c\odot d.$$

Then $\sim$ partitions S^2 into the subsets of pairs that are assigned equal values. To recover $\odot$ from $\sim$, all we need to know is which subset is associated with which value of S. But if we know that $\odot$ behaves neutrally with respect to a designated element 1 of S, this is clear: each class then contains exactly one element of the form $(1,a)$ and is associated with a. Consequently, a tomonoid $(S;\leqslant,\odot,1)$ can be identified with a chain S together with a certain partition on S^2 and the designated element 1.

To determine the one-element Rees coextensions means, in this picture, to replace the partition on S^2 by a suitable partition on $\bar{S}^2$, where $\bar{S}$ is the chain arising from S by a duplication of the bottom element. Thus, our topic is to specify a procedure leading exactly to those partitions of the enlarged set $\bar{S}^2$ that correspond to the one-element Rees coextensions.

We proceed as follows. In Section 2, we specify the structures under consideration and the particular type of quotients that we employ. In Section 3, we put up our basic framework, which relies on the level-set representation of binary operations. In Section 4, we start describing how the Rees coextensions of negative tomonoids can be determined. In this first step, we restrict to the case that the tomonoid is Archimedean. The general case is discussed in the subsequent Section 5. Section 6 contains some concluding remarks.

2 Totally ordered monoids

We investigate in this article the following structures.

Definition 2.1
A *totally ordered monoid*, or a *tomonoid* for short, is a structure $(S; \leqslant, \odot, 1)$ such that $(S; \odot, 1)$ is a monoid, $(S; \leqslant)$ is a chain, and $\leqslant$ is compatible with $\odot$, i.e., for any $a, b, c \in S$, $a \leqslant b$ implies $a \odot c \leqslant b \odot c$ and $c \odot a \leqslant c \odot b$. We call a tomonoid $(S; \leqslant, \odot, 1)$ *negative* if 1 is the top element, and we call S *commutative* if so is $\odot$.

We are exclusively interested in tomonoids that are finite and negative. We abbreviate these properties by 'f.n.'. We note that, in the context of residuated lattices, the notion 'integral' is commonly used instead of 'negative'. We further note that, in contrast to [7], we do not assume a tomonoid to be commutative and in fact we proceed without this assumption. However, the commutative case is without doubt important and will be considered as well.

The smallest tomonoid is the one that consists of the monoidal identity 1 alone, called the *trivial* tomonoid. Tomonoids with at least two elements are called *non-trivial*.

Congruences of tomonoids are defined as follows; cf. [7]. Here, a subset C of a poset is called *convex* if $a, c \in C$ and $a \leqslant b \leqslant c$ imply $b \in C$.

Definition 2.2
Let $(S; \leqslant, \odot, 1)$ be a tomonoid. A *tomonoid congruence* on S is an equivalence relation $\approx$ on S such that (i) $\approx$ is a congruence of S as a monoid and (ii) each $\approx$-class is convex. On the quotient $\langle S \rangle_{\approx}$, we then denote the operation induced by $\odot$ again by $\odot$ and, for $a, b \in S$, we let $\langle a \rangle_{\approx} \leqslant \langle b \rangle_{\approx}$ if $a \approx b$ or $a < b$.

If $\approx$ is a tomonoid congruence on a tomonoid S, we easily check that $(\langle S \rangle_{\approx}; \leqslant, \odot, \langle 1 \rangle_{\approx})$ is a tomonoid again, called the *quotient* of S by $\approx$. It is clear that the formation of a quotient preserves the properties of finiteness, negativity, and commutativity, respectively.

In [24], congruences of negative, commutative tomonoids are discussed that are induced by filters. Provided that a tomonoid is residuated, these congruences are precisely those that also preserve the residual implication. Here, we consider something different. A particularly simple type of congruences is the following; see, e.g. [7].

Lemma 2.3
Let $(S; \leqslant, \odot, 1)$ be a negative tomonoid and let $q \in S$. For $a, b \in S$, let $a \approx_q b$ if $a = b$ or $a, b \leqslant q$. Then $\approx_q$ is a tomonoid congruence.

Note that, because of the negativity of S, the set $\{a \in S : a \leqslant q\}$ in Lemma 2.3 is a semigroup ideal of S. This is why the indicated congruence is actually a Rees congruence of S, seen as a semigroup; see, e.g. [14].

For a finite chain S, let 0 denote the bottom element. We write $S^\star = S \setminus \{0\}$. Furthermore, we call the second smallest element of S, if it exists, the *atom* of S. The symbol α will be used in the sequel to denote it.

Definition 2.4
Let $(S; \leqslant, \odot, 1)$ be a f.n. tomonoid and let $q \in S$. Then we call $\approx_q$, as defined in Lemma 2.3, the *Rees congruence* by q. We denote the quotient by S/q and call it the *Rees quotient* of S by q.

Moreover, we call S a *Rees coextension* of S/q. We call S a *one-element* Rees coextension, or simply a *one-element coextension*, if S is non-trivial and q is the atom of S.

The problem that we rise in this article is: How can we determine all one-element coextensions of a f.n. tomonoid? Having defined a suitable such method, we will obviously be in the position to determine, starting from the trivial tomonoid, successively all f.n. tomonoids.

3 Tomonoid partitions

Let $\diamond$ be a binary operation on a set A. Then $\diamond$ gives rise to a partition of $A \times A$: the blocks of the partition are the subsets of all those pairs that are mapped by $\diamond$ to the same value. This partition, together with the assignment that associates with each block the respective element of A, specifies $\diamond$ uniquely.

The representation of tomonoids that we will employ in the sequel is based on this simple idea. From a geometric point of view, we will deal with a representation of tomonoids that comes along with two dimensions—in contrast to the commonly used graph of binary operations. For the case of triangular norms, the idea has been proposed in [22] and in this framework an open problem on the convex combinations of t-norms was solved [23]. An adaptation to the present context causes no difficulties.

We deal in the sequel with partitionings of posets. Let us fix some terminology. Let $(M;\leqslant)$ be a poset and let $\sim$ be an equivalence relation on M. Then $\leqslant$ induces the preorder $\leqslant_\sim$ on the set $\langle M\rangle_\sim$ of $\sim$-classes, where, for $a,b \in M$,

$$\langle a\rangle_\sim \leqslant_\sim \langle b\rangle_\sim \quad \text{if there are } c_0,\ldots,c_k \in M \text{ such that} \\ a \sim c_0 \leqslant c_1 \sim c_2 \leqslant \ldots \leqslant c_k \sim b.$$

We say that $\sim$ is *regular* for $\leqslant$ if the following condition, sometimes called the closed chain condition, is fulfilled: For any $c_0,\ldots,c_k \in M$ such that $c_0 \sim c_1 \leqslant c_2 \sim c_3 \leqslant \ldots \leqslant c_k \sim c_0$, we have $c_0 \sim \ldots \sim c_k$. In this case, $\leqslant_\sim$ is obviously antisymmetric and hence a partial order.

In other words, if an equivalence relation $\sim$ on a poset $(M;\leqslant)$ is regular for $\leqslant$, then $(\langle M\rangle_\sim;\leqslant_\sim)$ is a poset again and the natural surjection $a \mapsto \langle a\rangle_\sim$ is order-preserving. Our terminology originates from [5, Def. 1.7]. The paper [5] in fact contains a detailed discussion of partitions of posets and their relationship to the partial order.

We now turn to our actual objects of interest.

DEFINITION 3.1
Let $(S;\leqslant,\odot,1)$ be a tomonoid. For two pairs $(a,b),(c,d) \in S^2$ we define

$$(a,b) \sim (c,d) \quad \text{if} \quad a \odot b = c \odot d,$$

and we call $\sim$ the *level equivalence* of S.

The level equivalence of a tomonoid S defines a certain partition of S^2. We define a corresponding relational structure.

DEFINITION 3.2
Let $(S;\leqslant)$ be a chain and let $1 \in S$. By $\trianglelefteqslant$, we denote the componentwise order on S^2, that is, we put

$$(a,b) \trianglelefteqslant (c,d) \quad \text{if } a \leqslant c \text{ and } b \leqslant d$$

for $a,b,c,d \in S$. Moreover, let $\sim$ be an equivalence relation on S^2 such that the following conditions hold:

(P1) $\sim$ is regular for $\trianglelefteqslant$.
(P2) For any $(a,b) \in S^2$ there is exactly one $c \in S$ such that $(a,b) \sim (1,c) \sim (c,1)$.
(P3) For any $a,b,c,d,e \in S$, $(a,b) \sim (d,1)$ and $(b,c) \sim (1,e)$ imply $(d,c) \sim (a,e)$.

Then the structure $(S^2; \trianglelefteqslant, \sim, (1,1))$ is called a *tomonoid partition*.

PROPOSITION 3.3
Let $(S; \leqslant, \odot, 1)$ be a tomonoid and let $\sim$ be the level equivalence of S. Then $(S^2; \trianglelefteqslant, \sim, (1,1))$ is a tomonoid partition.

PROOF. Let $a,b,c,d \in S$. By the compatibility of $\leqslant$ with $\odot$, we have that $(a,b) \trianglelefteqslant (c,d)$ implies $a \odot b \leqslant c \odot d$. (P1) follows. Moreover, as 1 is the monoidal identity, we have that $(a,b) \sim (c,1)$ iff $(a,b) \sim (1,c)$ iff $a \odot b = c$. Hence also (P2) holds. Finally, (P3) is implied by the associativity of $\odot$. ∎

In the sequel, given a tomonoid $(S; \leqslant, \odot, 1)$ and its level equivalence $\sim$, we refer to $(S^2; \trianglelefteqslant, \sim, (1,1))$ as the tomonoid partition *associated with* S.

We note that, thanks to (P2), the regularity condition (P1) can be simplified.

LEMMA 3.4
Let $(S; \leqslant)$ be a chain, let $1 \in S$, and let $\sim$ be an equivalence relation on S^2 such that (P2) holds. Then (P1) is equivalent to each of following statements:

(P1') For any $a,a',b,b',c,c',d,d' \in S$, $(a,b) \sim (a',b') \trianglelefteqslant (c,d) \sim (c',d') \trianglelefteqslant (a,b)$ implies $(a,b) \sim (c,d)$.
(P1") For any $a,b,c,d,e,f \in S$, if $(1,e) \sim (a,b) \trianglelefteqslant (c,d) \sim (1,f)$, then $e \leqslant f$.

PROOF. (P1) trivially implies (P1').

Assume (P1') and let $(1,e) \sim (a,b) \trianglelefteqslant (c,d) \sim (1,f)$. Then $f < e$ implies $(1,f) \trianglelefteqslant (1,e)$ and hence, by (P1') and (P2), $e = f$, a contradiction. (P1") follows.

Assume (P1") and let $a,b,c_1,\ldots,c_k,d_1,\ldots,d_k$ be such that

$$(c_1,d_1) \sim (c_2,d_2) \trianglelefteqslant (c_3,d_3) \sim (c_4,d_4) \trianglelefteqslant \ldots \trianglelefteqslant (c_k,d_k) \sim (c_1,d_1).$$

By (P2), there are $e_1,\ldots,e_k$ such that $(c_i,d_i) \sim (1,e_i)$ for each i, and we conclude by (P1") that $e_1 = e_2 \leqslant e_3 = \ldots \leqslant e_k = e_1$. It follows that the (c_i,d_i) are pairwise $\sim$-equivalent, and (P1) is shown. ∎

Before establishing the converse direction of Proposition 3.3—in fact, tomonoid partitions can be identified with tomonoids—let us interpret the properties (P1)–(P3) in Definition 3.2 from a geometric point of view.

For a tomonoid S, let us view S^2 as a square array; cf. Figure 1. Let the columns and rows be indexed by the elements of S such that the order goes to the right and upwards, respectively. For two elements $(a,b),(c,d) \in S^2$, we then have $(a,b) \trianglelefteqslant (c,d)$ if (c,d) is on the right above (a,b). Moreover, S contains a designated element 1. In case of the tomonoids on which we focus here, 1 is the top element. In this case the element $(1,1)$ of S^2 is located in the upper right corner.

The level equivalence of S induces a partition of S^2. Condition (P2) implies that the blocks of this partition are in a one-to-one correspondence with the elements of the line indexed by 1. In fact, when passing through the line $(1,c)$, $c \in S$, from left to right, we meet each block exactly once. The same holds for the column indexed by 1. By (P2), $(c,1)$ and $(1,c)$ are for each $c \in S$ in the same block.

0	t	u	v	w	x	y	z	1	
0	t	u	v	w	x	y	z	1	1
0	0	t	u	u	v	w	x	z	z
0	0	0	t	t	u	u	v	y	y
0	0	0	0	t	t	t	v	x	x
0	0	0	0	0	0	t	u	w	w
0	0	0	0	0	0	0	t	v	v
0	0	0	0	0	0	0	0	u	u
0	0	0	0	0	0	0	0	t	t
0	0	0	0	0	0	0	0	0	0

FIGURE 1. A tomonoid partition associated with a 9-element negative tomonoid S. Rows and columns of the array correspond to the elements of S. Each square in the array thus corresponds to a pair $(a,b) \in S^2$, where a is the row index and b is the column index. Moreover, the element of S indicated in the square (a,b) is the product of a and b in S. For instance, $z \odot v = u$. Finally, let $\sim$ be the level equivalence. Then two squares belong to the same $\sim$-class iff they contain the same symbol. For instance, the $\sim$-class of t comprises ten elements and the $\sim$-class of 1 is just a singleton.

The partial order $\trianglelefteq$ induces a preorder $\trianglelefteq_\sim$ on the blocks. Condition (P1) ensures that $\trianglelefteq_\sim$ is a partial order as well. In fact, $\trianglelefteq_\sim$ is the total order inherited from S under the correspondence between the blocks and the line $(1,c)$, $c \in S$. A way to see the meaning of (P1) is thus the following: when switching from any element of a block containing $(1,c)$ to the right or upwards, then we arrive at a block containing $(1,d)$ such that $d \geqslant c$.

Finally, assuming that the 1 is the top element of S, also condition (P3) possesses an appealing interpretation in our geometric setting, a fact to which we will refer in the sequel repeatedly. Within the array representing S^2, consider two rectangles such that one hits the upper edge and the other one hits the right edge; cf. Figure 2. Assume that the upper left, upper right, and lower right vertices of these rectangles are in the same blocks, respectively. Then, by (P3), also the remaining pair, consisting of the lower left vertices, is in the same block. The corresponding property in web geometry is the *Reidemeister condition* [1, 2].

In the sequel, when working with a set S^2, where S is a chain with a designated element 1, we will identify the elements of the form $(1,c)$, $c \in S$, with c. It will be clear from the context if c denotes an element of S or of S^2. In particular, if $\sim$ is an equivalence relation on S^2, then $(a,b) \sim c$ means $(a,b) \sim (1,c)$. Moreover, the $\sim$-class of a $c \in S$ is meant to be the $\sim$-class containing $(1,c)$.

PROPOSITION 3.5
Let $(S^2; \trianglelefteq, \sim, (1,1))$ be a tomonoid partition. Let $\leqslant$ be the underlying total order of S. Moreover, for any $a, b \in S$, let

$$a \odot b = \text{ the unique } c \text{ such that } (a,b) \sim c. \tag{1}$$

Then $(S; \leqslant, \odot, 1)$ is the unique tomonoid such that $(S^2; \trianglelefteq, \sim, (1,1))$ is its associated tomonoid partition.

PROOF. By assumption, S is totally ordered and $\trianglelefteq$ is the induced componentwise order on S^2. Evidently, $\trianglelefteq$ determines the total order $\leqslant$ on S uniquely. It is furthermore clear from (P2) that $\odot$ can be defined by (1).

For $a \in S$, we have $1 \odot a = a$ by construction and $a \odot 1 = 1 \odot a$ by (P2). Furthermore, (P2) and (P3) imply the associativity of $\odot$. Thus $(S; \odot, 1)$ is a monoid. Recall next that (P1) and (P2) imply (P1") by

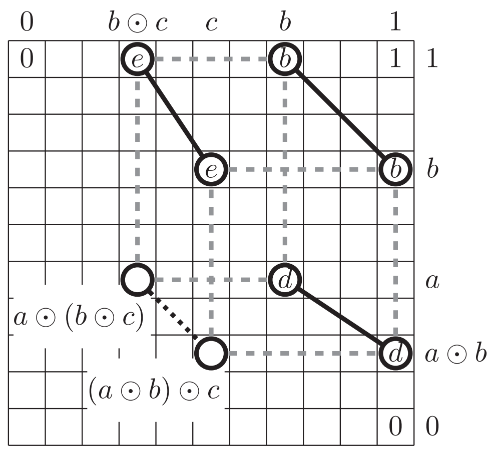

FIGURE 2. The 'Reidemeister' condition (P3). A (connected or broken) black line between two elements of the array indicates level equivalence; for instance, $(a,b) \sim (a \odot b, 1)$. By (P3), the equivalences of the pairs connected by a solid line imply the equivalence of the pair connected by a broken line.

Lemma 3.4. Let $a \leqslant b$. Then $(a,c) \trianglelefteq (b,c)$, and we conclude from (P1") that $a \odot c \leqslant b \odot c$. Similarly, we see that $c \odot a \leqslant c \odot b$. Thus $\leqslant$ is compatible with $\odot$ and $(S; \leqslant, \odot, 1)$ is a tomonoid. It is clear that $\sim$ is the level equivalence of S and we conclude that $(S^2; \trianglelefteq, \sim, (1,1))$ is its associated tomonoid partition.

Let $(S^2; \trianglelefteq, \sim, (1,1))$ be associated to another tomonoid $(S'; \leqslant', \odot', 1')$. Then, by the way in which a tomonoid partition is constructed from a tomonoid, $S' = S$, $\leqslant' = \leqslant$, and $1' = 1$. Furthermore, if for some $a, b, c \in S$ we have $a \odot' b = c$, then $(a,b) \sim (1,c)$ and hence $a \odot b = c$. We conclude $\odot' = \odot$. ■

By Propositions 3.3 and 3.5, tomonoids and tomonoid partitions are in a one-to-one correspondence. We will present our results in the sequel mostly with reference to the latter, i.e., with reference to tomonoid partitions.

Under this identification, we will apply properties, constructions, etc. defined for tomonoids to tomonoid partitions as well. For instance, a negative tomonoid partition is meant to be a tomonoid partition such that the corresponding tomonoid is negative.

Properties of tomonoids that are repeatedly addressed in this article are finiteness, negativity and commutativity. Finiteness has for tomonoids and their associated tomonoid partitions obviously the same meaning. Negativity and commutativity may be characterised as follows.

LEMMA 3.6
Let $(S^2; \trianglelefteq, \sim, (1,1))$ be a tomonoid partition.

(i) S^2 is negative if and only if $(1,1)$ is the top element of S^2 if and only if the $\sim$-class of any $c \in S$ is contained in $\{(a,b) \in S^2 : a, b \geqslant c\}$.
(ii) S^2 is commutative if and only if $(a,b) \sim (b,a)$ for any $a, b \in S$.

The following proposition is devoted to the structures in which we are actually interested: the f.n. tomonoid partitions. The slightly optimized characterization will be useful in subsequent proofs.

PROPOSITION 3.7
Let $(S;\leqslant)$ be a finite and at least two-element chain with the top element 1. Let 0 be the bottom element of S. Then $(S^2;\trianglelefteq,\sim,(1,1))$ is a tomonoid partition if and only if (P1”), (P2), and the following condition hold:

(P3’) For any $a,b,c,d,e\in S\setminus\{0,1\}$, $(a,b)\sim d$ and $(b,c)\sim e$ imply $(d,c)\sim(a,e)$.

In this case, $(S^2;\trianglelefteq,\sim,(1,1))$ is finite and negative.

PROOF. The ‘only if’ part is clear by definition and by Lemma 3.4.

To see the ‘if’ part, let $(S^2;\trianglelefteq,\sim,(1,1))$ fulfil (P1”), (P2), and (P3’). Then (P1) holds by Lemma 3.4. We next show that the negativity criterion of Lemma 3.6(i) holds:

($\star$) $(a,b)\sim(1,c)$ implies $c\leqslant a$ and $c\leqslant b$.

Indeed, in this case $(c,1)\sim(1,c)\sim(a,b)\trianglelefteq(a,1)$ by (P2) and the fact that 1 is the top element. Hence, by (P1”), $c\leqslant a$. Similarly, we see that $c\leqslant b$.

It remains to prove (P3). Let $a,b,c,d,e\in S$ be such that $(a,b)\sim d$ and $(b,c)\sim e$. We have to show $(d,c)\sim(a,e)$ if one of the five elements equals 0 or 1. We consider certain cases only, the remaining ones are seen similarly.

Let $a=1$. Then $(1,b)\sim(1,d)$, hence $b=d$ by (P2), and it follows $(d,c)=(b,c)\sim(1,e)=(a,e)$.

Let $d=1$. Then $(a,b)\sim(1,1)$, and by ($\star$), we conclude $a=b=1$. From $(b,c)\sim e$ it follows $e=c$. Hence $(d,c)=(a,e)$.

Note next that, for any $f\in S$, $(f,0)\sim 0$. This follows again from ($\star$).

Let $a=0$. Then $(a,b)=(0,b)\sim 0$ and hence $d=0$. Hence $(d,c)=(0,c)\sim 0\sim(0,e)=(a,e)$.

Let $d=0$. Then $(d,c)=(0,c)\sim 0$. From $(b,c)\sim e$, it follows by ($\star$) that $e\leqslant b$. Hence $(a,e)\trianglelefteq(a,b)\sim 0\sim(0,0)\trianglelefteq(a,e)$ and, by (P1), $(a,e)\sim 0$. In particular, $(a,e)\sim(d,c)$. ■

We finally see how Rees quotients are formed in our framework.

PROPOSITION 3.8
Let $(S^2;\trianglelefteq,\sim,(1,1))$ be a negative tomonoid partition and let $q\in S$. Let $S_q=\{a\in S:\ a>q\}\dot\cup\{0\}$, where 0 is a new element, and endow S_q with the total order extending the total order on $\{a\in S:\ a>q\}$ such that 0 is the bottom element. Then, for each $c\in {S_q}^\star$, the $\sim$-class of c is contained in $({S_q}^\star)^2$. Let $\sim_q$ be the equivalence relation on ${S_q}^2$ whose classes are the $\sim$-classes of each $c\in {S_q}^\star$ as well as the subset of ${S_q}^2$ containing the remaining elements. Then $({S_q}^2;\trianglelefteq,\sim_q,(1,1))$ is the Rees quotient of S^2 by q.

PROOF. Let $(S;\leqslant,\odot,1)$ be the corresponding negative tomonoid. Let $\odot_q$ be the binary operation on S_q such that $(S_q;\leqslant,\odot_q,1)$ is (under the obvious identifications) the Rees quotient of S by q. Let $({S_q}^2;\trianglelefteq,\sim'_q,(1,1))$ be the associated tomonoid partition.

Let $a,b,c\in S$ such that $c>q$ and $(a,b)\sim c$. Then $a,b\geqslant c$ by Lemma 3.6(i) and consequently $a,b>q$. We conclude that the $\sim$-class of each $c\in {S_q}^\star$ is contained in $({S_q}^\star)^2$.

We have to show $\sim'_q=\sim_q$. Let $a,b,c\in S_q$ such that $c\neq 0$. Then $(a,b)\sim'_q c$ iff $a\odot_q b=c$ iff $a\odot b=c$ iff $(a,b)\sim c$. Hence the $\sim'_q$-class of each $c\in {S_q}^\star$ coincides with the $\sim$-class of c. There is only one further $\sim'_q$-class, the $\sim'_q$-class of 0, which consequently consists of all elements of ${S_q}^2$ not belonging to the $\sim$-class of any $c\in {S_q}^\star$. ■

We may geometrically interpret Proposition 3.8 as follows. The Rees quotient by an element q arises from the partition on S^2 by removing all columns and rows indexed by elements $\leqslant q$ and by adding instead a single new column from left and a single new row from below. Moreover, all elements of the new column and the new row as well as all remaining elements that originally belonged to a

0	u	v	w	x	y	z	1	
0	u	v	w	x	y	z	1	1
0	0	u	u	v	w	x	z	z
0	0	0	0	u	u	v	y	y
0	0	0	0	0	0	v	x	x
0	0	0	0	0	0	u	w	w
0	0	0	0	0	0	0	v	v
0	0	0	0	0	0	0	u	u
0	0	0	0	0	0	0	0	0

0	v	w	x	y	z	1	
0	v	w	x	y	z	1	1
0	0	0	v	w	x	z	z
0	0	0	0	0	v	y	y
0	0	0	0	0	v	x	x
0	0	0	0	0	0	w	w
0	0	0	0	0	0	v	v
0	0	0	0	0	0	0	0

0	w	x	y	z	1	
0	w	x	y	z	1	1
0	0	0	w	x	z	z
0	0	0	0	0	y	y
0	0	0	0	0	x	x
0	0	0	0	0	w	w
0	0	0	0	0	0	0

0	x	y	z	1	
0	x	y	z	1	1
0	0	0	x	z	z
0	0	0	0	y	y
0	0	0	0	x	x
0	0	0	0	0	0

0	y	z	1	
0	y	z	1	1
0	0	0	z	z
0	0	0	y	y
0	0	0	0	0

0	z	1	
0	z	1	1
0	0	z	z
0	0	0	0

0	1	
0	1	1
0	0	0

1	
1	1

FIGURE 3. Beginning with the 9-element tomonoid shown in Figure 1, the successive formation of Rees quotients by the atom leads finally to the trivial tomonoid.

class of some $a \leqslant q$ are joined into a single class, which is the class of the new bottom element. The classes of elements strictly larger than q remain unchanged.

In the special case that q is the atom of a f.n. tomonoid, just the left-most two columns and the lowest two rows are merged in this way. Figure 3 shows the chain obtained from a 9-element tomonoid by applying this procedure repeatedly.

4 Rees coextensions: the Archimedean case

We now turn to the problem of determining all one-element coextensions of a finite, negative tomonoid. In this section, we will restrict to those tomonoids that fulfil the Archimedean property.

A negative tomonoid S is called *Archimedean* if, for any $a \leqslant b < 1$, there is an $n \geqslant 1$ such that $b^n \leqslant a$. Here, $b^n = b \odot \ldots \odot b$ (n factors). Note that, in the finite case, Archimedeanicity is obviously equivalent to nilpotency. Indeed, a f.n. tomonoid, whose bottom element is 0, is Archimedean if and only if there is an $n \geqslant 1$ such that $a^n = 0$ for all $a < 1$. Note furthermore that negative tomonoids with at most two elements are trivially Archimedean.

We begin by characterizing the Archimedean f.n. tomonoid partitions.

LEMMA 4.1
Let $(S^2; \trianglelefteq, \sim, (1,1))$ be a f.n. tomonoid partition. The following statements are pairwise equivalent:

(i) S^2 is Archimedean.
(ii) $(b,a) \not\sim (1,a)$ for any $a \in S^\star$ and $b < 1$.
(iii) $(a,b) \not\sim (a,1)$ for any $a \in S^\star$ and $b < 1$.

PROOF. Let $(S; \leqslant, \odot, 1)$ be the corresponding f.n. tomonoid and let 0 be the bottom element of S. W.l.o.g., we can assume $0 \neq 1$. We show that (i) and (ii) are equivalent. The equivalence of (i) and (iii) is seen similarly.

Assume that (ii) holds. By the negativity of S, we have $b \odot a < a$ for all $a \neq 0$ and $b < 1$. Let $a < 1$. Then, for any $n \geqslant 1$, either $a^{n+1} < a^n$ or $a^n = 0$. As S is finite, the latter possibility applies for a sufficiently large n. It follows that S is Archimedean.

Assume that (ii) does not hold. Let $a \neq 0$ and $b < 1$ such that $b \odot a = a$. As S is negative, we then have $a \leqslant b$ and it follows $b^n \geqslant b^{n-1} \odot a = a > 0$ for any $n \geqslant 2$. Hence S cannot be Archimedean. ∎

We shall construct coextensions of Archimedean f.n. tomonoids that are Archimedean again. Let us outline our procedure, adopting an intuitive point of view.

To begin with, we again identify the tomonoid partition with a partitioned square array; cf. Figure 1. We enlarge this square, doubling the lowest row and left-most column. The equivalence relation $\bar{\sim}$ making the enlarged square into a tomonoid partition will then be constructed in two steps. First, we determine what we call the ramification, which is based on an equivalence relation $\dot{\sim}$ contained in the level equivalence of any Archimedean one-element coextension. Secondly, we apply a simple procedure to choose the final equivalence relation $\bar{\sim}$. To this end, certain $\dot{\sim}$-classes have to be merged such that the part of the square containing the classes of the new tomonoid's bottom element and atom is divided up into exactly two $\bar{\sim}$-classes.

For a chain $(S; \leqslant)$, we denote by $(\bar{S}; \leqslant)$ its *zero doubling extension*: we put $\bar{S} = S^\star \dot{\cup} \{0, \alpha\}$, where $0, \alpha$ are new elements, and we endow $\bar{S}$ with the total order extending the total order on $S^\star$ such that $0 < \alpha < a$ for all $a \in S^\star$. Furthermore, let $(S; \leqslant, \odot, 1)$ be a f.n. tomonoid. Then we assume any one-element coextension of S to be of the form $(\bar{S}; \leqslant, \bar{\odot}, 1)$. In particular, the intersection of S and $\bar{S}$ is exactly $S^\star$ and $a \mathbin{\bar{\odot}} b = a \odot b = c$ whenever $a, b, c \in S^\star$.

Definition 4.2
Let $(S^2; \trianglelefteq, \sim, (1,1))$ be an Archimedean f.n. tomonoid partition. Let $\bar{S} = S^\star \dot{\cup} \{0, \alpha\}$ be the zero doubling extension of S. We define

$$\begin{aligned} \mathcal{P} &= \{(a,b) \in \bar{S}^2 \colon a, b \in S^\star \text{ and there is a } c \in S^\star \text{ such that } (a,b) \sim c\}, \\ \mathcal{Q} &= \bar{S}^2 \setminus \mathcal{P}. \end{aligned} \tag{2}$$

Let $\dot{\sim}$ be the smallest equivalence relation on $\bar{S}^2$ such that the following conditions hold:

(E1) For any $(a,b), (c,d) \in \mathcal{P}$ such that $(a,b) \sim (c,d)$, we have $(a,b) \mathrel{\dot{\sim}} (c,d)$.
(E2) For any $(a,b), (b,c) \in \mathcal{P}$ and $d, e \in S^\star$ such that $(d,c), (a,e) \in \mathcal{Q}$, $(a,b) \sim d$, and $(b,c) \sim e$, we have $(d,c) \mathrel{\dot{\sim}} (a,e)$.
(E3) For any $a, b, c, e \in S^\star$ such that $(a,b) \in \mathcal{Q}$, $(b,c) \sim e$, and $c < 1$, we have $(a,e) \mathrel{\dot{\sim}} 0$.
Moreover, for any $a, b, c, d \in S^\star$ such that $(b,c) \in \mathcal{Q}$, $(a,b) \sim d$, and $a < 1$, we have $(d,c) \mathrel{\dot{\sim}} 0$.
(E4) We have $(0,1) \mathrel{\dot{\sim}} (1,0) \mathrel{\dot{\sim}} (\alpha, b) \mathrel{\dot{\sim}} (b, \alpha)$ for any $b < 1$, and $(\alpha, 1) \mathrel{\dot{\sim}} (1, \alpha)$. Moreover, for any $(a,b), (c,d) \in \mathcal{Q}$ such that $(a,b) \trianglelefteq (c,d) \mathrel{\dot{\sim}} 0$, we have $(a,b) \mathrel{\dot{\sim}} 0$.

Then we call the structure $(\bar{S}^2; \trianglelefteq, \dot{\sim}, (1,1))$ the $(1,1)$-*ramification* of $(S^2; \trianglelefteq, \sim, (1,1))$.

In this section, we will refer to the $(1,1)$-ramification also simply as the 'ramification'. The reason of the reference to the pair $(1,1)$ will become clear only in the next section.

Let us review Definition 4.2 in order to elucidate how the ramification $(\bar{S}^2; \trianglelefteq, \dot{\sim}, (1,1))$ arises from a tomonoid partition $(S^2; \trianglelefteq, \sim, (1,1))$. We note first that $\mathcal{P}$ consists of all pairs $(a,b) \in S^2$ whose product in S is not the bottom element. Indeed, by the negativity of S, $(a,b) \sim c$ and $c \in S^\star$ implies $a, b \in S^\star$. In other words, $\mathcal{P}$ is the union of the $\sim$-classes of all $c \in S^\star$ and this union lies in ${S^\star}^2$. We furthermore note that $\mathcal{P}$ is an upwards closed subset of $\bar{S}^2$. This is a consequence of the regularity of $\sim$; cf. condition (P1") in Lemma 3.4. Consequently, $\mathcal{Q}$ is a downward closed subset of $\bar{S}^2$.

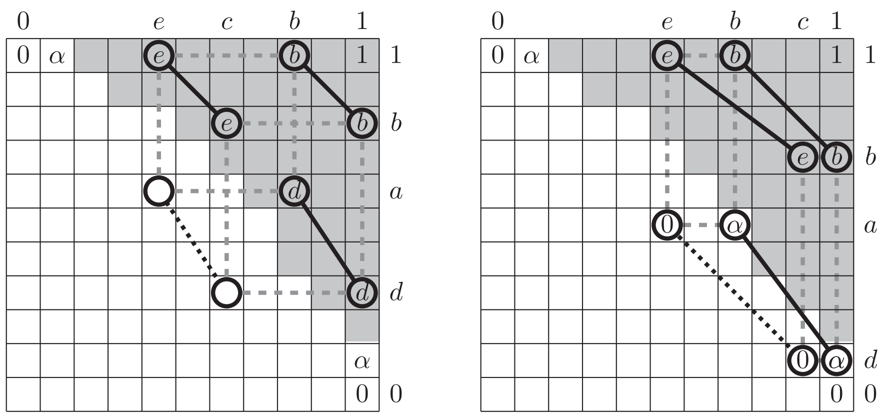

FIGURE 4. The prescriptions (E2) and (E3) in the construction of Rees coextensions for the Archimedean case.

Inspecting the conditions (E1)–(E4) defining $\dot{\sim}$, we next observe that $\dot{\sim}$-equivalences of elements of $\mathcal{P}$ are exclusively required by condition (E1). From this fact we conclude that the $\sim$-class of any $c \in S^\star$ is also a $\dot{\sim}$-class. Hence the $\sim$-classes contained in $\mathcal{P}$ are $\dot{\sim}$-classes as well. Note that this also means that the sets $\mathcal{P}$ and $\mathcal{Q}$ are uniquely determined by the ramification. In fact, $\mathcal{P}$ contains the $\dot{\sim}$-classes of all $c \in S^\star$ and $\mathcal{Q}$ contains all remaining $\dot{\sim}$-classes.

The $\dot{\sim}$-classes contained in $\mathcal{Q}$ are in turn defined by conditions (E2)–(E4). See Figure 4 for an illustration of conditions (E2) and (E3). Note that each prescription contained in (E2) and (E3) is of the form that certain $\sim$-equivalences imply that a certain pair of elements of $\mathcal{Q}$ is $\dot{\sim}$-equivalent. Finally, (E4) prescribes that the $\dot{\sim}$-class of 0 is downward closed. We remark that $\mathcal{Q}$ contains the $\dot{\sim}$-classes of the bottom element 0 and the atom α, but possibly further $\dot{\sim}$-classes, which do not contain $(1,c)$ or $(c,1)$ for any $c \in \bar{S}$.

For two equivalence relations $\sim_1$ and $\sim_2$ on a set A, we say that $\sim_2$ is a *coarsening* of $\sim_1$ if $\sim_1 \subseteq \sim_2$, that is, if each $\sim_2$-class is a union of $\sim_1$-classes.

LEMMA 4.3
Let $(S^2; \trianglelefteq, \sim, (1,1))$ be an Archimedean f.n. tomonoid partition and let $(\bar{S}^2; \trianglelefteq, \bar{\sim}, (1,1))$ be an Archimedean one-element coextension of S^2. Furthermore, let $(\bar{S}^2;\ \trianglelefteq, \dot{\sim}, (1,1))$ be the $(1,1)$-ramification of S^2. Then $\bar{\sim}$ is a coarsening of $\dot{\sim}$ such that the following holds: the $\bar{\sim}$-class of each $c \in S^\star$ coincides with the $\dot{\sim}$-class of c, the $\bar{\sim}$-class of 0 is downward closed, and each $\bar{\sim}$-class contains exactly one element of the form $(1,c)$ for some $c \in \bar{S}$.

PROOF. Let $(S; \leqslant, \odot, 1)$ and $(\bar{S}; \leqslant, \bar{\odot}, 1)$, where $\bar{S} = S^\star \dot{\cup} \{0, \alpha\}$, be the two tomonoids in question.

As noted above, condition (E1) requires $\dot{\sim}$-equivalences only between elements of $\mathcal{P}$ and the remaining conditions require $\dot{\sim}$-equivalences only between elements of $\mathcal{Q}$. Furthermore, $\mathcal{P}$ is the union of the $\sim$-classes of all $c \in S^\star$. By (E1), these $\sim$-classes are also $\dot{\sim}$-classes. Moreover, by Proposition 3.8, each $\sim$-class of a $c \in S^\star$ is a $\bar{\sim}$-class. We conclude that the $\bar{\sim}$-class of each $c \in S^\star$ coincides with the $\dot{\sim}$-class of c and $\mathcal{P}$ is the union of these subsets.

We next check that any two elements that are $\dot{\sim}$-equivalent according to one of the conditions (E2)–(E4) are also $\bar{\sim}$-equivalent. Since $\dot{\sim}$ is, by assumption, the smallest equivalence relation with the indicated properties, it will then follow that $\dot{\sim} \subseteq \bar{\sim}$.

Ad (E2): Let $(a,b),(b,c)\in\mathcal{P}$, $d,e\in S^\star$, $(a,b)\sim d$, and $(b,c)\sim e$. Then $a,b,c\in S^\star$, hence $a\mathbin{\bar\odot}b=a\odot b=d$ and $b\mathbin{\bar\odot}c=b\odot c=e$. Consequently, $d\mathbin{\bar\odot}c=(a\mathbin{\bar\odot}b)\mathbin{\bar\odot}c=a\mathbin{\bar\odot}(b\mathbin{\bar\odot}c)=a\mathbin{\bar\odot}e$, that is $(d,c)\mathrel{\tilde\sim}(a,e)$.

Ad (E3): Let $a,b,c,e\in S^\star$, $(a,b)\in\mathcal{Q}$, $(b,c)\sim e$, and $c<1$. Then $a\mathbin{\bar\odot}b\leqslant\alpha$ and hence $a\mathbin{\bar\odot}e=a\mathbin{\bar\odot}(b\mathbin{\bar\odot}c)=(a\mathbin{\bar\odot}b)\mathbin{\bar\odot}c\leqslant\alpha\mathbin{\bar\odot}c$. As S is assumed to be Archimedean, α is the atom of $\bar S$, and $c<1$, we conclude $\alpha\mathbin{\bar\odot}c=0$. Hence $(a,e)\mathrel{\tilde\sim}0$. Similarly, we argue for the second part of (E3).

Ad (E4): As S is Archimedean, we have, for any $b<1$, $0\mathbin{\bar\odot}1=1\mathbin{\bar\odot}0=\alpha\mathbin{\bar\odot}b=b\mathbin{\bar\odot}\alpha=0$ by Lemma 4.1 and hence $(0,1)\mathrel{\tilde\sim}(1,0)\mathrel{\tilde\sim}(\alpha,b)\mathrel{\tilde\sim}(b,\alpha)$. Furthermore, we have $(\alpha,1)\mathrel{\tilde\sim}(1,\alpha)$. Finally, let $(a,b),(c,d)\in\mathcal{Q}$ and assume $(a,b)\trianglelefteq(c,d)\mathrel{\tilde\sim}0$. Then $a\mathbin{\bar\odot}b\leqslant c\mathbin{\bar\odot}d=0$ and thus $(a,b)\mathrel{\tilde\sim}0$ as well.

It is finally clear that the $\tilde\sim$-class of 0 is downward closed. The last statement holds by condition (P2) of a tomonoid partition. ■

We are now ready to state the main result of this section.

THEOREM 4.4
Let $(S^2;\trianglelefteq,\sim,(1,1))$ be an Archimedean f.n. tomonoid partition and let $(\bar S^2;\trianglelefteq,\dot\sim,(1,1))$ be the $(1,1)$-ramification of S^2. Let $\tilde\sim$ be a coarsening of $\dot\sim$ such that the following holds: the $\tilde\sim$-class of each $c\in S^\star$ coincides with the $\dot\sim$-class of c, the $\tilde\sim$-class of 0 is downward closed, and each $\tilde\sim$-class contains exactly one element of the form $(1,c)$ for some $c\in\bar S$. Then $(\bar S^2;\trianglelefteq,\tilde\sim,(1,1))$ is an Archimedean one-element coextension of S^2.

Moreover, all Archimedean one-element coextensions of S^2 arise in this way.

PROOF. $\mathcal{P}$, defined by (2), is the union of the $\sim$-classes of all $c\in S^\star$. As we have seen in the proof of Lemma 4.3, these subsets of $\mathcal{P}$ are also $\dot\sim$-classes. Recall also that $\mathcal{P}$ is upwards closed and $\mathcal{Q}=\bar S^2\setminus\mathcal{P}$ is downward closed.

By (E4), we have $(1,0)\mathrel{\dot\sim}(0,1)$ and $(1,\alpha)\mathrel{\dot\sim}(\alpha,1)$. We claim that $(1,0)\mathrel{\dot\nsim}(1,\alpha)$. Indeed, (E1), (E2) and (E3) involve only elements (a,b) such that $a,b\in S^\star$. Hence, none of these prescriptions involves the elements $(1,\alpha)$ or $(\alpha,1)$. Moreover, by (E4), the elements $(a,0)$ and $(0,a)$ for any a as well as (a,α) and (α,a) for any $a\neq1$ belong to the $\dot\sim$-class of $(1,0)$. Again, $(1,\alpha)$ and $(\alpha,1)$ are not concerned. Finally, the $\dot\sim$-class of $(1,0)$ is a downward closed set. Also this prescription has no effect on $(1,\alpha)$ or $(\alpha,1)$ because there is no element in $\mathcal{Q}$ that is larger than $(1,\alpha)$ or $(\alpha,1)$. We conclude that $\{(1,\alpha),(\alpha,1)\}$ is an own $\dot\sim$-class and our claim is shown.

Let now $\tilde\sim\supseteq\dot\sim$ be as indicated. Note that, by what we have seen so far, at least one such equivalence relation exists. In accordance with Proposition 3.7, we will verify (P1”), (P2) and (P3’).

We have shown that $(1,c)\mathrel{\tilde\sim}(c,1)$ for all $c\in\bar S$. By construction, $\tilde\sim$ fulfils (P2). Furthermore, the $\tilde\sim$-class of 0 is downward closed and $\mathcal{Q}$, which is the union of the $\tilde\sim$-classes of 0 and α, is downward closed as well. We conclude that (P1”) holds for $\tilde\sim$.

It remains to show that $\tilde\sim$ fulfils (P3’). Let $a,b,c,d,e\in S\setminus\{0,1\}$ such that $(a,b)\mathrel{\tilde\sim}d$ and $(b,c)\mathrel{\tilde\sim}e$. We distinguish the following cases.

Case 1. Let $d,e\in S^\star$. Then $(a,b)\sim d$ and $(b,c)\sim e$. As $\sim$ fulfils (P3), we have $(d,c)\sim(a,e)$. In particular, it follows that $(d,c)\in\mathcal{P}$ iff $(a,e)\in\mathcal{P}$. If (d,c) and (a,e) are both in $\mathcal{P}$, we have $(d,c)\mathrel{\tilde\sim}(a,e)$ because the $\dot\sim$-classes contained in $\mathcal{P}$ are $\tilde\sim$-classes as well. If (d,c) and (a,e) are both in $\mathcal{Q}$, we have $(d,c)\mathrel{\dot\sim}(a,e)$ by (E2) and consequently also $(d,c)\mathrel{\tilde\sim}(a,e)$, because $\tilde\sim$ extends $\dot\sim$.

Case 2. Let $d=\alpha$ and $e\in S^\star$. Then $(d,c)\mathrel{\dot\sim}0$ by (E4). Furthermore, we have $a\in S^\star$ by (E4), $b,c\in S^\star$ because $(b,c)\in\mathcal{P}$, $(a,b)\in\mathcal{Q}$, and $(b,c)\sim e$. It follows $(a,e)\mathrel{\dot\sim}0$ by (E3). Consequently, $(d,c)\mathrel{\tilde\sim}0\mathrel{\tilde\sim}(a,e)$.

Case 3. Let $d\in S^\star$ and $e=\alpha$. We argue similarly to Case 2.

Case 4. Let $d=e=\alpha$. Then $(d,c)\mathrel{\dot\sim}(a,e)\mathrel{\dot\sim}0$ by (E4) and consequently also $(d,c)\mathrel{\tilde\sim}(a,e)$.

By Proposition 3.7, $(\bar{S}^2; \trianglelefteq, \bar{\sim}, (1,1))$ is a f.n. tomonoid partition, which is moreover Archimedean by (E4) and Lemma 4.1. It is finally clear from Proposition 3.8 that the Rees quotient of $\bar{S}^2$ by the atom α is S^2.

The final statement follows from Lemma 4.3. ■

Let us exhibit some features of our construction. Starting from a tomonoid partition $(S^2; \trianglelefteq, \sim, (1,1))$, we determine its $(1,1)$-ramification $(\bar{S}^2; \trianglelefteq, \dot{\sim}, (1,1))$ by applying conditions (E1)–(E4) from Definition 4.2. These prescriptions are largely independent. This is to say that, in order to determine $\dot{\sim}$, it is not necessary to use already obtained results in a recursive way. Furthermore, to obtain a coextension of the desired type, the set $\mathcal{Z} = \langle(1,0)\rangle_{\bar{\sim}}$, i.e. the $\bar{\sim}$-class of the bottom element, must be chosen. Theorem 4.4 characterizes $\mathcal{Z}$ as follows: $\mathcal{Z}$ is a union of $\dot{\sim}$-classes contained in $\mathcal{Q}$ including $\langle(1,0)\rangle_{\dot{\sim}}$ but excluding the $\dot{\sim}$-class $\{(1,\alpha),(\alpha,1)\}$, and $\mathcal{Z}$ is downward closed. Thus, to determine a specific one-element coextension, all we have to do is to select an arbitrary set of $\dot{\sim}$-classes different from $\{(\alpha,1),(1,\alpha)\}$ and $\mathcal{Z}$ will then be the smallest downward closed set containing them.

We may in particular mention a simple fact: the explained procedure of determining an Archimedean extension always leads to a result. That is, every Archimedean, finite, negative tomonoid has at least one Archimedean one-element coextension. Indeed, with respect to the above notation, we may always choose $\mathcal{Z} = \mathcal{Q} \setminus \{(\alpha,1),(1,\alpha)\}$. In general, it might be found interesting that the explained procedure never requires revisions. In fact, neither the determination of $\dot{\sim}$ nor of $\bar{\sim}$ involves decisions that lead to an impossible situation, we always end up with a structure of the desired type.

REMARK 4.5
We may characterise the set of Archimedean one-element coextensions of an Archimedean f.n. tomonoid $(S; \leqslant, \odot, 1)$ also as follows.

Recall first that with any preorder $\preccurlyeq$ on a set A we can associate a partial order, called its symmetrisation. Indeed, $\preccurlyeq$ gives rise to the equivalence relation $\approx$, where $a \approx b$ if $a \preccurlyeq b$ and $b \preccurlyeq a$, and on the quotient $\langle A\rangle_{\approx}$, $\preccurlyeq$ induces a partial order.

Referring to the notation of Lemma 4.3, let $\mathcal{E}$ be the set of all $\dot{\sim}$-classes contained in $\mathcal{Q}$. Then $\trianglelefteq$ induces on $\mathcal{E}$ the preorder $\trianglelefteq_{\dot{\sim}}$ (cf. Section 3). We can describe the extensions of S with exclusive reference to the preordered set $(\mathcal{E}; \trianglelefteq_{\dot{\sim}})$. Namely, by Theorem 4.4, there is one-to-one correspondence between the Archimedean one-element coextensions of S and the extensions of the preorder $\trianglelefteq_{\dot{\sim}}$ on $\mathcal{E}$ to a preorder whose symmetrisation consists of two elements, one of which contains $\langle(1,0)\rangle_{\dot{\sim}}$ and one of which contains $\langle(1,\alpha)\rangle_{\dot{\sim}}$.

The commutative case

We conclude this section by considering the commutative case. Given a commutative, Archimedean f.n. tomonoid S, our question is how to determine all its commutative, Archimedean one-element coextensions.

It is clear that we may apply to this end Theorem 4.4. Among the one-element coextensions of S we may simply select those that are commutative. An easy criterion of commutativity is stated in Lemma 3.6(ii).

However, it would be desirable to apply a more direct procedure, with the effect that no result must be discarded. This turns out to be easy. All we have to do is to adapt the notion of a $(1,1)$-ramification. We add in Definition 4.2 the following condition:

(E5) For any $a, b \in \bar{S}$ such that $(a,b),(b,a) \in \mathcal{Q}$, we have $(a,b) \mathrel{\dot{\sim}} (b,a)$.

On the basis of this modified notion of a $(1,1)$-ramification, we may reformulate Theorem 4.4 in order to deal with the commutative case only. We omit the straightforward details.

5 Rees coextensions: the general case

We now turn to the construction of one-element coextensions of finite, negative tomonoids without any further restriction. The procedure is slightly more involved than in the Archimedean case.

To see what makes the difference, recall that, by Lemma 4.1, a characteristic feature of the procedure explained in Theorem 4.4 was the following: both the column and the row indexed by the atom α contain, with the exception of $(1,\alpha)$ and $(\alpha,1)$, solely elements in the class of 0. In the general case, this line and row may contain further elements of the class of α. For our general construction, we have to make a decision about the division of this line and row into members of the classes of 0 and α.

An element ε of a tomonoid is called *idempotent* if $\varepsilon \odot \varepsilon = \varepsilon$.

LEMMA 5.1
Let $(S; \leqslant, \odot, 1)$ be a non-trivial f.n. tomonoid. Let $0, \alpha$ be its bottom element and its atom, respectively. Then there is an idempotent $\varepsilon_l \geqslant \alpha$ in S such that

$$a \odot \alpha = \begin{cases} 0 & \text{if } a < \varepsilon_l, \\ \alpha & \text{if } a \geqslant \varepsilon_l. \end{cases}$$

Similarly, there is an idempotent $\varepsilon_r \geqslant \alpha$ in S such that

$$\alpha \odot a = \begin{cases} 0 & \text{if } a < \varepsilon_r, \\ \alpha & \text{if } a \geqslant \varepsilon_r. \end{cases}$$

PROOF. We have $0 \odot \alpha = 0$ and $1 \odot \alpha = \alpha$. Let $\varepsilon_l \in S$ be the smallest element $a \in S$ such that $a \odot \alpha = \alpha$. Evidently, ε_l is not the bottom element. Furthermore, $\varepsilon_l \odot \varepsilon_l \odot \alpha = \alpha$ and, by the minimality of ε_l, it follows $\varepsilon_l \odot \varepsilon_l = \varepsilon_l$, i.e., ε_l is an idempotent. The second part is proved analogously. ∎

Given a tomonoid S, let us call the pair $(\varepsilon_l, \varepsilon_r)$ of idempotents, as specified in Lemma 5.1, *atom-characterizing*. The notion is to indicate that these two elements uniquely determine the inner left and right translation associated with the atom of S.

For the construction of a one-element coextension $\bar{S}$ of a f.n. tomonoid S, we will fix the atom-characterizing idempotents in advance. Note that we can identify each non-zero idempotent e of $\bar{S}$ with an idempotent element of S. In fact, either $e \in S^\star$ and hence e is a non-zero idempotent of S, or e is the atom of $\bar{S}$, in which case we can identify e with the bottom element of S. We say that $\bar{S}$ is a one-element coextension of S *with respect to* $(\varepsilon_l, \varepsilon_r)$ if $(\varepsilon_l, \varepsilon_r)$ is a pair of idempotents of S and, under the above identification, the atom-characterizing pair of idempotents of $\bar{S}$.

DEFINITION 5.2
Let $(S^2; \trianglelefteq, \sim, (1,1))$ be a f.n. tomonoid partition. Let $\bar{S} = S^\star \dot{\cup} \{0, \alpha\}$ be the zero doubling extension of S. Define $\mathcal{P}, \mathcal{Q} \subseteq \bar{S}^2$ according to (2).

Moreover, let $(\varepsilon_l, \varepsilon_r)$ be a pair of idempotents of S. Let $\dot{\sim}$ be the smallest equivalence relation on $\bar{S}^2$ such that (E1), (E2), as well as the following conditions hold:

(E3') (a) For any $a, b, c, e \in S^\star$ such that $(a,b) \in \mathcal{Q}$, $(b,c) \sim e$, and $c < \varepsilon_r$, we have $(a,e) \dot{\sim} 0$. Moreover, for any $a, b, c, d \in S^\star$ such that $(b,c) \in \mathcal{Q}$, $(a,b) \sim d$, and $a < \varepsilon_l$, we have $(d,c) \dot{\sim} 0$.

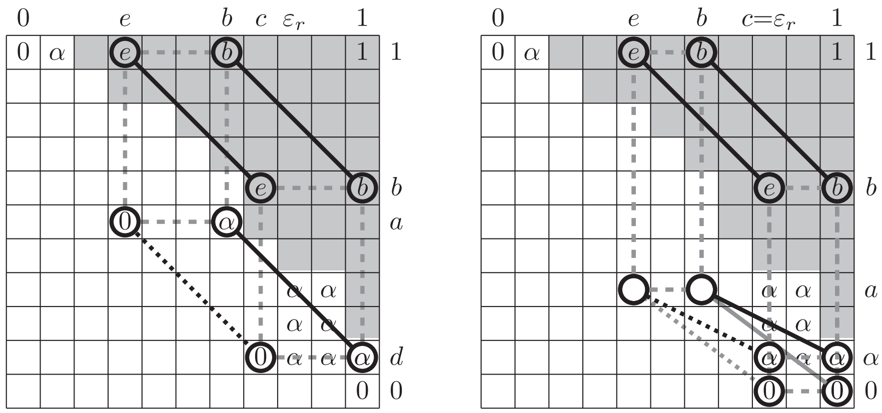

FIGURE 5. The prescriptions (E3')(a), (b) in the construction of one-element coextensions for the general case.

(b) For any $a,b,c,e \in S^\star$ such that $(a,b) \in \mathcal{Q}$, $(b,c) \sim e$, and $c \geqslant \varepsilon_r$, we have $(a,e) \dot{\sim} (a,b)$. Moreover, for any $a,b,c,d \in S^\star$ such that $(b,c) \in \mathcal{Q}$, $(a,b) \sim d$, and $a \geqslant \varepsilon_l$, we have $(d,c) \dot{\sim} (b,c)$.

(c) For any $a,b,c > 0$ such that $(a,b),(b,c) \in \mathcal{Q}$, $a < \varepsilon_l$, and $c \geqslant \varepsilon_r$ we have $(a,b) \dot{\sim} 0$. Moreover, for any $a,b,c > 0$ such that $(a,b),(b,c) \in \mathcal{Q}$, $a \geqslant \varepsilon_l$, and $c < \varepsilon_r$ we have $(b,c) \dot{\sim} 0$.

(E4') (a) We have $(1,0) \dot{\sim} (0,1) \dot{\sim} (a,\alpha) \dot{\sim} (\alpha,b)$ for any $a < \varepsilon_l$ and $b < \varepsilon_r$. Moreover, for any $(a,b),(c,d) \in \mathcal{Q}$ such that $(a,b) \trianglelefteq (c,d) \dot{\sim} 0$, we have $(a,b) \dot{\sim} 0$.

(b) We have $(1,\alpha) \dot{\sim} (\alpha,1) \dot{\sim} (\varepsilon_l,\alpha) \dot{\sim} (\alpha,\varepsilon_r)$. Moreover, for any $(a,b),(c,d) \in \mathcal{Q}$ such that $(a,b) \trianglerighteq (c,d) \dot{\sim} \alpha$, we have $(a,b) \dot{\sim} \alpha$.

Then we call the structure $(\bar{S}^2; \trianglelefteq, \dot{\sim}, (1,1))$ the $(\varepsilon_l,\varepsilon_r)$*-ramification* of $(S^2; \trianglelefteq, \sim, (1,1))$.

We see that Definition 5.2 is largely analogous to Definition 4.2. See Figure 5 for an illustration of conditions (E3')(a) and (b).

Note that Definition 4.2 is contained as a special case in Definition 5.2. To see that both definitions are consistent, assume that S is an Archimedean f.n. tomonoid. It is immediate that in case $\varepsilon_l = \varepsilon_r = 1$ most conditions of Definitions 5.2 coincide with those of Definition 4.2. The only difference is that condition (E4')(b) does not possess an analogue in (E4). In fact, (E4) does not require that the $\dot{\sim}$-class of the atom α of $\bar{S}$ is an upward closed subset of $\mathcal{Q}$. But we have seen in the proof of Theorem 4.4 that the $\dot{\sim}$-class of α is $\{(1,\alpha),(\alpha,1)\}$ and this set trivially fulfills the condition in question.

LEMMA 5.3
Let $(S^2; \trianglelefteq, \sim, (1,1))$ be a f.n. tomonoid partition and let $(\bar{S}^2; \trianglelefteq, \bar{\sim}, (1,1))$ be a one-element coextension of S^2 with respect to the idempotents $(\varepsilon_l,\varepsilon_r)$. Furthermore, let $(\bar{S}^2; \trianglelefteq, \dot{\sim}, (1,1))$ be the $(\varepsilon_l,\varepsilon_r)$/ramification of S. Then $\bar{\sim}$ is a coarsening of $\dot{\sim}$ such that the following holds: the $\bar{\sim}$-class of each $c \in S^\star$ coincides with the $\dot{\sim}$-class of c, the $\bar{\sim}$-class of 0 is downward closed, and each $\bar{\sim}$-class contains exactly one element of the form $(1,c)$ for some $c \in \bar{S}$.

PROOF. We denote again by $\odot$ and $\bar{\odot}$ the monoidal operations on S and $\bar{S} = S^\star \dot{\cup} \{0,\alpha\}$, respectively.

We proceed similarly as in the proof of Lemma 4.3 to see that, for each $c \in S^\star$, the $\sim$-class of c coincides with the $\dot{\sim}$-class as well as with the $\bar{\sim}$-class of c, and $\mathcal{P}$ is the union of these sets.

To show that $\bar{\sim}$ is a coarsening of $\dot{\sim}$, we check that any $\dot{\sim}$-equivalence according to (E2), (E3') or (E4') is an $\bar{\sim}$-equivalence as well. For (E2), see the proof of Lemma 4.3. Furthermore, for (E4'), the argument is obvious from Lemma 5.1. Thus we mention only the case of (E3'):

Ad (E3')(a): Let $a,b,c,e \in S^\star$, $(a,b) \in \mathcal{Q}$, $(b,c) \sim e$, and $c < \varepsilon_r$. Then $a \bar{\odot} b \leqslant \alpha$ and $b \bar{\odot} c = e$ and hence $a \bar{\odot} e = a \bar{\odot} (b \bar{\odot} c) = (a \bar{\odot} b) \bar{\odot} c \leqslant \alpha \bar{\odot} c = 0$. Hence $(a,e) \bar{\sim} 0$. Similarly, we argue for the second part of (E3')(a).

Ad (E3')(b): Let $a,b,c,e \in S^\star$, $(a,b) \in \mathcal{Q}$, $(b,c) \sim e$, and $c \geqslant \varepsilon_r$. Then $a \bar{\odot} b = 0$ or $a \bar{\odot} b = \alpha$. In the former case we have $a \bar{\odot} e = a \bar{\odot} b \bar{\odot} c = 0 \bar{\odot} c = 0$, and in the latter case we have $a \bar{\odot} e = a \bar{\odot} b \bar{\odot} c = \alpha \bar{\odot} c = \alpha$. We conclude $(a,e) \bar{\sim} (a,b)$. Similarly, we argue for the second part of (E3')(b).

Ad (E3')(c): Let $a,b,c \geqslant \alpha$, $(a,b),(b,c) \in \mathcal{Q}$, $a < \varepsilon_l$, and $c \geqslant \varepsilon_r$. Assume that $a \bar{\odot} b = \alpha$. Then $(a \bar{\odot} b) \bar{\odot} c = \alpha \bar{\odot} c = \alpha$, but $a \bar{\odot} (b \bar{\odot} c) \leqslant a \bar{\odot} \alpha = 0$. We conclude $a \bar{\odot} b = 0$, that is, $(a,b) \bar{\sim} 0$. Similarly, we argue for the second part of (E3')(c).

We complete the proof like in case of Lemma 4.3. ∎

THEOREM 5.4
Let $(S^2; \trianglelefteq, \sim, (1,1))$ be a f.n. tomonoid partition, let $(\varepsilon_l, \varepsilon_r)$ be a pair of idempotents of S, and let $(\bar{S}^2; \trianglelefteq, \dot{\sim}, (1,1))$ be the $(\varepsilon_l, \varepsilon_r)$-ramification of S^2.

If $(1,0) \dot{\sim} (1,\alpha)$, there is no one-element coextension of S^2 with respect to $(\varepsilon_l, \varepsilon_r)$.

Assume $(1,0) \dot{\nsim} (1,\alpha)$. Let $\bar{\sim}$ be a coarsening of $\dot{\sim}$ such that the following holds: the $\bar{\sim}$-class of each $c \in S^\star$ coincides with the $\dot{\sim}$-class of c, the $\bar{\sim}$-class of 0 is downward closed, and each $\bar{\sim}$-class contains exactly one element of the form $(1,c)$ for some $c \in \bar{S}$. Then $(\bar{S}^2; \trianglelefteq, \bar{\sim}, (1,1))$ is a one-element coextension of S^2 with respect to $(\varepsilon_l, \varepsilon_r)$.

Moreover, all one-element coextensions of S^2 with respect to $(\varepsilon_l, \varepsilon_r)$, if there are any, arise in this way.

PROOF. $\mathcal{P}$ is the union of the $\sim$-classes of all $c \in S^\star$. We argue like in the previous cases that these subsets of $\mathcal{P}$ are also $\dot{\sim}$-classes. Note again that $\mathcal{Q} = \bar{S}^2 \setminus \mathcal{P}$ is downward closed.

Assume that $(1,0) \dot{\sim} (1,\alpha)$. If there was a one-element coextension $(\bar{S}^2; \trianglelefteq, \bar{\sim}, (1,1))$ of S^2 with respect to $(\varepsilon_l, \varepsilon_r)$, we would have $\dot{\sim} \subseteq \bar{\sim}$ by Lemma 5.3. This means that $\bar{\sim}$ would violate (P2). Thus no such extension exists.

Assume now that $(1,0) \dot{\nsim} (1,\alpha)$. Then $(1,0)$ and $(0,1)$ on the one hand, and $(1,\alpha)$, $(\alpha,1)$, (ε_l,α), and (α,ε_r) on the other hand, are in distinct $\dot{\sim}$-classes. Obviously, an equivalence relation $\bar{\sim} \supseteq \dot{\sim}$ then exists as indicated. We readily see that $\bar{\sim}$ fulfils (P1") and (P2).

We shall show that $\bar{\sim}$ fulfils also (P3'). Let $a,b,c,d,e \in S \setminus \{0,1\}$ such that $(a,b) \bar{\sim} d$ and $(b,c) \bar{\sim} e$. We have to show $(d,c) \bar{\sim} (a,e)$. To this end, we distinguish a number of cases and subcases.

Case 1. Let $d,e \in S^\star$. Then we proceed like in the proof of Theorem 4.4 to conclude that $(d,c) \bar{\sim} (a,e)$.

Case 2. Let $d = \alpha$ and $e \in S^\star$. Then $(a,b) \in \mathcal{Q}$ and $(b,c) \in \mathcal{P}$. Furthermore, we have $(b,c) \sim e$ and $\alpha < e \leqslant b,c$. We distinguish four subcases.

> *Case a.* Let $a \in S^\star$ and $c < \varepsilon_r$. Then $(d,c) = (\alpha,c) \dot{\sim} 0$ by (E4')(a). Furthermore, $(a,e) \dot{\sim} 0$ by (E3')(a). It follows $(d,c) \bar{\sim} 0 \bar{\sim} (a,e)$.
>
> *Case b.* Let $a = \alpha$ and $c < \varepsilon_r$. Then, again by (E4')(a), $(d,c) = (\alpha,c) \dot{\sim} 0$. Moreover, $(a,e) = (\alpha,e) \dot{\sim} 0$ by (E4')(a), because $e \leqslant c$. It follows $(d,c) \bar{\sim} 0 \bar{\sim} (a,e)$.
>
> *Case c.* Let $a \in S^\star$ and $c \geqslant \varepsilon_r$. Then $(d,c) = (\alpha,c) \dot{\sim} \alpha$ by (E4')(b) and thus $(d,c) \bar{\sim} \alpha$. Furthermore, by (E3')(b), $(a,e) \dot{\sim} (a,b)$ and thus $(a,e) \bar{\sim} (a,b) \bar{\sim} d = \alpha$. In particular, $(d,c) \bar{\sim} (a,e)$.
>
> *Case d.* Let $a = \alpha$ and $c \geqslant \varepsilon_r$. Again, $(d,c) = (\alpha,c) \bar{\sim} \alpha$ by (E4')(b). Moreover, $(\alpha,b) = (a,b) \bar{\sim} d = \alpha$. It follows that $b \geqslant \varepsilon_r$ because otherwise we would have $(\alpha,b) \dot{\sim} 0$ by (E4')(a) and

0	y	z	1	
0	y	z	1	1
0	0	z	z	z
0	0	y	y	y
0	0	0	0	0

0	α	y	z	1	
0	α	y	z	1	1
0	α	α	z	z	z
0	0		y	y	y
0	0	0	0	α	α
0	0	0	0	0	0

0	α	y	z	1	
0	α	y	z	1	1
0	α	0	z	z	z
0	0		y	y	y
0	0	0	0	α	α
0	0	0	0	0	0

FIGURE 6. Left: A 4-element tomonoid S. The idempotents of S are 0, z, and 1. S does not possess a one-element coextension with respect to the pair of idempotents $(z,1)$. Middle: By (E4')(b), we have $(z,\alpha)\dot{\sim}\alpha$ and furthermore $(z,y)\dot{\sim}\alpha$. Right: By (E4')(a) we have $(\alpha,z)\dot{\sim}0$ and by (E3')(a) we have $(z,y)\dot{\sim}0$. We conclude that $(1,0)\dot{\sim}(1,\alpha)$ in this case.

thus $(\alpha,b)\bar{\sim}0$. In S, it follows from $\varepsilon_r \leqslant b,c$ that $\varepsilon_r = \varepsilon_r \odot \varepsilon_r \leqslant b \odot c = e$. Consequently, by (E4')(b), $(a,e)=(\alpha,e)\dot{\sim}\alpha$ and hence $(a,e)\bar{\sim}\alpha$. We have shown $(a,e)\bar{\sim}(d,c)$.

Case 3. Let $d \in S^\star$ and $e=\alpha$. Then we proceed analogously to Case 2.

Case 4. Let $d=e=\alpha$. Then $(a,b),(b,c)\in\mathcal{Q}$. Assume that $a<\varepsilon_l$ and $c \geqslant \varepsilon_r$. Then by (E3')(c) it follows $(a,b)\dot{\sim}0$, in contradiction to $(a,b)\bar{\sim}\alpha$. Similarly, we argue in case $a \geqslant \varepsilon_l$ and $c<\varepsilon_r$. We conclude that either $a<\varepsilon_l$ and $c<\varepsilon_r$, or $a \geqslant \varepsilon_l$ and $c \geqslant \varepsilon_r$. Thus $(a,e)=(a,\alpha)\dot{\sim}(\alpha,c)=(d,c)$ by (E4') and hence $(a,e)\bar{\sim}(d,c)$.

By Proposition 3.7, $(\bar{S}^2;\trianglelefteq,\bar{\sim},(1,1))$ is a f.n. tomonoid partition. Clearly, $\bar{S}^2$ is associated with a one-element coextension of S^2 with respect to $(\varepsilon_l,\varepsilon_r)$.

The final statement follows from Lemma 5.3. ■

We conclude that the construction of one-element coextensions works, on the whole, for general f.n. tomonoids similarly as for Archimedean f.n. tomonoids. In fact, the explanations given after Theorem 4.4 for the Archimedean case as well as Remark 4.5 apply, mutatis mutandis, for the general case as well. Maybe one point is worth mentioning. In order to determine the downward closed set $\mathcal{Z}$, the $\bar{\sim}$-class of 0, a set of $\dot{\sim}$-classes needs to be selected. In the Archimedean case, the $\dot{\sim}$-class $\{(\alpha,1),(1,\alpha)\}$ must be disregarded. Here, in the general case, the $\dot{\sim}$-class *containing* $(\alpha,1)$ and $(1,\alpha)$ must be disregarded instead.

The question remains if the pair of idempotents with respect to which we construct an one-element coextension can be chosen arbitrarily or not. The example shown in Figure 6 implies that the answer is negative. It is an open problem how to characterize those pairs of idempotents that can be used.

However, in two cases a coextension always exists. On the one hand, there is always at least one coextension w.r.t. $(1,1)$. In fact, in this case we can argue like in the proof of Theorem 4.4 to see that $\{(1,\alpha),(\alpha,1)\}$ is a $\dot{\sim}$-class and hence $(1,0)\dot{\nsim}(1,\alpha)$. We can consequently choose, e.g. $\mathcal{Z}=\mathcal{Q}\setminus\{(1,\alpha),(\alpha,1)\}$. On the other hand, there is always exactly one extension w.r.t. $(0,0)$, where 0 is the bottom element of S. Then both atom-characterizing idempotents of $\bar{S}$ are α and hence $\mathcal{Z}$ is necessarily the smallest possible set, namely, $\mathcal{Z}=\{(a,b)\colon\ a=0 \text{ or } b=0\}$. The new atom α will be idempotent and the construction may be regarded as the ordinal sum of the original tomonoid and the two-element tomonoid whose monoidal product is the infimum.

The commutative case

Again, let us check which modifications of our procedure are necessary to deal with the commutative case.

Let $\bar{S}$ be a commutative f.n. tomonoid. Then a one-element coextension of $\bar{S}$ with respect to a pair $(\varepsilon_l, \varepsilon_r)$ of idempotents can obviously be commutative only if $\varepsilon_l = \varepsilon_r$. Consequently, we have to restrict Definition 5.2 to this case. In Definition 5.2, we furthermore add again condition (E5). On the basis of these two modification, we may obtain an analogous version of Theorem 5.4, tailored to the commutative case.

6 Conclusion

In recent years, there has been a considerable interest in the structure of negative totally ordered monoids (tomonoids), which, for instance, occur in the context of fuzzy logic. Mostly, the commutative case has been studied. In particular, a classification of MTL-algebras is considered as an important aim. In the present article, we focus on the finite case. Moreover, commutativity is not assumed. Our aim is to contribute to a better understanding of finite, negative (f.n.) tomonoids.

To this end, we have described the set of one-element Rees coextensions of a f.n. tomonoid, i.e., the set of those f.n. tomonoids whose Rees quotient by the atom is the original tomonoid. It has turned out convenient to employ in this context the level-set representation of tomonoids. We have thus worked with certain partitions of a set S^2, where S is a finite chain. The construction consists of two steps and describes the extensions in a transparent, geometrically intuitive, and efficient way.

Among the open questions, we may mention the following. For an extension of a f.n. tomonoid, a pair of idempotents needs to be chosen in advance. Not all combinations, however, are possible and it is unclear how to characterize those pairs that are actually allowed.

Moreover, the tomonoids considered in this article are finite and negative. Remarkably, finiteness is not an essential condition of our method. Alternatively, we could restrict to the assumption that the tomonoid has a bottom element. The extended tomonoid would then again consist of the non-zero elements of the original tomonoid together with a new pair of elements. The situation is more difficult, however, with regard to negativity. To generalize our method to the non-negative case would require major modifications. A related problem is the extension of our method to totally ordered semigroups, which do not necessarily possess an identity.

Finally, our description of a finite, negative tomonoid is relative to a tomonoid that is by one element smaller. For the sake of a classification of all finite, negative tomonoids it would certainly be desirable to understand the construction process not just step by step, but as a whole. This concern certainly implies the need for an approach going well beyond of what we have proposed in the present work.

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RESEARCH ARTICLE

Rees coextensions of finite tomonoids and free pomonoids

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Abstract

A totally ordered monoid, or tomonoid for short, is a monoid endowed with a compatible total order. We reconsider in this paper the problem of describing the one-element Rees coextensions of a finite, negative tomonoid S, that is, those tomonoids that are by one element larger than S and whose Rees quotient by the poideal consisting of the two smallest elements is isomorphic to S. We show that any such coextension is a quotient of a pomonoid $\mathcal{R}(S)$, called the free one-element Rees coextension of S. We investigate the structure of $\mathcal{R}(S)$ and describe the relevant congruences. We moreover introduce a finite family of finite quotients of $\mathcal{R}(S)$ from which the coextensions arise in a particularly simple way.

Keywords Totally ordered monoid · Tomonoid · Rees congruence · Rees coextension · Free pomonoid · Finite-valued logic

1 Introduction

In fuzzy logic, the conjunction is typically interpreted by a binary operation making the chain of truth values into an integral residuated chain [6]. In case that only finitely many truth degrees are used, this means that we deal with finite, negative tomonoids. It is these latter structures that we investigate in the present paper. We recall that a tomonoid is a monoid endowed with a compatible total order [3,4] and negativity means that the identity is the top element. We note that we do not assume commutativity.

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Given a finite, negative tomonoid T, the set containing the smallest two elements is a poideal that induces a congruence on T. Following the terminology of semigroup theory, this is a Rees congruence. The quotient S is by one element smaller than T. We may repeat the process, to obtain a chain of successively smaller tomonoids, ending up eventually with the trivial tomonoid, which consists of one element only. The question obviously arises how to proceed in the opposite direction: given a finite negative tomonoid S, we may wonder how to determine all those tomonoids T that lead back to S by an identification of its two smallest elements. We call such tomonoids T the one-element Rees coextensions of S. We have dealt with the problem in our previous work [10], where we have proposed an algorithm to construct these coextensions in an effective way. Starting from the trivial tomonoid, the method defined in [10] can be applied to calculate successively all finite, negative tomonoids.

The topic of the present paper are the one-element Rees coextensions once more. We are interested this time, however, in algebraic rather than algorithmic aspects and to describe our procedure we have chosen a significantly different approach. Let S be a non-trivial finite, negative tomonoid. We define first what we call the one-element free Rees coextension of S. This is a pomonoid $\mathcal{R}(S)$ with the property that any one-element Rees coextension is among its quotients. The relevant congruences on $\mathcal{R}(S)$ are uniquely determined by the 0 class, hence to determine the one-element coextensions means to characterise those poideals that can assume this role. We provide a characterisation.

However, the pomonoid $\mathcal{R}(S)$ is infinite, in discrepancy with the fact that our intentions involve finite structures. The problem is solvable. The one-element Rees coextensions of S can be roughly classified by the pairs $(\varepsilon_l, \varepsilon_r)$ of two idempotent elements of the original tomonoid. We speak, accordingly, of $(\varepsilon_l, \varepsilon_r)$-coextensions. For a given such pair $(\varepsilon_l, \varepsilon_r)$, we define a pomonoid $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$, which is a quotient of $\mathcal{R}(S)$ and which is finite. There is an effective way of determining $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$. Moreover, each one-element $(\varepsilon_l, \varepsilon_r)$-coextension is in turn a quotient of $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$. To determine the relevant congruences on $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$ is very simple; they correspond to downsets within a certain interval of $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$.

We note that the new approach has enabled us to define a clear context for several notions and constructions of [10] that seemed to be chosen ad hoc. For instance, we used in [10] the somewhat technical notion of a ramification, which was in turn based on an intermediate equivalence relation on a certain subset of the enlarged so-called tomonoid partition. In the present approach, the pomonoids $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$ have taken over the role that the ramification played in the previous context. Furthermore, the present work answers several issues that were left open or vague in our previous work. For instance, the characterisation of the totality of one-element coextensions that we briefly announced in [10, Remark 4.5] is now stated in a precise form. We moreover provide one necessary and one sufficient condition regarding the difficult question whether, for given elements ε_l and ε_r, a one-element $(\varepsilon_l, \varepsilon_r)$-coextension exists at all.

The paper is structured as follows. In Sect. 2, we compile some basic facts about pomonoids, in particular those that concern congruences generated by inequalities, and we discuss free pomonoids. The lengthy Sect. 3 is devoted to the pomonoid $\mathcal{R}(S)$ and those congruences on $\mathcal{R}(S)$ that lead to one-element Rees coextensions of S. Section 4 describes the more specific procedure that results when assuming a pair $(\varepsilon_l, \varepsilon_r)$ to be associated with the coextension. Our concluding Sect. 5 contains a summary and an outlook to further work.

2 Preliminaries

Definition 2.1 A *partially ordered monoid*, or a *pomonoid* for short, is a structure $(S; \leqslant, \cdot, 1)$ such that $(S; \cdot, 1)$ is a monoid, $(S; \leqslant)$ is a poset, and $\leqslant$ is compatible with the monoidal product, that is, for any $a, b, c, d \in S$, $a \leqslant b$ and $c \leqslant d$ imply $a \cdot c \leqslant b \cdot d$.

A pomonoid S is called *negative* if the monoidal identity 1 is the top element. In case that the partial order is a chain, we refer to S as a *totally ordered monoid*, or *tomonoid* for short.

Our work focuses on finite negative tomonoids. We remark that the notion "tomonoid" is taken from [3]; we do not assume, however, a tomonoid to be commutative. The notion "negative" is chosen in accordance with [3] as well. We note that in [4], the notion "negatively ordered" is used instead and that in the context of residuated structures, usually the notion "integral" is preferred [5].

As usual, we will denote the monoidal product often just by juxtaposition.

A map $\varphi \colon S \to T$ between pomonoids is a *homomorphism* if φ is an order-preserving homomorphism of monoids. The homomorphism is called *order-determining* if the following holds: For any $c, d \in T$ such that $c \leqslant d$, there are $s_0, \ldots, s_k \in S$ such that

$$\begin{aligned}&c = \varphi(s_0),\ s_0 \leqslant s_1,\ \varphi(s_1) = \varphi(s_2),\ s_2 \leqslant s_3,\ \varphi(s_3) = \varphi(s_4), \ldots,\\ &s_{k-1} \leqslant s_k,\ \varphi(s_k) = d.\end{aligned}$$

If $\varphi \colon S \to T$ is a surjective, order-determining homomorphism of pomonoids, we call T a *homomorphic image* of S.

Congruences for pomonoids are defined in a way such that quotients correspond to homomorphic images. Let us make the conditions explicit [1]. Let $(S; \leqslant, \cdot, 1)$ be a pomonoid and let θ be a congruence of its monoidal reduct. In order to ensure that the monoid homomorphism $S \to S/\theta, a \mapsto a/\theta$ is order-preserving and order-determining, the partial order on S/θ must be the smallest preorder such that $a/\theta \leqslant b/\theta$ whenever $a \leqslant b$. Let us denote by $\leqslant_\theta$ the θ*-preorder* on S: for $a, b \in S$, we put $a \leqslant_\theta b$ if there is a θ-chain from a to b, that is, if there are $s_0, \ldots, s_k$, $k \geqslant 0$, such that

$$a = s_0 \leqslant s_1 \,\theta\, s_2 \leqslant \cdots \leqslant s_{k-1} \,\theta\, s_k = b.$$

Then the preorder induced by $\leqslant_\theta$ on S/θ is required to be a partial order, that is, antisymmetric. Accordingly, an equivalence relation θ on a pomonoid S is called a *congruence* if

(1) θ is a congruence on the monoid $(S; \cdot, 1)$ and
(2) for any $a, b \in S$, $a \leqslant_\theta b$ and $b \leqslant_\theta a$ imply $a \,\theta\, b$.

We readily check that congruences on pomonoids are in a one-to-one correspondence with homomorphic images.

Note that, for a congruence θ on a pomonoid, each θ-class is convex. In case of a total order, this property may replace condition (2). Indeed, in case that S is a tomonoid, θ is a congruence on S if and only if θ is a monoid congruence and each θ-class is convex.

A particularly simple type of congruence is the following [2]. Recall that a (two-sided) *ideal* of a monoid S is a set $I \subseteq S$ such that $b \in I$ and $a, c \in S$ imply $ab, bc \in I$. Moreover, a subset D of a poset P is a *downset* if $b \in D$ and $a \leqslant b$ imply $a \in D$, and D is an *upset* if $b \in D$ and $a \geqslant b$ imply $a \in D$. Given a pomonoid S, a *poideal* is a subset of S that is both an ideal and a downset.

Lemma 2.2 *Let $(S; \cdot, \leqslant, 1)$ be a pomonoid and I a poideal. For $a, b \in S$, let $a \; \rho_I \; b$ if $a = b$ or $a, b \in I$. Then ρ_I is a congruence.*

Proof Disregarding the order, we may easily verify the well-known fact that ρ_I is a monoid congruence [7]. As I is a downset, it is immediate that ρ_I is even a congruence of pomonoids [2]. □

We call ρ_I, as defined in Lemma 2.2, the *Rees congruence* induced by the poideal I and the quotient, denoted by S/I, is the *Rees quotient* by I. Note that S/I consists of the singletons $\{x\}$, $x \in S \setminus I$, and the poideal I. Following a common practice, we will assume in what follows that $S \setminus I$ is a subset of S/I and we will refer to the class I by (a variant of) a zero symbol, in accordance with the fact that this is an absorbing element.

Let us consider the case of a negative tomonoid S. As $ab, bc \leqslant b$ holds for any $a, b, c \in S$, a subset I is a poideal if and only if I is a downset. In particular, for any $q \in S$, the set $\{a \in S : a \leqslant q\}$ is a poideal and we shall write S/q for the corresponding Rees quotient. Given a finite pomonoid S, we call the pomonoid T a *one-element Rees coextension* of S, or *one-element coextension* for short, if T possesses a unique atom α and T/α is isomorphic to S.

The aim of this paper is to describe the one-element coextensions of a finite, negative tomonoid. Note that this is an instance of the ideal extension problem for posemigroups, see, e.g., [8].

Before we start the actual discussion, some further preparations are necessary. We will need to have ways of generating congruences at hand.

Given an arbitrary binary relation ρ on a pomonoid S, it is possible to construct the smallest pomonoid congruence containing ρ. This is, however, not our intention. Indeed, we do not want to consider congruences θ such that $a/\theta = b/\theta$ whenever $a \; \rho \; b$. We rather want to construct a congruence θ with the effect that, for all pairs of elements a and b such that $a \; \rho \; b$, we just have $a/\theta \leqslant b/\theta$.

The construction is due to Al Subaiei [1]. We outline a proof of the following proposition for later reference.

Proposition 2.3 *Let $(S; \cdot, \leqslant, 1)$ be a pomonoid and let $\trianglelefteq$ be any binary relation on S. Then there is a smallest congruence θ on S such that $a/\theta \leqslant b/\theta$ whenever $a \trianglelefteq b$.*

The congruence θ has the following universal property. Let $f : S \to T$ be a homomorphism from S to a further pomonoid T such that, for any $a, b \in S$, $a \trianglelefteq b$ implies $f(a) \leqslant f(b)$. Then there is a homomorphism $\tilde{f} : S/\theta \to T$ such that $\tilde{f}(a/\theta) = f(a)$.

Proof *(sketched)* For $a, b \in S$, let $a \leqslant_{\trianglelefteq} b$ if $a \leqslant b$ or there are $p_1, q_1, r_1, s_1, \ldots, p_k, q_k, r_k, s_k \in S$, $k \geqslant 1$, such that

$$\begin{array}{lll} a \leqslant p_1 r_1 q_1, & r_1 \trianglelefteq s_1, & p_1 s_1 q_1 \leqslant p_2 r_2 q_2, \\ & r_2 \trianglelefteq s_2, & p_2 s_2 q_2 \leqslant p_3 r_3 q_3, \\ & \ldots, & \\ & r_k \trianglelefteq s_k, & p_k s_k q_k \leqslant b. \end{array} \tag{1}$$

Furthermore, we put $a\,\theta\,b$ if $a \leqslant_{\trianglelefteq} b$ and $b \leqslant_{\trianglelefteq} a$, and we partially order the quotient S/θ by requiring that $a/\theta \leqslant b/\theta$ if $a \leqslant_{\trianglelefteq} b$. We readily check that θ is a pomonoid congruence. It is furthermore clear that θ is the smallest congruence with the property that $a/\theta \leqslant b/\theta$ for all $a, b \in S$ such that $a \trianglelefteq b$. Also the remaining part is shown by standard arguments. □

Given a relation $\trianglelefteq$ on a pomonoid S, we denote the congruence specified in Proposition 2.3 by $\Theta(\trianglelefteq)$ and we say that $\Theta(\trianglelefteq)$ is *generated* by $\trianglelefteq$.

We next turn to free constructions, for which we may once again refer to [1].

Proposition 2.4 *Let $(G; \leqslant)$ be a poset. Then there is a pomonoid $\mathcal{F}(G)$ and an order-preserving map $\iota\colon G \to \mathcal{F}(G)$ with the following universal property: for any pomonoid T and order-preserving map $f\colon G \to T$, there is a unique pomonoid homomorphism $\bar{f}\colon \mathcal{F}(G) \to T$ such that $f = \bar{f} \circ \iota$.*

Proof *(sketched)* We take $\mathcal{F}(G)$ as the monoid of words $a_1 \ldots a_n$, where $n \geqslant 0$ and $a_1, \ldots, a_n \in G$. The product is defined by concatenation and the identity is given by the empty word. We partially order $\mathcal{F}(G)$ be setting

$$a_1 \ldots a_m \leqslant b_1 \ldots b_n \quad \text{if} \quad n = m \text{ and } a_1 \leqslant b_1, \ldots, a_n \leqslant b_n. \tag{2}$$

Finally, we let $\iota\colon G \to \mathcal{F}(G)$ map each $a \in G$ to itself, considered as a word of length 1. □

The pomonoid $\mathcal{F}(G)$, that is, the monoid of words over G partially ordered by (2), will be called the *free pomonoid* on the poset G. We will denote its identity, the empty word, by ε. Note that the insertion map $\iota\colon G \to \mathcal{F}(G)$, which maps $a \in G$ to the corresponding word of length 1, is an embedding of posets: we have $a \leqslant b$ in G if and only if $\iota(a) \leqslant \iota(b)$ in $\mathcal{F}(G)$. In the sequel we can thus safely view G as a subposet of $\mathcal{F}(G)$. Accordingly, we will say that any order-preserving map from G to some pomonoid can be extended to $\mathcal{F}(G)$.

Combining Propositions 2.3 and 2.4, we see that we may specify pomonoids as quotients of free pomonoids in a common way: we indicate a poset G together with a set of inequalities among the elements of $\mathcal{F}(G)$. Identifying the inequalities with a binary relation $\trianglelefteq$ on $\mathcal{F}(G)$, the pomonoid defined in this way is the quotient of the free pomonoid on G induced by the congruence generated by $\trianglelefteq$, that is, $\mathcal{F}(G)/\Theta(\trianglelefteq)$. In practice, we will indicate the poset together with a set of equalities and inequalities; for, an equality $a = b$ can be identified with the two inequalities $a \leqslant b$ and $b \leqslant a$.

To avoid a cumbersome presentation, we will in this context abuse notation as follows. Unless otherwise noticed, we will make no difference in notation between the elements of $\mathcal{F}(G)$, and in particular the elements of G, on the one hand and their $\Theta(\trianglelefteq)$-classes on the other hand. For instance, let $a \in G$; then we will usually refer to the element $a/\Theta(\trianglelefteq)$ of $\mathcal{F}(G)/\Theta(\trianglelefteq)$ by the symbol a as well. It will be clear from the context what we mean.

3 The free one-element Rees coextension

In this section, $(S; \cdot, \leqslant, 1)$ is a finite negative tomonoid. We are interested in the one-element (Rees) coextensions of S. To this end, we will define a pomonoid of which any of the desired coextensions is a quotient.

If $S = \{1\}$, we call S the trivial tomonoid. In this case, S possesses exactly one one-element Rees-coextension. Indeed, the two element chain $\{0, 1\}$ can be made into a tomonoid in only one way: by defining $1 \cdot 1 = 1$ and $0 \cdot 1 = 1 \cdot 0 = 0 \cdot 0 = 0$. We will assume from now on that S is non-trivial, that is, S possesses at least two elements.

Denoting by $\dot{0}$ the smallest element of S, we define $S^\star = S \setminus \{\dot{0}\}$ and $\bar{S} = S^\star \dot{\cup} \{0, \alpha\}$, where 0 and α are new elements. We extend the total order on $S^\star$ to $\bar{S}$, letting $0 \leqslant \alpha \leqslant a$ for any $a \in S^\star$.

We view $S^\star$ as a partial monoid; cf., e.g., [11]. The multiplication on $S^\star$ is defined partially, the product of two elements $a, b \in S^\star$ being defined only if ab, calculated in S, is an element of $S^\star$. Then $1 \in S^\star$, the products $1\,a$ and $a\,1$ exist for all $a \in S^\star$, and $1\,a = a\,1 = a$. Furthermore, for $a, b, c \in S^\star$, the products ab and $(ab)c$ exist if and only if bc and $a(bc)$ exist, in which case $(ab)c = a(bc)$.

Our aim is to determine all possible ways of endowing $\bar{S}$ with a monoidal product such that $\bar{S}$ becomes a one-element coextension of S. Let us clarify what this means. Let $(\bar{S}; \bullet, \leqslant, 1)$ be a finite, negative tomonoid and assume that S is the Rees quotient of $\bar{S}$ by the ideal $\{0, \alpha\}$. Note that S consists of the elements of $S^\star$, which we identify with the singleton congruence classes, and of the element $\dot{0}$, which corresponds to the congruence class $\{0, \alpha\}$. The multiplication in S arises from the multiplication in $\bar{S}$ as follows: for $a, b \in S$, we have

$$a \cdot b \;=\; \begin{cases} a \bullet b & \text{if } a, b \neq \dot{0} \text{ and, in } \bar{S},\, a \bullet b \notin \{0, \alpha\}, \\ \dot{0} & \text{otherwise.} \end{cases} \tag{3}$$

In the present context, we consider $(S; \cdot, \leqslant, 1)$ as fixed and we intend to determine $(\bar{S}; \bullet, \leqslant, 1)$. From (3) we observe that the multiplication on S determines to a good extent the one on $\bar{S}$: for any two elements $a, b \in \bar{S} \setminus \{0, \alpha\}$ whose product is non-zero in S, this product is in $\bar{S}$ the same as in S. We may say that the partial monoid $S^\star$ is a substructure of $\bar{S}$: whenever, for $a, b \in S^\star$, ab is defined and equals c in $S^\star$, we have that $ab = c$ holds in $\bar{S}$ as well. Consequently, our aim is the extension of the partial tomonoid $S^\star$ to a (total) tomonoid based on the chain $\bar{S}$.

For a pair $(a, b) \in \bar{S}^2$ such that $a, b \in S^\star$ and ab is defined in $S^\star$, the product of a and b is in $\bar{S}$ thus determined from the outset. We denote the set of all remaining pairs of elements of $\bar{S}$ as follows:

$$\begin{aligned} \mathcal{N} = \{(a, b) \in \bar{S}^2 \colon a, b > \alpha \text{ and } ab = \dot{0} \text{ in } S\} \\ \cup\ \{0, \alpha\} \times \bar{S} \ \ \cup\ \ \bar{S} \times \{0, \alpha\}. \end{aligned}$$

For a pair $(a, b) \in \mathcal{N}$, it is in general not clear what the product of a and b in $\bar{S}$ is, but it is clear that ab equals either 0 or α. Indeed, if $a, b > \alpha$, then $ab = c > \alpha$ in $\bar{S}$ would imply that we have $ab = c$ in S as well and thus $ab \neq \dot{0}$ in S. Moreover, if $a \leqslant \alpha$ or $b \leqslant \alpha$, it follows $ab \leqslant \alpha$ by the negativity of $\bar{S}$, that is, $ab = 0$ or $ab = \alpha$. Determining the coextension $\bar{S}$ means accordingly that we have to suitably divide $\mathcal{N}$ into a set of pairs mapped to 0 and a set of pairs mapped to α.

We add that even the products of certain pairs in $\mathcal{N}$ are clear from the outset. Indeed, we have $0\,a = a\,0 = 0$ for any $a \in \bar{S}$ and $1\,\alpha = \alpha\,1 = \alpha$. But the following examples show that in all remaining cases the product may really be either 0 or α.

Example 3.1 For $a, b \in \bar{S}$, let us define

$$a \bullet b \;=\; \begin{cases} ab & \text{if } a, b \in S^\star \text{ and } ab \text{ exists in } S^\star, \\ \alpha & \text{if } a = \alpha \text{ and } b = 1, \text{ or } a = 1 \text{ and } b = \alpha, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

We readily verify that $(\bar{S}; \bullet, \leqslant, 1)$ is a one-element coextension of $(S; \cdot, \leqslant, 1)$.

Example 3.2 A further one-element coextension is $(\bar{S}; \bullet, \leqslant, 1)$, where, for $a, b \in \bar{S}$,

$$a \bullet b \;=\; \begin{cases} ab & \text{if } a, b \in S^\star \text{ and } ab \text{ exists in } S^\star, \\ 0 & \text{if } a = 0 \text{ or } b = 0, \\ \alpha & \text{otherwise.} \end{cases} \tag{5}$$

These examples are, in a sense, the extreme cases. Indeed, in case of Example 3.1, all products that are not determined from the outset are defined to be 0, whereas in case of Example 3.2 all these are α.

We now provide the key definition on which the present paper is based.

Definition 3.3 Let $\mathcal{R}(S)$ be the free pomonoid over the chain $\bar{S}$, subject to the following conditions:

(a) $a\,b = c$ for any $a, b, c \in S^\star$ fulfilling this equation in S,
(b) $\varepsilon = 1$.
(c) $a\,b \leqslant \alpha$ for any $(a, b) \in \mathcal{N}$,
(d) $0\,a = 0$ for any $a \in \bar{S}$,

We call $\mathcal{R}(S)$ the *free one-element Rees coextension*, or simply the *free one-element coextension* of S.

Proposition 3.4 *Let T be a one-element coextension of S. Then there is a congruence θ on $\mathcal{R}(S)$ such that each θ-class contains exactly one $a \in \bar{S}$ and $\mathcal{R}(S)/\theta$ is isomorphic to T.*

Proof Note that $\bar{S}$ and T are chains of equal size; let $f : \bar{S} \to T$ be the order isomorphism. By Proposition 2.4, f extends to a pomonoid homomorphism $\bar{f} : \mathcal{F}(\bar{S}) \to T$.

Let us identify the equalities and inequalities (a)–(d) in Definition 3.3 with the binary relation $\trianglelefteq$ on $\mathcal{F}(\bar{S})$; for instance, by (c) we require $ab \trianglelefteq \alpha$ for $(a, b) \in \mathcal{N}$.

Then $\mathcal{R}(S) = \mathcal{F}(\bar{S})/\Theta(\trianglelefteq)$. It is obvious that, for any $a, b \in \mathcal{F}(\bar{S})$, $a \trianglelefteq b$ implies $\bar{f}(a) \leqslant \bar{f}(b)$. Hence, by Proposition 2.3, there is a homomorphism $\bar{\bar{f}} \colon \mathcal{R}(S) \to T$ such that $\bar{\bar{f}}(a/\Theta(\trianglelefteq)) = \bar{f}(a)$ for any $a \in \mathcal{F}(\bar{S})$. In particular, for $a \in \bar{S}$, we have $\bar{\bar{f}}(a/\Theta(\trianglelefteq)) = f(a)$ and it follows that $\bar{\bar{f}}$ is order-determining.

Thus T is a homomorphic image of $\mathcal{R}(S)$ and the assertions follow. □

To shorten the subsequent statements, let us call a congruence on $\mathcal{R}(S)$ that, in the way indicated in Proposition 3.4, leads to a one-element coextension of S, a *coextension congruence*. Our aim is therefore to characterise these particular congruences on $\mathcal{R}(S)$.

We begin by describing $\mathcal{R}(S)$ itself. From now on we will make use of the simplified notation announced earlier: an element of $\bar{S}$ or a word in $\mathcal{F}(\bar{S})$ will also denote its congruence class in $\mathcal{R}(S)$.

Proposition 3.5 *Let β be the bottom element of $S^\star$, that is, the atom of S.*

(i) *$\mathcal{R}(S)$ is the union of the intervals $[0, \alpha]$ and $[\beta, 1]$, and we have $0 < \alpha < \beta$. Moreover, $[0, \alpha]$ is a poideal and $[\beta, 1]$ is a chain, consisting of the pairwise distinct elements $a \in S^\star$.*

(ii) *1 is the top element and the identity of $\mathcal{R}(S)$; in particular, $\mathcal{R}(S)$ is a negative pomonoid.*

(iii) *Let $a, b \in [\beta, 1]$. If $ab = c$ holds in $S^\star$, then we have $ab = c$ in $\mathcal{R}(S)$ as well. If ab is in $S^\star$ undefined, then we have $ab \leqslant \alpha$ in $\mathcal{R}(S)$.*

Proof We first show that the elements of $\bar{S}$ are in $\mathcal{R}(S)$ pairwise distinct. Indeed, for any one-element coextension T there is by Proposition 3.4 a congruence θ on $\mathcal{R}(S)$ such that $a \in \bar{S}$ are all in distinct classes and T is isomorphic to the quotient. Moreover, by Examples 3.1 and 3.2, a one-element coextension always exists. We conclude that distinct elements of $\bar{S}$ are indeed distinct in $\mathcal{R}(S)$.

By the definition of $\mathcal{R}(S)$, the elements of $\bar{S}$ moreover form a chain in $\mathcal{R}(S)$. In particular, we have $0 < \alpha < \beta \leqslant 1$, thus $[0, \alpha]$ and $[\beta, 1]$ are disjoint intervals of $\mathcal{R}(S)$.

We have $\varepsilon = 1$ by the defining equality (b) of Definition 3.3, thus 1 is the identity of $\mathcal{R}(S)$. We furthermore have $a \leqslant 1$ for any $a \in \bar{S}$, and $1 \cdot 1 = 1$. For $a_1, \ldots, a_k \in \bar{S}$, $k \geqslant 1$, it follows $a_1 \ldots a_k \leqslant 1 \cdot \cdots \cdot 1 = 1$. Hence 1 is the top element of $\mathcal{R}(S)$.

Furthermore, equality (d) of Definition 3.3 and the fact that $0 \leqslant a$ for any $\bar{S}$ imply that 0 is the bottom element of $\mathcal{R}(S)$. Hence $[0, \alpha]$ is a downset and since $\mathcal{R}(S)$ is negative, $[0, \alpha]$ is actually a poideal.

Let now $a, b \in S^\star$. Then $a, b \in [\beta, 1]$. Assume first that in S we have $ab = c \neq \dot{0}$. This means $ab = c$ holds in $S^\star$ and by equality (a) the same equality then holds in $\mathcal{R}(S)$. Assume second that in S we have $ab = \dot{0}$. This means that ab is in $S^\star$ undefined and by inequality (c) we have $ab \leqslant \alpha$.

As $\mathcal{R}(S)$ is (as a monoid) generated by $\bar{S}$ and $[0, \alpha]$ is a poideal, we conclude that $[\beta, 1]$ consists of the elements of $S^\star$ only and $\mathcal{R}(S) = [0, \alpha] \cup [\beta, 1]$. □

We thus see that the free one-element coextension $\mathcal{R}(S)$ consists of $S^\star$ and, strictly below this set, the poideal $[0, \alpha]$. We note that the latter set is infinite. Proposition 3.5 furthermore implies that $\mathcal{R}(S)/[0, \alpha]$ is (isomorphic to) S.

Assume that θ is a coextension congruence. Then each θ-class contains exactly one $a \in \bar{S}$. Hence, by Proposition 3.5, the quotient consists of the singletons $a/\theta = \{a\}$, $a \in S^\star$, as well as $0/\theta$ and α/θ, which partition $[0, \alpha]$. In particular, a coextension congruence is uniquely determined by the class $0/\theta$.

Proposition 3.6 *Let $Z \subseteq [0, \alpha]$ be a downset of $\mathcal{R}(S)$ that is non-empty but does not contain α. Assume that the following condition holds:*

$$\begin{aligned} &\text{For any } a, b \in [0, \alpha] \setminus Z \text{ and any } c \in \bar{S}, \quad ac \in Z \text{ if and only if } bc \in Z, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{and} \quad ca \in Z \text{ if and only if } cb \in Z. \end{aligned}$$

For $a, b \in \mathcal{R}(S)$, let

$$a \; \theta_Z \; b \quad \text{if and only if} \quad a = b \text{ or } a, b \in Z \text{ or } a, b \in [0, \alpha] \setminus Z. \tag{6}$$

Then θ_Z is a congruence on $\mathcal{R}(S)$ such that $Z = 0/\theta_Z$, and $\mathcal{R}(S)/\theta_Z$ is a one-element coextension of S.

Up to isomorphism, every one-element coextension of S arises in this way from a unique downset Z of $\mathcal{R}(S)$.

Proof Let Z be a downset of $[0, \alpha]$ and let θ_Z be given as indicated. The θ_Z-classes are then the singletons $\{a\}$ for each $a \in S^\star$, as well as Z and $[0, \alpha] \setminus Z$. To show that θ_Z is a congruence, let $a, b, c \in \mathcal{R}(S)$. Assume that $a \; \theta_Z \; b$; we have to show that $ac \; \theta_Z \; bc$. Obviously, we may restrict to the case $c \in \bar{S}$. If $a \in S^\star$, then $a = b$ and the assertion is clear. If $a \in Z$, then also $b \in Z$, and because Z is an ideal, we have $ac, bc \in Z$ and hence $ac \; \theta_Z \; bc$. Let $a \in [0, \alpha] \setminus Z$. Then also $b \in [0, \alpha] \setminus Z$. Because $[0, \alpha]$ is a poideal, we have that $ac, bc \in [0, \alpha]$. By assumption, ac and bc are either both in Z or both in $[0, \alpha] \setminus Z$, thus $ac \; \theta_Z \; bc$. A similar argument shows that also $ca \; \theta_Z \; cb$ holds, and we conclude that θ_Z is a monoid congruence.

To see that θ_Z is in fact a pomonoid congruence, assume $c_0 \leqslant c_1 \; \theta_Z \; c_2 \leqslant \ldots \theta_Z \; c_k = c_0$. As Z and $[0, \alpha]$ are downsets, it follows that $c_0, \ldots, c_k$ all lie in the same θ_Z-class.

Consider now the Rees quotient of $\mathcal{R}(S)/\theta_Z$ by the two-element poideal $[0/\theta_Z, \alpha/\theta_Z]$. The resulting tomonoid is obviously isomorphic to $\mathcal{R}(S)/[0, \alpha]$ and hence to S. This means that $\mathcal{R}(S)/\theta_Z$ is a one-element coextension of S.

We now turn to the last assertion. Let T be a one-element coextension of S. By Proposition 3.4, there is a congruence θ on $\mathcal{R}(S)$ such that $\mathcal{R}(S)/\theta$ is isomorphic with T. Identifying T with $\bar{S}$, the elements of T are the classes a/θ, $a \in \bar{S}$, and we have $a/\theta = \{a\}$ for $a \in S^\star$ and $0/\theta \cup \alpha/\theta = [0, \alpha]$.

Let $Z = 0/\theta$. As 0 is the bottom element of $\mathcal{R}(S)$ and the θ-classes are convex, Z is a downset of $\mathcal{R}(S)$ such that $\varnothing \subset Z \subset [0, \alpha]$. Furthermore, we have $[0, \alpha] \setminus Z = \alpha/\theta$. It follows that $\theta = \theta_Z$ according to (6). Moreover, for any $a, b \in [0, \alpha] \setminus Z$, we have $a \; \theta \; b \; \theta \; \alpha$ and hence $ac \; \theta \; bc \; \theta \; \alpha c$. Since $\alpha c \in [0, \alpha]$ it follows that ac and bc are both in Z or both in $[0, \alpha] \setminus Z$. This completes the proof that all one-element coextensions of S arise in the way asserted from a downset Z of $\mathcal{R}(S)$.

It remains to show that the downset giving rise to a one-element coextension is unique. But this is clear from the fact that distinct downsets $Z \subset [0, \alpha]$ induce distinct congruences θ, that is, distinct products on $\mathcal{R}(S)/\theta$. ◻

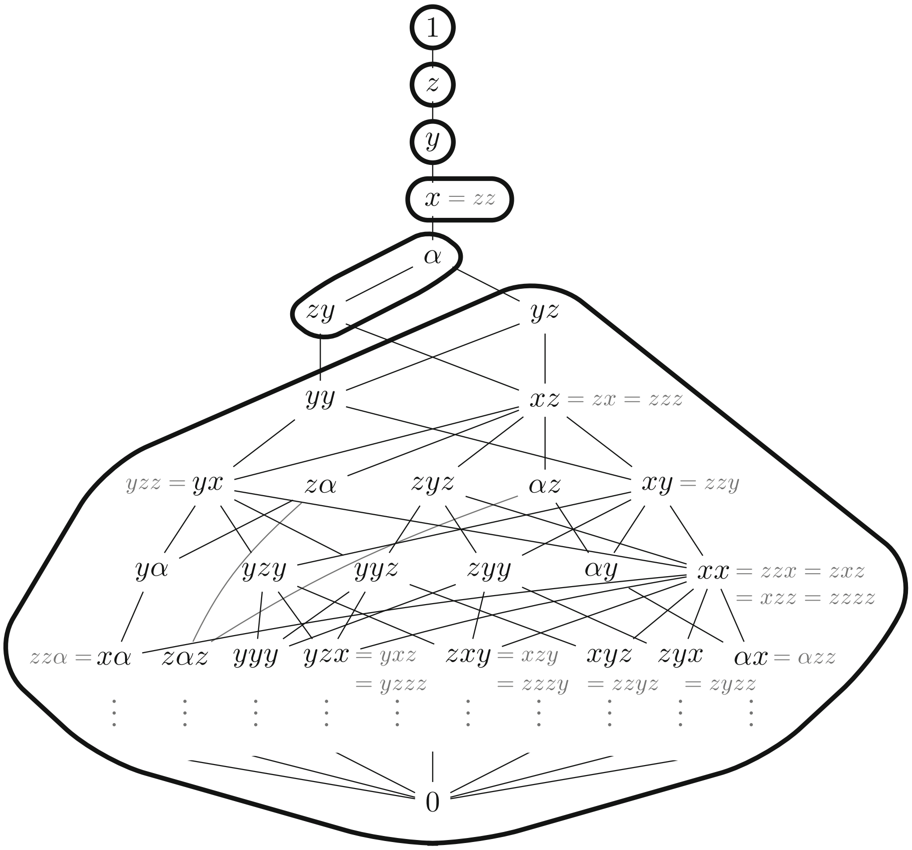

Fig. 1 The free one-element coextension $\mathcal{R}(S)$ for the tomonoid S from Example 3.7 and a coextension congruence

Example 3.7 We consider an example illustrating Proposition 3.6. Here as well as in case of subsequent examples of tomonoids, we denote the elements of the base set by lower case Latin letters from the end of the alphabet, except for the bottom and top elements, which are denoted by $\dot{0}$ and 1, respectively. Let S be the five-element chain, that is, let

$$S \;=\; \{\dot{0}, x, y, z, 1\},$$

understood to be totally ordered as indicated. We make S into a tomonoid by defining, for $a, b \in S$,

$$a \cdot b \;=\; \begin{cases} b & \text{if } a = 1, \\ a & \text{if } b = 1, \\ x & \text{if } a = b = z, \\ \dot{0} & \text{otherwise.} \end{cases}$$

The free one-element coextension $\mathcal{R}(S)$ is infinite; Fig. 1 shows a part of its poset reduct. Furthermore, an example of a coextension congruence is indicated; the congruence classes are encircled.

According to Proposition 3.6, it would now be natural to aim at a characterisation of the downset $Z = 0/\theta$, where θ is a coextension congruence. Although this is feasible, we proceed more conveniently as follows.

Lemma 3.8 *Let θ be a coextension congruence on $\mathcal{R}(S)$ and define*

$$\mathcal{Z}_\theta \;=\; \{(a, b) \in \bar{S}^2 \colon ab\,\theta\,0\}.$$

Then $\mathcal{Z}_\theta \subseteq \mathcal{N}$. Moreover, θ is the congruence generated by the equations

$$a\,b = 0 \;\text{ for any } (a, b) \in \mathcal{Z}_\theta, \quad a\,b = \alpha \;\text{ for any } (a, b) \in \mathcal{N} \setminus \mathcal{Z}_\theta.$$

Proof If $(a, b) \in \bar{S}^2 \setminus \mathcal{N}$, then $ab \in S^\star$ and hence $(a, b) \notin \mathcal{Z}_\theta$. That is, $\mathcal{Z}_\theta \subseteq \mathcal{N}$.

The only θ-classes that are not singletons are $0/\theta$ and α/θ. Let $a = a_1 \ldots a_k \in \mathcal{R}(S)$, where $a_1, \ldots, a_k \in \bar{S}$, $k \geqslant 0$. If $k = 0$, then $a = 1$ and hence neither $a\,\theta\,0$ nor $a\,\theta\,\alpha$ holds. If $k = 1$, then $a\,\theta\,0$ iff $a = 0$, and $a\,\theta\,\alpha$ iff $a = \alpha$. Let $k = 2$. Then $a\,\theta\,0$ iff $(a_1, a_2) \in \mathcal{Z}_\theta$ and $a\,\theta\,\alpha$ iff $(a_1, a_2) \in \mathcal{N} \setminus \mathcal{Z}_\theta$. Finally, assume that $k \geqslant 3$. Let $a_1 \ldots a_{k-1}\,\theta\,z$, where $z \in \bar{S}$. Then again, $a\,\theta\,0$ iff $(z, a_k) \in \mathcal{Z}_\theta$, and $a\,\theta\,\alpha$ iff $(z, a_k) \in \mathcal{N} \setminus \mathcal{Z}_\theta$. Hence the assertion follows by an inductive argument. □

We conclude that each coextension congruence θ on $\mathcal{R}(S)$ is determined by the set $\mathcal{Z}_\theta \subseteq \mathcal{N}$. The following lemma makes explicit how the class $0/\theta$ is determined by $\mathcal{Z}_\theta$.

Lemma 3.9 *Let θ be a coextension congruence on $\mathcal{R}(S)$. Let $a = a_1 \ldots a_k \in [0, \alpha]$, where $a_1, \ldots, a_k \in \bar{S}$, $k \geqslant 1$. Let $i \in \{1, \ldots, k\}$ be smallest such that the product $a_1 \ldots a_i$ is not in $S^\star$. We have $a\,\theta\,0$ if and only if either $a_i = 0$, or $i < k$ and $(\alpha, a_j) \in \mathcal{Z}_\theta$ for some $i < j \leqslant k$, or $i \geqslant 2$ and $(a_1 \ldots a_{i-1}, a_i) \in \mathcal{Z}_\theta$.*

Proof Let i be as indicated; then $a_1 \ldots a_i\,\theta\,0$ or $a_1 \ldots a_i\,\theta\,\alpha$. It is easily checked that $a\,\theta\,0$ holds under one of the indicated conditions.

Conversely, assume $a\,\theta\,0$. If $a_1 \ldots a_i\,\theta\,0$, then either $i = 1$ and $a_i = a_1 = 0$, or $i \geqslant 2$ and $(a_1 \ldots a_{i-1}, a_i) \in \mathcal{Z}_\theta$. If $a_1 \ldots a_i\,\theta\,\alpha$, then there is a smallest $j \in \{i+1, \ldots k\}$ such that $a_1 \ldots a_{j-1}\,\theta\,\alpha$, thus $a_1 \ldots a_j\,\theta\,0$ and consequently $(\alpha, a_j) \in \mathcal{Z}_\theta$. □

Our aim is the characterisation of the set $\mathcal{Z}_\theta \subseteq \mathcal{N}$. We need some preparations.

Definition 3.10 Let θ be a coextension congruence on $\mathcal{R}(S)$. Then we call the smallest element $\varepsilon_l \in S^\star \cup \{\alpha\}$ such that $\varepsilon_l\alpha\,\theta\,\alpha$ the *left border* for θ. Similarly, we call the smallest element $\varepsilon_r \in S^\star \cup \{\alpha\}$ such that $\alpha\varepsilon_r\,\theta\,\alpha$ the *right border* for θ.

We will denote the left and right border for a coextension congruence θ in the sequel always by ε_l and ε_r, respectively. The name is justified by the fact that ε_l and ε_r provide a limitation for the class $0/\theta$ and hence for $\mathcal{Z}_\theta$, as stated in the following remark.

Remark 3.11 Let θ be a coextension congruence on $\mathcal{R}(S)$. For any $a \in \bar{S}$, we have $0 = 0\,\alpha \leqslant a\,\alpha \leqslant 1\,\alpha = \alpha$ and thus either $a\,\alpha\,\theta\,0$ or $a\,\alpha\,\theta\,\alpha$. Hence ε_l is the smallest element in $\bar{S}$ such that $(\varepsilon_l, \alpha) \notin \mathcal{Z}_\theta$, that is, for $a \in \bar{S}$ we have $(a, \alpha) \in \mathcal{Z}_\theta$ iff $a < \varepsilon_l$. Similarly we may characterise ε_r.

Note furthermore that, for any $a, b \in \bar{S}$ such that $(a, b) \in \mathcal{Z}_\theta$, we have $a = 0$, or $b = 0$, or $a < \varepsilon_l$ and $b < \varepsilon_r$.

Lemma 3.12 *Let θ be a coextension congruence with the borders $\varepsilon_l, \varepsilon_r$.*

(i) *$\varepsilon_l = \alpha$ if and only if $\varepsilon_r = \alpha$. In this case, $0/\theta = \{0\}$ and the quotient is isomorphic to $(\bar{S}; \bullet, \leqslant, 1)$ from Example 3.2. In particular, α/θ is idempotent.*
(ii) *Let $\varepsilon_l, \varepsilon_r \in S^\star$. Then both ε_l and ε_r are idempotent elements of S.*

Proof (i) Assume that $\varepsilon_l = \alpha$. This means $\alpha^2 \,\theta\, \alpha$ and hence also $\varepsilon_r = \alpha$. Similarly, we see that $\varepsilon_r = \alpha$ implies $\varepsilon_l = \alpha$.
Moreover, for any $a, b \in \bar{S} \setminus \{0\}$, we have in this case $ab/\theta = a/\theta \cdot b/\theta \geqslant (\alpha/\theta)^2 = \alpha/\theta$. It follows $0/\theta = \{0\}$ and multiplication in $\mathcal{R}(S)/\theta$ is given according to (5).

(ii) By definition, $\varepsilon_l\alpha \,\theta\, \alpha$ and hence $\varepsilon_l^2\alpha \,\theta\, \varepsilon_l\alpha \,\theta\, \alpha$. We claim that $\varepsilon_l^2 \in S^\star$. Indeed, otherwise $\varepsilon_l^2 \leqslant \alpha < \varepsilon_l$ would imply $\varepsilon_l^2\alpha \leqslant \alpha^2 \,\theta\, 0$, a contradiction. From the minimality property of ε_l we conclude $\varepsilon_l \leqslant \varepsilon_l^2$. As we have $\varepsilon_l^2 \leqslant \varepsilon_l$ by the negativity of $\mathcal{R}(S)$, we have that $\varepsilon_l^2 = \varepsilon_l$ holds in $\mathcal{R}(S)$ and hence also in S.
The assertion concerning ε_r is seen analogously.

□

We conclude that the border elements $\varepsilon_l, \varepsilon_r$ either both equal α or are idempotent elements of the partial monoid $S^\star$.

Note that since $S^\star = S \setminus \{\dot{0}\}$ and $\dot{0}$ is an idempotent element of S, we can identify the pair $\varepsilon_l, \varepsilon_r$ with a pair of two idempotent elements of the original tomonoid S; in this case, $\dot{0} \in S$ corresponds to $\alpha \in \bar{S}$.

We are now ready to compile the characteristic properties of the set $\mathcal{Z}_\theta$.

Lemma 3.13 *Let θ be a coextension congruence on $\mathcal{R}(S)$ with the borders $\varepsilon_l, \varepsilon_r$. Then $\mathcal{Z} = \mathcal{Z}_\theta$ is a subset of $\mathcal{N}$ such that, for any $a, b, c, d \in \bar{S}$, the following holds:*

(Z1) *If $(b, d) \in \mathcal{Z}$ and $a \leqslant b$, $c \leqslant d$, then $(a, c) \in \mathcal{Z}$.*
(Z2) *$(a, 0), (0, b) \in \mathcal{Z}$ and $(\alpha, 1), (1, \alpha) \notin \mathcal{Z}$.*
(Z3) *Let $ab, bc \in S^\star$. Then $(a, bc) \in \mathcal{Z}$ if and only if $(ab, c) \in \mathcal{Z}$.*
(Z4) *If $(a, b), (b, c) \in \mathcal{N}$, $a \geqslant \varepsilon_l$, and $c < \varepsilon_r$, then $(b, c) \in \mathcal{Z}$.*
If $(a, b), (b, c) \in \mathcal{N}$, $a < \varepsilon_l$, and $c \geqslant \varepsilon_r$, then $(a, b) \in \mathcal{Z}$.
(Z5) *Let $bc \in S^\star$ or $b = 1$. If $(a, b) \in \mathcal{N}$, $c < \varepsilon_r$, then $(a, bc) \in \mathcal{Z}$.*
Let $ab \in S^\star$ or $b = 1$. If $(b, c) \in \mathcal{N}$, $a < \varepsilon_l$, then $(ab, c) \in \mathcal{Z}$.
(Z6) *Let $bc \in S^\star$ or $b = 1$. If $(a, b) \in \mathcal{N}$, $c \geqslant \varepsilon_r$, and $(a, bc) \in \mathcal{Z}$, then $(a, b) \in \mathcal{Z}$.*
Let $ab \in S^\star$ or $b = 1$. If $(b, c) \in \mathcal{N}$, $a \geqslant \varepsilon_l$, and $(ab, c) \in \mathcal{Z}$, then $(b, c) \in \mathcal{Z}$.

Proof (Z1), (Z2), and (Z3) are immediate from the definition of $\mathcal{Z}_\theta$.

In case of (Z4)–(Z6), we show the first halves only; the second ones are seen similarly.

Ad (Z4): Assume that $(a, b), (b, c) \in \mathcal{N}$, $a \geqslant \varepsilon_l$, and $c < \varepsilon_r$. Then $ab \leqslant \alpha$ and hence $abc \,\theta\, 0$ by the definition of ε_r. Moreover, $bc \leqslant \alpha$ and hence $bc \,\theta\, 0$ or $bc \,\theta\, \alpha$. In the latter case, it would follow $abc \,\theta\, \alpha$ by the definition of ε_l, hence $bc \,\theta\, 0$.

Ad (Z5): Assume that $(a, b) \in \mathcal{N}$ and $c < \varepsilon_r$. Then $abc\,\theta\,0$ and hence $(a, bc) \in \mathcal{Z}$.

Ad (Z6): Assume that $(a, b) \in \mathcal{N}$ and $c \geqslant \varepsilon_r$. Then $(a, b) \in \mathcal{N} \setminus \mathcal{Z}$ implies $abc\,\theta\,\alpha$ and hence $(a, bc) \in \mathcal{N} \setminus \mathcal{Z}$. □

We now turn to the converse direction: given a pair $\varepsilon_l, \varepsilon_r$ of border elements, we describe the possible subsets $\mathcal{Z}_\theta$ of $\mathcal{N}$ that determine a coextension congruence θ.

Lemma 3.14 *Let $\varepsilon_l, \varepsilon_r \in S^\star \cup \{\alpha\}$ and let $\mathcal{Z} \subseteq \mathcal{N}$ be such that, for any $a, b, c, d \in \bar{S}$, the conditions* (Z1)–(Z6) *hold. Then there is a uniquely determined coextension congruence θ with the borders $\varepsilon_l, \varepsilon_r$ such that $\mathcal{Z} = \mathcal{Z}_\theta$.*

Proof Before beginning the proof, let us show two additional properties of $\mathcal{Z}$.

(Z7) For any $(a, b) \in \mathcal{Z}$ such that $a, b \neq 0$, we have $a < \varepsilon_l$ and $b < \varepsilon_r$.

Indeed, we may apply the second half of (Z6) to $b = 1$ and $c = \alpha$, to conclude that $(a, \alpha) \notin \mathcal{Z}$ if $a \geqslant \varepsilon_l$. In view of (Z1), this shows one half of (Z7). The other one is seen similarly.

(Z8) $(a, \alpha) \in \mathcal{Z}$ for any $a < \varepsilon_l$, and $(a, \alpha) \in \mathcal{N} \setminus \mathcal{Z}$ for any $a \geqslant \varepsilon_l$. Similarly, $(\alpha, b) \in \mathcal{Z}$ for any $b < \varepsilon_r$, and $(\alpha, b) \in \mathcal{N} \setminus \mathcal{Z}$ for any $b \geqslant \varepsilon_r$.

Indeed, applying the second half of (Z5) to $b = 1$ and $c = \alpha$, we see that $(a, \alpha) \in \mathcal{Z}$ if $a < \varepsilon_l$. Furthermore, if $a \geqslant \varepsilon_l$, then $(a, \alpha) \notin \mathcal{Z}$ by (Z7). This shows one half of (Z8); the other one is seen similarly.

For $a, b \in \bar{S}$, let us define

$$a \bullet b = \begin{cases} ab & \text{if } a, b, ab \in S^\star, \\ 0 & \text{if } (a, b) \in \mathcal{Z}, \\ \alpha & \text{if } (a, b) \in \mathcal{N} \setminus \mathcal{Z}. \end{cases}$$

We shall show that the operation $\bullet$ makes $\bar{S}$ into a tomonoid. Indeed, 1 is an identity for $\bullet$, and it is clear that $\bullet$ is in both arguments order-preserving. It remains to prove the associativity.

Let $a, b, c \in \bar{S}$; we have to show that $(a \bullet b) \bullet c = a \bullet (b \bullet c)$. We distinguish several cases.

Case 1. Assume that $abc \in S^\star$. Then $ab, bc \in S^\star$ and hence $(a \bullet b) \bullet c = abc = a \bullet (b \bullet c)$.

Case 2. Assume that $ab, bc \in S^\star$ but $abc \notin S^\star$. Then $(ab, c), (a, bc) \in \mathcal{N}$ and, by (Z3), $(a \bullet b) \bullet c = ab \bullet c = a \bullet bc = a \bullet (b \bullet c)$.

Case 3. Assume that $(a, b) \in \mathcal{Z}$. Then $(a \bullet b) \bullet c = 0 \bullet c = 0$ by (Z2). Note, furthermore, that $b \bullet c \leqslant b$. Hence $(a, b \bullet c) \in \mathcal{Z}$ by (Z1) and we conclude $a \bullet (b \bullet c) = 0$.

Case 4. Assume that $(b, c) \in \mathcal{Z}$. We proceed analogously to Case 3.

Case 5. Assume that $(a, b), (b, c) \in \mathcal{N} \setminus \mathcal{Z}$. If $a < \varepsilon_l$ and $c \geqslant \varepsilon_r$, or $a \geqslant \varepsilon_l$ and $c < \varepsilon_r$, we get a contradiction by (Z4). Hence either $a < \varepsilon_l$ and $c < \varepsilon_r$, or $a \geqslant \varepsilon_l$ and $c \geqslant \varepsilon_r$. From (Z8), we derive $(a \bullet b) \bullet c = \alpha \bullet c = a \bullet \alpha = a \bullet (b \bullet c)$.

Case 6. Assume that $(a, b) \in \mathcal{N} \setminus \mathcal{Z}$ and $bc \in S^\star$. If $c < \varepsilon_r$, we have $(a \bullet b) \bullet c = \alpha \bullet c = 0$ and, by (Z5), $a \bullet (b \bullet c) = a \bullet bc = 0$ as well. If $c \geqslant \varepsilon_r$, we have $(a \bullet b) \bullet c = \alpha \bullet c = \alpha$ and, by (Z6), $a \bullet (b \bullet c) = a \bullet bc = \alpha$ as well.

Case 7. Assume that $(b, c) \in \mathcal{N} \setminus \mathcal{Z}$ and $ab \in S^\star$. We proceed analogously to Case 6.

We conclude that $(\bar{S}; \leqslant, \bullet, 1)$ is a tomonoid. Furthermore, $\bar{S}/\alpha$ is isomorphic to S, that is, $\bar{S}$ is a one-element coextension of S. Let θ be the coextension congruence on $\mathcal{R}(S)$ such that $\bar{S}$ is isomorphic to $\mathcal{R}(S)/\theta$. Then, for any $(a, b) \in \bar{S}^2$, we have

$$(a, b) \in \mathcal{Z}_\theta \text{ iff } ab \; \theta \; 0 \text{ iff } a/\theta \cdot b/\theta = 0/\theta \text{ iff } a \bullet b = 0 \text{ in } \bar{S} \text{ iff } (a, b) \in \mathcal{Z}.$$

Hence $\mathcal{Z} = \mathcal{Z}_\theta$. By (Z8), the border elements of θ are $\varepsilon_l, \varepsilon_r$.

Finally, let θ' be another coextension congruence on $\mathcal{R}(S)$ such that $\mathcal{Z} = \mathcal{Z}_{\theta'}$. But then $\mathcal{Z}_\theta = \mathcal{Z}_{\theta'}$ and hence, by Lemma 3.8, θ and θ' coincide. The uniqueness claim follows. ◻

By a *one-element* $(\varepsilon_l, \varepsilon_r)$*-coextension* of S, we mean the quotient of $\mathcal{R}(S)$ induced by a coextension congruence with borders $\varepsilon_l, \varepsilon_r \in S^\star \cup \{\alpha\}$. We may summarise our results as follows.

Theorem 3.15 *Let $\varepsilon_l, \varepsilon_r \in S^\star \cup \{\alpha\}$ and let $\mathcal{Z} \subseteq \mathcal{N}$ be such that, for any $a, b, c, d \in S^\star$, the conditions (Z1)–(Z6) hold. For $a, b \in \bar{S}$, let*

$$a \bullet b = \begin{cases} ab & \text{if } a, b, ab \in S^\star, \\ 0 & \text{if } (a, b) \in \mathcal{Z}, \\ \alpha & \text{if } (a, b) \in \mathcal{N} \setminus \mathcal{Z}. \end{cases} \tag{7}$$

Then $(\bar{S}; \leqslant, \bullet, 1)$ is a one-element $(\varepsilon_l, \varepsilon_r)$-coextension of S.

Up to isomorphism, any one-element $(\varepsilon_l, \varepsilon_r)$-coextension arises in this way from a unique set $\mathcal{Z} \subseteq \bar{S}^2$.

Proof The first part holds by Lemma 3.14; the second part holds by Lemma 3.13. ◻

In short, for a pair $\varepsilon_l, \varepsilon_r \in S^\star \cup \{\alpha\}$, there is a one-to-one correspondence between the one-element $(\varepsilon_l, \varepsilon_r)$-coextensions of S and the sets $\mathcal{Z} \subseteq \mathcal{N}$ subject to the conditions (Z1)–(Z6).

We can formulate Theorem 3.15 as follows in an algorithmic fashion. At this point, the present work becomes comparable with the approach chosen in [10].

In what follows, we endow $\mathcal{N}$ with the componentwise partial order $\trianglelefteq$, that is, for $(a, b), (c, d) \in \mathcal{N}$, we write $(a, b) \trianglelefteq (c, d)$ if $a \leqslant b$ and $c \leqslant d$.

Theorem 3.16 *Let $\varepsilon_l, \varepsilon_r \in S^\star \cup \{\alpha\}$ such that that either $\varepsilon_l = \varepsilon_r = \alpha$ or $\varepsilon_l, \varepsilon_r$ are idempotent elements of $S^\star$. Let $\sim$ be the smallest equivalence relation on $\mathcal{N}$ such that following holds:*

- (A1) *For any $a, b \in \bar{S}$, we have $(a, 0) \sim (0, b) \sim (1, 0)$. Moreover, for any $a \in \bar{S}$ such that $a < \varepsilon_l$, we have $(a, \alpha) \sim (1, 0)$. Similarly, for any $a \in \bar{S}$ such that $a < \varepsilon_r$, we have $(\alpha, a) \sim (1, 0)$.*
- (A2) *We have $(1, \alpha) \sim (\alpha, 1)$ and for any $(a, b) \in \mathcal{N}$ such that $a, b \neq 0$ and $a \geqslant \varepsilon_l$ or $b \geqslant \varepsilon_r$, we have $(a, b) \sim (1, \alpha)$.*
- (A3) *Let $a, b, c, ab, bc \in S^\star$ and $(a, bc), (ab, c) \in \mathcal{N}$. Then $(a, bc) \sim (ab, c)$.*
- (A4) *Let $(a, b), (b, c) \in \mathcal{N}$ such that $a \geqslant \varepsilon_l$ and $c < \varepsilon_r$. Then $(b, c) \sim (1, 0)$. Similarly, let $(a, b), (b, c) \in \mathcal{N}$ such that $a < \varepsilon_l$ and $c \geqslant \varepsilon_r$. Then $(a, b) \sim (1, 0)$.*

$\dot{0}$	x	y	z	1	
$\dot{0}$	x	y	z	1	1
$\dot{0}$	$\dot{0}$	$\dot{0}$	x	z	z
$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	y	y
$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	x	x
$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$

0	α	x	y	z	1	
0	α	x	y	z	1	1
0	0	A	C	x	z	z
0	0	0	B	D	y	y
0	0	0	0	A	x	x
0	0	0	0	0	α	α
0	0	0	0	0	0	0

Fig. 2 The coextension of S, where S is from Example 3.7. Left: The multiplication table of S. Right: The classes of the equivalence relation $\sim$ on $\bar{S}$

(A5) *Let* $a, b, c, bc \in S^\star$, $(a, b) \in \mathcal{N}$, *and* $c < \varepsilon_r$. *Then* $(a, bc) \sim (1, 0)$.
Similarly, let $a, b, c, ab \in S^\star$, $(b, c) \in \mathcal{N}$, *and* $a < \varepsilon_l$. *Then* $(ab, c) \sim (1, 0)$.

(A6) *Let* $a, b, c, bc \in S^\star$, $(a, b) \in \mathcal{N}$, *and* $c \geqslant \varepsilon_r$. *Then* $(a, b) \sim (a, bc)$.
Similarly, let $a, b, c, ab \in S^\star$, $(b, c) \in \mathcal{N}$, *and* $a \geqslant \varepsilon_l$. *Then* $(b, c) \sim (ab, c)$.

If $(1, 0) \sim (1, \alpha)$, *a one-element* $(\varepsilon_l, \varepsilon_r)$*-coextension of* S *does not exist. Otherwise, let* $\mathcal{Z} \subseteq \mathcal{N}$ *be a union of* $\sim$*-classes such that* $(1, 0) \in \mathcal{Z}$, $(1, \alpha) \notin \mathcal{Z}$, *and* $\mathcal{Z}$ *is a downset. Then* $(\bar{S}; \bullet, \leqslant, 1)$, *where* $\bullet$ *is defined by* (7), *is a one-element* $(\varepsilon_l, \varepsilon_r)$*-coextension of* S.

Every one-element coextension of S *arises in this way.*

Proof Assume $(1, 0) \nsim (1, \alpha)$ and let $\mathcal{Z} \subseteq \mathcal{N}$ arise in the indicated fashion. Then $\mathcal{Z}$ fulfils the properties (Z1)–(Z6) of Lemma 3.13; this is not difficult to check and we omit the details. Consequently, $\mathcal{Z}$ gives rise to a one-element coextension as indicated in Theorem 3.15.

Conversely, let $\mathcal{Z} = \mathcal{Z}_\theta$, where θ is a coextension congruence. Then $\mathcal{Z}$ is a subset of $\mathcal{N}$ that fulfils, by Lemma 3.13 and the proof of Lemma 3.14, the properties (Z1)–(Z8). By (Z2), $(1, 0) \in \mathcal{Z}$ but $(1, \alpha) \notin \mathcal{Z}$. It is now easily checked that, for any $a, b, c, d \in \bar{S}$ such that $(a, b) \sim (c, d)$, either $(a, b), (c, d) \in \mathcal{Z}$ or $(a, b), (c, d) \notin \mathcal{Z}$. Hence $\mathcal{Z}$ is a union of $\sim$-classes. Since $\mathcal{Z}$ contains $(1, 0)$ but not $(1, \alpha)$, we have $(1, 0) \nsim (1, \alpha)$. Finally, we have by (Z1) that $(c, d) \in \mathcal{Z}$, $a \leqslant c$, $b \leqslant d$ imply $(a, b) \in \mathcal{Z}$, that is, $\mathcal{Z}$ is a downset. We conclude that any one-element coextension arises in the indicated way. □

Example 3.17 Let us consider again the tomonoid S indicated in Example 3.7. Its multiplication table is given in Fig. 2(left). It is understood that the rows correspond to the first argument and the columns to the second one.

We determine the one-element coextensions of S for the case $\varepsilon_l = \varepsilon_r = 1$. The extended base set is $\bar{S} = \{0, \alpha, x, y, z, 1\}$. In accordance with Theorem 3.16, we have to calculate the equivalence relation $\sim$ on $\mathcal{N} \subseteq \bar{S}^2$.

The $\sim$-classes are indicated in Fig. 2(right). Here as well as in the subsequent figures, $\mathcal{N}$ is delimited against the remaining elements of $\bar{S}^2$ by a bold line. Moreover, we denote the $\sim$-class containing $(1, 0)$ by 0, the $\sim$-class containing $(1, \alpha)$ by α, and the remaining ones by Latin capital letters. It is indicated which element of $\mathcal{N}$ belongs to which $\sim$-class. We see that $\mathcal{N}$ contains, apart from the classes 0 and α, the four further classes A, B, C, and D.

By Theorem 3.16, a one-element coextension of S corresponds to a union $\mathcal{Z}$ of $\sim$-classes, including the class 0 but excluding α and such that $(c, d) \in \mathcal{Z}$, $a \leqslant c$, and

$b \leqslant d$ imply $(a, b) \in \mathcal{Z}$. For instance, let $\mathcal{Z}$ be the union of the classes 0, A, B, and D. Then we arrive at the coextension $(\bar{S}; \leqslant, \bullet, 1)$ that we have indicated in Fig. 1. The multiplication on $\bar{S}$ is in this case defined as follows:

$$a \bullet b \;=\; \begin{cases} b & \text{if } a = 1, \\ a & \text{if } b = 1, \\ x & \text{if } a = b = z, \\ \alpha & \text{if } a = z \text{ and } b = y, \\ 0 & \text{otherwise,} \end{cases}$$

where $a, b \in \bar{S}$.

4 One-element coextensions for given borders

Again, we fix in this section a non-trivial finite negative tomonoid $(S; \cdot, \leqslant, 1)$. We have specified in the previous section a method to determine the one-element coextensions of S. Recall that $\mathcal{N}$ consists of those pairs of elements of the extended base set $\bar{S}$ whose product is not defined in the partial monoid $S^\star$. Then the subsets of $\mathcal{N}$ that are subject to certain properties are in a one-to-one correspondence with the desired coextensions.

The one-element coextensions are roughly classified by the border elements $\varepsilon_l, \varepsilon_r \in S^\star \cup \{\alpha\}$ for the corresponding congruence on $\mathcal{R}(S)$. Here we reconsider the procedure of the previous section, but this time we assume a choice of border elements from the outset. Accordingly, let also $\varepsilon_l, \varepsilon_r \in S^\star \cup \{\alpha\}$ be fixed in this section, such that either $\varepsilon_l = \varepsilon_r = \alpha$ or $\varepsilon_l, \varepsilon_r$ are idempotent elements of $S^\star$.

Definition 4.1 Let $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$ be the free pomonoid over the chain $\bar{S}$, subject to the conditions (a)–(d) of Definition 3.3 as well as the following ones, for any $a, b, c \in \bar{S}$:

(e) $a\,b\,c = b\,c$ for any $(b, c) \in \mathcal{N}$ and $a \geqslant \varepsilon_l$.
$a\,b\,c = a\,b$ for any $(a, b) \in \mathcal{N}$ and $c \geqslant \varepsilon_r$.
(f) $a\,b\,c = 0$ for any $(b, c) \in \mathcal{N}$ and $a < \varepsilon_l$.
$a\,b\,c = 0$ for any $(a, b) \in \mathcal{N}$ and $c < \varepsilon_r$.

We call $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$ the *free one-element Rees coextension of S w.r.t.* $(\varepsilon_l, \varepsilon_r)$, or simply the *free one-element* $(\varepsilon_l, \varepsilon_r)$*-coextension of* S.

Proposition 4.2 *Let T be a one-element $(\varepsilon_l, \varepsilon_r)$-coextension of S. Then there is a congruence θ on $\mathcal{R}(S)$ such that each θ-class contains exactly one $a \in \bar{S}$ and $\mathcal{R}(S)/\theta$ is isomorphic to T.*

Proof This is seen similarly as Proposition 3.4. □

We note that a one-element $(\varepsilon_l, \varepsilon_r)$-coextension of S may not exist. We will discuss this point at the end of this section.

Lemma 4.3 *In $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$, the following holds, for any $a, b, c \in \bar{S}$:*

(g) $a\alpha = 0$ *for any* $a < \varepsilon_l$.
$\alpha b = 0$ *for any* $b < \varepsilon_r$.
(h) $ab = \alpha$ *for any* $(a, b) \in \mathcal{N}$ *such that* $a, b \neq 0$*, and* $a \geqslant \varepsilon_l$ *or* $b \geqslant \varepsilon_r$.

Proof Ad (g): Let $a < \varepsilon_l$. Since $(\alpha, 1) \in \mathcal{N}$, we have by (f) $a \cdot \alpha = a \cdot \alpha \cdot 1 = 0$. This is the first part of (g); the second one is seen similarly.

Ad (h): Let $a \geqslant \varepsilon_l$ and $b \neq 0$. Since $(\alpha, 1) \in \mathcal{N}$, we have by (e) that $a \cdot \alpha = a \cdot \alpha \cdot 1 = \alpha \cdot 1 = \alpha$. Moreover, $b \in \bar{S} \setminus \{0\}$, that is, $b \geqslant \alpha$ and hence $a \cdot b \geqslant \alpha$. By (c), $a \cdot b \leqslant \alpha$ if $(a, b) \in \mathcal{N}$. The first part of (h) is shown; similarly we see the second part. □

We continue with a description of $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$.

Proposition 4.4 *Let* β *be the bottom element of* $S^\star$*, that is, the atom of* S.

(i) $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$ *is the union of the intervals* $[0, \alpha]$ *and* $[\beta, 1]$*, and we have* $\alpha < \beta$. $[0, \alpha]$ *is a poideal and equals* $\{ab\colon (a, b) \in \mathcal{N}\}$*. Moreover,* $[\beta, 1]$ *is a chain, consisting of the pairwise distinct elements* $a \in S^\star$.
In particular, $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$ *is finite.*
(ii) 1 *is the top element and the identity of* $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$*; in particular,* $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$ *is a negative pomonoid.*
(iii) *Let* $a, b \in [\beta, 1]$*. If* $ab = c$ *holds in* $S^\star$*, then we have* $ab = c$ *in* $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$ *as well. If* ab *is in* $S^\star$ *undefined, then we have* $ab \leqslant \alpha$ *in* $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$.

Proof $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$ is a quotient of $\mathcal{R}(S)$, being subjected to the additional equalities (e) and (f). The latter involve elements of $[0, \alpha]$ only. Hence most statements follow from Proposition 3.5.

What remains to show is $[0, \alpha] = \{ab\colon (a, b) \in \mathcal{N}\}$. We will actually show that, in $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$, each word of length 3 equals a word of length 2. Let $a, b, c \in \bar{S}$. If one of a, b, or c equals 0, we have $abc = 0$ by equality (d). If one of a, b, or c equals α, abc equals 0 or α according to equalities (g) and (h). Let $a, b, c \in S^\star$. If $ab = d \in S^\star$, we have $abc = dc$ by equality (a). If $(a, b) \in \mathcal{N}$, we have by equality (f) that $abc = 0$ if $c < \varepsilon_r$, and by equality (e) $abc = ab$ otherwise. The assertion follows. □

We see that $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$ consists of the elements $a \in S^\star$ as well as those of the form ab, where $(a, b) \in \mathcal{N}$. Moreover, the multiplication among elements in $S^\star$ is like in S if the result is in $S^\star$ again, and the multiplication of elements of the form ab, $(a, b) \in \mathcal{N}$, with any $c \in \bar{S}$, as well as the multiplication of α with any $c \in \bar{S}$, is given by equalities (e)–(h).

In order to determine $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$ it is consequently necessary to know which among the elements ab, where $(a, b) \in \mathcal{N}$, coincide. It turns out that the corresponding partition of $\mathcal{N}$ is essentially the one occurring in Theorem 3.16.

Recall that $\mathcal{N}$ is endowed with the componentwise order $\trianglelefteq$. Given an equivalence relation $\sim$ on $\mathcal{N}$, let $\trianglelefteq_\sim$ be the smallest preorder on $\mathcal{N}$ such that $(a, b) \trianglelefteq_\sim (c, d)$ if $(a, b) \trianglelefteq (c, d)$ or $(a, b) \sim (c, d)$ or $(c, d) = (1, \alpha)$. Furthermore, let $(a, b) \approx_\sim (c, d)$ if $(a, b) \trianglelefteq_\sim (c, d)$ and $(c, d) \trianglelefteq_\sim (a, b)$.

Theorem 4.5 *Let* $\sim$ *be the equivalence relation on* $\mathcal{N}$ *specified in Theorem 3.16. Then, for any* $(a, b), (c, d) \in \mathcal{N}$*, we have* $(a, b) \approx_\sim (c, d)$ *if and only if* $a\,b = c\,d$ *in* $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$.

Proof We first consider the "only if" part. Our first aim is to show that, for any $(a, b), (c, d) \in \mathcal{N}$ such that $(a, b) \sim (c, d)$, we have $ab = cd$.

We have to check the conditions (A1)–(A6) of Theorem 3.16. We omit those cases that are seen similarly to those shown.

(A1) For $a, b \in \bar{S}$, we have $a \cdot 0 = 0 \cdot b = 0 = 1 \cdot 0$. If $a < \varepsilon_l$, we have $a \cdot \alpha = 0 = 1 \cdot 0$ by (g).
(A2) We have $1 \cdot \alpha = \alpha \cdot 1$. For $(a, b) \in \mathcal{N}$ such that $a, b \neq 0$ and $a \geqslant \varepsilon_l$ or $b \geqslant \varepsilon_r$, we have $ab = \alpha$ by (h).
(A3) Let $(a, bc), (ab, c) \in \mathcal{N}$. Obviously the products of the two entries are in both cases the same.
(A4) Let $(a, b), (b, c) \in \mathcal{N}$ such that $a \geqslant \varepsilon_l$ and $c < \varepsilon_r$. Then $ab \leqslant \alpha$ and, by (g), $\alpha\, c = 0$, hence $abc = 0$. Moreover, by (e), $abc = bc$. We conclude $bc = 0$.
(A5) Let $(a, b) \in \mathcal{N}$ and $c < \varepsilon_r$. Then, by (f), $abc = 0$.
(A6) Let $(a, b) \in \mathcal{N}$ and $c \geqslant \varepsilon_r$. Then, by (e), $abc = ab$.

It follows that, for $(a, b), (c, d) \in \mathcal{N}$, $(a, b) \sim (c, d)$ implies $ab = cd$. Moreover, $(a, b) \trianglelefteq (c, d)$ implies $ab \leqslant cd$, and we have $ab \leqslant \alpha$ for any $(a, b) \in \mathcal{N}$. The proof of the "only if" part is complete.

To see the "if" part, assume that $(a, b), (c, d) \in \mathcal{N}$ are such that $(a, b) \trianglelefteq_\sim (c, d)$ does not hold. Let then $\mathcal{Z} = \{(s, t) \in \mathcal{N} : (s, t) \trianglelefteq_\sim (c, d)\}$. Then $\mathcal{Z}$ is a union of $\sim$-classes and a downset, and we have $(c, d) \in \mathcal{Z}$ but $(a, b) \notin \mathcal{Z}$. Furthermore, $(1, 0) \in \mathcal{Z}$ because $(1, 0) \sim (0, 0) \trianglelefteq (c, d)$. But $(1, \alpha) \notin \mathcal{Z}$. Indeed, otherwise we would have $(1, \alpha) \trianglelefteq_\sim (c, d)$, and since $(a, b) \trianglelefteq_\sim (1, \alpha)$ it would follow $(a, b) \trianglelefteq_\sim (c, d)$, contrary to our assumption. Note in particular that $(1, \alpha) \nsim (1, 0)$.

By Theorem 3.16, there is a one-element $(\varepsilon_l, \varepsilon_r)$-coextension of S such that the products $a \bullet b$ and $c \bullet d$ are different. It follows $ab \neq cd$ in $\mathcal{R}_{\varepsilon_l, \varepsilon_r}(S)$. □

By Theorem 4.5, we are now in the position to determine $\mathcal{R}_{\varepsilon_l, \varepsilon_r}(S)$ in an effective way. We start with the equivalence relation $\sim$ on $\mathcal{N}$, as described by Theorem 3.16. Then we identify those $\sim$-classes forming $\trianglelefteq$-cycles and if necessary we enlarge the class of $(1, \alpha)$ to an upset of $\mathcal{N}$. The resulting $\approx_\sim$-classes can be considered as the elements of the poideal $[0, \alpha]$ of $\mathcal{R}_{\varepsilon_l, \varepsilon_r}(S)$; for each pair $(a, b) \in \mathcal{N}$, ab is the element of $[0, \alpha]$ associated with the $\approx_\sim$-class of (a, b). Moreover, the multiplication in $\mathcal{R}_{\varepsilon_l, \varepsilon_r}(S)$ is given by the product of S as regards $a, b \in S^\star$ such that $ab \in S^\star$; 0 is an absorbing element; and all remaining cases are covered by equalities (e)–(h).

To illustrate this procedure, we insert several examples.

Example 4.6 We consider once more the tomonoid S from Examples 3.7 and 3.17. We want to determine $\mathcal{R}_{1,1}(S)$. To this end, we have to calculate the equivalence relation $\approx_\sim$ on $\mathcal{N}$. Figure 2(right) shows $\sim$ and we see that there are no cycles w.r.t. $\trianglelefteq$. Hence $\approx_\sim$ coincides with $\sim$. Here and in the two following examples, we denote the elements of $[0, \alpha] \subseteq \mathcal{R}_{1,1}(S)$ by the associated subsets of $\mathcal{N}$. Then $\mathcal{R}_{1,1}(S) = \{0, A, B, C, D, \alpha, x, y, z, 1\}$. The multiplication and the order is specified in Fig. 3.

By Theorem 4.10, the one-element $(1, 1)$-coextensions of S correspond to the downsets J of the six-element interval $[0, \alpha]$ such that $0 \in J$ but $\alpha \notin J$. There are seven in total.

For $a, b \in \mathcal{R}_{1,1}(S)$, we have

$$a \cdot b = \begin{cases} b & \text{if } a = 1, \\ a & \text{if } b = 1, \\ x & \text{if } a = b = z, \\ A & \text{if } a = z \text{ and } b = x \\ & \text{or } a = x \text{ and } b = z, \\ B & \text{if } a = b = y, \\ C & \text{if } a = z \text{ and } b = y, \\ D & \text{if } a = y \text{ and } b = z, \\ 0 & \text{otherwise.} \end{cases}$$

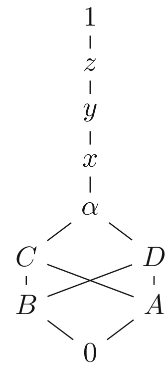

Fig. 3 The free one-element (1, 1)-coextension $\mathcal{R}_{1,1}(S)$ discussed in Example 4.6. Here, S is again the tomonoid from Example 3.7

$\dot{0}$	v	w	x	y	z	1	
$\dot{0}$	v	w	x	y	z	1	1
$\dot{0}$	$\dot{0}$	v	v	v	x	z	z
$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	v	y	y
$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	v	x	x
$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	v	w	w
$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	v	v
$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$

0	α	v	w	x	y	z	1	
0	α	v	w	x	y	z	1	1
0	0	A	v	v	v	x	z	z
0	0	0	C	A	E	v	y	y
0	0	0	A	A	A	v	x	x
0	0	0	B	A	D	v	w	w
0	0	0	0	0	0	A	v	v
0	0	0	0	0	0	0	α	α
0	0	0	0	0	0	0	0	0

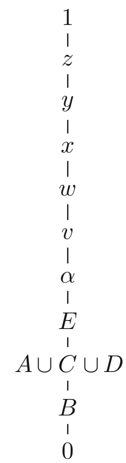

Fig. 4 The tomonoid from Example 4.7

Example 4.7 The next example is based on the work of Kozák [9]. Let $S = \{\dot{0}, v, w, x, y, z, 1\}$ be the seven-element chain, and define a product on S according to the table in Fig. 4(left). We determine the one-element (1, 1)-coextensions. The $\sim$-classes are indicated in Fig. 4(middle). This time, the equivalence relation $\sim$ does not coincide with $\approx_\sim$. Indeed, we have that $(x, w) \trianglelefteq (y, w) \trianglelefteq (y, x) \sim (x, w)$, consequently $(x, w) \trianglelefteq_\sim (y, w) \trianglelefteq_\sim (x, w)$, that is, $(x, w) \approx_\sim (y, w)$. Similarly, $(x, w) \approx_\sim (w, y)$ and we conclude that the union of the $\sim$-classes A, C, and D, forms a single $\approx_\sim$-class. Hence $\approx_\sim$ is in this case strictly coarser than $\sim$.

The interval $[0, \alpha]$ consists of five elements only, the Hasse diagram of the poset reduct of $\mathcal{R}_{1,1}(S)$ is as shown in Fig. 4(right). We see that S possesses four one-element (1, 1)-coextensions.

Example 4.8 Our last example shows the coextensions of a tomonoid such that the chosen border elements are not both 1. Let $S = \{\dot{0}, v, w, x, y, z, 1\}$ be the seven-element chain, and define a multiplication according to the table in Fig. 5(left). We determine the one-element $(1, z)$-coextensions. Again, the $\sim$-classes are shown, and we have that $\sim$ coincides with $\approx_\sim$. The Hasse diagram of the poset reduct of $\mathcal{R}_{1,z}(S)$ is as shown in Fig. 5(right). We observe that there are five one-element $(1, z)$-coextensions.

Let us now turn to the case that coextensions of the desired kind do not exist. Then the free one-element $(\varepsilon_l, \varepsilon_r)$-coextension collapses to S.

	$\dot{0}$	v	w	x	y	z	1
1	$\dot{0}$	v	w	x	y	z	1
z	$\dot{0}$	v	v	v	v	z	z
y	$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	x	y
x	$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	x	x
w	$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	w	w
v	$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	v	v
$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$	$\dot{0}$

	0	α	v	w	x	y	z	1
1	0	α	v	w	x	y	z	1
z	0	0	v	v	v	v	z	z
y	0	0	B	C	D	D	x	y
x	0	0	B	B	B	B	x	x
w	0	0	A	A	A	A	w	w
v	0	0	0	0	0	0	v	v
α	0	0	0	0	0	0	α	α
0	0	0	0	0	0	0	0	0

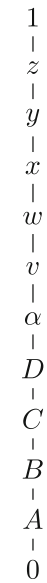

Fig. 5 The tomonoid from Example 4.8

Theorem 4.9 *S possesses a one-element $(\varepsilon_l, \varepsilon_r)$-coextension if and only if $0 \neq \alpha$ in $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$.*

Proof The "only if" is clear from Proposition 4.2.

To see the "if" part, assume that $0 \neq \alpha$. Let $\sim$ be the equivalence relation on $\mathcal{N}$ specified in Theorem 3.16. By Theorem 4.5, we have $(1, 0) \not\precsim_{\sim} (1, \alpha)$. Then there is a union $\mathcal{Z}$ of $\sim$-classes such that $\mathcal{Z}$ is a downset not containing $(1, \alpha)$. By Theorem 3.16, it follows that there is a one-element $(\varepsilon_l, \varepsilon_r)$-coextension of S. $\square$

If one-element $(\varepsilon_l, \varepsilon_r)$-coextensions of S do not exist, then by Theorem 4.9, $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$ is isomorphic to S. If such coextensions do exist, they are easily determined from $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$. As we will now see, it is indeed enough to choose a downset of $[0, \alpha]$ that contains 0 but not α.

Theorem 4.10 *Assume that $0 \neq \alpha$ and let $\varnothing \subset Z \subset [0, \alpha]$ be a downset of $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$. For $a, b \in \mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$, let*

$$a \,\theta_Z\, b \quad \text{if and only if} \quad a = b \text{ or } a, b \in Z \text{ or } a, b \in [0, \alpha] \setminus Z.$$

Then θ_Z is a congruence on $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$ and $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)/\theta_Z$ is a one-element $(\varepsilon_l, \varepsilon_r)$-coextension of S.

Up to isomorphism, every one-element $(\varepsilon_l, \varepsilon_r)$-coextension of S arises in this way from a unique downset Z of $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$.

Proof The one-element $(\varepsilon_l, \varepsilon_r)$-coextensions of S are in one-to-one correspondence with the sets $\mathcal{Z} \subseteq \mathcal{N}$ as specified in Theorem 3.16. Each such subset $\mathcal{Z}$ induces a congruence by requiring $ab = 0$ if $(a, b) \in \mathcal{Z}$, and $ab = \alpha$ if $(a, b) \in \mathcal{N} \setminus \mathcal{Z}$.

By Proposition 4.4, $[0, \alpha] = \{ab \colon (a, b) \in \mathcal{N}\}$. Hence, by Theorem 4.5, the sets $\mathcal{Z}$ are in a one-to-one correspondence with the downsets Z such that $\varnothing \subset Z \subset [0, \alpha]$.

Accordingly, each such downset Z induces a congruence θ by requiring $ab = 0$ if $ab \in Z$, and $ab = \alpha$ if $ab \in [0, \alpha] \setminus Z$. The assertions follow. □

In other words, the one-element $(\varepsilon_l, \varepsilon_r)$-coextensions of S are in a one-to-one correspondence with the downsets of the interval $[0, \alpha]$ of $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$ that contain 0 but not α.

We finally turn to the question how we can tell from the structure of S whether or not at least one one-element $(\varepsilon_l, \varepsilon_r)$-coextension exists. We know that if one of ε_l or ε_r equals α, also the respective other element equals α and the coextension exists. Otherwise, ε_l and ε_r must be idempotent elements of $S^\star$. However, not for every such pair a one-element $(\varepsilon_l, \varepsilon_r)$-coextension exists; see [10, Fig. 6].

The problem of an exact criterion, which was left unanswered in [10], turns out to be tricky and we conclude the paper by providing one necessary and one sufficient condition. Here, \ and / denote the residuals on S; we recall that this means

$$a \cdot b \leqslant c \quad \text{iff} \quad b \leqslant a \backslash c \quad \text{iff} \quad a \leqslant c/b$$

for $a, b, c \in s$. Moreover, for a non-zero element a of a finite chain, we denote by $\operatorname{pred} a$ the element preceding a.

Proposition 4.11 *Assume that S possesses a one-element $(\varepsilon_l, \varepsilon_r)$-coextension. Then the following holds in S:*

$$\begin{aligned} \operatorname{pred} \varepsilon_r &\leqslant (\varepsilon_l \backslash \dot{0}) \backslash \dot{0}, \\ \operatorname{pred} \varepsilon_l &\leqslant \dot{0}/(\dot{0}/\varepsilon_r). \end{aligned}$$

Proof We show the first part only; the second part follows by a dual argument.

Assume that $(\varepsilon_l \backslash \dot{0}) \backslash \dot{0} < \operatorname{pred} \varepsilon_r$ holds in S. This means $\varepsilon_l \backslash \dot{0} \cdot \operatorname{pred} \varepsilon_r > \dot{0}$. It follows that $b = \varepsilon_l \backslash \dot{0}$ and $\operatorname{pred} \varepsilon_r$ are elements of $S^\star$ such that, in $\bar{S}$, we have $b \cdot \operatorname{pred} \varepsilon_r > \alpha$. Furthermore, $\varepsilon_l \cdot b = \varepsilon_l \cdot (\varepsilon_l \backslash \dot{0}) = \dot{0}$ in S. Also $\varepsilon_l \in S^\star$, hence we have $\varepsilon_l \cdot b \leqslant \alpha$ in $\bar{S}$. But this is a contradiction: in $\bar{S}$, we have $(\varepsilon_l \cdot b) \cdot \operatorname{pred} \varepsilon_r \leqslant \alpha \operatorname{pred} \varepsilon_r = 0$ and $\varepsilon_l \cdot (b \cdot \operatorname{pred} \varepsilon_r) \geqslant \varepsilon_l \alpha = \alpha$. □

Proposition 4.12 *Assume that $\varepsilon_l, \varepsilon_r \in S^\star$ and let ε_m be the smallest non-zero idempotent of S. Assume that, for $a, b \in \bar{S}$, we have $ab = \dot{0}$ whenever either $a \leqslant \varepsilon_l$ and $b < \varepsilon_m$ or $a < \varepsilon_m$ and $b \leqslant \varepsilon_r$. Then S possesses a one-element $(\varepsilon_l, \varepsilon_r)$-coextension.*

Proof For $a, b \in \bar{S}$, we define

$$a \bullet b = \begin{cases} ab & \text{if } a, b \in S^\star \text{ and } ab \text{ exists in } S^\star, \\ \alpha & \text{if } (a, b) \in \mathcal{N},\ a, b \neq 0, \text{ and } a \geqslant \varepsilon_l \text{ or } b \geqslant \varepsilon_r, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that this product makes $\bar{S}$ into a tomonoid. The compatibility of the order and the fact that 1 is an identity are readily seen.

It remains to show the associativity. Let $a, b, c \in \bar{S}$; we have to show that $(a \bullet b) \bullet c = a \bullet (b \bullet c)$. We distinguish two cases.

Case 1. Let $(a \bullet b) \bullet c \in S^\star$. Then we have $a \bullet (b \bullet c) = abc = a \bullet (b \bullet c)$.

Case 2. Let $(a \bullet b) \bullet c \notin S^\star$, that is, $(a \bullet b, c) \in \mathcal{N}$. Then also $(a, b \bullet c) \in \mathcal{N}$. We have to show that $(a \bullet b) \bullet c = 0$ if and only if $a \bullet (b \bullet c) = 0$.

Assume that $(a \bullet b) \bullet c = 0$. In case that $a \bullet b = 0$ we have $a \bullet (b \bullet c) \leqslant a \bullet b = 0$, that is, $a \bullet (b \bullet c) = 0$. In case that $c = 0$ we have $a \bullet (b \bullet c) = 0$ as well. Assume now that $a \bullet b, c \neq 0$. Then $a \bullet b < \varepsilon_l$ and $c < \varepsilon_r$. We furthermore have that $a < \varepsilon_m$ or $b < \varepsilon_m$ or $c < \varepsilon_m$ because otherwise it would follow that $(a \bullet b) \bullet c \geqslant \varepsilon_m$. If $a < \varepsilon_m$, we have $a < \varepsilon_l$ and $b \bullet c \leqslant c < \varepsilon_r$ and hence $a \bullet (b \bullet c) = 0$. If $b < \varepsilon_m$, then $b < \varepsilon_l$ and hence $a \bullet (b \bullet c) = a \bullet 0 = 0$. Finally, if $c < \varepsilon_m$, we conclude from $a \bullet b < \varepsilon_l$ that $a < \varepsilon_l$ or $b < \varepsilon_l$, and we conclude in both cases $a \bullet (b \bullet c) = 0$ again. Similarly, we see that $a \bullet (b \bullet c) = 0$ implies $(a \bullet b) \bullet c = 0$.

We have shown the associativity of the product, thus $\bar{S}$ is indeed a tomonoid. Clearly, $\bar{S}/\alpha$ is isomorphic to S. □

5 Conclusion

We have proposed in this paper a new perspective on the problem of how to determine the one-element Rees coextensions of finite negative tomonoids. Given a finite, negative tomonoid S, the pomonoid $\mathcal{R}(S)$ is the free pomonoid generated by the extended chain $\bar{S}$ subject to conditions that hold in any one-element coextension of S. We thus get all coextensions of the latter kind as a quotient of $\mathcal{R}(S)$. The determination of the relevant congruence can most easily be done in a two-stage process: for elements $\varepsilon_l, \varepsilon_r$, which correspond to idempotent elements of S, we associate the pomonoid $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$, which is a quotient of $\mathcal{R}(S)$ and from which to get the actual one-element coextensions is straightforward.

Possibilities to elaborate on our work are numerous. A motivation of the present work has been the creation of a framework in which the generalisation of our method could be facilitated. In particular, we should get along without the condition of negativity. The situation has turned out to be tricky, it is nonetheless worth to be investigated. A possibly less difficult problem might be the generalisation to the case of an arbitrary instead of a total order.

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Algorithm to generate the Archimedean, finite, negative tomonoids

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***Abstract*—We study Archimedean, finite, negative totally ordered monoids. We describe an algorithm which generate the structures of this type in a step-wise fashion. Our approach benefits from the level set representation of monoids and is inspired by web geometry.**

I. Introduction

Monoids are important structures in many fields of mathematics, including logic, computer science, or the theory of languages. In fact, in non-classical logic, monoids are often used to interpret the conjunction. The monoids occurring in this context are then typically endowed with a translation-invariant partial order.

In this contribution, we focus on a special class of partially ordered monoids: we assume that the order is total; that the monoidal identity coincides with the top element; and finiteness. Our interest comes from the field of residuated lattices [GJKO]. Under the additional assumption of commutativity, the structures that we consider can in fact be identified with finite MTL-algebras [EsGo]; MTL-algebras are in turn the algebraic counterpart of the fuzzy logic MTL.

We utilize, as we call it, the *level set approach*, which is inspired by the field of web geometry [Acz], [BlBo]. A tomonoid can be represented by its level sets and associativity then corresponds to the so-called Reidemeister condition. The level-set approach has been applied already to triangular norms [PeSa1] and has been utilized to make a significant progress in some open problems on convex combinations of triangular norms [PeSa2].

Furthermore, our previous paper [PeVe] exploits this approach for the discussion of finite, negative tomonoids. In particular, we explain how to construct the elementary extensions of such tomonoids. An elementary extension is by one element larger and the identification of its two smallest elements leads back to the tomonoid we have started from. Starting from the one-element tomonoid, the successive formation of elementary extensions leads to any given finite, negative tomonoid.

In the present paper, we focus on the algorithmic aspects of the construction described in [PeVe]. We restrict, to this end, to the case that the tomonoids under consideration are Archimedean. For proofs and further details of the underlying theory, we refer to [PeVe].

II. Basic notions

Definition 2.1: A *monoid* is an algebra $(S; \odot, 1)$ of type $\langle 2, 0 \rangle$ such that, for any $a, b, c \in S$,

(T1) $(a \odot b) \odot c = a \odot (b \odot c)$,

(T2) $a \odot 1 = 1 \odot a = a$.

A total (linear) order $\leqslant$ on a monoid S is called *compatible* if, for every $a, b, c \in S$,

(T3) $a \leqslant b$ implies $a \odot c \leqslant b \odot c$ and $c \odot a \leqslant c \odot b$.

In this case, we call $(S; \leqslant, \odot, 1)$ a *totally ordered monoid*, or a *tomonoid* for short. Further, we call S *negative* if 1 is the top element and we call S *commutative* if, for every $a, b \in S$,

(T4) $a \odot b = b \odot a$.

In this paper, we are exclusively interested in finite, negative, tomonoids, abbreviated "*f. n. tomonoids*". Let us remark that, in contrast to [EKMMW], we do not assume commutativity, although we deal also with this case.

The smallest tomonoid, called the *trivial tomonoid*, is the one that consists of the monoidal identity 1 alone. Tomonoids with at least two elements are called *non-trivial*.

A negative tomonoid is called *Archimedean* if, for every $x, y \in S \setminus \{1\}$ such that $x \leq y$, there is an $n \in \mathbb{N}$ such that $y^n \leq x$. Here, we define

$$y^n \quad = \quad \underbrace{y \odot y \odot \ldots \odot y}_{n\text{-times}}.$$

We note that negative tomonoids with at most two elements are trivially Archimedean.

III. Level-set view on tomonoids

In this section we introduce the representation of tomonoids by level sets. Let $\odot \colon S \times S \to S$ be a binary operation on a totally ordered set S and let $\sim$ be a binary relation on $S \times S$ such that, for $a, b, c, d \in S$,

$$(a, b) \sim (c, d) \quad \text{iff} \quad a \odot b = c \odot d.$$

We can see that $\sim$ is an equivalence relation and that it partitions $S \times S$ into those subsets of pairs that are mapped by $\odot$ to equal values. When recovering $\odot$ from $\sim$ we need to know which equivalence class is associated with which value of S. But this is easy if $\odot$ possesses a neutral element $1 \in S$. In such a case each class contains exactly one pair of the form $(1, a)$ and one pair of the form $(a, 1)$. Furthermore, for every $a \in S$ there is exactly one class containing the pairs $(1, a)$ and $(a, 1)$. This gives us a one-to-one mapping between the equivalence classes and the elements from S.

Thus, a tomonoid $(S; \leqslant, \odot, 1)$ can be characterized by the totally ordered set $(S; \leqslant)$, the equivalence relation $\sim$ defining a partition on $S \times S$, and the designated element 1. This simple idea gives to our hand a tool that is geometric in nature and, in contrast with the graph of binary operations, gets along with two dimensions only.

Definition 3.1: Let $(S; \leqslant, \odot, 1)$ be a tomonoid. For two pairs $(a, b), (c, d) \in S \times S$ we define

$$(a, b) \sim (c, d) \quad \text{iff} \quad a \odot b = c \odot d$$

and we call $\sim$ the *level equivalence* associated with S.

Definition 3.2: We denote by $\trianglelefteq$ the componentwise order on $S \times S$ for some totally ordered set S, i.e., for every $a, b, c, d \in S$, we put

$$(a, b) \trianglelefteq (c, d) \quad \text{iff} \quad a \leqslant b \quad \text{and} \quad c \leqslant d.$$

Definition 3.3: Let $(S; \leqslant)$ be a totally ordered set, let $1 \in S$, and let $\sim$ be an equivalence relation on $S \times S$ such that the following holds.

(P1) For every $a, b, c, d, e \in S$,

$(a, b) \sim (1, d)$ and $(b, c) \sim (1, e)$ imply $(d, c) \sim (a, e)$.

(P2) For every $a, b \in S$ there is exactly one $c \in S$ such that

$$(a, b) \sim (1, c) \sim (c, 1).$$

(P3) For every $a, b, c, d, a', b', c', d' \in S$,

$$(a, b) \sim (a', b') \quad \trianglelefteq \quad (c, d) \sim (c', d') \quad \trianglelefteq \quad (a, b)$$

implies

$$(a, b) \sim (c, d).$$

Then we call $(S, \leqslant, 1, \sim)$ a *tomonoid partition*.

Proposition 3.4: [PeVe] Let $(S; \leqslant, \odot, 1)$ be a tomonoid and let $\sim$ be its level equivalence. Then $(S, \leqslant, 1, \sim)$ is a tomonoid partition.

Proposition 3.5: [PeVe] Let $(S, \leqslant, 1, \sim)$ be a tomonoid partition. For every $a, b \in S$, let

$$a \odot b := \text{ the unique } c \text{ such that } (a, b) \sim (1, c) \sim (c, 1).$$

Then $(S; \leqslant, \odot, 1)$ is the unique tomonoid such that $(S, \leqslant, 1, \sim)$ is its associated tomonoid partition.

We conclude from Proposition 3.4 and Proposition 3.5 that tomonoids and tomonoid partitions are in a one-to-one correspondence.

For a tomonoid S, the level equivalence $\sim$ partitions the set $S \times S$ into as many equivalence classes as there are elements of S. In fact, in view of (P2), the classes and the elements of S are in an one-to-one correspondence. Moreover, the equivalence classes inherit under this correspondence the total order from S.

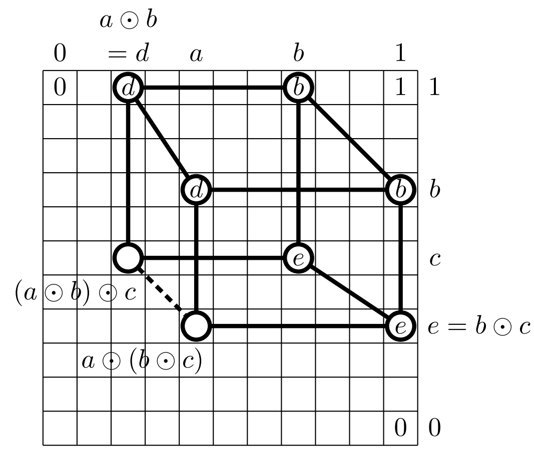

Fig. 1. Illustration of Property (P1) of a tomonoid partition. As depicted, this property corresponds with the associativity of the tmonoid.

Property (P1) is related to the associativity of the tomonoid and has the following geometric interpretation illustrated in Figure 1. On the square $S \times S$, consider two rectangles such that one hits the upper edge and the other one hits the right edge. Assume that the upper left, upper right, and lower right vertices of these rectangles are in the same equivalence classes, respectively. Then, by (P1), also the remaining lower left vertices are elements of the same equivalence class. The described property corresponds with the *Reidemeister condition* known web geometry [Acz], [BlBo].

In the sequel we will use the following simplified notation. Instead of $(a, b) \sim (1, c)$, or equivalently $(a, b) \sim (c, 1)$, we will simply write $(a, b) \sim c$.

Finally, let us specify the tomonoid partitions that correspond to additional properties of tomonoids.

Observation 3.6: Let $(S; \leqslant, \odot, 1)$ be a tomonoid and let $(S, \leqslant, 1, \sim)$ be its associated tomonoid partition.

- S is commutative if, and only if, the equivalence classes of $\sim$ are "mirrored by the diagonal", i.e., if $(a, b) \sim (b, a)$ for every $a, b \in S$.
- S is negative if, and only if, for every $c \in S$ the $\sim$-class of c is contained in $\{(a, b) \in S \times S \mid a, b \geq c\}$.
- S is finite if, and only if, $\sim$ is an equivalence relation on a finite set.

IV. Rees quotients and elementary extensions

For a finite totally ordered set, we will refer to the least element as the *zero*, to the second smallest element as its *atom*, and to the second greatest element as its *coatom*.

We now introduce the notion of an elementary extension of a f. n. tomonoid [PeVe]. A f. n. tomonoid $\bar{S}$ will be an elementary extension of a f. n. tomonoid S such that the cardinality of $\bar{S}$ is greater by one and S is a quotient of $\bar{S}$. Furthermore, this quotient will be the simplest non-trivial Rees

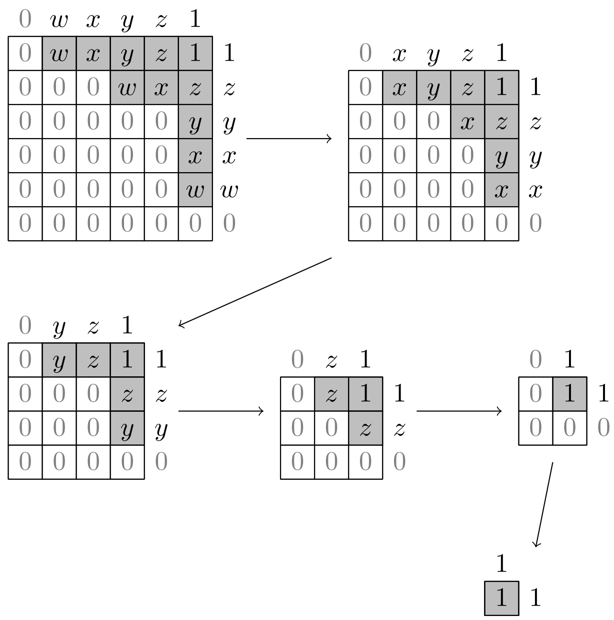

Fig. 2. Elementary quotients a f. n. tomonoid of size 6. The second tomonoid is the elementary quotient of the first one, the third is the elementary quotient of the second one, etc. Finally, we reach the trivial monoid.

quotient, called the elementary quotient of a f. n. tomonoid: it arises from merging the zero with the atom.

See an illustration in Figure 3 which depicts, successively, all the elementary quotients of a given f. n. tomonoid. As we can see in the tomonoid partition picture, the elementary quotient arises by "cutting off" the column and the row indexed by the zero and, further, the zero class and the atom class are joined to a new class which is evaluated to the new smallest element.

Exact definitions now follow. For unexplained general notions of see, e.g., [Gri].

Definition 4.1: Let $(S; \leqslant, \odot, 1)$ be a tomonoid. A *tomonoid congruence* on S is an equivalence relation $\approx$ on S such that

(i) $\approx$ is a congruence of S as a monoid and

(ii) each equivalence class is convex.

The operation induced by $\odot$ on the quotient $\langle S \rangle$ we denote again by $\odot$. For $a, b \in S$, we define $\langle a \rangle \leqslant \langle b \rangle$ if $a \approx b$ or $a < b$.

We may observe that $(\langle S \rangle; \leqslant, \odot, \langle 1 \rangle)$ is a tomonoid again and we call $\langle S \rangle$ the *tomonoid quotient* w.r.t. $\approx$. Clearly, the properties of finiteness, negativity, and commutativity are preserved by this procedure. What follows is a definition of the Rees congruence which is commonly used for semigroups [How].

Lemma 4.2: Let $(S; \leqslant, \odot, 1)$ be a negative tomonoid and let $q \in S$. For $a, b \in S$, let $a \approx_q b$ if $a = b$ or $a, b \leqslant q$. Then $\approx_q$ is a tomonoid congruence.

Definition 4.3: Let $(S; \leqslant, \odot, 1)$ be a f. n. tomonoid and let $q \in S$. We call $\approx_q$, as defined in Lemma 4.2, the *Rees congruence* w.r.t. q. Furthermore, we denote the quotient by S/q and we call it the *Rees quotient* of S w.r.t. q.

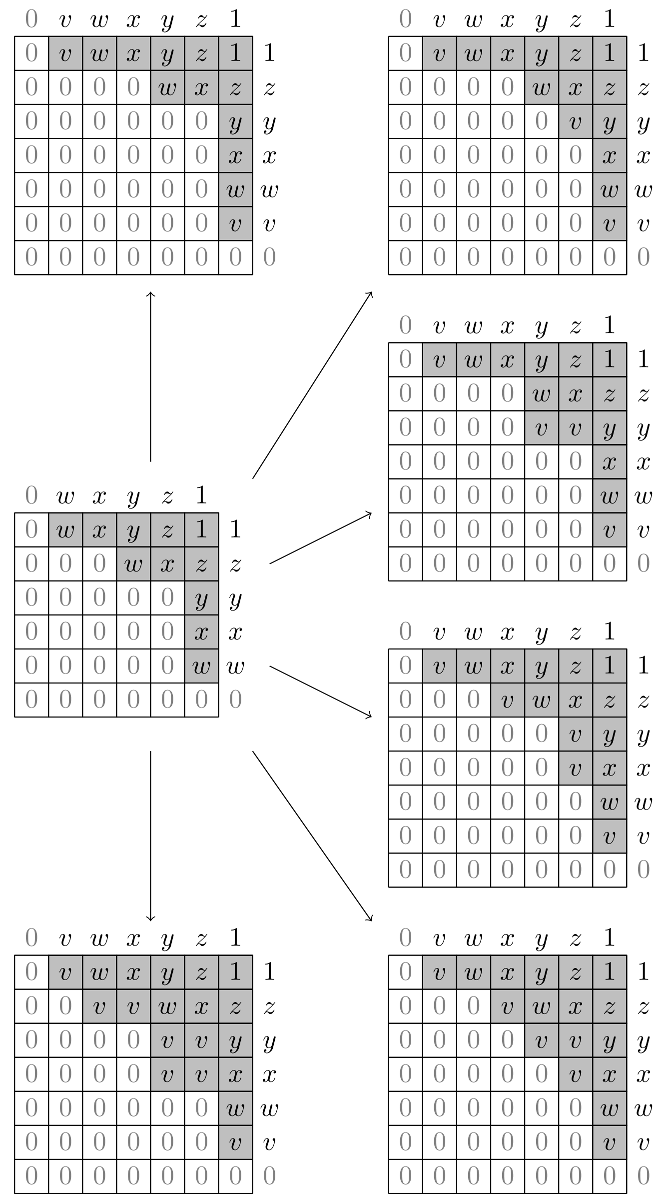

Fig. 3. All the elementary extensions of a f. n. tomonoid of size 6.

Further, let $\bar{S}$ be a non-trivial f. n. tomonoid and let α be the atom of $\bar{S}$. We call the Rees quotient S of $\bar{S}$ w.r.t. α the *elementary quotient* of $\bar{S}$ and, conversely, $\bar{S}$ an *elementary extension* of S.

V. Archimedean elementary extensions

We now turn to the main problem of determining the elementary extensions of an Archimedean f. n. tomonoid which we present in the form of an algorithm.

Algorithm 5.1:

Input: $(S, \leqslant, 1, \sim)$ ... tomonoid partition of an Archimedean f. n. tomonoid

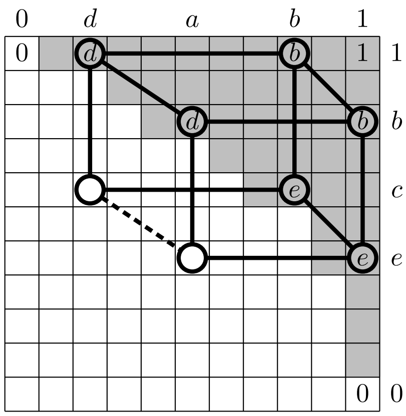

Fig. 4. Illustration of Step 8 of Algorithm 5.1.

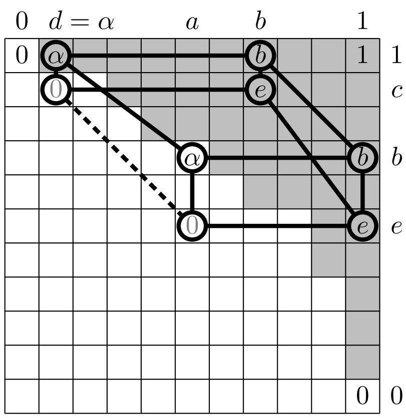

Fig. 5. Illustration of Step 9 and Step 10 of Algorithm 5.1.

Output: $(\bar{S}, \leqslant, 1, \bar{\sim})$ ... elementary extension of $(S, \leqslant, 1, \sim)$

1) Let $\bar{S} = S \,\dot{\cup}\, \{\bar{0}\}$, where $\bar{0}$ is a new element.
2) Endow $\bar{S}$ with the total order that extends the total order on S such that $\bar{0} < a$ for every $a \in S$.
3) Let α be the atom of $\bar{S}$ (i.e., the least element of S).
4) Let κ be the coatom of $\bar{S}$.
5) Let
$$P = \{(a,b) \in \bar{S} \times \bar{S} \mid \text{there is } c > \alpha \text{ such that } (a,b) \sim c\}.$$
6) Define a binary relation $\bar{\sim}$ on $\bar{S} \times \bar{S}$ such that
$$(a,b) \bar{\sim} (c,d) \text{ iff } (a,b) \sim (c,d) \sim e \text{ for some } e \in \bar{S} \setminus \{\bar{0}, \alpha\}$$
7) For every $a \in \bar{S} \setminus \{\bar{0}\}$:
   - define $(a, \bar{0}) \bar{\sim} (a, \alpha) \bar{\sim} (\bar{0}, a) \bar{\sim} (\alpha, a) \bar{\sim} \bar{0}$.
8) For every $(a,b), (b,c) \in P$:
   - let $d \in \bar{S}$ be such that $(a,b) \bar{\sim} d$,
   - let $e \in \bar{S}$ be such that $(b,c) \bar{\sim} e$,
   - define $(a,e) \bar{\sim} (d,c)$.
9) For every $a \in \bar{S} \setminus \{\bar{0}, \alpha, 1\}$:
   - let $b \in \bar{S}$ be the highest element such that $(a,b) \notin P$,
   - let $e \in \bar{S}$ be such that $(b, \kappa) \bar{\sim} e$,
   - for every $(x,y) \in \bar{S} \times \bar{S}$ such that $(x,y) \trianglelefteq (a,e)$:
     - define $(x,y) \bar{\sim} \bar{0}$.
10) For every $a \in \bar{S} \setminus \{\bar{0}, \alpha, 1\}$:
    - let $b \in \bar{S}$ be the highest element such that $(b,a) \notin P$,
    - let $e \in \bar{S}$ be such that $(\kappa, b) \bar{\sim} e$,
    - for every $(x,y) \in \bar{S} \times \bar{S}$ such that $(x,y) \trianglelefteq (e,a)$:
      - define $(x,y) \bar{\sim} \bar{0}$.
11) Let
$$R = \{(a,b) \in \bar{S} \times \bar{S} \mid \text{there is no } c \in \bar{S} \text{ such that } (a,b) \bar{\sim} c\}.$$
12) Relate each pair in R with either $\bar{0}$ or α regarding monotonicity and $\bar{\sim}$.

Remark 5.2: In Step 8, the pairs, where $a = 1$, $b = 1$, or $c = 1$, may be omitted as they bring no new information to $\bar{\sim}$. Step 8, Step 9, and Step 10 are illustrated by Figure 4 and Figure 5, respectively.

All the steps of the algorithm run in polynomial time except for Step 12 which has exponential complexity. This is, however, related to the fact that also the number of the results increases exponentially with the number of the pairs in R.

Theorem 5.3: Let $(S, \leqslant, 1, \sim)$ be an Archimedean f. n. tomonoid partition. The partition $(\bar{S}, \leqslant, 1, \bar{\sim})$ given by Algorithm 5.1 is an Archimedean elementary extension of $(S, \leqslant, 1, \sim)$ and, moreover, all its Archimedean elementary extension arise in this way.

See [PeVe] for a proof of this theorem.

VI. The Commutative Case

Under the assumption of commutativity, looking for an elementary extension of an Archimedean f. n. tomonoid can be performed analogously to Algorithm 5.1 with the following differences.

Step 9 and Step 10 can be merged to one single step:

- For every $a \in \bar{S} \setminus \{\bar{0}, \alpha, 1\}$:
  - let $b \in \bar{S}$ be the highest element such that $(a,b) \notin P$,
  - let $e \in \bar{S}$ be such that $(b, \kappa) \bar{\sim} e$,
  - for every $(x,y) \in \bar{S} \times \bar{S}$ such that $(x,y) \trianglelefteq (a,e)$:
    - define $(x,y) \bar{\sim} (y,x) \bar{\sim} \bar{0}$.

Step 12 needs to be changed in the way that when a pair $(a,b) \in R$ is related to $\bar{0}$-class or with α-class then the reversed pair (b,a) has to be related to the same class, as well.

VII. Lower bounds

In this section we want to show that the number of Archimedean f. n. tomonoid increases rapidly with the number of the elements and that it is lower-bounded by a function given by a bunomial coefficient in the general non-commutative case and by an exponential function in the commutative case.

Definition 7.1: A finite tomonoid $(S; \leq, \odot, 1)$ with the least element 0 is called *drastic* if, for every $a, b \in S$, $a, b \neq 1$, we have $a \odot b = 0$.

The following lemma is an easy observation.

Lemma 7.2: Let $(S; \leqslant)$ be a finite, totally ordered set with the least element 0, the greatest element 1, and the atom α. Let $\odot$ be a binary operation on S defined, for $a, b \in S$, by

$$\begin{array}{ll} a \odot b = a & \text{if} \quad b = 1, \\ a \odot b = b & \text{if} \quad a = 1, \\ a \odot b = 0 & \text{if} \quad a \in \{0, \alpha\} \text{ and } b < 1 \\ a \odot b = 0 & \text{if} \quad b \in \{0, \alpha\} \text{ and } a < 1, \\ a \odot b \in \{0, \alpha\} & \text{otherwise.} \end{array}$$

Then S is an Archimedean f. n. monoid.

If, moreover, the two following conditions are fulfilled:

(i) for every $a \in S$ there is $c \in S$ such that, for every $b \in S$, we have $a \odot b = 0$ if and only if $b \leqslant c$,

(ii) for every $b \in S$ there is $c \in S$ such that, for every $a \in S$, we have $a \odot b = 0$ if and only if $a \leqslant c$,

then $\odot$ is compatible with $\leqslant$ and S is an Archimedean f. n. tomonoid.

Proof: We start with the first part of the theorem. Clearly, 1 is a neutral element of S. Thus we only need to prove that the associativity equation $a \odot (b \odot c) = (a \odot b) \odot c$ holds for every $a, b, c \in S$. If any of a, b, c is equal to 1 then the equation is trivially satisfied. If this is not the case then it can be easily checked that both sides of the equation are equal to 0.

We are going to prove (T3) as stated in Definition 2.1. Take $a, b, c \in S$ such that $a \leqslant b$. If $c = 1$ then both $a \odot c \leqslant b \odot c$ and $c \odot a \leqslant c \odot b$ hold trivially. If this is not the case then, according to (i), we have exactly one of the following cases:

- $c \odot a = 0$ and $c \odot b = 0$,
- $c \odot a = 0$ and $c \odot b = \alpha$,
- $c \odot a = \alpha$ and $c \odot b = \alpha$

which implies $c \odot a \leqslant c \odot b$. In an analogous manner (ii) implies $a \odot c \leqslant b \odot c$. ∎

In an inspiration with this lemma we can see that, in order to obtain all the elementary extensions of a drastic tomonoid, we simply need to discover in how many ways we can place a block of zero elements and a block of atom elements to the multiplication table regarding only the monotonicity (and, eventually, also the commutativity). This brings the following statement.

Proposition 7.3: Let $(S; \leqslant, \odot, 1)$ be a drastic f. n. tomonoid of size $n+1$, $n \in \mathbb{N}$. There are $\binom{2n}{n}$ elementary extensions of S that are Archimedean f. n. tomonoids. There are 2^n elementary extensions of S that are commutative, Archimedean f. n. tomonoids.

Proof: Let $(\bar{S}; \leqslant, \bar{\odot}, 1)$ be an elementary extension of S; let $\bar{0}$ be the zero of $\bar{S}$, let α be the atom of $\bar{S}$. Since $\bar{S}$ is supposed to be Archimedean, we necessarily have $(a, b) \mathrel{\tilde{\sim}} 0$ if $a \in \{0, \alpha\}$ and $b < 1$ or if $b \in \{0, \alpha\}$ and $a < 1$ (cf. with Lemma 7.2). Denote $I = \bar{S} \setminus \{0, \alpha, 1\}$ and note that the cardinality of I is n. Since S is drastic, we need, in order to obtain an Archimedean f. n. tomonoid $\bar{S}$, to make a partition dividing $I \times I$ to Z and A such that

$$\begin{array}{lll} (a, b) \mathrel{\tilde{\sim}} 0 & \text{if} & (a, b) \in Z, \\ (a, b) \mathrel{\tilde{\sim}} \alpha & \text{if} & (a, b) \in A \end{array}$$

for every $(a, b) \in I \times I$. According to Lemma 7.2, regardless of the partition, $\bar{S}$ will be an Archimedean f. n. monoid. Thus the only thing we need to take into account is the compatibility of $\bar{\odot}$ with $\leqslant$. This will be satisfied if and only if Z and A are separated by a border that consists of n vertical and n horizontal segments. There are $\binom{2n}{n}$ such borders.

If $\bar{S}$ is, moreover, supposed to be commutative then the border separating Z and A needs to be symmetric according to the diagonal. A half of this border consists of n segments which are either vertical or horizontal. There are 2^n such borders. ∎

Thus, as we can see, the number of f. n. tomonoids of a given size is lower-bounded by a binomial coefficient in the non-commutative Archimedean case and by an exponential function in the commutative Archimedean case.

VIII. Comparison with brute-force methods

We give here a short comparison of our level-set based method with two brute-force methods. (The brute-force methods served us also to check out the correctness of the introduced level-set based method.) For this purpose, we have generated all the Archimedean and Archimedean, commutative f. n. tomonoids up to the size 10 and measured the times in which the methods have run. The times of running the level-set based method are, as expected, shorter which illustrate Tables I and II.

size	number	lvl	bru	sup	bru/lvl	sup/lvl
3	1	0.0002	0.0003	0.0003	1.45	1.46
4	2	0.0006	0.0008	0.0008	1.42	1.43
5	8	0.0028	0.0057	0.0328	2.08	11.9
6	44	0.0267	0.0700	77.472	2.63	2903
7	333	0.2343	1.6002		6.83	
8	3543	3.5066	53.173		15.2	
9	54954	63.676	2402.1		37.7	
10	1297705	2562.5				

TABLE I. Times to generate Archimedean f. n. tomonoids

Column "size" represents the sizes of the generated tomonoids and "number" represents their numbers. The used methods are the following.

Column "lvl" shows the times to run the algorithm based on our *level-set method*. In order to obtain all the tomonoids of the given size, first, the trivial tomonoid is created, then all its elementary extensions are computed, then the extensions of the extensions are computed, and so on. This way we are

size	number	lvl	bru	bru/lvl
3	1	0.0003	0.0004	1.46
4	2	0.0006	0.0009	1.44
5	6	0.0026	0.0032	1.20
6	22	0.0110	0.0223	2.03
7	95	0.0721	0.2155	2.99
8	471	0.5046	2.7271	5.40
9	2670	4.1314	29.211	7.07
10	17387	36.636		

TABLE II. TIMES TO GENERATE ARCHIMEDEAN, COMMUTATIVE F. N. TOMONOIDS

creating a tree of tomonoids until the level of the required size is reached.

Column "bru" represents, what we call, the *brute force method*. The idea is similar to the previous case and a tree of elementary extensions is created. However, to obtain all the extensions of a given tomonoid, we simply iterate through all the possible evaluations of the complement of the set P (see Step 5 of Algorithm 5.1) and we discard those cases that fail the tests on associativity, monotonicity, and Archimedeanicity.

Column "sup" contains times of, what we call, the *super-brute-force method*. In this case, in order to obtain all the tomonoids of the given size, we generate all the possible multiplication tables and test them on the requirements on associativity, monotonicity, and Archimedeanicity.

The algorithm has been implemented in Python and run on a personal computer with 1,3 GHz Intel Core i5 processor and 4GB of memory—this is evidently not a computer dedicated for such a task and thus the absolute times are not very significant. However, what might give an illustration are the ratios of the times of the different methods which are represented by the columns "bru/lvl" and "sup/lvl" in Tables I and II.

IX. CONCLUSION

An algorithm to give Archimedean elementary extensions and commutative Archimedean elementary extensions of a finite, negative, tomonoid has been presented. The authors, at the moment, are working also on the algorithm for the general, non-Archimedean, case. Further, a comparison of the introduced algorithm with already existing methods [BaNa], [BeVy], [DeMe] is planned.

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Algorithm for Generating Finite Totally Ordered Monoids

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Abstract. The semantics of fuzzy logic is typically based on negative totally ordered monoids. This contribution describes an algorithm generating in a step-wise fashion all finite structures of this kind.

Keywords: Discrete triangular norm · Finite negative totally ordered monoid · Rees coextension · Rees congruence · Reidemeister closure condition · Tomonoid partition

1 Introduction

Partially ordered monoids are structures occurring in several fields of mathematics and computer sciences, in particular in logic. In non-classical logic, the canonical set of truth values is often endowed with a binary operation making this set into a partially ordered monoid. The monoidal operation then corresponds to the conjunction.

The algebraic semantics of the fuzzy logic MTL is the variety of MTL-algebras—commutative residuated lattices where the top element is the monoidal identity—and every MTL algebra is a subdirect product of MTL chains (i.e., totally ordered MTL algebras). This makes negative totally ordered monoids (which are, in fact, monoidal reducts of MTL chains) important structures worth of studying.

In this contribution, we focus on finite structures as they may be used, e.g., in finite-valued fuzzy logics. We note that, under the additional assumption of commutativity, the structures that we consider can be identified with linearly ordered finite MTL-algebras; MTL-algebras are in turn the algebraic counterpart of the fuzzy logic MTL [8].

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Further, this contribution can be seen as a practical appendix to our previous paper [13] which has yielded a method to describe all the one-element Rees coextensions of a given finite, negative, totally ordered monoid (shortly a f.n. tomonoid) S, that is, all the f.n. tomonoids greater by one element such that S is their common Rees quotient. This way, starting from the trivial monoid, one can generate all the possible f.n. tomonoids up to a given finite size. While the cited paper has been focused on describing the coextensions and giving a proof that all the existing f.n. tomonoids are necessarily obtained this way, this paper intends to give a practical description of the algorithm that produces the tomonoids. This paper can be also viewed as a continuation of our previous result [12] where we have, however, dealt with the Archimedean case only.

2 Basic Notions

We begin with an introduction of the basic notions of the paper. A *monoid* is an algebra $(S; \odot, 1)$ of type $\langle 2, 0\rangle$ such that $(a \odot b) \odot c = a \odot (b \odot c)$ and $a \odot 1 = 1 \odot a = a$ for every $a, b, c \in S$. A total (linear) order $\leqslant$ on a monoid S is called *compatible* if $a \leqslant b$ implies both $a \odot c \leqslant b \odot c$ and $c \odot a \leqslant c \odot b$ for every $a, b, c \in S$. In such a case, we call $(S; \leqslant, \odot, 1)$ a *totally ordered monoid* or a *tomonoid*, for short. We also say that $\odot$ is *monotone* with respect to $\leqslant$. Further, S is called *commutative* if $a \odot b = b \odot a$ for every $a, b \in S$. Finally, S is

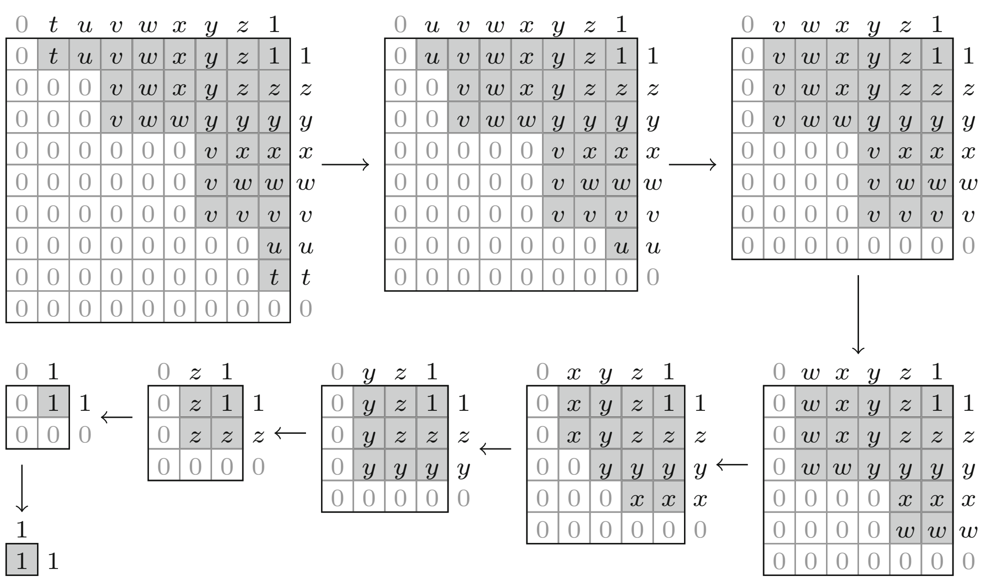

Fig. 1. Examples of f.n. tomonoids depicted by their multiplication tables. Seeing the cells of a table as ordered pairs from S^2, the level equivalence classes correspond with the maximal sets of the cells with the same symbol. The depicted f.n. tomonoids are actually created as one-element Rees quotients starting with the first f.n. tomonoid of size 9. As we may observe, the one-element Rees quotient arises by "cutting off" the column and the row indexed by the zero and by merging the zero and the atom classes into one. Finally, we reach the trivial monoid.

called *negative* if 1 is the top element. We note that in the context of residuated lattices, usually the notion "integral" is used instead.

This paper is focused mainly on finite, negative, totally ordered monoids which we abbreviate by "*f.n. tomonoids*". In general, we do not assume the monoids to be commutative [9], although, we deal with the commutativity, as well. Let us remark that commutative f.n. tomonoids correspond to *discrete triangular norms* [6]. The smallest monoid that consists of the monoidal identity 1 alone, is called the *trivial tomonoid*.

The illustrations in this paper depict f.n. tomonoids by their multiplication tables, see Fig. 1.

3 Level Set Representation of Tomonoids

We do not work with f.n. tomonoids directly but we rather work with their level set representations. In the the following text, by S^2 we denote the Cartesian product of the set S with itself, i.e., $S^2 = S \times S$.

For a tomonoid $(S; \leqslant, \odot, 1)$ and two pairs $(a, b), (c, d) \in S^2$ we define $(a, b) \sim (c, d)$ iff $a \odot b = c \odot d$ and we call $\sim$ the *level equivalence* associated with S.

Let $(S; \leqslant)$ be a totally ordered set. By $\trianglelefteq$ we denote the componentwise order on S^2, i.e., for every $a, b, c, d \in S$, we put $(a, b) \trianglelefteq (c, d)$ iff $a \leqslant b$ and $c \leqslant d$. Let $1 \in S$ and let $\sim$ be an equivalence on S^2 such that the following holds:

(P1) For every $a, b, c, d, e \in S$, $(a, b) \sim (1, d)$ and $(b, c) \sim (1, e)$ imply $(d, c) \sim (a, e)$. (See an illustration in Fig. 2-left.)

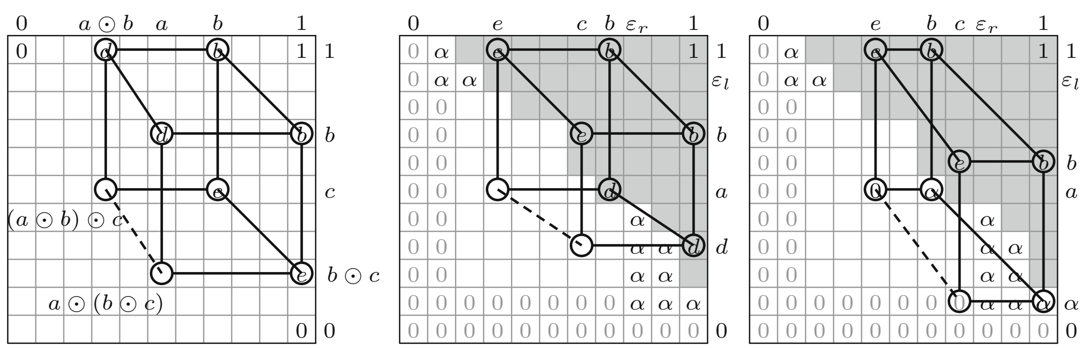

Fig. 2. Left: Illustration of Property (P1). Consider two rectangles such that the first one hits the upper edge and the second one hits the right edge of the multiplication table. Assume that the upper left, upper right, and lower right vertices are in the same level equivalence classes, respectively. Then also the remaining lower left vertices are elements of the same level equivalence class. This property is directly related to the associativity of the tomonoid and corresponds to the *Reidemeister condition* known from web geometry [1,5]. **Middle: Illustration of (E2).** For every two pairs $(a, b), (b, c) \in \mathcal{P}$ we relate $(a, e) \dot{\sim} (d, c)$. **Right: Illustration of (E3'a).** Let $(a, b) \in \mathcal{Q}$, let $c < \varepsilon_r$, and let $(b, c) \sim e$. If $(a, b) \dot{\sim} 0$ then also $(a, e) \dot{\sim} 0$ according to the monotonicity. If $(a, b) \dot{\sim} \alpha$ then $(a, e) \dot{\sim} 0$ according to (P1).

(P2) For every $a, b \in S$ there is exactly one $c \in S$ such that $(a,b) \sim (1,c) \sim (c,1)$.
(P3) For every $a, b, c, d, a', b', c', d' \in S$, $(a,b) \sim (a',b') \trianglelefteq (c,d) \sim (c',d') \trianglelefteq (a,b)$ implies $(a,b) \sim (c,d)$.

Then we call $(S^2; \trianglelefteq, \sim, (1{,}1))$ a *tomonoid partition*. The following two propositions show that tomonoids and tomonoid partitions are in a one-to-one correspondence.

Proposition 1 [13]. *Let $(S; \leqslant, \odot, 1)$ be a tomonoid and let $\sim$ be its level equivalence. Then $(S^2; \trianglelefteq, \sim, (1{,}1))$ is a tomonoid partition.*

Proposition 2 [13]. *Let $(S^2; \trianglelefteq, \sim, (1{,}1))$ be a tomonoid partition. For every $a, b \in S$, let $a \odot b$ be given as the unique c such that $(a,b) \sim (1,c) \sim (c,1)$. Then $(S; \leqslant, \odot, 1)$ is the unique tomonoid such that $(S^2; \trianglelefteq, \sim, (1{,}1))$ is its associated tomonoid partition.*

In the following text, we will write $(a,b) \sim c$ instead of $(a,b) \sim (1,c) \sim (c,1)$.

4 Rees Quotients and Coextensions

In this section we introduce the notion of a one-element coextension of a f.n. tomonoid.

Let $(S; \leqslant, \odot, 1)$ be a f.n. tomonoid. We call its least element the *zero* (and we denote it by 0), we call its second smallest element the *atom* (and we denote it by α), and we call its second greatest element the *coatom* (and we denote it by κ). Recall that 1 is the greatest element of S.

A *tomonoid congruence* on S is an equivalence relation $\approx$ on S such that

1. $\approx$ is a congruence [10] of S as a monoid and
2. each equivalence class is convex.

The operation induced by $\odot$ on the quotient $\langle S \rangle_\approx$ we denote again by $\odot$. For $a, b \in S$, we define $\langle a \rangle_\approx \leqslant \langle b \rangle_\approx$ if $a \approx b$ or $a < b$. We may observe that $(\langle S \rangle_\approx; \leqslant, \odot, \langle 1 \rangle_\approx)$ is a tomonoid again and we call $\langle S \rangle_\approx$ the *tomonoid quotient* with respect to $\approx$. This procedure preserves the properties of finiteness, negativity, and commutativity.

We proceed with the notion of the Rees congruence which is commonly used for semigroups [11]. Let $q \in S$. For $a, b \in S$ we define $a \approx_q b$ if $a = b$ or $a, b \leqslant q$. Then $\approx_q$ is a tomonoid congruence and we call it the *Rees congruence* with respect to q. We denote the corresponding quotient by S/q and we call it the *Rees quotient* of S with respect to q. Furthermore, we call S a *Rees coextension* of S/q [13]. If moreover $q = \alpha$, we call S/q the *one-element Rees quotient* of S and we call S the *one-element Rees coextension* (or, shortly, the *one-element coextension*) of S/q. See an illustration in Fig. 1.

5 One-Element Rees Coextensions of f.n. Tomonoids

The algorithm we are going to present is based on a theorem [13] which we briefly describe here. Let $(S; \leqslant, \odot, 1)$ be a f.n. tomonoid. We denote $S^\star = S \setminus \{0\}$. A *zero doubling extension* of S is a totally ordered set $\bar{S} = S^\star \dot{\cup} \{0, \alpha\}$ such that $0 < \alpha < a$ for every $a \in S^\star$. We call $a \in S$ an *idempotent* if $a \odot a = a$. Obviously, 0 and 1 are idempotents of every f.n. tomonoid.

Let $\sim_1$ and $\sim_2$ be two equivalence relations on S^2. We say that $\sim_2$ is a *coarsening* of $\sim_1$ if $\sim_1 \subseteq \sim_2$, that is, if each equivalence class of $\sim_2$ is a union of some equivalence classes of $\sim_1$.

Let $(S; \trianglelefteq, \sim, (1{,}1))$ be a f.n. tomonoid partition. Let $\bar{S}$ be the zero doubling extension of S. Define

$$\mathcal{P} = \{(a,b) \in \bar{S}^2 \mid a, b \in S^\star \text{ and there is } c \in S^\star \text{ s.t. } (a,b) \sim c\}, \tag{1}$$

$$\mathcal{Q} = \bar{S}^2 \setminus \mathcal{P}. \tag{2}$$

Let $(\varepsilon_l, \varepsilon_r)$ be a pair of non-zero idempotents of S and let $\dot{\sim}$ be the smallest equivalence relation on $\bar{S}^2$ such that the following conditions hold:

(E1) We have $(a,b) \dot{\sim} (c,d)$ for every $(a,b),(c,d) \in \mathcal{P}$ such that $(a,b) \sim (c,d)$.

(E2) We have $(d,c) \dot{\sim} (a,e)$ for every $(a,b),(b,c) \in \mathcal{P}$ and $(d,c),(a,e) \in \mathcal{Q}$ such that $(a,b) \sim d$ and $(b,c) \sim e$. (See an illustration in Fig. 2-middle.)

(E3'a) We have $(a,e) \dot{\sim} 0$ for every $a,b,c,e \in S^\star$ such that $(a,b) \in \mathcal{Q}$, $(b,c) \sim e$, and $c < \varepsilon_r$.

Furthermore, we have $(d,c) \dot{\sim} 0$ for any $a,b,c,d \in S^\star$ such that $(b,c) \in \mathcal{Q}$, $(a,b) \sim d$, and $a < \varepsilon_l$. (See an illustration in Fig. 2-right.)

(E3'b) We have $(a,e) \dot{\sim} (a,b)$ for every $a,b,c,e \in S^\star$ such that $(a,b) \in \mathcal{Q}$, $(b,c) \sim e$, and $c \geq \varepsilon_r$.

Furthermore, we have $(d,c) \dot{\sim} (b,c)$ for every $a,b,c,d \in S^\star$ such that $(b,c) \in \mathcal{Q}$, $(a,b) \sim d$, and $a \geq \varepsilon_l$. (See illustrations in Fig. 3-left and middle.)

(E3'c) We have $(a,b) \dot{\sim} 0$ for every $a,b,c > 0$ such that $(a,b),(b,c) \in \mathcal{Q}$, $a < \varepsilon_l$, and $c \geq \varepsilon_r$.

Furthermore, we have $(b,c) \dot{\sim} 0$ for every $a,b,c > 0$ such that $(a,b),(b,c) \in \mathcal{Q}$, $a \geq \varepsilon_l$, and $c < \varepsilon_r$. (See an illustration in Fig. 3-right.)

(E4'a) We have $(1,0) \dot{\sim} (0,1) \dot{\sim} (a,\alpha) \dot{\sim} (\alpha,b)$ for every $a < \varepsilon_l$ and $b < \varepsilon_r$.

Furthermore, we have $(a,b) \dot{\sim} 0$ for every $(a,b),(c,d) \in \mathcal{Q}$ such that $(a,b) \trianglelefteq (c,d) \dot{\sim} 0$.

(E4'b) We have $(1,\alpha) \dot{\sim} (\alpha,1) \dot{\sim} (\varepsilon_l,\alpha) \dot{\sim} (\alpha,\varepsilon_r)$.

Furthermore, we have $(a,b) \dot{\sim} \alpha$ for every $(a,b),(c,d) \in \mathcal{Q}$ such that $(a,b) \trianglerighteq (c,d) \dot{\sim} \alpha$.

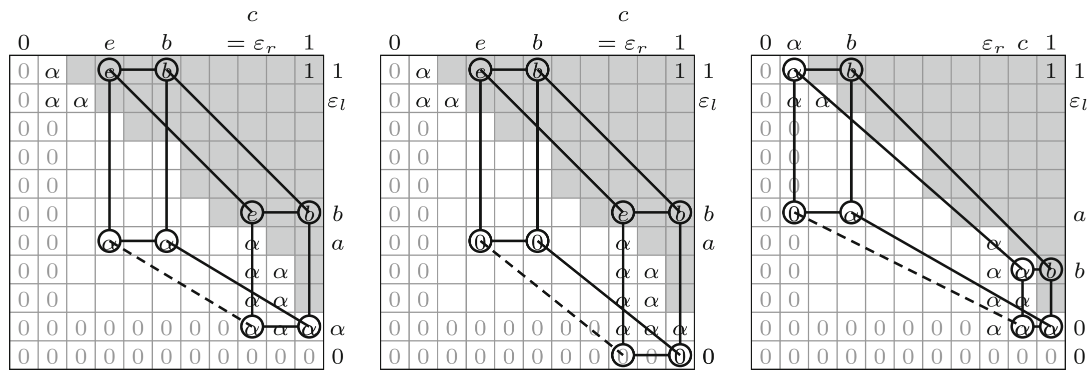

Fig. 3. Left and middle: Illustration of (E3'b). Let $(a,b) \in \mathcal{Q}$, let $c \geqslant \varepsilon_r$, and let $(b,c) \sim e$. If $(a,b) \mathrel{\dot{\sim}} \alpha$ then, according to (P1), $(a,e) \mathrel{\dot{\sim}} \alpha$, as well (left figure). If $(a,b) \mathrel{\dot{\sim}} 0$ then, according to (P1) or monotonicity, $(a,e) \mathrel{\dot{\sim}} 0$, as well (middle figure). **Right: Illustration of (E3'c).** Let $(a,b),(b,c) \in \mathcal{Q}$, let $c \geq \varepsilon_r$, and let $a < \varepsilon_l$. Then $(a,b) \mathrel{\dot{\sim}} 0$. Indeed, if we had $(a,b) \mathrel{\dot{\sim}} \alpha$ then, according to (P1), we would also have $(a,\alpha) \mathrel{\dot{\sim}} \alpha$ which is a contradiction.

We call the structure $(\bar{S}^2; \trianglelefteqslant, \dot{\sim}, (1{,}1))$ the $(\varepsilon_l, \varepsilon_r)$-*ramification* of $(S^2; \trianglelefteqslant, \sim, (1{,}1))$.

Theorem 1 [13]. *Let $(S; \trianglelefteqslant, \sim, (1{,}1))$ be a f.n. tomonoid partition and let $(\varepsilon_l, \varepsilon_r)$ be a pair of its non-zero idempotents. Let $(\bar{S}^2; \trianglelefteqslant, \dot{\sim}, (1{,}1))$ be the $(\varepsilon_l, \varepsilon_r)$-ramification of $(S^2; \trianglelefteqslant, \sim, (1{,}1))$.*

If $(1,0) \mathrel{\dot{\sim}} (1,\alpha)$ then there is no one-element coextension of S^2 with respect to $(\varepsilon_l, \varepsilon_r)$. Otherwise, let $\bar{\sim}$ be a coarsening of $\dot{\sim}$ such that the following holds: the $\bar{\sim}$-class of each $c \in S^\star$ coincides with the $\dot{\sim}$-class of c, the $\bar{\sim}$-class of 0 is downward closed, and each $\bar{\sim}$-class contains exactly one element of the form $(1,c)$ for some $c \in \bar{S}$. Then $(\bar{S}^2; \trianglelefteqslant, \bar{\sim}, (1{,}1))$ is a one-element coextension of S^2 with respect to $(\varepsilon_l, \varepsilon_r)$.

Moreover, all one-element coextensions of S^2 with respect to $(\varepsilon_l, \varepsilon_r)$, if there are any, arise in this way.

6 Representation of f.n. Tomonoids

The aim of this paper is to describe an algorithmic implementation of Theorem 1. The crucial part is to choose a suitable representation of the f.n. tomonoids (and the corresponding tomonoid partitions). F.n. tomonoids can be naturally represented by two-dimensional arrays representing their multiplication tables (see, e.g., Fig. 1). However, this approach has shown as unsuitable for the implementation. Performing the algorithm, we mainly need to work with the level equivalence classes; we need, for example, to add pairs to this classes or we need to merge two classes into one.

Therefore we have decided to represent a f.n. tomonoid $(S; \leqslant, \odot, 1)$ as a collection of level equivalence classes of pairs from S^2. Such a collection forms a partition of S^2, i.e., every pair belongs to an (exactly one) equivalence class. Each level equivalence class is either assigned to a unique value of the f.n. tomonoid (which means that it must contain two pairs of the form $(a, 1)$ and $(1, a)$ where $a \in S$) or it is "unassigned". An "unassigned" class can be a singleton.

7 Methods

Two methods, that recursively call each other, create the core of the implemented algorithm:

- a method that adds a pair (a, b) to a z-level equivalence class, we denote it by $(a, b) \dot{\sim} z$,
- a method that relates a pair (a, b) with a pair (c, d), we denote it by $(a, b) \dot{\sim} (c, d)$.

When implementing these two methods, it is first crucial that the transitivity of $\dot{\sim}$ is preserved. That is, when we add a pair (a, b) to a certain level equivalence class, we consequently need to add to the same class also all the pairs that are already related to (a, b).

Second, it is important that the monotonicity of the constructed tomonoid is not violated. This task is easier by the fact that (except for Part (E1), see below) we work only with pairs that are assigned to 0, α, or unassigned. Thus, when performing $(a, b) \dot{\sim} z$, z is either 0 or α. If $z = 0$ then we need to be sure that also all the pairs lower that (a, b) are assigned to 0. If $z = \alpha$ we proceed analogously for the pairs greater that (a, b). The details are described in the next two subsections.

Method Implementing $(a, b) \dot{\sim} z$

Recall that z is either 0 or α. We delete the whole level equivalence class containing (a, b) and we add all the deleted pairs to the z-level equivalence class. If $z = 0$ then for every pair (x, y) in the deleted class:

- for every pair $(u, v) \in \mathcal{Q}$ such that $(u, v) \trianglelefteq (x, y)$:
  - perform $(u, v) \dot{\sim} 0$.

If $z = \alpha$ then for every pair (x, y) in the deleted class:

- for every pair $(u, v) \in \mathcal{Q}$ such that $(u, v) \trianglerighteq (x, y)$:
  - perform $(u, v) \dot{\sim} \alpha$.

If (a, b) is already contained in a y-level equivalence class and $y \neq z$ an error is emitted signalizing that the constructed coextension is not possible.

Method Implementing $(a, b) \dot{\sim} (c, d)$

If both the pairs (a, b) and (c, d) belong to unassigned level equivalence classes, we simply delete one of the classes and add all its pairs to the second one.

If one of the pairs, say (a, b), belongs to a z-level equivalence class (z is either 0 or α), we perform $(c, d) \dot{\sim} z$.

If (a, b) belongs to a z-level equivalence class and (c, d) belongs to a y-level equivalence class then either $y = z$ which means that both (a, b) and (c, d) belong to the same level equivalence class and thus we do not perform anything, or $y \neq z$ which means that it is not possible to construct such a coextension. In the latter case an error is emitted stopping the process.

8 Algorithm

Input:

- $(S^2; \trianglelefteq, \sim, (1,1))$ ... tomonoid partition of a f.n. tomonoid $(S; \leqslant, \odot, 1)$
- $(\varepsilon_l, \varepsilon_r)$ ... pair of its non-zero idempotents

Output:

- $(\bar{S}^2; \trianglelefteq, \bar{\sim}, (1,1))$ ... a one-element coextension of $(S^2; \trianglelefteq, \sim, (1,1))$ with respect to $(\varepsilon_l, \varepsilon_r)$

Algorithm:

Initialization:

1. Let $\bar{S}$ be the zero doubling extension of S.
2. Let 0, α, and κ be the zero, the atom, and the coatom of $\bar{S}$, respectively. Let $\mathcal{P}$ and $\mathcal{Q}$ be given by (1) and (2), respectively.
3. Let $\dot{\sim}$ be an equivalence relation on $\bar{S}^2$. (The following steps are going to define this relation.)

Part (E1):

4. For every $(a, b), (c, d) \in \mathcal{P}$:
   - define $(a, b) \dot{\sim} (c, d)$
     if $(a, b) \sim (c, d) \sim e$ for some $e \in \bar{S} \setminus \{0, \alpha\}$.

Part (E2):

5. For every $(a, b), (b, c) \in \mathcal{P}$:
   - let $d \in \bar{S}$ be such that $(a, b) \sim d$,
   - let $e \in \bar{S}$ be such that $(b, c) \sim e$,
   - perform $(a, e) \dot{\sim} (d, c)$.

Part (E4'):

6. Perform $(1, 0) \dot{\sim} (0, 1) \dot{\sim} 0$.
7. Perform $(a, \alpha) \dot{\sim} (\alpha, b) \dot{\sim} 0$ for $a < \varepsilon_l$ and $b < \varepsilon_r$.
8. Perform $(\varepsilon_l, \alpha) \dot{\sim} (\alpha, \varepsilon_r) \dot{\sim} \alpha$.

Part (E3'a):

9. For every $a \in \bar{S}$ such that $\alpha < a < \varepsilon_l$:
   - let $b \in \bar{S}$ be the highest element such that $(a, b) \in \mathcal{Q}$,
   - let $c \in \bar{S}$ be the highest element such that $c < \varepsilon_r$,
   - let $e \in \bar{S}$ be such that $(b, c) \sim e$,
   - if $e > \alpha$ then perform $(a, e) \dot{\sim} 0$.
10. For every $c \in \bar{S}$ such that $\alpha < c < \varepsilon_r$:
   - let $b \in \bar{S}$ be the highest element such that $(b, c) \in \mathcal{Q}$,
   - let $a \in \bar{S}$ be the highest element such that $a < \varepsilon_l$,
   - let $d \in \bar{S}$ be such that $(a, b) \sim d$,
   - if $d > \alpha$ then perform $(d, c) \dot{\sim} 0$.

Part (E3'c):

11. For every $a \in \bar{S}$ such that $\varepsilon_l \leqslant a < 1$:
   - let $b \in \bar{S}$ be the highest element such that $(a, b) \in \mathcal{Q}$,
   - let $c \in \bar{S}$ be the highest element such that $(b, c) \in \mathcal{Q}$ and $c < \varepsilon_r$,
   - perform $(b, c) \dot{\sim} 0$.
12. For every $c \in \bar{S}$ such that $\varepsilon_r \leqslant c < 1$:
   - let $b \in \bar{S}$ be the highest element such that $(b, c) \in \mathcal{Q}$,
   - let $a \in \bar{S}$ be the highest element such that $(a, b) \in \mathcal{Q}$ and $a < \varepsilon_l$,
   - perform $(a, b) \dot{\sim} 0$.

Part (E3'b):

13. For every $b \in \bar{S}$ such that $\alpha < b < 1$:
   - let $e \in \bar{S}$ be such that $(b, \varepsilon_r) \sim e$,
   - if $e < b$ then:
     - for every $a \in \bar{S}$ s.t. $\alpha < a < \varepsilon_l$ and $(a, b) \in \mathcal{Q}$:
       * perform $(a, e) \dot{\sim} (a, b)$.
14. For every $b \in \bar{S}$ such that $\alpha < b < 1$:
   - let $d \in \bar{S}$ be such that $(\varepsilon_l, b) \sim d$,
   - if $d < b$ then:
     - for every $c \in \bar{S}$ s.t. $\alpha < c < \varepsilon_r$ and $(b, c) \in \mathcal{Q}$:
       * perform $(d, c) \dot{\sim} (b, c)$.

Coarsening:

15. Let $\bar{\sim} := \dot{\sim}$.
16. For every pair $(a, b) \in \bar{S}^2$, that belongs to an unassigned level equivalence class, perform arbitrarily either $(a, b) \bar{\sim} 0$ or $(a, b) \bar{\sim} \alpha$.

Remark 1. Let $\varphi \in S$ be the lowest non-zero idempotent of S. In Step 5 we may omit those pairs $(a, b), (b, c) \in \mathcal{P}$ where $(a, b), (b, c) \trianglerighteq (\varphi, \varphi)$ since, in such a case, $(a, e), (d, c) \in \mathcal{P}$.

Remark 2. In order to obtain all the one-element coextensions of S we simply repeat the procedure for every possible pair of its non-zero idempotents including $(1, 1)$. Furthermore, we create an additional coextension in the following way:

- Perform Steps 1 and 2.
- Perform $(1, 0) \bar{\sim} (0, 1) \bar{\sim} 0$.
- Perform $(\alpha, \alpha) \bar{\sim} \alpha$.

9 Example

Let us perform the algorithm taking the first f.n. tomonoid of size 9 in Fig. 1. As we can see, it has three non-zero idempotents: y, z, and 1. We are going to construct all the one-element coextensions with respect to (z, y).

- Initialization, Part (E1), and Part (E4'):
  - We obtain the values depicted in Fig. 4a.
- Part (E3'a) (see Fig. 4b):
  - Step 9:
    * For $(b, c) = (t, z)$ we perform $(y, t) \mathrel{\dot\sim} 0$.
    * For $(b, c) = (u, z)$ we perform $(y, u) \mathrel{\dot\sim} 0$.
    * For $(b, c) = (v, x)$ we perform $(v, v) \mathrel{\dot\sim} 0$.
    * For $(b, c) = (w, x)$ we perform $(v, w) \mathrel{\dot\sim} 0$.
    * For $(b, c) = (x, x)$ we perform $(v, x) \mathrel{\dot\sim} 0$.
    * For $(b, c) \in \{(y, u), (u, z)\}$ we do not perform anything.
  - Step 10:
    * for $(a, b) = (z, t)$ we perform $(x, t) \mathrel{\dot\sim} 0$.
    * for $(a, b) = (z, u)$ we perform $(x, u) \mathrel{\dot\sim} 0$.
    * for $(a, b) \in \{(x, v), (x, w), (x, x), (u, y), (x, z)\}$ we do not perform anything.
- Part (E3'c) (see Fig. 4c):
  - Step 11:
    * For $a = y$ we obtain $b = u$ and $c = y$. Thus we perform $(u, y) \mathrel{\dot\sim} 0$.
    * For $a = z$ we obtain $b = u$ and $c = y$. Thus we perform $(u, y) \mathrel{\dot\sim} 0$.

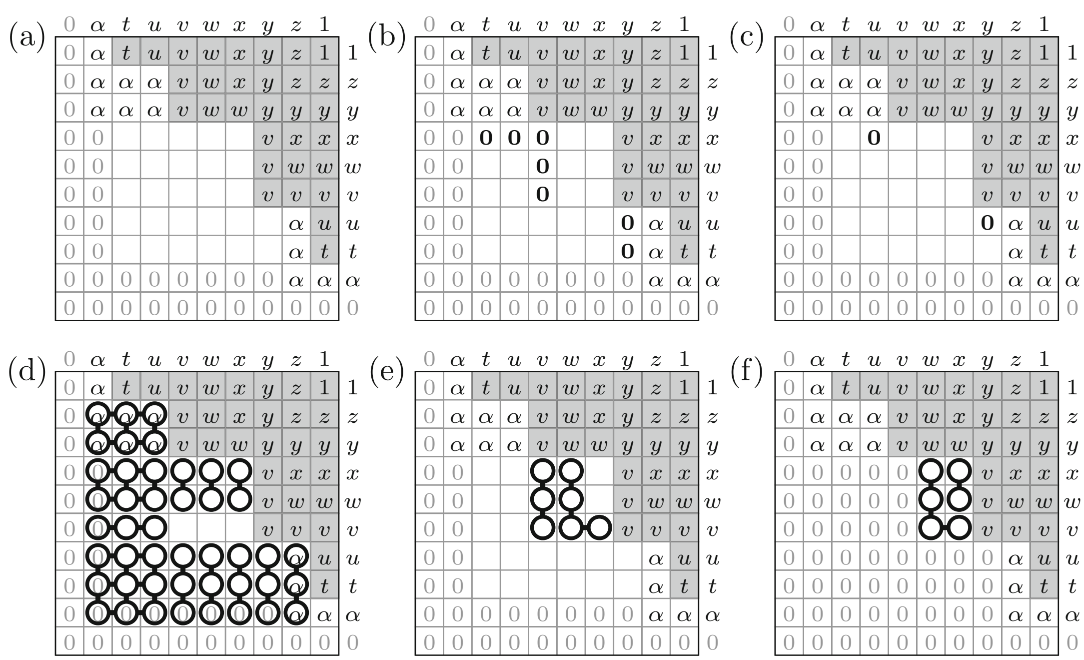

Fig. 4. Illustration of the algorithm.

- Step 12:
  - For $c = z$ we obtain $b = u$ and $a = x$. Thus we perform $(x, u) \dot\sim 0$.

– Part (E3'b) (see Fig. 4d):
- Step 13:
  - For $b = t$ we obtain $e = \alpha$ and we perform $(a, \alpha) \dot\sim (a, t)$ for every a from α to x.
  - For $b = u$ we obtain $e = \alpha$ and we perform $(a, \alpha) \dot\sim (a, u)$ for every a from α to x.
- Step 14:
  - For $b = t$ we obtain $d = \alpha$ and we perform $(\alpha, c) \dot\sim (t, c)$ for every c from α to y.
  - For $b = u$ we obtain $d = \alpha$ and we perform $(\alpha, c) \dot\sim (u, c)$ for every c from α to y.
  - For $b = x$ we obtain $d = w$ and we perform $(w, c) \dot\sim (x, c)$ for every c from α to x.
  - For $b = z$ we obtain $d = y$ and we perform $(y, c) \dot\sim (z, c)$ for every c from α to u.

– Part (E2) (see Fig. 4e):
- For $a = x$, $b = y$, and $c = x$ perform $(x, w) \dot\sim (v, x)$.
- For $a = w$, $b = y$, and $c = x$ perform $(w, w) \dot\sim (v, x)$.
- For $a = v$, $b = y$, and $c = x$ perform $(v, w) \dot\sim (v, x)$.
- For $a = x$, $b = y$, and $c = w$ perform $(x, w) \dot\sim (v, w)$.
- For $a = w$, $b = y$, and $c = w$ perform $(w, w) \dot\sim (v, w)$.
- For $a = x$, $b = y$, and $c = v$ perform $(x, v) \dot\sim (v, v)$.
- For $a = w$, $b = y$, and $c = v$ perform $(w, v) \dot\sim (v, v)$.

– Finally, we obtain the situation depicted in Fig. 4f. As we can see, there are three distinct one-element coextensions of the tomonoid.

10 Comparison with Existing Algorithms

We are aware that there already exists a number of results with similar goals [2–4,7]. We have, however, decided to introduce a new algorithm; firstly, since we wanted to have a practical support for our theoretical results [12], and secondly, since we believe that our approach is more effective. Let us make a short comparison.

The algorithm by Bartušek and Navara [2,3] is based on one-element Rees coextensions, as well, although this notion is not used. In this approach, all the existing coextensions of a given commutative f.n. tomonoid are obtained by checking the associativity for every newly defined $(x, y) \in \mathcal{Q}$.

The algorithm by Bělohlávek and Vychodil [4] uses a recursive backtracking procedure to test every possible tomonoid multiplication table on associativity.

As these two algorithms are both based on variants of the brute-force approach, our algorithm promises to give a better performance.

The comparison with the algorithm by De Baets and Mesiar [7] is planned to be made later when we have its description.

11 Conclusion

All the steps of the algorithm run in a polynomial time (with respect to the size of S) except for Step 16 where we, actually, obtain all the possible one-element coextensions with respect to the given pair of idempotents $(\varepsilon_l, \varepsilon_r)$. This step runs in exponential time since also the number of the coextensions is bounded from below by an exponential function depending on the size of S [12].

The validity of the algorithm's output is assured by Theorem 1. Remark that, since every f.n. tomonoid is a one-element Rees quotient of another f.n. tomonoid, it follows from Theorem 1 that every existing f.n. tomonoid can be obtained by the described procedure.

If we wish to obtain all the commutative one-element coextensions of a commutative f.n. tomonoid S, we simply perform $(a, b) \dot{\sim} (b, a)$ for every $(a, b) \in \mathcal{Q}$ right after Initialization.

The algorithm has been implemented and tested in the programming language Python; see http://cmp.felk.cvut.cz/~petrikm/extensions.php.

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Aequationes Mathematicae

New solutions to Mulholland inequality

Milan Petrík

Abstract. The paper gives answer to two open questions related to Mulholland's inequality. First, it is shown that there exists a larger set of solutions to Mulholland's inequality compared to the one delimited by Mulholland's condition. Second, it is demonstrated that the set of functions solving Mulholland's inequality is not closed with respect to compositions.

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Keywords. Convex function, dominance of strict triangular norms, geometrically convex function, Minkowski inequality, Mulholland inequality, probabilistic metric spaces.

1. Introduction

An increasing bijection $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$, where $\mathbb{R}_0^+$ denotes the set of positive real numbers with zero, is said to solve *Mulholland's inequality* if

$$\forall x, y, u, v \in \mathbb{R}_0^+ \colon f^{-1}\big(f(x+u) \\ + f(y+v)\big) \leq f^{-1}(f(x)+f(y)) + f^{-1}(f(u)+f(v)). \tag{1.1}$$

This inequality has been introduced by Mulholland [8] as a generalization of the *Minkowski inequality* which is obtained from (1.1) by setting $f(x) = x^p$, $p \geq 1$. Remark that the Minkowski inequality establishes the triangular inequality of p-norms (also L^p*-norms*).

Denote by MI the set of all the increasing bijections that solve Mulholland's inequality. In his paper [8, Theorem 1], Mulholland provided the following sufficient condition[1]:

Theorem 1.1. *Let* $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *be an increasing bijection. If both* f *and* $\log \circ f \circ \exp$ *are convex then* $f \in MI$*.*

[1] For the proof, see also the book by Kuczma [5, Theorem VIII.8.1].

This work was completed with the support of our TEX-pert.

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By MC we denote the set of all the bijections that comply with the assumptions of Theorem 1.2; thus we have $MC \subseteq MI$.

This paper intends to find answers to two open problems:

1. Is Mulholland's condition presented in Theorem 1.2 also necessary, i.e., is MC equal to MI or is it its proper subset?
2. Is the set of functions solving Mulholland's inequality closed with respect to their compositions?

The second problem is closely related to the question whether the dominance relation on the set of strict triangular norms [4] is transitive and thus an order [1, Problem 17] [11, Problem 12.11.3]. Recall that dominance plays a crucial role when constructing Cartesian products [13,14] of probabilistic metric spaces [7,11].

For our study, the first open problem emerged from the second one. It is known that the set MC is closed with respect to compositions. Thus, if $MC = MI$ then the answer to the second problem is positive. However, if there is a counter-example of two functions from MI such that their composition does not belong to MI then at least one of these functions must be outside MC.

As far as we know, the first open problem is stated in no paper explicitely, however, investigations on a representation of the set MI have been done already and several papers have appeared. In 1984, Tardiff [13] provided an alternative sufficient condition (and posed a question whether it is also a necessary one); the result can be formulated in the following way:

Theorem 1.2. *Let $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a differentiable increasing bijection. If both f and $\log \circ f' \circ \exp$ are convex then $f \in MI$.*

In 1999 Schweizer posed a question [12] on comparing the results of Mulholland and Tardiff. In 2002, Jarczyk and Matkowski [3] demonstrated that Tardiff's condition actually implies Mulholland's condition. An alternative proof of this statement was also given by Baricz [2] in 2010.

Throughout the paper we will mostly work with increasing bijections on $\mathbb{R}_0^+$. Note that if $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is an increasing bijection then it is, necessarily, continuous, $f(0) = 0$, and $\lim_{x\to\infty} f(x) = \infty$.

2. Pseudo-addition

We do not investigate Mulholland's inequality directly but we rather investigate the pseudo-additions generated by the corresponding functions. Particularly, we benefit from their level set plots.

Let $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be an increasing bijection. The *pseudo-addition* generated by f is a binary operation $*_f\colon \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ given, for every $x, y \in \mathbb{R}_0^+$, by

$$x *_f y = f^{-1}(f(x) + f(y)).$$

Using this notion, Mulholland's inequality can be reformulated to

$$\forall x, y, u, v \in \mathbb{R}_0^+ : \quad (x+u) *_f (y+v) \leq (x *_f y) + (u *_f v).$$

A *level set* of $*_f$ at the level $a \in \mathbb{R}_0^+$ is the set

$$L_a^f = \{(x, y) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ \mid x *_f y = a\}.$$

A *level cut* of $*_f$ at the level $a \in \mathbb{R}_0^+$ is the set

$$\Lambda_a^f = \{(x, y) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ \mid x *_f y \leq a\}.$$

Note that $(x, y) \in \Lambda_a^f$ if, and only if, $f(x) + f(y) \leq f(a)$.

In the sequel it will be usually clear from the context which increasing bijection f we are considering. Therefore we will often omit the index denoting f in the notation of $*_f$, L_a^f, and Λ_a^f.

What follows is an easy observation.

Lemma 2.1. *If an increasing bijection $f: \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is convex then, for every $a \in \mathbb{R}_0^+$, Λ_a is a convex set.*

For $A, B \subseteq \mathbb{R}_0^+ \times \mathbb{R}_0^+$ and $\alpha \in \mathbb{R}_0^+$, the *Minkowski sum* and *positive real multiple* are respectively defined by

$$A + B = \{(x+u, y+v) \mid (x, y) \in A, (u, v) \in B\},$$
$$\alpha A = \{(\alpha x, \alpha y) \mid (x, y) \in A\}.$$

Recall that for every $A, B, C, D \subseteq \mathbb{R}_0^+ \times \mathbb{R}_0^+$ and for every $\alpha, \beta \in \mathbb{R}_0^+$ we have

$$A \subseteq C \quad \text{and} \quad B \subseteq D \Rightarrow A + B \subseteq C + D. \tag{2.1}$$

If, moreover, A is a convex set, we have

$$(\alpha + \beta)A = \alpha A + \beta A. \tag{2.2}$$

With the introduced notions, Mulholland's inequality can be reformulated to a form which is quantified by two variables only:

$$\forall a, b \in \mathbb{R}_0^+ : \quad \Lambda_a + \Lambda_b \subseteq \Lambda_{a+b}. \tag{2.3}$$

This formulation allows an intuitive geometric interpretation which is illustrated in Fig. 1.

Observe, further, that if A is a singleton then $A + B = B$; this trivially implies:

Lemma 2.2. *Let $f: \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a convex increasing bijection and let $a, b \in \mathbb{R}_0^+$. If $a = 0$ or $b = 0$ then $\Lambda_a + \Lambda_b = \Lambda_{a+b}$ and thus $\Lambda_a + \Lambda_b \subseteq \Lambda_{a+b}$.*

For this reason we will be mainly discussing the cases of (2.3) when $a, b > 0$.

As the last notion of this section, we define a relation $\leq$ on the set of the level cuts at positive levels of a pseudo-addition $*$ by

$$\Lambda_a \leq \Lambda_b \text{ iff } \frac{1}{a}\Lambda_a \subseteq \frac{1}{b}\Lambda_b.$$

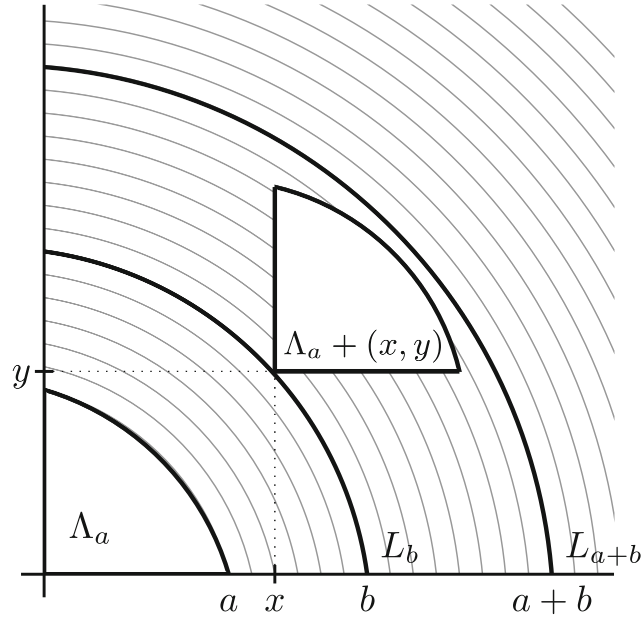

Figure 1. Geometric representation of the Mulholland inequality on the level set plot of the corresponding pseudo-addition. If we take a level cut Λ_a and shift it so that its bottom-left corner coincides with L_b then this shifted level cut must remain within the borders of the level cut Λ_{a+b}

This relation is reflexive and transitive, however, it is not anti-symmetric; thus it is a pre-order.

3. Geometric convexity and *k*-subscalability

Denote by $\mathbb{R}^{-\infty}$ the set of real numbers enriched by the least element $-\infty$ with $\log 0 = -\infty$ and $\exp(-\infty) = 0$.

Let $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be an increasing bijection. If $F = \log \circ f \circ \exp$ is convex then we say, following the terminology of Matkowski [6], that f is *geometrically convex*. We can observe that F is an increasing bijection on $\mathbb{R}^{-\infty}$. Thus it is convex if, and only if,

$$\forall A, B \in \mathbb{R}, X \in \mathbb{R}_0^+, A \leq B\colon \quad F(A) - F(A - X) \leq F(B) - F(B - X).$$

From this formula, using the substitutions $a = \exp A$, $b = \exp B$, and $x = \exp(-X)$, we obtain that f is geometrically convex if, and only if,

$$\forall a, b \in \mathbb{R}^+, x \in [0, 1], a \leq b\colon \quad \frac{f(ax)}{f(a)} \geq \frac{f(bx)}{f(b)} \tag{3.1}$$

where $\mathbb{R}^+$ denotes the set of positive real numbers and $[0,1]$ denotes the closed real unit interval.

Another characterization of geometric convexity is due to Sarkoci [10, Proposition 13]:

Proposition 3.1. *An increasing bijection* $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *is geometrically convex if, and only if, there exists a sequence* $(g_i\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+)_{i\in\mathbb{N}}$ *of power functions* $g_i\colon x \mapsto q_i\, x^{p_i}$, $p_i \geq 1$, $q_i > 0$, *such that* $f = \sup_{i\in\mathbb{N}} g_i$.

Remark that the condition of f being convex in Theorem 1.2 is necessary [8]. Thus, in order to answer negatively the first open question, we need to find a way how to weaken the geometric convexity of f while preserving the fact that $f \in MI$. The following two lemmas give an idea how to achieve this goal.

Lemma 3.2. *Let* $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *be a convex increasing bijection. If, for some fixed* $a, b \in \mathbb{R}^+$, *we have*

$$\forall x \in [0,1]: \quad \frac{f(ax)}{f(a)} \geq \frac{f(bx)}{f(b)}$$

then $\Lambda_a \leq \Lambda_b$.

Proof. Observe that $(x,y) \in \frac{1}{a}\Lambda_a$ resp. $(x,y) \in \frac{1}{b}\Lambda_b$ if, and only if,

$$\frac{f(ax)}{f(a)} + \frac{f(ay)}{f(a)} \leq 1 \quad \text{resp.} \quad \frac{f(bx)}{f(b)} + \frac{f(by)}{f(b)} \leq 1.$$

Therefore, according to the assumptions, $(x,y) \in \frac{1}{a}\Lambda_a$ implies $(x,y) \in \frac{1}{b}\Lambda_b$. □

Lemma 3.3. *Let* $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *be a convex increasing bijection. If, for some fixed* $a, b \in \mathbb{R}^+$, *we have* $\Lambda_a \leq \Lambda_{a+b}$ *and* $\Lambda_b \leq \Lambda_{a+b}$ *then* $\Lambda_a + \Lambda_b \subseteq \Lambda_{a+b}$.

Proof. From $\Lambda_a \leq \Lambda_{a+b}$ and $\Lambda_b \leq \Lambda_{a+b}$ we have

$$\Lambda_a \subseteq \frac{a}{a+b}\Lambda_{a+b}, \quad \text{and} \quad \Lambda_b \subseteq \frac{b}{a+b}\Lambda_{a+b}.$$

We finish the proof by summing these two inequalities [confer with (2.1), (2.2), and Lemma 2.1]. □

These results inspire us to introduce the following new notion which generalizes the notion of geometric convexity represented by (3.1).

Definition 3.4. *An increasing bijection* $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *is said to be* k-subscalable *for some given* $k \in \mathbb{R}_0^+$ *if*

$$\forall a, b \in \mathbb{R}^+, x \in [0,1], b - a \geq k: \quad \frac{f(ax)}{f(a)} \geq \frac{f(bx)}{f(b)}.$$

Observe that f is geometrically convex if, and only if, it is 0-subscalable.

Definition 3.5. *Let* $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *be a convex increasing bijection. The function* $f_\downarrow^{a,b}\colon [0,a] \to [0,f(a)]$, $a,b \in \mathbb{R}^+$, *is given, for* $x \in [0,a]$, *by*

$$f_\downarrow^{a,b}(x) = \frac{f(a)}{f(b)} f\left(\frac{b}{a}x\right).$$

We can see that $f_\downarrow^{a,b}$ is a convex increasing bijection. This new notion allows a reformulation of Definition 3.4:

Lemma 3.6. *A convex increasing bijection* $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *is* k*-subscalable for* $k \in \mathbb{R}_0^+$ *if, and only if,*

$$\forall a,b \in \mathbb{R}^+, y \in [0,a], b-a \geq k\colon \quad f_\downarrow^{a,b}(y) \leq f(y). \tag{3.2}$$

Proof. The inequality in Definition 3.4 can be rewritten as: $\frac{f(a)}{f(b)} f(bx) \leq f(ax)$. The substitution $y = ax$ finishes the proof. □

Remark 3.7. As we can observe, the graph of $f_\downarrow^{a,b}$ on $[0,a]$ is actually a scaled version of the graph of f on $[0,b]$ such that $f_\downarrow^{a,b}(0) = 0$ and $f_\downarrow^{a,b}(a) = f(a)$. Further, this scaled graph is expected to be under the graph of f whenever f is k-subscalable. This is why we call this property "subscalability".

Lemmas 2.2, 3.2, 3.3, and Definition 3.4 imply the following statement.

Corollary 3.8. *Let* $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *be a convex increasing bijection that is* k*-subscalable for some* $k \in \mathbb{R}_0^+$. *Then*

$$\forall a,b \in \mathbb{R}_0^+, a \geq k, b \geq k\colon \quad \Lambda_a + \Lambda_b \subseteq \Lambda_{a+b}.$$

The next section introduces another condition which assures that the inequality holds also for $a < k$ or $b < k$.

4. Increasing bijections linear on $[0, k]$

Definition 4.1. *We say that a function* $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *is* linear on an interval $I \subseteq \mathbb{R}_0^+$ *if there is* $r \in \mathbb{R}_0^+$ *such that*

$$\forall x \in I\colon \quad f(x) = rx.$$

We say that f *is* affine on I *if there are* $s,t \in \mathbb{R}$ *such that*

$$\forall x \in I\colon \quad f(x) = sx + t.$$

We are going to work with increasing bijections that are linear on $[0,k]$. Note that we consider also the extreme case $k = 0$ requiring no partial linearity of f.

By Δ_a we denote the following subset of $\mathbb{R}_0^+ \times \mathbb{R}_0^+$:

$$\Delta_a = \{(x,y) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ \mid x + y \leq a\}.$$

Clearly, if f is linear on $[0,k]$ then $\Lambda_a = \Delta_a$ for every $a \in [0,k]$.

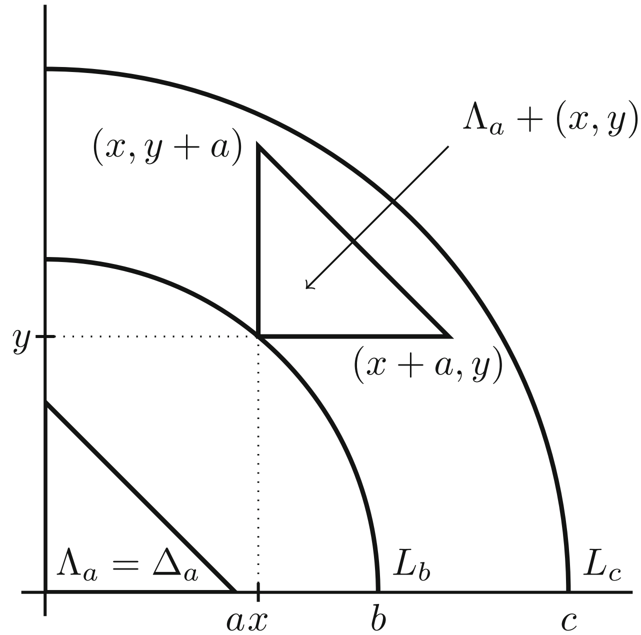

Figure 2. If $\Lambda_a = \Delta_a$ then $\Lambda_a + \Lambda_b \subseteq \Lambda_{a+b}$ always holds for pseudo-additions generated by convex increasing bijections

Lemma 4.2. *Let* $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *be a convex increasing bijection. Then for every* $a, b \in \mathbb{R}_0^+$ *and for every point* $(x, y) \in \Lambda_b$ *we have* $(x + a, y) \in \Lambda_{a+b}$ *and* $(x, y + a) \in \Lambda_{a+b}$.

Proof. We prove only $(x + a, y) \in \Lambda_{a+b}$, the proof of $(x, y + a) \in \Lambda_{a+b}$ is analogous. As $(x, y) \in \Lambda_b$, we have $f(x)+f(y) \leq f(b)$ and, thus, $x \leq b$. Because of this, and since f is convex, we have $f(x+a)-f(x) \leq f(b+a)-f(b)$. Summing the two inequalities we obtain $f(x + a) + f(y) \leq f(a + b)$ which concludes the proof. □

Lemma 4.3. *Let* $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *be a convex increasing bijection. Then for every* $a, b \in \mathbb{R}_0^+$ *we have* $\Delta_a + \Lambda_b \subseteq \Lambda_{a+b}$.

Proof. Observe that (by Lemma 2.1) Λ_{a+b} is a convex set and that Δ_a is a triangle as illustrated in Fig. 2. Take any $(x, y) \in \Lambda_b$. According to the assumptions and to Lemma 4.2, all the three vertices of $\Delta_a + (x, y)$ are contained in Λ_{a+b}. Therefore $\Delta_a + (x, y) \subseteq \Lambda_{a+b}$ which concludes the proof. □

The latter lemma constitutes the following statement.

Corollary 4.4. *Let* $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *be a convex increasing bijection that is linear on* $[0, k]$ *for some* $k \in \mathbb{R}_0^+$. *Then*

$$\forall a, b \in \mathbb{R}_0^+, a \leq k \text{ or } b \leq k\colon \quad \Lambda_a + \Lambda_b \subseteq \Lambda_{a+b}.$$

Finally, the following theorem gives a new class of functions that solve Mulholland's inequality. Its proof follows directly from Corollary 3.8 and Corollary 4.4.

Theorem 4.5. *Let $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a convex increasing bijection that, for some $k \in \mathbb{R}_0^+$, is k-subscalable and linear on $[0,k]$. Then f solves Mulholland's inequality, i.e., $f \in MI$.*

For a given $k \in \mathbb{R}_0^+$, we denote by LS_k the set of all convex increasing bijections on $\mathbb{R}_0^+$ that are k-subscalable and linear on $[0,k]$. Further, the set LS represents all the functions that accords with the assumptions of Theorem 4.5, i.e.,

$$LS = \bigcup_{k \in \mathbb{R}_0^+} LS_k.$$

It is worth mentioning that if we fix $k = 0$ then Theorem 4.5 reduces to Theorem 1.2. Thus we have $LS_0 = MC$ and $MC \subseteq LS \subseteq MI$. The next section shows that $MC \neq LS$ by giving an example of a bijection that belongs to LS but not to MC.

5. Mulholland's condition is not necessary to solve the Mulholland inequality

An increasing bijection which solves the Mulholland inequality but does not accord with Mulholland's condition is presented in the following example.

Example 5.1. The function $g\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is defined, for every $x \in \mathbb{R}_0^+$, by

$$g(x) = \begin{cases} \frac{5}{3}x & \text{if} \quad x \in [0,1[, \\ \frac{7}{3}x - \frac{2}{3} & \text{if} \quad x \in [1,2[\,, \\ x^2 & \text{if} \quad x \in [2,\infty[\,. \end{cases}$$

This bijection is convex, linear on $[0,1]$, and 1-subscalable. Therefore it belongs to LS_1 and thus also to MI. However, since its definition on $[1,2[$ does not accord with the form $x \mapsto q\,x^p$, $p \geq 1$, $q > 0$, we have, according to Proposition 3.1, $g \notin MC$.

The convexity of g and its linearity on $[0,1]$ can be easily seen. To show that g is also 1-subscalable requires a long and technical proof which is presented in Appendix A.

6. The set MI is not closed with respect to compositions

Before we start to present the result of this section, we would like to make some easy observations on the level set plot of a pseudo-addition generated by a bijection of the form $\varphi_p\colon x \mapsto x^p$, where $p > 1$. As we can observe, such a pseudo-addition actually accords with the p-norm

$$\|(x, y)\|_p = (x^p + y^p)^{\frac{1}{p}} \tag{6.1}$$

and, as a vector norm, it satisfies the *homogeneity*:

$$\forall x, y, \alpha \in \mathbb{R}_0^+ : \quad \|(\alpha x, \alpha y)\|_p = \alpha \|(x, y)\|_p .$$

As a consequence, all the level sets of the p-norm, as well as all its level cuts, are identical up to a scale; this can be expressed as

$$\forall a, b \in \mathbb{R}_0^+ : \quad \frac{1}{a} L_a = \frac{1}{b} L_b,$$

$$\forall a, b \in \mathbb{R}_0^+ : \quad \frac{1}{a} \Lambda_a = \frac{1}{b} \Lambda_b.$$

Another consequence is the following. Focus on levels $a, b, c \in \mathbb{R}_0^+$ such that $c - b = a$ and the respective level sets L_a, L_b, L_c. If we draw an arbitrary line passing through the point $(0, 0)$ and we denote the intersections with L_a, L_b, L_c respectively by A, B, C we can observe that the distance between B and C is exactly the same as the distance between $(0, 0)$ and A. Furthermore, as all the level cuts are convex sets, if we shift the level cut Λ_a so that its bottom-left corner coincides with some point of L_b then it will “fit exactly” between L_b and L_c, i.e., it will have a common point with L_c but it will not overlap L_c; all the points of the shifted Λ_a will be contained in the area delimited by the borders L_b and L_c. (The situation is illustrated in Fig. 3.) This also means that if $c - b < a$ then Λ_a will not “fit” between L_b and L_c no matter to which point of L_b it will be shifted; it will always overlap with L_c.

Notation 6.1. *The notation $g \circ h$ denotes the composition of the two functions $g, h\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ so that $(g \circ h)(x) = g(h(x))$ holds for every $x \in \mathbb{R}_0^+$.*

Proposition 6.2. *Let $m, n \in \mathbb{R}^+$, $m < n$, and let $g\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be an increasing bijection linear on $[0, m]$ and affine on $[m, n]$ such that $2g(m) < g(n)$. Let $p > 1$ and let $h\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be given by $h\colon x \mapsto x^p$. Then $f = g \circ h \notin MI$.*

Proof. As illustrated in Figs. 4 and 5, the level sets of $*_f$ are obtained from the level sets of $*_g$ by performing the mapping

$$(x, y) \mapsto \left(x^{\frac{1}{p}}, y^{\frac{1}{p}}\right) \tag{6.2}$$

on the set $\mathbb{R}_0^+ \times \mathbb{R}_0^+$. That is to say that

$$\left(x^{\frac{1}{p}}, y^{\frac{1}{p}}\right) \in \Lambda^f_{a^{1/p}} \text{ iff } (x, y) \in \Lambda^g_a \tag{6.3}$$

and we consider $\Lambda^f_{a^{1/p}}$ as the image of Λ^g_a. Thus, every part of a level set of $*_g$ that is identical to a part of a level set of the addition (i.e. a line segment that

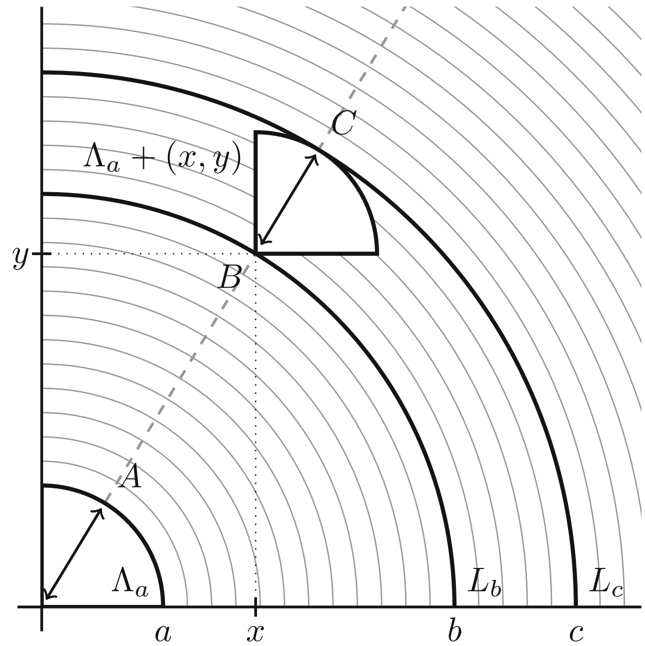

Figure 3. Level set plot of a p-norm. We can observe that, for $a = c - b$, the level cut Λ_a will always "fit exactly" between the level sets L_b and L_c

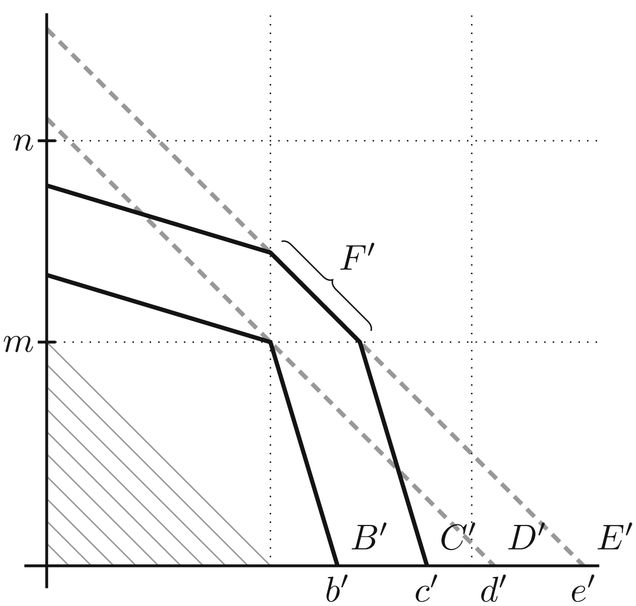

Figure 4. Level set plot of the pseudo-addition $*_g$ highlighting the level sets important for the proof of Proposition 6.2

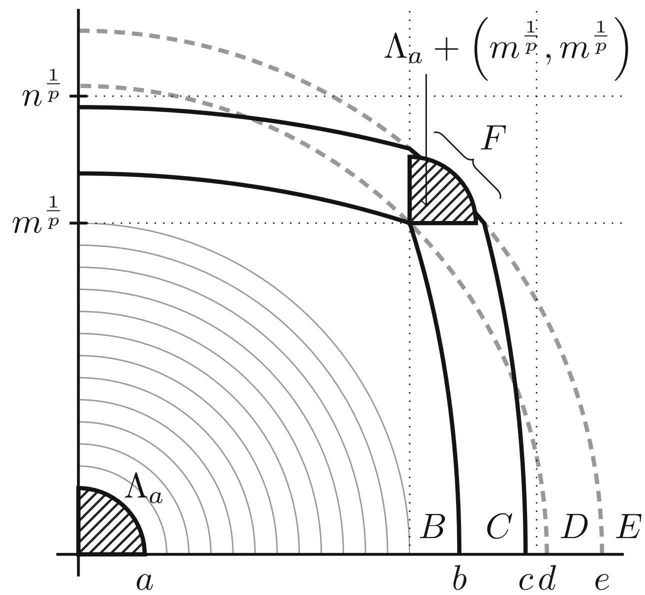

Figure 5. Level set plot of the pseudo-addition $*_f$. Its level sets are obtained from the level sets of $*_g$ by performing the mapping $(x, y) \mapsto \left(x^{\frac{1}{p}}, y^{\frac{1}{p}}\right)$

forms a 135° angle with the horizontal line) is mapped to a part of a level set of a p-norm (here represented by $*_h$).

To obtain the proof, we take the level set of $*_g$ that passes through the point (m, m) and we denote it by B'; its level we denote by b'. Note that thanks to $2g(m) < g(n)$ we have $b' < n$. Further we take any level set at the level $c' \in \,]b', n[$ and we denote it by C'. We can observe that C' consists of exactly three line segments as depicted in Fig. 4; two of them are parallel with the line segments of B' and the middle one is identical to a part of a level set of the addition.

Taking (6.2) and (6.3) into account, the levels b', c' are respectively mapped to b, c and the level sets B', C' are respectively mapped to B, C as it is illustrated in Fig. 5. Note that since C' is partially identical to a level set of the addition, C is partially identical to a level set of the p-norm; we denote this part of C' resp. C by F' resp. F. Further, we denote the level cut of $*_f$ at the level $a = c - b$ by Λ_a.

Now we turn back to the level set plot of $*_g$ and we draw two auxiliary lines, D' and E', that form a 135° angle with the horizontal line and that respectively pass through the point (m, m) and through F'; see Fig. 4. Their intersections with the axis we denote by d' resp. e'. They are mapped by (6.2) to the curves D resp. E which are identical to the level sets of $*_h$ (of the p-norm) at the levels d resp. e; see Fig. 5.

We point out that

$$b = (b')^{\frac{1}{p}}, \quad c = (c')^{\frac{1}{p}}, \quad d = (d')^{\frac{1}{p}}, \quad e = (e')^{\frac{1}{p}}.$$

Since $p > 1$, $x \mapsto x^{\frac{1}{p}}$ is a concave function. Therefore from $c' - b' = e' - d'$ and $c' < e'$ we have $c - b > e - d$ which implies $a > e - d$.

Thus, according to the observations made just before Proposition 6.2, the level cut at a of $*_f$, which is identical to the level cut at a of $*_h$, cannot "fit" between D and E. As B is at the point $\left(m^{1/p}, m^{1/p}\right)$ identical to D and C is in F identical to E, we can see that Λ_a shifted to the point $\left(m^{1/p}, m^{1/p}\right)$ will overlap the level set C which means a violation of Mulholland's inequality. □

Theorem 6.3. *The set of solutions to Mulholland's inequality is not closed with respect to their compositions.*

Proof. Taking the function g from Example 5.1 and the function h from Proposition 6.2 we can see that $g, h \in MI$ but $g \circ h \notin MI$. □

7. Concluding remarks

A new, larger, set of functions solving Mulholland's inequality has been introduced. However, most probably it does not cover all the possible solutions. The structure of this set remains unexplored, as well.

Further, it has been shown that the set of solutions to Mulholland's inequality is not closed with respect to compositions. This result also gives a negative answer to the question whether the dominance relation on the set of strict triangular norms is transitive and thus an order [1,11]. We remark that the same question has been recently answered recently by Sarkoci [9] for the set of non-strict continuous triangular norms; also negatively.

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Appendix A. Proof of 1-subscalability of g

Lemma A.1. *The function g from Example 5.1 is 1-subscalable.*

Proof. We need to show that

$$\forall a, b \in \mathbb{R}^+, x \in [0, a], b - a \geq 1: \quad g_{\downarrow}^{a,b}(x) \leq g(x). \tag{A.1}$$

Since $b - a \geq 1$, we can omit the cases when $a, b \in \,]0, 1[$ and when $a, b \in [1, 2[$.

Recall that, according to Remark 3.7, $g_{\downarrow}^{a,b}$ is a convex function and we have

$$g_{\downarrow}^{a,b}(0) = g(0), \tag{A.2}$$

$$g_{\downarrow}^{a,b}(a) = g(a). \tag{A.3}$$

It is easy to prove the case when $a \in \,]0, 1[$; in this case g is linear on $[0, a]$, $g_{\downarrow}^{a,b}$ is convex on $[0, a]$, and these two functions are identical on the border points of $[0, a]$ thanks to (A.2) and (A.3). Also, the case when $a = 0$ is trivial.

Thus we only need to deal with the case when $a, b \in [2, \infty[$ and the case when $a \in [1, 2[$, $b \in [2, \infty[$.

(i) If $a, b \in [2, \infty[$ then $g_{\downarrow}^{a,b}(x) = g(x) = x^2$ for $x \in [2, a]$ and thus

$$g_{\downarrow}^{a,b}(2) = g(2). \tag{A.4}$$

Moreover, g is linear on $[0, 1]$ and affine on $[1, 2]$ while $g_{\downarrow}^{a,b}$ is convex on $[0, 2]$ with (A.2) and (A.4). Therefore, if $g_{\downarrow}^{a,b}(x) > g(x)$ for some $x \in \,]1, 2[$ then, necessarily, $g_{\downarrow}^{a,b}(1) > g(1)$. Thus, in order to prove (A.1), it is enough to show that

$$g_{\downarrow}^{a,b}(1) \leq g(1). \tag{A.5}$$

The value of $g(1)$ is $\frac{5}{3}$; to evaluate $g_{\downarrow}^{a,b}(1)$ we distinguish between two cases:

(a) If $2\frac{a}{b} \leq 1$ then

$$g_{\downarrow}^{a,b}(1) = \frac{a^2}{b^2}\left(\frac{b}{a} \cdot 1\right)^2 = 1.$$

(b) If $2\frac{a}{b} > 1$ then

$$g_{\downarrow}^{a,b}(1) = \frac{a^2}{b^2}\left(\frac{7}{3}\frac{b}{a} \cdot 1 - \frac{2}{3}\right).$$

We rewrite (A.5) to

$$\frac{a^2}{b^2}\left(\frac{7}{3}\frac{b}{a} - \frac{2}{3}\right) \leq \frac{5}{3}.$$

Let us introduce a new variable $c = \frac{a}{b}$. Since $a, b \in [2, \infty[$ and $a + 1 \leq b < 2a$, we have $c \in \left]\frac{1}{2}, 1\right[$. We rewrite the latter inequality as:

$$\begin{aligned} \frac{7}{3}c - \frac{2}{3}c^2 &\leq \frac{5}{3}, \\ 0 &\leq 2c^2 - 7c + 5, \\ 0 &\leq 2\,(c-1)\left(c - \frac{5}{2}\right). \end{aligned}$$

It can be checked easily that this is true for $c \in \left]\frac{1}{2}, 1\right[$.

(ii) If $a \in [1, 2[$ and $b \in [2, \infty[$ then, similarly, g is linear on $[0, 1]$ and affine on $[1, a]$ while $g_{\downarrow}^{a,b}$ is convex on $[0, a]$ with (A.2) and (A.3). Thus again it is enough to show (A.5).

(a) If $2\frac{a}{b} \leq 1$ then

$$g_{\downarrow}^{a,b}(1) = \frac{\frac{7}{3}a - \frac{2}{3}}{b^2}\left(\frac{b}{a} \cdot 1\right)^2 = \frac{\frac{7}{3}a - \frac{2}{3}}{a^2} = \frac{7a-2}{3a^2}.$$

We rewrite (A.5) to

$$\begin{aligned} \frac{7a-2}{3a^2} &\leq \frac{5}{3}, \\ 7a - 2 &\leq 5a^2, \\ 0 &\leq 5a^2 - 7a + 2, \\ 0 &\leq 5\,(a-1)\left(a - \frac{2}{5}\right). \end{aligned}$$

We can see that this holds true for $a \in [1, 2[$.

(b) If $2\frac{a}{b} > 1$ then

$$g_{\downarrow}^{a,b}(1) = \frac{\frac{7}{3}a - \frac{2}{3}}{b^2}\left(\frac{7}{3}\frac{b}{a} \cdot 1 - \frac{2}{3}\right) = \frac{(7a-2)(7b-2a)}{9ab^2}.$$

We rewrite (A.5) to

$$\begin{aligned} \frac{(7a-2)(7b-2a)}{9ab^2} &\leq \frac{5}{3}, \\ 49ab - 14a^2 - 14b + 4a &\leq 15ab^2, \\ 15ab^2 - 49ab + 14a^2 + 14b - 4a &\geq 0. \end{aligned}$$

The left part of the inequality is a two-variable polynomial

$$P(a, b) = 15ab^2 - 49ab + 14a^2 + 14b - 4a$$

defined on the domain

$$D = \{(a, b) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid 1 \leq a < 2, a + 1 \leq b < 2a\}$$

having a shape of a triangle with vertices $(1,2)$, $(2,3)$, and $(2,4)$. We are going to show that $P(a,b) \geq 0$ for every $(a,b) \in D$. First we compute

$$\begin{aligned} P(1,2) &= 0, \\ P(2,3) &= 66, \\ P(2,4) &= 192. \end{aligned}$$

Further, we compute the first partial derivatives

$$\begin{aligned} \frac{\partial P}{\partial a}(a,b) &= 15b^2 - 49b + 28a - 4, \\ \frac{\partial P}{\partial b}(a,b) &= 30ab - 49a + 14. \end{aligned}$$

We can see that these two polynomials are never zero for any $(a,b) \in D$. Therefore P has no local extremes in the interior of D.
Restricting P to the borders of its domain, we obtain the functions

$$\begin{aligned} P_1(a) &= P(a, a+1) &&= 15a^3 - 5a^2 - 24a + 14, \\ P_2(a) &= P(a, 2a) &&= 60a^3 - 84a^2 + 24a, \\ P_3(b) &= P(2, b) &&= 30b^2 - 84b + 48 \end{aligned}$$

defined on $]1,2]$, $]1,2]$, and $[3,4[$, respectively. Their first derivatives are

$$\begin{aligned} g_1'(a) &= 45a^2 - 10a - 24, \\ g_2'(a) &= 180a^2 - 168a + 24, \\ g_3'(b) &= 60b - 84 \end{aligned}$$

and we can observe that they are all positive on their domains.
Thus P is positive in D.

□

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Dominance on strict triangular norms and Mulholland inequality

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Abstract

This paper is devoted to the question of transitivity of the dominance relation defined on the set of strict triangular norms and to the connection between dominance and Mulholland inequality. A summary of the results reached in this field is given and a new result for the strict triangular norms of Family 19 is shown. Further, some parametric classes of solutions of Mulholland inequality, that do not comply with the Mulholland's condition, are presented and, finally, it is shown that the dominance relation is, in general, not transitive on the set of strict triangular norms.

Keywords: Dominance relation; Mulholland inequality; Strict triangular norm; Transitivity

1. Introduction

As a recent result [14] it has been shown that the set of the solutions of Mulholland inequality is strictly larger than the set delimited by the Mulholland's condition. This fact has been proven by introducing a new sufficient condition which is stronger than the Mulholland's one and also by providing an example of a function that does not comply with the Mulholland's condition but does satisfy the new one. Furthermore, it has been shown that the result of a composition of two functions that solve Mulholland inequality need not be a solution, again.

All these results have direct impact on an area of mathematics that studies the dominance relation on the set of strict triangular norms.

The paper starts by introducing the basic notions regarding the triangular norms and the dominance relation. It continues by summarizing the results that have been achieved in this area, so far. Further, Mulholland inequality is presented and its connection with the dominance relation is shown. The paper then summarizes the families of strict triangular norms for which the question of the transitivity of the dominance relation has not been answered, yet, and provides an answer for Family 19. Finally, the new sufficient condition for Mulholland inequality, together with

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Table 1
Prototypical examples of triangular norms.

Product:	$x *_P y = x \cdot y$
Łukasiewicz t-norm:	$x *_L y = \max(x + y - 1, 0)$
Minimum:	$x *_M y = \min(x, y)$
Drastic t-norm:	$x *_D y = \begin{cases} 0 & \text{if } x, y < 1 \\ min(x, y) & \text{otherwise} \end{cases}$

parametric families of solutions, is presented and it is demonstrated that the dominance is generally not transitive on the set of strict triangular norms.

2. Strict triangular norms

Definition 2.1. A *triangular norm* [2,8] (or, shortly, a *t-norm*) is a binary operation $*\colon [0, 1] \times [0, 1] \to [0, 1]$ such that, for every $x, y, z \in [0, 1]$,

1. $x * y = y * x$ *(commutativity)*,
2. $(x * y) * z = x * (y * z)$ *(associativity)*,
3. $x * 1 = x$ *(unit element)*,
4. if $y \le z$ then $x * y \le x * z$ *(non-decreasingness)*.

Originally, the notion of a t-norm has been introduced within the framework of probabilistic metric spaces [12,23] where it helps to describe the triangular inequality of the probabilistic metrics. However, nowadays t-norms also play the role of the logical conjunction in the semantics of the *basic logic* [6,7] and the *monoidal t-norm based logic* [4] which are both prototypical logics of graded truth (fuzzy logics). Furthermore, many t-norms are actually copulas [25], [8, Definition 9.4] and thus can express dependence of random variables [23].

The most prominent examples of t-norms are listed in Table 1.

Definition 2.2. A t-norm is said to be *continuous* if it is continuous as a two-variable real function. A t-norm is said to be *strict* if it is continuous and if its restriction to $]0, 1] \times]0, 1]$ is strictly increasing in each variable.

Remark 2.3. By $[0, \infty]$ we denote the chain of non-negative real numbers with the least element 0, the greatest element ∞, and we assume $x + \infty = \infty$ for any $x \in [0, \infty]$.

It can be observed that the structure $([0, 1], *, 1, \le)$, where $*$ is a t-norm, is an integral, commutative, totally ordered monoid [5]. The following proposition [8, Section 3.2] shows that if $*$ is a strict t-norm then $([0, 1], *, 1, \le)$ is isomorphic to the additive monoid of $[0, \infty]$.

Proposition 2.4. *A t-norm $*$ is strict if, and only if, there is a decreasing bijection $t\colon [0, 1] \to [0, \infty]$ such that, for every $x, y \in [0, 1]$,*

$$x * y = t^{-1}\big(t(x) + t(y)\big).$$

The function t is called the *additive generator* of the strict t-norm $*$ [8, Definition 3.25]. Remark that t is, for a given strict t-norm, unique up to a multiplicative constant. Observe that $*_P$ is a strict t-norm and that its additive generator is the function $x \mapsto -\log x$.

3. Dominance relation

What follows, is a definition of the dominance relation on the set of t-norms [26, Definition 3.4]. Remark that this binary relation can be defined in a more general setting [16].

Definition 3.1. A t-norm $*_1$ is said to *dominate* a t-norm $*_2$ (and we denote it by $*_1 \gg *_2$) if, for every $x, y, u, v \in [0, 1]$,

$$(x *_2 y) *_1 (u *_2 v) \geq (x *_1 u) *_2 (y *_1 v). \tag{1}$$

The motivation to study the dominance of t-norms comes from Tardiff who has shown that this notion plays a crucial role when constructing Cartesian products of probabilistic metric spaces [26, Theorem 3.5]. Later, this relation has shown to be significant when working with t-norm based fuzzy equivalences and partitions [3, Theorem 2] and with their refinements [3, Theorem 5]. Let us summarize some known properties of this binary relation.

Lemma 3.2. *For every two t-norms, $*_1$ and $*_2$,*

$$\text{if} \quad *_1 \gg *_2 \quad \text{then} \quad *_1 \geq *_2.$$

Proof. The proof is done by setting $y = u = 1$ in (1). $\square$

It follows from Lemma 3.2 that the dominance is an anti-symmetric relation. Further, it can be easily observed that $*_M \gg * \gg *_D$ for every t-norm $*$. Finally, for any t-norm $*$ we have $* \gg *$ which follows from the associativity and commutativity of $*$. Therefore, the dominance is a reflexive relation.

Since the dominance relation on the set of t-norms is anti-symmetric and reflexive, it is natural to ask, whether it is also transitive and thus an order with a least and a greatest element. This question has been stated as an open problem [1, Problem 17], [23, Problem 12.11.3]:

Problem 3.3. Is the dominance relation transitive, and hence a partial order, on the set of all t-norms? If not, for what subsets is this the case?

In the rest of this section we will be referring to the following result [8, Theorem 3.43]:

Theorem 3.4. *Let I be an at most countable index set, $(*_i)_{i\in I}$ a sequence of t-norms, and $(]a_i, b_i[)_{i\in I}$ a sequence of mutually disjoint subintervals of $[0, 1]$. Then*

$$*\colon\ [0,1] \to [0,1] :\ (x, y) \mapsto \begin{cases} a_i + (b_i - a_i)\left(\frac{x-a_i}{b_i-a_i} *_i \frac{y-a_i}{b_i-a_i}\right) & \text{if } x, y \in\,]a_i, b_i[\,, \\ \min\{x, y\} & \text{otherwise}, \end{cases}$$

is a t-norm.

We call the t-norm $*$ from Theorem 3.4 the *ordinal sum* of the t-norms $(*_i)_{i\in I}$; the t-norms $(*_i)_{i\in I}$ are called *summands* and the intervals $[a_i, b_i]$ are called *summand carriers*. Remark that the ordinal sum construction gives a characterization of the continuous t-norms since a t-norm is continuous if, and only if, it is an ordinal sum of strict and nilpotent [8, Definition 2.13] t-norms. For such t-norms the question of the transitivity of the dominance relation has been answered negatively in 2008 by Sarkoci [18,21]. To see his counter-example, take the following three t-norms: the t-norm $*_1$ is an ordinal sum where the only summand is the Łukasiewicz t-norm on the interval $\left[0, \frac{1}{2}\right]$, the t-norm $*_2$ is an ordinal sum with two summands, both the Łukasiewicz t-norm, on the intervals $\left[0, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, 1\right]$, and the t-norm $*_3$ is the Łukasiewicz t-norm. Then we have $*_1 \gg *_2$ and $*_2 \gg *_3$ but $*_1 \not\gg *_3$.

Further, Sarkoci has given a characterization of the dominance on the class of ordinal sum t-norms that use the Łukasiewicz t-norm as the only summand operation and the class of ordinal sum t-norms that use the product t-norm as the only summand operation [22].

Thus, it has been shown that the dominance relation is in general not transitive on the set of continuous t-norms. However, the question whether this relation is transitive on the set of strict (or nilpotent) t-norms has remained open.

Table 2
Summary of families of strict triangular norms for which the question of transitivity of the dominance relation has been already answered. We use the description by their additive generators and we use the numbering proposed in the book by Alsina, Frank, and Schweizer [2, Table 2.6] and the names used in the book by Klement, Mesiar, and Pap [8, Chapter 4].

	Name	$t_\lambda(x) =$	$\lambda \in$	$*_\alpha \gg *_\beta \Leftrightarrow$
1.	Schweizer–Sklar	$\frac{x^{-\lambda}-1}{\lambda}$	$]0, \infty[$	$\alpha \geq \beta$
		$-\log x$	$\{0\}$	
3.	Hamacher	$\frac{1-x}{x}$	$\{0\}$	$\alpha = 0, \alpha = \beta$,
		$\log \frac{\lambda+(1-\lambda)x}{x}$	$]0, \infty[$	or $\beta = \infty$
4.	Aczél-Alsina	$(-\log x)^\lambda$	$]0, \infty[$	$\alpha \geq \beta$
5.	Frank	$-\log x$	$\{1\}$	$\alpha = 0, \alpha = \beta$,
		$-\log \frac{\lambda^x-1}{\lambda-1}$	$]0, 1[\cup]1, \infty[$	or $\beta = \infty$
9.		$\log(1 - \lambda \log x)$	$]0, \infty[$	$\alpha = \infty, \alpha = \beta$
		$-\log x$	$\{\infty\}$	or $\beta = 0$
12.	Dombi	$\left(\frac{1-x}{x}\right)^\lambda$	$]0, \infty[$	$\alpha \geq \beta$
15.		$\log(1 - \log x)$	$\{0\}$	$\alpha \geq \beta$
		$(1 - \log x)^\lambda - 1$	$]0, \infty[$	
22.		$\frac{1-x}{x}$	$\{0\}$	$\alpha \geq \beta$
		$e^{\frac{\lambda}{x}} - e^\lambda$	$]0, \infty[$	
23.		$-\log x$	$\{0\}$	$\alpha \geq \beta$
		$e^{x^{-\lambda}} - e$	$]0, \infty[$	

4. Families of strict t-norms: solved cases

In the last decades, it has been demonstrated by several authors that the dominance relation is transitive on single-parametric families of strict t-norms known from the literature [2,8]. The families, as well as the results on transitivity of the dominance, are summarized in Table 2.

Remark that Schweizer–Sklar t-norms are defined also for $\lambda \in]-\infty, 0[$, but is such a case we obtain nilpotent t-norms, and that their limit cases are $*_D$ for $\lambda = -\infty$ and $*_M$ for $\lambda = \infty$. Further, the limit case for Hamacher t-norms is $*_D$ for $\lambda = \infty$, for Aczél-Alsina and Dombi t-norms it is $*_D$ for $\lambda = 0$ and $*_M$ for $\lambda = \infty$, and for Frank t-norms it is $*_M$ for $\lambda = 0$ and $*_L$ for $\lambda = \infty$.

Note that if λ is the value of the parameter, we denote the t-norm generated by the corresponding additive generator by:

$$*_\lambda : \quad (x, y) \mapsto \quad t_\lambda^{-1}(t_\lambda(x) + t_\lambda(y)).$$

Transitivity of the dominance can be proven easily in the case of the families where the additive generators do not differ up to a positive power [8, Example 6.17.v]. In such a case $\alpha \geq \beta$ implies $*_\alpha \gg *_\beta$; this is the case of Families 4 and 12.

In 1984, Sherwood has given the historically first result proving that the dominance on Family 1 is transitive [24].

Results for Families 3 and 5 have been given in 2005 by Sarkoci [19].

In 2009, Saminger-Platz [15] has proven transitivity of the dominance relation on Families 9, 15, 22, and 23.

5. Mulholland inequality

Mulholland inequality has been introduced in 1947 by Mulholland [13] as a generalization of Minkowski inequality which establishes the triangular inequality of p-norms (or L^p-*norms*) and has the form

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}$$

where $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ and $p \ge 1$. In his work, Mulholland has replaced the power function by an arbitrary increasing bijection of $[0, \infty[$ and asked, under which conditions the inequality will be preserved:

Definition 5.1. An increasing bijection $f\colon\ [0,\infty[\to [0,\infty[$ is said to *solve Mulholland inequality* if

$$f^{-1}\left(\sum_{i=1}^{n} f(x_i + y_i)\right) \le f^{-1}\left(\sum_{i=1}^{n} f(x_i)\right) + f^{-1}\left(\sum_{i=1}^{n} f(y_i)\right)$$

for every $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in [0, \infty[^n$.

Mulholland has also provided the following sufficient condition [13, Theorem 1] (for the proof, see also Kuczma [9, Theorem VIII.8.1]):

Theorem 5.2. *Let $f\colon\ [0,\infty[\to [0,\infty[$ be an increasing bijection. If both f and $\log \circ f \circ \exp$ are convex then f solves Mulholland inequality.*

Note that if $\log \circ f \circ \exp$ is convex then, following the terminology of Matkowski [10], we say that f is *geometrically convex*. A characterization of increasing bijections of $[0, \infty[$ that are both convex and geometrically convex has been provided by Sarkoci [20, Proposition 13]:

Proposition 5.3. *An increasing bijection $f\colon\ [0,\infty[\to [0,\infty[$ is both convex and geometrically convex if, and only if, there exists a sequence*

$$\left(g_i\colon\quad [0,\infty[\to [0,\infty[:\quad x \mapsto q_i\, x^{p_i}\right)_{i\in\mathbb{N}}$$

of power functions with $p_i \ge 1$ and $q_i \ge 0$, such that

$$f \;=\; \sup_{i\in\mathbb{N}} g_i\,.$$

In 1993, Matkowski and Świątkowski have provided a necessary condition on functions that solve Mulholland inequality [11].

Lemma 5.4. *If an increasing bijection $f\colon\ [0,\infty[\to [0,\infty[$ solves Mulholland inequality then it is necessarily convex.*

We end this section with two easy observations.

Lemma 5.5. *Let $f\colon\ [0,\infty[\to [0,\infty[: x \mapsto x^p$ with $p > 0$. Then f solves Mulholland inequality if, and only if, $p \ge 1$.*

Proof. If $p < 1$ then the proof is done by Lemma 5.4. If $p \ge 1$ then the proof is done by Proposition 5.3. □

Observation 5.6. If an increasing bijection $f\colon\ [0,\infty[\to [0,\infty[$ solves Mulholland inequality then $a \cdot f$, where $a \in\,]0,\infty[$, does as well.

6. Correspondence with the dominance relation

In 1984, Tardiff has shown that Mulholland inequality is in a close correspondence with dominance of strict t-norms [27, Theorem 3].

Notation 6.1. The notation $f \circ g$ denotes the composition of two functions, f and g, such that it holds $(f \circ g)(x) = f(g(x))$ for every x in the domain of g.

Theorem 6.2. *Let $*_1$ and $*_2$ be two strict t-norms defined by their additive generators t_1 and t_2, for every $x, y \in [0, 1]$, as*

$$x *_1 y = t_1^{-1}\big(t_1(x) + t_1(y)\big),$$
$$x *_2 y = t_2^{-1}\big(t_2(x) + t_2(y)\big).$$

Then $*_1$ dominates $*_2$ if, and only if, $f = t_1 \circ t_2^{-1}$ restricted to $[0, \infty[$ solves Mulholland inequality.

To demonstrate the proof, observe that, since $t_1, t_2\colon\ [0, 1] \to [0, \infty]$ are both decreasing bijections, $f = t_1 \circ t_2^{-1}$ is an increasing bijection of $[0, \infty]$. In the sequel, the variables x, y, u, v take any values from $[0, 1]$, and we introduce $X = t_2(x)$, $Y = t_2(y)$, $U = t_2(u)$, $V = t_2(v)$; thus $X, Y, U, V \in [0, \infty]$. The following formulas are equivalent:

$$*_1 \gg *_2,$$
$$(x *_2 y) *_1 (u *_2 v) \ \geq\ (x *_1 u) *_2 (y *_1 v),$$

$$t_1^{-1}\Big(t_1\Big(t_2^{-1}\big(t_2(x) + t_2(y)\big)\Big) + t_1\Big(t_2^{-1}\big(t_2(u) + t_2(v)\big)\Big)\Big) \quad \geq \quad t_2^{-1}\Big(t_2\Big(t_1^{-1}\big(t_1(x) + t_1(u)\big)\Big) + t_2\Big(t_1^{-1}\big(t_1(y) + t_1(v)\big)\Big)\Big),$$

$$t_2\Big(t_1^{-1}\Big(t_1\big(t_2^{-1}(X + Y)\big) + t_1\big(t_2^{-1}(U + V)\big)\Big)\Big) \quad \geq \quad t_2\Big(t_1^{-1}\Big(t_1\big(t_2^{-1}(X)\big) + t_1\big(t_2^{-1}(U)\big)\Big)\Big) + t_2\Big(t_1^{-1}\Big(t_1\big(t_2^{-1}(Y)\big) + t_1\big(t_2^{-1}(V)\big)\Big)\Big),$$

$$f^{-1}\Big(f\big(X + Y\big) + f\big(U + V\big)\Big) \leq f^{-1}\big(f(X) + f(U)\big) + f^{-1}\big(f(Y) + f(V)\big). \tag{2}$$

Observe that (2) is, actually, Mulholland inequality from Definition 5.1 for the case when $n = 2$. As this paper deals mainly with the dominance of strict t-norms, in the sequel we will refer to Mulholland inequality in the form (2), solely.

Remark that in 2008, Saminger-Platz, De Baets, and De Meyer have enlarged this correspondence to the set of all continuous Archimedean t-norms [8, Definition 2.9.iv] introducing the notion of *generalized Mulholland inequality* [17].

7. Families of strict t-norms: unsolved cases and result for Family 19

The families of strict t-norms for which the question of transitivity of the dominance has not been yet answered are summarized in Table 3. Observe that, using the same reasoning as in the case of Families 4 and 12 in Section 4, or utilizing Mulholland inequality, the result for Family 19 can be achieved easily.

Proposition 7.1. *A strict t-norm of Family 19 is defined by*

$$*_\lambda\colon\quad (x, y) \mapsto \quad 1 - \mathrm{e}^{-\left((-\log(1-x))^{-\lambda} + (-\log(1-y))^{-\lambda}\right)^{-\frac{1}{\lambda}}},$$

Table 3
Summary of families of strict triangular norms for which the question of transitivity of the dominance relation has not been answered, yet.

	$t_\lambda(x) =$	$\lambda \in$
6.	$-\log\left(1-(1-x)^\lambda\right)$	$]0,\infty[$
10.	$\log(2x^{-\lambda}-1)$	$]0,\infty[$
16.	$(x^{-\frac{1}{\lambda}}-1)^\lambda$	$]0,\infty[$
18.	$(\frac{\lambda}{x}+1)(1-x)$	$[0,\infty[$
19.	$\left(-\log(1-x)\right)^{-\alpha}$	$]0,\infty[$
20.	$\log\frac{(1+x)^{-\lambda}-1}{2^{-\lambda}-1}$	$\mathbb{R}\setminus\{0\}$

where $\lambda \in]0,\infty[$ is a parameter. For two such t-norms, $*_\alpha$ and $*_\beta$, $\alpha,\beta \in]0,\infty[$, we have

$$*_\alpha \gg *_\beta \quad \text{if, and only if,} \quad \alpha \geq \beta.$$

Proof. The first implication is proven by Lemma 3.2. To prove the reverse one, let us take two t-norms, $*_\alpha$ and $*_\beta$, of Family 19, and their respective additive generators t_α and t_β with $\alpha,\beta \in]0,\infty[$. According to Theorem 6.2, $*_\alpha$ dominates $*_\beta$ if, and only if, $t_\alpha \circ t_\beta^{-1}$ solves Mulholland inequality. From Table 3 we have

$$(t_\alpha \circ t_\beta^{-1})(x) \;=\; \left(-\log\left(1-\left(1-\mathrm{e}^{-x^{\frac{1}{\beta}}}\right)\right)\right)^{-\alpha} \;=\; x^{\frac{\alpha}{\beta}}.$$

Thus $t_\alpha \circ t_\beta^{-1}$ is a power function with power $\frac{\alpha}{\beta}$. According to Lemma 5.5, it solves the Mulholland inequality if, and only if,

$$\frac{\alpha}{\beta} \geq 1,$$
$$\alpha \geq \beta. \qquad \square$$

8. New sufficient condition for Mulholland inequality

As a recent result [14] it has been shown that the set of the solutions of Mulholland inequality is strictly larger than the set delimited by the Mulholland's condition. It has been done by introducing a new sufficient condition which we present here, briefly.

Definition 8.1. Let $f\colon [0,\infty[\to [0,\infty[$ be an increasing bijection. For $a,b>0$, such that $a \leq b$, the function $f_\downarrow^{a,b}$ is given by

$$f_\downarrow^{a,b}\colon \quad [0,a] \to \left[0,f(a)\right]\colon \quad x \mapsto \frac{f(a)}{f(b)} f\left(\frac{b}{a}x\right).$$

Definition 8.2. For a given $k \geq 0$, an increasing bijection $f\colon [0,\infty[\to [0,\infty[$ is said to be k-*subscalable* if

$$f_\downarrow^{a,b}(x) \;\leq\; f(x)$$

for every $a,b>0$ and $x \in [0,a]$ such that

$$b-a \;\geq\; k.$$

Remark 8.3. Observe that the graph of $f_\downarrow^{a,b}$ is equal to the graph of f that has been scaled by the mapping

$$S^{a,b}\colon \quad [0,\infty[^2 \to [0,\infty[^2\colon \quad (x,y) \mapsto \left(\frac{a}{b}x, \frac{A}{B}y\right)$$

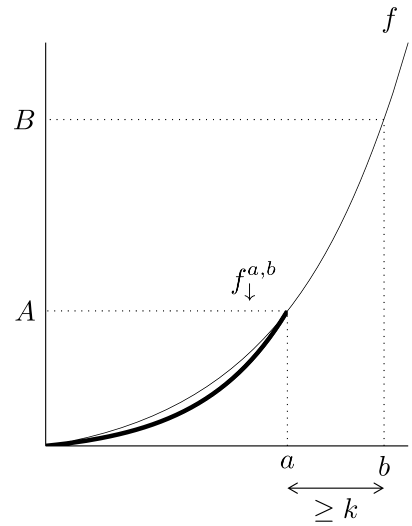


Fig. 1. Illustration of a k-subscalable convex increasing bijection. Its scaled graph is supposed to be situated "under" the original graph.

where $A = f(a)$ and $B = f(b)$; see an illustration in Fig. 1. (It is the linear scaling that maps the rectangle $[0, b] \times [0, B]$ to the rectangle $[0, a] \times [0, A]$.)

Further, according to Definition 8.2, the graph of the scaled function is supposed to be situated under the graph of the original function f. This is why the notion "subscalable" has been chosen.

Definition 8.4. Let $I \subseteq [0, \infty[$ be an interval. An increasing bijection $f\colon\ [0, \infty[\to [0, \infty[$ is said to be *affine on* I if there is $r \in\,]0, \infty[$ and $s \in \mathbb{R}$ such that, for every $x \in I$,

$$f(x) \quad = \quad r\,x + s.$$

If $s = 0$ then f is said to be *linear on* I.

Remark 8.5. Every increasing bijection of $[0, \infty[$ is linear on $[0, 0] = \{0\}$.

Theorem 8.6. *[14, Theorem 4.5] Let* $f\colon\ [0, \infty[\to [0, \infty[$ *be an increasing bijection and let* $k \in [0, \infty[$*. If* f *is convex,* k*-subscalable, and linear on* $[0, k]$ *then it solves Mulholland inequality.*

As it can be shown [14, Section 3], function f from Theorem 8.6 satisfies the Mulholland's condition (i.e., it is both convex and geometrically convex) if, and only if, $k = 0$. Actually, in the case of convex increasing bijections, geometric convexity is equivalent to 0-subscalability. Therefore the new condition, when compared to the Mulholland's one, does not delimit a smaller set of functions. To prove that it delimits even a strictly larger set, the following example of an increasing bijection has been presented [14, Example 5.1]:

Example 8.7. The function $g\colon\ [0, \infty[\to [0, \infty[$ (see Fig. 2-left) is defined by

$$g\colon\quad [0, \infty[\to [0, \infty[\,:\quad x \mapsto \begin{cases} \frac{5}{3}x & \text{if } x \in [0, 1[\,, \\ \frac{7}{3}x - \frac{2}{3} & \text{if } x \in [1, 2[\,, \\ x^2 & \text{if } x \in [2, \infty[\,. \end{cases}$$

It can be shown easily, that g is a convex increasing bijection of $[0, \infty[$ which is linear on $[0, 1]$, and not so easily that it is also 1-subscalable [14, Lemma A.1]. Therefore g solves Mulholland inequality. However, according to Proposition 5.3, g is not geometrically convex and thus does not accord with the Mulholland's condition.

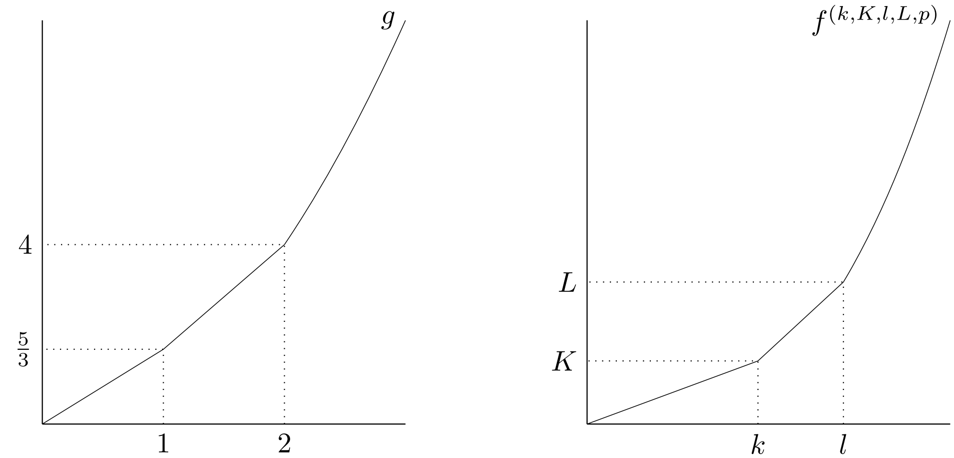


Fig. 2. Left: graph of a function that solves Mulholland inequality but does not satisfy the Mulholland's condition. Right: Graph of a general function that is partially linear, partially affine, and partially a power function.

9. More new solutions to Mulholland inequality

As we can see, function g in Example 8.7 is linear on $[0, 1]$, affine on $[1, 2]$, and it is a power function on $[2, \infty[$. In order to provide more such counter-examples, let us introduce a general function of this type (confer with Fig. 2-right) in the following way:

Definition 9.1. Let $k, K, l, L \in [0, \infty[$ and $p \in [1, \infty[$ such that $k \le l$, $K \le L$, and

$$\begin{aligned} k = 0 \quad &\text{if, and only if,} \quad K = 0, \\ l = 0 \quad &\text{if, and only if,} \quad L = 0. \end{aligned}$$

A parametric function $f^{(k,K,l,L,p)}\colon [0, \infty[\to [0, \infty[$ is given,

1. for $0 < k < K$ and $0 < l < L$ by

$$x \mapsto \begin{cases} \frac{K}{k} x & \text{if} \quad x \in [0, k[\,, \\ \frac{L-K}{l-k} x - \frac{Lk-Kl}{l-k} & \text{if} \quad x \in [k, l[\,, \\ \frac{L}{l^p} x^p & \text{if} \quad x \in [l, \infty[\,, \end{cases} \tag{3}$$

2. for $k = K = 0$ and $l, L > 0$, or for $k = l > 0$ and $K = L > 0$, by

$$x \mapsto \begin{cases} \frac{L}{l} x & \text{if} \quad x \in [0, l[\,, \\ \frac{L}{l^p} x^p & \text{if} \quad x \in [l, \infty[\,, \end{cases}$$

3. and, for $k = K = l = L = 0$, by $x \mapsto x^p$.

Observe that in Cases 2 and 3 the function $f^{(k,K,l,L,p)}$ is, according to Proposition 5.3, an increasing bijection which is both convex and geometrically convex. Observe also that, referring to Example 8.7, $g = f^{\left(1, \frac{5}{3}, 2, 4, 2\right)}$.

Lemma 9.2. *The function $f^{(k,K,l,L,p)}$ is a convex increasing bijection if, and only if,*

$$\frac{K}{k} \le \frac{L-K}{l-k}, \tag{4}$$

$$\textit{and} \quad \frac{L-K}{l-k} \le \frac{L}{l}\, p. \tag{5}$$

Proof. Observe first that the left and the right first derivative of $f^{(k,K,l,L,p)}$ is defined in every point of $]0,\infty[$. Therefore the function is strictly increasing (and thus an increasing bijection) if, and only if, the left and right first derivative in every point of $]0,\infty[$ are positive. This is, however, assured by the definition.

Further, $f^{(k,K,l,L,p)}$ is clearly convex on $[0,k]$, on $[k,l]$, and on $[l,\infty[$. Therefore it will be convex on its whole domain $[0,\infty[$ if, and only if,

- the left first derivative in k is not greater than the right one, i.e., (4)
- and the left first derivative in l is not greater than the right one, i.e., (5). □

It is clear that $f^{(k,K,l,L,p)}$ is linear on $[0,k]$. Thus, if it is also k-subscalable then it solves Mulholland inequality. In the sequel, we are going to introduce a sufficient condition for the k-subscalability of $f^{(k,K,l,L,p)}$.

Proposition 9.3. *Let $f^{(k,K,l,L,p)}$ be given according to Definition 9.1 such that $0<k<l$ and $0<K<L$. If*

$$\frac{\sqrt{5}-1}{2}\,l \le k, \tag{6}$$

$$\frac{K}{k} \le \frac{L}{l}, \tag{7}$$

$$\text{and} \quad \frac{L-K}{l-k} \le \frac{K}{k}\,p \tag{8}$$

then $f^{(k,K,l,L,p)}$ is a convex k-subscalable increasing bijection of $[0,\infty[$ which is linear on $[0,k]$ (and thus it solves Mulholland inequality).

Remark 9.4. Moreover, referring to Proposition 5.3, $f^{(k,K,l,L,p)}$ does not satisfy the Mulholland's condition.

Proof. Let $f^{(k,K,l,L,p)}$ satisfy (6), (7), and (8).

From (7) and (8) we immediately have (5). Further, (7) is equivalent to (4):

$$\begin{aligned} \frac{K}{k} &\le \frac{L}{l}, \\ K\,l - K\,k &\le L\,k - K\,k, \\ \frac{K}{k} &\le \frac{L-K}{l-k}. \end{aligned}$$

Therefore, according to Lemma 9.2, $f^{(k,K,l,L,p)}$ is a convex increasing bijection.

To show that it is also k-subscalable, we need to show that

$$f_{\downarrow}^{a,b}(x) \quad \le \quad f(x) \tag{9}$$

holds for every $a,b>0$ and $x\in[0,a]$ such that

$$b-a \quad \ge \quad k. \tag{10}$$

Notice that $f_{\downarrow}^{a,b}(0)=f(0)=0$ and that $f_{\downarrow}^{a,b}(a)=f(a)$.

Case 1. If $a\in\,]0,k[$ then (9) holds trivially since $f_{\downarrow}^{a,b}$ is convex and f is linear on $[0,a]$; see Fig. 3.

Case 2. The case $a,b\in[k,l[$ may never happen since (6) and (10).

In the following two cases (Case 3 and Case 4), since f is convex, linear on $[0,k]$, and affine on $[k,l]$, and since $a\ge k$ and $b\ge l$ (confer with Figs. 3 and 4), the condition (9) will be violated for some $x\in[0,a]$ if, and only if, it will be violated also for $x=k$, i.e., if, and only if,

$$f_{\downarrow}^{a,b}(k) \quad > \quad f(k) \quad = \quad K. \tag{11}$$

Let us show that this may not happen.

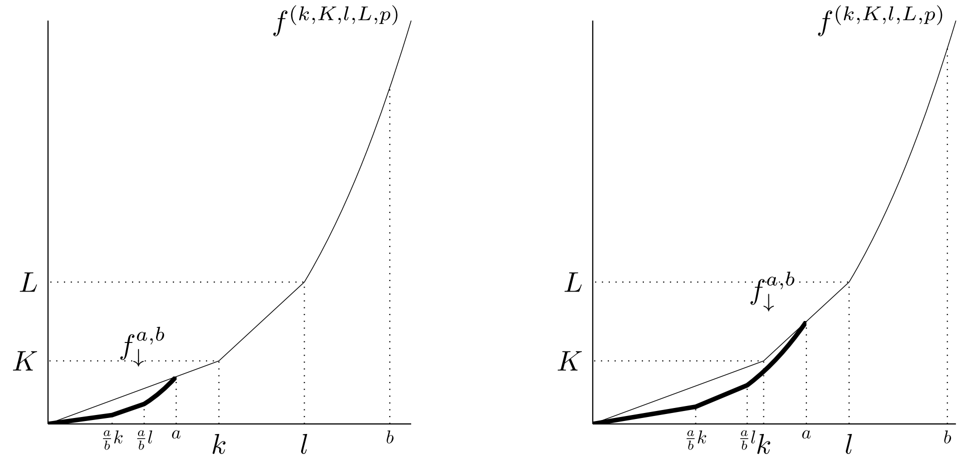


Fig. 3. Illustration of the proof of Proposition 9.3: graph of $f_{\downarrow}^{a,b}$ when $b \in [l, \infty[$ and $a \in [0, k[$ (left) or $a \in [k, l[$ (right).

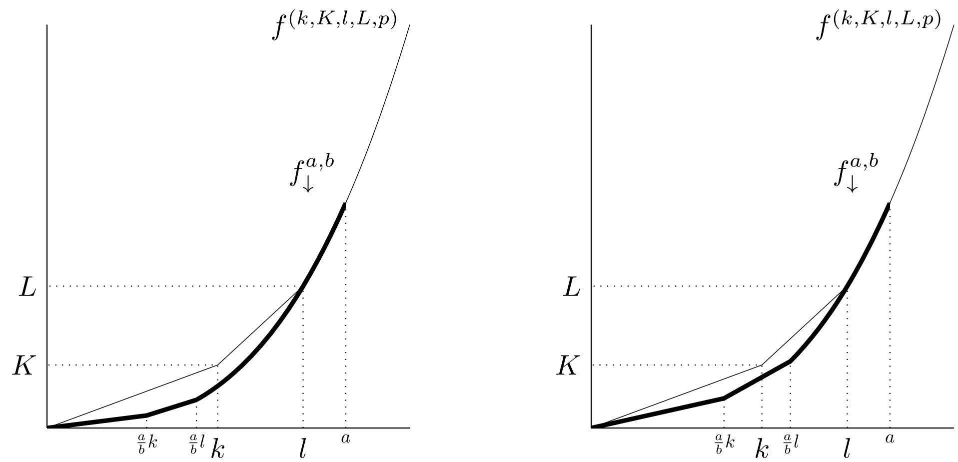


Fig. 4. Illustration of the proof of Proposition 9.3: graph of $f_{\downarrow}^{a,b}$ when $a, b \in [l, \infty[$ and $l' = \frac{a}{b}\, l \leq k$ (left) or $l' = \frac{a}{b}\, l > k$ (right).

Case 3. Suppose that $a \in [k, l[$ and $b \in [l, \infty[$; this case is illustrated in Fig. 3-right. We have

$$\text{either} \quad \frac{b}{a}\, k \geq l \tag{12}$$

$$\text{or} \quad \frac{b}{a}\, k < l. \tag{13}$$

Thanks to (6) the case (13) may never happen. Indeed, in such a case, from $a \in [k, l[$, (10), and (13), the domain of the points (a, b) is the triangle depicted in Fig. 5-left. We can see that this triangle will become an empty set if

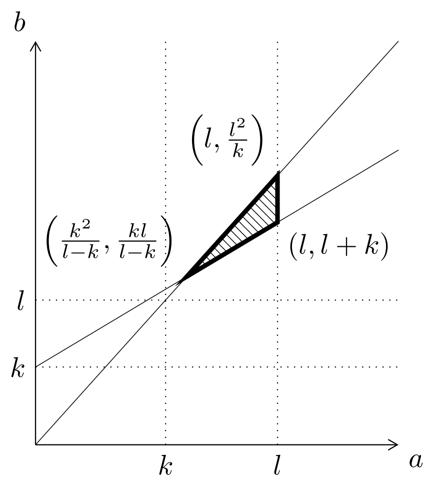
$$\frac{k^2}{l-k} \geq l,$$

$$k^2 + l\,k - l^2 \geq 0.$$

This inequality is, however, equivalent to (6) and thus true under our assumptions. Therefore (12) is the only possible case. We have

$$f_{\downarrow}^{a,b}(k) \;=\; \frac{f(a)}{f(b)}\, f\left(\frac{b}{a}\, k\right) \;=\; \frac{f(a)}{\frac{L}{l^p} b^p}\, \frac{L}{l^p} \left(\frac{b}{a}\, k\right)^p \;=\; \frac{f(a)}{a^p}\, k^p \tag{14}$$

and, therefore, $f_{\downarrow}^{a,b}(k) \leq f(k)$ (the contradiction to (11)) will be valid if, and only if,



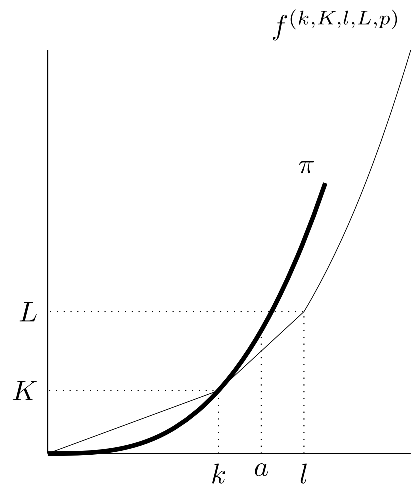


Fig. 5. Illustration of the proof of Proposition 9.3. Left: domain of the values a and b in Case 3 of the proof. Right: Comparison of the functions π and $f^{(k,K,l,L,p)}$ in Case 3 of the proof.

$$\begin{aligned} &\frac{f(a)}{a^p} k^p \le K, \\ &\frac{K}{k^p} a^p \ge f(a). \end{aligned} \tag{15}$$

The expression on the left hand side of (15) is the value in the point a of the power function $\pi : x \mapsto \frac{K}{k^p} x^p$; notice that its graph contains the point (k, K) as illustrated in Fig. 5-right. Since π is a smooth increasing convex function, and since the point $(a, f(a))$ lays on the affine part of the graph of f, the inequality (15) will be true if, and only if, the first derivative of π in k is greater or equal to the first derivative of the affine part of f, i.e.,

$$\frac{K}{k^p} p k^{p-1} \ge \frac{L-K}{l-k}$$

which is, however, assured by (8).

Case 4. If $a, b \in [l, \infty[$ then $f_\downarrow^{a,b}$ and f are identical on $[l, a]$ as illustrated in Fig. 4; thus $f_\downarrow^{a,b}(l) = f(l)$. As in Case 3, we have either (12) or (13).

If (12) is the case then

$$f_\downarrow^{a,b}(k) = \frac{f(a)}{f(b)} f\left(\frac{b}{a} k\right) = \frac{\frac{L}{l^p} a^p}{\frac{L}{l^p} b^p} \frac{L}{l^p} \left(\frac{b}{a} k\right)^p = \frac{L}{l^p} k^p.$$

Making the same reasoning as in Case 3 (set $a = l$ in (14)), we disprove (11).

If (13) is the case then, denoting $u = \frac{b}{a} k$,

$$\begin{aligned} f_\downarrow^{a,b}(k) = \frac{f(a)}{f(b)} f\left(\frac{b}{a} k\right) &= \frac{\frac{L}{l^p} a^p}{\frac{L}{l^p} b^p} f\left(\frac{b}{a} k\right) = \frac{a^p}{b^p} f(u) \\ = \frac{k^p}{k^p} \frac{a^p}{b^p} f(u) &= \frac{f(u)}{u^p} k^p. \end{aligned}$$

To disprove (11) we need to show that

$$\frac{f(u)}{u^p} k^p \le K$$

which can be done in the same way as in Case 3 (set $a = u$ in (15)). $\square$

In order to show that there are functions of the type $f^{(k,K,l,L,p)}$ that do solve Mulholland inequality, let us introduce the following family of increasing bijections, given by two parameters.

Definition 9.5. A parametric function $f^{(l,p)}\colon [0,\infty[\to [0,\infty[$ is given, for $l \in [0,\infty[$ and $p \in [1,\infty[$ by $f^{(l,p)} = f^{\left(\frac{2}{3}l,\frac{2l^p}{p+2},l,l^p,p\right)}$. More explicitly,

$$f^{(l,p)}\colon \quad x \mapsto \begin{cases} \frac{3\,l^{p-1}}{p+2}\,x & \text{if} \quad x \in \left[0, \frac{2}{3}l\right[, \\ \frac{3\,p\,l^{p-1}}{p+2}\,x - 2\,\frac{p-1}{p+2}\,l^p & \text{if} \quad x \in \left[\frac{2}{3}l, l\right[, \\ x^p & \text{if} \quad x \in [l,\infty[\,. \end{cases}$$

Proposition 9.6. *Every member of the family in Definition 9.5 solves the Mulholland inequality. Moreover, if $l > 0$ and $p > 1$ then $f^{(l,p)}$ does not satisfy the Mulholland's condition while, for $l = 0$, $f^{(l,p)}$ becomes a power function with power p.*

Proof. It is easy to check that if $l = 0$ then $f^{(l,p)}$ becomes a power function with power p, and if $p = 1$ then $f^{(l,p)}$ becomes a linear function. Therefore, referring to Proposition 5.3, $f^{(l,p)}$ satisfies both the Mulholland's condition and Mulholland inequality.

Suppose that $l > 0$ and $p > 1$. Recall that we have

$$k = \frac{2}{3}\,l, \tag{16}$$

$$K = \frac{2l^p}{p+2}, \tag{17}$$

$$L = l^p. \tag{18}$$

From (16) we have the validity of (6). Further, from (7) we have:

$$\frac{K}{k} \le \frac{L}{l},$$

$$\frac{\frac{2\,l^p}{p+2}}{\frac{2}{3}\,l} \le \frac{l^p}{l},$$

$$3 \le p+2.$$

The last inequality holds because $p \ge 1$ and, since all the three inequalities are equivalent, we have the validity of (7). Finally, from (8) we have

$$\frac{L-K}{l-k} \le \frac{K}{k}\,p,$$

$$\frac{l^p - \frac{2l^p}{p+2}}{l - \frac{2}{3}\,l} \le \frac{\frac{2l^p}{p+2}}{\frac{2}{3}\,l}\,p,$$

$$\frac{2\,p\,l^p}{p+2} \le \frac{2\,p\,l^p}{p+2}.$$

Again, since all the three inequalities are equivalent, we have the validity of (8).

Since (6), (7), and (8) are valid, the function $f^{(l,p)}$ is, according to Proposition 9.3, k-subscalable and thus solves Mulholland inequality. Furthermore, since $f^{(l,p)}$ is affine, but not linear, on $\left[\frac{2}{3}l, l\right]$, it does not satisfy, according to Proposition 5.3, the Mulholland's condition. □

10. Dominance is not transitive on strict t-norms

We refer to a recent result [14, Proposition 6.2] which can be reformulated in the following way:

Proposition 10.1. *Let $f^{(k,K,l,L,p)}\colon [0,\infty[\to [0,\infty[$ be an increasing bijection, given according to Definition 9.1, with*

$$2K \quad < \quad L. \tag{19}$$

Let $h\colon [0,\infty[\to [0,\infty[\,:\, x \mapsto x^q$ be a power function with $q > 1$. Then the composition $f^{(k,K,l,L,p)} \circ h$ does not satisfy the Mulholland inequality.

Remark 10.2. Observe that (19) holds for every member of the family introduced in Definition 9.5 with $l > 0$ and $p > 2$.

Theorem 10.3. *The dominance relation is in general not transitive on the set of strict triangular norms.*

Proof. To demonstrate this fact, let f be any member of the family introduced in Definition 9.5 with $l > 0$ and $p > 2$. Further, let $t\colon [0,1] \to [0,\infty]$ be a decreasing bijection and let $h\colon [0,\infty[\to [0,\infty[\,:\, x \mapsto x^q$ be a power function with $q > 1$. We introduce the following three additive generators,

$$\begin{aligned} t_1 &= f \circ t, \\ t_2 &= t, \\ \text{and} \quad t_3 &= t^{1/q}, \end{aligned}$$

of, respectively, three strict t-norms, $*_1$, $*_2$, and $*_3$. We have, referring to Lemma 5.5, Proposition 10.1, and Remark 10.2,

$$\begin{aligned} t_1 \circ t_2^{-1} &= f && \Rightarrow *_1 \gg *_2, \\ t_2 \circ t_3^{-1} &= h && \Rightarrow *_2 \gg *_3, \\ t_1 \circ t_3^{-1} &= f \circ h && \Rightarrow *_1 \not\gg *_3. \quad \square \end{aligned}$$

11. Conclusion

Although it has been shown that the dominance relation is not transitive on the set of strict t-norms, the question of a nice characterization of this relation still remains open. Recall that the families for which the question of transitivity is still not answered are summarized in Table 3. What remains open, as well, is the question of the structure of the solutions of Mulholland inequality. We conjecture that the new condition, presented in Theorem 8.6, is not a necessary one and, therefore, a condition, that would entirely delimit all the solutions of Mulholland inequality, is still unknown.

Acknowledgements

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Dominance on continuous Archimedean triangular norms and generalized Mulholland inequality

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Abstract

As a preceding result, it has been shown that the dominance relation is not transitive on the set of strict triangular norms. This result has been achieved thanks to new results on Mulholland inequality. Recently, Saminger-Platz, De Baets, and De Meyer have introduced the generalized Mulholland inequality which characterizes the dominance on all continuous Archimedean triangular norms in an analogous way as does Mulholland inequality on the strict triangular norms. Based on these new results, the present paper shows that the dominance relation is not transitive on the set of nilpotent triangular norms and, consequently, on the set of continuous Archimedean triangular norms. This result is achieved by introducing a new sufficient condition under which a given function solves the generalized Mulholland inequality and by showing that the set of the functions that solve the inequality is not closed with respect to compositions.

Keywords: Dominance relation; Generalized Mulholland inequality; Nilpotent triangular norm; Transitivity

1. Introduction

Mulholland inequality is a functional inequality which has been introduced in a paper by Mulholland [11] in 1947 as a generalization of Minkowski inequality which describes the triangular inequality of L^p-norms. In Minkowski inequality, Mulholland has replaced the power functions $x \mapsto x^p$, $p > 1$, by an arbitrary increasing bijection obtaining a generalization.

Few decades later, in 1984, Tardiff has revealed a close connection between Mulholland inequality and the dominance relation defined on the set of triangular norms [22] (shortly t-norms). He has shown that Mulholland inequality directly characterizes the dominance on a subset of t-norms called strict t-norms.

Recall that both the notions of dominance and t-norms [2,8] have been introduced within the framework of probabilistic metric spaces [10,19]; t-norms describe the triangular inequality of the probabilistic metrics while a satisfaction of the dominance relation is crucial when constructing Cartesian products of probabilistic metric spaces [21]. Later on, t-norms have found their role also as the interpretation of the logical conjunction in the semantics of fuzzy log-

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ics [4–6,12]. Recall that the dominance has found its use also when working with t-norm based fuzzy equivalences and partitions [3, Theorem 2] and with their refinements [3, Theorem 5].

It is easy to show that the relation of dominance is both reflexive and anti-symmetric. However, for a long time the question, whether this relation is also transitive, and thus an order, was open. This question has been even stated as an open problem in the monograph by Schweizer and Sklar [19, Problem 12.11.3] as well as in the list of open problems by Alsina, Frank, and Schweizer [1, Problem 17]. Recently, in 2008, it has been answered negatively for the class of continuous t-norms in a paper by Sarkoci [18]. A few years later, a negative answer has been given also for strict t-norms [14]. This answer has been achieved thanks to new results on Mulholland inequality [13]. Remark that the question has remained open for the class of nilpotent t-norms, which also form an important sub-class of continuous t-norms, although some parametric sub-classes of this class have been already studied [7,8,15,20].

In 2008, Saminger-Platz, De Baets, and De Meyer have introduced the generalized Mulholland inequality [17] which, analogously to the original Mulholland inequality, characterizes the dominance on the class of all continuous Archimedean t-norms, i.e., on both strict and nilpotent t-norms. It is therefore natural to ask whether this generalized inequality can help to answer the question of the transitivity of the dominance also for nilpotent t-norms and, generally, for all continuous Archimedean t-norms.

This research follows up on the latter cited paper stating some new results on the generalized Mulholland inequality. Further, as a corollary, the question of the transitivity of the dominance on nilpotent t-norms is answered. The paper is similar to previous ones [14,13] stating analogous results which have been already made for the original Mulholland inequality and giving analogous proofs. However, as the assumptions are different to the previous case, it has shown as a better option to write all the proofs again. Moreover, despite the analogous approach we end up this time with a different solution—an answer for the case of nilpotent t-norms.

2. Triangular norms and dominance relation

A *triangular norm* [2,8] (shortly a *t-norm*) is a commutative, associative, and monotone binary operation $*\colon [0,1]\times[0,1]\to[0,1]$ with a neutral element 1, i.e., we have $x*1=1*x=x$ for every $x\in[0,1]$.

For $n\in\mathbb{N}$, we define the n-th natural power of $x\in[0,1]$, according to a t-norm $*$, by $x_*^{(n)}=x_*^{(n-1)}*x$ and $x_*^{(0)}=1$. A t-norm $*$ is said to be *continuous* if it is continuous as a two-variable real function. It is said to be *Archimedean* if, for every $x,y\in\,]0,1[$, such that $x<y$, there is $n\in\mathbb{N}$ such that $y_*^{(n)}<x$. A continuous t-norm $*$ is Archimedean if, and only if, $x*x<x$ for every $x\in\,]0,1[$.

A continuous t-norm $*$ is said to be *nilpotent* if, for every $x\in\,]0,1[$ there is $n\in\mathbb{N}$ such that $x_*^{(n)}=0$. It is said to be *strict* if its restriction to $]0,1]^2$ is strictly increasing in each variable. A continuous Archimedean t-norm is either nilpotent or strict.

By $[0,\infty]$ we denote the interval of non-negative real numbers enriched by the top element ∞ with $x+\infty=\infty$ for any $x\in[0,\infty]$.

A t-norm $*$ is continuous Archimedean if, and only if, there is a continuous decreasing injection $t\colon [0,1]\to[0,\infty]$ with $t(1)=0$ such that

$$x*y \;=\; t^{(-1)}\big(t(x)+t(y)\big)$$

for every $x,y\in[0,1]$. Here, $t^{(-1)}$ denotes the *pseudo-inverse* of t defined, in this particular case, by $t^{(-1)}(x)=t^{-1}(x)$ if $x\le t(0)$ and $t^{(-1)}(x)=0$ otherwise. The mapping t is called the *additive generator* of the continuous Archimedean t-norm $*$ and this generator is unique up to a multiplication by a positive real constant. The t-norm generated by an additive generator t is nilpotent if $t(0)<\infty$; if $t(0)=\infty$ then $*$ is strict.

Dominance is a binary relation defined on the set of t-norms by Tardiff [21, Definition 3.4]. Remark that this relation can be defined also in a more general setting [16]. A t-norm $*_1$ is said to *dominate* a t-norm $*_2$ (and we write $*_1\gg *_2$) if

$$\forall x,y,u,v\in[0,1]:\quad (x*_2 y)*_1(u*_2 v)\quad\ge\quad (x*_1 u)*_2(y*_1 v). \tag{1}$$

By setting $y=u=1$ in (1), we have that $*_1\gg *_2$ implies $*_1\ge *_2$ for every two t-norms $*_1$ and $*_2$. From this fact it follows that dominance is an anti-symmetric relation. Dominance is, furthermore, also reflexive since for any t-norm $*$ we have $*\gg *$ from the associativity and commutativity of $*$. In this paper, we are going to deal with the question whether, on certain classes of t-norms, the relation of dominance is also transitive, i.e., whether

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$$*_1 \gg *_2 \quad \text{and} \quad *_2 \gg *_3 \quad \text{implies} \quad *_1 \gg *_3$$

for every three t-norms $*_1$, $*_2$, and $*_3$ from a given class.

3. Generalized Mulholland inequality

An increasing bijection $f\colon\ [0, \infty[\to [0, \infty[$ is said to *solve Mulholland inequality* if

$$f^{-1}\left(\sum_{i=1}^{n} f(x_i + y_i)\right) \le f^{-1}\left(\sum_{i=1}^{n} f(x_i)\right) + f^{-1}\left(\sum_{i=1}^{n} f(y_i)\right)$$

holds for every $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in [0, \infty[^n$. In this paper we will be exclusively using the two-dimensional ($n = 2$) variant of Mulholland inequality, i.e., f solves Mulholland inequality if

$$\begin{aligned}&\forall x, y, u, v \in [0, \infty[\,:\\ &f^{-1}\Big(f(x+u) + f(y+v)\Big) \le f^{-1}\Big(f(x) + f(y)\Big) + f^{-1}\Big(f(u) + f(v)\Big).\end{aligned} \tag{MI}$$

Mulholland inequality is in a close correspondence with the dominance of strict t-norms which has been shown in 1984 by Tardiff [22, Theorem 3].

Notation 3.1. The notation $f \circ g$ denotes the composition of two functions, f and g, such that it holds $(f \circ g)(x) = f(g(x))$ for every x in the domain of g.

Theorem 3.2. *[22] Let $*_1$ and $*_2$ be two strict t-norms defined respectively by their additive generators t_1 and t_2, for every $x, y \in [0, 1]$, as*

$$\begin{aligned}x *_1 y &= t_1^{-1}\big(t_1(x) + t_1(y)\big),\\ x *_2 y &= t_2^{-1}\big(t_2(x) + t_2(y)\big).\end{aligned}$$

Then $*_1$ dominates $*_2$ if, and only if, $f = t_1 \circ t_2^{-1}$ restricted to $[0, \infty[$ solves Mulholland inequality.

In 2008, Saminger-Platz, De Baets, and De Meyer have enlarged this correspondence to the set of all continuous Archimedean t-norms introducing the following result [17, Theorem 1]:

Theorem 3.3. *[17] Let $*_1$ and $*_2$ be two continuous Archimedean t-norms defined respectively by their additive generators t_1 and t_2, for every $x, y \in [0, 1]$, as*

$$\begin{aligned}x *_1 y &= t_1^{(-1)}\big(t_1(x) + t_1(y)\big),\\ x *_2 y &= t_2^{(-1)}\big(t_2(x) + t_2(y)\big).\end{aligned}$$

Then $*_1$ dominates $*_2$ if, and only if, the functions $f = t_1 \circ t_2^{(-1)}$ and $f^{(-1)} = t_1 \circ t_2^{(-1)}$ satisfy

$$\begin{aligned}&\forall x, y, u, v \in [0, t_2(0)]\,:\\ &f^{(-1)}\Big(f(x+y) + f(u+v)\Big) \le f^{(-1)}\Big(f(x) + f(u)\Big) + f^{(-1)}\Big(f(y) + f(v)\Big).\end{aligned} \tag{GMI}$$

If (GMI) holds we say that f *solves the generalized Mulholland inequality*. Analogously to the case of strict t-norms and Mulholland inequality, the generalized Mulholland inequality gives a characterization of the dominance on all continuous Archimedean t-norms which involve both strict and nilpotent t-norms. Notice that if we deal with strict t-norms and thus $t_1(0) = t_2(0) = \infty$ then (GMI) becomes equivalent to (MI). As the case of Mulholland inequality has been already described in the previous papers [13,14], in this paper we will mostly focus on the case when $t_1(0) < \infty$ and $t_2(0) < \infty$, which describes the dominance on nilpotent t-norms.

We can see that, according to the properties of additive generators, f and $f^{(-1)}$ in Theorem 3.3 satisfy the following.

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Assumptions 3.4. Assume a function $f\colon\ [0,\infty]\to[0,\infty]$ and fixed values $d, e\in\,]0,\infty]$ such that:

1. $f(0)=0$ and $f(d)=e$,
2. f is continuous and strictly increasing on $[0,d]$,
3. $f(x)=e$ for $x\geq d$.

Assume, further, the function $f^{(-1)}\colon\ [0,\infty]\to[0,\infty]$ defined by

$$f^{(-1)}\colon\quad x\mapsto\begin{cases} f^{-1}(x) & \text{if } x\in[0,e]\,,\\ d & \text{otherwise.}\end{cases}$$

Note that if $f=t_1\circ t_2^{(-1)}$ and $f^{(-1)}=t_1\circ t_2^{(-1)}$ then we have $d=t_2(0)$ and $e=t_1(0)$.

The same paper, which has introduced the generalized Mulholland inequality [17], has presented also sufficient conditions under which a given function solves this inequality. One of them [17, Theorem 6] we are going to present here. According to the terminology of Matkowski [9], we define a function $f\colon\ [0,\infty]\to[0,\infty]$ to be *geometrically convex on an interval* $I\subseteq[0,\infty]$ if

$$f\left(x^{1-\alpha}\cdot y^{\alpha}\right)\quad\leq\quad f^{1-\alpha}(x)\cdot f^{\alpha}(y)$$

for every $x, y\in I$ and $\alpha\in[0,1]$.

Theorem 3.5. *[17] Consider functions f and $f^{(-1)}$, which comply with Assumptions 3.4, such that f is*

- *convex on* $]0,d[$,
- *geometrically convex on* $]0,d[$.

Then f solves the generalized Mulholland inequality.

Note that f is geometrically convex on $]0,d[$ if, and only if, the function $F=\log\circ f\circ\exp$, is convex on $\big]{-\infty},\log d\big[$. We finish this section by one more result from the mentioned paper [17, Proposition 10]:

Proposition 3.6. *[17] Consider functions f and $f^{(-1)}$, which comply with Assumptions 3.4. If f solves the generalized Mulholland inequality then it is convex on* $]0,d[$.

Remark that if f is convex resp. geometrically convex on $]0,d[$ then, since it is continuous, it is convex resp. geometrically convex also on $[0,d]$.

4. Bounded pseudo-addition

Throughout this section, f and $f^{(-1)}$ are given according to Assumptions 3.4 and f is assumed to be convex on $[0,d]$. Let $\oplus_f$ be a binary operation on $[0,\infty]$ defined by

$$x\oplus_f y\quad=\quad f^{(-1)}\Big(f(x)+f(y)\Big) \tag{2}$$

for every $x, y\in[0,\infty]$. We call $\oplus_f$ the *bounded pseudo-addition* generated by f. For $z\in[0,\infty]$, we define the *z-level set* of $\oplus_f$ as the set

$$L_z^f=\{(x,y)\in[0,\infty]^2\mid x\oplus_f y=z\}$$

and the *z-level cut* of $\oplus_f$ as the set

$$\Lambda_z^f=\{(x,y)\in[0,\infty]^2\mid x\oplus_f y\leq z\}.$$

Note that $(x,y)\in\Lambda_z^f$ if, and only if, $f(x)+f(y)\leq f(z)$. Finally, we define the *support of* $\oplus_f$ as the set

$$\operatorname{supp} \oplus_f = \{(x, y) \in [0, \infty]^2 \mid x \oplus_f y < d\}.$$

In this paper we will often omit the index f in the notation of $\oplus_f$, L_z^f, and Λ_z^f as it will be clear from the context which function f we are dealing with.

We can observe the following properties of $\oplus$.

- The operation is continuous, commutative, and associative.
- We have $x \oplus 0 = 0 \oplus x = x$ for every $x \in [0, d]$.
- The operation is strictly increasing in both variables if its domain is restricted to its support. On the rest of the domain, the operation $\oplus$ is the constant d.
- We have $x \oplus y \le d$ for every $x, y \in [0, \infty]$; this gives the reason for "bounded" in the name of $\oplus$. (Be, however, aware that we can also have $d = \infty$; in such a case $x \oplus y$ is not "bounded".)

We call the collection of all the level sets (and the level cuts) of $\oplus$ the *level set plot of* $\oplus$. We can observe the following structure of such level set plots.

- For $z < d$, Λ_z is a convex set and L_z is its border in $[0, \infty]^2$.
- For $z = d$, $\Lambda_z = [0, \infty]^2$ and L_z is the complement of $\operatorname{supp} \oplus$.
- For $z > d$, $\Lambda_z = [0, \infty]^2$ and L_z is an empty set.

The operation of pseudo-addition allows us to rewrite (GMI) to the form:

$$\forall x, y, u, v \in [0, d]: \quad (x + u) \oplus (y + v) \;\; \le \;\; (x \oplus y) + (u \oplus v). \tag{GMI$_\oplus$}$$

Observe that we have actually obtained the dominance inequality; thus a function f solves (GMI) if, and only if, the standard addition dominates the bounded pseudo-addition generated by f. Furthermore, (GMI$_\oplus$) can be also understood as sub-additivity of $\oplus$.

Minkowski sum of $A, B \in [0, \infty]^2$ is defined by

$$A + B \;\; = \;\; \{(x + u, y + v) \mid (x, y) \in A, (u, v) \in B\}.$$

Theorem 4.1. *Let f and f^{-1} be two functions given according to Assumptions 3.4. Then they satisfy the generalized Mulholland inequality if, and only if,*

$$\forall a, b \in [0, \infty], \;\; a + b < d: \quad \Lambda_a + \Lambda_b \;\; \subseteq \;\; \Lambda_{a+b} \tag{GMI$_\Lambda$}$$

Proof. First, we prove (GMI$_\oplus$) $\Rightarrow$ (GMI$_\Lambda$). Denote $a = x \oplus y$ and $b = u \oplus v$. Observe that (GMI$_\Lambda$) is apparently satisfied if $a + b \ge d$ since, in such a case, $\Lambda_{a+b} = [0, \infty]^2$. Assume therefore $a + b < d$ and assume a, b to be fixed and $x, y, u, v \in [0, \infty]$ to be variable. The left-hand side of (GMI$_\oplus$), which is the value of $\oplus$ in the point $(x + u, y + v)$, is supposed to be less than or equal to the right-hand side, which is equal to $a + b$. This means, due to the monotonicity of $\oplus$, that the point $(x + u, y + v)$ must be located inside the area of Λ_{a+b}; confer with Fig. 1. This fact must hold true not only for every $(x, y) \in L_a$ and every $(u, v) \in L_b$ but, apparently, also for every $(x, y) \in \Lambda_a$ and $(u, v) \in \Lambda_b$. Thus, if we sum every point of Λ_a with every point of Λ_b we have to obtain a subset of Λ_{a+b}; this is what (GMI$_\Lambda$) expresses.

Now, we prove (GMI$_\Lambda$) $\Rightarrow$ (GMI$_\oplus$). Suppose $a, b \in [0, \infty]$ and $x, y, u, v \in [0, \infty]$ such that $a = x \oplus y$ and $b = u \oplus v$. If $a + b \ge d$ then (GMI$_\oplus$) is apparently satisfied since, thanks to the properties of $\oplus$, its left-hand side is always less than or equal to d. Assume therefore $a + b < d$. Equation (GMI$_\Lambda$) states that, for every $(x, y) \in \Lambda_a$ and $(u, v) \in \Lambda_b$, we have

$$(x + u, y + v) \in \Lambda_{a+b},$$
$$(x + u) \oplus (y + v) \le a + b.$$

Hence we obtain (GMI$_\oplus$). ■

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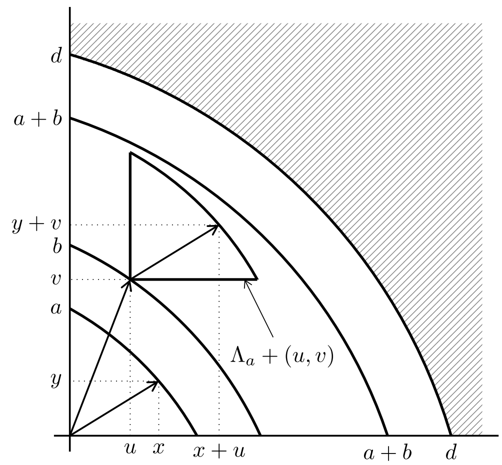

Fig. 1. Geometric interpretation of the generalized Mulholland inequality.

Let us introduce some more notions and facts related to the level set plot of $\oplus$ and to Minkowski sum. Assume $A, B, C, D \in [0, \infty]^2$ and $\alpha, \beta \in [0, \infty[$. We have that

$$A \subseteq C \quad \text{and} \quad B \subseteq D \quad \text{implies} \quad A + B \subseteq C + D.$$

The scalar multiple of A by α is defined by

$$\alpha A \;=\; \{(\alpha x, \alpha y) \mid (x, y) \in A\}.$$

If A is a convex set then

$$(\alpha + \beta)A \;=\; \alpha A + \beta A.$$

For $a, b \in [0, \infty]$ and the two corresponding level cuts of $\oplus$, Λ_a and Λ_b respectively, we define

$$\Lambda_a \;\le\; \Lambda_b \quad \text{iff} \quad \frac{1}{a}\Lambda_a \;\subseteq\; \frac{1}{b}\Lambda_b.$$

Observe that the latter defined binary relation $\le$ is both reflexive and transitive. However, it is generally neither anti-symmetric nor symmetric. Hence it is a pre-order.

Observation 4.2. *If $a = 0$ or $b = 0$ then $\Lambda_a + \Lambda_b = \Lambda_{a+b}$; hence $\Lambda_a + \Lambda_b \subseteq \Lambda_{a+b}$.*

Lemma 4.3. *If, for some $a, b \in [0, \infty]$, $\Lambda_a \le \Lambda_{a+b}$ and $\Lambda_b \le \Lambda_{a+b}$ then $\Lambda_a + \Lambda_b \subseteq \Lambda_{a+b}$.*

Proof. From the assumptions we have $\Lambda_a \subseteq \frac{a}{a+b}\Lambda_{a+b}$ and $\Lambda_b \subseteq \frac{b}{a+b}\Lambda_{a+b}$. The proof is finished by summing these two inequalities. ∎

5. Sufficient condition

Assume f and $f^{(-1)}$ according to Assumptions 3.4, assume f to be convex on $[0, d]$, and let $\oplus$ be given according to (2).

Lemma 5.1. *If, for some $a, b \in [0, d]$, we have*

$$\forall x \in [0, 1] : \quad \frac{f(ax)}{f(a)} \;\ge\; \frac{f(bx)}{f(b)}$$

then $\Lambda_a \le \Lambda_b$.

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Proof. Observe that $(x, y) \in \frac{1}{a}\Lambda_a$ resp. $(x, y) \in \frac{1}{b}\Lambda_b$ if, and only if,

$$\begin{aligned} (ax, ay) \in \Lambda_a \quad &\text{resp.} \quad (bx, by) \in \Lambda_b, \\ f(ax) + f(ay) \leq f(a) \quad &\text{resp.} \quad f(bx) + f(by) \leq f(b), \\ \frac{f(ax)}{f(a)} + \frac{f(ay)}{f(a)} \leq 1 \quad &\text{resp.} \quad \frac{f(bx)}{f(b)} + \frac{f(by)}{f(b)} \leq 1. \end{aligned}$$

Hence $(x, y) \in \frac{1}{a}\Lambda_a$ implies $(x, y) \in \frac{1}{b}\Lambda_b$. ■

The function f is said to be, for a given $k \in [0, d]$, *k-subscalable* on $[0, d]$ if

$$\forall a, b \in [0, d], x \in [0, 1], b - a \geq k: \quad \frac{f(ax)}{f(a)} \geq \frac{f(bx)}{f(b)}.$$

For $a, b \in [0, d]$ we define $f_\downarrow^{a,b}\colon [0, a] \to [0, f(a)]$ by

$$f_\downarrow^{a,b}(x) = \frac{f(a)}{f(b)} f\left(\frac{b}{a}x\right).$$

Apparently, $f_\downarrow^{a,b}$ is an increasing bijection and f is k-subscalable on $[0, d]$ if, and only if,

$$\forall a, b \in [0, d], x \in [0, a], b - a \geq k: \quad f_\downarrow^{a,b}(x) \leq f(x).$$

Lemma 5.2. *If f is, for some $k \in [0, d]$, k-subscalable on $[0, d]$ then*

$$\forall a, b \in [0, \infty[, \quad a \geq k, \quad b \geq k: \quad \Lambda_a + \Lambda_b \subseteq \Lambda_{a+b}.$$

Proof. If $a + b \geq d$ then apparently $\Lambda_a + \Lambda_b \subseteq \Lambda_{a+b}$ since $\Lambda_{a+b} = [0, \infty]^2$. If $a + b < d$ then, by the definition of k-subscalability and by Lemma 5.1, we have $\Lambda_a \leq \Lambda_{a+b}$ and $\Lambda_b \leq \Lambda_{a+b}$. Lemma 4.3 finishes the proof. ■

The function f is said to be, for a given $k \in [0, d]$, *linear on* $[0, k]$ if there is $r \in]0, \infty[$ such that $f(x) = rx$ for every $x \in [0, k]$. Note that we admit also the extreme case $k = 0$ when no linearity is required. We denote

$$\Delta_a = \{(x, y) \in [0, \infty]^2 \mid x + y \leq a\}.$$

Clearly, if f is linear on $[0, k]$ then $\Lambda_a = \Delta_a$ for every $a \in [0, k]$.

Lemma 5.3. *For every $a, b \in [0, \infty[$ and for every $(x, y) \in \Lambda_b$ we have $(x + a, y) \in \Lambda_{a+b}$ and $(x, y + a) \in \Lambda_{a+b}$.*

Proof. If $a + b \geq d$ then the proof is apparent since $\Lambda_{a+b} = [0, \infty]^2$. Assume $a + b < d$. We prove $(x + a, y) \in \Lambda_{a+b}$; the proof of $(x, y + a) \in \Lambda_{a+b}$ is analogous. From $(x, y) \in \Lambda_b$ we have $f(x) + f(y) \leq f(b)$; hence $x \leq b$. Since f is strictly increasing and convex on $[0, d]$ we have $f(x + a) - f(x) \leq f(b + a) - f(b)$. The proof is concluded by summing these two inequalities. ■

Lemma 5.4. *For every $a, b \in [0, \infty[$ we have $\Delta_a + \Lambda_b \subseteq \Lambda_{a+b}$.*

Proof. If $a + b \geq d$ then apparently $\Delta_a + \Lambda_b \subseteq \Lambda_{a+b}$ since $\Lambda_{a+b} = [0, \infty]^2$. Assume $a + b < d$. Since f is convex on $[0, d]$, Λ_{a+b} is a convex set. Take any $(x, y) \in \Lambda_b$. Then $\Delta_a + (x, y)$ is a triangle with the vertices (x, y), $(x + a, y)$, and $(x, y + a)$. According to Lemma 5.3, all these vertices are contained in Λ_{a+b} as well as the whole triangle. ■

Lemma 5.5. *If f is, for some $k \in [0, d]$, linear on $[0, k]$ then*

$$\forall a, b \in [0, \infty[, \quad a \leq k \ \text{ or } \ b \leq k: \quad \Lambda_a + \Lambda_b \subseteq \Lambda_{a+b}.$$

Proof. The lemma is a corollary of Lemma 5.4. ■

The new sufficient condition now follows.

Theorem 5.6. *Consider functions f and $f^{(-1)}$, which comply with Assumptions 3.4, such that f is, for some $k \in [0, d]$,*

- *convex on* $[0, d]$,
- *k-subscalable on* $[0, d]$,
- *linear on* $[0, k]$.

Then f solves the generalized Mulholland inequality.

Proof. The proof is done invoking Lemma 5.2 and Lemma 5.5. ■

The introduced sufficient condition is not weaker than the one presented by Theorem 3.5 as the following proposition states.

Proposition 5.7. *The assumptions of Theorem 3.5 imply the assumptions of Theorem 5.6.*

Proof. We are going to show that, under the assumptions of Theorem 3.5, f is both 0-subscalable on $[0, d]$ and linear on the one-point interval $[0, 0]$. The latter statement is apparent, let us focus on the first one.

The function f is geometrically convex on $]0, d[$ if, and only if, the function $F\colon \left]-\infty, \log d\right[\to \left]-\infty, \log e\right[$, $F = \log \circ f \circ \exp$, is convex. Observe that F is an increasing bijection. Therefore, it is convex if, and only if,

$$F(A) - F(A + X) \quad \leq \quad F(B) - F(B + X)$$

for every $X \in]-\infty, 0[$ and $A, B \in \left]-\infty, \log d\right[$ such that $A \leq B$. Using the substitutions $a = \exp A$, $b = \exp B$, and $x = \exp X$ we obtain

$$\forall a, b \in [0, d]\,, \quad x \in [0, 1]\,, \quad a \leq b\colon \qquad \frac{f(ax)}{f(a)} \geq \frac{f(bx)}{f(b)}\,.$$

The latter formula represents the 0-subscalability of f. ■

The introduced sufficient condition is even strictly stronger than the one presented by Theorem 3.5. In order to prove this, we are going to present a parametric class of functions which comply with the assumptions of Theorem 5.6 but not with the assumptions of Theorem 3.5.

Example 5.8. Let $s \in \left[\frac{1}{2}, 1\right[$ and $r \in]0, s[$ be a pair of parameters. Function $g_{r,s}\colon [0, \infty] \to [0, \infty]$ is defined, for $x \in [0, \infty]$, by

$$g_{r,s}(x) = \begin{cases} \frac{r}{s} x & \text{if } x \in [0, s[\,, \\ \frac{1-r}{1-s} x - \frac{s-r}{1-s} & \text{if } x \in [s, 1[\,, \\ 1 & \text{if } x \in [1, \infty]\,. \end{cases}$$

Observe that $g_{r,s}$ is a function convex on $[0, 1]$ which complies with Assumptions 3.4 with $d = e = 1$.

Lemma 5.9. *The function $g_{r,s}$ is not geometrically convex on $]0, 1[$ for any admissible pair (r, s).*

Proof. If $g_{r,s}$ is geometrically convex on $]0, 1[$ then $G_{r,s} = \log \circ g_{r,s} \circ \exp$ is convex on $]-\infty, 0[$. For $x \in \left]\log s, 0\right[$ we have

$$G_{r,s}(x) = \log\left(\frac{1-r}{1-s} e^x - \frac{s-r}{1-s}\right),$$

$$G'_{r,s}(x) = \frac{\frac{1-r}{1-s}\mathrm{e}^x}{\frac{1-r}{1-s}\mathrm{e}^x - \frac{s-r}{1-s}},$$
$$G''_{r,s}(x) = -\frac{\frac{s-r}{1-s}\frac{1-r}{1-s}\mathrm{e}^x}{\left(\frac{1-r}{1-s}\mathrm{e}^x - \frac{s-r}{1-s}\right)^2}.$$

However, as we can see, $G''_{r,s} < 0$ on the whole $\left]\log s, 0\right[$. ■

Lemma 5.10. *The function $g_{r,s}$ is s-subscalable on* $[0, 1]$ *and linear on* $[0, s]$.

Proof. The linearity on $[0, s]$ is apparent. To show the s-subscalability on $[0, 1]$, take $a, b \in [0, 1]$ such that $b - a \geq s$. Observe that, necessarily, $a \in [0, s]$ and $b \in [s, 1]$. We need to show that

$$g_{\downarrow}^{a,b}(x) \quad = \quad \frac{g_{r,s}(a)}{g_{r,s}(b)} g_{r,s}\left(\frac{b}{a}x\right) \quad \leq \quad g_{r,s}(x)$$

holds for every $x \in [0, a]$. This is however true since $g_{\downarrow}^{a,b}(0) = g_{r,s}(0)$, $g_{\downarrow}^{a,b}(a) = g_{r,s}(a)$, $g_{\downarrow}^{a,b}(x)$ is a convex function, and $g_{r,s}(x)$ is a linear function on $[0, a]$. ■

6. Necessary condition

By Theorem 4.1, we have obtained a reformulation of the generalized Mulholland inequality which is quantified by two variables only. This reformulation has, moreover, the following geometric interpretation. Let us take $a, b \in [0, \infty]$ such that $a + b < d$. If we shift the level cut Λ_a such that its bottom-left corner coincides with the level set L_b then this shifted level cut must remain contained in the level cut Λ_{a+b}, i.e., "below" the level set L_{a+b}; see an illustration in Fig. 1. If $a + b \geq d$ then nothing is required. This geometric interpretation gives us an inspiration for a necessary condition based on directional derivatives.

Let $\varphi\colon [0, \infty]^2 \to [0, \infty]$ be a function of two variables. By $\frac{\partial\varphi}{\partial\mathbf{w}}(x, y)$ we denote, if it is defined, the directional derivative of φ along a given positive unit vector $\mathbf{w} = (w_1, w_2)$, $w_1, w_2 \in [0, 1]$, $\sqrt{w_1^2 + w_2^2} = 1$, at a given point $(x, y) \in [0, \infty[^2$. Let us recall the definition which is

$$\frac{\partial\varphi}{\partial\mathbf{w}}(x, y) \quad = \quad \lim_{t\to 0_+} \frac{\varphi\big((x, y) + t\mathbf{w}\big) - \varphi(x, y)}{t}.$$

The partial derivative of φ with respect to a variable t we denote by $\frac{\partial\varphi}{\partial t}$.

Lemma 6.1. *Assume f and $f^{(-1)}$ according to Assumptions 3.4 and let $\oplus$ be the pseudo-addition generated by f according to* (2).

If f and $f^{(-1)}$ satisfy the generalized Mulholland inequality then, for every positive unit vector $\mathbf{w} = (w_1, w_2)$, $w_1, w_2 \in [0, 1]$, $\sqrt{w_1^2 + w_2^2} = 1$, and for every point $(x, y) \in [0, \infty[^2$ we have

$$\frac{\partial\oplus}{\partial\mathbf{w}}(0, 0) \quad \geq \quad \frac{\partial\oplus}{\partial\mathbf{w}}(x, y)$$

if the corresponding derivatives are defined.

Proof. Since $\oplus$ is continuous and increasing in both variables, we have $\frac{\partial\oplus}{\partial\mathbf{w}}(x, y) \geq 0$ in every point $(x, y) \in [0, \infty[^2$ where it is defined. If $(x, y) \notin \operatorname{supp}\oplus$ then $\frac{\partial\oplus}{\partial\mathbf{w}}(x, y) = 0$ and the conclusion of the lemma is apparent. Suppose $(x, y) \in \operatorname{supp}\oplus$. By the definition of directional derivative we have

$$\frac{\partial\oplus}{\partial\mathbf{w}}(0, 0) = \lim_{t\to 0_+} \frac{(0 + t\,w_1) \oplus (0 + t\,w_2) - (0 \oplus 0)}{t}, \tag{3}$$
$$\frac{\partial\oplus}{\partial\mathbf{w}}(x, y) = \lim_{t\to 0_+} \frac{(x + t\,w_1) \oplus (y + t\,w_2) - (x \oplus y)}{t}. \tag{4}$$

Since $0 \oplus 0 = 0$, (3) can be simplified to

$$\frac{\partial \oplus}{\partial \mathbf{w}}(0,0) = \lim_{t \to 0_+} \frac{t\,w_1 \oplus t\,w_2}{t} \tag{5}$$

which is, actually, the right derivative of the expression $t\,w_1 \oplus t\,w_2$ according to t in zero:

$$\frac{\partial \oplus}{\partial \mathbf{w}}(0,0) = \frac{\partial}{\partial t}\Big[t\,w_1 \oplus t\,w_2\Big]_{t=0_+}. \tag{6}$$

Similarly, the right hand part of (4) is the right derivative of the expression $(x + t\,w_1) \oplus (y + t\,w_2) - (x \oplus y)$ according to t in zero. Since, in such a case, $x \oplus y$ is considered as a constant, we obtain:

$$\begin{aligned}\frac{\partial \oplus}{\partial \mathbf{w}}(x,y) &= \frac{\partial}{\partial t}\Big[(x + t\,w_1) \oplus (y + t\,w_2) - (x \oplus y)\Big]_{t=0_+} \\ &= \frac{\partial}{\partial t}\Big[(x + t\,w_1) \oplus (y + t\,w_2)\Big]_{t=0_+}.\end{aligned} \tag{7}$$

By ($\mathrm{GMI}_\oplus$) we have, for every $t \geq 0$,

$$(x \oplus y) + (t\,w_1 \oplus t\,w_2) \geq (x + t\,w_1) \oplus (y + t\,w_2).$$

Observe that both sides of the inequality are equal when $t = 0$. Therefore, their right partial derivatives with respect to t at zero must satisfy the same:

$$\frac{\partial}{\partial t}\Big[(x \oplus y) + (t\,w_1 \oplus t\,w_2)\Big]_{t=0_+} \geq \frac{\partial}{\partial t}\Big[(x + t\,w_1) \oplus (y + t\,w_2)\Big]_{t=0_+}.$$

As $x \oplus y$ in the left-hand side derivative plays the role of a constant, we obtain that (6) is greater than or equal to (7) which finishes the proof. ■

7. Compositions of solutions of the generalized Mulholland inequality

In this section we are going to show that the set of functions that solve the generalized Mulholland inequality is not closed with respect to compositions. This fact has been, actually, already proven by a previous paper [13] in which case, however, the counter-example has been based on functions that were bijections of $[0, \infty]$. Here we are going to show that a similar counter-example can be made also on those solutions of the generalized Mulholland inequality that are *not* bijections of $[0, \infty]$. As a corollary of this fact, the next section will state that the dominance relation is *not* transitive on the set of nilpotent t-norms.

Observation 7.1. *It can be easily checked that the function* $h\colon [0,\infty] \to [0,\infty]$, *defined for* $p > 1$ *by*

$$h(x) = \begin{cases} x^p & \text{if } x \in [0,1]\,, \\ 1 & \text{if } x > 1, \end{cases}$$

is both convex and geometrically convex on $]0, 1[$. *Hence by Theorem 3.5 it satisfies the generalized Mulholland inequality.*

Lemma 7.2. *Assume, for some fixed* $s \in \left]\frac{1}{2}, 1\right[$ *and* $r \in\,]0, s[$, *the function* $g_{r,s}$ *from Example 5.8; denote it just by* g. *Assume further, for some fixed* $p > 1$, *the function* h *from Observation 7.1. While both* g *and* h *satisfy the generalized Mulholland inequality,* $f = g \circ h$ *does not.*

Proof. Let $\oplus$ be the pseudo-addition generated by f according to (2). We are going to show that it does not satisfy ($\mathrm{GMI}_\oplus$) by showing that the conclusion of Lemma 6.1 is violated.

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Observe that we have, for $x, y \in [0, \infty]$,

$$f(x) = \begin{cases} \frac{r}{s} x^p & \text{if } x \in \left[0, s^{\frac{1}{p}}\right[, \\ \frac{1-r}{1-s} x^p - \frac{s-r}{1-s} & \text{if } x \in \left[s^{\frac{1}{p}}, 1\right[, \\ 1 & \text{if } x \geq 1, \end{cases}$$

$$f^{(-1)}(x) = \begin{cases} \left(\frac{s}{r} x\right)^{\frac{1}{p}} & \text{if } x \in [0, r[\,, \\ \left(\frac{1-s}{1-r} x + \frac{s-r}{1-r}\right)^{\frac{1}{p}} & \text{if } x \in [r, 1[\,, \\ 1 & \text{if } x \geq 1, \end{cases}$$

$$x \oplus y = \begin{cases} (x^p + y^p)^{\frac{1}{p}} & \text{if } x, y \geq 0 \text{ and } x^p + y^p < s, \\ \left(x^p + y^p - \frac{s-r}{1-r}\right)^{\frac{1}{p}} & \text{if } x, y \geq s^{\frac{1}{p}} \text{ and } \left(x^p + y^p - \frac{s-r}{1-r}\right)^{\frac{1}{p}} < 1, \\ \ldots & \text{(we omit the rest).} \end{cases}$$

Let us take the unit vector $\mathbf{w} = (1/\sqrt{2}, 1/\sqrt{2})$ and observe that $\oplus$ is a smooth function when restricted to $\left\{(x, y) \in [0, \infty]^2 \mid x^p + y^p < s\right\}$. Therefore the directional derivative of $\oplus$ along $\mathbf{w}$ at $(0, 0)$ exists and is given by (confer with (6)):

$$\begin{aligned} \frac{\partial \oplus}{\partial \mathbf{w}}(0,0) &= \frac{\partial}{\partial t}\left[\frac{t}{\sqrt{2}} \oplus \frac{t}{\sqrt{2}}\right]_{t=0_+} \\ &= \frac{\partial}{\partial t}\left[\left(\left(\frac{t}{\sqrt{2}}\right)^p + \left(\frac{t}{\sqrt{2}}\right)^p\right)^{\frac{1}{p}}\right]_{t=0_+} = \frac{2^{\frac{1}{p}}}{\sqrt{2}}. \end{aligned}$$

Observe that the area $\left\{(x, y) \in \left[s^{\frac{1}{p}}, \infty\right]^2 \mid \left(x^p + y^p - \frac{s-r}{1-r}\right)^{\frac{1}{p}} < 1\right\}$ in the above definition of $\oplus$ is non-empty. Indeed, since $s < 1$ and $r < \frac{1}{2}$ we can derive

$$\begin{aligned} 2r &< 1, \\ 2r(1-s) &< 1 - s, \\ s + r - 2rs &< 1 - r, \\ 2s - 2rs + r - s &< 1 - r, \\ 2s - \frac{s-r}{1-r} &< 1, \\ \left(\left(s^{\frac{1}{p}}\right)^p + \left(s^{\frac{1}{p}}\right)^p - \frac{s-r}{1-r}\right)^{\frac{1}{p}} &< 1. \end{aligned}$$

Furthermore, when restricted to this area, $\oplus$ is obviously a smooth function.

Therefore the directional derivative of $\oplus$ along $\mathbf{w}$ at $(s^{\frac{1}{p}}, s^{\frac{1}{p}})$ exists, as well, and is given by (confer with (7)):

$$\begin{aligned} \frac{\partial \oplus}{\partial \mathbf{w}}\left(s^{\frac{1}{p}}, s^{\frac{1}{p}}\right) &= \frac{\partial}{\partial t}\left[\left(s^{\frac{1}{p}} + \frac{t}{\sqrt{2}}\right) \oplus \left(s^{\frac{1}{p}} + \frac{t}{\sqrt{2}}\right)\right]_{t=0_+} \\ &= \frac{\partial}{\partial t}\left[\left(\left(s^{\frac{1}{p}} + \frac{t}{\sqrt{2}}\right)^p + \left(s^{\frac{1}{p}} + \frac{t}{\sqrt{2}}\right)^p - \frac{s-r}{1-r}\right)^{\frac{1}{p}}\right]_{t=0_+} \\ &= \left[\frac{2}{\sqrt{2}}\left(2\left(s^{\frac{1}{p}} + \frac{t}{\sqrt{2}}\right)^p - \frac{s-r}{1-r}\right)^{\frac{1-p}{p}}\left(s^{\frac{1}{p}} + \frac{t}{\sqrt{2}}\right)^{p-1}\right]_{t=0_+} \end{aligned}$$

$$= \frac{2}{\sqrt{2}}\left(2s - \frac{s-r}{1-r}\right)^{\frac{1-p}{p}} s^{\frac{p-1}{p}} = \frac{2}{\sqrt{2}}\left(1 + \frac{r}{s}\cdot\frac{1-s}{1-r}\right)^{\frac{1-p}{p}}.$$

Since the expression $\frac{r}{s}\cdot\frac{1-s}{1-r}$ is always strictly greater than zero and since $p > 1$, we have

$$\frac{\partial\oplus}{\partial \mathbf{w}}(0,0) < \frac{\partial\oplus}{\partial \mathbf{w}}\left(s^{\frac{1}{p}}, s^{\frac{1}{p}}\right).$$

This however, together with Lemma 6.1, means that $\oplus$ does not satisfy ($\mathrm{GMI}_\oplus$); hence f does not satisfy the generalized Mulholland inequality. ■

8. Transitivity of dominance on nilpotent t-norms

The result of Lemma 7.2 gives us the following corollary.

Theorem 8.1. *The relation of dominance is not transitive on the set of nilpotent t-norms.*

Proof. Define functions g, h, and $g\circ h$ as in the proof of Lemma 7.2. Let t_2 be the additive generator of any nilpotent t-norm such that $t_2(0) = 1$; denote the t-norm generated by t_2 as $*_2$. Define $t_1 = g\circ t_2$ and $t_3 = h^{(-1)}\circ t_2$ and observe that both t_1 and t_3 are generators of nilpotent t-norms; denote these t-norms by $*_1$ and $*_3$, respectively. We have $t_1 \circ t_2^{(-1)} = g$, hence $*_1$ dominates $*_2$, and we have $t_2\circ t_3^{(-1)} = h$, hence $*_2$ dominates $*_3$. However, since $t_1\circ t_3^{(-1)} = g\circ h$, which by Lemma 7.2 does not solve the generalized Mulholland inequality, $*_1$ does not dominate $*_3$. ■

Thus we can construct a counter-example of three nilpotent t-norms that violate the transitivity of the dominance relation. The recent result [14] has shown that we can do the same for strict t-norms. Hence we can claim that the dominance relation is generally not transitive on continuous Archimedean t-norms.

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