Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

# Real Options Valuation: The Binomial Model

# Oceňování projektů metodou reálných opcí: Binomický model

Bachelor's Degree Project

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# ZADÁNÍ BAKALÁŘSKÉ PRÁCE

| | |
|---|---|
| Student: | Yevgeniy Nazarenko |
| Studijní program: | Aplikace přírodních věd |
| Obor: | Aplikovaná informatika |
| Název práce (česky): | Oceňování projektů metodou reálných opcí: Binomický model |
| Název práce (anglicky): | Real Options Valuation: The Binomial Model |

Pokyny pro vypracování:

1. Seznamte se s binomickým modelem oceňování projektů metodou reálných opcí.
2. Shrňte klíčové předpoklady modelu, popište jeho strukturu a vysvětlete jeho použití v kontextu manažerského rozhodování za neurčitosti.
3. Zvolte si jednu ze základních kategorií reálných opcí (timing options, switching options, learning options) a popište obecný přístup k jejich oceňování a algoritmus výpočtu hodnoty projektu.
4. Implementujte algoritmus oceňování ve Vámi zvoleném výpočetním nástroji a ověřte jej na simulovaných (hypotetických) nebo reálných datech.
5. Analyzujte možnosti a limity použití binomického modelu při oceňování projektů.

Doporučená literatura:

1. T. E. Copeland, V. Antikarov, Real Options: A Practitioner's Guide. Revised ed., Texere, New York, 2003.
2. F. K. Crundwell, Finance for Engineers: Evaluation and Funding of Capital Projects. Springer, London, 2008.
3. G. Guthrie, Real Options in Theory and Practice. Oxford University Press, Oxford, 2009.
4. S. M. Ross, An Elementary Introduction to Mathematical Finance. 3rd ed., Cambridge University Press, New York, 2011.
5. S. E. Shreve, Stochastic Calculus for Finance I: The Binomial Asset Pricing Model. Springer, New York, 2004.

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Jméno a pracoviště konzultanta:

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Doba platnosti zadání je dva roky od data zadání.

V Praze dne 23. října 2018

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*Author's declaration:*
I declare that this Bachelor's Degree Project is entirely my own work and I have listed all the used sources in the bibliography.

Prague, July 8, 2020

Yevgeniy Nazarenko

*Název práce:*
**Oceňování projektů metodou reálných opcí: Binomický model**

*Autor:* Yevgeniy Nazarenko

*Obor:* Aplikovaná Informatika

*Druh práce:* Bakalářská práce

*Vedoucí práce:* Ing. Rudolf Kulhavý, DrSc.

*Abstrakt:* Tato práce zkoumá oceňování projektů metodou reálných opcí pomocí binomického modelu. Tradiční metoda 'discounted cash flow' je popsána včetně jejích omezení. Binomický model a jeho předpoklady jsou detailně popsány. Aplikace reálných opcí je také zkoumaná na praktických příkladech.

*Klíová slova:* Binomický model, reálné opce

*Title:*
**Real Options Valuation: The Binomial Model**

*Author:* Yevgeniy Nazarenko

*Abstract:* This project investigates the framework of real options and its valuation using binomial model. Traditional discounted cash flow method and its limitations are discussed. Binomial model is described in detail including its assumptions. Application of real options framework is also examined and illustrated with examples.

*Key words:* binomial model, real options

# Contents

# Chapter 1

# Real Options

In pursuit of a better understanding of complex real-world economic processes, mathematicians and economists of the second half of the last century have developed numerous innovative economical models and concepts. Those models included the famous Nobel Prize (1997) winning Black-Scholes-Merton (BSM) model derived from heat equation, first introduced to the world in paper called "The Pricing of Options and Corporate Liabilities" (1973) by Fischer Black and Myron Scholes [9] and "Theory of Rational Option Pricing" by Robert Merton (1973) [10]. One of its primary applications is to value financial options and it is still being used for this matter. It is an arguably difficult partial differential equation, which stimulated development of simpler methods. John Carrington Cox, Stephen Ross and Mark Rubinstein invented the binomial model in their paper called "Option pricing: A simplified approach" from 1979 [11], a discrete approximation of the continuous BSM.

Real Options Analysis is a relatively new discipline. First books on this topic include work by Avinash Dixit and Robert Pindyck called "Investment under Uncertainty" (1994) [12] and Lenos Trigeorgis called "Real Options" [13] from 1996. For this bachelor's project, however, Graeme Guthrie's book called "Real Options in Theory and Practice" [1] has been chosed as the main source of knowledge and inspiration. The rich theory of valuating financial options has been modified to valuate strategic investments; the idea was popularized by Timothy Luehrman in his Harvard Business Review papers ("Whats It Worth? A General Managers Guide to Valuation.", 1997; [14] "Investment Opportunities as Real Options; Getting Started on the Numbers", 1998a; [15] "Strategy as a Portfolio of Real Options", 1998b [16]).

A real option is, generally speaking, a possibility to make a decision but not necessarily an obligation. Real options are particularly useful in high-risk projects, where classic Discounted Cash Flow (DCF) may undervalue possible profits due to inability to consider dynamic decision making.

## 1.1 Domain Terminology

Let us briefly introduce some of the economic and mathematical concepts and notations used in this work, should the reader be unfamiliar with them:

- **Cash Flow** $Y$ (or **CF**) is the difference between aggregated (typically, monthly or quarterly) revenues and costs.
- **Free Cash Flow** (**FCF**) is the cash flow remained after covering the company's expenses and investing in new projects.
- **Net Present Value (NPV)** is the expected return of the investment. In the case of real options, it is sometimes called *market value* and denoted $V$. [1]
- **Discounted Cash Flow (DCF)** is the sum of cash flows over multiple time periods adjusted for the time value of money. It is the standard method to calculate the NPV of cash flows.
- **Financial Option.** An option traded in financial markets is a contract providing the right to buy or sell an underlying asset, but not an obligation to do so, for a specified strike price before or on the expiration date. The most basic example could be as simple as an investor buying an option for $10 to obtain the right to buy stocks of a company for a fixed price of $100 one year later. If the market price will be $200, the investor will have spent $10 + $100 instead of $200.
- **Real Option** is the right, but not the obligation, to make a certain managerial decision. For example, opening one restaurant and already valuing possibilities of opening several more depending on the success of the first one could be a problem solved by real options.
- **Capital Asset Pricing Model** is a model used to determine an adequate change in expected profits depending on the market risk. Heavily used by Guthrie in his book [1, p. 42].

## 1.2 Discounted Cash Flow Valuation

Any investor will always want to have an idea of how profitable a project is going to be. But before we introduce options in decision making, let us first consider the conventional Discounted Cash Flow (DCF) method. Simple budgeting problems may not necessarily benefit from real options analysis [3, p. 23].

*Cash Flow* (CF) is an essential term in solving any economical problem. Its method of valuation is different for any practical problem, but it always indicates the difference between money received and money spent over a time horizon. The conventional Discounted Cash Flow (DCF) method sums cash flows over multiple time periods with every cash flow adjusted by some interest rate [3, p. 23]:

$$\mathrm{DCF} = \sum_{n=1}^{N} \frac{\mathrm{CF}_n}{(1+r)^n}, \tag{1.1}$$

The time horizon is split into $N$ periods of equal length labeled $n = 1, 2, ..., N$. The cash flow value at period $n$ is denoted $\mathrm{CF}_n$, $r > 0$ is the interest rate.

If there are no considerable uncertainties in the project implementation and if cash flows are estimated correctly, DCF analysis will provide a good answer to the question if the project is worth implementing.

| DCF is... | Meaning |
|---|---|
| negative | The investment is likely to be unprofitable. |
| close to zero | The investment will neither add or lose money.<br>Decision should be based on other criteria. |
| positive | The project is likely to be profitable. |

**Ex 1.2.1** (DCF example)**.** An investor considers opening a car wash station. Initial investment $I$ will cost \$$150,000$ with resale value $R$ of \$$100,000$ and each month they would expect to spend \$$10,000$ (monthly cost) and to receive \$$30,000$ (monthly revenue). Assuming $r = 0.1$, cash flows (revenue minus cost) for the next $6$ months are:

| Month ($m$) | $m=0$ | $m=1$ | $m=2$ | $m=3$ | $m=4$ | $m=6$ |
|---|---|---|---|---|---|---|
| Discounting factor ($(1+r)^m$) | $1.0$ | $1.1$ | $1.21$ | $1.33$ | $1.46$ | $1.61$ |
| Operating costs (\$) | $10,000$ | $10,000$ | $10,000$ | $10,000$ | $10,000$ | $10,000$ |
| Revenue (\$) | $30,000$ | $30,000$ | $30,000$ | $30,000$ | $30,000$ | $30,000$ |
| Resale Value (\$) | $0$ | $0$ | $0$ | $0$ | $0$ | $100,000$ |
| Cash Flow (\$) | $20,000$ | $20,000$ | $20,000$ | $20,000$ | $20,000$ | $120,000$ |
| $\mathrm{CF}_m/(1+r)^m$ (\$) | $20,000$ | $18,181$ | $16,528$ | $15,026$ | $13,660$ | $74,510$ |

DCF (Eq. 1.1) equals \$$157,905$. Net Present Value (NPV), the actual revenue from the investment assuming 6 months is the lifetime of the project, would be $\mathrm{NPV} = -I + \mathrm{DCF} =$ \$$7905$, which means it is a hypothetically attractive invesment.

□

This approach, however, does not account for the possibility of optimizing the project implementation dynamically, in response to the actual performance data as they become available. later as the project unfolds. What if there is a potentially significant possibility to improve the project's cash flow one year from the start? Or, to avert further losses after it becomes clear that the investment does not promise sufficient return?

## 1.3 Optimizing Project Value

At the start of a project, investors are primarily interested in having established the main risks and potential rewards. Different factors, such as significant uncertainties, may contribute to the project success or failure. It is certainly preferable for every shareholder to calculate potential returns of the project before it actually starts. However, there are projects where dynamic decision making is possible, thus, it is not obvious at all how to determine the final value of the project. If we embed these choices into our calculation, investors will have a better understanding of the potential benefits in the case of the positive decision of the investment.

Suppose a simple example where a company is presented with an investment opportunity. If they choose not to execute this option, the current NPV of the investment is $V$, debt is $D$ (e.g. a loan) and equity is $E$. Then it follows that:

$$E = V - D \tag{1.2}$$

If they choose instead to exercise this option, it will require $I$ amount of cash. It will also require $N$ amount of securities, while $N = I$. In this scenario, the equity will equal the following:

$$E' = V' - D' - N \tag{1.3}$$

where $E'$ is the new equity, $V'$ is the new market value and $D'$ is the new debt

The option is worth exercising only when $E' \geqslant E$, thus:

$$V' - D' - N \geqslant V - D \tag{1.4}$$

Now we can make this equation more logical if two statements will be assumed. First one is that $N = I$, meaning that the cash amount will equal to the securities amount. It may not be true. For example, raising funds from foreign markets may add certain inertia. The second assumption is that the new debt will equal the existing debt, that is $D' = D$. With these assumptions, the previous equation reduces to:

$$V' - V \geqslant I \tag{1.5}$$

In other words, the increase in market value must be larger than the amount of money invested.

Option value, or the NPV of the investment in this particular case, at the very moment of valuation is given by this equation [2, p.3]:

$$\text{NPV} = \max\{\ V' - V - I,\ 0\ \} \tag{1.6}$$

An option cannot be of a negative value. If the project is going to yield negative revenue, a corresponding option is worthless costing exactly zero.

## 1.4 Decision Tree Analysis

Real options analysis must start with exploring particular project implementation options. We define what kind of options we want to valuate and then use methods described in the next chapters. At this stage the analyst must determine the most significant possibilities for altering an existing strategy, abandoning the project or any other form of decision making.

Consider the hypothetical decision tree displayed at 1.1. The investor must decide if they want to develop a proof of concept (PoC) of a piece of software for $50 thousand or rather "Wait" and observe the market development and spend $10 thousand. If their PoC is successful, they can invest more into a proper solution, or they can completely abandon the project, should the market not accept the PoC well enough. At the stage of developing their PoC, their biggest uncertainty is how many clients they will be able to attract.

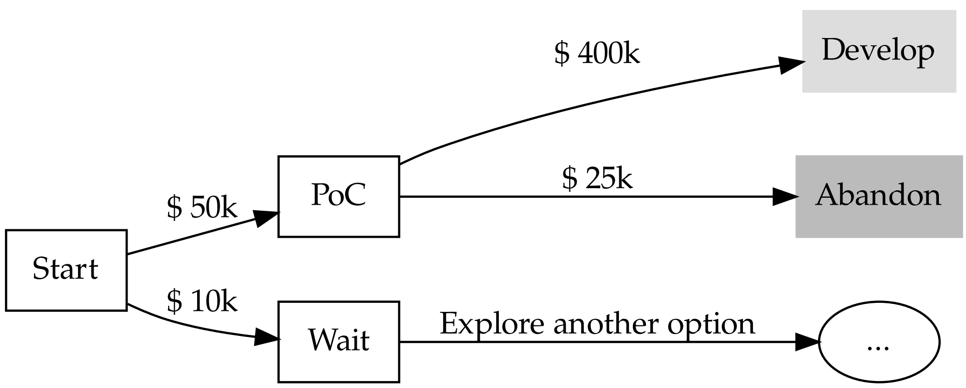


Figure 1.1: Example of a decision tree

Usually it means that the management will have, at each node, the right *but not the obligation* to exercise a particular option or a combination of them. A real option is any deferred decision that will be made during the project lifetime.

Let us at least try to solve this example using non-dynamic DCF method. We cannot embed dynamic decision making so we need to calculate a DCF for every possible outcome. Then, as Guthrie proposes [1, 1.2.2], we could simply choose the best DCF at each node of the tree.

This is where real options analysis allows us to valuate each future decision before we commit to one of them.

## 1.5 Applying Real Options

The whole ROA approach is not particularly useful when the option value is already very high. In this case, there is a chance that DCF will also provide a high resulting number, which will eventually lead to the same decision to invest. However, when the risk is significant, and there are many possible outcomes of the project, ROA is more convenient than classic DCF. Real options are not meant to discard traditional DCF, but rather to complement them, when it is unclear whether DCF is applicable to a problem. If the project does not allow for dynamic decision making and uncertainties are small, ROA may not provide a significant value.

It often far from trivial to forecast future revenues of a project, especially if no other company has attempted to develop a similar project. Competition can also be a big factor in this valuation, but it is difficult to embed it into ROA. Predicting the future usually requires a certain amount of data provided by a similar project. However, a reference project does not have to exist, if the valuated project is highly innovative and unique.

Actual valuation can be accomplished using several techniques. One of the methods of real options valuation is the Black-Scholes partial differential equation, which is also often used in financial options. While it is relatively easy to use, the underlying mathematics is quite complex. A real-world application may require some adjustments to the Black-Scholes equation, and even though it is possible, the resulting model will become even more complicated.

Binomial model is the method described and used in this project. While being close to Black-Scholes equation in the terms of inputs and the result, the mechanics of this method are much more simple, which will be described in a following chapter. It is also easier to modify this method for special problems.

## 1.6 Real Options Categories

Different real options can solve different real-world investment problems. Some categories in the literature include:

- **Timing Options** [1, ch. 7]

  This type of options investigates the possibility of finding a good occasion to execute an option. For example, investors would prefer to wait a year before launching a new product, if there is a high probability the market situation will be much more favorable.

- **Switching Options** [1, ch. 10]

  As market conditions change, it may be useful to adjust management strategy in order to maximize the cash flow. For example, it may be beneficial to run a factory only when the production cost is minimal.

- **Compound Options** [1, ch. 8]

  A chain of options, where each step is optional, is a compound option. It is useful, when there are multiple similar possibilities of project development, while all of them depend on the successful execution of the previous options.

- **Learning Options** [1, ch. 11]

  At the start of the project, it may be difficult to predict the resulting value. This type of options describes possibilities of investing into eliminating uncertainties and refining the expected value.

- **Other**

  The whole concept of real options does not strictly define what can or cannot be calculated using this approach. Virtually every real-world event or decision can be incorporated into the valuation. For example, Schulmerich [3, p.26] provides additional types of options.

# Chapter 2

# Valuation Using Binomial Model

This chapter introduces the binomial process and its application for real options valuation. We use the framework proposed by Guthrie [1], which usually includes choosing a variable for the binomial model, calculating cash flows and market value.

## 2.1 Single-Step Binomial Model

In the single-step model, the random process representing the "state of the world" starts from the value $X \in \mathbb{R}$ at period $i = 0$ and develops into "up" value $X_u > X$ or "down" value $X_d < X$ at period $i = 1$ (Fig. 2.2). For convenience, we introduce the sizes of up and down moves, respectively, relative to the initial level of the random process, as:

$$u = \frac{X_u}{X} \qquad d = \frac{X_d}{X}, \tag{2.1}$$

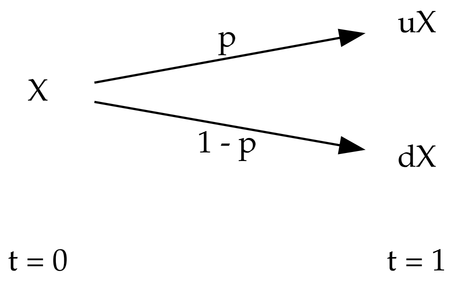


Figure 2.1: Development of underlying asset value in 1-period binomial tree

The factors $u$ and $d$ are calculated using information on the initial time $0$ and estimation of the upper and lower boundaries at the last time period $t = 1$. Probabilities of increasing and decreasing in value are $p$ and $1-p$ respectively, and $p$ must be in interval of $[0,1]$. Although it may seem unrealistic that there can be only two possible outcomes on each subsequent time period, this issue becomes less and less significant as we increase the number of periods in our valuation. We can use $u$ and $d$ to construct the binomial tree (lattice).

The state variable $X$ must represent the most impactful source of uncertainty. Having several significant sources of uncertainty may render this model inaccurate. There are methods to include more than one state variable into valuation, and even though this scenario is not covered in this work, the main concepts will remain the same. Our ultimate goal is, however, obtaining the estimated market value $V$ of future cash flows. In this model, we have two future cash flows, $Y_u$ and $Y_d$ at period $i=1$.

In order to estimate $V$, we will assume $A$ amount of *stocks*, underlying variable, and $B$ amount of *bonds*, riskless assets, with an interest rate of $r>0$ per period. Combination of $(A,B)$ is called *portfolio*. We also introduce *spanning asset* $Z$, a risky asset, which price lies between $X_u$ and $X_d$. Assuming $V=AZ+B$, we obtain the market value defined as (Eq. 2.2) as shown in subsection (2.1.1).

**Non-arbitrage assumption** is crucial in this model. An arbitrage itself is a possibility to buy some stocks and immediately sell it for a bigger price in another market. The absence of such a possibility implies the existence of the *law of one price*: portfolios of the same future cash flows must have the same price. If this were not true, this model would allow for generating "free" money by selling a portfolio for more money than the investor has bought it for.

$$V=\frac{\pi_u Y_u+\pi_d Y_d}{1+r}, \tag{2.2}$$

where $Y_u$ and $Y_d$ are cash flows at the next step, $r>0$ to be *one-period risk-free interest rate* and $\pi_u$ and $\pi_d$ are so-called *risk-neutral probabilities* (Eq. 2.3):

$$\pi_u=\frac{(1+r)Z-X_d}{X_u-X_d} \qquad \pi_d=\frac{X_u-(1+r)Z}{X_u-X_d} \tag{2.3}$$

By introducing a *risk-adjusted growth factor* $K=\frac{Z(1+r)}{X}$ we can further simplify risk-neutral probabilities (Eq. 2.4):

$$\pi_u=\frac{K-d}{u-d} \qquad \pi_d=\frac{u-K}{u-d}, \tag{2.4}$$

where $u$ and $d$ are growth factors and $K$ is *risk-adjusted growth factor* defined as (Eq. 2.6).

Guthrie proposes using CAPM (Capital Asset Pricing Model) for risk-adjusted growth factors. In his work [1, 3.4.2] he demonstrates that:

$$Z = \frac{E[\widetilde{X}] - (E[\widetilde{R}_m] - R_f)\beta}{1+r}, \tag{2.5}$$

where $\beta = \left(\frac{\text{Cov}(\widetilde{X}, \widetilde{R}_m)}{\text{Var}(\widetilde{R}_m)}\right)$, $\widetilde{R}_m$ is the total return on the market portfolio, $\widetilde{X} \in \{X_u, X_d\}$, $R_f = 1 + r$ and $(E[\widetilde{R}_m] - R_f)$ is so-called *market risk premium*.

$$K = E[\widetilde{R}_x] - (E[\widetilde{R}_m] - R_f)\beta_x, \tag{2.6}$$

where $\widetilde{R}_x = \frac{\widetilde{X}}{X}$, and $\beta_x$ is called "usage beta" and defined as $\beta_x = \left(\frac{\text{Cov}(\widetilde{R}_x, \widetilde{R}_m)}{\text{Var}(\widetilde{R}_m)}\right)$

We can already solve a problem using this formula (Eq. 2.2) for market value $V$. Let us consider an example (Ex. 2.1.2) to see how this model can be applied to a simple practical problem.

### 2.1.1 Deriving Risk-Neutral Probabilities

Consider a portfolio of $A$ units of a risky "spanning" asset generating a payoff after one period equal to $X_u$ in the "up" state and $X_d$ in the "down" state, with current price of $Z$, and $B$ units of the one-period risk-free bond. Market value $V$ is defined as $AZ + B$. Risk premium $r > 0$.

We assume $A$ and $B$ replicate the cash flow after one period $(Y_u, Y_d)$:

$$\begin{aligned} A \cdot X_u + B \cdot (1+r) &= Y_u \\ A \cdot X_d + B \cdot (1+r) &= Y_d \end{aligned} \tag{2.7}$$

This is a system of two linear equations for two unknowns, $A$ and $B$. It is easy to calculate A by subtracting the second equation from the first one:

$$A = \frac{Y_u - Y_d}{X_u - X_d} \tag{2.8}$$

The value of $B$ follows after substituting for $A$:

$$
\begin{aligned}
B &= \frac{Y_u - AX_u}{1+r} \\
&= \frac{Y_u - \frac{Y_u - Y_d}{X_u - X_d} X_u}{1+r} \\
&= \frac{Y_u X_u - Y_u X_d - Y_u X_u + Y_d X_u}{(1+r)(X_u - X_d)} \\
&= \frac{1}{1+r} \cdot \frac{Y_d X_u - Y_u X_d}{X_u - X_d}
\end{aligned}
\tag{2.9}
$$

Having determined values of $A$ and $B$, market value $V$ at the initial time period $0$ transforms into the following equation:

$$
\begin{aligned}
V = AZ + B &= \frac{Y_u - Y_d}{X_u - X_d} Z + \frac{Y_d X_u - Y_u X_d}{X_u - X_d} \\
&= \frac{1}{1+r} \cdot \left( Y_u \frac{(1+r)Z - X_d}{X_u - X_d} + Y_d \frac{X_u - Z(1+r)}{X_u - X_d} \right)
\end{aligned}
\tag{2.10}
$$

This solution is called *replicating* portfolio, which means that this specific portfolio replicates the value of cash flow $Y$ after one period.

Predicting asset value in the future may be troublesome, since it is entirely possible the investor has little information about the market state. In this case, *risk-neutral probability* becomes particularly useful, allowing us to estimate probabilities from asset value rather than actual probabilities. We have already derived them in 2.10, and their values are:

$$
\pi_d = \frac{X_u - (1+r)Z}{X_u - X_d} \qquad \pi_u = \frac{(1+r)Z - X_d}{X_u - X_d} \tag{2.11}
$$

It is worth noting that $\pi_d + \pi_u$ always equals $1$ and those two probabilities are always constant for all valuation steps thanks to the replicating portfolio assumptions. Should the investor prefer a high risk premium, $\pi_u$ will quickly decrease, while $\pi_d$ will increase accordingly. Therefore, the formula for replicating portfolio is:

$$
V = \frac{\pi_u Y_u + \pi_d Y_d}{1+r} \tag{2.12}
$$

which proves the formula (2.1.2).

### 2.1.2 Example (Single-Period)

An electronics manufacturer produces 5000 items of specific sensors each year and all of them are always sold at $5 each. The demand for the sensors may change next year. In case of favorable market conditions, 6250 units will be sold with $p = 0.6$, otherwise the company will distribute only 4000 sensors. Fixed costs will always remain $22000. The manufacturer assumes market risk premium defined by CAPM is $E[\widetilde{R}_m] - R_f = 0.1$ and CAPM $\beta_x$ is $0.8$.

**Solution.** The underlying variable $X$ is clearly the number of sold sensors.

$$X_u = 6250 \text{ unit/year} \qquad X_d = 4000 \text{ unit/year}$$

Those values will produce incomes of $6250 \cdot \$5$ and $4000 \cdot \$5$ accordingly. The sizes of up and down moves are $U = 6250/5000 = 1.25$ and $D = 0.8$. Respectively, actual income, or cash flow $Y$, will then be:

$$Y_u = \$5 \cdot X_u - FC = 32250 \text{ \$/year} - 22000 \text{ \$/year} = 10250 \text{ \$/year}$$
$$Y_d = \$5 \cdot X_d - FC = 20000 \text{ \$/year} - 22000 \text{ \$/year} = -2000 \text{ \$/year}$$

$K$ from equation (2.6) and risk-neutral probabilities are:

$$K = \frac{(0.6 \cdot \frac{6250}{5000} + 0.4 \cdot \frac{4000}{5000}) - 0.1 \cdot 0.8}{(1 + 0.1)} = \frac{(0.75 + 0.32) - 0.08}{1.1} = 0.9$$

$$\pi_u = \frac{K - D}{U - D} = \frac{0.9 - 0.8}{1.25 - 0.8} = 0.2222$$
$$\pi_d = \frac{U - K}{U - D} = \frac{1.25 - 0.9}{1.25 - 0.8} = 0.7777$$

Now we have all the necessary information to calculate market value $V$, assuming that market value at the last node equals its cash flow (in this example $V_d = Y_d$ and $V_u = Y_u$):

$$V = \frac{\pi_u Y_u + \pi_d Y_d}{1 + r} = \frac{0.2222 \cdot 10250 + 0.7777 \cdot (-2000)}{1 + 0.1} = \frac{722.15}{1.1} = \$656.5$$

This number, $\$656.5$, reflects how much money the manufacturer can expect to receive from the sensor sales.

## 2.2 Multi-Step Valuation

Practical problems would apparently be difficult to solve while valuating only a single period in the future. Not only because it would be impossible to incorporate custom logic at each node, but also due to smaller valuation precision with only one time period in the future. Fortunately, it is rather straightforward to extend the single-period model into multi-period.

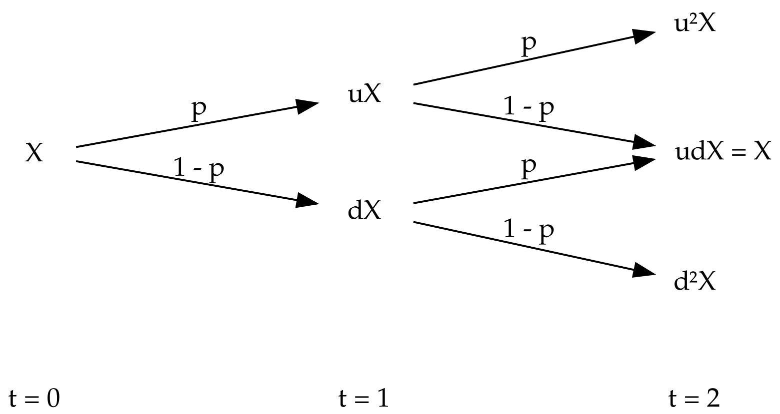


Figure 2.2: Development of underlying asset value in 2-period binomial tree

In case of one-off cash flow generated at the end of valuation horizon, we can calculate the value of cash flows using *backwards induction* ([1, Ch. 4]), starting from $V(n,i) = Y(n,i)$. We still have the same variables and assumptions, however, we must perform the valuation using backwards induction, starting at the last period and finishing at the beginning ($t = 0$) using the following formula:

$$V(n,i) = \frac{\pi_u(n,i)\ V(n+1,i) + \pi_d(n,i)\ V(n+1,i+1)}{1+r}$$

Where:

| | | |
|---|---|---|
| $V(n,i)$ | ... | market value for the option at time $n$ after $i$ down steps. |
| $\pi(n,i)$ | ... | risk-neutral probabilities. |

In case a general cash flow that is distributed along the valuation horizon, we need to include cash flow $Y(n,i)$ at each step (Eq. 2.13):

$$V(n,i) = Y(n,i) + \frac{\pi_u(n,i)\ V(n+1,i) + \pi_d(n,i)\ V(n+1,i+1)}{1+r} \tag{2.13}$$

This amount of flexibility is a major advantage of the binomial model. It is possible to compose the market value using effectively any information we believe to be important. That said, there is a number of required steps:

1. Estimating risk premium, market risk premium and usage beta.
2. Constructing a binomial lattice of values of the underlying process $X(n, i)$, such as (figure 2.2), and populating it with appropriate mutations of the variable.
3. Calculating cash flows at each node using information carried by $X(n, i)$.
4. Calculating market value at each node using the recursive equation (Eq. 2.13), where $n$ and $i$ denote total number of steps performed and number of moves down accordingly. We start at the final time period, where we assign cash flow to market value, and then, step by step, perform the backwards induction until we reach the beginning.

Extending the number of valuation steps may greatly improve accuracy of the market value. Provided the number of periods is high enough, discrete binomial model will deliver practically the same result as continuous models (Black-Scholes) within an insignificant margin [8, Chance], while being more flexible and easier to adjust to a particular problem. Let us consider a modified example from the previous section (Ex. 2.1.2).

**Ex 2.2.1.** The same firm considers launching another product into the market. They expect to produce precisely 10000 units each quarter. Expenses related to the potential production are \$35 per unit and they will increase by \$1 each quarter.

The selling price will, however, fluctuate. At the starting moment, it is possible to sell 10000 items for \$50 each. The price will either increase by the factor of $u = 1.1$ or decrease by $d = 1/u$ with the same probability. What would be the cash flow generated by this option for the next year assuming $r = 0.05$, market risk premium $0.1$ and usage beta $\beta_x = 0.7$?

**Solution.** In this case, the underlying variable must be the selling price. Let us begin by exploring its development in the table (2.1). This method of describing binomial lattice is equivalent to the graph representation displayed above. We use $X(n, i)$ notation as the value at the specific node in the lattice, where $n$ is the total number of steps and $i$ is the number of moves down ($i \leq n$ and $0 \leq i$).

| | $n = 0$ | $n = 1$ | $n = 2$ | $n = 3$ |
|---|---|---|---|---|
| $X(n, 0)$ | $50.00$ | $55.00$ | $60.50$ | $66.55$ |
| $X(n, 1)$ | | $45.55$ | $50.00$ | $55.00$ |
| $X(n, 2)$ | | | $41.32$ | $45.45$ |
| $X(n, 3)$ | | | | $37.57$ |

Table 2.1: Development of market price for the product, the underlying variable

The next logical step is to calculate cash flow at each step. Our revenue is the number of produced units multiplied by selling price, which results in $10000 \cdot X(n,i)$. Therefore revenues Rev$(n,i)$ for unit price $X(n,i)$ will be as shown in (2.2).

| | $n=0$ | $n=1$ | $n=2$ | $n=3$ |
|---|---|---|---|---|
| Rev$(n,0)$ | 500000 | 550000 | 605000 | 665500 |
| Rev$(n,1)$ | | 455500 | 500000 | 550000 |
| Rev$(n,2)$ | | | 413200 | 454545 |
| Rev$(n,3)$ | | | | 375757 |

Table 2.2: Development of revenues.

Expenses (Cost) defined as Cost$(n,i) = (35+n) \cdot 10000$ for the same periods are listed in (2.3).

| | $n=0$ | $n=1$ | $n=2$ | $n=3$ |
|---|---|---|---|---|
| Cost$(n,0)$ | 350000 | 360000 | 370000 | 380000 |
| Cost$(n,1)$ | | 360000 | 370000 | 380000 |
| Cost$(n,2)$ | | | 370000 | 380000 |
| Cost$(n,3)$ | | | | 380000 |

Table 2.3: Development of expenses.

Using the cash flow formula (eq. 2.14) we then calculate corresponding cash flow values $Y$ into the cash flow table (Table 2.4).

$$Y(n,i) = \text{Rev}(n,i) - \text{Cost}(n,i) \tag{2.14}$$

| | $n=0$ | $n=1$ | $n=2$ | $n=3$ |
|---|---|---|---|---|
| $Y(n,0)$ | $150,000$ | $190,000$ | $235,000$ | $285,500$ |
| $Y(n,1)$ | | $94,545$ | $130,000$ | $170,000$ |
| $Y(n,2)$ | | | $43,223$ | $74,545$ |
| $Y(n,3)$ | | | | $-4,342$ |

Table 2.4: Cash flow

Having obtained cash flow values, we apply the recursive formula of market value (2.13) in order to acquire the value of the opportunity. The first step is to calculate $K$ and risk-neutral probabilities:

$$\begin{aligned} K &= E[\widetilde{R}_x] - (E[\widetilde{R}_m] - R_f)\beta_x \\ &= (p_u \cdot u + p_d \cdot d) - (E[\widetilde{R}_m] - R_f)\beta_x \\ &= (0.5 \cdot 1.1 + 0.5 \cdot \frac{1}{1.1}) - 0.1 \cdot 0.7 = 0.9345 \end{aligned}$$

Therefore, risk-neutral probabilities are:

$$\pi_u = \frac{K - d}{u - d} = 0.1333 \qquad \pi_d = \frac{u - K}{u - d} = 0.8666 \tag{2.15}$$

The next step is to assume that market value equals cash flow at the end nodes, that is $V(3,i) = Y(3,i)$. (Table 2.5).

| | $n=0$ | $n=1$ | $n=2$ | $n=3$ |
|---|---|---|---|---|
| $V(n,0)$ | ... | ... | ... | $285,500$ |
| $V(n,1)$ | | ... | ... | $170,000$ |
| $V(n,2)$ | | | ... | $74,545$ |
| $V(n,3)$ | | | | $-4,342$ |

Table 2.5: First step: assigning $V(n,i) = Y(n,i)$ with $n = 3$ for every possible $i$.

Applying recursive formula for $V(n,i)$ (2.13 will eventually provide us the market value $V(0,0) = 336,929$, which is the value of this option (Table 2.6):

| | $n=0$ | $n=1$ | $n=2$ | $n=3$ |
|---|---|---|---|---|
| $V(n,0)$ | $336,929$ | $418,169$ | $411,571$ | $285,500$ |
| $V(n,1)$ | | $162,138$ | $213,116$ | $170,000$ |
| $V(n,2)$ | | | $49,104$ | $74,545$ |
| $V(n,3)$ | | | | $-4,342$ |

Table 2.6: Complete $V$ valuation

# Chapter 3

# Application of Real Options

Each problem solved by ROA must be treated differently. There is no generic sequence of steps to solve a strategical problem such as dynamic decision making. However, we may begin solving a problem by treating it like an economical problem. We define costs and revenue of the project, so that we can calculate cash flows and market values. No matter which type of option is being considered, there must be a clear decision tree upon which a specific algorithm is built. The source code is provided and described at the end of this chapter.

## 3.1 Timing Options

Suppose a company provides a cloud computing platform and manages several computational clusters of various purposes. In order to guarantee certain capacity for clients, the company must think in advance about its machinery capacity. When they decide to expand the amount of available power, it might be extremely hard to reverse the investment. For sure they would prefer to know for certain that if they invest more money then they will be able to utilise their newly acquired hardware. We suppose the investment is irreversible.

The question to ask is how the company decides if they should expand or wait. If they wait, they may lose potential customers due to being unable to satisfy the demand. On the other hand, if they expand and clients will suddenly lose interest in their offerings, they will end up operating expensive hardware with high operational costs. The company certainly realizes that doubling the capacity by erecting another cluster will not immediately imply they will have double the clients. How will the demand will be developing in the future? What if it becomes stagnant due to market saturation? It would not be a difficult decision if this decision was reversible with little additional cost, however, that is not the case.

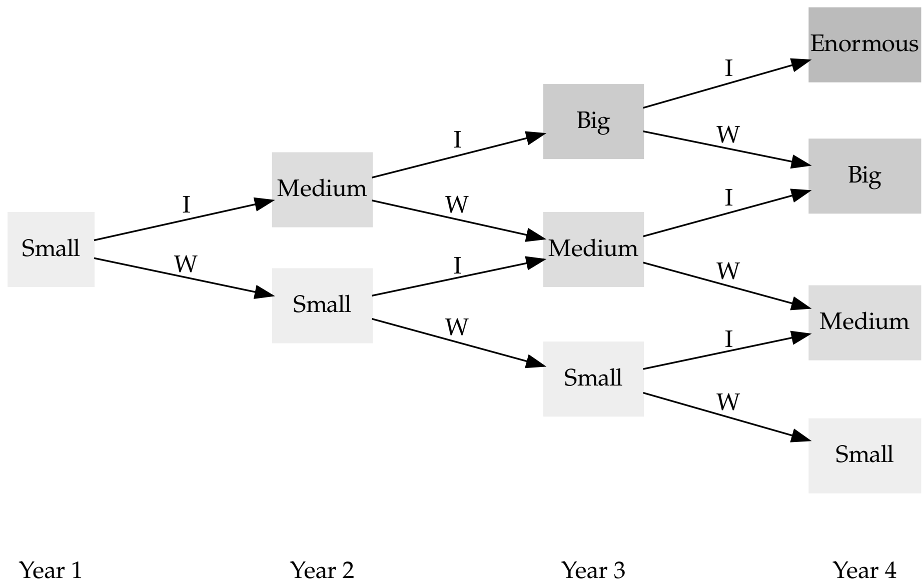


Figure 3.1: Decision Tree. Company's timing options. At each investment opportunity occuring once per year they may either invest (I) or wait (W).

At each investment opportunity (Fig. 3.1) we decide between waiting ($W$) and doing nothing or investing ($I$) and expanding our computational capabilities. Each expansion costs the company additional one-time investment $I$ and will contribute to operating costs by $C$. Our state variable $X$ is the demand for computational power, which we will measure in PFLOPS (floating point operations per second).

By the end of each quarter, the company will have revenue $X \cdot P$, where $P$ is the price per one PFLOPS, whereas fixed costs must be $C$. Therefore, cash flow $Y$ is

$$Y(n,i) = X(n,i) \cdot P - C \tag{3.1}$$

We also suppose that one cluster can maintain a load of maximum 2 PFLOPS. So, if the company decides to wait, its future cash flows will be capped by this amount instead of $X$. Let us consider the situation, when the company decides to wait until another period.

Suppose we start with 1 PFLOPS and the demand may rise up to 2 PFLOPS by the end of the year, while the demand $X$ will change every quarter. The "up" factor is $U = 1.26$. The development of $X$ is displayed in the Table (3.1). We also assume $r = 0.1$, market risk premium $0.1$ and usage beta $\beta_x = 0.65$. Price per 1 PFLOPS is \$100000, fixed costs $C$ are \$$50000$ per cluster, price of a new cluster $C_N$ is \$200000.

| | $n=0$ | $n=1$ | $n=2$ | $n=3$ (Year's end) |
|---|---|---|---|---|
| $X(n,0)$ | $1.00$ | $1.26$ | $1.59$ | $2.00$ |
| $X(n,1)$ | | $0.79$ | $1.00$ | $1.26$ |
| $X(n,2)$ | | | $0.63$ | $0.79$ |
| $X(n,3)$ | | | | $0.50$ |

| | $n=0$ | $n=1$ | $n=2$ | $n=3$ (Year's end) |
|---|---|---|---|---|
| $Y(n,0)$ | $50,000$ | $76,000$ | $108,760$ | $150,000$ |
| $Y(n,1)$ | | $29,365$ | $50,000$ | $76,000$ |
| $Y(n,2)$ | | | $12,988$ | $29,365$ |
| $Y(n,3)$ | | | | $-9$ |

| | $n=0$ | $n=1$ | $n=2$ | $n=3$ (Year's end) |
|---|---|---|---|---|
| $V(n,0)$ | $156,401$ | $195,725$ | $202,110$ | $150,000$ |
| $V(n,1)$ | | $72,662$ | $91,984$ | $76,000$ |
| $V(n,2)$ | | | $22,609$ | $29,365$ |
| $V(n,3)$ | | | | $-9$ |

Table 3.1: Demand $X$, cash flow $Y$ and market value $V$ when no timing option is considered.

We calculate cash flows of the first year (Table 3.1) using cash flow formula (eq. 3.1). Then we apply the recursive equation for the market value (eq. 2.13) to obtain market value without exercising real options. For $V(0,0)$, the value we are most interested in, we also have to subtract one cluster cost $C_N$ of $200,000, which will result in $156,401 - 200,000 = -43,599$.

Of course, we want to investigate possibilities of exercising "Wait" and "Invest" options. At the end of each year we construct two additional binomial models, first for "Wait" option, the other one for "Invest" option. Let us consider our position at $n = 3$ with $X = 1.26$ and our cash flows if we choose to wait for the next year (Table 3.2). Notice that $Y_W(2,0) = Y_W(3,0)$ because we still have only one cluster and cannot serve more clients than $X = 2.0$.

| | $n=0$ | $n=1$ | $n=2$ | $n=3$ |
|---|---|---|---|---|
| $X(n,0)$ | $1.26$ | $1.59$ | $2.00$ | $2.52$ |
| $X(n,1)$ | | $1.00$ | $1.26$ | $1.59$ |
| $X(n,2)$ | | | $0.79$ | $1.00$ |
| $X(n,3)$ | | | | $0.63$ |

| | $n=0$ | $n=1$ | $n=2$ | $n=3$ |
|---|---|---|---|---|
| $Y_W(n,0)$ | $76,000$ | $108,760$ | $150,000$ | $150,000$ |
| $Y_W(n,1)$ | | $50,000$ | $76,000$ | $108,760$ |
| $Y_W(n,2)$ | | | $29,365$ | $50,000$ |
| $Y_W(n,3)$ | | | | $12,988$ |

Table 3.2: Company decides to wait. $Y_W$ denotes cash flow for the "Wait" decision

If we, however, decide to invest, it will cost additional $C_N$, but we will not be limited by offering only one cluster. So, if it is unclear which of $I$ or $W$ is a more efficient strategy in terms of expected returns, we calculate both values and compare them using market value. When it is time to make a decision, we can alter the formula from the previous chapter (Eq. 2.13) the following way:

$$V(n,i) = \max\{\ -C_N + V_{\mathrm{I}}(n,i),\ V_{\mathrm{W}}(n,i)\ \}, \tag{3.2}$$

where $V_I$ is the market value with an additional cluster ("Invest") and $V_W$ is the market value of "Wait".

Both $V_{\mathrm{I}}$ and $V_{\mathrm{W}}$ are calculated by building another binomial tree which starts with the final state variable $X$ of the previous binomial tree. Calculating market value for the next four years will yield these results (Table 3.3):

| | $n=0$ | $n=1$ | $n=2$ | $n=3$ |
|---|---|---|---|---|
| $V(n,0)$ | $320,033$ | $542,643$ | $809,895$ | $1,111,386$ |
| $V(n,1)$ | | $158,512$ | $34,6028$ | $579,406$ |
| $V(n,2)$ | | | $27,021$ | $182,498$ |
| $V(n,3)$ | | | | $-78,787$ |

Table 3.3: Market value with timing options.

This particular combination of options and fictional parameters increased the market value of the project. Subtracting $C_N$ results in $V(0,0) = \$120,033$, which is already profitable. But can we improve it even further? Let us suppose we can abandon the company when it is unprofitable, so that instead of having to choose between I and W, we may completely stop further valuation. Adding this property leads us to equation (Eq. 3.3).

$$V(i,n) = \max\{\ Y(i,n),\ -C_N + V_{\mathrm{I}}(i,n),\ V_{\mathrm{W}}(i,n)\ \}, \tag{3.3}$$

| | $t=0$ | $t=1$ | $t=2$ | $t=3$ |
|---|---|---|---|---|
| $V(t,0)$ | $350,293$ | $558,364$ | $816,007$ | $1,113,386$ |
| $V(t,1)$ | | $201,703$ | $369,627$ | $588,793$ |
| $V(t,2)$ | | | $88,019$ | $217,804$ |
| $V(t,3)$ | | | | $6,240$ |

Table 3.4: Market value with timing options without an obligation to either "Wait" or "Invest".

Even better, the possibility to abandon the project in case of its failure increases the whole market value of cash flow streams (Table 3.4).

## 3.2 Switching Options

Software companies developing their own products have a variety of possible distribution and selling strategies. One classic way to sell software is to charge one-time payment which is supposed to allow the user to use the software perpetually including all future updates. While it is simple and comprehensible for end-users, companies tend to use SaaS (Software-as-a-service) model. Basically it means that users may use the software as long as their pay a subscription fee. This pricing model somewhat guarantees a stable revenue flow as long as the software remains in demand.

Switching from one-time payment model to a SaaS model might be beneficial for the company, however, they probably should expect a certain resistance from the existing client base, knowing that they have already paid a fixed amount for a perpetual license. Sudden denial of future updates will definitely anger a number of users. Assuming the sales numbers are well below being profitable, the company may have no other choice.

One thing the company can do to smoothen the transition is to switch their customers to the new model gradually.

The company has chosen SaaS as the method of product delivery. Its key properties include the following:

- **Pricing.** Customer are usually charged once per a certain time period, typically once either a month or a year. Once the customer stops paying for the subscription, access to the software can be denied or limited.
- **Delivery.** Software is deployed and maintained by the supplier. Customers then may access it using a client application. However, corporate clients may want to self-host the solution on their own hardware, which the supplier may or may not allow.
- **Updating.** The supplier typically has to support only one version of the product. If the client application is developed as a web application, then customers will always have access to the latest version.

Success of this project depends on several variables. The main uncertainty is the number of clients of the product. Main variables of this project are:

$$
\begin{aligned}
X &\;\dots\; \text{Number of clients} \\
P &\;\dots\; \text{Pricing plans}
\end{aligned}
$$

The project is planned to be monetized using several subscription plans. Business analysts have prepared a decision tree (Fig. 3.2) of different plans based on the number of active customers, denoted $X$, which is the main source of uncertainty. The initial plan consists of three stages:

1. **User adoption.** The main goal at this stage is to gather as many clients as possible. In order to achieve this, the company will provide free access to the service free of charge to everyone. They believe, once they have at least 25,000 users, it is possible to introduce a paid subscription plan.

2. **First monetization.** Once the company have acquired the necessary number of active users, they will attempt to monetize a part of the userbase. It is uncertain how many users will want to continue using the product with a paid subscription. Further pricing adjustments may be helpful in finding a good pricing strategy.

3. **Pricing adjustments.** When and if there is an adequate number of paid users, it is possible to optimize subscription plans. Otherwise, it might be necessary to lower the price.

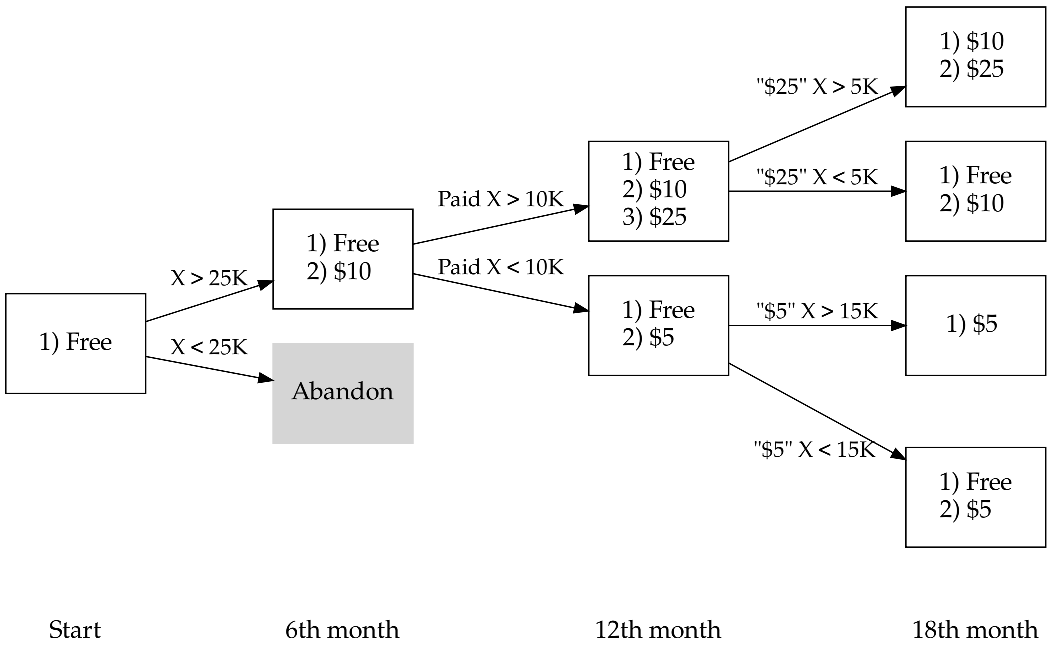


Figure 3.2: Subscription plans prices

When budgeting a project, investors will want to know about important steps of the project development. Specifically, the company considers the following implementation strategy:

1. **Start.** Considering the highly competing market, first 6 months are devoted to attracting as many clients as possible, even though certain features of the product are

not finished. No monetization is planned. Business analysts believe it is crucial to provide a free version for marketing reasons.

2. **6th month.**
   (a) **Success.** If there will be 25,000 or more active clients, a subscription plan will be introduced with exclusive features. The upper bound is estimated at 40,000. 25% of clients are expected to buy the subscription.
   (b) **Failure.** If there are less than 25,000 active clients, the project will be abandoned. The management believes more interesting opportunities will arise in other projects.
3. **12th month.**
   (a) **Success.** If there will be at least 10,000 clients with a subscription plan, then a more expensive subscription plan will be developed alongside the existing one. 40% of clients of the '$10' subscription are expected to buy the '$25' subscription.
   (b) **Failure.** A cheaper subscription plan will replace the current one, if there are less than 10,000 active clients, as an attempt to convert more users to a paid subscription plan.
4. **18th month.**
   (a) **Success of $25 plan.** At least 5,000 users will use the new $25 plan in the case of success. The free subscription will be cancelled. A certain amount of 'free' users will decide to use either of the two paid subscriptions.
   (b) **Failure of $25 plan.** Should there be less than 5,000 users of the new plan, it must be discarded.
   (c) **Success of $5 plan.** If there will be at least 15,000 clients with the $5 subscription plan, then the free subscription will be cancelled. One third of existing free users are expected to buy the subscription.
   (d) **Failure of $5 plan.** No change will be made and the company will continue its course from the previous phase.

We must begin by defining costs and revenues of the project so that we may easily calculate the market value at each step. No matter which type of real option we consider, the total cost TC will be calculated in the same way, which is the sum of FC (fixed) and VC (variable).

$$FC = \$10{,}000 \text{ /month} \tag{3.4}$$

$$VC = U \cdot C_v \tag{3.5}$$

In the case of this software company, FC primarily corresponds to the annual salaries of employees and renting an office space, however, in real world, salaries are unlikely to be constant through a whole year. VC depends on the number of active users.

Revenues are constituted by the amount of money paid by users ($U$) of paid subscription plans ($P$), specifically:

$$\text{Rev} = \sum_{i=1}^{P} U_i \cdot P_i \tag{3.6}$$

Cash flow is therefore:

$$Y = \text{Rev} - TC \tag{3.7}$$

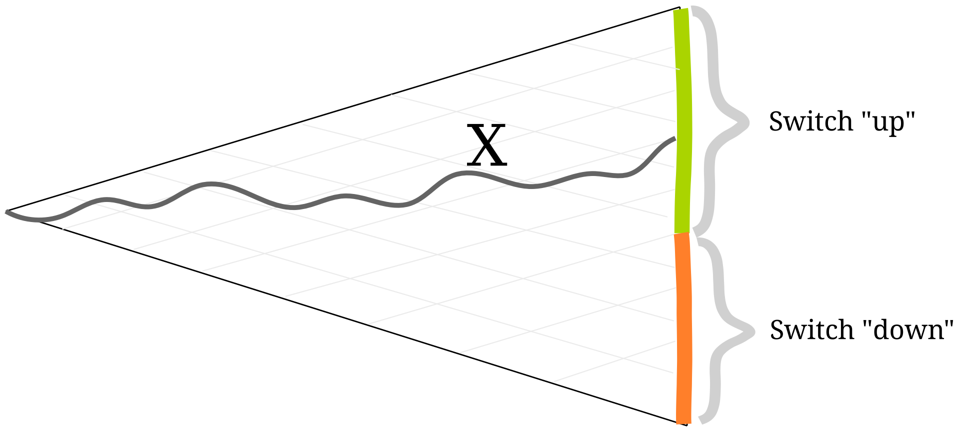


Figure 3.3: Switch possibility

This type of real options describes the possibility to switch between two or more states of the project. For example, changing the pricing model will affect the number of users and the eventual revenue.

Suppose at some point we have $X_i$ users using pricing plan $P_i$. Introducing another plan or modifying the existing one will for sure alter the project state. Certain users will switch to another subscription plan. A good choice of pricing may attract additional users, who have not used the software before. Of course, it can work the other way, a bad strategy will encourage users to have a look at some other solutions from competitors.

When choosing between two or more strategies, we might want to use the following equation to determine the optimal one:

$$V(0,0) = \max\{\ V_1(0,0)\ ,\ \ \dots\ ,\ V_j(0,0)\ \} \tag{3.8}$$

where $V_j$ corresponds to a pricing strategy. Once the company chooses a strategy, they will have to commit to it for some time. How do valuate the project at this moment? Our steps will be the following:

- It is necessary to determine variables needed to construct the binomial tree. However, in this case we must simultaneously build multiple trees, because there are multiple groups of users with different subscription plans.
- Each pricing strategy may open different subsequent options, so there is hardly a generic way to define their valuation formula. That said, assuming there are no other options, we simply valuate the end result of the binomial tree

Let us consider the company's situation at the very beginning. We do not have any paid subscription plans so all the cash flow will be generated by the subsequent options. Therefore, at date 0, we begin by building the tree for $X$, where it starts at 25,000 and may grow each month by $u = 1.1$.

| | $n=1$ | $n=2$ | $n=3$ | $n=4$ | $n=5$ | $n=6$ |
|---|---|---|---|---|---|---|
| $X(n,1)$ | $25,000$ | $27,500$ | $30,250$ | $33,275$ | $36,602$ | $40,262$ |
| $X(n,2)$ | | $22,727$ | $25,000$ | $27,500$ | $30,250$ | $33,275$ |
| $X(n,3)$ | | | $20,661$ | $22,727$ | $25,000$ | $27,500$ |
| $X(n,4)$ | | | | $18,782$ | $20,661$ | $22,727$ |
| $X(n,5)$ | | | | | $17,075$ | $18,782$ |
| $X(n,6)$ | | | | | | $15,523$ |

Table 3.5: Development of users number during first phase.

| | $n=1$ | $n=2$ | $n=3$ | $n=4$ | $n=5$ | $n=6$ |
|---|---|---|---|---|---|---|
| $Y(n,1)$ | $-20,000$ | $-21,000$ | $-22,100$ | $-23,310$ | $-24,641$ | $-26,105$ |
| $Y(n,2)$ | | $-19,090$ | $-20,000$ | $-21,000$ | $-22,100$ | $-23,310$ |
| $Y(n,3)$ | | | $-18,264$ | $-19,090$ | $-20,000$ | $-21,000$ |
| $Y(n,4)$ | | | | $-17,513$ | $-20,000$ | $-19,090$ |
| $Y(n,5)$ | | | | | $-16,830$ | $-17,513$ |
| $Y(n,6)$ | | | | | | $-16,209$ |

Table 3.6: Development of cash flow during first phase (no paid plans).

Apparently our cash flow equals our total cost $TC$ since all users are provided the service free of charge. However, options open by this initial development may be highly rewarding. If our $X$ will rise above 25,000, we may safely assume that 25% of those users will pay for our first subscription plan at the next stage . So, in the best case, we will start with 10,000 users, when each of them will pay $10 per month. However, we may also have to abandon the project, if our first phase ends with a low number of users.

At the start of $6^{th}$ month we will be monitoring the number of our paid users. Depending on that number we will adjust the pricing model even further. Let us construct a generic equation for market value. Cash flow therefore equals:

$$Y(i,n) = X_1 \cdot P_1 + \cdots + X_j \cdot P_j - U \cdot C_v \tag{3.9}$$

where $P$ is the price of a particular pricing plan. If we are at the point where we must make a decision, then the market value for the next pricing plan for a particular $X$ is:

$$V(i,n) = Y(i,n) + V_{\mathrm{X}}(i,n) \tag{3.10}$$

Otherwise it is the same as it is in the previous chapter:

$$V(i,n) = Y(i,n) + \frac{\pi_u(n,i)\ V(n+1,i) + \pi_d(n,i)\ V(n+1,i+1)}{1+r} \tag{3.11}$$

Let us calculate the company's state between $6^{th}$ and $12^{th}$ months. As we see from the previous table (Table 3.5), there are three possible values ([40262, 33275, 27500]) leading to introducing a paid pricing plan. Each of them will generate different cash flow and market value. Let us demonstrate the cash flow for $X = 40262$:

| | $n=1$ | $n=2$ | $n=3$ | $n=4$ | $n=5$ | $n=6$ |
|---|---|---|---|---|---|---|
| $Y(n,1)$ | $74,551$ | $83,006$ | $92,307$ | $102,538$ | $113,792$ | $126,171$ |
| $Y(n,2)$ | | $66,865$ | $74,551$ | $83,006$ | $92,307$ | $102,538$ |
| $Y(n,3)$ | | | $59,877$ | $66865$ | $74,551$ | $83,006$ |
| $Y(n,4)$ | | | | $53,525$ | $59,877$ | $66,865$ |
| $Y(n,5)$ | | | | | $47,750$ | $53,525$ |
| $Y(n,6)$ | | | | | | $42,500$ |

Table 3.7: Cash flow between $6^{th}$ and $12^{th}$ months when previous $X$ is $40262$ and 1/4 users pay $10 per month.

We can see that at the first node the value is calculated the following way:

$$
\begin{aligned}
Y(1,1) &= X(1,1) \cdot 0.25 \cdot \$10 - VC - FC \\
&= 40262 \cdot 0.25 \cdot \$10 - 40262 \cdot \$0.4 - \$10000 \\
&= \$74551
\end{aligned}
$$

Market value $V$ for the whole project is shown in table 3.8, $-40385$, which suggests that the investment may not return any profit, however, there is a noticeable difference between cash flows with and without switch options. A different composition of business decisions could provide a positive value. Adding more plans and valuating the project for even longer may allow us to obtain a positive value of $V$.

| | $n = 1$ | $n = 2$ | $n = 3$ | $n = 4$ | $n = 5$ | $n = 6$ |
|---|---|---|---|---|---|---|
| $V(n, 1)$ | $-40,385$ | $29,064$ | $135,886$ | $274,371$ | $414,318$ | $649,576$ |
| $V(n, 2)$ | | $-50,924$ | $3,834$ | $102,431$ | $253,050$ | $350,553$ |
| $V(n, 3)$ | | | $-55,219$ | $-20,247$ | $57,398$ | $252,850$ |
| $V(n, 4)$ | | | | $-49,654$ | $-35,498$ | $-19,090$ |
| $V(n, 5)$ | | | | | $-32,725$ | $-17,513$ |
| $V(n, 6)$ | | | | | | $-16,209$ |

Table 3.8: Market value when previous $X$ is $40262$.

## 3.3 Source Code

Examples were programmed and solved in TypeScript. Having installed NodeJS v12 is a prerequisite. Visit `https://nodejs.org/` for installation instructions. Best viewed in Visual Studio Code or WebStorm. Source code is located at:

`https://github.com/nazaryev-fjfi/nazaryev-fjfi-real-options`

Use these commands to clone repository and install dependencies:

```
$ git clone https://github.com/nazaryev-fjfi/nazaryev-fjfi-real-options.git
$ cd nazaryev-fjfi-real-options
$ npm install
```

Run examples using:

```
$ npm run example:ch2
$ npm run example:timing
$ npm run example:switch
```

### 3.3.1 Overview of the Implemetation

BinomialModel class contains low-level logic responsible for constructing three binomial lattices, that is, for $X$, $Y$ and $V$. In order to prevent the user from performing tiresome work with indexes of the lattices, it does not expose them directly. Instead, it requires the client to provide several dependencies described by BinomialModelProps:

1. `steps` (integer): depth of the binomial tree.
2. `riskPremium` (decimal): denoted $r$ in this work.
3. `marketRiskPremium` (decimal): denoted $(E[\widetilde{R}_m] - R_f)$.
4. `capmBeta` (decimal): denoted $\beta$ in this work.
5. `underlyingVariable` (function): Function accepting two arguments, number of steps $n$ and number of moves down $i$. It has to return $X(n, i)$.
6. `cashFlow` (function): Returns $Y(n, i)$. This function should additionally accept the value of the corresponding $X(n, i)$ to calculate cash flow.
7. `marketValue` (function): Calculates $V(n, i)$. This function is provided with the current and the "next" values of $X(n, i)$ and $Y(n, i)$. It is responsible for correct calculation of risk-neutral probabilities, however, there is a helper class called CAPM containing all the necessary logic for it.

Specific examples are solved by implementing BinomialModelProps properties with inclusion of custom logic and creating nested binomial models if necessary during $V(n, i)$ calculation.

# Conclusion

The recombining binomial model is commonly used to valuate any types of real options. While being easier to comprehend than Black-Scholes model, it also is more flexible and easily applicable. By using programming techniques we can model complex business decisions at any time of the projects, which is harder to do with continuous models.

Several hypothetical project development problems were analyzed using the real options approach. Timing options method is very useful in the case when certain decisions can either be cancelled or postponed, as it was demonstrated in the according chapter 3.1. Switching options 3.2 may benefit projects with highly unpredictable future. Embedding managerial flexibility into option valuation is probably the most prominent feature of real options approach.

Future work may first of all include analyzing other types of options. For example, learning options could be an interesting tool for projects where some parameters could be dramatically refined during the project lifetime. Another interesting area of potential future research is analyzing how market competition affects the real option framework application, because projects rarely develop without any competition. Using just one state variable could be limiting for some projects and embedding several sources of uncertainty could provide even more flexibility at the cost of higher complexity of the valuation.

# Bibliography


[1] G. Guthrie: *Real Options in Theory and Practice*. Oxford: Oxford University Press, Oxford, 2009.

[2] S. E. Shreve: *Stochastic Calculus for Finance I: The Binomial Asset Pricing Model.*. Springer, New York, 2004.

[3] M. Schulmerich: *Real Options Valuation: The Importance of Interest Rate Modelling in Theory and Practice*. Springer, New York, 2010.

[4] K. Sullivan, P. Chalasani, S. Jha, V. Sazawal: *Software Design as an Investment Activity: A Real Options Perspective*. 1998.

[5] S. Benninga: *Financial Modeling*. The MIT Press, 2014.

[6] H. Erdogmus: *Valuation of Learning Options in Software Development Under Private and Market Risk*. The Engineering Economist, 2002.

[7] W. Whitt: *IEOR 4701: Stochastic Models in Financial Engineering; Lecture notes on the Binomial Lattice Model, August 13*. Summer 2007, Professor Whitt.

[8] Chance, Don M.: *A Synthesis of Binomial Option Pricing Models for Lognormally Distributed Assets*. March 2008, Journal of Applied Finance, Vol. 18

[9] Black, Fischer, and Myron Scholes: *The Pricing of Options and Corporate Liabilities.*. The Journal of Political Economy 81 (3): 637654. 1973.

[10] Merton, Robert C.: *Theory of Rational Option Pricing.*. The Bell Journal of Economics and Management Science 4 (1): 141183. 1973.

[11] Cox, John C., Stephen A. Ross, and Mark Rubinstein: *Option Pricing: A Simplified Approach.*. Journal of Financial Economics 7: 229263. 1979.

[12] Dixit, Avinash K., and Robert S. Pindyck: *Investment under Uncertainty*. Princeton, NJ: Princeton University Press. 1994.

[13] Trigeorgis, Lenos: *Real Options: Managerial Flexibility and Strategy in Resource Allocation*. Cambridge, MA: MIT Press. 1996.

[14] Luehrman, Timothy A: *What's It Worth? A General Manager's Guide to Valuation*. Harvard Business Review 75 (3): 132142. 1997.

[15] Luehrman, Timothy A: *Investment Opportunities as Real Options: Getting Started on the Numbers*. Harvard Business Review 76 (4): 5167. 1998a.

[16] Luehrman, Timothy A: *Strategy as a Portfolio of Real Options*. Harvard Business Review 76 (5): 8999. 1998b.