Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Spectrum of Schrödinger operators via the method of multipliers

Spektrum Schrödingerových operátorů metodou násobitelů

Master's Thesis

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Prohlášení:

Prohlašuji, že jsem svou diplomovou práci vypracoval samostatně a použil jsem pouze podklady (literaturu, projekty, SW atd.) uvedené v přiloženém seznamu.

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V Praze dne Vojtěch Šmíd

Název práce:
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Abstrakt:

Studujeme spektrum Schrödingerových operátorů pomocí metody násobitelů. Nacházíme postačující podmínky pro absenci bodového spektra pro operátor s komplexním potenciálem na poloprostoru s různými hraničními podmínkami a pro operátor s metrikou. Uvádíme stručný úvod do metody násobitelů.
Klíčová slova: metoda násobitelů, Schrödingerův operátor, spektrální analýza, sektoriální forma, nesamosdružený operátor

Title:
Spectrum of Schrödinger operators via the method of multipliers

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Abstract: We study the spectrum of Schrödinger operators using the method of multipliers. We

find sufficient conditions for the absence of the point spectrum of an operator with complex potential on a half-space with various boundary conditions and for an operator with a metric. We give a brief introduction to the method of multipliers
Key words: method of multipliers, Schrödinger operator, spectral analysis, sectorial form, non-selfadjoint operator

Contents

Chapter 1

Introduction

Let H_V be a Schrödinger operator in the Hilbert space $L^2(\Omega)$ where Ω is a domain in $\mathbb{R}^d$. We consider that the potential V is complex thus the operator H_V is non-self-adjoint. We study the spectrum of such operator and discuss how it can be affected by imposing various boundary conditions. Namely, we will find suitable conditions on the potential V under which the point spectrum is empty. In order to do so, we study the equation for eigenvalues

$$\Delta u + \lambda u = Vu \quad \text{in} \quad \Omega \subset \mathbb{R}^d, \tag{1.1}$$

where we assume that its solution u satisfies one of the following boundary conditions:

C1 Dirichlet: $u = 0$ on $\partial\Omega$,

C2 Neumann: $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$,

C3 Robin: $\alpha u + \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$,

where $\alpha : \partial\Omega \to [0, +\infty)$ satisfies $\nabla'\alpha \cdot x' \leq 0$ for $x' := (x_2, \cdots, x_d)$ and $\nabla'\alpha := (\frac{\partial\alpha}{\partial x_2}, \cdots, \frac{\partial\alpha}{\partial x_d})$. We will denote the operator $H_V^\iota = -\Delta + V$, where ι represents the boundary conditions which functions from its domain satisfy. We set ι as D for Dirichlet, N for Neumann and R for Robin boundary conditions.

More specifically, we are interested in the solutions in the half-space $\mathbb{R}^d_+$ which we define as

$$\mathbb{R}^d_+ := \{x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d | x_1 \geq 0\}.$$

This is an example of very simple domain. In fact, its boundary is $\partial\mathbb{R}^d = \mathbb{R}^{d-1}$. It is the first step towards our main goal which are waveguides in $\mathbb{R}^d$.

The inspiration for this thesis comes from the paper [4]. Here the the whole three-dimensional space $\Omega = \mathbb{R}^3$ was considered and the following condition on the potential V was derived

$$\exists a < 1, \qquad \forall \psi \in H^1(\mathbb{R}^d), \qquad \int_{\mathbb{R}^d} |V||\psi|^2 \leq a \int_{\mathbb{R}^d} |\nabla\psi|^2. \tag{1.2}$$

Apart from guaranteeing the emptiness of the point spectrum it also implies that the spectrum of an Schrödinger operator with such potential coincides with that of the free Hamiltonian $H_0 = -\Delta$

which is $\sigma(H_0) = \sigma_c(H_0) = [0, +\infty)$. The free Hamiltonian can be associated with the quadratic form

$$h_0[\psi] := \int_{\mathbb{R}^d} |\nabla\psi|^2, \qquad D(h_0) := H^1(\mathbb{R}^d). \tag{1.3}$$

Analogously, the potential V can be associated with the form

$$v[\psi] = \int_{\mathbb{R}^d} V|\psi|^2, \quad D(v) = \left\{\psi \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} |V||\psi|^2 < \infty\right\}$$

In another words, the condition (1.1) means that the potential V is form-subordinated to H_0 with the subordination bound less than one. This feature allows us to define the corresponding Schrödinger operator as an m-sectorial operator by associating it with the form $h_V = h_0 + v$ via the First Representation Theorem which can be found in [5].

However, the proof of this result is based on Birman-Schwinger principle and it fails in higher dimensions. In order to generalize this result to dimensions higher than three, the authors had to utilize a different approach which was the method of multipliers. This technique was developed by Morawetz in [6] to study the non-linear Klein Gordon equation. Over the course of time, this method was used in various contexts as in dispersive equations, kinetic equations, Helmholtz equatoin, etc. We refer to papers [2,7,8]. In the setting corresponding to our problem this technique was used apart from our main inspiration [4] also in [1] where electromagnetic potential in exteriors of domains were studied. Another recent paper [3] uses this method to prove the absence of eigenvalues for magnetic Schrödinger operator in two dimensions. Our main interest lies in the following result from the article [4].

Theorem 1. *Let $d \geq 3$ and suppose*

$$\forall \psi \in H^1(\mathbb{R}^d), \qquad \int_{\mathbb{R}^d} |x|^2 |V(x)|^2 |\psi(x)|^2 \mathrm{d}x \leq \Lambda \int_{\mathbb{R}^d} |\nabla\psi(x)|^2 \mathrm{d}x,$$

where Λ satisfies

$$\frac{2(2d-3)}{d-2}\Lambda + \frac{\sqrt{2}}{\sqrt{d-2}}\Lambda^{\frac{3}{2}} < 1.$$

Then $\sigma_p(H_V) = \emptyset$.

It can be shown that the condition in this theorem also implies the form-subordination (1.2). However, the article also contains a generalization of this result for which the form-subordination does not hold. To this extent, we define the functions $f_\pm$ as

$$f_\pm(x) := \max\{\pm f(x), 0\}. \tag{1.4}$$

Using this formalism, we can rewrite the form h_V as

$$h_V = h_V^{(1)} + h_V^{(2)},$$

where $h_V^{(1)}[\psi] := \int_{\mathbb{R}^d} |\nabla\psi|^2 + \int_{\mathbb{R}^d} (\Re V)_+ |\psi|^2$ and $H_V^{(2)}[\psi] := -\int_{\mathbb{R}^d} (\Re V)_- |\psi|^2 + \int_{\mathbb{R}^d} \Im V |\psi|^2$. Now, we assume that only $(\Re V)_-$ and $\Im V$ are form-subordinated to H_0 with the subordination bound less than one. Again, the form h_V gives rise to an m-sectorial operator H_V. For this case the authors derive the following result

Theorem 2. *Let $d \geq 3$ and assume that there exist non-negative numbers b_1, b_2 and b_3 satisfying*

$$b_1^2 < 1 - \frac{2b_3}{d-2}, \qquad b_2^2 + 2b_3 + b_3^{\frac{1}{2}} \left(\frac{2}{d-2}\right)^{\frac{3}{2}} < 1$$

such that for all $\psi \in D(h_V)$

$$\int_{\mathbb{R}^d} (V_1)_- |\psi|^2 \leq b_1^2 \int_{\mathbb{R}^d} |\nabla\psi|^2,$$

$$\int_{\mathbb{R}^d_+} [\partial_r (|x|V_1)]_+ |\psi|^2 \leq b_2^2 \int_{\mathbb{R}^d} |\nabla\psi|^2,$$

$$\int_{\mathbb{R}^d} |x|^2 |V_2|^2 |\psi|^2 \leq b_3^2 \int_{\mathbb{R}^d} |\nabla\psi|^2,$$

where $\partial_r f(x) := \frac{x}{|x|} \cdot \nabla f(x)$. Then $\sigma_p(H_V) = \emptyset$.

Our main goal is to generalize the Theorems 1 and 2 to the half-space. In fact, we will use the same technique and the main difference is appearance of integrals over the boundary. We also derive results for real potentials and thus self-adjoint operators H_V which can be obtained by more straightforward method. For the moment, we give only formal results. This thesis gives only formal results therefore we merely denote the space of test functions as $\mathcal{D}(\mathbb{R}^d_+)$ and abuse the notation and use it for all the cases.

Our first result is an analogue to the Theorem 1 and leaves the conditions unchanged for the boundary conditions **C1** to **C3**.

Theorem 3. *Let $d \geq 3$ and assume that*

$$\forall \psi \in \mathcal{D}\left(\mathbb{R}^d_+\right) \qquad \int_{\mathbb{R}^d_+} |x|^2 |V(x)|^2 |\psi(x)|^2 \mathrm{d}x \leq \Lambda \int_{\mathbb{R}^d_+} |\nabla\psi(x)|^2 \mathrm{d}x, \tag{1.5}$$

*where Λ satisfies (2.31). Then $\sigma_p\left(H_V^\iota\right) = \emptyset$, where ι represents the boundary conditions **C1** to **C3**.*

The analogue to the Theorem 2 also leaves the conditions unchanged and we obtain

Theorem 4. *Let $d \geq 3$ and assume that there exist non-negative numbers b_1, b_2 and b_3 satisfying*

$$b_1^2 < 1 - \frac{2b_3}{d-2}, \qquad b_2^2 + 2b_3 + b_3^{\frac{1}{2}} \left(\frac{2}{d-2}\right)^{\frac{3}{2}} < 1 \tag{1.6}$$

such that for all $\psi \in \mathcal{D}(\mathbb{R}^d_+)$

$$\int\limits_{\mathbb{R}^d_+} (V_1)_- \, |\psi|^2 \leq b_1^2 \int\limits_{\mathbb{R}^d_+} |\nabla\psi|^2, \tag{1.7}$$

$$\int\limits_{\mathbb{R}^d_+} \left[\partial_r \left(|x| V_1\right)\right]_+ |\psi|^2 \leq b_2^2 \int\limits_{\mathbb{R}^d_+} |\nabla\psi|^2, \tag{1.8}$$

$$\int\limits_{\mathbb{R}^d_+} |x|^2 |V_2|^2 |\psi|^2 \leq b_3^2 \int\limits_{\mathbb{R}^d_+} |\nabla\psi|^2, \tag{1.9}$$

$$\tag{1.10}$$

where $\partial_r f(x) := \frac{x}{|x|} \cdot \nabla f(x)$*. Then* $\sigma_p(H_V^\iota) = \emptyset$*, where* ι *represents the boundary conditions* ***C1*** *to* ***C3****.*

An example of potential satisfying these conditions for $\Im V = 0$ is the large class of repulsive potentials of Coulomb-type interaction $V(x) = c|x|^{-1}$ with $c > 0$. However, these potential do not fulfil the form-subordination. We say that u is a solution of (1.1) if $u \in H^1(\Omega)$ and

The second part of this thesis generalizes [4] by introducing a new operator. Let $a : \mathbb{R}^d \to \mathbb{C}^{d,d}$ be a measurable function whose real part satisfies the uniform elliptic condition

$$\operatorname{Re} a = \frac{a + a^*}{2} \geq cI > 0. \tag{1.11}$$

Note that this inequality compares hermitian matrices so in fact it means

$$\forall \xi \in C_0^\infty(\mathbb{R}^d, \mathbb{C}^d) \qquad 0 < c|\xi|^2 \leq \bar{\xi} \frac{a + a^*}{2} \xi.$$

This condition ensures that the following operator is elliptic

$$H\psi = -\nabla(a \nabla\psi) \tag{1.12}$$

We assume no potential and as can be seen in Theorem 9 the necessary condition is imposed on the function a.

Chapter 2

Half-space

In this chapter, we prove the Theorems 3 and 4 using the method of multipliers. We also derive analogous results for real potentials. In this case the operator H_V is self-adjoint and the proofs are therefore easier. These results are described in Theorems 5 and 6.

The structure of this chapter is as follows. In the first section we establish lemmas which are essential for the proofs of the main theorems. We give separate lemmas for self-adjoint and non-self-adjoint cases. In the sections 2.2 and 2.3 we prove the main results for real and complex potentials respectively. Finally, in the last section of this chapter we discuss the results for complex Robin boundary conditions.

2.1 Fundamental Lemmas

In this section we establish the essential lemmas for the proofs of the main results. We consider a general domain $\Omega \subset \mathbb{R}^d$. The method of multipliers is based on the equation (1.1) which we multiply by a suitable test function v and integrate over the domain to obtain

$$\forall v \in \mathcal{D}(\Omega), \quad \int\limits_{\Omega} \nabla u \cdot \nabla \bar{v} + \lambda \int\limits_{\Omega} u\bar{v} + \int\limits_{\partial\Omega} \frac{\partial u}{\partial n}\bar{v} = \int\limits_{\Omega} V u\bar{v}. \tag{2.1}$$

This equation can be regarded as a weak formulation of the problem (1.1). We proceed by choosing suitable test functions v and generate elementary identities which are described in the first lemma.

Lemma 1. *Let u be a solution of (1.1) and let $G_1, G_2, G_3 : \mathbb{R}^d \to \mathbb{R}$ be three smooth functions. Then the following identities hold:*

$$\lambda_1 \int\limits_{\Omega} G_1|u|^2 - \int\limits_{\Omega} G_1|\nabla u|^2 + \frac{1}{2}\int\limits_{\Omega} \Delta G_1|u|^2 + \Re\int\limits_{\partial\Omega} G_1\frac{\partial u}{\partial n}\bar{u} - \frac{1}{2}\int\limits_{\partial\Omega} |u|^2\frac{\partial G_1}{\partial n} = \Re\int\limits_{\Omega} G_1V|u|^2, \tag{2.2}$$

$$\lambda_2 \int\limits_{\Omega} G_2|u|^2 - \Im\int\limits_{\Omega} \nabla G_2 \cdot \bar{u}\nabla u + \Im\int\limits_{\partial\Omega} G_2\frac{\partial u}{\partial n}\bar{u} = \Im\int\limits_{\Omega} G_2V|u|^2, \tag{2.3}$$

$$\int_\Omega \nabla u\cdot\nabla^2 G_3\cdot\nabla\bar u-\frac14\int_\Omega \Delta^2 G_3|u|^2+\lambda_2\Im\int_\Omega \nabla G_3\cdot u\nabla\bar u-\Re\int_{\partial\Omega}\nabla G_3\cdot\nabla\bar u\frac{\partial u}{\partial n}$$
$$-\frac12\Re\int_{\partial\Omega}\Delta G_3\bar u\frac{\partial u}{\partial n}+\frac12\int_{\partial\Omega}|\nabla u|^2\frac{\partial G_3}{\partial n}+\frac14\int_{\partial\Omega}|u|^2\frac{\partial(\Delta G_3)}{\partial n}-\frac{\lambda_1}{2}\int_{\partial\Omega}|u|^2\frac{\partial G_3}{\partial n} \tag{2.4}$$
$$=-\Re\int_\Omega V\nabla G_3\cdot u\nabla\bar u-\frac12\Re\int_\Omega \Delta G_3 V|u|^2,$$

where n is the outward unit normal vector, ∇^2 is the Hessian matrix and $\Delta^2 := \Delta\Delta$ denotes the bi-Laplacian.

Proof. In order to prove the first identity, we choose $v = G_1 u$ in (2.1) which leads us to

$$-\int_\Omega G_1|\nabla u|^2-\int_\Omega \nabla G_1\cdot\bar u\nabla u+\lambda\int_\Omega G_1|u|^2+\int_{\partial\Omega}G_1\bar u\frac{\partial u}{\partial n}=\int_\Omega G_1 V|u|^2.$$

The real part of the second term can rewritten using the useful formula

$$2\Re(\bar u\nabla u)=\nabla|u|^2. \tag{2.5}$$

Then, we can use the first Green's identity on the real part of the second integral to obtain

$$-\frac12\int_\Omega \nabla G_1\cdot\nabla|u|^2=\frac12\int_\Omega \Delta G_1|u|^2-\frac12\int_{\partial\Omega}|u|^2\frac{\partial G_1}{\partial n}.$$

Thus, we get

$$-\int_\Omega G_1|\nabla u|^2-i\Im\int_\Omega \nabla G_1\cdot\bar u\nabla u+\frac12\int_\Omega \Delta G_1|u|^2-\frac12\int_{\partial\Omega}|u|^2\frac{\partial G_1}{\partial n}+\lambda\int_\Omega G_1|u|^2+\int_{\partial\Omega}G_1\bar u\frac{\partial u}{\partial n}$$
$$=\int_\Omega G_1 V|u|^2. \tag{2.6}$$

The identity (2.2) is then the real part of (2.6).

Analogously, choosing $v = G_2 u$ in (2.1) and taking the imaginary part of the resulting identity, we obtain (2.3).

Finally, the choice $v = 2\nabla G_3\cdot\nabla u + \Delta G_3 u$ in (2.1) leads us to

$$-2\int_\Omega \nabla u\cdot\nabla^2 G_3\cdot\nabla\bar u-2\int_\Omega \nabla G_3\cdot\nabla^2\bar u\cdot\nabla u-\int_\Omega \nabla(\Delta G_3)\cdot\nabla u\bar u-\int_\Omega \Delta G_3|\nabla u|^2$$
$$+2\lambda\int_\Omega \nabla G_3\cdot\nabla\bar u u+\lambda\int_\Omega \Delta G_3|u|^2+\int_{\partial\Omega}(2\nabla G_3\cdot\nabla\bar u+\Delta G_3\bar u)\frac{\partial u}{\partial n}$$
$$=2\int_\Omega V\nabla G_3\cdot\nabla\bar u u+\int_\Omega V\Delta G_3|u|^2.$$

Now, we use the identity (2.5) and the same identity for ∇u, which then reads $\nabla|\nabla u|^2 = 2\Re\nabla^2 u\cdot\nabla\bar{u}$, to get

$$
\begin{aligned}
&-2\int\limits_\Omega \nabla u\cdot\nabla^2 G_3\cdot\nabla\bar{u} - \int\limits_\Omega \nabla G_3\cdot\nabla|\nabla u|^2 - 2i\Im\int\limits_\Omega \nabla G_3\cdot\nabla^2\bar{u}\cdot\nabla u - \frac{1}{2}\int\limits_\Omega \nabla(\Delta G_3)\cdot\nabla|u|^2\\
&-i\Im\int\limits_\Omega \nabla(\Delta G_3)\cdot\nabla u\bar{u} - \int\limits_\Omega \Delta G_3|\nabla u|^2 + \lambda\int\limits_\Omega \nabla G_3\cdot\nabla|u|^2 + 2i\lambda\Im\int\limits_\Omega \nabla G_3\cdot\nabla\bar{u}u + \lambda\int\limits_\Omega \Delta G_3|u|^2\\
&+\int\limits_{\partial\Omega}(2\nabla G_3\cdot\nabla\bar{u} + \Delta G_3\bar{u})\frac{\partial u}{\partial n}\\
&= 2\int\limits_\Omega V\nabla G_3\cdot\nabla\bar{u}u + \int\limits_\Omega V\Delta G_3|u|^2.
\end{aligned}
$$

Using the first Green's formula and dividing the whole identity by -2, we arrive at

$$
\begin{aligned}
&\int\limits_\Omega \nabla u\cdot\nabla^2 G_3\cdot\nabla\bar{u} + i\Im\int\limits_\Omega \nabla G_3\cdot\nabla^2\bar{u}\cdot\nabla u - \frac{1}{4}\int\limits_\Omega \Delta^2 G_3|u|^2 + \frac{i}{2}\Im\int\limits_\Omega \nabla(\Delta G_3)\cdot\nabla u\bar{u}\\
&-i\lambda\Im\int\limits_\Omega \nabla G_3\cdot\nabla\bar{u}u - \int\limits_{\partial\Omega}\nabla G_3\cdot\nabla\bar{u}\frac{\partial u}{\partial n} - \frac{1}{2}\int\limits_{\partial\Omega}\Delta G_3\bar{u}\frac{\partial u}{\partial n} + \frac{1}{2}\int\limits_{\partial\Omega}|\nabla u|^2\frac{\partial G_3}{\partial n}\\
&+\frac{1}{4}\int\limits_{\partial\Omega}|u|^2\frac{\partial(\Delta G_3)}{\partial n} - \frac{\lambda}{2}\int\limits_{\partial\Omega}|u|^2\frac{\partial G_3}{\partial n}\\
&= -\int\limits_\Omega V\nabla G_3\cdot\nabla\bar{u}u - \frac{1}{2}\int\limits_\Omega V\Delta G_3|u|^2.
\end{aligned}
$$

The equation (2.4) is then the real part of this identity. □

In what follows, we will assume that the functions G_1, G_2, G_3 are radial, *i.e.* there exist smooth functions $g_1, g_2, g_3 : (0,\infty)\to\mathbb{R}$ such that $G_i(x) = g_i(|x|)$ for all $x\in\mathbb{R}^d$ and $i\in\{1,2,3\}$. In this case, we have

$$
\nabla G_i(x) = g_i'(|x|)\frac{x}{|x|},\quad \Delta G_i(x) = g_i''(|x|) + g_i'(|x|)\frac{d-1}{|x|},
$$

$$
\nabla^2 G_i(x) = g_i''(|x|)\frac{xx}{|x|^2} + \frac{g_i'(|x|)}{|x|}\left(I - \frac{xx}{|x|^2}\right),
$$

where xx denotes the dyadic product of x and x. In the following, for any $g:\mathbb{R}^d\to\mathbb{C}$ we denote by

$$
\partial_r g(x) := \frac{x}{|x|}\cdot\nabla g(x),\quad \nabla_\tau g(x) := \left(I - \frac{xx}{|x|^2}\right)\cdot\nabla g(x)
$$

the radial derivative and the angular gradient of g, respectively. Therefore, we can write

$$
|\nabla g|^2 = |\partial_r g|^2 + |\nabla_\tau g|^2. \tag{2.7}
$$

Naturally, if the identities (2.2), (2.3) and (2.4) hold then any combination of their multiples holds as well. We will use this property together with the radiality of the functions G_1, G_2 and G_3 to prove the essential identities in the following two lemmas. In the self-adjoint case it will be sufficient to combine only the identities (2.2) and (2.3) which will be described in the lemma 2. Including also the identity (2.4), we obtain a result which enables us to treat m-sectorial operators with complex potentials. This result is described in the Lemma 3.

Lemma 2. *Let u be the solution of* (1.1)*. Then the following identity holds:*

$$\begin{aligned} &\lambda_1 \int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 + 2\lambda_2 \Im \int_\Omega |x| u \partial_r \bar{u} \\ &+ \Re \int_{\partial\Omega} (1-d) \frac{\partial u}{\partial n} \bar{u} - 2\Re \int_{\partial\Omega} |x| \partial_r \bar{u} \frac{\partial u}{\partial n} + \int_{\partial\Omega} |\nabla u|^2 x \cdot n - \lambda_1 \int_{\partial\Omega} |u|^2 x \cdot n \\ &= \int_\Omega (1-d)\, V_1 |u|^2 - 2\Re \int_\Omega V |x| u \partial_r \bar{u}. \end{aligned} \tag{2.8}$$

Proof. The desired identity follows immediately from the combination of (2.2) with $G_1 = 1$ and (2.4) with $G_3 = |x|^2$. □

In the next lemma we use u to define a new function $u^-(x) := u(x) e^{-i \operatorname{sgn}(\lambda_2) \lambda_1^{\frac{1}{2}} |x|}$. The convenience of this formalism is that u^- has the same norm as u but considering that $|a - ib|^2 = |a|^2 + |b|^2 - 2\Im a \cdot \bar{b}$, we derive the identity for norm of its derivative which reads

$$\begin{aligned} |\nabla u^-|^2 &= \left| \nabla u - i \operatorname{sgn}(\lambda_2) \lambda_1^{\frac{1}{2}} |^2 \frac{x}{|x|} u \right|^2 \\ &= |\nabla u|^2 + \lambda_1 |u|^2 - 2 \operatorname{sgn}(\lambda_2) \lambda_1^{\frac{1}{2}} \Im\left(\bar{u} \partial_r u\right). \end{aligned} \tag{2.9}$$

In the proofs of the main results it is important to know which terms are positive. Using the identity (2.9), we can include the term involving $\operatorname{sgn}(\lambda_2)$, whose sign we are not able to determine, in a non-negative term.

Lemma 3. *Let u be a solution of* (1.1)*. Then the following identity holds:*

$$\begin{aligned} &\int_\Omega |\nabla u^-|^2 + \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \int_\Omega |x| |\nabla u^-|^2 - \frac{d-1}{2} \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \int_\Omega \frac{|u|^2}{|x|} + 2\lambda_1^{\frac{1}{2}} \operatorname{sgn}(\lambda_2) \Im \int_{\partial\Omega} |x| \frac{\partial u}{\partial n} \bar{u} - 2\Re \int_{\partial\Omega} |x| \partial_r \bar{u} \frac{\partial u}{\partial n} \\ &- \Re \int_{\partial\Omega} (d-1) \bar{u} \frac{\partial u}{\partial n} - \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \Re \int_{\partial\Omega} |x| \frac{\partial u}{\partial n} \bar{u} + \int_{\partial\Omega} |\nabla u|^2 x \cdot n - \lambda_1 \int_{\partial\Omega} |u|^2 x \cdot n + \frac{|\lambda_2|}{2\lambda_1^{\frac{1}{2}}} \int_{\partial\Omega} |u|^2 \frac{x \cdot n}{|x|} \\ &= 2\Im \int_\Omega |x| V_2 u \left(i \lambda_1^{\frac{1}{2}} \operatorname{sgn}(\lambda_2) \bar{u} + \partial_r \bar{u} \right) + \int_\Omega \partial_r \left(|x| V_1 \right) |u|^2 - \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \int_\Omega V_1 |x| |u|^2 - \int_{\partial\Omega} V_1 |u|^2 x \cdot n. \end{aligned} \tag{2.10}$$

Proof. Since u is a solution of (1.1), the identities in Lemma 1 hold. We take the sum (2.2) + $\lambda_1^{\frac{1}{2}}$ (2.3) + (2.4) and assume that the functions G_1, G_2, G_3 are radial to get

$$
\begin{aligned}
&\int\limits_{\Omega} \left(g_3'' - g_1\right)|\partial_r u|^2 + \int\limits_{\Omega} \left(\frac{g_3'}{|x|} - g_1\right)|\nabla_\tau u|^2 + \int\limits_{\Omega} \left(\lambda_1 g_1 + \lambda_1^{\frac{1}{2}}\lambda_2\right)|u|^2 \\
&+ \int\limits_{\Omega} \left(\frac{1}{2}\Delta G_1 - \frac{1}{4}\Delta^2 G_3\right)|u|^2 - \lambda_1^{\frac{1}{2}} \Im \int\limits_{\Omega} g_2' \bar{u}\partial_r u + \lambda_2 \Im \int\limits_{\Omega} g_3' u \partial_r \bar{u} - \frac{1}{2}\int\limits_{\partial\Omega} g_1' |u|^2 \frac{x\cdot n}{|x|} \\
&\Re \int\limits_{\partial\Omega} g_1 \frac{\partial u}{\partial n}\bar{u} + \lambda_1^{\frac{1}{2}} \Im \int\limits_{\partial\Omega} g_2 \frac{\partial u}{\partial n}\bar{u} - \Re \int\limits_{\partial\Omega} g_3' \partial_r \bar{u} \frac{\partial u}{\partial n} - \frac{1}{2}\Re \int\limits_{\partial\Omega} g_3'' \bar{u} \frac{\partial u}{\partial n} - \frac{1}{2}\Re \int\limits_{\partial\Omega} g_3' \frac{d-1}{|x|}\bar{u}\frac{\partial u}{\partial n} \\
&+ \frac{1}{2}\int\limits_{\partial\Omega} g_3' |\nabla u|^2 \frac{x\cdot n}{|x|} + \frac{1}{4}\int\limits_{\partial\Omega} |u|^2 \frac{\partial(\Delta G_3)}{\partial n} - \frac{\lambda_1}{2}\int\limits_{\partial\Omega} g_3' |u|^2 \frac{x\cdot n}{|x|} \\
&= \int\limits_{\Omega} g_1 V_1 |u|^2 + \lambda_1^{\frac{1}{2}} \int\limits_{\Omega} g_2 V_2 |u|^2 - \Re \int\limits_{\Omega} g_3' V u \partial_r \bar{u} - \frac{1}{2}\int\limits_{\Omega} g_3'' V_1 |u|^2 - \frac{1}{2}\int\limits_{\Omega} g_3' V_1 \frac{d-1}{|x|}|u|^2.
\end{aligned}
$$

Choosing $g_1 = \frac{1}{2}g_3''$ and $g_2 = \operatorname{sgn}(\lambda_2) g_3'$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\int\limits_{\Omega} g_3'' \left(|\partial_r u|^2 + \lambda_1 |u|^2\right) - \operatorname{sgn}(\lambda_2)\lambda_1^{\frac{1}{2}} \int\limits_{\Omega} g_3'' \bar{u}\partial_r u + \int\limits_{\Omega} |\nabla_\tau u|^2 \left(\frac{g_3'}{|x|} - \frac{g_3''}{2}\right) + \frac{1}{4}\int\limits_{\Omega} \left(\Delta G_3'' - \Delta^2 G_3\right)|u|^2 \\
&+ |\lambda_2|\lambda_1^{\frac{1}{2}} \int\limits_{\Omega} g_3' |u|^2 + \lambda_2 \Im \int\limits_{\Omega} g_3' u \partial_r \bar{u} + \lambda_1^{\frac{1}{2}} \operatorname{sgn}(\lambda_2) \Im \int\limits_{\partial\Omega} g_3' \frac{\partial u}{\partial n}\bar{u} - \Re \int\limits_{\partial\Omega} g_3' \partial_r \bar{u}\frac{\partial u}{\partial n} - \frac{1}{2}\Re \int\limits_{\partial\Omega} g_3' \frac{d-1}{|x|}\bar{u}\frac{\partial u}{\partial n} \\
&- \frac{1}{4}\int\limits_{\partial\Omega} g_3''' |u|^2 \frac{x\cdot n}{|x|} + \frac{1}{2}\int\limits_{\partial\Omega} g_3' |\nabla u|^2 \frac{x\cdot n}{|x|} + \frac{1}{4}\int\limits_{\partial\Omega} |u|^2 \frac{\partial(\Delta G_3)}{\partial n} - \frac{\lambda_1}{2}\int\limits_{\partial\Omega} g_3' |u|^2 \frac{x\cdot n}{|x|} \\
&= \lambda_1^{\frac{1}{2}} \operatorname{sgn}(\lambda_2) \int\limits_{\Omega} g_3' V_2 |u|^2 - \Re \int\limits_{\Omega} V g_3' u \partial_r \bar{u} - \frac{1}{2}\int\limits_{\Omega} V_1 g_3' \frac{d-1}{|x|}|u|^2,
\end{aligned}
$$

where $G_3''(x) := g_3''(|x|)$. Choosing $G_3(|x|) = |x|^2$ and using (2.7), we obtain

$$
\begin{aligned}
&\int\limits_{\Omega} \left(|\nabla u|^2 + \lambda_1 |u|^2\right) - 2\operatorname{sgn}(\lambda_2)\lambda_1^{\frac{1}{2}} \int\limits_{\Omega} \bar{u}\partial_r u + 2|\lambda_2|\lambda_1^{\frac{1}{2}} \int\limits_{\Omega} |x||u|^2 + 2\lambda_2 \Im \int\limits_{\Omega} |x| u \partial_r \bar{u} \\
&+ 2\lambda_1^{\frac{1}{2}} \operatorname{sgn}(\lambda_2) \Im \int\limits_{\partial\Omega} |x| \frac{\partial u}{\partial n}\bar{u} - 2\Re \int\limits_{\partial\Omega} |x| \partial_r \bar{u}\frac{\partial u}{\partial n} - \Re \int\limits_{\partial\Omega} (d-1)\bar{u}\frac{\partial u}{\partial n} + \int\limits_{\partial\Omega} |\nabla u|^2 x\cdot n - \lambda_1 \int\limits_{\partial\Omega} |u|^2 x\cdot n \\
&= 2\lambda_1^{\frac{1}{2}} \operatorname{sgn}(\lambda_2) \int\limits_{\Omega} |x| V_2 |u|^2 - 2\Re \int\limits_{\Omega} V |x| u \partial_r \bar{u} - \int\limits_{\Omega} V_1 (d-1)|u|^2,
\end{aligned}
$$

where $V_1 := \Re V$ and $V_2 := \Im V$. Using the identity (2.9), we get

$$
\begin{aligned}
&\int_\Omega |\nabla u^-|^2 + 2|\lambda_2|\lambda_1^{\frac12}\int_\Omega |x||u|^2 + 2\Im\lambda_2\int_\Omega |x|u\partial_r\bar u + 2\lambda_1^{\frac12}\operatorname{sgn}(\lambda_2)\Im\int_{\partial\Omega}|x|\frac{\partial u}{\partial n}\bar u - 2\Re\int_{\partial\Omega}|x|\partial_r\bar u\frac{\partial u}{\partial n}\\
&-\Re\int_{\partial\Omega}(d-1)\bar u\frac{\partial u}{\partial n} + \int_{\partial\Omega}|\nabla u|^2 x\cdot n - \lambda_1\int_{\partial\Omega}|u|^2 x\cdot n\\
&= 2\lambda_1^{\frac12}\operatorname{sgn}(\lambda_2)\int_\Omega |x|V_2|u|^2 - 2\Re\int_\Omega V|x|u\partial_r\bar u - \int_\Omega V_1(d-1)|u|^2.
\end{aligned}
\tag{2.11}
$$

We will use the identity (2.2) once again, but this time with the choice $G_1 = \lambda_1^{-\frac12}|\lambda_2||x|$ so that it reads

$$
\begin{aligned}
&\lambda_1^{\frac12}|\lambda_2|\int_\Omega |x||u|^2 - \lambda_1^{-\frac12}|\lambda_2|\int_\Omega |x||\nabla u|^2 + \frac12\lambda_1^{-\frac12}|\lambda_2|\int_\Omega \frac{d-1}{|x|}|u|^2 + \lambda_1^{-\frac12}|\lambda_2|\Re\int_{\partial\Omega}|x|\frac{\partial u}{\partial n}\bar u - \frac{\lambda_1^{-\frac12}|\lambda_2|}{2}\int_{\partial\Omega}|u|^2\frac{x\cdot n}{|x|}\\
&= \lambda_1^{-\frac12}|\lambda_2|\int_\Omega V_1|x||u|^2.
\end{aligned}
$$

Subtracting it from (2.11), we obtain

$$
\begin{aligned}
&\int_\Omega |\nabla u^-|^2 - \lambda_1^{\frac12}|\lambda_2|\int_\Omega |x||u|^2 + \lambda_1^{-\frac12}|\lambda_2|\int_\Omega |x||\nabla u|^2 + 2\lambda_2\Im\int_\Omega |x|u\partial_r\bar u - \frac{d-1}{2}\lambda_1^{-\frac12}|\lambda_2|\int_\Omega \frac{|u|^2}{|x|}\\
&+ 2\lambda_1^{\frac12}\operatorname{sgn}(\lambda_2)\Im\int_{\partial\Omega}|x|\frac{\partial u}{\partial n}\bar u - 2\Re\int_{\partial\Omega}|x|\partial_r\bar u\frac{\partial u}{\partial n} - \Re\int_{\partial\Omega}(d-1)\bar u\frac{\partial u}{\partial n} - \lambda_1^{-\frac12}|\lambda_2|\Re\int_{\partial\Omega}|x|\frac{\partial u}{\partial n}\bar u\\
&+ \int_{\partial\Omega}|\nabla u|^2 x\cdot n - \lambda_1\int_{\partial\Omega}|u|^2 x\cdot n + \frac{\lambda_1^{-\frac12}|\lambda_2|}{2}\int_{\partial\Omega}|u|^2\frac{x\cdot n}{|x|}\\
&= 2\lambda_1^{\frac12}\operatorname{sgn}(\lambda_2)\int_\Omega |x|V_2|u|^2 - 2\Re\int_\Omega V|x|u\partial_r\bar u + \int_\Omega V_1(1-d)|u|^2 - \lambda_1^{-\frac12}|\lambda_2|\int_\Omega V_1|x||u|^2.
\end{aligned}
\tag{2.12}
$$

Using the divergence theorem and (2.5), we can rewrite the third term on the right hand side as follows:

$$
\int_\Omega V_1(1-d)|u|^2 = \int_\Omega V_1(1-\nabla x)|u|^2 = \int_\Omega V_1|u|^2 + 2\Re\int_\Omega V_1|x|u\partial_r\bar u + \int_\Omega |x|\partial_r V_1|u|^2 - \int_{\partial\Omega} V_1|u|^2 x\cdot n.
\tag{2.13}
$$

The right hand side of (2.12) can then be rewritten as

$$2\Im\int_{\Omega}|x|V_2u\left(i\lambda_1^{\frac{1}{2}}\operatorname{sgn}(\lambda_2)\bar{u}+\partial_r\bar{u}\right)+\int_{\Omega}\left(V_1+|x|\partial_rV_1\right)|u|^2-\lambda_1^{-\frac{1}{2}}|\lambda_2|\int_{\Omega}V_1|x||u|^2-\int_{\partial\Omega}V_1|u|^2x\cdot n$$
$$=2\Im\int_{\Omega}|x|V_2u\left(i\lambda_1^{\frac{1}{2}}\operatorname{sgn}(\lambda_2)\bar{u}+\partial_r\bar{u}\right)+\int_{\Omega}\partial_r\left(|x|V_1\right)|u|^2-\lambda_1^{-\frac{1}{2}}|\lambda_2|\int_{\Omega}V_1|x||u|^2-\int_{\partial\Omega}V_1|u|^2x\cdot n.$$

Finally, using the identity (2.9) on (2.12), we get the desired result. □

2.2 Real potential

In this section we use the method of multipliers to derive results for real potentials. In this case the operator H_V with conditions **C1** to **C3** is self-adjoint and the proofs are therefore more straightforward than in the case of complex potentials. The results from section 2.3 are a generalization of the theorems which we present here. The purpose of this section is to demonstrate the method of multipliers on a simpler problem.

We start with the following lemma which under certain condition enables us to eliminate positive eigenvalues.

Lemma 4. *Let $d\geq 3$. Let u be a solution of (1.1) with $\lambda_1>0$ on the half-space $\mathbb{R}^d_+$. Assume that u satisfies one of the boundary conditions **C1** to **C3**. Let the potential V be real and satisfy*

$$\|xVu\|\leq\Lambda\|\nabla u\|, \tag{2.14}$$

where Λ is determined by

$$\frac{2(2d-3)}{d-2}\Lambda\leq 1. \tag{2.15}$$

Then $u=0$.

Proof. The proof is based on the identity (2.8) which under the assumptions of this theorem reads

$$\lambda_1\int_{\mathbb{R}^d_+}|u|^2+\int_{\mathbb{R}^d_+}|\nabla u|^2+\Re\int_{\partial\mathbb{R}^d_+}(1-d)\frac{\partial u}{\partial n}\bar{u}-2\Re\int_{\partial\mathbb{R}^d_+}|x|\partial_r\bar{u}\frac{\partial u}{\partial n}+\int_{\partial\mathbb{R}^d_+}|\nabla u|^2x\cdot n-\lambda_1\int_{\partial\mathbb{R}^d_+}|u|^2x\cdot n$$
$$=\int_{\mathbb{R}^d_+}(1-d)\,V|u|^2-2\Re\int_{\mathbb{R}^d_+}V|x|u\partial_r\bar{u}.$$

Since the boundary of the half-space is $\partial\mathbb{R}^d_+=\{(x_1,\cdots,x_d)\in\mathbb{R}^d\mid x_1=0\}$, the vector x is the tangent vector multiplied by $|x|$. Hence, the dot product $x\cdot n$ is equal to zero and all the terms involving such product vanish and we are left with

$$\lambda_1\int_{\mathbb{R}^d_+}|u|^2+\int_{\mathbb{R}^d_+}|\nabla u|^2+\Re\int_{\partial\mathbb{R}^d_+}(1-d)\frac{\partial u}{\partial n}\bar{u}-2\Re\int_{\partial\mathbb{R}^d_+}|x|\partial_r\bar{u}\frac{\partial u}{\partial n} \tag{2.16}$$
$$=\int_{\mathbb{R}^d_+}(1-d)\,V|u|^2-2\Re\int_{\mathbb{R}^d_+}V|x|u\partial_r\bar{u}. \tag{2.17}$$

We now proceed by estimating the right-hand side from above. Recall the classical Hardy inequality

$$\forall d \geq 3, \qquad \forall \psi \in H^1(\mathbb{R}^d), \qquad \int\limits_{\mathbb{R}^d} |\nabla \psi(x)|^2 \mathrm{d}x \geq \left(\frac{d-2}{2}\right)^2 \int\limits_{\mathbb{R}^d} \frac{|\psi(x)|^2}{|x|^2} \mathrm{d}x. \tag{2.18}$$

Using this inequality together with the Schwarz inequality and (2.14), we estimate the first term as

$$\int\limits_{\mathbb{R}^d_+} (1-d)\, V|u|^2 \leq (d-1) \left| \int\limits_{\mathbb{R}^d_+} V|u|^2 \right| \leq (d-1)\, \|xVu\| \left\| \frac{u}{|x|} \right\| \leq \frac{2(d-1)}{d-2} \|\nabla u\|^2. \tag{2.19}$$

The other integral can be estimated as

$$-2\Re \int\limits_{\mathbb{R}^d_+} V|x|u\partial_r \bar{u} \leq 2 \left| \int\limits_{\mathbb{R}^d_+} V|x|u\partial_r \bar{u} \right| \leq \|xVu\| \|\partial_r u\| \leq 2\|xVu\| \|\nabla u\| \leq 2\Lambda \|\nabla u\|, \tag{2.20}$$

where we use the Schwarz inequality, (2.7) and (2.14). Combining the inequalities (2.19) and (2.20) together with the assumption that $\lambda_1 > 0$, we can estimate (2.16) as

$$\left(1 - \frac{2(2d-3)}{d-2}\Lambda\right) \int\limits_{\mathbb{R}^d_+} |\nabla u|^2 + \Re \int\limits_{\partial\mathbb{R}^d_+} (1-d)\, \frac{\partial u}{\partial n} \bar{u} - 2\Re \int\limits_{\partial\mathbb{R}^d_+} |x| \partial_r \bar{u} \frac{\partial u}{\partial n} \leq 0. \tag{2.21}$$

Applying the Robin boundary conditions, we may rewrite this inequality as

$$\left(1 - \frac{2(2d-3)}{d-2}\Lambda\right) \int\limits_{\mathbb{R}^d_+} |\nabla u|^2 + \int\limits_{\partial\mathbb{R}^d_+} (d-1)\, \alpha|u|^2 + 2\Re \int\limits_{\partial\mathbb{R}^d_+} |x| \alpha u \partial_r \bar{u} \leq 0.$$

Integrating by parts, we can rewrite the second integral over the boundary as

$$2\Re \int\limits_{\partial\mathbb{R}^d_+} \alpha|x|u\partial_r \bar{u} = 2\Re \int\limits_{\partial\mathbb{R}^d_+} \alpha u \sum_{i=2}^d \frac{\partial \bar{u}}{\partial x_i} x_i = \int\limits_{\partial\mathbb{R}^d_+} \alpha \sum_{i=2}^d \frac{\partial |u|^2}{\partial x_i} x_i = -\int\limits_{\partial\mathbb{R}^d_+} \left((d-1)\alpha + x' \cdot \nabla' \alpha\right) |u|^2. \tag{2.22}$$

Since we assume $x' \cdot \nabla' \alpha \leq 0$, we obtain

$$\left(1 - \frac{2(2d-3)}{d-2}\Lambda\right) \int\limits_{\mathbb{R}^d_+} |\nabla u|^2 \leq 0. \tag{2.23}$$

We get the same result for Dirichlet and Neumann boundary conditions since both the integrals over the boundary in (2.21) disappear. Indeed, since $u = 0$ along the whole boundary in the Dirichlet case, the radial derivative of u is also equal to zero. The remaining terms vanish immediately by applying the relevant condition.

The condition (2.15) implies that the bracket in front of the integral in (2.23) is strictly positive, thus the only case in which this inequality holds is $\nabla u = 0$. Being unbounded, the half-space $\mathbb{R}^d_+$ restricts u to be equal to zero. □

We are now in position to prove the first main theorem of this section.

Theorem 5. *Let $d \geq 3$ and assume that the potential V is real. Suppose*

$$\forall \psi \in H^1\left(\mathbb{R}^d_+\right) \qquad \int\limits_{\mathbb{R}^d_+} |x|^2 |V(x)|^2 |\psi(x)|^2 \mathrm{d}x \leq \Lambda \int\limits_{\mathbb{R}^d_+} |\nabla \psi(x)|^2 \mathrm{d}x, \tag{2.24}$$

*where Λ satisfies (2.15). Then $\sigma_p(H_V) = \emptyset$ for one of the boundary conditions **C1** to **C3**.*

Proof. The Lemma 4 implies that H_V has no positive eigenvalues. However, (2.24) together with (2.15) implies (1.2), which, as we will see, yields that H_V can only have positive eigenvalues. Taking (2.2) with $G_1 = 1$, we get

$$\lambda_1 \int\limits_{\mathbb{R}^d_+} |u|^2 = \int\limits_{\mathbb{R}^d_+} V|u|^2 + \int\limits_{\mathbb{R}^d_+} |\nabla u|^2 - \Re \int\limits_{\partial\mathbb{R}^d_+} \frac{\partial u}{\partial n}\bar{u} \geq \int\limits_{\mathbb{R}^d_+} |\nabla u|^2 - \int\limits_{\mathbb{R}^d_+} |V||u|^2 \geq (1-a) \int\limits_{\mathbb{R}^d_+} |\nabla u|^2,$$

where the last inequality follows from (1.2). The integral over the boundary vanishes immediately in the Dirichlet and Neumann case. Applying the Robin boundary conditions, we can rewrite the integral as $\int\limits_{\partial\mathbb{R}^d_+} \alpha|u|^2$ and estimate it from below by zero, since we assume that α is non-negative. By (1.2), the bracket on the right-hand side is strictly positive, thus this inequality hold for a non-zero u only if $\lambda > 0$, *i.e.* only positive eigenvalues are permitted. □

We proceed by proving an analogous result to the Theorem 4 where alternative conditions on the potential are imposed. But first, we prove the following lemma which similarly as Lemma 4 eliminates positive eigenvalues.

Lemma 5. *Let u be a solution of (1.1) where $\lambda_1 > 0$ and $\mathbb{R}^d_+$ is the half-space $\mathbb{R}^d_+$ with a real potential V. Assume that there exist non-negative numbers $b_1 < 1, b_2 < 1$ such that*

$$\int\limits_{\mathbb{R}^d_+} V_- |\psi|^2 \leq b_1^2 \int\limits_{\mathbb{R}^d_+} |\nabla \psi|^2, \tag{2.25}$$

$$\int\limits_{\mathbb{R}^d_+} \left[\partial_r (rV)\right]_+ |\psi|^2 \leq b_2^2 \int\limits_{\mathbb{R}^d_+} |\nabla \psi|^2, \tag{2.26}$$

*where $\partial_r = \nabla \cdot \frac{x}{|x|}$. Let u satisfy one of the boundary conditions **C1** to **C3**. Then $u = 0$.*

Proof. The proof is based on the identity (2.8) from the Lemma 2 but many terms will drop out under the assumptions of this theorem. The operator H_V with a real potential V and one of assumed boundary conditions on $\mathbb{R}^d_+$ is self-adjoint, thus its spectrum is real and the term involving λ_2 will vanish. The identity then reads

$$\lambda_1 \int\limits_{\mathbb{R}^d_+} |u|^2 + \int\limits_{\mathbb{R}^d_+} |\nabla u|^2 + \Re \int\limits_{\partial\mathbb{R}^d_+} (1-d) \frac{\partial u}{\partial n}\bar{u} - 2\Re \int\limits_{\partial\mathbb{R}^d_+} |x| \partial_r \bar{u} \frac{\partial u}{\partial n} + \int\limits_{\partial\mathbb{R}^d_+} |\nabla u|^2 x \cdot n - \lambda_1 \int\limits_{\partial\mathbb{R}^d_+} |u|^2 x \cdot n$$
$$= \int\limits_{\mathbb{R}^d_+} (1-d) V_1 |u|^2 - 2\Re \int\limits_{\mathbb{R}^d_+} V|x| u \partial_r \bar{u}.$$

The first integral on the right hand side can be rewritten as in (2.13) so that we obtain

$$\lambda_1 \int\limits_{\mathbb{R}^d_+} |u|^2 + \int\limits_{\mathbb{R}^d_+} |\nabla u|^2 + \Re \int\limits_{\partial\mathbb{R}^d_+} (1-d)\frac{\partial u}{\partial n}\bar{u} - 2\Re \int\limits_{\partial\mathbb{R}^d_+} |x|\partial_r \bar{u}\frac{\partial u}{\partial n} + \int\limits_{\partial\mathbb{R}^d_+} |\nabla u|^2 x\cdot n - \lambda_1 \int\limits_{\partial\mathbb{R}^d_+} |u|^2 x\cdot n$$
$$= \int\limits_{\mathbb{R}^d_+} \partial_r\left(|x|V_1\right)|u|^2. \tag{2.27}$$

Let us start with Dirichlet and Neumann boundary conditions. As we will see, all the integrals over the boundary will disappear. Indeed, in the Dirichlet case, we can discard the term involving u. Additionally, the term containing $\partial_r u$ will disappear as well, since $u = 0$ along the whole boundary so its radial derivative is also equal to zero on the boundary. In the Neumann case, the first two integrals over the boundary vanish immediately.

Since the boundary of the half-space is $\partial\mathbb{R}^d_+ = \{(x_1, \cdots, x_d) \in \mathbb{R}^d \mid x_1 = 0\}$, the vector x is the tangent vector multiplied by $|x|$. Hence, the dot product $x \cdot n$ is equal to zero and all the terms involving such product vanish. After applying this reasoning we are left with

$$\lambda_1 \int\limits_{\mathbb{R}^d_+} |u|^2 + \int\limits_{\mathbb{R}^d_+} |\nabla u|^2 = \int\limits_{\mathbb{R}^d_+} \partial_r\left(|x|V_1\right)|u|^2.$$

Using (2.26) and considering that λ_1 is positive, we obtain

$$\left(1-b_2^2\right)\int\limits_{\mathbb{R}^d_+} |\nabla u|^2 \le 0.$$

Since $b_2 < 1$, the only case in which this inequality holds is $\nabla u = 0$. The half-space $\mathbb{R}^d_+$ is an unbounded domain thus $u = 0$.

Applying the Robin boundary conditions to (2.27) and using that $x \cdot n = 0$ on $\partial\mathbb{R}^d_+$, we get

$$\lambda_1 \int\limits_{\mathbb{R}^d_+} |u|^2 + \int\limits_{\mathbb{R}^d_+} |\nabla u|^2 - \int\limits_{\partial\mathbb{R}^d_+} (1-d)\,\alpha|u|^2 + 2\Re \int\limits_{\partial\mathbb{R}^d_+} \alpha|x|u\partial_r\bar{u} = \int\limits_{\mathbb{R}^d_+} \partial_r\left(|x|V_1\right)|u|^2. \tag{2.28}$$

Repeating the procedure done in the Dirichlet and Neumann case, we use (2.26) and $\lambda_1 > 0$ to estimate (2.28) as

$$\left(1-b_2^2\right)\int\limits_{\mathbb{R}^d_+} |\nabla u|^2 - \int\limits_{\partial\mathbb{R}^d_+} |u|^2 x' \cdot \nabla'\alpha \le 0.$$

This inequality gives rise to the condition $\nabla'\alpha \cdot x' \le 0$ under which $u = 0$ holds. □

Finally, we are in position to prove the last theorem of this section.

Theorem 6. *Let $d \ge 3$ and assume that conditions (2.25) and (2.26) hold. Then $\sigma_p(H_V) = \emptyset$ for one of the boundary conditions **C1** to **C3**.*

Proof. As we have shown in Lemma 5, $\sigma_p(H_V) \cap \{\lambda_1 > 0\} = \emptyset$. In order to prove that there are no eigenvalues $\lambda_1 \leq 0$, we take (2.2) with $G_1 = 1$ to obtain

$$\begin{aligned}\lambda_1 \int_{\mathbb{R}^d_+} |u|^2 &= \int_{\mathbb{R}^d_+} |\nabla u|^2 + \int_{\mathbb{R}^d_+} V_1 |u|^2 - \Re \int_{\partial\mathbb{R}^d_+} \frac{\partial u}{\partial n}\bar{u} \\ &\geq \int_{\mathbb{R}^d_+} |\nabla u|^2 + \int_{\mathbb{R}^d_+} (V_1)_- |u|^2 - \Re \int_{\partial\mathbb{R}^d_+} \frac{\partial u}{\partial n}\bar{u} \\ &\geq (1-b_1)\int_{\mathbb{R}^d_+} |\nabla u|^2, \end{aligned} \tag{2.29}$$

where the last inequality arises from (2.25) and by applying the boundary conditions. In the Dirchlet and Neumann case, the integral over the boundary vanishes immediately. In the Robin case, we estimate the integral over the boundary as

$$-\Re \int_{\partial\mathbb{R}^d_+} \frac{\partial u}{\partial n}\bar{u} = \int_{\partial\mathbb{R}^d_+} \alpha |u|^2 \geq 0,$$

where we used the assumption that α is non-negative.

Since we assume that $\lambda_1 \leq 0$ and $b_1 < 1$, the inequality (2.29) holds only for $u = 0$. □

2.3 Complex potential

In contrast with the previous section, we are no longer dealing with a self-adjoint operator. Due to the complex potential, the operator H_V is now m-sectorial. The eigenvalues are no longer real and additional terms involving imaginary parts of eigenvalues and potential appear. As a result, we shift form Lemma 2 to Lemma 1 where we combined three identities instead of only two. We also utilize the function $u^-(x) := ue^{-i\,\mathrm{sgn}(\lambda_2)\lambda_1^{\frac{1}{2}}|x|}$ which is useful for the estimates due to the identity (2.9).

Let us start with the following lemma which under certain condition enables us to eliminate eigenvalues with positive real part.

Lemma 6. *Let $d \geq 3$. Let u be a solution of (1.1) with $\lambda_1 > 0$ on the half-space $\mathbb{R}^d_+$ and let it satisfy one of the boundary conditions **C1** to **C3**. Assume that the potential V satisfies*

$$\|xVu\| \leq \Lambda \|\nabla u\|, \qquad \|xVu^-\| \leq \Lambda \|\nabla u^-\|, \tag{2.30}$$

where Λ is determined by

$$\frac{2(2d-3)}{d-2}\Lambda + \frac{\sqrt{2}}{\sqrt{d-2}}\Lambda^{\frac{3}{2}} < 1. \tag{2.31}$$

Then $u = 0$.

Proof. The proof of this theorem is analogous to that of the Lemma 4. The difference lies in allowing complex potentials which leads to H_V being a m-sectorial operator as opposed to the self-adjoint one in the case of a real potential. Hence, the eigenvalues are complex numbers and additional terms involving λ_2 will appear. Inspired by [4], we divide the proof into two cases: $|\lambda_2|\leq \lambda_1$ and $|\lambda_2|> \lambda_1$.

- $|\lambda_2|\leq \lambda_1$

  Let us start by taking the identity (2.12) which in the case of the half-space $\mathbb{R}^d_+$ reads

$$
\begin{aligned}
I := &\int\limits_{\mathbb{R}^d_+} |\nabla u^-|^2 + \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \int\limits_{\mathbb{R}^d_+} |x||\nabla u^-|^2 - \frac{d-1}{2}\frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \int\limits_{\mathbb{R}^d_+} \frac{|u|^2}{|x|} \\
&+ 2\lambda_1^{\frac{1}{2}}\operatorname{sgn}(\lambda_2)\Im \int\limits_{\partial\mathbb{R}^d_+} |x|\frac{\partial u}{\partial n}\bar{u} - 2\Re \int\limits_{\partial\mathbb{R}^d_+} |x|\partial_r\bar{u}\frac{\partial u}{\partial n} - \Re \int\limits_{\partial\mathbb{R}^d_+} (d-1)\bar{u}\frac{\partial u}{\partial n} - \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}}\Re \int\limits_{\partial\mathbb{R}^d_+} |x|\frac{\partial u}{\partial n}\bar{u} \\
= &\underbrace{(1-d)\int\limits_{\mathbb{R}^d_+} V_1|u|^2}_{I_1} \underbrace{-2\Re \int\limits_{\mathbb{R}^d_+} |x|Vu\left(i\lambda_1^{\frac{1}{2}}\operatorname{sgn}(\lambda_2)\bar{u} + \partial_r\bar{u}\right)}_{I_2} \underbrace{- \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}}\int\limits_{\mathbb{R}^d_+} V_1|x||u|^2}_{I_3},
\end{aligned}
\tag{2.32}
$$

  where we used that $x\cdot n = 0$ since x on the boundary of the half-space $\mathbb{R}^d_+$ is simply tangential vector multiplied by $|x|$. By the weighted Hardy inequality

$$
\int\limits_{\mathbb{R}^d} \frac{|\psi|^2}{x} \leq \frac{4}{(d-1)^2}\int\limits_{\mathbb{R}^d} |x||\nabla\psi|^2
$$

  and the fact that $|u|= |u^-|$, we can estimate the left-hand side of (2.32) from below as follows:

$$
\begin{aligned}
I \geq &\int\limits_{\mathbb{R}^d_+} |\nabla u^-|^2 + \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}}\frac{d-3}{d-1}\int\limits_{\mathbb{R}^d_+} |x||\nabla u^-|^2 \\
&+ 2\lambda_1^{\frac{1}{2}}\operatorname{sgn}(\lambda_2)\Im \int\limits_{\partial\mathbb{R}^d_+} |x|\frac{\partial u}{\partial n}\bar{u} - 2\Re \int\limits_{\partial\mathbb{R}^d_+} |x|\partial_r\bar{u}\frac{\partial u}{\partial n} - \Re \int\limits_{\partial\mathbb{R}^d_+} (d-1)\bar{u}\frac{\partial u}{\partial n} - \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}}\Re \int\limits_{\partial\mathbb{R}^d_+} |x|\frac{\partial u}{\partial n}\bar{u}.
\end{aligned}
$$

  The boundary integrals vanish immediately in the case of Neumann boundary conditions. In order to get the same result for the Dirichlet case, we have to use additional reasoning that $u = 0$ along the whole boundary, thus $\frac{x}{|x|}\cdot\nabla u = |x|\partial_r u = 0$. Finally, the Robin boundary conditions **C3** lead to the following estimate of the integrals over the boundary:

$$
2\Re \int\limits_{\partial\mathbb{R}^d_+} \alpha|x|u\partial_r\bar{u} + (d-1)\int\limits_{\partial\mathbb{R}^d_+} \alpha|u|^2 + \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}}\int\limits_{\partial\mathbb{R}^d_+} \alpha|x||u|^2 = -\int\limits_{\partial\mathbb{R}^d_+} x'\cdot\nabla'\alpha|u|^2 + \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}}\int\limits_{\partial\mathbb{R}^d_+} \alpha|x||u|^2 \geq 0,
$$

  where the equality follows from (2.22) and the estimate is the result of the assumptions in **C3**. In conclusion, we can estimate the left-hand side of (2.32) with any of the boundary conditions considered in this theorem as

$$
I \geq \int\limits_{\mathbb{R}^d_+} |\nabla u^-|^2 + \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}}\frac{d-3}{d-1}\int\limits_{\mathbb{R}^d_+} |x||\nabla u^-|^2. \tag{2.33}
$$

The first term on the right-hand side can be estimated using the Hardy inequality (2.18), Schwarz inequality and (2.30) as

$$|I_1| \leq (d-1)\|xVu\| \left\| \frac{u}{|x|} \right\| \leq \frac{2(d-1)}{d-2} \Lambda \|\nabla u\|^2. \tag{2.34}$$

The second term can be rewritten using $i\lambda_1^{\frac{1}{2}} \operatorname{sgn}(\lambda_2)\bar{u} + \partial_r \bar{u} = \overline{\partial_r u^-} \exp\Big(-i \operatorname{sgn}(\lambda_2)\lambda_1^{\frac{1}{2}} |x|\Big)$ and then estimated by the Schwarz inequality and (2.30) as

$$|I_2| \leq 2\big\|xVu^-\big\| \big\|\partial_r u^-\big\| \leq 2\big\|xVu^-\big\| \big\|\nabla u^-\big\| \leq 2\Lambda \big\|\nabla u^-\big\|^2. \tag{2.35}$$

In order to estimate the last term, we return to the identity (2.3) and choose $G_2 = \frac{\lambda_2}{|\lambda_2|}$ so that we obtain

$$\|u\|^2 = |\lambda_2|^{-1} \Im \int_{\mathbb{R}^d_+} V|u|^2 \leq |\lambda_2|^{-1} \int_{\mathbb{R}^d_+} |Vu||u|. \tag{2.36}$$

Using this upper bound together with $|\lambda_2| \leq \lambda_1$ and (2.30), we get

$$\begin{aligned} |I_3| &\leq \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \|xVu\|\|u\| \leq \left(\frac{|\lambda_2|}{\lambda_1}\right)^{\frac{1}{2}} \Lambda \big\|\nabla u^-\big\| \sqrt{\int_{\mathbb{R}^d_+} |Vu||u|} \\ &\leq \Lambda \big\|\nabla u^-\big\| \|xVu\|^{\frac{1}{2}} \left\| \frac{u}{|x|} \right\|^{\frac{1}{2}} \\ &\leq \Lambda^{\frac{3}{2}} \frac{\sqrt{2}}{\sqrt{d-2}} \big\|\nabla u^-\big\|^2, \end{aligned} \tag{2.37}$$

where the last inequality follows from the Hardy inequality (2.18). Now that we have treated all the terms, we can estimate (2.32) using (2.33), (2.34), (2.35) and (2.37) as

$$\left(1 - \frac{2(2d-3)}{d-2}\Lambda - \frac{\sqrt{2}}{\sqrt{d-2}}\Lambda^{\frac{3}{2}}\right) \int_{\mathbb{R}^d_+} |\nabla u^-|^2 + \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \frac{d-3}{d-1} \int_{\mathbb{R}^d_+} |x||\nabla u^-|^2 \leq 0. \tag{2.38}$$

The bracket in front of the first integral is strictly positive due to (2.31), thus $\|\nabla u^-\| = 0$. This implies that u and therefore also u^- are equal to zero, since any other constant would lead to an infinite norm.

- $|\lambda_2| > \lambda_1$

  Let us take the real part of (2.1) with $v = u$ and the imaginary part of the same identity with $v = \pm u$. Combining the two results, we have

$$(\lambda_1 \pm \lambda_2) \int_{\mathbb{R}^d_+} |u|^2 = \int_{\mathbb{R}^d_+} |\nabla u|^2 + \int_{\mathbb{R}^d_+} (V_1 \pm V_2)|u|^2. \tag{2.39}$$

By the Schwarz inequality, Hardy inequality (2.18) and assumption (2.31), we estimate the second term on the right-hand side as

$$\int_{\mathbb{R}^d_+} (V_1 \pm V_2)|u|^2 \leq 2 \int_{\mathbb{R}^d_+} |Vu||u| \leq \|xVu\| \left\| \frac{u}{|x|} \right\| \leq \frac{4}{d-2}\Lambda \int_{\mathbb{R}^d_+} |\nabla u|^2.$$

The identity (2.39) can then be estimated as

$$(\lambda_1 \pm \lambda_2) \int_{\mathbb{R}^d_+} |u|^2 \geq \left(1 - \frac{4}{d-2}\Lambda\right) \int_{\mathbb{R}^d_+} |\nabla u|^2. \tag{2.40}$$

The assumption (2.31) implies that the bracket on the right-hand side is strictly positive. Conversely, $\lambda_1 \pm \lambda_2$ is necessarily strictly negative in one of the cases since $|\lambda_2| > \lambda_1$. In conclusion, the inequality (2.40) holds only for $u = 0$.

□

Having eliminated the eigenvalues with positive real part, we are only one step from proving the Theorem 3.

Proof of the Theorem 3. The Lemma 6 states that no eigenvalues of H_V with $\lambda_1 > 0$ are allowed. We proceed by showing that the assumptions of this theorem restrict the eigenvalues to have a positive real part thus the point spectrum is empty.

The identity (2.2) with the constant choice $G_1 = 1$ reads

$$\lambda_1 \int_{\mathbb{R}^d_+} |u|^2 = \int_{\mathbb{R}^d_+} V|u|^2 + \int_{\mathbb{R}^d_+} |\nabla u|^2 - \Re \int_{\partial\mathbb{R}^d_+} \frac{\partial u}{\partial n}\bar{u}. \tag{2.41}$$

Using the Hardy inequality (2.18), the Schwarz inequality and (1.5), we obtain the following inequality:

$$\int_{\mathbb{R}^d_+} |V||u|^2 \leq \|xVu\| \left\| \frac{u}{|x|} \right\| \leq \frac{2\Lambda}{d-2} \|\nabla u\|,$$

where Λ satisfies (2.31) which in turn implies $\Lambda < \frac{d-2}{2}$. Hence, the potential V satisfies (1.2) and we can estimate (2.41) as

$$\lambda_1 \int_{\mathbb{R}^d_+} |u|^2 = \int_{\mathbb{R}^d_+} V|u|^2 + \int_{\mathbb{R}^d_+} |\nabla u|^2 - \Re \int_{\partial\mathbb{R}^d_+} \frac{\partial u}{\partial n}\bar{u} \geq \int_{\mathbb{R}^d_+} |\nabla u|^2 - \int_{\mathbb{R}^d_+} |V||u|^2 \geq (1-a) \int_{\mathbb{R}^d_+} |\nabla u|^2.$$

By (1.2), the right-hand side is positive for a non-zero u thus $\lambda_1 > 0$. □

Let us now shift our attention to the Theorem 4 whose proof is analogous to that of Theorem 6. We start again with the lemma which eliminates eigenvalues with a positive real part.

Lemma 7. *Let $d \geq 3$. Let $u \in \mathcal{D}(\mathbb{R}^d_+)$ be a solution of (1.1) with $\lambda_1 > 0$ on the half-space $\mathbb{R}^d_+$. Assume that the potential V satisfies (1.7), (1.8), (1.9) and (1.6). Let u satisfy one of the boundary conditions **C1** to **C3**. Then $u = 0$.*

Proof. Analogously to the proof of the Lemma 6, we split the proof into the cases $|\lambda_2|\le \lambda_1$ and $|\lambda_2|> \lambda_1$.

- $|\lambda_2|\le \lambda_1$

  We take the identity (2.10) from the Lemma 3 which in the case of the half-space $\mathbb{R}^d_+$ reads

$$\begin{aligned} I :=& \int\limits_{\mathbb{R}^d_+} |\nabla u^-|^2 + \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \int\limits_{\mathbb{R}^d_+} |x||\nabla u^-|^2 - \frac{d-1}{2}\frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \int\limits_{\mathbb{R}^d_+} \frac{|u|^2}{|x|} \\ &+ 2\lambda_1^{\frac{1}{2}} \operatorname{sgn}(\lambda_2)\Im \int\limits_{\partial\mathbb{R}^d_+} |x|\frac{\partial u}{\partial n}\bar{u} - 2\Re \int\limits_{\partial\mathbb{R}^d_+} |x|\partial_r \bar{u}\frac{\partial u}{\partial n} - \Re \int\limits_{\partial\mathbb{R}^d_+} (d-1)\bar{u}\frac{\partial u}{\partial n} - \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}}\Re \int\limits_{\partial\mathbb{R}^d_+} |x|\frac{\partial u}{\partial n}\bar{u} \\ =& \underbrace{2\Im \int\limits_{\mathbb{R}^d_+} |x| V_2 u \left(i\lambda_1^{\frac{1}{2}} \operatorname{sgn}(\lambda_2)\bar{u} + \partial_r \bar{u}\right)}_{I_1} + \underbrace{\int\limits_{\mathbb{R}^d_+} \partial_r\left(|x|V_1\right)|u|^2}_{I_2} \underbrace{- \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \int\limits_{\mathbb{R}^d_+} V_1|x||u|^2}_{I_3}, \end{aligned} \tag{2.42}$$

  where we used that $x\cdot n = 0$ since x on the boundary of the half-space $\mathbb{R}^d_+$ is simply tangential vector multiplied by $|x|$. We proceed by estimating the individual terms on the right-hand side of this equation.

  Notice that $i\lambda_1^{\frac{1}{2}} \operatorname{sgn}(\lambda_2)\bar{u} + \partial_r\bar{u} = \overline{\partial_r u^-} e^{-i\operatorname{sgn}(\lambda_2)\lambda_1^{\frac{1}{2}}|x|}$. Hence, we get

$$|I_1|\le 2\|xV_2u\|\left\|\partial_r u^-\right\| \le 2\left\|xV_2u^-\right\|\left\|\nabla u^-\right\| \le 2b_3\left\|\nabla u^-\right\|^2, \tag{2.43}$$

  where the last inequality follows from (1.9). By the assumption (1.8), we can estimate the second term as

$$I_2 = \int\limits_{\mathbb{R}^d_+} \partial_r\left(|x|V_1\right)|u^-|\le \int\limits_{\mathbb{R}^d_+} \left[\partial_r\left(|x|V_1\right)\right]_+ |u^-|\le \int\limits_{\mathbb{R}^d_+} b_2^2|\nabla u^-|^2. \tag{2.44}$$

  Finally, using (1.7), we estimate the last integral as

$$I_3 \le \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \int\limits_{\mathbb{R}^d_+} (V_1)_- \left||x|^{\frac{1}{2}}u^-\right|^2 \le b_1^2\frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \int\limits_{\mathbb{R}^d_+} \left|\nabla\left(|x|^{\frac{1}{2}}u^-\right)\right|^2. \tag{2.45}$$

  Additionally, the resulting integral can be rewritten as

$$\int\limits_{\mathbb{R}^d_+} \left|\nabla\left(|x|^{\frac{1}{2}}u^-\right)\right|^2 = \int\limits_{\mathbb{R}^d_+} \left|\frac{1}{2}\frac{x}{|x|^{\frac{3}{2}}}u^- + |x|^{\frac{1}{2}}\nabla u^-\right|^2 = \frac{1}{4}\int\limits_{\mathbb{R}^d_+} \frac{|u^-|^2}{|x|} + \int\limits_{\mathbb{R}^d_+} |x||\nabla u^-|^2 + \Re\int\limits_{\mathbb{R}^d_+} \overline{u^-}\partial_r u^-.$$

  Integrating by parts the last integral and using (2.5), we see that the second and third term on the left-hand side of (2.42) can be rewritten as

$$\frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \int\limits_{\mathbb{R}^d_+} |x||\nabla u^-|^2 - \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}}\frac{d-1}{2} \int\limits_{\mathbb{R}^d_+} \frac{|u^-|^2}{|x|} = \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \int\limits_{\mathbb{R}^d_+} \left|\nabla\left(|x|^{\frac{1}{2}}u^-\right)\right|^2 - \frac{1}{4}\frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \int\limits_{\mathbb{R}^d_+} \frac{|u^-|^2}{|x|}. \tag{2.46}$$

Taking the identity (2.3) with the constant choice $G_2 = \frac{\lambda_2}{|\lambda_2|}$, we obtain

$$\int_{\mathbb{R}^d_+} |u|^2 = |\lambda_2|^{-1} \int_{\mathbb{R}^d_+} V_2 |u^-|^2 \leq |\lambda_2|^{-1} \|xV_2 u^-\| \left\| \frac{u^-}{|x|} \right\| \leq |\lambda_2|^{-1} \frac{2b_3}{d-2} \|\nabla u^-\|^2, \tag{2.47}$$

where we used the Hardy inequality (2.18) and (1.9) in the last inequality. As a consequence, the second term on the right-hand side of (2.46) can be estimated as

$$\frac{1}{4} \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \int_{\mathbb{R}^d_+} \frac{|u^-|^2}{|x|} \leq \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \left\| \frac{u^-}{|x|} \right\| \|u^-\| \leq b_3^{\frac{1}{2}} \left(\frac{2}{d-2} \right)^{\frac{3}{2}} \|\nabla u^-\|^2, \tag{2.48}$$

where the last inequality follows from the Hardy inequality (2.18) and $|\lambda_2| \leq \lambda_1$.

Moreover, all the integrals over the boundary in (2.42) can be estimated from below by zero. Indeed, in the Neumann case, all the boundary terms vanish immediately. To obtain the same result in the Dirichlet case, we note that $u = 0$ along the whole boundary and $\frac{x}{|x|}$ represents the tangential vector on the bounary of the half-space $\mathbb{R}^d_+$, thus $\partial_r u = 0$. In the Robin case **C3**, the integrals over the boundary can be estimated as

$$2\Re \int_{\partial\mathbb{R}^d_+} \alpha |x| u \partial_r \bar{u} + \int_{\partial\mathbb{R}^d_+} (d-1)\alpha |u|^2 + \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \int_{\partial\mathbb{R}^d_+} \alpha |x| |u|^2 = - \int_{\partial\mathbb{R}^d_+} x' \cdot \nabla' \alpha |u|^2 + \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \int_{\partial\mathbb{R}^d_+} \alpha |x| |u|^2 \geq 0,$$

where the equality follows from (2.22) and the inequality is the result of the assumption in **C3**.

In conclusion, using (2.46) and (2.48), we estimate the left-hand side of (2.42) as

$$I \geq \left[1 - b_3^{\frac{1}{2}} \left(\frac{2}{d-2} \right)^{\frac{3}{2}} \right] \int_{\mathbb{R}^d_+} |\nabla u^-|^2 + \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \int_{\mathbb{R}^d_+} \left| \nabla \left(|x|^{\frac{1}{2}} u^- \right) \right|^2.$$

Applying this result together with the estimates (2.43), (2.44) and (2.45) on (2.42), we obtain

$$\left[1 - b_2^2 - 2b_3 - b_3^{\frac{1}{2}} \left(\frac{2}{d-2} \right)^{\frac{3}{2}} \right] \int_{\mathbb{R}^d_+} |\nabla u^-|^2 + \left(1 - b_1^2\right) \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \int_{\mathbb{R}^d_+} \left| \nabla \left(|x|^{\frac{1}{2}} u^- \right) \right|^2 \leq 0. \tag{2.49}$$

Both the brackets in front of the integrals are strictly positive due to the assumption (1.6) therefore this inequality can only hold if $\|\nabla u^-\| = 0$. This implies that $u^- = 0$ and thus also $u = 0$ since any other constant would lead to an infinite norm.

- $|\lambda_2| > \lambda_1$

  The proof of this case is analogous to that of the Lemma 6. We take the identity (2.39) and make the estimate from below as follows:

$$\begin{aligned}(\lambda_1 \pm \lambda_2) \int_{\mathbb{R}^d_+} |u|^2 &= \int_{\mathbb{R}^d_+} |\nabla u|^2 + \int_{\mathbb{R}^d_+} V_1 |u|^2 \pm \int_{\mathbb{R}^d_+} V_2 |u|^2 \geq \int_{\mathbb{R}^d_+} |\nabla u|^2 - \int_{\mathbb{R}^d_+} (V_1)_- |u|^2 - \left| \int_{\mathbb{R}^d_+} V_2 |u|^2 \right| \\ &\geq \int_{\mathbb{R}^d_+} |\nabla u|^2 - \int_{\mathbb{R}^d_+} (V_1)_- |u|^2 - \|x V_2 u\| \left\| \frac{u}{|x|} \right\| \\ &\geq \left[1 - b_1^2 - \frac{2b_3}{d-2}\right] \int_{\mathbb{R}^d_+} |\nabla u|^2,\end{aligned} \tag{2.50}$$

where the last inequality follows from the Hardy inequality and the assumptions (1.7) and (1.9). The bracket on the right-hand side of (2.50) is strictly positive by (1.6). However, since $|\lambda_2| > \lambda_1$, the sum $\lambda_1 \pm \lambda_2$ is necessarily strictly negative in one of the cases. Hence, the inequality (2.50) holds only for $u = 0$.

□

We are now in position to prove the last main result of this chapter.

Proof of the Theorem 4. By the Lemma 7, we have $\sigma_p(H_V) \cap \{\lambda_1 > 0\} = \emptyset$. If we now consider that $\lambda_1 \leq 0$, then choosing $v := u$ in (2.1) and taking the resulting real part, we obtain

$$\lambda_1 \int_{\mathbb{R}^d_+} |u|^2 = \int_{\mathbb{R}^d_+} |\nabla u|^2 + \int_{\mathbb{R}^d_+} V_1 |u|^2 - \Re \int_{\partial\mathbb{R}^d_+} \frac{\partial u}{\partial n} \bar{u} \geq \int_{\mathbb{R}^d_+} |\nabla u|^2 - \int_{\mathbb{R}^d_+} (V_1)_- |u|^2 \geq (1 - b_1^2) \int_{\mathbb{R}^d_+} |\nabla u|^2, \tag{2.51}$$

where the last inequality follows from (1.7). The integral over the boundary vanishes in Dirichlet and Neumann case and it can be estimated from below by zero in the Robin case due to the assumption in **C3**. The assumption (1.6) implies that $1 - b_1^2 > 0$, but $\lambda_1 \leq 0$, therefore only $u = 0$ satisfies the inequality (2.51), *i.e.* $\sigma_p(H_V) \cap \{\lambda_1 \leq 0\} = \emptyset$. □

We conclude this section by comparing the results for real and complex potentials. The Theorems 5 and 3 differ in the conditions on Λ (2.15) and (2.31). There appears an additional non-negative term in the non-self-adjoint case thus (2.31) is a stronger version of (2.15).

If we consider the Theorem 4 with $V_2 = 0$ we can put $b_3 = 0$ and the conditions (1.6) to (1.9) match those of the Theorem 6.s

2.4 Complex Robin boundary conditions

In this section we discuss the result for the complex counterpart of the Robin boundary conditions **C3** on the half-space $\mathbb{R}^d_+$, *i.e.*

$$\alpha u + \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\mathbb{R}^d_+,$$

where $\alpha : \partial\mathbb{R}^d_+ \to \mathbb{C}$. We will impose additional conditions on α as we proceed. Our aim is to obtain similar results as Lemma 6 and Lemma 7.

We start with an analogue to the Lemma 6 and the case $|\lambda_2| \le \lambda_1$. The identity (2.32) in our case reads

$$\begin{aligned} I := &\int_{\mathbb{R}^d_+} |\nabla u^-|^2 + \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \int_{\mathbb{R}^d_+} |x||\nabla u^-|^2 - \frac{d-1}{2}\frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \int_{\mathbb{R}^d_+} \frac{|u|^2}{|x|} \\ &- 2\lambda_1^{\frac{1}{2}} \operatorname{sgn}(\lambda_2) \int_{\partial\mathbb{R}^d_+} \alpha_2 |x||u|^2 + 2\Re \int_{\partial\mathbb{R}^d_+} \alpha |x| u \partial_r \bar{u} + (d-1) \int_{\partial\mathbb{R}^d_+} \alpha_1 |u|^2 + \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \Re \int_{\partial\mathbb{R}^d_+} \alpha_1 |x||u|^2 \\ = &\underbrace{(1-d)\int_{\mathbb{R}^d_+} V_1 |u|^2}_{I_1} \underbrace{-2\Re \int_{\mathbb{R}^d_+} |x| V u \left(i\lambda_1^{\frac{1}{2}} \operatorname{sgn}(\lambda_2)\bar{u} + \partial_r \bar{u}\right)}_{I_2} \underbrace{- \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \int_{\mathbb{R}^d_+} V_1 |x||u|^2}_{I_3}. \end{aligned} \tag{2.52}$$

Now, we proceed in the same fashion as we did in the real case but we will keep all the boundary terms. However, additional boundary term involving α_2 will arise. Namely, the bound (2.36) will read

$$\|u\| = |\lambda_2|^{-\frac{1}{2}} \left(\Im \int_{\mathbb{R}^d_+} V|u|^2 + \int_{\partial\mathbb{R}^d_+} \alpha_2 |u|^2 \right)^{\frac{1}{2}} \le |\lambda_2|^{-\frac{1}{2}} \left(\int_{\mathbb{R}^d_+} |Vu||u| \right)^{\frac{1}{2}} + |\lambda_2|^{-\frac{1}{2}} \left(\int_{\partial\mathbb{R}^d_+} |\alpha_2||u|^2 \right)^{\frac{1}{2}}. \tag{2.53}$$

Hence, repeating the estimates in (2.37), we obtain

$$|I_3| \le \Lambda^{\frac{3}{2}} \frac{\sqrt{2}}{\sqrt{d-2}} \int_{\mathbb{R}^d_+} |\nabla u^-|^2 + \Lambda \sqrt{\int_{\mathbb{R}^d_+} |\nabla u^-|^2} \sqrt{\int_{\partial\mathbb{R}^d_+} |\alpha_2||u|^2} \le \left(\Lambda^{\frac{3}{2}} \frac{\sqrt{2}}{\sqrt{d-2}} + \frac{\Lambda}{2} \right) \int_{\mathbb{R}^d_+} |\nabla u^-|^2 + \frac{\Lambda}{2} \int_{\partial\mathbb{R}^d_+} |\alpha_2||u|^2,$$

where the last inequality follows from the Young inequality. Thus, the key inequality (2.38) in our case reads

$$\begin{aligned} &\left(1 - \frac{9d-14}{2(d-2)}\Lambda - \frac{\sqrt{2}}{\sqrt{d-2}}\Lambda^{\frac{3}{2}}\right) \int_{\mathbb{R}^d_+} |\nabla u^-|^2 + \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \frac{d-3}{d-1} \int_{\mathbb{R}^d_+} |x||\nabla u^-|^2 \\ &- 2\lambda_1^{\frac{1}{2}} \operatorname{sgn}(\lambda_2) \int_{\partial\mathbb{R}^d_+} \alpha_2 |x||u|^2 + 2\Re \int_{\partial\mathbb{R}^d_+} \alpha |x| u \partial_r \bar{u} + (d-1) \int_{\partial\mathbb{R}^d_+} \alpha_1 |u|^2 + \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \int_{\partial\mathbb{R}^d_+} \alpha_1 |x||u|^2 - \frac{\Lambda}{2} \int_{\partial\mathbb{R}^d_+} |\alpha_2||u|^2 \\ = &\left(1 - \frac{9d-14}{2(d-2)}\Lambda - \frac{\sqrt{2}}{\sqrt{d-2}}\Lambda^{\frac{3}{2}}\right) \int_{\mathbb{R}^d_+} |\nabla u^-|^2 + \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \frac{d-3}{d-1} \int_{\mathbb{R}^d_+} |x||\nabla u^-|^2 \\ &- 2 \int_{\partial\mathbb{R}^d_+} \alpha_2 |x| \Im(u^- \partial_r \overline{u^-}) - \int_{\partial\mathbb{R}^d_+} x' \cdot \nabla' \alpha_1 |u|^2 + \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}} \int_{\partial\mathbb{R}^d_+} \alpha_1 |x||u|^2 - \frac{\Lambda}{2} \int_{\partial\mathbb{R}^d_+} |\alpha_2||u|^2 \le 0, \end{aligned} \tag{2.54}$$

where the equality follows form (2.22). The bracket in front of the first integral can be made strictly positive by imposing modified condition (2.31). It also implies $\Lambda < \frac{d-2}{4}$ thus we have

$$-\int_{\partial\mathbb{R}^d_+} x'\cdot\nabla'\alpha_1|u|^2 - \frac{\Lambda}{2}\int_{\partial\mathbb{R}^d_+}|\alpha_2||u|^2 \geq -\int_{\partial\mathbb{R}^d_+}\left(x'\cdot\nabla'\alpha_1 + \frac{d-2}{8}|\alpha_2|\right)|u|^2.$$

This term can be made positive by imposing $x'\cdot\nabla'\alpha_1 \leq \frac{2-d}{8}|\alpha_2|$. The integral $\frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}}\int_{\partial\mathbb{R}^d_+}\alpha_1|x||u|^2$ can be estimated from below by zero if we assume that α_1 is non-negative. However, the integral $-2\int_{\partial\mathbb{R}^d_+}\alpha_2|x|\Im(u\partial_r\bar{u})$ remains a problem. There is no other integral to compare it with and there is no suitable condition on α_2 independent of u.

The case $|\lambda_2| > \lambda_1$ yields another condition on α. Indeed, the inequality (2.40) in our case reads

$$(\lambda_1 \pm \lambda_2)\int_{\mathbb{R}^d_+}|u|^2 \geq \left(1 - \frac{4}{d-2}\Lambda\right)\int_{\mathbb{R}^d_+}|\nabla u|^2 + \int_{\partial\mathbb{R}^d_+}(\alpha_1 \pm \alpha_2)|u|^2, \tag{2.55}$$

where the integral over the boundary can be estimated from below by zero if we assume $|\alpha_2| \leq \alpha_1$. In conclusion, notwithstanding the problematic term $-2\int_{\partial\mathbb{R}^d_+}\alpha_2|x|\Im(u\partial_r\bar{u})$, we would impose the condition on α as follows:

$$|\alpha_2| \leq \alpha_1, \qquad x'\cdot\nabla'\alpha_1 \leq \frac{2-d}{8}|\alpha_2|. \tag{2.56}$$

If the function α is real, this result is consistent with **C3**.

Alternatively, we could assume that α_2 is negative which would result in (2.53) being the same as (2.36). The inequality (2.54) would then read

$$\left(1 - \frac{2(2d-3)}{d-2}\Lambda - \frac{\sqrt{2}}{\sqrt{d-2}}\Lambda^{\frac{3}{2}}\right)\int_{\mathbb{R}^d_+}|\nabla u^-|^2 + \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}}\frac{d-3}{d-1}\int_{\mathbb{R}^d_+}|x||\nabla u^-|^2$$
$$-2\int_{\partial\mathbb{R}^d_+}\alpha_2|x|\Im(u^-\partial_r\overline{u^-}) - \int_{\partial\mathbb{R}^d_+}x'\cdot\nabla'\alpha_1|u|^2 + \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}}\int_{\partial\mathbb{R}^d_+}\alpha_1|x||u|^2 \leq 0.$$

In this case we could assume that α_1 would satisfy the conditions **C3** imposed on α. Nonetheless, the term $-2\int_{\partial\mathbb{R}^d_+}\alpha_2|x|\Im(u^-\partial_r\overline{u^-})$ would still remain a problem.

The analogue to the Lemma 7 would also involve the problematic integral over the boundary. Indeed, starting with the case $|\lambda_2| \leq \lambda_1$, we repeat the proof of Lemma 7 with complex Robin boundary conditions. The identity (2.42) differs only in the integrals over the boundary which are the same as in (2.52). We obtain additional boundary term involving α_2 also from (2.47) which reads

$$\|u\| = |\lambda_2|^{-\frac{1}{2}}\left(\Im\int_{\mathbb{R}^d_+}V|u|^2 + \int_{\partial\mathbb{R}^d_+}\alpha_2|u|^2\right)^{\frac{1}{2}} \leq |\lambda_2|^{-\frac{1}{2}}\left(\frac{2b_3}{d-2}\int_{\mathbb{R}^d_+}|\nabla u|^2\right)^{\frac{1}{2}} + |\lambda_2|^{-\frac{1}{2}}\left(\int_{\partial\mathbb{R}^d_+}|\alpha_2||u|^2\right)^{\frac{1}{2}}. \tag{2.57}$$

Repeating the steps of the proof of Lemma 7, we obtain the following inequality:

$$\left[\frac{d-3}{d-2} - b_2^2 - 2b_3 - b_3^{\frac{1}{2}}\left(\frac{2}{d-2}\right)^{\frac{3}{2}}\right]\int_{\mathbb{R}^d_+}|\nabla u^-|^2 + \left(1-b_1^2\right)\frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}}\int_{\mathbb{R}^d_+}\left|\nabla\left(|x|^{\frac{1}{2}}u^-\right)\right|^2$$
$$-2\int_{\partial\mathbb{R}^d_+}\alpha_2|x|\Im(u^-\partial_r\overline{u^-}) - \int_{\partial\mathbb{R}^d_+}x'\cdot\nabla'\alpha_1|u|^2 + \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}}\int_{\partial\mathbb{R}^d_+}\alpha_1|x||u|^2 - \frac{1}{d-2}\int_{\partial\mathbb{R}^d_+}|\alpha_2||u|^2 \leq 0.$$

This result is even worse than (2.54) since we are not able to make the bracket in front of the first integral positive for $d=3$. This problem could be solved by assuming that α_2 is negative which would imply that (2.57) is the same as (2.47) and we would obtain

$$\left[1 - b_2^2 - 2b_3 - b_3^{\frac{1}{2}}\left(\frac{2}{d-2}\right)^{\frac{3}{2}}\right]\int_{\mathbb{R}^d_+}|\nabla u^-|^2 + \left(1-b_1^2\right)\frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}}\int_{\mathbb{R}^d_+}\left|\nabla\left(|x|^{\frac{1}{2}}u^-\right)\right|^2$$
$$-2\int_{\partial\mathbb{R}^d_+}\alpha_2|x|\Im(u^-\partial_r\overline{u^-}) - \int_{\partial\mathbb{R}^d_+}x'\cdot\nabla'\alpha_1|u|^2 + \frac{|\lambda_2|}{\lambda_1^{\frac{1}{2}}}\int_{\partial\mathbb{R}^d_+}\alpha_1|x||u|^2 \leq 0,$$

which differs from (2.49) only in the boundary terms. The integrals involving α_1 could be estimated from below by zero if we assumed that α_1 satisfies the conditions imposed on α in **C3**. However, we would be still left with the problematic term $-2\int_{\partial\mathbb{R}^d_+}\alpha_2|x|\Im(u^-\partial_r\overline{u^-})$.

Chapter 3

The operator $-\nabla(a\nabla)$

3.1 Definition of the operator

In this section we summarize the theory of sectorial forms and we use it to define the operator (1.12) via the Friedrichs equation. Let us start with some elementary definitions.

Definition 3.1.1. A map $t : D(t) \times D(t) \to \mathbb{C}$ is called a sesquilinear form in $\mathscr{H}$ if it is conjugate linear in the first argument and linear in the second. The function $t[u] := t(u, u)$ is called a quadratic form.

Contrary to linear operators, it is not difficult to find the adjoint form which is defined by $t^*(\psi, \varphi) := \overline{t(\varphi, \psi)}$, $D(t^*) = D(t)$. We say that a form is symmetric if $t(\psi, \varphi) = t^*(\psi, \varphi)$. Equipped with the adjoint form, we can define its real and imaginary part as $\Re t := \frac{t+t^*}{2}$ and $\Im t := \frac{t-t^*}{2i}$. Note that neither $\Re t$ nor $\Im t$ are real-valued, however, it holds that $\Re t[\psi] = \Re(t[\psi])$, $\Im t[\psi] = \Im(t[\psi])$ and we can also write $t = \Re t + i\Im t$. The following notion is important for the definition of the sectorial form.

Definition 3.1.2. Let t be a sesquilinear form in $\mathscr{H}$. Its numerical range is defined by

$$\Theta(t) := \{t[\phi] \mid \phi \in D(t), \|\phi\| = 1\}.$$

The numerical range of an operator T in $\mathscr{H}$ is defined by

$$\Theta(T) := \{(\phi, T\phi) \mid \phi \in D(t), \|\phi\| = 1\}.$$

In general, a numerical range need not to be closed or open, nonetheless, it is always a convex subset of the complex plane. Now we are in position to define the sectorial form.

Definition 3.1.3. A sesquilinear form t in $\mathscr{H}$ is called sectorial if its numerical range is a subset of a sector, *i.e.*

$$\Theta(h) \subset S_{\gamma,\theta} := \{\lambda \in \mathbb{C} \mid |arg(\lambda - \gamma)| \leq \theta\} \tag{3.1}$$

with a vertex $\gamma \in \mathbb{R}$ and a semi-angle θ such that $0 \leq \theta < \frac{\pi}{2}$.

Note that the parameters γ and θ are not uniquely determined by the form t. Indeed, a reduction of the semi-angle θ can be compensated by a reduction of the vertex γ. Every symmetric form is real-valued and if it is also bounded from below, then it is sectorial with $\gamma = 0$. Hence, the sectorial forms can be regarded as a generalization of symmetric forms bounded from below. We introduce the notion of relative boundedness which specifies a relation between two forms.

Definition 3.1.4. Let t be a sectorial form in $\mathscr{H}$. A form t' in $\mathscr{H}$ is said to be relatively bounded with respect to t (or t-bounded), if $D(t') \supset D(t)$ and

$$|t'[u]| \leq a\|u\|^2 + b|t[u]|, \tag{3.2}$$

where $u \in D(t)$ and a, b are nonnegative constants.

This property is useful for studying sums of a sectorial forms.

Theorem 7. *([5, Theorem VI-1.33]) Let t be a sectorial form and let t' be t-bounded with $b < 1$ in (3.2). Then $t + t'$ is sectorial. The form $t + t'$ is closed, if and only if t is closed.*

The notion of sectoriality becomes more complicated for linear operators. Let us start with the following definitions.

Definition 3.1.5. A linear operator T in $\mathscr{H}$ is said to be accretive if $Re(\psi, T\psi) \geq 0$ for all $\phi \in D(T)$, and quasi-accretive if $T + \alpha I$ is accretive for some $\alpha > 0$.

Definition 3.1.6. A closed linear operator T in $\mathscr{H}$ is said to be m-accretive if it satisfies

$$\{\lambda \in \mathbb{C} \mid \Re\lambda < 0\} \subset \rho(T),$$

$$\|(T - \lambda I)^{-1}\| \leq \frac{1}{|\Re\lambda|} \qquad \textit{for} \quad \Re\lambda < 0$$

If $T + \alpha I$ is m-accretive for some $\alpha > 0$, then T is said to be quasi-m-accretive.

If an operator is m-accretive it means that it is *maximal accretive* in the sense that it is accretive and there is no proper accretive extension.

Definition 3.1.7. A linear operator T in $\mathscr{H}$ is said to be sectorial if its numerical range lies in a sector defined by (3.1). We say that T is m-sectorial if it is sectorial and quasi-m-accretive

An important property of m-sectorial operators is that they are closed and densely defined. If a form t is bounded we can associate with it a bounded operator T so that $t(\psi, \varphi) = (\psi, T\varphi)$. This claim can be extended to densely defined, sectorial and closed form. In this case the associated operator is m-sectorial.

Theorem 8. *(The first representation theorem, [5, Theorem VI-2.1]) Let t be a densely defined, closed, sectorial sesquilinear form in $\mathscr{H}$. There exists an m-sectorial operator T such that*

i) *$D(T) \subset D(t)$ and*

$$t(u, v) = (u, Tv)$$

for every $u \in D(t)$ and $v \in D(T)$;

ii) *$D(T)$ is a core of t;*

iii) *if $v \in D(t)$, $w \in \mathscr{H}$ and*

$$t(u,v) = (w,v)$$

holds for every u belonging to a core of t, then $v \in D(T)$ and $Tv = w$.

The m-sectorial operator T is uniquely defined by the condition i

Furthermore, from the above theorem follows that there is a one-to-one correspondence between the set of all m-sectorial operators and the set of all densely defined, closed and sectorial sesquilinear forms.

Let us proceed with the definition of the operator (1.12). We start with the minimal operator

$$\tilde{H}\psi = -\nabla(a\nabla\psi)$$
$$D(\tilde{H}) = C_0^\infty(\mathbb{R}^d),$$

where $C_0^\infty(\mathbb{R}^d)$ consists of smooth functions on $\mathbb{R}^d$ with compact support. The minimal operator is associated with the quadratic form

$$\begin{aligned}\tilde{h}[\psi] :=& (\psi, \tilde{H}\psi) = \int_{\mathbb{R}^d} \nabla\bar{\psi} a \nabla\psi = \int_{\mathbb{R}^d} \nabla\bar{\psi} \mathrm{Re}\, a \nabla\psi + \int_{\mathbb{R}^d} \nabla\bar{\psi} \mathrm{Im}\, a \nabla\psi \\ D(\tilde{h}) =& C_0^\infty(\mathbb{R}^d),\end{aligned}$$

where $\mathrm{Re}\, a = \frac{a+a^*}{2}$, $\mathrm{Im}\, a = \frac{a-a^*}{2i}$. Let us now concentrate on the real part of this form. We assume $\mathrm{Im}\, a = 0$ hence the operator $\tilde{H}$ is symmetric and bounded from below. By [5, Corollary VI-1.28], the associated form $\tilde{h}$ is closable. Its closure is

$$\mathrm{Re}\, h[\psi] = \int_{\mathbb{R}^d} \nabla\bar{\psi} \mathrm{Re}\, a \nabla\psi$$
$$D(\mathrm{Re}\, h) = \overline{C_0^\infty(\mathbb{R}^d)}^{\|.\|_{\mathrm{Re}\, \tilde{h}}},$$

where $\|.\|_{\mathrm{Re}\, \tilde{h}}$ is the norm induced by $\mathrm{Re}\, \tilde{h}$. Assuming $\mathrm{Re}\, a < C$ together with (1.11), we see that the norm $\|.\|_{\mathrm{Re}\, \tilde{h}}$ is equivalent to the norm $\|f\|_{H^1(\mathbb{R}^d)} = \left(\sum_{|\alpha|\le 1} \|D^\alpha f\|^2\right)^{\frac{1}{2}}$ corresponding to the Sobolev space $H^1(\mathbb{R}^d)$. Hence, the domain of the form h is by definition equal to $H_0^1(\mathbb{R}^d)$. In the case of the wole space $\mathbb{R}^d$ we can write

$$D(\mathrm{Re}\, h) = H^1(\mathbb{R}^d).$$

The form h is closed, symmetric and bounded from below, thus we can use the First representation theorem to associate it with the operator

$$\begin{aligned}\mathrm{Re}\, H\psi =& \eta \\ D(\mathrm{Re}\, H) =& \left\{\psi \in H^1(\mathbb{R}^d) \mid \exists \eta \in L^2(\mathbb{R}^d) \quad \forall \varphi \in H^1(\mathbb{R}^d) \quad \mathrm{Re}\, h(\varphi,\psi) = (\varphi,\eta)\right\}.\end{aligned}$$

The operator (1.12) with $\mathrm{Im}\, a \neq 0$ can be defined using the perturbation theory. Let us assume $\mathrm{Im}\, a < cI$, where c is defined in (1.11). The form $\int_{\mathbb{R}^d} \nabla\bar{\psi} \mathrm{Im}\, a \nabla\psi$ now satisfies (3.2) and we can regard it as a perturbation of the real part of the form h. By the Theorem 7, the form h is closed and sectorial. Using the First representation theorem, we associate this form with the operator

$$\begin{aligned} H\psi &= \eta \\ D(H) &= \left\{\psi \in H^1(\mathbb{R}^d) \mid \exists \eta \in L^2(\mathbb{R}^d) \quad \forall \varphi \in H^1(\mathbb{R}^d) \quad h(\varphi, \psi) = (\varphi, \eta)\right\}. \end{aligned}$$

3.2 Toy model

Let us start by studying the point spectrum of a simple one-dimensional operator. Assume that $A \neq 1$ and d are positive constants. We define real function

$$a(x) = \begin{cases} A & x \in (-d, d) \\ 1 & \text{otherwise.} \end{cases}$$

This function models a potential well if $A < 1$ whereas in the case $A > 1$ it represents a potential barrier. Let us consider the operator $H := -\frac{d}{dx} a(x) \frac{d}{dx}$. Our aim is to explicitly find its point spectrum hence we are interested the eigenvalue equation

$$-\frac{d}{dx} a(x) \frac{d}{dx} \psi - \lambda \psi = 0. \tag{3.3}$$

Let us start with negative eigenvalues. We split the real axis into three intervals in which the above equation reads

$$\begin{aligned} -\psi'' - \lambda\psi &= 0 \quad \text{in } (-\infty, -d) \\ -A\psi'' - \lambda\psi &= 0 \quad \text{in } (-d, d) \\ -\psi'' - \lambda\psi &= 0 \quad \text{in } (d, \infty). \end{aligned} \tag{3.4}$$

For each of the intervals we have the solution

$$\begin{aligned} \psi_{\mathrm{I}}(x) &= A_{\mathrm{I}} e^{\sqrt{-\lambda}x} + B_{\mathrm{I}} e^{-\sqrt{-\lambda}x} && \text{in } (-\infty, -d) \\ \psi_{\mathrm{II}}(x) &= A_{\mathrm{II}} e^{\sqrt{-\frac{\lambda}{a}}x} + B_{\mathrm{II}} e^{-\sqrt{-\frac{\lambda}{a}}x} && \text{in } (-d, d) \\ \psi_{\mathrm{III}}(x) &= A_{\mathrm{III}} e^{\sqrt{-\lambda}x} + B_{\mathrm{III}} e^{-\sqrt{-\lambda}x} && \text{in } (d, \infty). \end{aligned}$$

The solutions lie in $L^2(\mathbb{R}^d)$ hence we impose the conditions

$$\lim_{x \to -\infty} \psi_{\mathrm{I}}(x) = 0, \qquad \lim_{x \to \infty} \psi_{\mathrm{III}}(x) = 0 \tag{3.5}$$

which in turn imply

$$B_{\mathrm{I}} = 0, \qquad A_{\mathrm{III}} = 0.$$

We assume that the final solution ψ is continuous therefore we impose conditions $\psi_{\mathrm{I}}(-d) = \psi_{\mathrm{II}}(-d)$ and $\psi_{\mathrm{II}}(d) = \psi_{\mathrm{III}}(d)$ which read

$$A_{\mathrm{I}} e^{-\sqrt{-\lambda}d} = A_{\mathrm{II}} e^{-\sqrt{-\frac{\lambda}{A}}d} + B_{\mathrm{II}} e^{\sqrt{-\frac{\lambda}{A}}d}, \tag{3.6}$$

$$B_{\mathrm{III}} e^{\sqrt{-\lambda}d} = A_{\mathrm{II}} e^{\sqrt{-\frac{\lambda}{A}}d} + B_{\mathrm{II}} e^{-\sqrt{-\frac{\lambda}{A}}d}. \tag{3.7}$$

Additionally, we impose the conditions $\psi_{\mathrm{I}}(-d) = A\psi_{\mathrm{II}}(-d)$ and $A\psi'_{\mathrm{II}}(d) = \psi'_{\mathrm{III}}(d)$ which yield

$$\sqrt{-\lambda}A_{\mathrm{I}}e^{-\sqrt{-\lambda}d} = A\sqrt{-\frac{\lambda}{A}}A_{\mathrm{II}}e^{-\sqrt{-\frac{\lambda}{A}}d} - A\sqrt{-\frac{\lambda}{A}}B_{\mathrm{II}}e^{\sqrt{-\frac{\lambda}{A}}d}, \tag{3.8}$$

$$-\sqrt{-\lambda}B_{\mathrm{III}}e^{-\sqrt{-\lambda}d} = A\sqrt{-\frac{\lambda}{A}}A_{\mathrm{II}}e^{\sqrt{-\frac{\lambda}{A}}d} - A\sqrt{-\frac{\lambda}{A}}B_{\mathrm{II}}e^{-\sqrt{-\frac{\lambda}{A}}d}. \tag{3.9}$$

These conditions can be compactly rewritten as a matrix equation

$$\mathbb{M}\begin{pmatrix} A_{\mathrm{I}} \\ B_{\mathrm{III}} \\ A_{\mathrm{II}} \\ B_{\mathrm{II}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where

$$\mathbb{M} = \begin{pmatrix} e^{-\sqrt{-\lambda}d} & 0 & -e^{-\sqrt{-\frac{\lambda}{A}}d} & -e^{\sqrt{-\frac{\lambda}{A}}d} \\ 0 & e^{-\sqrt{-\lambda}d} & -e^{\sqrt{-\frac{\lambda}{A}}d} & -e^{-\sqrt{-\frac{\lambda}{A}}d} \\ \sqrt{-\lambda}e^{-\sqrt{-\lambda}d} & 0 & -A\sqrt{-\frac{\lambda}{A}}e^{-\sqrt{-\frac{\lambda}{A}}d} & A\sqrt{-\frac{\lambda}{A}}e^{\sqrt{-\frac{\lambda}{A}}d} \\ 0 & -\sqrt{-\lambda}e^{-\sqrt{-\lambda}d} & -A\sqrt{-\frac{\lambda}{A}}e^{\sqrt{-\frac{\lambda}{A}}d} & A\sqrt{-\frac{\lambda}{A}}e^{-\sqrt{-\frac{\lambda}{A}}d} \end{pmatrix}.$$

Since this equation is homogeneous, there exists a non-trivial solution only for $\det\mathbb{M} = 0$ which implies

$$-\lambda e^{-2\sqrt{-\lambda}d}\left\{(A+1)\left(e^{2\sqrt{-\frac{\lambda}{A}}d} - e^{-2\sqrt{-\frac{\lambda}{A}}d}\right) + 2\sqrt{A}\lambda\left(e^{2\sqrt{-\frac{\lambda}{A}}d} + e^{-2\sqrt{-\frac{\lambda}{A}}d}\right)\right\} = 0.$$

Doing some basic algebraic operations, we can rewrite the equation above as

$$\left(1-\sqrt{A}+e^{2\sqrt{-\frac{\lambda}{A}}d}\left(1+\sqrt{A}\right)\right)\left(-1+\sqrt{A}+e^{2\sqrt{-\frac{\lambda}{A}}d}\left(1+\sqrt{A}\right)\right) = 0.$$

This condition consequently implies that the only negative eigenvalue is

$$-\frac{A}{4d}\log^2\left(\frac{|1-\sqrt{A}|}{1+\sqrt{A}}\right).$$

This result remains the same for the potential well $A < 1$ and for the potential barrier $A > 1$.

For $\lambda = 0$ the eigenvalue equations read

$$\begin{aligned} -\psi'' &= 0 \quad \text{in } (-\infty, -d) \\ -A\psi'' &= 0 \quad \text{in } (-d, d) \\ -\psi'' &= 0 \quad \text{in } (d, \infty). \end{aligned}$$

The results are linear functions and due to integrability condition we deduce that zero is not included in the point spectrum.

Lastly, the equations (3.4) for positive eigenvalues yield

$$\begin{aligned} \psi_{\mathrm{I}}(x) &= A_{\mathrm{I}}e^{i\sqrt{\lambda}x} + B_{\mathrm{I}}e^{-i\sqrt{\lambda}x} && \text{in } (-\infty, -d) \\ \psi_{\mathrm{II}}(x) &= A_{\mathrm{II}}e^{i\sqrt{\frac{\lambda}{d}}x} + B_{\mathrm{II}}e^{-i\sqrt{\frac{\lambda}{d}}x} && \text{in } (-d, d) \\ \psi_{\mathrm{III}}(x) &= A_{\mathrm{III}}e^{i\sqrt{\lambda}x} + B_{\mathrm{III}}e^{-i\sqrt{\lambda}x} && \text{in } (d, \infty). \end{aligned}$$

These functions do not belong to L^2 therefore

$$\sigma_p(H) = -\frac{A}{4d}\log^2\left(\frac{|1-\sqrt{A}|}{1+\sqrt{A}}\right).$$

3.3 Self-adjoint operator

Let us assume that the matrix a is hermitian and the operator H is therefore self-adjoint. We will introduce two multipliers and for both of them we will derive conditions under which the point spectrum of H is empty, *i.e* there is no $\lambda \in \mathbb{C}$ satisfying the eigenvalue equation

$$-\nabla\cdot(a\nabla)u - \lambda u = 0. \tag{3.10}$$

We reformulate this equation in the weak sense so it reads

$$\forall v \in H^1(\mathbb{R}^d) \qquad -\int\limits_{\mathbb{R}^d} \nabla\bar{v}a\nabla u + \lambda\int\limits_{\mathbb{R}^d}\bar{v}u = 0. \tag{3.11}$$

The following identity will render itself useful in the forthcoming proofs.

Lemma 8. *Let $u \in \mathbb{C}^n$ and let $a \in \mathbb{C}^{n,n}$ be a hermitian matrix. Then*

$$2\mathrm{Re}\,\bar{u}_{,ij}a_{jk}u_{,k} = (\bar{u}_{,j}a_{jk}u_{,k})_{,i} - \bar{u}_{,j}a_{jk,i}u_{,k}. \tag{3.12}$$

Proof. Differentiating the first term on the right hand side, we get

$$\begin{aligned}(\bar{u}_{,j}a_{jk}u_{,k})_{,i} =& \bar{u}_{,ij}a_{jk}u_{,k} + \bar{u}_{,j}a_{jk,i}u_{,k} + \bar{u}_{,j}a_{jk}u_{,ki}\\ =&\bar{u}_{,ij}a_{jk}u_{,k} + \bar{u}_{,j}a_{jk,i}u_{,k} + u_{,ik}\bar{a}_{kj}\bar{u}_{,j}\\ =&\bar{u}_{,ij}a_{jk}u_{,k} + \overline{\bar{u}_{,ij}a_{jk}u_{,k}} + \bar{u}_{,j}a_{jk,i}u_{,k}\\ =&2\mathrm{Re}\,\bar{u}_{,ij}a_{jk}u_{,k} + \bar{u}_{,j}a_{jk,i}u_{,k}.\end{aligned}$$

The second equality follows from the fact that the matrix of second derivative is symmetric and a is hermitian. □

Let us now consider the multiplier

$$v = \frac{1}{2}[\Delta,\phi]u = \nabla\phi\cdot\nabla u + \frac{1}{2}u\Delta\phi, \tag{3.13}$$

where $\phi = |x|^2$. Such choice of the multiplier in (3.11) yields the following result.

Theorem 9. *Assume that there exists $\alpha < 1$ satisfying*

$$|x|(\partial_r a)_+ \le 2\alpha a, \tag{3.14}$$

where $(\partial_r a)_{ij} = \frac{x_k}{|x|}a_{ij,k}$. Then $\sigma_p(H) = \emptyset$.

Proof. Using the multiplier (3.13) in the eigenvalue equation (3.11) and taking the real part, we obtain

$$-(d+2)\int_{\mathbb{R}^d} \nabla\bar{u}a\nabla u - 2\mathrm{Re}\int_{\mathbb{R}^d} x\nabla^2\bar{u}a\nabla u + 2\mathrm{Re}\,\lambda\int_{\mathbb{R}^d} xu\nabla\bar{u} + d\mathrm{Re}\,\lambda\int_{\mathbb{R}^d}|u|^2 = 0. \tag{3.15}$$

Using the identity (3.12) and integrating by parts, we can rewrite the second term as

$$-2\mathrm{Re}\int_{\mathbb{R}^d} x\nabla^2\bar{u}a\nabla u = -\int_{\mathbb{R}^d} x\nabla(\nabla\bar{u}a\nabla u) + \int_{\mathbb{R}^d}\nabla\bar{u}|x|\partial_r a\nabla u = d\int_{\mathbb{R}^d}\nabla\bar{u}a\nabla u + \int_{\mathbb{R}^d}\nabla\bar{u}|x|\partial_r a\nabla u. \tag{3.16}$$

The terms containing eigenvalues are treated in a similar way. We use the identity $2\mathrm{Re}\,\bar{u}\nabla u = \nabla|u|^2$ and integrate by parts to see that the terms cancel out and we are left with

$$-2\int_{\mathbb{R}^d}\nabla\bar{u}a\nabla u + \int_{\mathbb{R}^d}\nabla\bar{u}|x|\partial_r a\nabla u = 0. \tag{3.17}$$

Note that the first term is real and negative due to a being hermitian and $0 \le c < a$. Our aim is to compare these terms and in order to so we establish the condition (3.14). Applying this condition and multiplying the equation (3.17) by $-\frac{1}{2}$, we obtain

$$(1-\alpha)\int_{\mathbb{R}^d}\nabla\bar{u}a\nabla u \le 0.$$

By $\alpha < 1$ and the uniform ellipticity condition (1.11), the integral is non-negative therefore ∇u needs to be equal to zero almost everywhere. Hence, the above inequality can be attained only for constant u. However, the only acceptable constant is zero, since u would not be integrable on $\mathbb{R}^d$ otherwise. In conclusion, there are no eigenvalues because there are no non-zero eigenvectors. ☐

Let us now consider the following multiplier.

$$v = [\nabla\cdot(a\nabla), \phi]u = \nabla u(a+\bar{a})\nabla\phi + u\nabla\cdot(a\nabla\phi), \tag{3.18}$$

where $\phi = |x|^2$. The conditions resulting from using this multiplier are as follows.

Theorem 10. *Assume that there exist constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ satisfying*

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 < 1, \tag{3.19}$$

such that for all $x \in \mathbb{R}^d$ and for all $u \in H^1(\mathbb{R}^d)$

$$x_l(a+\bar{a})_{lk,i}a_{ij} \le \alpha_1 a_{ki}a_{ij}, \tag{3.20}$$

$$\left[\frac{a\bar{a}+\bar{a}a}{2}\right]_- \le \alpha_2 aa, \tag{3.21}$$

$$[x_l(a+\bar{a})_{kl,i}a_{ij} + x_l(a+\bar{a})_{lj,i}a_{ki}]_- \le \alpha_3 a_{ki}a_{ij}, \tag{3.22}$$

$$\left|\mathrm{Re}\int_{\mathbb{R}^d}(a_{kl}x_l)_{,ki}a_{ij}\bar{u}u_{,j}\right| \le \alpha_4\|a\nabla u\|^2. \tag{3.23}$$

Then $\sigma_p(H) = 0$.

Proof. Applying this multiplier to the eigenvalue equation (3.11) and taking the real part, we get

$$-\mathrm{Re}\int_{\mathbb{R}^d}\left[\bar{u}_{,ki}(a+\bar{a})_{kl}x_l+\bar{u}_{,k}(a+\bar{a})_{kl,i}x_l+\bar{u}_{,k}(a+\bar{a})_{ki}\right]a_{ij}u_{,j}-\mathrm{Re}\int_{\mathbb{R}^d}(a_{kl}x_l)_{,k}\bar{u}_{,i}a_{ij}u_{,j}$$
$$-\mathrm{Re}\int_{\mathbb{R}^d}(a_{kl}x_l)_{,ki}a_{ij}\bar{u}u_{,j}+\lambda\mathrm{Re}\int_{\mathbb{R}^d}\bar{u}_{,k}(a+\bar{a})_{kl}x_l u+\lambda\mathrm{Re}\int_{\mathbb{R}^d}(a_{kl}x_l)_{,k}|u|^2=0.$$

Firstly, let us concentrate on the terms containing the eigenvalue λ. Using the identity $2\mathrm{Re}\,\bar{u}\nabla u=\nabla|u|^2$ and integrating by parts, we can write

$$\lambda\mathrm{Re}\int_{\mathbb{R}^d}\bar{u}_{,k}(a+\bar{a})_{kl}x_l u=\frac{\lambda}{2}\int_{\mathbb{R}^d}(a+\bar{a})_{kl}x_l|u|^2_{,k}=-\frac{\lambda}{2}\int_{\mathbb{R}^d}\left((a+\bar{a})_{kl}x_l\right)_{,k}|u|^2.$$

Considering that $\mathrm{Re}\,a_{kl}=\left(\frac{a+\bar{a}}{2}\right)_{kl}$, we see that the terms containing λ cancel out and we are left with

$$-\mathrm{Re}\int_{\mathbb{R}^d}x_l(a+\bar{a})_{lk}\bar{u}_{,ki}a_{ij}u_{,j}-\mathrm{Re}\int_{\mathbb{R}^d}\bar{u}_{,k}(a+\bar{a})_{kl,i}x_l a_{ij}u_{,j}-\mathrm{Re}\int_{\mathbb{R}^d}\bar{u}_{,k}(a+\bar{a})_{ki}a_{ij}u_{,j}$$
$$-\mathrm{Re}\int_{\mathbb{R}^d}(a_{kl}x_l)_{,k}\bar{u}_{,i}a_{ij}u_{,j}-\mathrm{Re}\int_{\mathbb{R}^d}(a_{kl}x_l)_{,ki}a_{ij}\bar{u}u_{,j}=0.$$

Using the identity (3.12) and integrating by parts, we can rewrite the first term as

$$-\mathrm{Re}\int_{\mathbb{R}^d}x_l(a+\bar{a})_{lk}\bar{u}_{,ki}a_{ij}u_{,j}=-\frac{1}{2}\int_{\mathbb{R}^d}x_l(a+\bar{a})_{lk}(\bar{u}_{,i}a_{ij}u_{,j})_{,k}+\frac{1}{2}\int_{\mathbb{R}^d}x_l(a+\bar{a})_{lk}\bar{u}_{,i}a_{ij,k}u_{,j}$$
$$=\frac{1}{2}\int_{\mathbb{R}^d}((a+\bar{a})_{lk}x_l)_{,k}\bar{u}_{,i}a_{ij}u_{,j}+\frac{1}{2}\int_{\mathbb{R}^d}x_l(a+\bar{a})_{lk}\bar{u}_{,i}a_{ij,k}u_{,j}.$$

Hence, we obtain the main identity

$$\|a\nabla u\|^2-\frac{1}{2}\int_{\mathbb{R}^d}\bar{u}_{,i}x_l(a+\bar{a})_{lk}a_{ij,k}u_{,j}+\mathrm{Re}\int_{\mathbb{R}^d}\bar{u}_{,k}(a+\bar{a})_{kl,i}x_l a_{ij}u_{,j}$$
$$+\mathrm{Re}\int_{\mathbb{R}^d}\bar{u}_{,k}\bar{a}_{ki}a_{ij}u_{,j}+\mathrm{Re}\int_{\mathbb{R}^d}(a_{kl}x_l)_{,ki}a_{ij}\bar{u}u_{,j}=0.$$

Considering the conditions (3.20) to (3.23), we obtain the inequality

$$(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\|a\nabla u\|^2\leq 0.$$

The condition (3.19) implies that the above inequality can be attained only for $\|a\nabla u\|=0$. This statement holds for all a therefore we can choose it to be scalar, *i.e.* $a=aI$. In this case we have $0=\|a\nabla u\|\geq c\|\nabla u\|$ hence u needs to be constant almost everywhere and due to integrability we conclude $u=0$. □

3.4 Non-selfadjoint operator

In this section we will discuss the results of using the method of multipliers in the non-selfadjoint case. Since we did not obtain any reasonable conditions which could be summarized in a theorem, this section will rather illustrate our endeavours.

We start by using the multiplier (3.13) in the eigenvalue equation (3.11) so that we receive the same identity (3.15) as in the self-adjoint case. However, this time a is not hermitian so we divide it into hermitian and skew-hermitian part, *i.e.* $a = a^1 + ia^2$, where

$$a^1 := \frac{a + a^*}{2}, \qquad a^2 := \frac{a - a^*}{2i}. \tag{3.24}$$

Using this formalism, we can rewrite the identity (3.15) as

$$-(d+2)\int\limits_{\mathbb{R}^d} \nabla\bar{u}a^1\nabla u - 2\mathrm{Re}\int\limits_{\mathbb{R}^d} x\nabla^2\bar{u}a^1\nabla u + 2\mathrm{Im}\int\limits_{\mathbb{R}^d} x\nabla^2\bar{u}a^2\nabla u + 2\mathrm{Re}\,\lambda\int\limits_{\mathbb{R}^d} xu\nabla\bar{u} + d\mathrm{Re}\,\lambda\int\limits_{\mathbb{R}^d}|u|^2 = 0.$$

The second term is treated in the same way as in (3.16) and the terms containing the eigenvalue λ are treated similarly as in the self-adjoint case. However, since we are dealing with a non-selfadjoint operator, the eigenvalues λ are now not necessarily real and we are left with

$$-2\int\limits_{\mathbb{R}^d} \nabla\bar{u}a^1\nabla u + \int\limits_{\mathbb{R}^d} \nabla\bar{u}|x|\partial_r a^1\nabla u + 2\mathrm{Im}\int\limits_{\mathbb{R}^d} x\nabla^2\bar{u}a^2\nabla u - 2\lambda_2\int\limits_{\mathbb{R}^d}|x|\mathrm{Im}\,(u\partial_r\bar{u}) = 0, \tag{3.25}$$

where we denote $(\partial_r a)_{ij} := \frac{x_k}{|x|}a_{ij,k}$ and $\lambda_2 := \mathrm{Im}\,\lambda$. In this identity we only know the sign of the first term and our aim is to compare the remaining terms with it. Analogously to the self-adjoint case, we deal with the second term by imposing a condition $|x|\partial_r a^1 \leq 2\alpha a^1$. However, we are not able to compare the two remaining integrals with the first term.

The idea is to include the problematic terms into a sum for which we are able to determine its sign. For this purpose, we establish additional identities which we consequently sum up together with (3.25) in order to receive a new main identity. Let $G_1, G_2 : \mathbb{R}^d \to \mathbb{R}$ be two smooth functions. Choosing the multiplier $v = G_1 u$ in (3.11) and taking the real part, we obtain

$$-\int\limits_{\mathbb{R}^d} G_1\nabla\bar{u}a^1\nabla u - \mathrm{Re}\int\limits_{\mathbb{R}^d} u\nabla G_1 a\nabla u + \lambda_1\int\limits_{\mathbb{R}^d} G_1|u|^2 = 0.$$

Similarly, choosing the multiplier $v = G_2 u$ and taking the imaginary part leads to

$$-\int\limits_{\mathbb{R}^d} G_2\nabla\bar{u}a^2\nabla u - \mathrm{Im}\int\limits_{\mathbb{R}^d} u\nabla G_2 a\nabla u + \lambda_2\int\limits_{\mathbb{R}^d} G_2|u|^2 = 0.$$

Following [4], we choose $G_1 = 1$ and $G_2 = 2\lambda_1^{\frac{1}{2}}\,\mathrm{sgn}(\lambda_2)|x|$ so that we get

$$-\int\limits_{\mathbb{R}^d} \nabla\bar{u}a^1\nabla u + \lambda_1\int\limits_{\mathbb{R}^d}|u|^2 = 0, \tag{3.26}$$

$$-2\lambda_1^{\frac{1}{2}}\,\mathrm{sgn}(\lambda_2)\int\limits_{\mathbb{R}^d}|x|\nabla\bar{u}a^2\nabla u - -2\lambda_1^{\frac{1}{2}}\,\mathrm{sgn}(\lambda_2)\mathrm{Im}\int\limits_{\mathbb{R}^d} u\frac{x}{|x|}a\nabla u + 2\lambda_1^{\frac{1}{2}}|\lambda_2|\int\limits_{\mathbb{R}^d}|x||u|^2 = 0. \tag{3.27}$$

Summing up (3.26) + (3.27) − (3.25), we obtain

$$\begin{aligned}&\int\limits_{\mathbb{R}^d} \nabla\bar{u}a^1\nabla u + \lambda_1\int\limits_{\mathbb{R}^d}|u|^2 - 2\lambda_1^{\frac{1}{2}}\operatorname{sgn}\lambda_2 \mathrm{Im}\int\limits_{\mathbb{R}^d}\bar{u}\frac{x}{|x|}a\nabla u - 2\lambda_1^{\frac{1}{2}}\operatorname{sgn}\lambda_2\int\limits_{\mathbb{R}^d}|x|\nabla\bar{u}a^2\nabla u\\ &+2\lambda_1^{\frac{1}{2}}|\lambda_2|\int\limits_{\mathbb{R}^d}|x||u|^2 - \int\limits_{\mathbb{R}^d}\nabla\bar{u}|x|\partial_r a^1\nabla u - 2\mathrm{Im}\int\limits_{\mathbb{R}^d}x\nabla^2\bar{u}a^2\nabla u + 2\lambda_2\int\limits_{\mathbb{R}^d}|x|\mathrm{Im}\,(u\partial_r\bar{u}) = 0.\end{aligned} \tag{3.28}$$

For a solution u of (3.11) we put

$$u^-(x) := u(x)e^{-i\operatorname{sgn}(\lambda_2)\lambda_1^{\frac{1}{2}}|x|}. \tag{3.29}$$

This definition is inspired by [4], where it was used to deal with terms containing $\mathrm{Im}\,(\bar{u}\partial_r u)$, since it is included in

$$|\nabla u^-|^2 = |\nabla u|^2 + \lambda_1|u|^2 - 2\lambda_1^{\frac{1}{2}}\operatorname{sgn}(\lambda_2)\mathrm{Im}\,(\bar{u}\partial_r u). \tag{3.30}$$

However, the identity (3.28) contains $\nabla\bar{u}a^1\nabla u$ instead of $|\nabla u|^2$ therefore we expect that the identity above should be generalized to

$$\nabla\bar{u}^-a^1\nabla u^- = \nabla\bar{u}a^1\nabla u + \lambda_1\frac{x}{|x|}a^1\frac{x}{|x|}|u|^2 - 2\lambda_1^{\frac{1}{2}}\operatorname{sgn}(\lambda_2)\mathrm{Im}\left(\bar{u}\frac{x}{|x|}a^1\nabla u\right).$$

The above identity coincides with (3.30) for $a^1 = I$ but it does not contain the term containing $\mathrm{Im}\,(\bar{u}\partial_r u)$ hence we are not able to use it to sum up the first three terms of (3.28) as we could have in the case $a^1 = I$ using (3.30).

Similarly, we are not able to deal with the last term of (3.28) as in the case $a^1 = I$. Choosing $G_1 = \lambda_1^{-\frac{1}{2}}|\lambda_2||x|$, we obtain

$$-\lambda_1^{-\frac{1}{2}}|\lambda_2|\mathrm{Re}\int\limits_{\mathbb{R}^d}\frac{x}{|x|}a\bar{u}\nabla u - \lambda_1^{-\frac{1}{2}}|\lambda_2|\mathrm{Re}\int\limits_{\mathbb{R}^d}|x|\nabla\bar{u}a\nabla u + \lambda_1^{-\frac{1}{2}}|\lambda_2|\int\limits_{\mathbb{R}^d}|x||u|^2 = 0.$$

Subtracting this identity from (3.28), we arrive at

$$\begin{aligned}&\int\limits_{\mathbb{R}^d}\nabla\bar{u}a^1\nabla u + \lambda_1\int\limits_{\mathbb{R}^d}|u|^2 - 2\lambda_1^{\frac{1}{2}}\operatorname{sgn}\lambda_2\mathrm{Im}\int\limits_{\mathbb{R}^d}\bar{u}\frac{x}{|x|}a\nabla u - 2\lambda_1^{\frac{1}{2}}\operatorname{sgn}\lambda_2\int\limits_{\mathbb{R}^d}|x|\nabla\bar{u}a^2\nabla u\\ &+\lambda_1^{-\frac{1}{2}}|\lambda_2|\int\limits_{\mathbb{R}^d}|x|\Big[\nabla\bar{u}a^1\nabla u + \lambda_1|u|^2 + 2\lambda_1^{\frac{1}{2}}\operatorname{sgn}(\lambda_2)\mathrm{Im}\,(u\partial_r\bar{u})\Big]\\ &-\int\limits_{\mathbb{R}^d}\nabla\bar{u}|x|\partial_r a^1\nabla u - 2\mathrm{Im}\int\limits_{\mathbb{R}^d}x\nabla^2\bar{u}a^2\nabla u + \lambda_1^{-\frac{1}{2}}|\lambda_2|\mathrm{Re}\int\limits_{\mathbb{R}^d}\frac{x}{|x|}a\bar{u}\nabla u = 0.\end{aligned}$$

Chapter 4

Conclusion

We used the method of multipliers to study the absence of point spectrum of the Schrödinger operators. We generalized the result of Theorems 1 and 2 to the half-space. We proved that the conditions stated there remain unchanged for the boundary conditions **C1** to **C3**. Nevertheless, we were unable to prove the same result for complex Robin boundary conditions which is described in the section 2.4. Possible solution could be to try non-radial multipliers. Radial multipliers were chosen in the case of the whole space due to the symmetry of the problem. However, in our situation are not able to rely on the symmetry. This obstacle restrained us from studying the main goal which we the waveguides.

We also generalized the result of Theorems 1 and 2 by introducing the new operator $-\nabla(a\nabla)$. We used two different multipliers in the self-adjoint case which resulted in Theorems 9 and 10. We did not manage to find any results for the non-selfadjoint operator with non-hermitian a.

Results given in this research project are only formal. We did not specify the space of test functions which should be followed by regularization of the functions we worked with.

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