ČESKÉ VYSOKÉ UČENÍ TECHNICKÉ V PRAZE

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Mathematical modeling of coupled transport processes in porous media

DISERTAČNÍ PRÁCE

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Prohlášení

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V Praze dne 10. října 2019 ....

Lukáš Krupička

Poděkování

Na tomto místě bych rád poděkoval svému školiteli Michalu Benešovi, katedře matematiky FSv ČVUT a všem, kteří mě v průběhu studia podporovali.

Abstrakt

Tato práce se zabývá teoretickou analýzou sdruženého transportu vody a tepla v nehomogenním částečně saturovaném pórovitém prostředí. V práci je uvedeno podrobné odvození modelu, který je popsán dvěma evolučními nelineárními diferenciálními rovnicemi s degeneracemi ve všech transportních koeficientech. V hlavní části textu je analyzován model se smíšenými Dirichletovými a Neumannovými okrajovými podmínkami. Numerické řešení je založeno na semi-implicitní časové diskretizaci, která vede na soustavu nelineárních stacionárních okrajových úloh s neznámým rozložením teploty a tlakové výšky. Pro popsanou úlohu je v práci dokázána existence a regularita řešení stacionární úlohy v každém časovém kroku. Dále je pomocí vhodných apriorních odhadů pro časové interpolace neznámých funkcí ukázána existence slabého řešení nestacionární úlohy a za dodatečných předpokladů i její jednoznačnost.

Dále je v práci stručně analyzován takzvaný duální model, zahrnující odlišný přístup k popisu porézního prostředí. Na závěr je představen duální model s obecnými nelineárními okrajovými podmínkami a model obsahují disperzní rovnici, popisující transport rozpuštěných látek v proudící tekutině.

Klíčová slova: sdružený transport, pórovité prostředí, nelineární diferenciální rovnice, Rotheho metoda, apriorní odhady, existence, slabé řešení

Abstract

This thesis deals with a theoretical analysis of a coupled heat and water transport in partially saturated porous media. In the first part of this work, we derive a model, which consists of two evolution nonlinear partial differential equations with degeneracies in all transport coefficients. In the main part of the work we analyze the single porosity model with mixed boundary conditions of Dirichlet and Neumann type. Employed numerical procedure is based on a semi-implicit time discretization, which leads to a system of coupled nonlinear stationary equations with unknown temperature and pressure head. We prove the existence and regularity of the solution to the stationary problem in each time step. Further, by deriving suitable a-priori estimates for the time interpolants of the unknown functions, we prove the existence and uniqueness of the weak solution to the nonstationary problem.

Further, we briefly analyze the dual model, arising from a dual porosity approach to the porous media description. Finally, we also present a model with general nonlinear boudary conditions and a coupled diffusion-dispersion-convection model.

Keywords: coupled transport, porous medium, nonlinear differential equation, Rothe's method, a-priori estimates, existence, weak solution

Contents

List of symbols

Mathematical symbols

Symbol	**Description**
$\mathbb{R}^N$	N-dimensional Euclidean space
Ω	domain in $\mathbb{R}^N$
$\partial\Omega$	boundary of domain Ω
Γ	part of boundary $\partial\Omega$
$\overline{\mathcal{M}}$	closure of a set $\mathcal{M}$
$\boldsymbol{n}$	outward unit vector
$L^p(\Omega)$	Lebesgue's space
$\\|\cdot\\|_{L^p(\Omega)}$	norm in the space $L^p(\Omega)$
$W^{k,p}(\Omega)$	Sobolev's space
$\\|\cdot\\|_{W^{k,p}(\Omega)}$	norm in the space $W^{k,p}(\Omega)$
$W^{k,p}(\Omega)^*$	dual space to $W^{k,p}(\Omega)$
$\hookrightarrow$	continuous embedding
$\rightharpoonup$	weak convergence
$C^p(\Omega)$	space of functions with a continuous derivative up to order p
X	Banach's space
X^*	dual space to X
$\langle\cdot,\cdot\rangle$	symbol for duality between X a X^*
$\partial_t u$	time derivative of u
∇	nabla operator
$\frac{Du}{Dt}$	material derivative of u
r'	conjugate index for r, $r' = r/(r-1)$
$\cap$	intersection

Nomenclature

Latin letters

Symbol	Unit	Description
a	[m^{-1}]	van Genuchten's parameter
C	[J kg^{-1} K^{-1}]	specific heat per unit of mass
e	[J m^{-3}]	internal energy per unit of volume
e_m	[-]	relative surface emissivity
g	[m s^{-2}]	acceleration due to gravity
k	[m s^{-1}]	hydraulic permeability
k_s	[m s^{-1}]	saturated hydraulic permeability
n	[-]	porosity
n_1	[-]	van Genuchten's parameter
n_2	[-]	van Genuchten's parameter
p	[Pa]	pressure
$\boldsymbol{q}_T$	[W m^{-2}]	heat flux
$\boldsymbol{q}$	[W m^{-2}]	conductive heat flux
s	[kg s^{-1}]	source of mass
u	[m]	pressure head
$\boldsymbol{v}$	[m s^{-1}]	velocity
V	[m^{-3}]	volume

Greek letters

Symbol	Unit	Description
α	[-]	material index
α_c	[W m^{-2}K^{-1}]	heat transfer coefficient
γ_u	[m s^{-1}]	prescribed liquid flux
γ_θ	[W m^{-2}]	prescribed heat flux
ε	[W m^{-3}]	exchange heat
κ	[m s^{-1}]	relative hydraulic conductivity
Λ	[W m^{-1} K^{-1}]	thermal conductivity
σ_{SB}	[W m^{-2} K^{-4}]	Stefan-Boltzmann constant
ν	[m s^{-2}]	kinematic viscosity
ρ	[kg m^{-3}]	density
θ	[K]	temperature
Θ	[-]	volume fraction
χ	[W m^{-2}]	prescribed heat energy flux
υ	[kg m-2 s^{-1}]	prescribed flux of a dissolved polute

Miscellaneous

Symbol	**Unit**	**Description**
$\mathcal{E}$	[J kg^{-1}]	internal energy per unit of mass
$\overline{\mathcal{H}}$	[J]	enthalpy
$\mathcal{H}$	[J kg^{-1}]	enthalpy per unit of mass
$\mathcal{Q}$	[W m^{-3}]	heat source
$\mathcal{U}$	[J]	total internal energy

Part

Introduction

1 Motivation

Modeling of coupled moisture and heat transport through partially saturated porous media plays a very important role in many agricultural, biological, environmental and civil engineering problems. Transport models may help with describing problems such as concrete degradation due to elevated temperatures or corrosion, concrete carbonation, contamination of soil areas due to infiltration of pollutants under the surface, water infiltration into subsurface structures, e.g. tunnels, radioactive waste repositories, subsurface pipelines etc., geothermal energy analyses, the groundwater distribution analyses and prediction of a drug delivery through biological tissues etc. Models including phase changes may help with the prediction of thawing of permafrost and its affects, degradation of railroad structures due to frost action, and many others.

Transport models are based on the conservation laws and the difficulty of analysis of these models lies in non-linear dependence of the transport coefficients on the solution, which arises from the complex microstructure of various porous materials. Moreover, different approaches need to be utilized to describe various porous materials since their structure and performance differs significantly.

2 Thesis outline

This work is organized as follows. In Part I, we discuss the main aspects of mathematical description of porous media and we briefly introduce the most frequent approaches to the porous media description. We also derive the equations describing transport of mass and energy, which are based on basic conservation laws.

In Part II, we briefly summarize the results, regarding mathematical analysis of the coupled transport models, which can be found in literature by various authors. We also describe where the main difficulty of analyzing these models lies.

In Part III, we present a single porosity model with mixed boundary conditions of Dirichlet and Neumann type. The formulation of the problem in a variational sense is introduced, and later its existence and uniqueness is analyzed under physically relevant assumptions, e.g. the transport coefficients degenerate in both elliptic and parabolic part. The proof of the existence theorem can be separated in several substeps. First, we approximate the evolution problem by means of a semi-implicit time scheme, and we prove the existence and regularity of the solution to a steady problem. Further, we derive suitable a-priori estimates to show that the solutions of the steady problem converge and that the limit is a solution to the original problem. Under some additional assumptions, we also show the uniqueness of the solution.

In Part IV, we present a dual porosity model, which may be more suitable for various engineering and ecological applications. Since the structure of the mathematical analysis remains the same, we focus our attention only on the main differences between the dual porosity model and the single porosity model.

In Part V, a general model with nonlinear boundary conditions and a coupled diffusion-dispersion-convection model are introduced. These models have been analyzed in detail in papers [6] and [7], which have been attached to this work in Appendices A and B.

Finally, in the last part of Appendices, we summarize some well known relations and theorems which have been used throughout the work.

Let us mention that the main results of this work are subject to the following papers which have been published throughout my doctoral studies at the Department of Mathematics under the supervision of Michal Beneš.

- [5] M. Beneš, L. Krupička, R. Štefan: *On coupled heat transport and water flow in unsaturated partially frozen porous media*, Applied Mathematics and Modelling, 39 (**2015**) 6580–3598.
- [6] M. Beneš, L. Krupička: *Weak solutions of coupled dual porosity flows in fractured rock mass and structured porous media*, Journal od Mathematical Analysis and Applications, 433 (**2016**) 543–565.
- [7] M. Beneš, L. Krupička: *Global weak solutions to degenerate coupled diffusion-convection-dispersion processed and heat transport in porous media*, International Conference on Applications of Mathematics to Nonlinear Sciences, Electronic Journal of Differential Equations, Conf. 24 (**2017**) 11–22.
- [8] M. Beneš, L. Krupička: *On coupled dual porosity flows in structured porous media*, AIP conference proceedings, 1978 (**2018**).

Part I

Physical background

The porous medium generally consists of the solid matrix and a void space occupied partially or fully by one or more fluid phases. The solid matrix and the void space are both distributed throughout all the medium, e.g. wherever we take a sufficiently large sample of the domain, it will always contain the matrix and the void space.

3 Porous media description

In order to describe any phenomena in the porous media, the real porous medium is represented by a conceptual model supposing a set of simplifying assumptions. Based on them we may formulate a mathematical model describing the performance of the sample. In this section we will present tha main ideas of the most commonly used approaches for porous media description.

3.1 Volume averaging

Generally the transport phenomena in porous media can be described at the microscopic level at each point. But due to our inability to describe the exact microscopic complex geometry of the porous media structure we have to search an appropriate model to be able to pass from the microscopic level to the macroscopic level. The most frequent approach to that is to average the physical quantity over a representative elementary volume (abbreviated REV). The representative elementary volume is the smallest volume over which we are allowed to make a measurement to obtain a representative value, i.e. a subvolume of a porous medium that has the same geometric configuration as the medium at a macroscopic scale (see Figure 1). Without the loss of the generality suppose the REV_0 is a circle with a center x_0. In order to evaluate the macroscopic value $\mathcal{V}(x_0)_{macro}$ of any physical quantity representing the porous medium, we average its microscopic value $\mathcal{V}(\boldsymbol{x})_{micro}$ over the representative elementary volume (for details see [3])

$$\mathcal{V}(x_0)_{macro} = \frac{1}{\operatorname{meas}(REV_0)} \int_{REV_0} \mathcal{V}(\boldsymbol{x})_{micro} \, \mathrm{d}x. \tag{3.1}$$

Using this procedure each point in the domain is assigned apropriate average values of physical quantities.

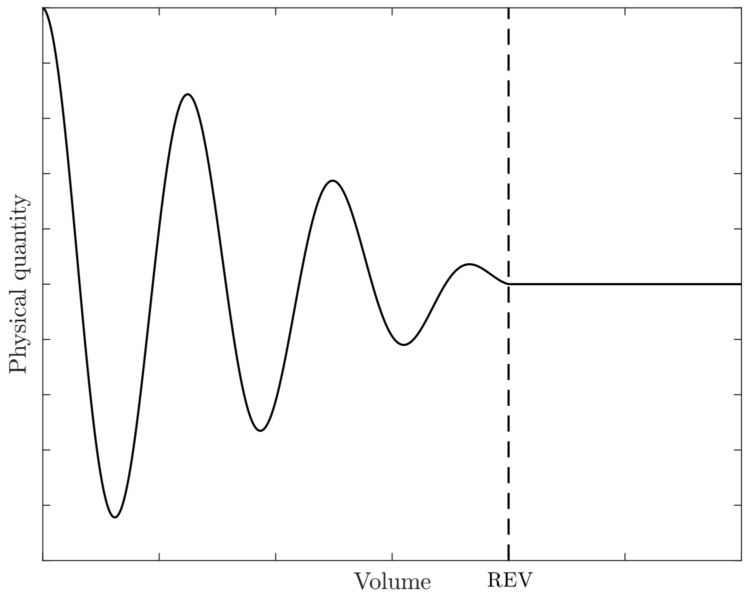

Figure 1: The representative elementary volume determination.

3.2 Continuum approach vs discrete approach

At the micro scale a porous medium is heterogeneous and possess a very complex structure. To describe transport phenomena in such a medium, various approaches may be introduced depending on the scale of application. The complex system description can be obtained applying discrete approach. However, this approach has limitations due to the fact, that for representative modeling the exact geometry of all individual fractures must be determined and the very high computational effort is needed. For more information see [9]. For this reason the discrete approach is limited only to the local studies with well known and described geometry. If a porous medium can be described without detailed knowledge of the fractured system, the continuum approach may be used. The continuum approach assumes that all phases are continuous within a representative elementary volume. However, in the literature could be found two different ways to mathematically describe the porous system using a continuum approach [14].

Single porosity/permeability continuum approach. The most common way of modeling the porous media is to assign a value of any physical quantity, that is averaged over REV, to each point of the domain of interest (see Figure 2). This is sometimes called the single porosity continuum approach.

Dual porosity/permeability continuum approach. An alternative option is to introduce the dual porosity continuum approach. This approach assumes, that

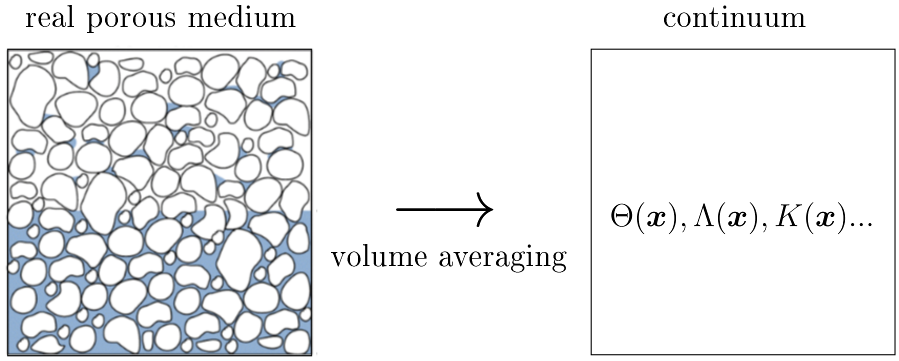

Figure 2: Representation of domain of interest with the single continuum approach.

the porous medium consisting of fractures and matrices, is represented by two overlapping interacting continua. (see Figure 3) One representing fractures, other representing matrix, respectively. These two continua possess different physical characteristics, e.g. hydraulic conductivity, thermal conductivity, diffusivity etc., and they are connected by appropriate exchange terms. The critical aspect of using this approach lies in determining these exchange terms.

Several articles dealing with the numerical comparison of presented approaches in various fields of interest can be found, for instance see [22], [37] and [48].

3.3 Saturated vs unsaturated zone

In terms of moisture retention, the porous medium can be divided into two regions. First, the zone where all available spaces are filled with water, e.g. saturated zone. The pressure in the saturated zone is greater than the atmospheric pressure, e.g. the gauge pressure is greater than zero. The surface where the pressure is equal to zero is called the water table. The zone where all available spaces are not filled with water is called unsaturated or partially saturated zone, sometimes also called vadose zone, from latin word "vadus" meaning shallow (see Figure 4). The water here is held by the surface adhesive forces and it is sucked above the water table level by the negative gauge pressure which is caused by capillary action. The capillary pressure in the porous medium depends mostly on pore size, hence in media with larger pores such as sand the capillary pressure is less than in clay soils with small pore size.

Let us note that the commonly used physical quantity describing pressure in the porous media is the pressure head u [m] which is a height of a liquid column corresponding to a particular pressure p [Pa]. This may be expressed mathematically as

$$u = \frac{p}{\rho g},$$

where ρ [kg m^{-3}] is the liquid density and g [m s^{-2}] is the acceleration due to gravity. The moisure retention dependance on the pressure head in the vadose zone

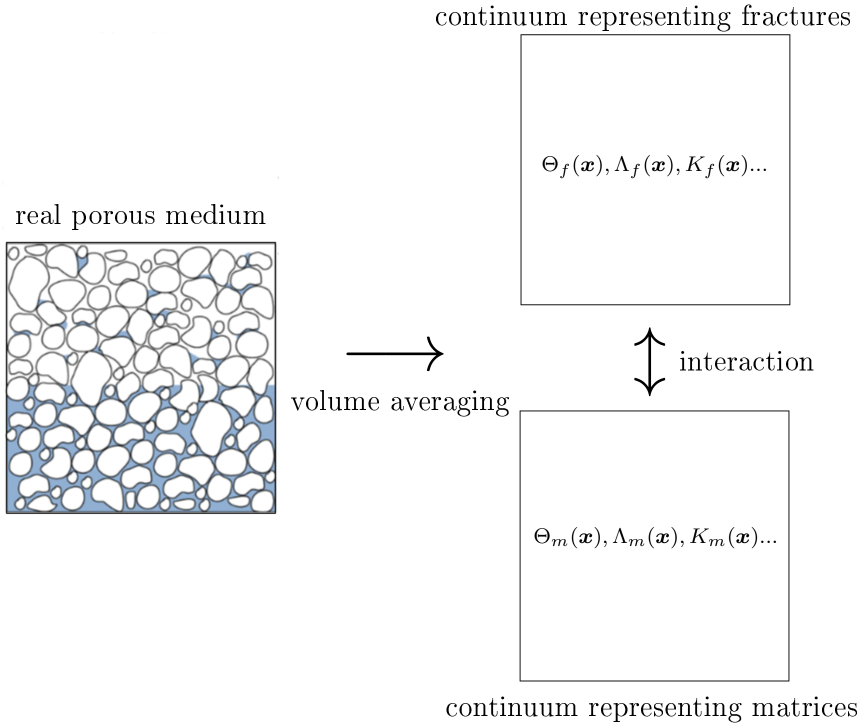

Figure 3: Representation of domain of interest with the dual continuum approach.

is described by the water retention curves (4.24) and the flow of water is commonly mathematically described by the Richards equation (see more in Section 4.1), which is based on Darcy's law (4.25).

4 Theory of mass and energy transport in partially saturated porous media

The transport processes in porous media are described by the basic conservation laws, namely the conservation of mass and the conservation of heat energy.

4.1 Fluid mass conservation law

In mixture theory, the derivation of the equation describing fluid transport in a variably saturated porous media is based on mass conservation of fluid α-phase in the domain Ω. A general form of a mass balance law is [50]

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{B}} \rho^{\alpha}\,\mathrm{d}x + \int_{\partial\mathcal{B}} \rho^{\alpha}\boldsymbol{v}_{\alpha}\cdot\boldsymbol{n}\,\mathrm{d}S = \int_{\mathcal{B}} s_{\alpha}\,\mathrm{d}x. \tag{4.1}$$

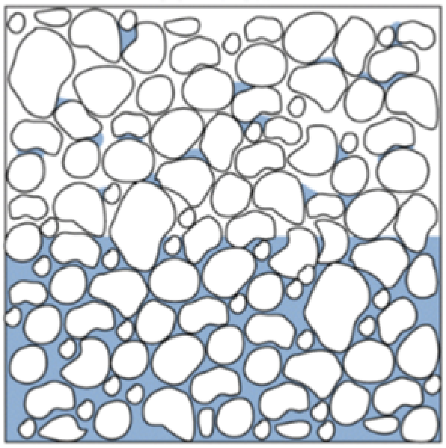

Figure 4: Variably saturated porous medium sample.

The equation has to be satisfied for any domain $\mathcal{B} \subset \overline{\mathcal{B}} \subset \Omega$. In (4.1) ρ^α [$\mathrm{kg\,m^{-3}}$] represents the phase averaged density and s_α [$\mathrm{kg\,m^{-3}\,s^{-1}}$] is a term representing production. Further, $\boldsymbol{v}_\alpha$ [$\mathrm{m\ s^{-1}}$] is the velocity of α-phase and $\boldsymbol{n}$ represents an outward unit normal vector to the boundary $\partial\mathcal{B}$. The phase averaged density ρ^α can be expressed as follows

$$\rho^\alpha = \Theta_\alpha \rho_\alpha, \tag{4.2}$$

where Θ_α [-] is the volume fraction of the α-phase and ρ_α [$\mathrm{kg\,m^{-3}}$] stands for the intrinsic phase averaged density. Let us note that

$$\sum_\alpha \Theta_\alpha = 1.$$

Considering $\mathcal{B}$ is an arbitrary subdomain within Ω one is allowed to use Green's theorem on (4.1) and remove the integrals to obtain

$$\frac{\partial \rho^\alpha}{\partial t} + \nabla \cdot (\rho^\alpha \boldsymbol{v}_\alpha) = s_\alpha. \tag{4.3}$$

Introducing the material derivative we may rewrite (4.3) as

$$\frac{\mathrm{D}^\alpha \rho^\alpha}{\mathrm{D}t} + \rho^\alpha \nabla \cdot \boldsymbol{v}_\alpha = s_\alpha, \tag{4.4}$$

where $\frac{\mathrm{D}^\alpha(.)}{\mathrm{D}t} = \frac{\partial(.)}{\partial t} + \nabla(.) \cdot \boldsymbol{v}_\alpha$ denotes the material derivative.

4.2 Heat energy transport conservation law

Derivation of heat equation is based on energy conservation law inside any arbitrary volume $\mathcal{B}$ in a porous domain of interest. The rate of temporary energy change in $\mathcal{B}$ plus the net rate of energy loss due to flow across the surface $\partial\mathcal{B}$ of $\mathcal{B}$ must be equal to rate of energy increase due to sources and interactions between phases. A general form of the heat energy balance law reads [39]

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{B}} e_\alpha \mathrm{d}x + \int_{\partial\mathcal{B}} (\boldsymbol{q}_T)_\alpha \cdot \boldsymbol{n}\,\mathrm{d}S = \int_{\mathcal{B}} \mathcal{Q}_\alpha \mathrm{d}x + \int_{\mathcal{B}} \varepsilon_\alpha \mathrm{d}x, \tag{4.5}$$

where e_α [$\mathrm{J\,m^{-3}}$] is the internal energy of the α-phase in $\mathcal{B}$ per unit of volume, $(\boldsymbol{q}_T)_\alpha$ [$\mathrm{W\,m^{-2}}$] is the heat flux, $\mathcal{Q}_\alpha$ [$\mathrm{W\,m^{-3}}$] stands for the volumetric heat source, ε_α [$\mathrm{W\,m^{-3}}$] represents the term expressing energy exchange with the other phases. For the internal energy per unit of volume we assume

$$e_\alpha = \rho^\alpha C_\alpha \theta_\alpha, \tag{4.6}$$

θ_α [K] is the absolute temperature and C_α [$\mathrm{J\,kg^{-1}\,K^{-1}}$] represents the specific isobaric heat of the α-phase. Further in (4.5) the heat flux vector $(\boldsymbol{q}_T)_\alpha$ includes the conductive flux $\boldsymbol{q}_\alpha$ [$\mathrm{W\,m^{-2}}$] and convection

$$(\boldsymbol{q}_T)_\alpha = \boldsymbol{q}_\alpha + \rho^\alpha C_\alpha \theta_\alpha \boldsymbol{v}_\alpha. \tag{4.7}$$

Let us note that besides the internal energy per unit of volume e_α given by (4.6) we may also use the internal energy per unit of mass $\mathcal{E}_\alpha$ [$\mathrm{J\ kg^{-1}}$] defined as

$$\mathcal{E}_\alpha = C_\alpha \theta_\alpha. \tag{4.8}$$

Considering $\mathcal{B}$ is an arbitrary subdomain within Ω we are allowed to use Green's theorem on (4.5) and remove the integrals to obtain

$$\frac{\partial e_\alpha}{\partial t} + \nabla \cdot (\boldsymbol{q}_T)_\alpha = \mathcal{Q}_\alpha + \varepsilon_\alpha. \tag{4.9}$$

Combining (4.6), (4.7), (4.8), and (4.9) we have

$$\mathcal{E}_\alpha \frac{\partial \rho^\alpha}{\partial t} + \rho^\alpha \frac{\partial \mathcal{E}_\alpha}{\partial t} + \nabla \cdot (\rho^\alpha \mathcal{E}_\alpha \boldsymbol{v}_\alpha) = -\nabla \cdot \boldsymbol{q}_\alpha + \mathcal{Q}_\alpha + \varepsilon_\alpha. \tag{4.10}$$

Further we replace $\partial_t \rho^\alpha$ in (4.10) using (4.3) to obtain

$$\rho^\alpha \frac{\partial \mathcal{E}_\alpha}{\partial t} - \mathcal{E}_\alpha \nabla \cdot (\rho^\alpha \boldsymbol{v}_\alpha) + \nabla \cdot (\rho^\alpha \mathcal{E}_\alpha \boldsymbol{v}_\alpha) = -\nabla \cdot \boldsymbol{q}_\alpha + \mathcal{Q}_\alpha + \varepsilon_\alpha - \mathcal{E}_\alpha Q^\alpha. \tag{4.11}$$

Since $\nabla \cdot (\rho^\alpha \mathcal{E}_\alpha \boldsymbol{v}_\alpha) - \mathcal{E}_\alpha \nabla \cdot (\rho^\alpha \boldsymbol{v}_\alpha) = \rho^\alpha \boldsymbol{v}_\alpha \cdot \nabla \mathcal{E}_\alpha$ (4.11) becomes

$$\rho^\alpha \frac{\partial \mathcal{E}_\alpha}{\partial t} + \rho^\alpha \boldsymbol{v}_\alpha \cdot \nabla \mathcal{E}_\alpha = -\nabla \cdot \boldsymbol{q}_\alpha + \mathcal{Q}_\alpha + \varepsilon_\alpha - \mathcal{E}_\alpha s_\alpha. \tag{4.12}$$

Using the material derivative we have

$$\rho^\alpha \frac{D^\alpha \mathcal{E}_\alpha}{Dt} = -\nabla \cdot \boldsymbol{q}_\alpha + \mathcal{Q}_\alpha + \varepsilon_\alpha - C_\alpha \theta_\alpha s_\alpha. \tag{4.13}$$

Enthalpy. To describe the property of a thermodynamic system we introduce enthalpy $\overline{\mathcal{H}}$ [J], which is equal to system's total internal energy $\mathcal{U}$ [J] and product of its pressure p [Pa] and volume V [kg m^{-3}] (see [39])

$$\overline{\mathcal{H}} = \mathcal{U} + pV.$$

In the following text, we will use the specific enthalpy within α-phase per unit of mass $\mathcal{H}_\alpha$ [J kg^{-1}] defined as

$$\mathcal{H}_\alpha = \mathcal{E}_\alpha + \frac{p_\alpha}{\rho_\alpha}, \tag{4.14}$$

where $\mathcal{E}_\alpha$ is the internal energy per unit of mass defined in (4.8). Let us specifically mention that $\mathcal{H}_\alpha$ is a function of pressure and temperature [39].

Now we express the material derivative of internal energy $\mathcal{E}_\alpha$ in (4.13) with specific enthalpy (4.14)

$$\begin{aligned}
\frac{D^\alpha \mathcal{E}_\alpha}{Dt} &= \frac{D^\alpha}{Dt}\left(\mathcal{H}_\alpha - \frac{p_\alpha}{\rho_\alpha}\right) \\
&= \frac{D^\alpha \mathcal{H}_\alpha}{Dt} - \left[\frac{\partial}{\partial p_\alpha}\left(\frac{p_\alpha}{\rho_\alpha}\right)\right]\frac{D^\alpha p_\alpha}{Dt} - \left[\frac{\partial}{\partial \rho_\alpha}\left(\frac{p_\alpha}{\rho_\alpha}\right)\right]\frac{D^\alpha \rho_\alpha}{Dt} \\
&= \frac{D^\alpha \mathcal{H}_\alpha}{Dt} - \frac{1}{\rho_\alpha}\frac{D^\alpha p_\alpha}{Dt} + \frac{p_\alpha}{\rho_\alpha^2}\frac{D^\alpha \rho_\alpha}{Dt} \\
&= \left(\frac{\partial \mathcal{H}_\alpha}{\partial \theta_\alpha}\right)_p \frac{D^\alpha T_\alpha}{Dt} + \left(\frac{\partial \mathcal{H}_\alpha}{\partial p_\alpha}\right)_\theta \frac{D^\alpha p_\alpha}{Dt} - \frac{1}{\rho_\alpha}\frac{D^\alpha p_\alpha}{Dt} + \frac{p_\alpha}{\rho_\alpha^2}\frac{D^\alpha \rho_\alpha}{Dt},
\end{aligned}$$

hence

$$\frac{D^\alpha \mathcal{E}_\alpha}{Dt} = \left(\frac{\partial \mathcal{H}_\alpha}{\partial T_\alpha}\right)_p \frac{D^\alpha T_\alpha}{Dt} + \frac{p_\alpha}{\rho_\alpha^2}\frac{D^\alpha \rho_\alpha}{Dt} + \left[\left(\frac{\partial \mathcal{H}_\alpha}{\partial p_\alpha}\right)_\theta - \frac{1}{\rho_\alpha}\right]\frac{D^\alpha p_\alpha}{Dt}. \tag{4.15}$$

Now we will express the term $\frac{D^\alpha \rho_\alpha}{Dt}$ from the mass transport equation. Putting $\rho^\alpha = \Theta_\alpha \rho_\alpha$ in (4.4) we obtain

$$\frac{\partial(\rho_\alpha \Theta_\alpha)}{\partial t} + \rho_\alpha \Theta_\alpha \nabla \cdot \boldsymbol{v}_\alpha + \boldsymbol{v}_\alpha \cdot \nabla(\Theta_\alpha \rho_\alpha) - s_\alpha = 0,$$

which becomes

$$\rho_\alpha\left(\frac{\partial \Theta_\alpha}{\partial t} + \boldsymbol{v}_\alpha \cdot \nabla \Theta_\alpha\right) + \Theta_\alpha\left(\frac{\partial \rho_\alpha}{\partial t} + \boldsymbol{v}_\alpha \cdot \nabla \rho_\alpha\right) + \rho_\alpha \Theta_\alpha \nabla \cdot \boldsymbol{v}_\alpha - s_\alpha = 0. \tag{4.16}$$

We rewrite (4.16) using the material derivative we get

$$\rho_\alpha \frac{D^\alpha \Theta_\alpha}{Dt} + \Theta_\alpha \frac{D^\alpha \rho_\alpha}{Dt} + \rho_\alpha \Theta_\alpha \nabla \cdot \boldsymbol{v}_\alpha - s_\alpha = 0.$$

Now we can express the material derivative of intristic phase averaged density as

$$\frac{D^\alpha \rho_\alpha}{Dt} = -\frac{\rho_\alpha}{\Theta_\alpha}\frac{D^\alpha \theta_\alpha}{Dt} - \rho_\alpha \nabla \cdot \boldsymbol{v}_\alpha + \frac{1}{\Theta_\alpha}s_\alpha. \tag{4.17}$$

Combining (4.17) and (4.15) we arrive at

$$\frac{D^\alpha \mathcal{E}_\alpha}{Dt} = \left(\frac{\partial \mathcal{H}_\alpha}{\partial \theta_\alpha}\right)_p \frac{D^\alpha \theta_\alpha}{Dt} + \left[\left(\frac{\partial \mathcal{E}_\alpha}{\partial p_\alpha}\right)_\vartheta - \frac{1}{\rho_\alpha}\right] \frac{D^\alpha p_\alpha}{Dt} - \frac{p_\alpha}{\rho_\alpha \Theta_\alpha} \frac{D^\alpha \Theta_\alpha}{Dt} + \frac{p_\alpha}{\rho_\alpha^2 \Theta_\alpha} s_\alpha - \frac{p_\alpha}{\rho_\alpha} \nabla \cdot \boldsymbol{v}_\alpha. \quad (4.18)$$

Now we put the expression for the material derivative of internal energy per unit of mass $\mathcal{E}_\alpha$ (4.18) in (4.13) and we have

$$\rho_\alpha \Theta_\alpha \left[\left(\frac{\partial \mathcal{H}_\alpha}{\partial T_\alpha}\right)_p \frac{D^\alpha \theta_\alpha}{Dt} + \left[\left(\frac{\partial \mathcal{H}_\alpha}{\partial p_\alpha}\right)_{\theta_\alpha} - \frac{1}{\rho_\alpha}\right] \frac{D^\alpha p_\alpha}{Dt}\right] - \rho_\alpha \Theta_\alpha \left[\frac{p_\alpha}{\rho_\alpha \Theta_\alpha} \frac{D^\alpha \Theta_\alpha}{Dt} - \frac{p_\alpha}{\rho_\alpha^2 \Theta_\alpha} s_\alpha + \frac{p_\alpha}{\rho_\alpha} \nabla \cdot \boldsymbol{v}_\alpha\right] = -\nabla \cdot \boldsymbol{q}_\alpha + \mathcal{Q}_\alpha + \varepsilon_\alpha - \mathcal{E}_\alpha \mathcal{Q}_\alpha.$$

Hence we obtain

$$\rho_\alpha \Theta_\alpha \left(\frac{\partial \mathcal{H}_\alpha}{\partial \theta_\alpha}\right)_p \frac{D^\alpha \theta_\alpha}{Dt} + \nabla \cdot \boldsymbol{q}_\alpha = \mathcal{Q}_\alpha + \varepsilon_\alpha - \mathcal{E}_\alpha s_\alpha - \frac{p_\alpha}{\rho_\alpha} s_\alpha + p_\alpha \frac{D^\alpha \Theta_\alpha}{Dt} + \Theta_\alpha \frac{D^\alpha p_\alpha}{Dt} + p_\alpha \Theta_\alpha \nabla \cdot \boldsymbol{v}_\alpha - \rho_\alpha \Theta_\alpha \left(\frac{\partial \mathcal{H}_\alpha}{\partial p_\alpha}\right)_\theta \frac{D^\alpha p_\alpha}{Dt}. \quad (4.19)$$

Considering (4.14) we arrive at general form of heat equation for α-phase

$$\rho_\alpha \Theta_\alpha \left(\frac{\partial \mathcal{H}_\alpha}{\partial \theta_\alpha}\right)_p \frac{D^\alpha \theta_\alpha}{Dt} + \nabla \cdot \boldsymbol{q}_\alpha = \mathcal{Q}_\alpha + \varepsilon_\alpha - \mathcal{H}_\alpha s_\alpha + \frac{D^\alpha (p_\alpha \Theta_\alpha)}{Dt} + p_\alpha \Theta_\alpha \nabla \cdot \boldsymbol{v}_\alpha - \rho_\alpha \Theta_\alpha \left(\frac{\partial \mathcal{H}\alpha}{\partial p_\alpha}\right)_\theta \frac{D^\alpha p_\alpha}{Dt}. \quad (4.20)$$

After neglecting some small terms related to viscous dissipation and mechanical work, caused by density variation due to temperature changes and caused by volume fraction changes (for details see [39]) we get

$$\rho_\alpha \Theta_\alpha \left(\frac{\partial \mathcal{H}_\alpha}{\partial \theta_\alpha}\right)_p \frac{D^\alpha \theta_\alpha}{Dt} + \nabla \cdot \boldsymbol{q}_\alpha = \mathcal{Q}_\alpha + \varepsilon_\alpha - \mathcal{H}_\alpha s_\alpha, \quad (4.21)$$

which becomes, considering (4.14)

$$\rho_\alpha \Theta_\alpha C_\alpha \frac{\partial \theta_\alpha}{\partial t} + \rho_\alpha C_\alpha \Theta_\alpha \boldsymbol{v}_\alpha \cdot \nabla \theta_\alpha + \nabla \cdot \boldsymbol{q}_\alpha = \mathcal{Q}_\alpha + \varepsilon_\alpha - \mathcal{H}_\alpha s_\alpha. \quad (4.22)$$

The total energy balance within a multiphase system consists of contribution from each phase. Considering the total amount of heat energy exchange within phases remains in the system, we can write for the multiphase system heat equation in general form

$$\sum_\alpha \rho_\alpha \Theta_\alpha C_\alpha \frac{\partial \theta_\alpha}{\partial t} + \sum_\alpha \rho_\alpha C_\alpha \Theta_\alpha \boldsymbol{v}_\alpha \cdot \nabla \theta_\alpha + \sum_\alpha \nabla \cdot \boldsymbol{q}_\alpha = \sum_\alpha \mathcal{Q}_\alpha - \sum_\alpha \mathcal{H}_\alpha s_\alpha. \quad (4.23)$$

4.3 Constitutive relationships and hydraulic characteristics

Retention curves. Concerning retention curves of the matrix pore systems, we present here the commonly used relation proposed by van Genuchten and Mualem (see, for instance, [21])

$$\Theta(u) = \Theta_r + (\Theta_s - \Theta_r)[1 + |au|^{n_1}]^{-n_2}, \tag{4.24}$$

where Θ_s is the soil saturated water content [-], Θ_r is the soil residual water content [-], α [m^{-1}], n_1 and n_2 are parameters. For an example of a retention curve see Figure 5.

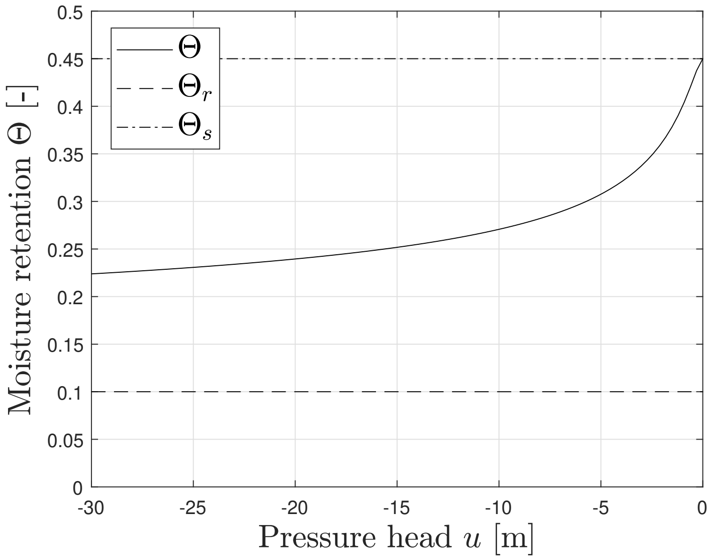

Figure 5: The retention curve given by (4.24).

Darcy's law. The moisture flux through the variably saturated porous system is determined by Darcy's constitutive law

$$\Theta \boldsymbol{v} = -k(u,\theta)(\nabla u + \boldsymbol{e}_z), \tag{4.25}$$

where u [m] is the pressure head, $\boldsymbol{e}_z$ stands for the vertical unit vector and k [m s^{-1}] represents the hydraulic permeability of the porous medium. The temperature-pressure head dependence of the hydraulic conductivity is given by [13, 54]

$$k(\theta, u) = k_s\, \nu_0\, \frac{\kappa(u)}{\nu(\theta)}, \tag{4.26}$$

where k_s [m s^{-1}] is the saturated hydraulic conductivity at the reference temperature T_0 [K], κ [m s^{-1}] is the h-dependent relative hydraulic conductivity,

$$\kappa(u) = \sqrt{S(u)}\left(1 - \left(1 - S(u)^{1/n_2}\right)^{n_2}\right)^2 \tag{4.27}$$

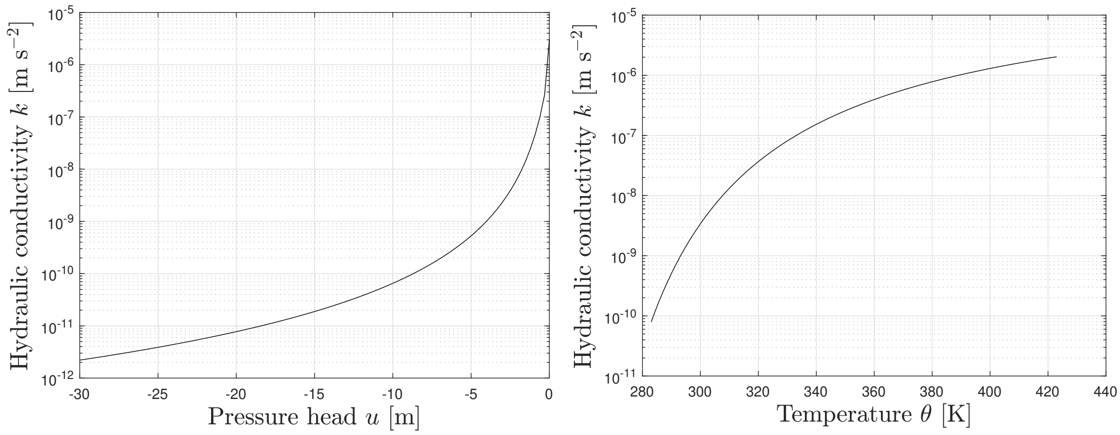

Figure 6: An example of the hydraulic conductivity k $[\mathrm{m\,s^{-1}}]$ at constant temperature $\theta = 300\mathrm{K}$ (left) and constant pressure head $u = -5\mathrm{m}$ (right) given by (4.26).

for $h < 0$ (unsaturated porous media), $S(u) = \frac{\Theta(u)-\Theta_r}{\Theta_s-\Theta_r}$. Finally, ν [m s^{-2}] is the temperature dependent kinematic viscosity for water given by

$$\nu(\theta) = 2.414 \times 10^{-5} \times 10^{247.8/(\theta-140)} \tag{4.28}$$

and $\nu_0 := \nu(\kappa_0)$. For an example of the hydraulic conductivity see Figure 6.

Fourier's law. We assume the conductive heat flux $\boldsymbol{q}$ to be given by Fourier's law

$$\boldsymbol{q} = -\Lambda(u,\theta)\nabla\theta \tag{4.29}$$

with the thermal conductivity function Λ $[\mathrm{W\,m^{-1}\,K^{-1}}]$. The thermal conductivity for porous media may be given by [20]

$$\Lambda(u,\theta) = \Lambda_d(\theta)\Lambda_t(u). \tag{4.30}$$

In (4.30) n [-] is porosity, and Λ_d is the thermal conductivity of a dry sample given by

$$\Lambda_d(\theta) = \Lambda_{d,ref}\left[1 + A_\Lambda(\theta - \theta_{ref})\right], \tag{4.31}$$

where $\Lambda_{d,ref}$ [W m^{-1} K^{-1}] is the reference thermal conductivity of a dry sample at a reference temperature θ_{ref} [K] and A_Λ is a parameter [K^{-1}]. And Λ_t in (4.30) is the reference thermal conductivity of a sample at a reference temperature θ_{ref} given by

$$\Lambda_t(u) = 1 + \frac{4n\rho_f\Theta(u)}{(1-n)\rho_m}. \tag{4.32}$$

For an example of the thermal conductivity see Figure 7.

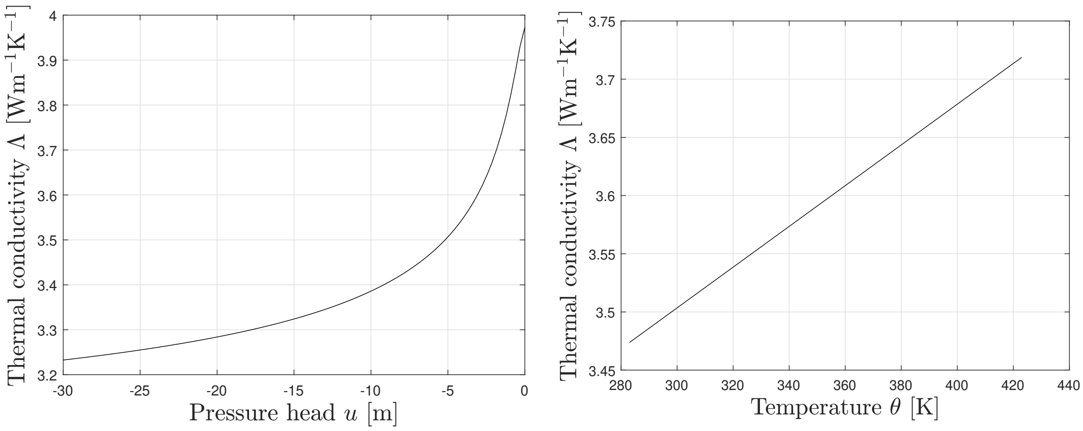

Figure 7: An example of the thermal conductivity Λ [W m^{-1} K^{-1}] at constant temperature $\theta = 300$K (left) and constant pressure head $u = -5$m (right) given by (4.30).

4.4 Boundary and initial conditions

The fluid flux across the boundary is given by the Neumann type boundary condition

$$\Theta(u)\boldsymbol{v} \cdot \boldsymbol{n} = \gamma_u,$$

where γ_u [m s^{-1}] represents the liquid flux imposed on the boundary. Considering Darcy's law (4.25) we may write

$$-k(u, \theta)(\nabla u + \boldsymbol{e}_z) \cdot \boldsymbol{n} = \gamma_u.$$

For the heat flux, we may use the natural boundary condition given by

$$\boldsymbol{q} \cdot \boldsymbol{n} = \alpha_c(\theta - \theta_\infty) + e_m \sigma_{SB}(\theta^4 - \theta_\infty^4) + \gamma_\theta,$$

where α_c [W m^{-2}K^{-1}] is the heat transfer coefficient, e_m [-] stands for the relative surface emissivity, σ_{SB} [W m^{-2}K^{-4}] represents the Stefan-Boltzmann constant, T_∞ [K] is the temperature of the environment and γ_θ [W m^{-2}] represents the heat flux imposed on the boundary. Considering Fourier's law (4.29) we have

$$-\Lambda(u, \theta)\nabla\theta \cdot \boldsymbol{n} = \alpha_c(\theta - \theta_\infty) + e\sigma_{SB}(\theta^4 - \theta_\infty^4) + \gamma_\theta.$$

The Dirichlet boundary conditions are given by prescribed values of the pressure head u_D [m] and temperature θ_D [K] on the boundary

$$u = u_D, \quad \theta = \theta_D.$$

The initial conditions are set as follows:

$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \quad \theta(\boldsymbol{x}, 0) = \theta_0(\boldsymbol{x}),$$

where, u_0 [m] and θ_0 [K] represent the initial distributions of the pressure head and temperature.

4.5 Summary of transport equations

Physical assumptions. In order to model effectively coupled transport of moisture and heat energy, we present a set of simplifying assumptions based on the physical reality.

(A) The porous medium consists of flowing liquid water (index f) and solid matrix (index m);

(B) solid phase is incompressible and immobile, hence

$$\boldsymbol{v}_m = 0, \quad \Theta_m = \text{const};$$

(C) liquid phase is incompressible, hence

$$\nabla \cdot \boldsymbol{v}_f = 0;$$

(D) hysteresis is not present;

(E) porous medium is not deformable;

(F) there are neither external sources of heat and mass, nor the phase changes;

(G) the medium is isotropic;

(H) the intristic phase averaged density ρ_α and the specific isobaric heat C_α are constant.

Simplified equations. Taking into account the set of simplifying assumptions (A)–(H) the basic general conservation equations introduced in the Sections 4.1 and 4.2 may be simplified. The moisture transport equation (4.3) becomes

$$\frac{\partial \Theta_f(u)}{\partial t} + \nabla \cdot (\Theta_f \boldsymbol{v}_f) = 0, \tag{4.33}$$

hence, using Darcy's law (4.25), we obtain

$$\frac{\partial \Theta_f(u)}{\partial t} - \nabla \cdot K(u, \theta)(\nabla u + \boldsymbol{e}_z) = 0. \tag{4.34}$$

Considering the set of simplifying assumptions (A)–(H) the heat conservation equation (4.22) becomes

$$\rho_f \Theta_f(u) C_f \frac{\partial T_f}{\partial t} + \rho_m \Theta_m C_m \frac{\partial \theta_m}{\partial t} + \rho_f C_f \Theta_f(h) \boldsymbol{v}_f \cdot \nabla \theta_f + \nabla \cdot \boldsymbol{q}_f + \nabla \cdot \boldsymbol{q}_m = 0. \tag{4.35}$$

For the first term we can write

$$\rho_f \Theta_f(u) C_f \frac{\partial \theta_f}{\partial t} = \rho_f C_f \frac{\partial [\Theta_f(u) \theta_f]}{\partial t} - \rho_f C_f \frac{\partial \Theta_f(u)}{\partial t} \theta_f. \tag{4.36}$$

Now combining (4.36), (4.33) and putting into (4.35) we obtain

$$\rho_f C_f \frac{\partial[\Theta_f(h)\theta_f]}{\partial t} + \rho_m \Theta_m C_m \frac{\partial \theta_m}{\partial t} + \rho_f C_f \nabla \cdot [\theta_f \Theta_f(u) \boldsymbol{v}_f] + \nabla \cdot \boldsymbol{q}_f + \nabla \cdot \boldsymbol{q}_m = 0. \quad (4.37)$$

Hence, taking into account the Darcy's law (4.25) and Fourier's law (4.29), we obtain

$$\begin{aligned} \rho_f C_f \frac{\partial[\Theta_f(u)\theta_f]}{\partial t} + \rho_m \Theta_m C_m \frac{\partial \theta_m}{\partial t} - \rho_f C_f \nabla \cdot [T_f k(h, \theta_f)(\nabla h + \boldsymbol{e}_z)] \\ - \nabla \cdot \Lambda_f(u, \theta_f) \nabla \theta_f - \nabla \cdot \Lambda_m(u, \theta_m) \nabla \theta_m = 0. \quad (4.38) \end{aligned}$$

4.5.1 Single porosity continuum mathematical model

Let the domain of interest Ω be a part of a variably saturated porous medium partially filled with water. We assume the domain as a continuum described in Section 3.2, hence one domain is continuously filled in each point by both water and skeleton. For further text we denote

$$b(u) := \Theta_f(u).$$

The moisture transport in such a domain is described by Richards equation completed by Darcy's law in form

$$\frac{\partial b(u)}{\partial t} - \nabla \cdot k(u, \theta)(\nabla u + \boldsymbol{e}_z) = 0. \quad (4.39)$$

Further we assume the thermal equilibrium in each point of the continuum, i.e. $\theta_f = \theta_m$. Now let us denote

$$\begin{aligned} \varrho &:= \frac{\rho_m \Theta_m C_m}{\rho_f C_f}, \\ \lambda(u, \theta) &:= \frac{\Lambda_f(u, \theta) + \Lambda_m(u, \theta)}{\rho_f C_f} \end{aligned}$$

to obtain the heat balance equation in the form

$$\frac{\partial[b(u)\theta + \varrho\theta]}{\partial t} - \nabla \cdot [\theta k(u, \theta)(\nabla u + \boldsymbol{e}_z)] - \nabla \cdot \lambda(u, \theta) \nabla \theta = 0. \quad (4.40)$$

Part II

Literature review

In mathematics, a wide variety of phenomena can be described by partial differential equations, for instance heat transport, moisture transport, fluid dynamics, electrostatics, elasticity, quantum mechanics and many others. Unfortunately, in general, we cannot expect that a partial differential equation or a system of them has a classical, i.e. strong solution. The existence of a classical solution requires sufficiently smooth parameters and certain strict regularity conditions on the domain of interest. However, these conditions are often not satisfied in various applications and phenomena. Therefore, we often deal with so called weak formulation of the problem described by the partial differential equations. A weak solution, sometimes also called a generalized solution, is a solution for which the derivatives may not but in spite of that it satisfies the equation in some precisely defined sense. This approach is widely used in application of mathematics in various fields of interest in order to solve systems of equations describing the real nature or technical phenomena. Sometimes it is even convenient to prove the existence of a weak solution of the problem and after that show that this solution is smooth enough.

As already mentioned above, one of the important applications of the partial differential equations is modelling transport processes of heat energy and moisture within porous media. In the following paragraphs we will briefly summarize the existing results regarding this topic, which can be found in literature and we will also demonstrate where lies the main difficulty of these problems.

We shall rewrite the general model describing the coupled transport phenomena in porous media in terms of vector operators $\boldsymbol{A} = A_{ij}$, $\boldsymbol{\Psi} = \Psi_i$, $\boldsymbol{F} = F_i$ in the form

$$\partial_t \boldsymbol{\Psi}(\boldsymbol{u}) - \nabla \cdot \boldsymbol{A}(\boldsymbol{u}, \nabla \boldsymbol{u}) = \boldsymbol{F}(\boldsymbol{u}), \tag{4.41}$$

where $\boldsymbol{u}$ stands for the unknown vector of state variables corresponding here to the matric potential, temperature and concentration of the dissolved species.

The difficulty in predicting the transport phenomena and in analyzing the model (4.41) remains in non-linear dependence of the transport coefficients on temperature and moisture retention, which have been observed in laboratories. This is a result of the complex microstructure of porous media or fractured rock masses. For reasonable applications these nonlinearities cannot be ignored. Therefore, problems of this type equipped with the appropriate initial and boundary conditions are too complex to be solved analytically. However, they may be solved in a weak sense using spacial discretization of the domain by means of the finite element or finite volume method and time discretization of the time interval.

In this text we will study some qualitative properties of the system (4.41) in order to prove the existence and regularity of the variational solution. Let us note that there is no complete theory for such general problem. However, we might find some results considering special structures of operators $\boldsymbol{A}$, $\boldsymbol{\Psi}$ and growth conditions of $\boldsymbol{F}$.

The first related work, where is proven the existence, regularity and uniqueness of solutions to (4.41) is by H. W. Alt and S. Luckhaus in 1983 [2]. The authors obtained results for the problem

$$\partial_t \boldsymbol{\Psi}(\boldsymbol{u}) - \nabla \cdot \boldsymbol{A}(\boldsymbol{\Psi}(\boldsymbol{u}), \nabla \boldsymbol{u}) = \boldsymbol{F}(\boldsymbol{\Psi}(\boldsymbol{u})) \qquad \text{in } \Omega \times [0;T], \tag{4.42}$$
$$\boldsymbol{\Psi}(\boldsymbol{u}) = \boldsymbol{\Psi}_0 \qquad \text{in } \Omega \times 0, \tag{4.43}$$
$$\boldsymbol{u} = \boldsymbol{u}_D \qquad \text{on } \Gamma_{DT} \times [0;T], \tag{4.44}$$
$$\boldsymbol{A}(\boldsymbol{u}, \nabla \boldsymbol{u}) \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma_{NT} \times [0;T]. \tag{4.45}$$

Assuming the following assumptions on the coefficients in (4.42)-(4.45)

- $\boldsymbol{\Psi}$ is monotone;
- $\boldsymbol{\Psi}$ has a gradient structure;
- $\boldsymbol{A}$ is continuous and elliptic;
- $\boldsymbol{F}$ is continuous and satisfies general growth condition (for details see [2], Section 1.1. condition 4).

In 1987 G. Modica and M. Giaquinta [24] proved local solvability in classical sense of a quasilinear parabolic system with nonlinear Neumann boundary conditions without assuming any growth conditions

$$\partial_t \boldsymbol{u} - \nabla \cdot \boldsymbol{A}(\boldsymbol{u}, \nabla \boldsymbol{u}) = \boldsymbol{F}(\boldsymbol{u}) \qquad \text{in } \Omega \times [0;\infty), \tag{4.46}$$
$$\boldsymbol{u} = 0 \qquad \text{in } \Omega \times 0, \tag{4.47}$$
$$\boldsymbol{A}(\boldsymbol{u}, \nabla \boldsymbol{u}) \cdot \boldsymbol{n} = \boldsymbol{g} \qquad \text{on } \Gamma \times [0;\infty). \tag{4.48}$$

There are no further assumptions on $\boldsymbol{F}$ in (4.46)–(4.48). Strong solutions to the Richards equation are further analysed in [44] by Rybka and Merz.

Later in 1995 J. Filo and J. Kačur [17] extended previous results by proving the local existence of the weak solution assuming nonlinear boundary condition of Neumann type and more general growth conditions on $\boldsymbol{F}$.

$$\partial_t \boldsymbol{\Psi}(\boldsymbol{u}) - \nabla \cdot \boldsymbol{A}(\boldsymbol{u}, \nabla \boldsymbol{u}) = \boldsymbol{F}(\boldsymbol{u}) \qquad \text{in } \Omega \times [0;T], \tag{4.49}$$
$$\boldsymbol{u} = \boldsymbol{u}_0 \qquad \text{in } \Omega \times 0, \tag{4.50}$$
$$\boldsymbol{A}(\boldsymbol{u}, \nabla \boldsymbol{u}) \cdot \boldsymbol{n} = \boldsymbol{g} \qquad \text{on } \Gamma \times [0;T]. \tag{4.51}$$

- $\boldsymbol{\Psi}$ is monotone;
- $\boldsymbol{\Psi}$ has a gradient structure;
- $\boldsymbol{A}$ is continuous, monotone and coercive;

- $\boldsymbol{F}$ is continuous and its growth is bounded by polynomial (for details see [17], equation (1.4));
- $\boldsymbol{g}$ is continuous and its growth is bounded by polynomial (for details see [17], equation (1.5)).

Nevertheless, mentioned results assume either linear parabolic part or the gradient structure of Ψ. In 2002 J. Vala [58] analyzed model with an approach allowing non-symmetry in the parabolic term although requiring unrealistic symmetry in the elliptic term. The same author in [59] presented some other special transformations preserving the symmetry in the elliptic part.

However, most theoretical results exclude the non-symmetrical parabolic part. These models are applicable in various issues within engineering, ecology and biology. Let us mention several examples of articles related to such topics. In [45] and [46] authors analyze degradation processes in concrete due to chemical corrosion caused by concrete carbonation. In [33], [34] and [35] deal with model describing the interplay between fluxes of a colloidal population and heat flux. Such models are applicable in predicting the concrete performance to high temperatures due to explosions or predicting the drug delivery through biological tissues. Further in [40], [41] and [42] B. Li and W. Sun analyze the specific model arising from coupled moisture and heat transport with phase change in fibrous textile material. In [12] and [28] is analysed solvability, uniqueness and regularity of a solution to a quasilinear model of fluid/gas transport exposed to an electric field, thermal and diffusive forces. In [32] the authors deal with parabolic variational inequalities solved by means of a combined relaxation method and method of characteristic. A similar problem was discussed by Hornung in [27].

Further, many results concerning the practical aspects and the physical relevance of mathematical modelling of transport processes in porous media can be found in literature. In 1989 [38] Lewis, Roberts and Schrefler introduced the finite element scheme solving the system describing coupled heat and fluid flow in deforming porous media. Authors also discussed its advantages and disadvantages concerning the numerical stability during the initial time steps and also present the comparison of obtained results with practical experiments.

Similar problem is dealt with in [57], where authors compare obtained results with measurement on the fractured rock mass. Next numerical scheme is introduced in [55], where Simunek and Saito introduced a numerical model solving the equations governing liquid water and water vapor movement under the soil surface in the vadose zone. Various research and commercial numerical tools are compared in [53].

In [43] Liu and Yu present a model including freezing processes as well as Hansson and Simunek in [25]. Such models are applicable for predicting the transport processes in freezing soils which is one of the most important issues in transport engineering, e.g. railway structures or subsurface structures. Another authors compare in their publications various approaches to the modelling, e.g. single porosity approach and double porosity approach and others. See for instance [11] and [26].

As mentioned before many results concerning numerical modelling can be found in literature, however, the main issue in modelling transport processes in porous media is, that some of the key components of the model, such as the hydraulic conductivity or parameters for the water retention curves often rely on empirical relations whose predictions may not be physically reliable since the structure of the porous media is usually very complex and very variable. Because of that the complex geological research is necessary in order to be able to predict the performance of the porous medium with a required accuracy.

In this work we focus our attention on the theoretical aspects of (4.41) concerning the degenerate doubly nonlinear elliptic-parabolic system with a specific structure arising from diffusion-convection-dispersion processes in partially saturated porous media. Degenerations occur in all transport coefficients according to the physical background. The transport coefficients are not assumed to be bounded by positive constants. We consider a weak formulation of (4.41) in the integral form

$$\int_{Q_T} \partial_t \boldsymbol{\Psi}(\boldsymbol{u}) \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}\mathrm{d}t + \int_{Q_T} \boldsymbol{A}(\boldsymbol{u}, \nabla \boldsymbol{u}) \cdot \nabla \boldsymbol{v} = \int_{Q_T} \boldsymbol{F}(\boldsymbol{u}) \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}\mathrm{d}t, \tag{4.52}$$

where $\boldsymbol{v}$ is a test function from an appropriate Sobolev space, Q_T denotes a space time cylinder $\Omega \times I$, (Ω is a domain in $\mathbb{R}^2$ and I denotes a time interval). The aim of the existence and convergence analysis of (4.52) is to find some $\boldsymbol{u} \in L^\infty(Q_T)$ with $\boldsymbol{u} \in L^2\left(I, W^{1,2}(\Omega)\right)$ satisfying (4.52) in some time interval $I = \{t \in \mathbb{R} : 0 \leq t \leq T\}$.

The presented text is based on our previous results [5], [6], [7] and [8] published during the Ph.D. studies.

In [5] we deal with a model decribing coupled transport processes including freezing and thawing phenomena. In the paper we proved the existence of a weak solution of the steady problem and we presented a numerical example documenting its physical relevance. In [7] we analyzed qualitative properties of the single porosity model of a coupled transport of heat, moisture and dissolved species. And finally, in [6] and [8] we dealt with a dual porosity approach model of coupled heat and moisture transport. The papers [6] and [7] are attached to this text in Appendices A and B.

Part III

Mathematical analysis of the single porosity model

Let Ω be a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. Let Γ_D and Γ_N be open disjoint subsets of $\partial\Omega$ such that $\Gamma_D \neq \emptyset$ and $\partial\Omega\backslash(\Gamma_D \cup \Gamma_N)$ is a finite set. Let $T \in (0,\infty)$ be fixed throughout the text, $I = (0,T)$ and $Q_T = \Omega \times I$ denotes the space-time cylinder, $\Gamma_{DT} = \Gamma_D \times I$ and $\Gamma_{NT} = \Gamma_N \times I$.

5 Strong Formulation of the Problem

We will analyze the initial boundary value problem in Q_T arising from a coupled moisture-heat transport through a partially saturated porous media as described in the previous text. We consider the following system

$$\partial_t b(u) = \nabla \cdot (k(\theta,u)\,(\nabla u + \boldsymbol{e}_z)) \quad \text{in } Q_T, \tag{5.1}$$

$$\partial_t\,[b(u)\theta + \varrho\theta] = \nabla \cdot (\lambda(\theta,u)\nabla\theta) + \nabla \cdot (\theta k(\theta,u)\,(\nabla u + \boldsymbol{e}_z))) \quad \text{in } Q_T, \tag{5.2}$$

$$u = 0 \quad \text{on } \Gamma_{DT}, \tag{5.3}$$

$$\theta = 0 \quad \text{on } \Gamma_{DT}, \tag{5.4}$$

$$(\nabla u + \boldsymbol{e}_z) \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma_{NT}, \tag{5.5}$$

$$\nabla\theta \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma_{NT}, \tag{5.6}$$

$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) \quad \text{in } \Omega, \tag{5.7}$$

$$\theta(\boldsymbol{x},0) = \theta_0(\boldsymbol{x}) \quad \text{in } \Omega. \tag{5.8}$$

In (5.1)–(5.8) $u : Q_T \to \mathbb{R}$ and $\theta : Q_T \to \mathbb{R}$ are the unknown functions representing pressure head and temperature of the porous medium. Further $k : \mathbb{R}^2 \to \mathbb{R}$, $b : \mathbb{R} \to \mathbb{R}$, $\lambda : \mathbb{R}^2 \to \mathbb{R}$, $u_0 : \Omega \to \mathbb{R}$, and $\theta_0 : \Omega \to \mathbb{R}$ are given functions, ϱ is a real constant, $\boldsymbol{e}_z$ is the vertical unit vector.

Let us note that in order to avoid technicalities we present in this part the problem with homogeneous boundary conditions. However, the presented procedure can be extended to the problem with nonhomogeneous boundary conditions.

6 Preliminaries

Here we present some helpful auxiliary results and remarks concerning notation and structure and data properties.

6.1 Notation

Remark 6.1 *Throughout this part of the text we suppose that r is a fixed number, such that (δ is some sufficiently small positive number)*

$$r = 2 + \delta. \tag{6.1}$$

Remark 6.2 *(Sobolev space $W_D^{1,p}(\Omega)$) By the symbol $W_D^{1,p}(\Omega)$ with some $p \geq 1$, we denote the Sobolev space $W^{1,p}(\Omega)$ with zero trace on Γ_D (see Appendix* C.6).

6.2 Structure and data properties

According to the physical background we present the following assumptions on functions in (5.1)–(5.8):

(i) b is a positive lipschitz continuous strictly monotone function such that

$$\begin{aligned} &0 < b(\xi) \leq b_2 < +\infty && \forall \xi \in \mathbb{R} \quad (b_2 = \text{const}), \\ &(b(\xi_1) - b(\xi_2))(\xi_1 - \xi_2) > 0 && \forall \xi_1, \xi_2 \in \mathbb{R},\ \xi_1 \neq \xi_2; \end{aligned}$$

(ii) k and λ are positive continuous functions;

(iii) ϱ is a real positive constant and $\boldsymbol{e}_z$ is a vertical unit vector;

(iv) $u_0, \theta_0 \in L^\infty(\Omega)$.

Let us note that the assumption (i) is physically relevant since the function $b(u)$ corresponds to the moisture retention given by van Genuchten's relation (4.24), hence for negative pressure head (e.g. unsaturated zone which is subject to our work) the function is positive and increasing. Further, assumption (ii) is physically relevant as well since the transport coefficients k (hydraulic conductivity), and λ (thermal conductivity) are positive continuous functions (see also Figures 6 and 7).

6.3 Auxiliary results

Here we present some useful auxiliary results which can be found in literature.

Remark 6.3 *([2], Section 1.1) Let us note that* (i) *implies that there is a (strictly) convex C^1-function $\Phi : \mathbb{R} \to \mathbb{R}$ such that*

$$b(z) - b(0) = \Phi'(z) \quad \forall z \in \mathbb{R}. \tag{6.2}$$

Introduce the Legendre transform

$$B(z) := \int_0^1 (b(z) - b(sz))z \,\mathrm{d}s = \int_0^z (b(z) - b(s)) \,\mathrm{d}s. \tag{6.3}$$

Corollary 6.4 *Let us present some properties of B [2]:*

$$B(z) := \int_0^1 (b(z) - b(sz))z\mathrm{d}s = \int_0^z (b(z) - b(s))\ \mathrm{d}s \geq 0 \qquad \forall z \in \mathbb{R}, \tag{6.4}$$

$$B(s) - B(p) \geq (b(s) - b(p))p \qquad \forall p, s \in \mathbb{R}, \tag{6.5}$$

$$b(z)z - \Phi(z) + \Phi(0) = B(z) \leq (b(z) - b(0))\, z \qquad \forall z \in \mathbb{R}. \tag{6.6}$$

Proof. Since b is a positive increasing function (recall (i)), it is obvious that

$$\int_0^z b(z)\mathrm{d}s = zb(z) \geq \int_0^z b(s)\mathrm{d}s$$

and hence (6.4) holds. Further we rewrite (6.5) and use (6.3), to get

$$\int_0^s (b(s) - b(x))\,\mathrm{d}x - \int_0^p (b(p) - b(x))\,\mathrm{d}x \geq (b(s) - b(p))p.$$

Subtracting the identical terms and modifying the inequality we arrive at

$$b(s)(s - r) \geq \int_0^s b(x)\,\mathrm{d}x - \int_0^p b(x)\,\mathrm{d}x. \tag{6.7}$$

For the case $s > p$ we may rewrite (6.7) as

$$b(s)(s - p) \geq \int_p^s b(x)\,\mathrm{d}x. \tag{6.8}$$

Since b is an increasing function (recall (i)) the inequality (6.8) holds. Similarly for $p > s$ we rewrite (6.7) as

$$b(s)(s - p) \leq \int_s^p b(x)\,\mathrm{d}x. \tag{6.9}$$

The inequality (6.9) is satisfied thanks to monotonicity of b. Finally the case $s = p$ is trivial. Now we integrate (6.2) from 0 to z to get

$$\int_0^z (b(s) - b(0))\ \mathrm{d}s = \int_0^z \Phi'(s)\,\mathrm{d}s \tag{6.10}$$

and hence

$$-\int_0^z (b(s) - b(0))\ \mathrm{d}s = -\Phi(z) + \Phi(0). \tag{6.11}$$

We put (6.11) in (6.6) to obtain

$$b(z)z - \int_0^z (b(s) - b(0))\ \mathrm{d}s = B(z), \tag{6.12}$$

which becomes

$$\int_0^z (b(z) - b(s) - b(0))\ \mathrm{d}s = B(z). \tag{6.13}$$

Hence the equality in (6.6) is satisfied. Finally we put (6.13) into (6.6) to obtain

$$\int_0^z \left(b(z) - b(s) - b(0)\right) \mathrm{d}s \le \left(b(z) - b(0)\right) z. \tag{6.14}$$

Subtracting the identical terms in (6.14) we have

$$-\int_0^z b(s)\,\mathrm{d}s \le 0. \tag{6.15}$$

Note that since b is a positive function, (6.15) holds for all $z \in \mathbb{R}$. This proves (6.6) and the proof of Corollary 6.4 is complete. □

Lemma 6.5 *Let $g \in L^\infty(\Omega)$, then* (i) *implies, that there exist positive constants b_1, b_2 such that*

$$0 < b_1 \le b(g) \le b_2 < +\infty \quad \text{a.e. in } \Omega, \tag{6.16}$$

and further, let $g_1, g_2 \in L^\infty(\Omega)$, (ii) *implies, that there exist positive constants $k_1, k_2, \lambda_1, \lambda_2$ such that*

$$0 < k_1 \le k(g_1, g_2) \le k_2 < +\infty \quad \text{a.e. in } \Omega, \tag{6.17}$$

$$0 < \lambda_1 \le \lambda(g_1, g_2) \le \lambda_2 < +\infty \quad \text{a.e. in } \Omega. \tag{6.18}$$

7 Existence result

The aim of this part is to prove the existence of a weak solution to the problem (5.1)–(5.8). First we formulate our problem in a variational sense.

Definition 7.1 *A weak solution of* (5.1)–(5.8) *is a pair*

$$u \in L^2(I; W_D^{1,2}(\Omega)) \cap L^\infty(Q_T),$$
$$\theta \in L^2(I; W_D^{1,2}(\Omega)) \cap L^\infty(Q_T),$$

which satisfies

$$-\int_{Q_T} b(u)\partial_t\phi\,\mathrm{d}x\mathrm{d}t + \int_{Q_T} k(\theta, u)\left(\nabla u + \boldsymbol{e}_z\right)\cdot\nabla\phi\,\mathrm{d}x\mathrm{d}t = \int_\Omega b(u_0)\phi(0)\,\mathrm{d}x \tag{7.1}$$

for any $\phi \in L^2(I; W_D^{1,2}(\Omega)) \cap W^{1,1}(I; L^1(\Omega))$ and $\phi(T) = 0$;

$$\begin{aligned}-\int_{Q_T} \left(b(u)\theta + \varrho\theta\right)\partial_t\psi\,\mathrm{d}x\mathrm{d}t &+ \int_{Q_T} \lambda(\theta, u)\nabla\theta\cdot\nabla\psi\,\mathrm{d}x\mathrm{d}t \\ &+ \int_{Q_T} \left(\theta\, k(\theta, u)\left(\nabla u + \boldsymbol{e}_z\right)\right)\cdot\nabla\psi\,\mathrm{d}x\mathrm{d}t = \int_\Omega \left(b(u_0)\theta_0 + \varrho\theta_0\right)\psi(0)\,\mathrm{d}x\end{aligned} \tag{7.2}$$

for any $\psi \in L^2(I; W_D^{1,2}(\Omega)) \cap W^{1,1}(I; L^1(\Omega))$ and $\psi(T) = 0$.

Remark 7.2 *([31], Remark 1.19) There exists $\partial_t b(u) \in L^2(I; W_D^{1,2}(\Omega)^*)$ and*

$$\int_{Q_T} [b(u_0) - b(u)]\, \partial_t \phi \,\mathrm{d}x\mathrm{d}t = \int_0^T \langle \partial_t b(u), \phi \rangle \,\mathrm{d}t$$

holds for any $\phi \in L^2(I; W_D^{1,2}(\Omega)) \cap W^{1,1}(I; L^1(\Omega))$ and $\phi(T) = 0$, then in the place of (7.1) we have

$$\int_0^T \langle \partial_t b(u), \phi \rangle \,\mathrm{d}t + \int_{Q_T} k(\theta, u)\, (\nabla u + \boldsymbol{e}_z) \cdot \nabla \phi \,\mathrm{d}x\mathrm{d}t = 0$$

for any $\phi \in L^2(I; W_D^{1,2}(\Omega))$.
Similarly, there exists $\partial_t\, (b(u)\theta + \varrho\theta) \in L^2(I; W_D^{1,2}(\Omega)^*)$ and

$$\int_{Q_T} [(b(u_0)\theta_0 + \varrho\theta_0) - (b(u)\theta - \varrho\theta)]\, \partial_t \phi \,\mathrm{d}x\mathrm{d}t = \int_0^T \langle \partial_t\, (b(u)\theta + \varrho\theta)\,, \psi \rangle \,\mathrm{d}t$$

holds for any $\psi \in L^2(I; W_D^{1,2}(\Omega)) \cap W^{1,1}(I; L^1(\Omega))$ and $\eta(T) = 0$, then in the place of (7.2) we have

$$\int_0^T \langle \partial_t\, (b(u)\theta + \varrho\theta)\,, \psi \rangle \,\mathrm{d}t + \int_{Q_T} \lambda(\theta, u) \nabla\theta \cdot \nabla\psi \,\mathrm{d}x\mathrm{d}t + \int_{Q_T} (\theta\, k(\theta, u)\, (\nabla u + \boldsymbol{e}_z)) \cdot \nabla\psi \,\mathrm{d}x\mathrm{d}t = 0$$

for any $\psi \in L^2(I; W_D^{1,2}(\Omega))$.

Theorem 7.3 *(Existence of the weak solution) Let the assumptions* (i)–(iv) *be satisfied. Then there exists at least one weak solution of the system* (5.1)–(5.8).

8 Proof of the existence result

To prove Theorem 7.3 we will use the method of semidiscretization in time. The proof is divided into three steps. In the first step we approximate our problem by means of a semi-implicit time discretization scheme and prove the existence and regularity of the solution to the steady problem in each time step. In the second step we show some suitable a-priori estimates and finally in the third step we pass to the limit from discrete approximations to obtain the weak solution of the original continuous problem.

8.1 Steady problem

Fix $p \in \mathbb{N}$ and let $\tau := T/p$ be a time step. Let

$$\left.\begin{array}{l} u_p^0 := u_0, \\ \theta_p^0 := \theta_0. \end{array}\right\} \quad \text{a.e. in } \Omega. \tag{8.1}$$

We approximate the evolution problem by a semi-implicit time discretization scheme and define functions u_p^n and θ_p^n as a solution of the following recurrence steady problem in each time step.

Problem 8.1 *Find $u_p^n \in W_D^{1,2}(\Omega) \cap L^\infty(\Omega)$ and $\theta_p^n \in W_D^{1,2}(\Omega) \cap L^\infty(\Omega)$ to be solutions of the recurrence system*

$$\int_\Omega \frac{b(u_p^n) - b(u_p^{n-1})}{\tau} \phi \,\mathrm{d}x + \int_\Omega k(\theta_p^{n-1}, u_p^{n-1}) \left(\nabla u_p^n + \boldsymbol{e}_z\right) \cdot \nabla\phi \,\mathrm{d}x = 0 \tag{8.2}$$

for any $\phi \in W_D^{1,2}(\Omega)$;

$$\int_\Omega \frac{b(u_p^n)\theta_p^n - b(u_p^{n-1})\theta_p^{n-1}}{\tau} \psi \,\mathrm{d}x + \varrho \int_\Omega \frac{\theta_p^n - \theta_p^{n-1}}{\tau} \psi \,\mathrm{d}x + \int_\Omega \lambda(\theta_p^{n-1}, u_p^{n-1}) \nabla\theta_p^n \cdot \nabla\psi \mathrm{d}x$$
$$+ \int_\Omega \theta_p^n k(\theta_p^{n-1}, u_p^{n-1}) \left(\nabla u_p^n + \boldsymbol{e}_z\right) \cdot \nabla\psi \mathrm{d}x = 0 \tag{8.3}$$

for any $\psi \in W_D^{1,2}(\Omega)$, where u_p^0 and θ_p^0 are initial functions from (8.1) *satisfying the assumption* (iv).

In what follows we will prove the existence and some regularity of the solution to the problem (8.2)–(8.3).

8.2 Approximate solution to the moisture equation

8.2.1 Existence of the approximate solution to the moisture equation

Theorem 8.2 *[Existence of the solution to* (8.2)*] Consider $n \in \mathbb{N}$, $1 \le n \le p$, and let $[u_p^{n-1}, \theta_p^{n-1}] \in L^\infty(\Omega)^2$ be given and the assumptions* (i)–(iv) *be satisfied. Then there exists $u_p^n \in W_D^{1,2}(\Omega)$ the solution to the problem* (8.2).

In order to prove the existence of the approximate solution to the moisture equation (8.2) we define the functional μ_u and the operator $\mathcal{A}_u$ corresponding to the problem. Next, we show some important properties of the operator $\mathcal{A}_u$, which yield the existence of the approximate solution.

Define the functional $\mu_u \in [W_D^{1,2}(\Omega)]^*$ by the equation

$$\langle \mu_u, \phi \rangle = \frac{1}{\tau} \int_\Omega b(u_p^{n-1}) \phi \,\mathrm{d}x - \int_\Omega k(\theta_p^{n-1}, u_p^{n-1}) \boldsymbol{e}_z \cdot \nabla\phi \,\mathrm{d}x \tag{8.4}$$

for all $\phi \in W_D^{1,2}(\Omega)$.

Further, define the operator $\mathcal{A}_u : W_D^{1,2}(\Omega) \to [W_D^{1,2}(\Omega)]^*$ by the equation

$$\langle \mathcal{A}_u(u_p^n), \phi \rangle = \int_\Omega k(\theta_p^{n-1}, u_p^{n-1}) \nabla u_p^n \cdot \nabla\phi \,\mathrm{d}x + \frac{1}{\tau} \int_\Omega b(u_p^n) \phi \,\mathrm{d}x \tag{8.5}$$

for all $\phi \in W_D^{1,2}(\Omega)$.

A function u_p^n is a solution of the operator equation $\mathcal{A}_u(u_p^n) = \mu_u$ if and only if u_p^n solves (8.2). Now we present some properties of the operator $\mathcal{A}_u$.

Lemma 8.3 *The operator $\mathcal{A}_u : W_D^{1,2}(\Omega) \to [W_D^{1,2}(\Omega)]^*$ is bounded.*

Proof. Taking into account (i)-(iv) and using Hölder's inequality to each term on the right-hand side of (8.5), we deduce

$$\begin{aligned}\langle \mathcal{A}_u(u_p^n), \phi \rangle &\leq c_1 \|u_p^n\|_{W_D^{1,2}(\Omega)} \|\phi\|_{W_D^{1,2}(\Omega)} + c_2 \|\phi\|_{W_D^{1,2}(\Omega)}, \\ &\leq \|\phi\|_{W_D^{1,2}(\Omega)} \left(c_1 \|u_p^n\|_{W_D^{1,2}(\Omega)} + c_2 \right),\end{aligned}$$

which yields

$$\|\mathcal{A}_u(u_p^n)\|_{[W_D^{1,2}(\Omega)]^*} = \sup_{\phi \in W_D^{1,2}(\Omega), \|\phi\| \neq 0} \frac{|\langle \mathcal{A}_u(u_p^n), \phi \rangle|}{\|\phi\|_{W_D^{1,2}(\Omega)}} \leq c_1 \|u_p^n\|_{W_D^{1,2}(\Omega)} + c_2.$$

Hence the operator $\mathcal{A}_u : W_D^{1,2}(\Omega) \to [W_D^{1,2}(\Omega)]^*$ is bounded. $\square$

Lemma 8.4 *The operator $\mathcal{A}_u : W_D^{1,2}(\Omega) \to [W_D^{1,2}(\Omega)]^*$ is coercive.*

Proof. The operator $\mathcal{A}_u$ is coercive iff

$$\lim_{\|u_p^n\|_{W_D^{1,2}(\Omega)} \to \infty} \frac{\langle \mathcal{A}_u(u_p^n), u_p^n \rangle}{\|u_p^n\|_{W_D^{1,2}(\Omega)}} = +\infty. \tag{8.6}$$

Using (6.6) we have

$$\frac{1}{\tau} \int_\Omega b(u_p^n)\, u_p^n \,\mathrm{d}x \geq \frac{1}{\tau} \int_\Omega \left(B(u_p^n) + b(0) u_p^n \right) \,\mathrm{d}x. \tag{8.7}$$

Next, using Young's inequality we obtain

$$\frac{1}{\tau} \int_\Omega \left(B(u_p^n) + b(0) u_p^n \right) \,\mathrm{d}x \geq \frac{1}{\tau} \int_\Omega B(u_p^n) \,\mathrm{d}x - \frac{1}{\tau} \int_\Omega \epsilon\, u_p^n \,\mathrm{d}x - c(\epsilon). \tag{8.8}$$

And further, using Friedrichs inequality we arrive at

$$\frac{1}{\tau} \int_\Omega \left(B(u_p^n) + b(0) u_p^n \right) \,\mathrm{d}x \geq \frac{1}{\tau} \int_\Omega B(u_p^n) \,\mathrm{d}x - \frac{1}{\tau} \epsilon c_\Omega \|u_p^n\|^2_{W_D^{1,2}(\Omega)} - c(\epsilon). \tag{8.9}$$

Using Friedrichs inequality and due to (6.17) we have for the elliptic term

$$\int_\Omega \left| k(\theta_p^{n-1}, u_p^{n-1}) \nabla u_p^n \cdot \nabla u_p^n \right| \,\mathrm{d}x \geq c \|u_p^n\|^2_{W_D^{1,2}(\Omega)}. \tag{8.10}$$

Combining estimates (8.9) and (8.10) and choosing ϵ sufficiently small, we arrive at

$$\langle \mathcal{A}_u(u_p^n), u_p^n \rangle \geq c_1 \|u_p^n\|^2_{W_D^{1,2}(\Omega)} - c_2. \tag{8.11}$$

Hence the identity (8.6) holds and the operator $\mathcal{A}_u : W_D^{1,2}(\Omega) \to [W_D^{1,2}(\Omega)]^*$ is coercive. □

Lemma 8.5 *Operator $\mathcal{A}_u : W_D^{1,2}(\Omega) \to [W_D^{1,2}(\Omega)]^*$ is monotone in the main part. (For the definition of monotonicity in the main part see [52], Chapter 2.)*

Proof. Let us define the operator $\hat{\mathcal{A}}_u : W_D^{1,2}(\Omega) \to [W_D^{1,2}(\Omega)]^*$ by the equation

$$\langle \hat{\mathcal{A}}_u(u_p^n), \phi \rangle = \int_\Omega k(\theta_p^{n-1}, u_p^{n-1}) \nabla u_p^n \cdot \nabla \phi \, \mathrm{d}x. \tag{8.12}$$

Since k is a nonnegative continuous function (recall (ii)), we have

$$\langle \hat{\mathcal{A}}_u(u_{p1}^n) - \hat{\mathcal{A}}_u(u_{p2}^n), u_{p1}^n - u_{p2}^n \rangle = \int_\Omega k(\theta_p^{n-1}, u_p^{n-1}) \left| \left(\nabla u_{p1}^n - \nabla u_{p2}^n \right) \right|^2 \, \mathrm{d}x \geq 0 \tag{8.13}$$

for all $u_{p1}^n, u_{p2}^n \in W_D^{1,2}(\Omega)$. Hence the operator $\hat{\mathcal{A}}_u : W_D^{1,2}(\Omega) \to [W_D^{1,2}(\Omega)]^*$ is monotone and the operator $\mathcal{A}_u : W_D^{1,2}(\Omega) \to [W_D^{1,2}(\Omega)]^*$ is monotone in the main part. □

Proof of Theorem 8.2. We have shown that the operator $\mathcal{A}_u$ defined by equation (8.5) is bounded, coercive and monotone in the main part. Now by [47, Theorem 3.3.42] the operator $\mathcal{A}_u$ is surjective, which yields the existence of the solution $u_p^n \in W_D^{1,2}(\Omega)$ to the equation $\mathcal{A}_u(u_p^n) = \mu_u$. This completes the proof of Theorem 8.2. □

8.2.2 Regularity of the approximate solution to the moisture equation

Theorem 8.6 *($W_D^{1,s}$-regularity of the solution to* (8.2)*) Let $u_p^n \in W_D^{1,2}(\Omega)$ be the weak solution to the discrete problem* (8.2)*. Then $u_p^n \in W_D^{1,s}(\Omega)$, with some $s > 2$.*

In order to show $W_D^{1,s}$-regularity of the solution to (8.2) we use the following lemma.

Lemma 8.7 ([19, Theorem 4], [15]) *Let Ω be a bounded connected open set with a Lipschitz continuous boundary of $\mathbb{R}^N$. Let Γ be a regular part of $\partial\Omega$ and $\widetilde{\Gamma} = \partial\Omega \backslash \Gamma$. Suppose $\widetilde{\Gamma}$ has a non-null $(N-1)$-dimensional measure. There is a real number s_0,*

$2^* \geq s_0 > 2$, *such that, if u is the weak solution of (A represents a function from $L^\infty(\Omega)$ satisfying the ellipticity condition)*

$$\begin{cases} u \in W_D^{1,2}(\Omega), \\ \int_\Omega A(x)\nabla u(x) \cdot \nabla\varphi(x)\,\mathrm{d}\Omega = \langle f, \varphi\rangle_{W_D^{1,2}(\Omega)^*, W_D^{1,2}(\Omega)} \quad \forall \varphi \in W_D^{1,2}(\Omega), \end{cases}$$

where $f \in W_D^{1,s'}(\Omega)^*$, $s' = s/(s-1)$, $s \in [2, s_0)$. Then u belongs to $W_D^{1,s}(\Omega)$ and there exists a real number $C(s)$ such that

$$\|u\|_{W_D^{1,s}(\Omega)} \leq C(s)\|f\|_{W_D^{1,s'}(\Omega)^*}.$$

Moreover, s_0 only depends on A and Ω and $C(s)$ on A, Ω, s, not on f.

Proof of Theorem 8.6. Following Theorem 8.2, $u_p^n \in W_D^{1,2}(\Omega)$ solves the equation

$$\int_\Omega k(\theta_p^{n-1}, u_p^{n-1})\nabla u_p^n \cdot \nabla\phi\,\mathrm{d}x = \langle\tilde{\mu}_u, \phi\rangle, \tag{8.14}$$

for all $\phi \in W_D^{1,2}(\Omega)$, where $\langle\tilde{\mu}_u, \phi\rangle$ is given by the equation

$$\langle\tilde{\mu}_u, \phi\rangle = -\int_\Omega k(\theta_p^{n-1}, u_p^{n-1})\boldsymbol{e}_z \cdot \nabla\phi\,\mathrm{d}x - \frac{1}{\tau}\int_\Omega \tilde{b}(u_p^n)\phi\,\mathrm{d}x + \frac{1}{\tau}\int_\Omega b(u_p^{n-1})\phi\,\mathrm{d}x. \tag{8.15}$$

Provided $[u_p^{n-1}, \theta_p^{n-1}] \in [W_D^{1,r}(\Omega)]^2$ we may use (6.17) to conclude that

$$k_1 \leq k(\theta_p^{n-1}, u_p^{n-1}) \leq k_2.$$

Further thanks to (i), b is a bounded function, which guarantees $\tilde{\mu}_u \in [W_D^{1,r'}(\Omega)]^*$, $r' = r/(r-1)$. We can directly apply Lemma 8.7 to conclude the $W_D^{1,s}$ regularity of the solution with some $s > 2$.

8.3 Approximate solution to the heat equation

8.3.1 Existence of the approximate solution to the heat equation

Theorem 8.8 *(Existence of the solution to (8.3)) Let $[u_p^{n-1}, \theta_p^{n-1}] \in L^\infty(\Omega)^2$ and $u_p^n \in W_D^{1,r}(\Omega)$ be the solution to (8.2) and the assumptions (i)–(iv) be satisfied. Then there exists the solution $\theta_p^n \in W_D^{1,2}(\Omega)$ to the discrete problem (8.3).*

We proceed in the similar way as in the proof of Theorem 8.2. We define the functional μ_θ and the operator $\mathcal{A}_\theta$ corresponding to the problem. And further, we show some important properties of the operator $\mathcal{A}_\theta$, which yield the existence of the approximate solution. First, we define the functional $\mu_\theta \in [W_D^{1,2}(\Omega)]^*$ by the equation

$$\langle\mu_\theta, \psi\rangle = \frac{1}{\tau}\int_\Omega b(u_p^{n-1})\theta_p^{n-1}\psi\mathrm{d}x + \frac{1}{\tau}\int_\Omega \varrho\,\theta_p^{n-1}\psi\mathrm{d}x \tag{8.16}$$

for all $\psi \in W_D^{1,2}(\Omega)$. Further, define the operator $\mathcal{A}_\theta : W_D^{1,2}(\Omega) \to [W_D^{1,2}(\Omega)]^*$ by the equation

$$\begin{aligned}\langle \mathcal{A}_\theta(\theta_p^n), \psi\rangle = \frac{1}{\tau}\int_\Omega \left[b(u_p^n) + \varrho\right] \theta_p^n \psi \,\mathrm{d}x \\ + \int_\Omega \lambda(\theta_p^{n-1}, u_p^{n-1}) \nabla\theta_p^n \cdot \nabla\psi \,\mathrm{d}x \\ + \int_\Omega \theta_p^n k(\theta_p^{n-1}, u_p^{n-1}) \left(\nabla u_p^n + \boldsymbol{e}_z\right) \cdot \nabla\psi \,\mathrm{d}x \end{aligned} \tag{8.17}$$

for all $\psi \in W_D^{1,2}(\Omega)$. Now, we will present properties of the operator $\mathcal{A}_\theta$.

Lemma 8.9 *The operator $\mathcal{A}_\theta : W_D^{1,2}(\Omega) \to [W_D^{1,2}(\Omega)]^*$ is bounded.*

Proof. Let us estimate all terms on the right-hand side of (8.17). Recall that $u_p^n \in W_D^{1,r}(\Omega)$. Using Hölder's inequality we can write for the convective term

$$\int_\Omega \theta_p^n k(\theta_p^{n-1}, u_p^{n-1}) \nabla u_p^n \cdot \nabla\psi \,\mathrm{d}x \le c\|\theta_p^n\|_{L^q(\Omega)} \|\nabla u_p^n\|_{L^r(\Omega)^2} \|\nabla\psi\|_{L^2(\Omega)^2}, \tag{8.18}$$

where $1/q + 1/r + 1/2 = 1$. Considering $r = 2 + \delta$, $\delta > 0$ we have $q > 1$. Since $W_D^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \in [1;\infty)$ we have $\theta_p^n \in W_D^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$. Hence we can write

$$\int_\Omega \theta_p^n k(\theta_p^{n-1}, u_p^{n-1}) \nabla u_p^n \cdot \nabla\psi \,\mathrm{d}x \le c\|\theta_p^n\|_{W_D^{1,2}(\Omega)} \|u_p^n\|_{W_D^{1,r}(\Omega)} \|\psi\|_{W_D^{1,2}(\Omega)}. \tag{8.19}$$

Thus and in view of (6.16) and (6.17), we can write for any $\psi \in W_D^{1,2}(\Omega)$

$$|\langle \mathcal{A}_\theta(\theta_p^n), \psi\rangle| \le \left(c_1\|\theta_p^n\|_{W_D^{1,2}(\Omega)} + c_2\right) \|\psi\|_{W_D^{1,2}(\Omega)}.$$

Therefore we have

$$\|\mathcal{A}_\theta(\theta_p^n)\|_{[W_D^{1,2}(\Omega)]^*} = \sup_{\psi \in W_D^{1,2}(\Omega), \|\psi\| \ne 0} \frac{|\langle \mathcal{A}_\theta(\theta_p^n), \psi\rangle|}{\|\psi\|_{W_D^{1,2}(\Omega)}} \le c_1\|\theta_p^n\|_{W_D^{1,2}(\Omega)} + c_2.$$

Hence the operator $\mathcal{A}_\theta : W_D^{1,2}(\Omega) \to [W_D^{1,2}(\Omega)]^*$ is bounded. □

Lemma 8.10 *The operator $\mathcal{A}_\theta : W_D^{1,2}(\Omega) \to [W_D^{1,2}(\Omega)]^*$ is coercive.*

Proof. We use $\psi = (\theta_p^n)^2$ as a test function in (8.2) to obtain

$$\int_\Omega k(\theta_p^{n-1}, u_p^{n-1}) \left(\nabla u_p^n + \boldsymbol{e}_z\right) \cdot \theta_p^n \nabla\theta_p^n \,\mathrm{d}x = -\frac{1}{2}\int_\Omega \frac{b(u_p^n) - b(u_p^{n-1})}{\tau} (\theta_p^n)^2 \,\mathrm{d}x. \tag{8.20}$$

Let us note that all integrals in (8.20) are well defined. Since $\theta_p^{n-1}, u_p^{n-1} \in L^\infty(\Omega)$ we may use (6.17) to conclude that $k_1 \le k(\theta_p^{n-1}, u_p^{n-1}) \le k_2$. Further $\nabla u_p^n \in L^{2+\delta}(\Omega)^2$, $\nabla \theta_p^n \in L^2(\Omega)^2$ and $\theta_p^n \in W_D^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$, where $p \in [1; \infty)$.

From the definition of the operator $\mathcal{A}_\theta : W_D^{1,2}(\Omega) \to [W_D^{1,2}(\Omega)]^*$ we can write

$$\langle \mathcal{A}_\theta(\theta_p^n), \theta_p^n \rangle = \frac{1}{\tau} \int_\Omega \left[b(u_p^n) + \varrho \right] (\theta_p^n)^2 \mathrm{d}x + \int_\Omega \lambda(\theta_p^{n-1}, u_p^{n-1}) |\nabla \theta_p^n|^2 \mathrm{d}x \\ + \int_\Omega k(\theta_p^{n-1}, u_p^{n-1}) \left(\nabla u_p^n + \boldsymbol{e}_z \right) \cdot \theta_p^n \nabla \theta_p^n \, \mathrm{d}x. \quad (8.21)$$

Using (8.20) in (8.21) we get

$$\langle \mathcal{A}_\theta(\theta_p^n), \theta_p^n \rangle = \frac{1}{2\tau} \int_\Omega \left[b(u_p^n) + b(u_p^{n-1}) + 2\varrho \right] (\theta_p^n)^2 \, \mathrm{d}x \\ + \int_\Omega \lambda(\theta_p^{n-1}, u_p^{n-1}) |\nabla \theta_p^n|^2 \, \mathrm{d}x. \quad (8.22)$$

Considering $\theta_p^{n-1}, u_p^{n-1} \in L^\infty(\Omega)$ we may use (6.18) to conclude that $\lambda_1 \le \lambda(\theta_p^{n-1}, u_p^{n-1})$. Similarly we use (6.16) to conclude that $b_1 \le b(u_p^{n-1})$ and finally thanks to $u_p^n \in W_D^{1,r}(\Omega)$ we have $b_1 \le b(u_p^n)$. Now since ϱ is a positive constant, using Friedrichs inequality, we can write

$$\langle \mathcal{A}(\theta_p^n), \theta_p^n \rangle \ge c \|\theta_p^n\|^2_{W_D^{1,2}(\Omega)}.$$

Hence the operator $\mathcal{A}_\theta : W_D^{1,2}(\Omega) \to [W_D^{1,2}(\Omega)]^*$ is coercive. $\Box$

Lemma 8.11 *The operator $\mathcal{A}_\theta : W_D^{1,2}(\Omega) \to [W_D^{1,2}(\Omega)]^*$ is monotone in the main part.*

Proof. Let us define the operator $\hat{\mathcal{A}}_\theta : W_D^{1,2}(\Omega) \to [W_D^{1,2}(\Omega)]^*$ by the equation

$$\langle \hat{\mathcal{A}}_\theta(\theta^n), \psi \rangle = \int_\Omega \lambda(\theta_p^{n-1}, u_p^{n-1}) \nabla \theta_p^n \cdot \nabla \psi \, \mathrm{d}x. \quad (8.23)$$

Obviously

$$\langle \hat{\mathcal{A}}_\theta(\tilde{\theta}_{p1}) - \mathcal{A}_\theta(\theta_{p2}^n), \theta_{p1}^n - \theta_{p2}^n \rangle = \int_\Omega \lambda(\theta_p^{n-1}, u_p^{n-1}) \left(\nabla \theta_{p1}^n - \nabla \theta_{p2}^n \right)^2 \, \mathrm{d}x \ge 0 \quad (8.24)$$

for all $\theta_{p1}^n, \theta_{p2}^n \in W_D^{1,2}(\Omega)$.

Hence the operator $\hat{\mathcal{A}}_\theta : W_D^{1,2}(\Omega) \to [W_D^{1,2}(\Omega)]^*$ is monotone and the operator $\mathcal{A}_\theta : W_D^{1,2}(\Omega) \to [W_D^{1,2}(\Omega)]^*$ is monotone in the main part. $\Box$

Proof of Theorem 8.8. With the same arguments as in the proof of Theorem 8.2 we conclude that the operator $\mathcal{A}_\theta : W_D^{1,2}(\Omega) \to [W_D^{1,2}(\Omega)]^*$ is surjective. The operator equation $\mathcal{A}_\theta(\theta_p^n) = \mu_\theta$ has a solution if and only if the function $\theta_p^n \in W_D^{1,2}(\Omega)$ is the solution to the variational equation (8.3) and the proof of Theorem 8.8 is complete. $\square$

8.3.2 Regularity of the approximate solution to the heat equation

Similarly as for pressure head u_p^n we have the following regularity result for temperature θ_p^n.

Theorem 8.12 *($W_D^{1,s}$-regularity of the solution to (8.3)) Let $\theta_p^n \in W_D^{1,2}(\Omega)$ be the weak solution to the discrete problem (8.3), and $u_p^n \in W_D^{1,r}(\Omega)$ be the weak solution to the discrete problem (8.2). Then $\theta_p^n \in W_D^{1,s}(\Omega)$ with some $s > 2$.*

Proof of Theorem 8.12 Let $\theta_p^n \in W_D^{1,2}(\Omega)$ be the solution of the equation $\mathcal{A}_\theta(\theta_p^n) = \mu_\theta$, i.e.

$$\int_\Omega \lambda(\theta_p^{n-1}, u_p^{n-1}) \nabla\theta_p^n \cdot \nabla\psi \, \mathrm{d}x = \langle \tilde{\mu}_\theta, \psi \rangle \tag{8.25}$$

for all $\psi \in W_D^{1,2}(\Omega)$, where

$$\begin{aligned} \langle \tilde{\mu}_\theta, \psi \rangle = \frac{1}{\tau} \int_\Omega \left[b(u_p^n)\theta_p^n - b(u_p^{n-1})\theta_p^{n-1} \right] \psi \mathrm{d}x + \varrho \frac{1}{\tau} \int_\Omega (\theta_p^n - \theta_p^{n-1}) \psi \mathrm{d}x \\ - \int_\Omega \theta_p^n k(\theta_p^{n-1}, u_p^{n-1}) \left(\nabla u_p^n + \boldsymbol{e}_z \right) \cdot \nabla\psi \, \mathrm{d}x \\ + \frac{1}{\tau} \int_\Omega b(u_p^{n-1}) \theta_p^{n-1} \psi \mathrm{d}x + \frac{1}{\tau} \int_\Omega \varrho \, \theta_p^{n-1} \psi \mathrm{d}x. \end{aligned} \tag{8.26}$$

Let us focus our attention on the critical convective term on the second line of (8.26). Recall that we have $u_p^n \in W_D^{1,2+\delta}(\Omega)$ and $\theta_p^n \in W_D^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$ with arbitrary $q \in [1; +\infty)$ [52, Theorem 1.20]. Then we have, using Hölder's inequality,

$$\int_\Omega \theta_p^n k(\theta_p^{n-1}, u_p^{n-1}) \nabla u_p^n \cdot \nabla\psi \, \mathrm{d}x \le c \|\theta_p^n\|_{L^q(\Omega)} \|\nabla u_p^n\|_{L^{2+\delta}(\Omega)} \|\nabla\psi\|_{L^{s'}(\Omega)}, \tag{8.27}$$

where

$$\frac{1}{q} + \frac{1}{2+\delta} + \frac{1}{s'} = 1. \tag{8.28}$$

Hence

$$s' = \frac{-q\delta + 4 + 2\delta}{q(1+\delta) - 2 - \delta}. \tag{8.29}$$

Let $s_0 > 2$. Now we see that for arbitrary small $\delta > 0$ there exists q large enough such that

$$s' \in \left(\frac{s_0}{s_0 - 1}; 2 \right].$$

Hence, $\tilde{\mu}_\theta \in \left(W_D^{1,s'}(\Omega)\right)^*$ and the conditions of Lemma 8.7 are satisfied. This ensures the required regularity of u_p^n. □

Summary of Section 8.1. In this section we have proven, at first, the existence of the approximate solution $u_p^n \in W_D^{1,2}(\Omega)$ to the moisture equation (8.2). Next, thanks to Lemma 8.7 we have shown also the $W^{1,r}$-regularity of the solution. Hence, due to the embedding $W_D^{1,r}\Omega \hookrightarrow L^\infty(\Omega)$ we have $u_p^n \in W_D^{1,2}(\Omega) \cap L^\infty(\Omega)$. Similarly, we have shown the existence of the solution $\theta_p^n \in W_D^{1,2}(\Omega)$ to the heat equation (8.3) and its $W^{1,r}$-regularity. With the same argument we can conclude $\theta_p^n \in W_D^{1,2}(\Omega) \cap L^\infty(\Omega)$. Hence we have solution of the recurrence Problem 8.1 for $n = 1, \dots, p$.

8.4 A-priori estimates

In this section, we prove some uniform estimates with respect to p for the time interpolants of the solution.

8.4.1 Construction of time interpolants

Let us define the piecewise constant interpolant functions ($n = 1, 2, \dots, p$)

$$\bar{u}_p(t) = u_p^n \qquad \text{for } t \in ((n-1)\tau, n\tau], \tag{8.30}$$

$$\bar{u}_p(t) = u_0 \qquad \text{for } t \in (-\tau, 0], \tag{8.31}$$

$$\bar{\theta}_p(t) = \theta_p^n \qquad \text{for } t \in ((n-1)\tau, n\tau], \tag{8.32}$$

$$\bar{\theta}_p(t) = \theta_0 \qquad \text{for } t \in (-\tau, 0]. \tag{8.33}$$

The piecewise constant interpolants $\bar{u}_p(t) \in L^\infty(I; W_D^{1,2}(\Omega)) \cap L^\infty(I; L^\infty(\Omega))$ and $\bar{\theta}_p(t) \in L^\infty(I; W_D^{1,2}(\Omega)) \cap L^\infty(I; L^\infty(\Omega))$ satisfy for all $t \in (0; T]$ the equations

$$\int_\Omega \frac{b(\bar{u}_p(t)) - b(\bar{u}_p(t-\tau))}{\tau}\phi(t)\,\mathrm{d}x + \int_\Omega k(\bar{\theta}_p(t-\tau), \bar{u}_p(t-\tau))\,[\nabla \bar{u}_p(t) + \boldsymbol{e}_z]\cdot\nabla\phi(t)\,\mathrm{d}x = 0 \tag{8.34}$$

for any $\phi \in L^2(I; W_D^{1,2}(\Omega))$;

$$\begin{aligned}
&\int_\Omega \frac{b(\bar{u}_p(t))\bar{\theta}_p(t) - b(\bar{u}_p(t-\tau))\bar{\theta}_p(t-\tau)}{\tau}\psi(t)\,\mathrm{d}x \\
&\quad + \varrho\int_\Omega \frac{\bar{\theta}_p(t) - \bar{\theta}_p(t-\tau)}{\tau}\psi(t)\,\mathrm{d}x + \int_\Omega \lambda(\bar{\theta}_p(t-\tau), \bar{u}_p(t-\tau))\nabla\bar{\theta}_p(t)\cdot\nabla\psi(t)\mathrm{d}x \\
&\qquad + \int_\Omega \bar{\theta}_p(t) k(\bar{\theta}_p(t-\tau), \bar{u}_p(t-\tau))\,[\nabla\bar{u}_p(t) + \boldsymbol{e}_z]\cdot\nabla\psi(t)\mathrm{d}x = 0
\end{aligned} \tag{8.35}$$

for any $\psi \in L^2(I; W_D^{1,2}(\Omega))$.

8.4.2 L^∞-estimates

In this section we will derive the L^∞-estimates for time interpolants $\overline{u}_p$ and $\overline{\theta}_p$.

L^∞-bound for $\bar{u}_p$. Let $\kappa \in \mathbb{R}$, $\xi \in \mathbb{R}$ and

$$\beta_\kappa^- (\xi) := \int_\kappa^\xi b'(s)(s-\kappa)_- \,\mathrm{d}s \tag{8.36}$$

and

$$\beta_\kappa^+ (\xi) := \int_\kappa^\xi b'(s)(s-\kappa)_+ \,\mathrm{d}s, \tag{8.37}$$

where symbols $-$ and $+$ denote negative and positive part of a function, i.e. $(s-\kappa)_- = \min\{s-\kappa, 0\}$ and $(s-\kappa)_+ = \max\{s-\kappa, 0\}$.

Lemma 8.13 *Let us present some properties of β_κ^- and β_κ^+:*

$$\beta_\kappa^-(\xi_1) - \beta_\kappa^-(\xi_2) \le (b(\xi_1) - b(\xi_2))(\xi_1 - \kappa)_- \quad \forall \xi_1, \xi_2 \in \mathbb{R}, \tag{8.38}$$

$$\beta_\kappa^+(\xi_1) - \beta_\kappa^+(\xi_2) \le (b(\xi_1) - b(\xi_2))(\xi_1 - \kappa)_+ \quad \forall \xi_1, \xi_2 \in \mathbb{R}. \tag{8.39}$$

Proof. From (8.37) we have

$$\beta_\kappa^+(\xi_1) - \beta_\kappa^+(\xi_2) = \int_\kappa^{\xi_1} b'(s)(s-\kappa)_+ \,\mathrm{d}s - \int_\kappa^{\xi_2} b'(s)(s-\kappa)_+ \,\mathrm{d}s. \tag{8.40}$$

Substituting (8.40) in (8.39) we obtain

$$\int_\kappa^{\xi_1} b'(s)(s-\kappa)_+ \,\mathrm{d}s - \int_\kappa^{\xi_2} b'(s)(s-\kappa)_+ \,\mathrm{d}s \le (b(\xi_1) - b(\xi_2))(\xi_1 - \kappa)_+. \tag{8.41}$$

Let us first consider the case when $\xi_1 > \kappa$ and $\xi_2 > \kappa$. Integrating by parts the left-hand side of (8.41) we get

$$b(\xi_1)(\xi_1-\kappa) - b(\xi_2)(\xi_2-\kappa) - \int_\kappa^{\xi_1} b(s)\,\mathrm{d}s + \int_\kappa^{\xi_2} b(s)\,\mathrm{d}s \le (b(\xi_1) - b(\xi_2))(\xi_1-\kappa).$$

Hence

$$-b(\xi_2)\xi_2 + \int_{\xi_1}^{\xi_2} b(s)\,\mathrm{d}s \le -b(\xi_2)\xi_1.$$

If $\xi_1 \le \xi_2$, considering b is a positive function, the integral is nonnegative and the inequality (8.39) holds. If $\xi_1 > \xi_2$ we have

$$b(\xi_2)(\xi_1 - \xi_2) \le \int_{\xi_2}^{\xi_1} b(s)\,\mathrm{d}s.$$

Since b is an increasing function, the inequality holds. Further, cases with $\xi_1 < \kappa$ or $\xi_2 < \kappa$ can be handled in the same manner. In the same way we would prove (8.38). $\square$

Let $\kappa_\sharp \in \mathbb{R}$, such that $\kappa_\sharp \leq u_0 + x_2$ a.e. in Ω. In order to show the L^∞-bound for $\bar{u}_p$ we set

$$\phi := [u_p^n + x_2 - \kappa_\sharp]_- = \begin{cases} u_p^n + x_2 - \kappa_\sharp, & \text{for } u_p^n < \kappa_\sharp - x_2, \\ 0, & \text{for } u_p^n \geq \kappa_\sharp - x_2, \end{cases} \tag{8.42}$$

as a test function in (8.2) to obtain

$$\int_\Omega \frac{b(u_p^n) - b(u_p^{n-1})}{\tau} (u_p^n + x_2 - \kappa_\sharp)_- \,\mathrm{d}x \\ + \int_\Omega k(\theta_p^n, u_p^{n-1}) \nabla \left(u_p^n + x_2 - \kappa_\sharp\right)_- \cdot \nabla (u_p^n + x_2 - \kappa_\sharp)_- = 0. \tag{8.43}$$

The second integral in (8.43) is clearly nonnegative and we can write

$$\frac{1}{\tau} \int_\Omega \left(b(u_p^n) - b(u_p^{n-1})\right) (u_p^n + x_2 - \kappa_\sharp)_- \,\mathrm{d}x \leq 0. \tag{8.44}$$

Let us set $\tilde{\kappa}_\sharp = \max_{x_2 \in \Omega}(\kappa_\sharp - x_2)$. In view of (8.38) we may write

$$\int_\Omega \beta_{\tilde{\kappa}_\sharp}^-(u_p^n) - \beta_{\tilde{\kappa}_\sharp}^-(u_p^{n-1}) \leq 0. \tag{8.45}$$

Let us now consider the case $n = 1$. Since $u_0 \in L^\infty$, there exist $\tilde{\kappa}_\sharp$ such that $\tilde{\kappa}_\sharp \leq u_0$ almost everywhere in Ω. Hence

$$\beta_{\tilde{\kappa}_\sharp}^-(u_0(x)) := \int_{\tilde{\kappa}_\sharp}^{u_0(x)} b'(s)(s - \tilde{\kappa}_\sharp)_- \,\mathrm{d}s = 0. \tag{8.46}$$

And from (8.45) we get

$$\int_\Omega \beta_{\tilde{\kappa}_\sharp}^-(u_p^1) \leq 0. \tag{8.47}$$

Considering the definition of β^- in (8.36), this implies

$$u_p^1 \geq \tilde{\kappa}_\sharp. \tag{8.48}$$

Repeating the described procedure successively for $n = 2, 3, \cdots, p$ we conclude that

$$u_p^n \geq \tilde{\kappa}_\sharp \tag{8.49}$$

for all $n = 1, \ldots, p$.

In the same manner we will search the upper bound. Let $\kappa^\sharp \in \mathbb{R}$, such that $\kappa^\sharp \geq u_0 + x_2$ a.e. in Ω. We set

$$\phi := [u_p^n + x_2 - \kappa^\sharp]_+ = \begin{cases} u_p^n + x_2 - \kappa^\sharp, & u_p^n > \kappa^\sharp - x_2, \\ 0, & u_p^n \leq \kappa^\sharp - x_2, \end{cases} \tag{8.50}$$

as a test function in (8.2) to obtain

$$\int_\Omega \frac{b(u_p^n)-b(u_p^{n-1})}{\tau}(u_p^n+x_2-\kappa^\sharp)_+\,\mathrm{d}x \\ +\int_\Omega k(\theta_p^n,u_p^{n-1})\nabla\left(u_p^n+x_2\right)\cdot\nabla(u_p^n+x_2-\kappa^\sharp)_+=0. \quad (8.51)$$

The elliptic term is nonnegative, hence we get

$$\frac{1}{\tau}\int_\Omega \left(b(u_p^n)-b(u_p^{n-1})\right)(u_p^n+x_2-\kappa^\sharp)_+\,\mathrm{d}x\le 0. \quad (8.52)$$

Let us set $\tilde\kappa^\sharp=\min_{x_2\in\Omega}(\kappa^\sharp-x_2)$, now considering (8.38) we may write

$$\int_\Omega \beta^+_{\tilde\kappa^\sharp}(u_p^n)-\beta^+_{\tilde\kappa^\sharp}(u_p^{n-1})\le 0. \quad (8.53)$$

Similarly as before, for $n=1$, we obtain (by similar arguments)

$$\beta^+_{\tilde\kappa^\sharp}(u_0(x)):=\int_{\tilde\kappa^\sharp}^{u_0(x)} b'(s)(s-\tilde\kappa_\sharp)_+\,\mathrm{d}s=0 \quad (8.54)$$

and from (8.53) we deduce

$$u_p^1\le\tilde\kappa^\sharp. \quad (8.55)$$

Hence we may conclude, successively for $n=2,3,\cdots,p$, that

$$u_p^n\le\tilde\kappa^\sharp. \quad (8.56)$$

Combining (8.49) and (8.56), for $n=1,\dots,p$, we arrive at

$$\tilde\kappa_\sharp\le u_p^n\le\tilde\kappa^\sharp, \quad (8.57)$$

which becomes

$$\|\bar u_p\|_{L^\infty(Q_T)}\le c, \quad (8.58)$$

where c is independent of p.

The a-priori estimate (8.58) allows us to conclude that there exists $u\in L^\infty(Q_T)$ such that, letting $p\to+\infty$ (along a selected subsequence),

$$\bar u_p\rightharpoonup u \qquad \text{weakly star in } L^\infty(Q_T). \quad (8.59)$$

L^∞-bound for $\bar\theta_p$. Let ℓ be an odd integer. Using $\phi=[\ell/(\ell+1)][\bar\theta_p(s)]^{\ell+1}$ as a test function in (8.34) we have

$$\frac{\ell}{\ell+1}\int_\Omega \frac{b(\bar u_p(s))-b(\bar u_p(s-\tau))}{\tau}[\bar\theta_p(s)]^{\ell+1}\,\mathrm{d}x \\ +\frac{\ell}{\ell+1}\int_\Omega k(\bar\theta_p(s-\tau),\bar u_p(s-\tau))\left[\nabla\bar u_p(s)+\boldsymbol{e}_z\right]\cdot\nabla[\bar\theta_p(s)]^{\ell+1}\,\mathrm{d}x=0. \quad (8.60)$$

Further, we use $\psi = [\bar{\theta}_p(s)]^\ell$ as a test function in (8.35) to obtain

$$\begin{aligned}
&\int_\Omega \frac{b(\bar{u}_p(s))\bar{\theta}_p(s) - b(\bar{u}_p(s-\tau))\bar{\theta}_p(s-\tau)}{\tau}[\bar{\theta}_p(s)]^\ell \,\mathrm{d}x + \varrho \int_\Omega \frac{\bar{\theta}_p(s) - \bar{\theta}_p(s-\tau)}{\tau}[\bar{\theta}_p(s)]^\ell \,\mathrm{d}x \\
&\quad + \int_\Omega \lambda(\bar{\theta}_p(s-\tau), \bar{u}_p(s-\tau)) \nabla \bar{\theta}_p(s) \cdot \nabla [\bar{\theta}_p(s)]^\ell \mathrm{d}x \\
&\quad\quad + \int_\Omega \bar{\theta}_p(s) k(\bar{\theta}_p(s-\tau), \bar{u}_p(s-\tau)) \left[\nabla \bar{u}_p(s) + \boldsymbol{e}_z\right] \cdot \nabla [\bar{\theta}_p(s)]^\ell \mathrm{d}x = 0. \qquad (8.61)
\end{aligned}$$

Now subtracting (8.60) from (8.61) we obtain

$$\begin{aligned}
&\frac{1}{\tau}\frac{1}{\ell+1}\int_\Omega b(\bar{u}_p(s))[\bar{\theta}_p(s)]^{\ell+1}\,\mathrm{d}x - \frac{1}{\tau}\frac{1}{\ell+1}\int_\Omega b(\bar{u}_p(s-\tau))[\bar{\theta}_p(s-\tau)]^{\ell+1}\,\mathrm{d}x \\
&\quad + \frac{1}{\tau}\frac{1}{\ell+1}\int_\Omega b(\bar{u}_p(s-\tau))[\bar{\theta}_p(s-\tau)]^{\ell+1}\,\mathrm{d}x + \frac{1}{\tau}\frac{\ell}{\ell+1}\int_\Omega b(\bar{u}_p(s-\tau))[\bar{\theta}_p(s)]^{\ell+1}\,\mathrm{d}x \\
&\quad - \frac{1}{\tau}\int_\Omega b(\bar{u}_p(s-\tau))\bar{\theta}_p(s-\tau)[\bar{\theta}_p(s)]^\ell\,\mathrm{d}x \\
&\quad\quad + \varrho\frac{1}{\tau}\int_\Omega \left[\bar{\theta}_p(s) - \bar{\theta}_p(s-\tau)\right][\bar{\theta}_p(s)]^\ell\,\mathrm{d}x \\
&\quad\quad + \int_\Omega \lambda(\bar{\theta}_p(s-\tau), \bar{u}_p(s-\tau)) \nabla \bar{\theta}_p(s) \cdot \nabla [\bar{\theta}_p(s)]^\ell \mathrm{d}x = 0. \qquad (8.62)
\end{aligned}$$

Rearranging the terms on the third and fourth line we obtain

$$\begin{aligned}
&\frac{1}{\tau}\frac{1}{\ell+1}\int_\Omega b(\bar{u}_p(s))[\bar{\theta}_p(s)]^{\ell+1}\,\mathrm{d}x - \frac{1}{\tau}\frac{1}{\ell+1}\int_\Omega b(\bar{u}_p(s-\tau))[\bar{\theta}_p(s-\tau)]^{\ell+1}\,\mathrm{d}x \\
&\quad + \frac{1}{\tau}\frac{1}{\ell+1}\int_\Omega b(\bar{u}_p(s-\tau))[\bar{\theta}_p(s-\tau)]^{\ell+1}\,\mathrm{d}x + \frac{1}{\tau}\frac{\ell}{\ell+1}\int_\Omega b(\bar{u}_p(s-\tau))[\bar{\theta}_p(s)]^{\ell+1}\,\mathrm{d}x \\
&\quad - \frac{1}{\tau}\int_\Omega \left[b(\bar{u}_p(s-\tau)) + \varrho\right]\bar{\theta}_p(s-\tau)[\bar{\theta}_p(s)]^\ell\,\mathrm{d}x + \varrho\frac{1}{\tau}\int_\Omega [\bar{\theta}_p(s)]^{\ell+1}\,\mathrm{d}x \\
&\quad\quad + \int_\Omega \lambda(\bar{\theta}_p(s-\tau), \bar{u}_p(s-\tau)) \nabla \bar{\theta}_p(s) \cdot \nabla [\bar{\theta}_p(s)]^\ell \mathrm{d}x = 0. \qquad (8.63)
\end{aligned}$$

Applying Young's inequality on the first term in the third line we can write

$$\begin{aligned}
&\frac{1}{\tau}\int_\Omega \left[b(\bar{u}_p(s-\tau)) + \varrho\right]\bar{\theta}_p(s-\tau)[\bar{\theta}_p(s)]^\ell\,\mathrm{d}x \\
&\qquad\qquad \le \frac{1}{\tau}\frac{1}{\ell+1}\int_\Omega \left[b(\bar{u}_p(s-\tau)) + \varrho\right][\bar{\theta}_p(s-\tau)]^{\ell+1}\,\mathrm{d}x \\
&\qquad\qquad\qquad + \frac{1}{\tau}\frac{\ell}{\ell+1}\int_\Omega \left[b(\bar{u}_p(s-\tau)) + \varrho\right][\bar{\theta}_p(s)]^{\ell+1}\,\mathrm{d}x. \qquad (8.64)
\end{aligned}$$

Combining (8.63) and (8.64) we deduce

$$\frac{1}{\tau}\frac{1}{\ell+1}\int_\Omega [b(\bar{u}_p(s-\tau))+\varrho]\,[\bar{\theta}_p(s)]^{\ell+1}\,\mathrm{d}x \\ -\frac{1}{\tau}\frac{1}{\ell+1}\int_\Omega [b(\bar{u}_p(s-\tau))+\varrho]\,[\bar{\theta}_p(s-\tau)]^{\ell+1}\,\mathrm{d}x \\ +\int_\Omega \lambda(\bar{\theta}_p(s-\tau),\bar{u}_p(s-\tau))\nabla\bar{\theta}_p(s)\cdot\nabla[\bar{\theta}_p(s)]^\ell \mathrm{d}x \le 0. \quad (8.65)$$

The elliptic term is nonnegative since

$$\int_\Omega \lambda(\bar{\theta}_p(s),\bar{u}_p(s-\tau))\nabla\bar{\theta}_p(s)\cdot\nabla[\bar{\theta}_p(s)]^\ell \mathrm{d}x \ge \int_\Omega c\,\ell\,\bar{\theta}_p(s)^{\ell-1}|\nabla\bar{\theta}_p(s)|^2\mathrm{d}x \ge 0. \quad (8.66)$$

From (8.66) and (8.65) we have

$$\int_\Omega (\bar{\theta}_p(t))^{\ell+1}\,[b(\bar{u}_p(t))+\varrho]\,\mathrm{d}x \le \int_\Omega (\theta_0)^{\ell+1}\,[b\,(u_0)+\varrho]\,\mathrm{d}x. \quad (8.67)$$

Considering ϱ is a positive constant and b is a nonnegative function we obtain

$$\sup_{0\le t\le T}\int_\Omega [\bar{\theta}_p(t)]^{\ell+1}\,[b(\bar{u}_p(t))+\varrho]\,\mathrm{d}x \le C. \quad (8.68)$$

Hence

$$\|\bar{\theta}_p\|_{L^\infty(I;L^{\ell+1}(\Omega))} \le C, \quad \ell\in\mathbb{N}, \quad (8.69)$$

where the constant C is independent of ℓ and p. Now, let $\ell\to+\infty$ in (8.69). We get

$$\|\bar{\theta}_p\|_{L^\infty(Q_T)} \le C. \quad (8.70)$$

From the a-priori estimate (8.70), we conclude, that there exists $\theta\in L^\infty(Q_T)$ such that, letting $p\to+\infty$ (along a selected subsequence),

$$\bar{\theta}_p \rightharpoonup \theta \qquad \text{weakly star in } L^\infty(Q_T). \quad (8.71)$$

8.4.3 Energy estimates

In this section we will derive the energy estimates for time interpolants $\overline{u}_p$ and $\overline{\theta}_p$.

Energy estimate for $\bar{u}_p$. To derive the suitable a-priori estimate for $\bar{u}_p$ we test the equation (8.34) with $\phi=\bar{u}_p(t)$ to obtain

$$\int_\Omega \frac{b(\bar{u}_p(t))-b(\bar{u}_p(t-\tau))}{\tau}\bar{u}_p(t)\,\mathrm{d}x \\ +\int_\Omega k(\bar{\theta}_p(t-\tau),\bar{u}_p(t-\tau))\,[\nabla\bar{u}_p(t)+\boldsymbol{e}_z]\cdot\nabla\bar{u}_p(t)\,\mathrm{d}x = 0 \quad (8.72)$$

for all $t \in (0;T]$.

Now, we will deal with each term of the above equation separately. First, in view of Remark 6.3, we estimate the first term as

$$\int_\Omega \frac{b(\bar{u}_p(t)) - b(\bar{u}_p(t-\tau))}{\tau} \bar{u}_p(t) \,\mathrm{d}x \geq c \int_\Omega \frac{B(\bar{u}_p(t)) - B(\bar{u}_p(t-\tau))}{\tau} \,\mathrm{d}x. \tag{8.73}$$

Further, using Friedrich's inequality, we have for the elliptic term the estimate

$$\int_\Omega k(\bar{\theta}_p(t-\tau), \bar{u}_p(t-\tau)) |\nabla \bar{u}_p(t)|^2 \,\mathrm{d}x \geq c \|\bar{u}_p(t)\|^2_{W^{1,2}_D(\Omega)}. \tag{8.74}$$

Note that, thanks to (6.17), the constant c in (8.74) does not depend on p.

Taking into account Cauchy's inequality $\frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2 \geq ab$ and Hölder's inequality we have

$$\begin{aligned}\int_\Omega \boldsymbol{e}_z \cdot \nabla \bar{u}_p(t) \,\mathrm{d}x &\leq \|\boldsymbol{e}_z\|_{L^2(\Omega)} \|\bar{u}_p(t)\|_{W^{1,2}_D(\Omega)^2}, \\ &\leq \frac{\epsilon}{2}\|\boldsymbol{e}_z\|^2_{L^2(\Omega)^2} + \frac{1}{2\epsilon}\|\bar{u}_p(t)\|^2_{W^{1,2}_D(\Omega)}. \end{aligned} \tag{8.75}$$

Combining (8.73), (8.74) and (8.75) we obtain

$$\begin{aligned}\frac{1}{\tau}\int_\Omega B(\bar{u}_p(t)) - B(\bar{u}_p(t-\tau)) \,\mathrm{d}x + c_1 \|\bar{u}_p(t)\|^2_{W^{1,2}_D(\Omega)} \\ \leq \frac{\epsilon}{2}\|\boldsymbol{e}_z\|^2_{L^2(\Omega)^2} + \frac{1}{2\epsilon}\|\bar{u}_p(t)\|^2_{W^{1,2}_D(\Omega)}. \end{aligned} \tag{8.76}$$

Further, we can write

$$\frac{1}{\tau}\int_\Omega B(\bar{u}_p(t)) - B(\bar{u}_p(t-\tau)) \,\mathrm{d}x + \left(c - \frac{1}{2\epsilon}\right) \|\bar{u}_p(t)\|^2_{W^{1,2}_D(\Omega)} \leq c \tag{8.77}$$

and therefore, for sufficiently large ϵ, we have

$$\frac{1}{\tau}\int_\Omega B(\bar{u}_p(t)) - B(\bar{u}_p(t-\tau)) \,\mathrm{d}x + \|\bar{u}_p(t)\|^2_{W^{1,2}_D(\Omega)} \leq c. \tag{8.78}$$

We now integrate (8.78) with respect to time from 0 to s ($0 \leq s \leq T$). Without loss of generality suppose $s = k\tau, k = 1, \dots, p$. We arrive at

$$\sum_{i=1}^{k} \frac{1}{\tau} \int_{(i-1)\tau}^{i\tau} \int_\Omega B(\bar{u}_p(t)) - B(\bar{u}_p(t-\tau)) \,\mathrm{d}x \,\mathrm{d}t + \sum_{i=1}^{k} \int_{(i-1)\tau}^{i\tau} \|\bar{u}_p(t)\|^2_{W^{1,2}_D(\Omega)} \,\mathrm{d}t \leq k\tau c. \tag{8.79}$$

Evaluating the integrals we have

$$-\int_\Omega B(\bar{u}_p(0)) \,\mathrm{d}x + \int_\Omega B(\bar{u}_p(k\tau)) \,\mathrm{d}x + \int_0^{k\tau} \|\bar{u}_p(t)\|^2_{W^{1,2}_D(\Omega)} \,\mathrm{d}t \leq k\tau c. \tag{8.80}$$

Hence

$$\sup_{0\le t\le T}\int_\Omega B(\bar{u}_p(t))\,\mathrm{d}x + \int_0^T \|\bar{u}_p(t)\|^2_{W_D^{1,2}}\,\mathrm{d}t \le Tc. \tag{8.81}$$

Let us note that (8.81) becomes

$$\|\bar{u}_p\|_{L_2(I,W_D^{1,2}(\Omega))} \le Tc. \tag{8.82}$$

As a consequence of the a-priori estimate (8.82) we see ([49], Section 10.26) that there exists a function $u \in L^2(I; W_D^{1,2}(\Omega))$ such that, along a selected subsequence (letting $p \to \infty$), we have

$$\bar{u}_p \rightharpoonup u \qquad \text{weakly in } L^2(I; W_D^{1,2}(\Omega)). \tag{8.83}$$

Energy estimate for $\bar{\theta}_p$. In order to derive the energy estimate for $\bar{\theta}_p$ we use $\psi(t) = 2\bar{\theta}_p(t)$ as a test function in (8.35) to obtain

$$\begin{aligned}\int_\Omega \frac{b(\bar{u}_p(t))\bar{\theta}_p(t) - b(\bar{u}_p(t-\tau))\bar{\theta}_p(t-\tau)}{\tau}2\bar{\theta}_p(t)\,\mathrm{d}x + \varrho\int_\Omega \frac{\bar{\theta}_p(t) - \bar{\theta}_p(t-\tau)}{\tau}2\bar{\theta}_p(t)\,\mathrm{d}x \\ +\int_\Omega \lambda(\bar{\theta}_p(t-\tau), \bar{u}_p(t-\tau))\nabla\bar{\theta}_p(t)\cdot\nabla 2\bar{\theta}_p(t)\mathrm{d}x \\ +\int_\Omega \bar{\theta}_p(t)k(\bar{\theta}_p(t-\tau), \bar{u}_p(t-\tau))\left[\nabla\bar{u}_p(t) + \boldsymbol{e}_z\right]\cdot\nabla 2\bar{\theta}_p(t)\mathrm{d}x = 0.\end{aligned}$$

We modify the above equation to get

$$\begin{aligned}\int_\Omega \frac{b(\bar{u}_p(t)) - b(\bar{u}_p(t-\tau))}{\tau}2\bar{\theta}_p(t)^2\,\mathrm{d}x + \int_\Omega \frac{\bar{\theta}_p(t) - \bar{\theta}_p(t-\tau)}{\tau}2\bar{\theta}_p(t)\left[b(\bar{u}_p(t-\tau) + \varrho\right]\,\mathrm{d}x \\ +\int_\Omega 2\lambda(\bar{\theta}_p(t-\tau), \bar{u}_p(t-\tau))\nabla\bar{\theta}_p(t)\cdot\nabla\bar{\theta}_p(t)\mathrm{d}x \\ +\int_\Omega \bar{\theta}_p(t)k(\bar{\theta}_p(t-\tau), \bar{u}_p(t-\tau))\left[\nabla\bar{u}_p(t) + \boldsymbol{e}_z\right]\cdot\nabla 2\bar{\theta}_p(t)\mathrm{d}x = 0.\end{aligned} \tag{8.84}$$

Further, we use $\phi(t) = \bar{\theta}_p(t)^2$ as a test function in (8.34) to obtain

$$\begin{aligned}\int_\Omega \frac{b(\bar{u}_p(t)) - b(\bar{u}_p(t-\tau))}{\tau}\bar{\theta}_p(t)^2\,\mathrm{d}x \\ + \int_\Omega 2\bar{\theta}_p(t)k(\bar{\theta}_p(t-\tau), \bar{u}_p(t-\tau))\left[\nabla\bar{u}_p(t) + \boldsymbol{e}_z\right]\cdot\nabla\bar{\theta}_p(t)\,\mathrm{d}x = 0.\end{aligned} \tag{8.85}$$

Substituting (8.85) in (8.84) we have

$$\int_\Omega \frac{\bar\theta_p(t)^2\,(b(\bar u_p(t))+\varrho)-\bar\theta_p(t-\tau)^2\,(b(\bar u_p(t-\tau))+\varrho)}{\tau}\,\mathrm{d}x \\ +\int_\Omega \frac{1}{\tau}\left[\left(\bar\theta_p(t)\right)-\left(\bar\theta_p(t-\tau)\right)\right]^2(b(\bar u_p(t-\tau))+\varrho)\,\mathrm{d}x \\ +2\int_\Omega \lambda(\bar\theta_p(t-\tau),\bar u_p(t-\tau))\nabla\bar\theta_p(t)\cdot\nabla\bar\theta_p(t)\,\mathrm{d}x=0. \quad (8.86)$$

Since b is a positive function and ϱ is a positive constant, the second integral is nonnegative. Further, we use Friedrich's inequality for the elliptic term and integrate with respect to time from 0 to s ($0\le s\le T, s=k\tau, k\in\mathbb{N}$). Hence we obtain

$$2\int_0^{k\tau}\int_\Omega \lambda(\bar\theta_p(t-\tau),\bar u_p(t-\tau))\nabla\bar\theta_p(t)\cdot\nabla\bar\theta_p(t)\,\mathrm{d}x\,\mathrm{d}t \ge c\int_0^{k\tau}\|\bar\theta_p(t)\|^2_{W_D^{1,2}(\Omega)}\,\mathrm{d}t. \quad (8.87)$$

In the same way we integrate the first term in (8.86) with respect to time from 0 to s, $s=k\tau$, $k\in\mathbb{N}$, to obtain

$$\int_0^{k\tau}\int_\Omega \frac{1}{\tau}\left(\bar\theta_p(t)^2\,(b(\bar u_p(t))+\varrho)-\bar\theta_p(t-\tau)^2\,(b(\bar u_p(t-\tau))+\varrho)\right)\,\mathrm{d}x\,\mathrm{d}t \\ =\int_\Omega \frac{1}{\tau}\sum_{i=1}^{k}\int_{(i-1)\tau}^{i\tau}\left(\bar\theta_p(t)^2\,(b(\bar u_p(t))+\varrho)-\bar\theta_p(t-\tau)^2\,(b(\bar u_p(t-\tau))+\varrho)\right)\,\mathrm{d}t\,\mathrm{d}x \\ =\int_\Omega -\left(\bar\theta_p(0)\right)^2(b(\bar u_p(0))+\varrho)\,\mathrm{d}x+\int_\Omega\left(\bar\theta_p(k\tau)\right)^2(b(\bar u_p(k\tau))+\varrho)\,\mathrm{d}x. \quad (8.88)$$

We integrate (8.86) with respect to time from 0 to s, $s=k\tau$, $k\in\mathbb{N}$ and we substitute (8.87) and (8.88) in (8.86) to obtain

$$\int_\Omega\left(\bar\theta_p(k\tau)\right)^2(b(\bar u_p(k\tau))+\varrho)\,\mathrm{d}x+c\int_0^{k\tau}\|\bar\theta_p(t)\|^2_{W_D^{1,2}(\Omega)}\,\mathrm{d}t \\ \le\int_\Omega\left(\bar\theta_p(0)\right)^2(b(\bar u_p(0))+\varrho)\,\mathrm{d}x. \quad (8.89)$$

Considering b is a bounded function and ϱ is a positive constant we can write

$$\int_\Omega\left|\bar\theta_p(k\tau)\right|^2\,\mathrm{d}x+c\int_0^T\|\bar\theta_p(t)\|^2_{W_D^{1,2}(\Omega)}\,\mathrm{d}t\le c. \quad (8.90)$$

Hence, since $\bar\theta_p(t)$ is a piecewise constant function, we have also

$$\sup_{0\le t\le T}\int_\Omega\left|\bar\theta_p(t)\right|^2\,\mathrm{d}x+\int_0^T\|\bar\theta_p(t)\|^2_{W_D^{1,2}}\,\mathrm{d}t\le c. \quad (8.91)$$

Let us mention that (8.91) becomes

$$\|\bar{\theta}_p\|_{L^2(I;W_D^{1,2}(\Omega))} \leq c. \tag{8.92}$$

Now, from the a-priori estimate (8.92), we conclude ([49], Section 10.26) that there exists $\theta \in L^2(I;W_D^{1,2}(\Omega))$ such that, letting $p \to +\infty$ (along a selected subsequence),

$$\bar{\theta}_p \rightharpoonup \theta \qquad \text{weakly in } L^2(I;W_D^{1,2}(\Omega)). \tag{8.93}$$

8.4.4 Further estimates

Due to the nonlinearities in the model we need some further estimates in order to show the convergence of time interpolants almost everywhere. In what follows we use the procedure proposed by Alt and Luckhaus in [2].

Theorem 8.14 *(Convergence almost everywhere of $\bar{u}_p$ and $\bar{\theta}_p$) Let the assumptions* (i)–(iv) *be satisfied, then*

$$\bar{u}_p \to u \qquad \textit{almost everywhere on } Q_T, \tag{8.94}$$

$$\bar{\theta}_p \to \theta \qquad \textit{almost everywhere on } Q_T. \tag{8.95}$$

To show (8.94) and (8.95) we use the following lemma:

Lemma 8.15 *(see [2], Lemma 1.9) Suppose u_ε converge weakly in $L^r\left(0,T;H^{1,r}(\Omega)\right)$ to u with the estimates*

$$\frac{1}{\tau}\int_0^{T-h}\int_\Omega \left[b\left(u_\varepsilon(t+h)\right) - b\left(u_\varepsilon(t)\right)\right]\left(u_\varepsilon(t+h) - u_\varepsilon(t)\right)\,\mathrm{d}x\,\mathrm{d}t \leqq c, \tag{8.96}$$

and

$$\int_\Omega B(u_\varepsilon(t))\,\mathrm{d}x \leq c \quad \text{for} \quad 0 < t < T, \tag{8.97}$$

then

$$b(u_\varepsilon) \to b(u) \qquad \textit{in } L^1([0,T]\times\Omega), \tag{8.98}$$

and

$$B(u_\varepsilon) \to B(u) \qquad \textit{almost everywhere.} \tag{8.99}$$

Proof of Theorem 8.14 Let $k \in \mathbb{N}$. We use

$$\phi(t) = \frac{1}{k\tau}\left(\bar{u}_p(s_j + k\tau) - \bar{u}_p(s_j)\right)$$

for $j\tau \leq t \leq (j+k)\tau$ with $(j-1)\tau \leq s_j \leq j\tau$ and $1 \leq j \leq \frac{T}{\tau} - k$, as a test function in (8.34). For the parabolic term, we can write

$$
\begin{aligned}
&\frac{1}{k\tau^2} \int_{j\tau}^{(j+k)\tau} \int_\Omega [b(\bar{u}_p(t)) - b(\bar{u}_p(t-\tau))]\, [\bar{u}_p(s_j + k\tau) - \bar{u}_p(s_j)]\, \mathrm{d}x\mathrm{d}t \\
&= \frac{1}{k\tau^2} \int_\Omega \tau [b(\bar{u}_p(j+k\tau)) - b(\bar{u}_p(j\tau))]\, [\bar{u}_p(j+k\tau) - \bar{u}_p(j\tau)]\, \mathrm{d}x \\
&= \frac{1}{k\tau^2} \int_{(j-1)\tau}^{j\tau} \int_\Omega [b(\bar{u}_p(t+k\tau)) - b(\bar{u}_p(t))]\, [\bar{u}_p(t+k\tau) - \bar{u}_p(t)]\, \mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Hence, summing over $j = 1, \ldots, p-k$ we get

$$
\begin{aligned}
&\sum_{j=1}^{p-k} \frac{1}{k\tau^2} \int_{j\tau}^{(j+k)\tau} \int_\Omega [b(\bar{u}_p(t)) - b(\bar{u}_p(t-\tau))]\, [\bar{u}_p(s_j + k\tau) - \bar{u}_p(s_j)]\, \mathrm{d}x\mathrm{d}t \\
&\geq \frac{1}{k\tau^2} \int_0^{T-k\tau} \int_\Omega [b(\bar{u}_p(t+k\tau)) - b(\bar{u}_p(t))]\, [\bar{u}_p(t+k\tau) - \bar{u}_p(t)]\, \mathrm{d}x\mathrm{d}t. \qquad (8.100)
\end{aligned}
$$

For the elliptic term we use a similar approach. To simplify the procedure, let us introduce the following notation

$$
\begin{aligned}
\xi_j(t) &:= \frac{1}{k\tau} \left(\bar{u}_p(s_j + k\tau) - \bar{u}_p(s_j) \right), \\
\bar{\boldsymbol{q}}(t) &:= k(\bar{\theta}_p(t-\tau), \bar{u}_p(t-\tau)) \left[\nabla \bar{u}_p(t) + \boldsymbol{e}_z \right],
\end{aligned}
$$

to obtain

$$
\sum_{j=1}^{p-k} \int_{j\tau}^{(j+k)\tau} \int_\Omega \bar{\boldsymbol{q}}(t) \cdot \nabla \xi_j(t)\, \mathrm{d}x\mathrm{d}t.
$$

One is allowed to divide the time interval for integration to get

$$
\sum_{j=1}^{p-k} \sum_{i=1}^{k} \int_{(j+i-1)\tau}^{(j+i)\tau} \int_\Omega \bar{\boldsymbol{q}}(t) \cdot \nabla \xi_j(t)\, \mathrm{d}x\mathrm{d}t.
$$

Expanding the first sum, we have

$$
\begin{aligned}
&\sum_{i=1}^{k} \int_{(i)\tau}^{(1+i)\tau} \int_\Omega \bar{\boldsymbol{q}}(t) \cdot \nabla \xi_1(t)\, \mathrm{d}x\mathrm{d}t \\
&+ \sum_{i=1}^{k} \int_{(1+i)\tau}^{(2+i)\tau} \int_\Omega \bar{\boldsymbol{q}}(t) \cdot \nabla \xi_2(t)\, \mathrm{d}x\mathrm{d}t \\
&+ \ldots \\
&+ \sum_{i=1}^{k} \int_{(p-k+i-1)\tau}^{(p-k+i)\tau} \int_\Omega \bar{\boldsymbol{q}}(t) \cdot \nabla \xi_{p-k}(t)\, \mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

which becomes

$$
\begin{aligned}
&\int_{\tau}^{2\tau}\int_{\Omega}\bar{\boldsymbol{q}}(t)\cdot\nabla\,\xi_1(t)\,\mathrm{d}x\mathrm{d}t+\int_{2\tau}^{3\tau}\int_{\Omega}\bar{\boldsymbol{q}}(t)\cdot\nabla\,\xi_1(t)\,\mathrm{d}x\mathrm{d}t+...\\
&\qquad\qquad\qquad\qquad\qquad\qquad+\int_{k\tau}^{(k+1)\tau}\int_{\Omega}\bar{\boldsymbol{q}}(t)\cdot\nabla\,\xi_1(t)\,\mathrm{d}x\mathrm{d}t\\
&+\int_{2\tau}^{3\tau}\int_{\Omega}\bar{\boldsymbol{q}}(t)\cdot\nabla\,\xi_2(t)\,\mathrm{d}x\mathrm{d}t+\int_{3\tau}^{4\tau}\int_{\Omega}\bar{\boldsymbol{q}}(t)\cdot\nabla\,\xi_2(t)\,\mathrm{d}x\mathrm{d}t+...\\
&\qquad\qquad\qquad\qquad\qquad\qquad+\int_{(k+1)\tau}^{(k+2)\tau}\int_{\Omega}\bar{\boldsymbol{q}}(t)\cdot\nabla\,\xi_2(t)\,\mathrm{d}x\mathrm{d}t\\
&\qquad\qquad\qquad\vdots\\
&\int_{(p-k)\tau}^{(p-k+1)\tau}\int_{\Omega}\bar{\boldsymbol{q}}(t)\cdot\nabla\,\xi_{p-k}(t)\,\mathrm{d}x\mathrm{d}t+\int_{(p-k+1)\tau}^{(p-k+2)\tau}\int_{\Omega}\bar{\boldsymbol{q}}(t)\cdot\nabla\,\xi_{p-k}(t)\,\mathrm{d}x\mathrm{d}t+...\\
&\qquad\qquad\qquad\qquad\qquad\qquad+\int_{(p-1)\tau}^{p\tau}\int_{\Omega}\bar{\boldsymbol{q}}(t)\cdot\nabla\,\xi_{p-k}(t)\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{8.101}
$$

For the first term on the first line we can write

$$
\begin{aligned}
\int_{\tau}^{2\tau}\int_{\Omega}\bar{\boldsymbol{q}}(t)\cdot\nabla\,\xi_1(t)\,\mathrm{d}x\mathrm{d}t&=\frac{1}{\tau}\int_{\tau}^{2\tau}\int_{\Omega}\bar{\boldsymbol{q}}(t)\cdot\nabla\,\left[\bar{u}_p(\tau+k\tau)-\bar{u}_p(\tau)\right]\,\mathrm{d}x\mathrm{d}t\\
&=\frac{1}{\tau}\int_{\tau}^{2\tau}\int_{\Omega}\bar{\boldsymbol{q}}(t)\cdot\nabla\,\left[\bar{u}_p(2\tau+k\tau-\tau)-\bar{u}_p(2\tau-\tau)\right]\,\mathrm{d}x\mathrm{d}t\\
&=\frac{1}{\tau}\int_{\tau}^{2\tau}\int_{\Omega}\bar{\boldsymbol{q}}(t)\cdot\nabla\,\left[\bar{u}_p(t+k\tau-\tau)-\bar{u}_p(t-\tau)\right]\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Similarly, for the second term on the first line we have

$$
\begin{aligned}
\int_{2\tau}^{3\tau}\int_{\Omega}\bar{\boldsymbol{q}}(t)\cdot\nabla\,\xi_1(t)\,\mathrm{d}x\mathrm{d}t,&=\frac{1}{\tau}\int_{\tau}^{2\tau}\int_{\Omega}\bar{\boldsymbol{q}}(t)\cdot\nabla\,\left[\bar{u}_p(\tau+k\tau)-\bar{u}_p(\tau)\right]\,\mathrm{d}x\mathrm{d}t\\
&=\frac{1}{\tau}\int_{2\tau}^{3\tau}\int_{\Omega}\bar{\boldsymbol{q}}(t)\cdot\nabla\,\left[\bar{u}_p(3\tau+k\tau-2\tau)-\bar{u}_p(3\tau-2\tau)\right]\,\mathrm{d}x\mathrm{d}t\\
&=\frac{1}{\tau}\int_{2\tau}^{3\tau}\int_{\Omega}\bar{\boldsymbol{q}}(t)\cdot\nabla\,\left[\bar{u}_p(t+k\tau-2\tau)-\bar{u}_p(t-2\tau)\right]\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Hence, for each line in (8.101) we have

$$\begin{aligned}
&\text{first} && \sum_{i=1}^{k}\frac{1}{\tau}\int_{i\tau}^{(i+1)\tau}\int_{\Omega}\bar{\boldsymbol{q}}(t)\cdot\nabla\left[\bar{u}_p(t-i\tau+k\tau)-\bar{u}_p(t-i\tau)\right]\,\mathrm{d}x\mathrm{d}t;\\
&\text{second} && \sum_{i=1}^{k}\frac{1}{\tau}\int_{i\tau}^{(i+1)\tau}\int_{\Omega}\bar{\boldsymbol{q}}(t)\cdot\nabla\left[\bar{u}_p(t-i\tau+k\tau)-\bar{u}_p(t-i\tau)\right]\,\mathrm{d}x\mathrm{d}t;\\
&(p-k)\text{−th} && \sum_{i=1}^{k}\frac{1}{\tau}\int_{(i+p-k-1)\tau}^{(i+p-k)\tau}\int_{\Omega}\bar{\boldsymbol{q}}(t)\cdot\nabla\left[\bar{u}_p(t-i\tau+k\tau)-\bar{u}_p(t-i\tau)\right]\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Summing over the lines we obtain

$$\begin{aligned}
&\sum_{i=1}^{k}\frac{1}{\tau}\int_{i\tau}^{(i+p-k)\tau}\int_{\Omega}\bar{\boldsymbol{q}}(t)\cdot\nabla\left[\bar{u}_p(t-i\tau+k\tau)-\bar{u}_p(t-i\tau)\right]\,\mathrm{d}x\mathrm{d}t\\
&\quad=\sum_{i=1}^{k}\frac{1}{\tau}\int_{i\tau}^{T+i\tau-k\tau}\int_{\Omega}\bar{\boldsymbol{q}}(t)\cdot\nabla\left[\bar{u}_p(t-i\tau+k\tau)-\bar{u}_p(t-i\tau)\right]\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Now, we use Hölder's inequality and after a straightforward computation we obtain

$$\begin{aligned}
&=\sum_{i=1}^{k}\frac{1}{\tau}\int_{i\tau}^{T+i\tau-k\tau}\|\bar{\boldsymbol{q}}(t)\|_{L^2(\Omega)}\|\nabla\left[\bar{u}_p(t-i\tau+k\tau)-\bar{u}_p(t-i\tau)\right]\|_{L^2(\Omega)}\mathrm{d}t\\
&\leq\sum_{i=1}^{k}\frac{1}{\tau}\int_{i\tau}^{T+i\tau-k\tau}\|\bar{\boldsymbol{q}}(t)\|^2_{L^2(\Omega)}+\|\nabla\left[\bar{u}_p(t-i\tau+k\tau)-\bar{u}_p(t-i\tau)\right]\|^2_{L^2(\Omega)}\mathrm{d}t\\
&=\sum_{i=1}^{k}\frac{1}{\tau}\int_{i\tau}^{T+i\tau-k\tau}\|\bar{\boldsymbol{q}}(t)\|^2_{L^2(\Omega)}\mathrm{d}t+\sum_{i=1}^{k}\frac{1}{\tau}\int_{0}^{T-k\tau}\|\nabla\,\bar{u}_p(s+k\tau)-\nabla\bar{u}_p(s)\|^2_{L^2(\Omega)}\mathrm{d}s\\
&\leq\sum_{i=1}^{k}\frac{1}{\tau}\int_{i\tau}^{T+i\tau-k\tau}\|\bar{\boldsymbol{q}}(t)\|^2_{L^2(\Omega)}\mathrm{d}t+\sum_{i=1}^{k}\frac{1}{\tau}\int_{0}^{T-k\tau}\|\nabla\,\bar{u}_p(s+k\tau)\|^2_{L^2(\Omega)}+\|\nabla\bar{u}_p(s)\|^2_{L^2(\Omega)}\mathrm{d}s\\
&\leq\frac{c_1}{\tau}+\frac{c_2}{\tau}\leq\frac{c}{\tau}. \qquad (8.102)
\end{aligned}$$

Combining (8.100) and (8.102) we arrive at

$$\int_0^{T-k\tau}\int_{\Omega}\left[b(\bar{u}_p(t+k\tau))-b(\bar{u}_p(t))\right]\left[\bar{u}_p(t+k\tau)-\bar{u}_p(t)\right]\,\mathrm{d}x\mathrm{d}t\leq ck\tau. \qquad (8.103)$$

Further, in much the same way as in (8.103), we can show

$$\int_0^{T-k\tau}\int_{\Omega}\left|b(\bar{u}_p(t+k\tau))\bar{\theta}_p(t+k\tau)-b(\bar{u}_p(t))\bar{\theta}_p(t)\right|\leq ck\tau. \qquad (8.104)$$

From (8.104) we conclude, using (8.70), that

$$\frac{1}{\tau}\int_0^{T-\tau}\int_\Omega \left[\bar{\theta}_p(t+\tau)-\bar{\theta}_p(t)\right]^2 \,\mathrm{d}x\,\mathrm{d}t \le c. \tag{8.105}$$

Now, in view of (8.70), (8.81), (8.83), (8.93), (8.103), (8.104) and (8.105) we employ Lemma 8.15 and ([30], Proposition 3.35) to conclude

$$\bar{u}_p \to u \qquad \text{almost everywhere on } Q_T, \tag{8.106}$$

$$\bar{\theta}_p \to \theta \qquad \text{almost everywhere on } Q_T. \tag{8.107}$$

This completes the proof of Theorem 8.14. □

Summary of Section 8.4. Let us summarize that the a-priori estimates (8.58), (8.70), (8.81), and (8.91) allow us to conclude that there exist $u \in L^2(I; W_D^{1,2}(\Omega)) \cap L^\infty(Q_T)$ and $\theta \in L^2(I; W_D^{1,2}(\Omega)) \cap L^\infty(Q_T)$ such that, letting $p \to +\infty$ (along a selected subsequence), we have

$$\begin{aligned}
\bar{u}_p &\rightharpoonup u && \text{weakly in } L^2(I; W_D^{1,2}(\Omega)),\\
\bar{u}_p &\rightharpoonup u && \text{weakly star in } L^\infty(Q_T),\\
\bar{\theta}_p &\rightharpoonup \theta && \text{weakly in } L^2(I; W_D^{1,2}(\Omega)),\\
\bar{\theta}_p &\rightharpoonup \theta && \text{weakly star in } L^\infty(Q_T).
\end{aligned}$$

Further, we also have proven

$$\begin{aligned}
\bar{u}_p &\to u && \text{almost everywhere on } Q_T,\\
\bar{\theta}_p &\to \theta && \text{almost everywhere on } Q_T.
\end{aligned}$$

8.5 Passage to the limit for $p \to \infty$

The moisture equation. We define the sequence of functionals $\mathcal{X}_p \in L^2\left(I, W_D^{1,2}(\Omega)^*\right)$ such that

$$\begin{aligned}
\int_0^T \langle \mathcal{X}_p, \phi\rangle &= \int_0^T\int_\Omega \frac{b(\bar{u}_p(t))-b(\bar{u}_p(t-\tau))}{\tau}\phi(t)\,\mathrm{d}x\,\mathrm{d}t\\
&= -\int_0^T\int_\Omega k(\bar{\theta}_p(t-\tau),\bar{u}_p(t-\tau))\left[\nabla\bar{u}_p(t)+\boldsymbol{e}_z\right]\cdot\nabla\phi(t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{8.108}$$

The parabolic term in (8.108) can be rewritten, for $\phi \in L^2\left(I, W_D^{1,2}(\Omega)\right)$ and $\phi(T) = 0$, as

$$\begin{aligned}
&\int_0^T\int_\Omega \frac{b(\bar{u}_p(t))-b(\bar{u}_p(t-\tau))}{\tau}\phi(t)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad = -\int_0^{T-\tau}\int_\Omega \left[b(\bar{u}_p(t))-b(u(0))\right]\frac{\phi(t+\tau)-\phi(t)}{\tau}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{8.109}$$

Further, thanks to the energy estimate (8.82), we have

$$\int_0^T \langle \mathcal{X}_p, \phi\rangle \leq \|k(\bar{\theta}_p(t-\tau), \bar{u}_p(t-\tau))\|_{L^\infty(Q_T)} \|\nabla \bar{u}_p(t)\|_{L^2(Q_T)} \cdot \|\phi(t)\|_{L^2\left(I, W_D^{1,2}(\Omega)\right)}, \tag{8.110}$$

hence the functionals $\mathcal{X}_p$ are bounded in $L^2\left(I, W_D^{1,2}(\Omega)^*\right)$, therefore, for a selected subsequence

$$\mathcal{X}_p \rightharpoonup \mathcal{X} \quad \text{weakly in } L^2(I; W_D^{1,2}(\Omega)^*). \tag{8.111}$$

Hence, (8.111) implies $\mathcal{X} = \partial_t b(u)$. Further, for the elliptic term in (8.108), we have, thanks to (8.83), (8.106) and (8.107)

$$\lim_{\tau \to 0} \int_0^T \int_\Omega k(\bar{\theta}_p(t-\tau), \bar{u}_p(t-\tau)) \left[\nabla \bar{u}_p(t) + \boldsymbol{e}_z\right] \cdot \nabla \phi(t) \,\mathrm{d}x \,\mathrm{d}t$$
$$= \int_\Omega k(\theta(t), u(t)) \left[\nabla u(t) + \boldsymbol{e}_z\right] \cdot \nabla \phi(t) \,\mathrm{d}x \,\mathrm{d}t. \tag{8.112}$$

The heat equation. Let us note that above established convergences (8.83), (8.93), (8.106) and (8.107) are sufficient for repeating the same procedure, which has been presented in the paragraph before, also for the heat equation. Hence the functions u and θ are a weak solution of the problem (5.1)–(5.8). This completes the proof of the main result stated in Theorem 7.3.

9 Uniqueness of the solution

In this section we will prove the uniqueness of the solution, under some additional, but still physically relevant assumptions.

9.1 Additional assumptions

We present some additional assumptions:

(a) the hydraulic conductivity k does not depend on temperature, hence (4.26) becomes

$$k(u) = \kappa(u)\, \nu_0\, k_s; \tag{9.1}$$

(b) the thermal conductivity does not depend on temperature, hence (4.30) becomes

$$\lambda(u) = \Lambda_t(u); \tag{9.2}$$

(c) assume

$$|\boldsymbol{e}(u_1) - \boldsymbol{e}(u_2)|^2 \leq |b(u_1) - b(u_2)|\,(u_1 - u_2). \tag{9.3}$$

Taking into account the additional assumptions mentioned above we can write the moisture equation (4.39) in the form

$$\partial_t b(u) - \nabla \cdot (k(u)(\nabla u + \boldsymbol{e}_z)) = 0. \tag{9.4}$$

The heat equation (4.40) becomes

$$\frac{\partial[b(u)\theta + \varrho\theta]}{\partial t} - \nabla \cdot [\theta k(u)(\nabla u + \boldsymbol{e}_z)] - \nabla \cdot \lambda(u)\nabla\theta = 0. \tag{9.5}$$

9.2 Kirchhoff transformation

In order to eliminate the nonlinearities in the elliptic part of the moisture equation we introduce the Kirchhoff transformation (see e.g. [2]). Define the function $\beta : \mathbb{R} \to \mathbb{R}$, by

$$\beta(\xi) = \int_0^\xi \kappa(s)\mathrm{d}s.$$

Hence

$$\nabla\beta(u) = \nabla u \,\frac{\mathrm{d}\beta}{\mathrm{d}u} = \kappa(u)\nabla u.$$

Further let us introduce

$$\tilde{u} := \beta(u).$$

Provided κ is an increasing function with respect to u, which is physically relevant, we have

$$\beta^{-1}(\tilde{u}) = u. \tag{9.6}$$

Putting (9.6) in the equation (9.4) we obtain

$$\partial_t b(\beta^{-1}(\tilde{u})) - \nabla \cdot \left[k_s\,\nu_0\nabla\tilde{u} + k(\beta^{-1}(\tilde{u}))\boldsymbol{e}_z)\right] = 0. \tag{9.7}$$

Similarly using the transformation for the equation (9.4) we obtain

$$\frac{\partial[b(\beta^{-1}(\tilde{u}))\theta + \varrho\theta]}{\partial t} - \nabla\cdot[\theta\left(k_s\,\nu_0\nabla\tilde{u} + k(\beta^{-1}(\tilde{u})\boldsymbol{e}_z)\right)] - \nabla\cdot\left[\lambda(\beta^{-1}(\tilde{u}))\nabla\theta\right] = 0. \tag{9.8}$$

Without loss of generality let us assume that the physical constants $k_s = 1$, $\nu_0 = 1$ and $\varrho = 1$. Finally, in order to simplify mathematical formulations, let us introduce the following notation:

$$\begin{aligned}
\tilde{b}(\tilde{u}) &:= b(\beta^{-1}(\tilde{u})),\\
\boldsymbol{e}(\tilde{u}) &:= k(\beta^{-1}(\tilde{u}))\boldsymbol{e}_z,\\
\tilde{\lambda}(\tilde{u}) &:= \lambda(\beta^{-1}(\tilde{u})).
\end{aligned}$$

9.3 The transformed problem

Strong formulation of the transformed problem. In terms of the notation which has been introduced above, we introduce the following initial boundary problem

$$\partial_t \tilde{b}(\tilde{u}) - \nabla \cdot (\nabla \tilde{u} + \boldsymbol{e}(\tilde{u}))) = 0 \quad \text{in } Q_T, \tag{9.9}$$

$$\partial_t [\tilde{b}(\tilde{u})\theta + \theta] - \nabla \cdot [\theta \, (\nabla \tilde{u} + \boldsymbol{e}(\tilde{u})))] - \nabla \cdot \tilde{\lambda}(\tilde{u}) \nabla \theta = 0 \quad \text{in } Q_T. \tag{9.10}$$

$$\begin{aligned}
\tilde{u} &= 0 && \text{on } \Gamma_{DT}, && (9.11)\\
\theta &= 0 && \text{on } \Gamma_{DT}, && (9.12)\\
(\nabla \tilde{u} + \boldsymbol{e}_z) \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma_{NT}, && (9.13)\\
\nabla \theta \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma_{NT}, && (9.14)\\
\tilde{u}(\boldsymbol{x}, 0) &= \tilde{u}_0(\boldsymbol{x}) && \text{in } \Omega, && (9.15)\\
\theta(\boldsymbol{x}, 0) &= \theta_0(\boldsymbol{x}) && \text{in } \Omega. && (9.16)
\end{aligned}$$

In (9.9)–(9.16) $\tilde{u} : Q_T \to \mathbb{R}$ and $\theta : Q_T \to \mathbb{R}$ are the unknown functions.

Variational formulation of the transformed problem. The variational formulation of the system (9.9)–(9.16) with homogeneous boundary conditions reads as follows.

Definition 9.1 *A weak solution of* (9.9)–(9.16) *is a pair*

$$\begin{aligned}
\tilde{u} &\in L^2(I; W_D^{1,2}(\Omega)) \cap L^\infty(Q_T),\\
\theta &\in L^2(I; W_D^{1,2}(\Omega)) \cap L^\infty(Q_T),
\end{aligned}$$

which satisfies

$$-\int_{Q_T} \tilde{b}(\tilde{u}) \partial_t \phi \, \mathrm{d}x\mathrm{d}t + \int_{Q_T} (\nabla \tilde{u} + \boldsymbol{e}(\tilde{u})) \cdot \nabla \phi \, \mathrm{d}x\mathrm{d}t = \int_\Omega \tilde{b}(\tilde{u}_0) \phi(0) \, \mathrm{d}x \tag{9.17}$$

holds for any $\phi \in L^2(I; W_D^{1,2}(\Omega)) \cap W^{1,1}(I; L^1(\Omega))$ *and* $\phi(T) = 0$*;*

$$\begin{aligned}
-\int_{Q_T} \left(\tilde{b}(\tilde{u})\theta + \varrho\theta \right) \partial_t \psi \, \mathrm{d}x\mathrm{d}t + \int_{Q_T} \tilde{\lambda}(\tilde{u}) \nabla \theta \cdot \nabla \psi \, \mathrm{d}x\mathrm{d}t&\\
+ \int_{Q_T} (\theta \, (\nabla \tilde{u} + \boldsymbol{e}(\tilde{u}))) \cdot \nabla \psi \, \mathrm{d}x\mathrm{d}t = \int_\Omega \left(\tilde{b}(\tilde{u}_0))\theta_0 + \varrho\theta_0 \right) \psi(0) \, \mathrm{d}x&
\end{aligned} \tag{9.18}$$

holds for any $\psi \in L^2(I; W_D^{1,2}(\Omega)) \cap W^{1,1}(I; L^1(\Omega))$ *and* $\psi(T) = 0$.

Remark 9.2 *([31], Remark 1.19) There exists* $\partial_t \tilde{b}(\tilde{u}) \in L^2(I; W_D^{1,2}(\Omega)^*)$ *and*

$$\int_{Q_T} \left[\tilde{b}(\tilde{u}_0) - \tilde{b}(\tilde{u}) \right] \partial_t \phi \, \mathrm{d}x\mathrm{d}t = \int_0^T \left\langle \partial_t \tilde{b}(\tilde{u}), \phi \right\rangle \, \mathrm{d}t$$

holds for any $\phi \in L^2(I;W_D^{1,2}(\Omega)) \cap W^{1,1}(I;L^1(\Omega))$ *and* $\phi(T)=0$*, then in the place of* (7.1) *we can have*

$$\int_0^T \left\langle \partial_t \tilde{b}(\tilde{u}), \phi \right\rangle \mathrm{d}t + \int_{Q_T} (\nabla \tilde{u} + \boldsymbol{e}_z) \cdot \nabla \phi \,\mathrm{d}x\mathrm{d}t = 0$$

for any $\phi \in L^2(I;W_D^{1,2}(\Omega))$*;*

Similarly, there exists $\partial_t \left(\tilde{b}(\tilde{u})\theta + \varrho\theta\right) \in L^2(I;W_D^{1,2}(\Omega)^*)$ *and*

$$\int_{Q_T} \left[(\tilde{b}(\tilde{u}_0)\theta_0 + \varrho\theta_0) - (\tilde{b}(\tilde{u})\theta - \varrho\theta)\right] \partial_t\phi \,\mathrm{d}x\mathrm{d}t = \int_0^T \left\langle \partial_t \left(\tilde{b}(\tilde{u})\theta + \varrho\theta\right), \psi \right\rangle \mathrm{d}t$$

holds for any $\psi \in L^2(I;W_D^{1,2}(\Omega)) \cap W^{1,1}(I;L^1(\Omega))$ *and* $\phi(T)=0$*, then in the place of* (7.2) *we have*

$$\int_0^T \left\langle \partial_t \left(\tilde{b}(\tilde{u})\theta + \varrho\theta\right), \psi \right\rangle \mathrm{d}t + \int_{Q_T} \tilde{\lambda}(\tilde{u}) \nabla\theta \cdot \nabla\psi \,\mathrm{d}x\mathrm{d}t + \int_{Q_T} (\theta \; (\nabla\tilde{u} + \boldsymbol{e}_z)) \cdot \nabla\psi \,\mathrm{d}x\mathrm{d}t = 0$$

for any $\phi \in L^2(I;W_D^{1,2}(\Omega))$*.*

Theorem 9.3 *(Existence of the solution to the transformed problem). Let the assumptions* (i)–(iv) *and* (a)–(c) *be satisfied, then there exists a solution to* (9.17)–(9.18)*.*

Proof. The proof can be realized in the way as described in Section 8. □

Theorem 9.4 *(Uniqueness of the solution to the moisture equation) There exists a unique solution to* (9.17)*.*

Proof. We follow [2]. Suppose there exist two solutions $\tilde{u}_1, \tilde{u}_2$ to (9.17). Hence

$$\int_0^T \left\langle \partial_t \tilde{b}(\tilde{u}_1) - \partial_t \tilde{b}(\tilde{u}_2), \phi \right\rangle \mathrm{d}t + \int_{Q_T} (\nabla(\tilde{u}_1 - \tilde{u}_2) + \boldsymbol{e}(\tilde{u}_1) - \boldsymbol{e}(\tilde{u}_2)) \cdot \nabla\phi \,\mathrm{d}x\mathrm{d}t = 0 \tag{9.19}$$

for all $\phi \in L^2(I;W_D^{1,2}(\Omega))$. Introduce the function $\beta \in L^2(I;W_D^{1,2}(\Omega)^*)$ such that

$$\beta := \tilde{b}(\tilde{u}_1) - \tilde{b}(\tilde{u}_2).$$

Now Lax-Milgram's theorem yields the existence of the unique function $w_u \in \phi \in L^2(I;W_D^{1,2}(\Omega))$ such that

$$\int_0^T \int_\Omega \nabla w_u \cdot \nabla\Phi \,\mathrm{d}x\mathrm{d}t = \int_0^T \langle \beta, \Phi \rangle \,\mathrm{d}t. \tag{9.20}$$

From [2] we have

$$\frac{1}{2}\int_\Omega \nabla w_u(t)\cdot\nabla w_u(t) = \int_0^t \langle \partial_s\beta, w_u\rangle \,\mathrm{d}s \tag{9.21}$$

for all $t\in I$. Introduce the function $\chi(t)_{[0,\tau]}$ such that

$$\chi(t)_{[0,\tau]} = \left\{\begin{array}{lll} 0 & \text{if} & t\notin[0;\tau], \\ 1 & \text{if} & t\in[0;\tau], \end{array}\right\} \forall\tau\in I. \tag{9.22}$$

Now let us set $\phi = \chi(t)_{[0,\tau]}w_u$ as a test function in (9.19) to obtain

$$\int_0^\tau \left\langle \partial_t\tilde{b}(\tilde{u}_1) - \partial_t\tilde{b}(\tilde{u}_2), w_u\right\rangle \mathrm{d}t + \int_0^\tau\int_\Omega (\nabla(\tilde{u}_1-\tilde{u}_2) + \boldsymbol{e}(\tilde{u}_1) - \boldsymbol{e}(\tilde{u}_2))\cdot\nabla w_u \,\mathrm{d}x\mathrm{d}t = 0, \tag{9.23}$$

which becomes

$$\int_0^\tau \langle\partial_t\beta, w_u\rangle \,\mathrm{d}t + \int_0^\tau\int_\Omega (\nabla(\tilde{u}_1-\tilde{u}_2) + \boldsymbol{e}(\tilde{u}_1) - \boldsymbol{e}(\tilde{u}_2))\cdot\nabla w_u \,\mathrm{d}x\mathrm{d}t = 0. \tag{9.24}$$

Using (9.21) we have

$$\frac{1}{2}\int_\Omega \nabla w_u(\tau)\cdot\nabla w_u(\tau)\,\mathrm{d}x + \int_0^\tau\int_\Omega (\nabla(\tilde{u}_1-\tilde{u}_2) + \boldsymbol{e}(\tilde{u}_1) - \boldsymbol{e}(\tilde{u}_2))\cdot\nabla w_u \,\mathrm{d}x\mathrm{d}t = 0. \tag{9.25}$$

Further we set $\Phi = \chi(t)_{[0,\tau]}(\tilde{u}_1-\tilde{u}_2)$ as a test function in (9.20) to obtain

$$\int_0^\tau\int_\Omega \nabla w_u\cdot\nabla(\tilde{u}_1-\tilde{u}_2)\,\mathrm{d}x\mathrm{d}t = \int_0^\tau \langle\beta, (\tilde{u}_1-\tilde{u}_2)\rangle\,\mathrm{d}t. \tag{9.26}$$

Combining (9.25) and (9.26) we have

$$\frac{1}{2}\int_\Omega \nabla w_u(\tau)\cdot\nabla w_u(\tau)\,\mathrm{d}x + \int_0^\tau \langle\beta, (u_1-u_2)\rangle\,\mathrm{d}t + \int_0^\tau\int_\Omega [\boldsymbol{e}(\tilde{u}_1) - \boldsymbol{e}(\tilde{u}_2)]\cdot\nabla w_u\,\mathrm{d}x\mathrm{d}t = 0. \tag{9.27}$$

Let us now focus on the third term in (9.27), using Young's inequality with ϵ we can write

$$\int_0^\tau\int_\Omega [\boldsymbol{e}(\tilde{u}_1) - \boldsymbol{e}(\tilde{u}_2)]\cdot\nabla w_u\,\mathrm{d}x\mathrm{d}t \le \epsilon\int_0^\tau\int_\Omega |\boldsymbol{e}(\tilde{u}_1) - \boldsymbol{e}(\tilde{u}_2)|^2\,\mathrm{d}x\mathrm{d}t + \int_0^\tau c(\epsilon)\,|\nabla w_u|^2\,\mathrm{d}x\mathrm{d}t. \tag{9.28}$$

Moreover, using (c), we can write

$$\frac{1}{2}\int_\Omega |\nabla w_u(\tau)|^2\,\mathrm{d}x + (1-\epsilon)\int_0^\tau \langle\beta, (u_1-u_2)\rangle\,\mathrm{d}t \le c(\epsilon)\int_0^\tau |\nabla w_u|^2\,\mathrm{d}x\mathrm{d}t \tag{9.29}$$

Since
$$\int_0^\tau \langle \beta, (u_1 - u_2)\rangle \,\mathrm{d}t = \int_0^\tau \langle b(u_1) - b(u_2), (u_1 - u_2)\rangle \,\mathrm{d}t \geq 0,$$
taking ϵ sufficiently small, (9.29) becomes
$$\frac{1}{2}\int_\Omega |\nabla w_u(\tau)|^2 \,\mathrm{d}x \leq \int_0^\tau c(\epsilon)\, |\nabla w_u|^2 \,\mathrm{d}x\mathrm{d}t. \tag{9.30}$$
Now Gronwall's lemma yields $w_u(\tau) = 0$ almost everywhere.

Lax Milgram's theorem yields the existence of a unique function w_u satisfying (9.20), now we proved that this function equals zero almost everywhere. This implies that $\beta = 0$ almost everywhere. More over since b is a monotone function,
$$\beta = b(u_1) - b(u_2) = 0$$
implies
$$u_1 = u_2 \quad \text{a.e. in } Q_T.$$
The proof of Theorem 9.4 is complete. $\square$

Theorem 9.5 *(Uniqueness of the solution to the heat equation) There exists a unique solution to* (9.18)*, in the class of weak solutions such that $\partial_t b(u) \in L^2(Q_T)$ and $\partial_t \theta \in L^2(Q_T)$.*

Proof. Suppose there exist two solutions θ_1, θ_2 to (9.18). We have
$$\int_0^T \int_\Omega \partial_t \left(\tilde{b}(\tilde{u})(\theta_1 - \theta_2) + (\theta_1 - \theta_2)\right) \psi \,\mathrm{d}x\mathrm{d}t + \int_{Q_T} \tilde{\lambda}(\tilde{u})\nabla(\theta_1 - \theta_2)\cdot\nabla\psi \,\mathrm{d}x\mathrm{d}t$$
$$+ \int_{Q_T} ((\theta_1 - \theta_2)\ (\nabla\tilde{u} + \boldsymbol{e}(\tilde{u})))\cdot\nabla\psi \,\mathrm{d}x\mathrm{d}t = 0. \tag{9.31}$$

We set $\psi = \theta_1 - \theta_2$ as a test function in (9.31) to obtain
$$\int_0^T \int_\Omega \left(\partial_t \left[\tilde{b}(\tilde{u})(\theta_1 - \theta_2)\right] (\theta_1 - \theta_2)\right\rangle + \langle \partial_t[\theta_1 - \theta_2](\theta_1 - \theta_2))\,\mathrm{d}x\mathrm{d}t$$
$$+ \int_{Q_T} \tilde{\lambda}(\tilde{u})\, |\nabla(\theta_1 - \theta_2)|^2 \,\mathrm{d}x\mathrm{d}t$$
$$+ \int_{Q_T} ((\theta_1 - \theta_2)\ (\nabla\tilde{u} + \boldsymbol{e}(\tilde{u})))\cdot\nabla(\theta_1 - \theta_2)\,\mathrm{d}x\mathrm{d}t = 0. \tag{9.32}$$

For the first term in (9.32) we can write

$$\int_0^T \int_\Omega \partial_t \left[\tilde{b}(\tilde{u})(\theta_1 - \theta_2)\right](\theta_1 - \theta_2) = \int_{Q_T} \partial_t \tilde{b}(\tilde{u})(\theta_1 - \theta_2)^2 \,\mathrm{d}x\mathrm{d}t$$
$$+ \int_{Q_T} \tilde{b}(\tilde{u})\partial_t[\theta_1 - \theta_2](\theta_1 - \theta_2)\,\mathrm{d}x\mathrm{d}t$$
$$= \frac{1}{2}\int_{Q_T} \partial_t \tilde{b}(\tilde{u})(\theta_1 - \theta_2)^2 \,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_{Q_T} \partial_t[\tilde{b}(\tilde{u})(\theta_1 - \theta_2)^2]\,\mathrm{d}x\mathrm{d}t$$
$$= \frac{1}{2}\int_\Omega \tilde{b}(\tilde{u}(T))(\theta_1(T) - \theta_2(T))^2 \,\mathrm{d}x + \frac{1}{2}\int_{Q_T} \partial_t[\tilde{b}(\tilde{u})](\theta_1 - \theta_2)^2 \,\mathrm{d}x\mathrm{d}t. \quad (9.33)$$

Hence (9.32) becomes

$$\frac{1}{2}\int_\Omega \tilde{b}(\tilde{u}(T))(\theta_1(T) - \theta_2(T))^2 \,\mathrm{d}x + \frac{1}{2}\int_{Q_T} \partial_t[\tilde{b}(\tilde{u})](\theta_1 - \theta_2)^2 \,\mathrm{d}x\mathrm{d}t$$
$$+ \frac{1}{2}\int_{Q_T} \partial_t\left[(\theta_1 - \theta_2)^2\right]\,\mathrm{d}x\mathrm{d}t + \int_{Q_T} \tilde{\lambda}(\tilde{u})\left|\nabla(\theta_1 - \theta_2)\right|^2 \,\mathrm{d}x\mathrm{d}t$$
$$+ \frac{1}{2}\int_{Q_T} (\nabla\tilde{u} + \boldsymbol{e}(\tilde{u}))\cdot\nabla(\theta_1 - \theta_2)^2 \,\mathrm{d}x\mathrm{d}t = 0. \quad (9.34)$$

Now we set $\phi = \frac{1}{2}(\theta_1 - \theta_2)^2$ as a test function in (9.17) to obtain

$$\frac{1}{2}\int_{Q_T} \partial_t\,[\tilde{b}(\tilde{u})](\theta_1 - \theta_2)^2 \,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_{Q_T} (\nabla\tilde{u} + \boldsymbol{e}(\tilde{u}))\cdot\nabla(\theta_1 - \theta_2)^2 \,\mathrm{d}x\mathrm{d}t = 0 \quad (9.35)$$

and subtracting (9.35) from (9.34) we obtain

$$\frac{1}{2}\int_\Omega \tilde{b}(\tilde{u}(T))(\theta_1(T) - \theta_2(T))^2 \,\mathrm{d}x + \frac{1}{2}\int_{Q_T} \partial_t\left[(\theta_1 - \theta_2)^2\right]\,\mathrm{d}x\mathrm{d}t$$
$$+ \int_{Q_T} \tilde{\lambda}(\tilde{u})\left|\nabla(\theta_1 - \theta_2)\right|^2 \,\mathrm{d}x\mathrm{d}t = 0. \quad (9.36)$$

Since $\tilde{b}$ and $\tilde{\lambda}$ are positive functions we can write

$$\frac{1}{2}\int_\Omega (\theta_1 - \theta_2)^2 \,\mathrm{d}x \leq 0 \quad (9.37)$$

and hence $\theta_1 = \theta_2$ a.e. in Q_T. This completes the proof of Theorem 9.5. □

10 Conclusion

In this section of the work, we have proven the existence of the weak solution

$$u \in L^2(I; W_D^{1,2}(\Omega)) \cap L^\infty(Q_T),$$
$$\theta \in L^2(I; W_D^{1,2}(\Omega)) \cap L^\infty(Q_T)$$

to the problem (5.1)-(5.8) describing coupled moisture transport and heat transfer through a partially saturated porous media. The model is describing porous media performance by means of a single porosity approach described in Section 3.2. In order to avoid unnecessary technicalities we have analyzed in this section a model with homogenous boundary conditions of a Dirichlet and Neumann type. The presented analysis can be straightforwardly extended to a setting with general boundary conditions.

We have also shown the uniqueness of the obtained solution, prescribing some additional, but still physically relevant, assumptions on the transport coefficients.

Part IV

Mathematical analysis of the dual porosity model

11 Dual porosity approach

The single porosity model which has been presented in the previous part of the text is the most frequently used approach. However, some physical and engineering issues may require different approach, i.e. dual porosity model (see Section 3.2). The dual porosity model consists of two sets of equations, representing two overlapping continua corresponding to the fracture system and matrix system, respectively. The system is completed by the coupling terms providing the communication between these two continua. The structure of the analysis is realized in the same manner, therefore, in the text we will focus our attention on the differences since the main ideas of the analysis remain the same.

11.1 Strong formulation

Let Ω be a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary Γ. Let $T \in (0, \infty)$ be fixed throughout the paper, $I = (0, T)$ and $Q_T = \Omega \times I$ denotes the space-time cylinder, $\Gamma_T = \Gamma \times I$. We introduce the following dual porosity model ($i = 1, 2$)

$$\begin{aligned} \partial_t b_i(u_i) &= \nabla \cdot \left(k_i(\theta_i, u_i)\left(\nabla u_i + \boldsymbol{e}_z\right)\right) \\ &\qquad + \omega_i \alpha_\omega(u_1, u_2)(u_1 - u_2) \qquad \text{in } Q_T, \qquad (11.1) \\ \partial_t \left[b_i(u_i)\theta_i + \varrho_i \theta_i\right] &= \nabla \cdot (\lambda_i(\theta_i, u_i)\nabla\theta_i) \\ &\qquad + \nabla \cdot \left(\theta_i k_i(\theta_i, u_i)\left(\nabla u_i + \boldsymbol{e}_z\right)\right) \\ &\qquad\qquad + F_i(u_1, u_2, \theta_1, \theta_2) \qquad \text{in } Q_T, \qquad (11.2) \end{aligned}$$

completed by boundary and initial conditions

$$\begin{aligned} u_i &= 0 && \text{on } \Gamma_T, && (11.3) \\ \theta_i &= 0 && \text{on } \Gamma_T, && (11.4) \\ u_i(x, 0) &= u_{i0}(x) && \text{in } \Omega, && (11.5) \\ \theta_i(x, 0) &= \theta_{i0}(x) && \text{in } \Omega. && (11.6) \end{aligned}$$

In (11.1)–(11.6), $u_i : Q_T \to \mathbb{R}$ and $\theta_i : Q_T \to \mathbb{R}$ are the unknown functions representing pressure head and temperature. Further $k_i : \mathbb{R}^2 \to \mathbb{R}$, $b_i : \mathbb{R} \to \mathbb{R}$, $\lambda_i : \mathbb{R}^2 \to \mathbb{R}$, $u_{i0} : \Omega \to \mathbb{R}$ and $\theta_{i0} : \Omega \to \mathbb{R}$ are given functions, and $\boldsymbol{e}_z$ is the vertical unit vector. Further $\omega_1 = 1/\omega$, $\omega_2 = 1/(\omega - 1)$, $\alpha_i > 0$, $\beta_i > 0$, ϱ_i and $\omega \in (0, 1)$ are given material constants. Finally $\alpha_\omega : \mathbb{R}^2 \to \mathbb{R}$ is a first order mass transfer coefficient function for water, $F_i : \mathbb{R}^4 \to \mathbb{R}$ represents the exchange term for heat exchange between two components.

Remark 11.1 *(Sobolev space $W_0^{1,p}(\Omega)$). By the symbol $W_0^{1,p}(\Omega)$, with some $p \geq 1$, we denote the Sobolev space with zero trace on the boundary Γ. (See C.5.)*

11.2 Structure and data properties

According to the physical background we present the following assumptions on functions in (11.1)–(11.6):

(I) b_i is a positive lipschitz continuous strictly monotone function such that

$$\begin{aligned} &0 < b_i(\xi) \leq b_2 < +\infty && \forall \xi \in \mathbb{R} \quad (b_2 = \text{const}), \\ &(b_i(\xi_1) - b_i(\xi_2))\,(\xi_1 - \xi_2) > 0 && \forall \xi_1, \xi_2 \in \mathbb{R},\ \xi_1 \neq \xi_2; \end{aligned}$$

(II) α_ω, k_i and λ_i are positive continuous functions;

(III) ϱ_i is a real positive constant and $\boldsymbol{e}_z$ is a vertical unit vector;

(IV) $F_i(\xi_1, \xi_2, \zeta_1, \zeta_2)$ is continuous on ξ_1, ξ_2 and lipschitz continuous with respect to ζ_1, ζ_2;

(V) $u_{i0}, \theta_{i0} \in L^\infty(\Omega)$.

Remark 11.2 *Similarly as in Lemma 6.5,* (I) *implies, that there exist positive constants b_1 and b_2 such that*

$$b_1 < b_i(g) \leq b_2 < +\infty, \tag{11.7}$$

for all $g \in L^\infty(\Omega)$.

Further (II) *implies that there exist positive constants $k_1, k_2, \lambda_1, \lambda_2, \alpha_1$ and α_2 such that*

$$0 < k_1 < k_i(g_1, g_2) \leq k_2 < +\infty, \tag{11.8}$$

$$0 < \lambda_1 < \lambda_i(g_1, g_2) \leq \lambda_2 < +\infty, \tag{11.9}$$

$$0 < \alpha_1 < \alpha_\omega(g_1, g_2) \leq \alpha_2 < +\infty \tag{11.10}$$

for all $g_1, g_2 \in L^\infty(\Omega)$.

11.3 Weak formulation

We now formulate the problem (11.1)–(11.6) in a variational sense.

Definition 11.3 *By a weak solution of* (11.1)–(11.6) *we mean a pair* $[\boldsymbol{u}, \boldsymbol{\theta}]$, $\boldsymbol{u} = (u_1, u_2)$, $\boldsymbol{\theta} = (\theta_1, \theta_2)$, *such that*

$$u_i \in L^2(I; W_0^{1,2}(\Omega)) \cap L^\infty(Q_T),$$
$$\theta_i \in L^2(I; W_0^{1,2}(\Omega)) \cap L^\infty(Q_T),$$

which satisfy

$$-\int_{Q_T} b_i(u_i)\partial_t\phi_i \,\mathrm{d}x\mathrm{d}t + \int_{Q_T} k_i(\theta_i, u_i)\left(\nabla u_i + \boldsymbol{e}_z\right)\cdot\nabla\phi_i \,\mathrm{d}x\mathrm{d}t + \int_{Q_T} \omega_i\alpha_\omega(u_1, u_2)(u_1 - u_2)\phi_i \,\mathrm{d}x\mathrm{d}t = \int_\Omega b_i(u_{i0})\phi_i(0)\,\mathrm{d}x \quad (11.11)$$

for any $\phi_i \in L^2(I; W_0^{1,2}(\Omega)) \cap W^{1,1}(I; L^1(\Omega))$ *and* $\phi_i(T) = 0$, i=1,2*;*

$$-\int_{Q_T} \left(b_i(u_i)\theta + \varrho_i\theta_i\right)\partial_t\psi_i \,\mathrm{d}x\mathrm{d}t + \int_{Q_T} \lambda_i(\theta_i, u_i)\nabla\theta_i\cdot\nabla\psi_i \,\mathrm{d}x\mathrm{d}t + \int_{Q_T} \left(\theta_i\, k_i(\theta_i, u_i)\left(\nabla u_i + \boldsymbol{e}_z\right)\right)\cdot\nabla\psi_i \,\mathrm{d}x\mathrm{d}t = \int_{Q_T} F_i(u_1, u_2, \theta_1, \theta_2)\psi_i \,\mathrm{d}x\mathrm{d}t + \int_\Omega \left(b_i(u_{i0})\theta_{i0} + \varrho_i\theta_{i0}\right)\psi_i(0)\,\mathrm{d}x \quad (11.12)$$

for any $\psi_i \in L^2(I; W_0^{1,2}(\Omega)) \cap W^{1,1}(I; L^1(\Omega))$ *and* $\psi_i(T) = 0$, i=1,2.

Theorem 11.4 *(Existence of the weak solution) Let the assumptions* (I)–(V) *be satisfied. Then there exists at least one weak solution of the system* (11.1)–(11.6).

In the following text we will deal with the proof of Theorem 11.4 using the similar procedure as in Section 8.

11.4 Steady problem

Fix $p \in \mathbb{N}$ and set $\tau := T/p$ be a time step. We use a semi-implicit time discretization, further define functions $[\boldsymbol{u}_p^n, \boldsymbol{\theta}_p^n]$, $\boldsymbol{u}_p^n = (u_{1p}^n, u_{2p}^n)$, $\boldsymbol{\theta}_p^n = (\theta_{1p}^n, \theta_{2p}^n)$ as solutions of the steady problem in each time step.

Problem 11.5 *Find a pair* $[\boldsymbol{u}_p^n, \boldsymbol{\theta}_p^n]$ *such that*

$$u_{ip}^n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega),$$
$$\theta_{ip}^n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega),$$

$$\int_\Omega \frac{b_i(u_{ip}^n)-b_i(u_{ip}^{n-1})}{\tau}\phi_i\,\mathrm{d}x+\int_\Omega k_i(\theta_{ip}^{n-1},u_{ip}^{n-1})\left(\nabla u_{ip}^n+\boldsymbol{e}_z\right)\cdot\nabla\phi_i\,\mathrm{d}x$$
$$+\int_\Omega \omega_i\alpha_\omega(u_1^{n-1},u_2^{n-1})(u_1^n-u_2^n)\phi_i\,\mathrm{d}x=0 \quad (11.13)$$

holds for any $\phi_i\in W_0^{1,2}(\Omega)$, *i*=1,2;

$$\int_\Omega \frac{b_i(u_{ip}^n)\theta_{ip}^n-b_i(u_{ip}^{n-1})\theta_{ip}^{n-1}}{\tau}\psi_i\,\mathrm{d}x+\varrho_i\int_\Omega \frac{\theta_{ip}^n-\theta_{ip}^{n-1}}{\tau}\psi_i\,\mathrm{d}x+\int_\Omega \lambda_i(\theta_{ip}^{n-1},u_{ip}^{n-1})\nabla\theta_{ip}^n\cdot\nabla\psi_i\mathrm{d}x$$
$$+\int_\Omega \theta_{ip}^n k_i(\theta_{ip}^{n-1},u_{ip}^{n-1})\left(\nabla u_{ip}^n+\boldsymbol{e}_z\right)\cdot\nabla\psi_i\mathrm{d}x=\int_\Omega F_i(u_1^n,u_2^n,\theta_1^n,\theta_2^n)\psi_i\,\mathrm{d}x \quad (11.14)$$

holds for any $\psi_i\in W_0^{1,2}(\Omega)$, *i*=1,2.

11.4.1 Existence and regularity of the approximate solutions

Theorem 11.6 *[Existence of the solution to* (11.13)*] Let* u_{ip}^{n-1} *and* $\theta_{ip}^{n-1}\in L^\infty(\Omega)$, *be given and the assumptions* (I)–(V) *be satisfied. Then there exist* $u_{ip}^n\in W_0^{1,2}(\Omega)$, *i*=1,2, *the solution to the discrete problem* (11.13).

Proof. Let us introduce $\boldsymbol{\phi}:=[\phi_1,\phi_2]\in W_0^{1,2}(\Omega)^2$. Define the functional $\boldsymbol{\mu}_u\in[W_0^{1,2}(\Omega)^2]^*$ by

$$\langle\boldsymbol{\mu}_u,\boldsymbol{\phi}\rangle=\sum_{i=1}^2\frac{1}{\tau}\int_\Omega b_i(u_{ip}^{n-1})\phi_i\,\mathrm{d}x-\sum_{i=1}^2\int_\Omega k_i(\theta_{ip}^{n-1},u_{ip}^{n-1})\boldsymbol{e}_z\cdot\nabla\phi_i\,\mathrm{d}x \quad (11.15)$$

for all $\boldsymbol{\phi}\in W_0^{1,2}(\Omega)^2$.

Further, define the operator $\mathcal{A}_u:W_0^{1,2}(\Omega)^2\to[W_0^{1,2}(\Omega)^2]^*$ by the equation

$$\langle\mathcal{A}_u(\boldsymbol{u}_p^n),\boldsymbol{\phi}\rangle=\sum_{i=1}^2\int_\Omega k_i(\theta_{ip}^{n-1},u_{ip}^{n-1})\nabla u_{ip}^n\cdot\nabla\phi_i\,\mathrm{d}x+\sum_{i=1}^2\frac{1}{\tau}\int_\Omega b_i(u_{ip}^n)\phi_i\,\mathrm{d}x$$
$$+\sum_{i=1}^2\int_\Omega \omega_i\alpha_\omega(u_1^{n-1},u_2^{n-1})(u_1^n-u_2^n)\phi_i\,\mathrm{d}x \quad (11.16)$$

for all $\boldsymbol{\phi}\in W_0^{1,2}(\Omega)^2$.

The operator $\mathcal{A}_u$ is monotone in the main part. Further, for any $\boldsymbol{u}_p\in W_0^{1,2}(\Omega)^2$ we have, taking into account (I)–(III),

$$\langle\mathcal{A}_u(\boldsymbol{u}_p^n),\boldsymbol{\phi}\rangle\le c_1\|\boldsymbol{u}_p^n\|_{W_0^{1,2}(\Omega)^2}\|\boldsymbol{\phi}\|_{W_D^{1,2}(\Omega)^2}+c_2\|\boldsymbol{\phi}\|_{W_0^{1,2}(\Omega)^2},$$
$$\le\|\boldsymbol{\phi}\|_{W_D^{1,2}(\Omega)^2}\left(c_1\|\boldsymbol{u}_p^n\|_{W_0^{1,2}(\Omega)^2}+c_2\right).$$

Therefore, we have

$$\|\mathcal{A}_u(\boldsymbol{u}_p^n)\|_{[W_0^{1,2}(\Omega)^2]^*} = \sup_{\boldsymbol{\phi}\in W_0^{1,2}(\Omega)^2, \|\boldsymbol{\phi}\|\neq 0} \frac{|\langle \mathcal{A}_u(\boldsymbol{u}_p^n), \boldsymbol{\phi}\rangle|}{\|\boldsymbol{\phi}\|_{W_0^{1,2}(\Omega)^2}} \leq c_1\|\boldsymbol{u}_p^n\|_{W_0^{1,2}(\Omega)^2} + c_2. \quad (11.17)$$

And further, applying Young's inequality, we derive

$$\langle \mathcal{A}_u(\boldsymbol{u}_p^n), \boldsymbol{u}_p^n\rangle \geq c_1\|\boldsymbol{u}_p^n\|^2_{W_0^{1,2}(\Omega)^2} - c_2. \quad (11.18)$$

Now, we can conclude from (11.17) and (11.18), with the same arguments as in the Section 8.2.1, that above shown properties of the operator yield together with ([47], Theorem 3.3.42) the existence of the solution $\boldsymbol{u}_p \in W_0^{1,2}(\Omega)^2$ to the problem (11.13). □

Theorem 11.7 *($W_0^{1,r}$-regularity of the solution to* (11.13)*) Let $\boldsymbol{u}_p^n \in W_0^{1,2}(\Omega)^2$ be the weak solution to the discrete problem* (11.13)*. Then $\boldsymbol{u}_p^n \in W_0^{1,r}(\Omega)^2$ with some $r > 2$.*

Proof. The proof of Theorem 11.7 can be realized in the same way as the proof of Theorem 8.6. □

Theorem 11.8 *[Existence of the solution to* (11.14)*] Let u_{ip}^{n-1} and $\theta_{ip}^{n-1} \in L^\infty(\Omega)$, be given, let $u_{ip}^n \in W_0^{1,r}(\Omega)$ and the assumptions* (I)–(V) *be satisfied. Then there exist $\theta_{ip}^n \in W_0^{1,2}(\Omega)$, i=1,2, the solution to the discrete problem* (11.14)*.*

Proof. We denote $\boldsymbol{\psi} := [\psi_1, \psi_2] \in W_0^{1,2}(\Omega)^2$ and we define the functional $\boldsymbol{\mu}_\theta \in [W_0^{1,2}(\Omega)^2]^*$ by

$$\langle \boldsymbol{\mu}_\theta, \boldsymbol{\psi}\rangle = \sum_{i=1}^{2} \frac{1}{\tau}\int_\Omega b_i(u_{ip}^{n-1})\theta_{ip}^{n-1}\psi_i \,\mathrm{d}x - \sum_{i=1}^{2}\int_\Omega \varrho\theta_{ip}^{n-1}\psi_i \,\mathrm{d}x \quad (11.19)$$

for all $\boldsymbol{\psi} \in W_0^{1,2}(\Omega)^2$.

Further, define the operator $\mathcal{A}_\theta : W_0^{1,2}(\Omega)^2 \to [W_0^{1,2}(\Omega)^2]^*$ by the equation

$$\begin{aligned}\langle \mathcal{A}_\theta(\boldsymbol{u}_p^n), \boldsymbol{\psi}\rangle = &\sum_{i=1}^{2} \frac{1}{\tau}\int_\Omega \left[b_i(u_{ip}^n) + \varrho\right]\psi_i \,\mathrm{d}x + \sum_{i=1}^{2}\int_\Omega \lambda_i(\theta_{ip}^{n-1}, u_{ip}^{n-1})\nabla\theta_{ip}^n \cdot \nabla\psi_i \\ &+ \sum_{i=1}^{2}\int_\Omega \theta_{ip}^n k_i(\theta_{ip}^{n-1}, u_{ip}^{n-1})\left(\nabla u_{ip}^n + \boldsymbol{e}_z\right)\cdot\nabla\psi_i \,\mathrm{d}x \\ &- \sum_{i=1}^{2}\int_\Omega F_i(u_1^n, u_2^n, \theta_1^n, \theta_2^n)\psi_i \,\mathrm{d}x \quad (11.20)\end{aligned}$$

for all $\boldsymbol{\psi} \in W_0^{1,2}(\Omega)^2$.

In view of (I)–(IV), the operator is bounded and monotone in the main part. Further, we may write

$$\langle \mathcal{A}_\theta(\boldsymbol{\theta}_p^n), \boldsymbol{\theta}_p^n \rangle \geq c_1 \|\boldsymbol{\theta}_p^n\|^2_{W_0^{1,2}(\Omega)^2} - \sum_{i=1}^{2} \int_\Omega c_2 \left(1 + |\theta_{1p}^n| + |\theta_{2p}^n|\right) \theta_{ip}^n \, \mathrm{d}x. \tag{11.21}$$

Hence

$$\langle \mathcal{A}_\theta(\boldsymbol{\theta}_p^n), \boldsymbol{\theta}_p^n \rangle \geq c_1 \|\boldsymbol{\theta}_p^n\|^2_{W_0^{1,2}(\Omega)^2} - \int_\Omega c_2 \left(\theta_{1p}^n + \theta_{2p}^n + 2|\theta_{1p}^n \theta_{2p}^n| + |\theta_{1p}^n|^2 + |\theta_{2p}^n|^2\right) \mathrm{d}x. \tag{11.22}$$

Applying Young's inequality on the second term on the right-hand side of (11.22), we arrive at

$$\langle \mathcal{A}_\theta(\boldsymbol{\theta}_p^n), \boldsymbol{\theta}_p^n \rangle \geq \left(c_1 - c_2(\epsilon)\right) \|\boldsymbol{\theta}_p^n\|^2_{W_0^{1,2}(\Omega)^2}. \tag{11.23}$$

Therefore, we may conclude, that the operator $\mathcal{A}_\theta$ is coercive. Hence, the properties of the operator yield the existence of the weak solution $\theta_{ip}^n \in W_0^{1,2}\Omega$ to the steady problem (11.14) with the same arguments as in Theorem 11.6. □

Theorem 11.9 *($W_0^{1,s}$-regularity of the solution to* (11.14)*) Let $\boldsymbol{u}_p^n \in W_0^{1,r}(\Omega)^2$ be the weak solution to the discrete problem* (11.13)*, let $\boldsymbol{\theta}_p^n \in W_0^{1,2}(\Omega)^2$ be the weak solution to the discrete problem* (11.14)*. Then $\boldsymbol{\theta}_p^n \in W_0^{1,s}(\Omega)^2$ with some $s > 2$.*

Proof. The proof can be realized in the same way as in Section 8.3.2 since the structure of the critical convective term remains the same. □

Now, let us summarize, that we have shown the existence of the solutions $\boldsymbol{u}_p^n$ and $\boldsymbol{\theta}_p^n \in W_0^{1,2}(\Omega)^2$ to the discrete system (11.13)–(11.14) and their $W^{1,r}$-regularity. Since $W_0^{1,r}(\Omega) \hookrightarrow L^\infty(\Omega)$, we may conclude that $\boldsymbol{u}_p^n$ and $\boldsymbol{\theta}_p^n \in W_0^{1,2}(\Omega)^2 \cap L^\infty(\Omega)^2$. This proves the existence of the solution to the recurrence Problem (11.13)–(11.14) for $n = 1, \ldots, p$.

11.5 Time interpolants

Let us define the piecewise constant interpolant functions ($n = 1, 2, \ldots, p$)

$$\begin{aligned}
\bar{u}_{ip}(t) &= u_{ip}^n && \text{for } t \in ((n-1)\tau, n\tau], \\
\bar{u}_{ip}(t) &= u_{i0} && \text{for } t \in (-\tau, 0], \\
\bar{\theta}_{ip}(t) &= \theta_{ip}^n && \text{for } t \in ((n-1)\tau, n\tau], \\
\bar{\theta}_{ip}(t) &= \theta_{i0} && \text{for } t \in (-\tau, 0].
\end{aligned}$$

The piecewise constant interpolants $\bar{u}_{ip}(t) \in L^\infty(I; W_0^{1,2}(\Omega)) \cap L^\infty(I; L^\infty(\Omega))$ and $\bar{\theta}_{ip}(t) \in L^\infty(I; W_0^{1,2}(\Omega)) \cap L^\infty(I; L^\infty(\Omega))$ satisfy for all $t \in (0; T]$ the equations

$$\int_\Omega \frac{b_i(\bar{u}_{ip}(t)) - b_i(\bar{u}_{ip}(t-\tau))}{\tau}\phi_i \,\mathrm{d}x + \int_\Omega k(\bar{\theta}_{ip}(t-\tau), \bar{u}_{ip}(t-\tau)) \left[\nabla \bar{u}_{ip}(t) + \boldsymbol{e}_z\right] \cdot \nabla \phi_i \,\mathrm{d}x$$
$$+ \int_\Omega \omega_i \alpha_\omega(\bar{u}_{1p}(t-\tau), \bar{u}_{2p}(t-\tau))[\bar{u}_{1p}(t) - \bar{u}_{2p}(t)]\phi_i \,\mathrm{d}x = 0 \quad (11.24)$$

for any $\phi_i \in L^2(I; W_0^{1,2}(\Omega))$, $\phi_i(T) = 0$, $i = 1, 2$;

$$\int_\Omega \frac{b_i(\bar{u}_{ip}(t))\bar{\theta}_{ip}(t) - b_i(\bar{u}_{ip}(t-\tau))\bar{\theta}_{ip}(t-\tau)}{\tau}\psi \,\mathrm{d}x$$
$$+ \varrho_i \int_\Omega \frac{\bar{\theta}_{ip}(t) - \bar{\theta}_{ip}(t-\tau)}{\tau}\psi_i \,\mathrm{d}x + \int_\Omega \lambda_i(\bar{\theta}_{ip}(t-\tau), \bar{u}_{ip}(t-\tau))\nabla\bar{\theta}_{ip}(t) \cdot \nabla\psi_i \mathrm{d}x$$
$$+ \int_\Omega \bar{\theta}_{ip}(t) k_i(\bar{\theta}_{ip}(t-\tau), \bar{u}_{ip}(t-\tau)) \left[\nabla \bar{u}_{ip}(t) + \boldsymbol{e}_z\right] \cdot \nabla\psi_i \mathrm{d}x$$
$$= F_i(\bar{u}_{1p}(t), \bar{u}_{2p}(t), \bar{\theta}_{1p}(t), \bar{\theta}_{2p}(t))\psi_i \quad (11.25)$$

for any $\psi_i \in L^2(I; W_0^{1,2}(\Omega))$, $\psi_i(T) = 0$, $i = 1, 2$.

11.6 A-priori estimates

In this section we briefly introduce the main ideas of deriving the suitable a-priori estimates for the time interpolants.

L^∞-bound for $\bar{u}_{ip}$. Let $\kappa \in \mathbb{R}$, $\xi \in \mathbb{R}$. Introduce the functions

$$\varepsilon_\kappa^-(\xi) := \int_\kappa^\xi (s-\kappa)_- \,\mathrm{d}s, \quad (11.26)$$
$$\varepsilon_\kappa^+(\xi) := \int_\kappa^\xi (s-\kappa)_+ \,\mathrm{d}s. \quad (11.27)$$

Recall that symbols $-$ and $+$ denote negative and positive part of a function. Hence we can write $(s-\kappa)_- = \min(s-\kappa, 0)$ and $(s-\kappa)_+ = \max(s-\kappa, 0)$.

Lemma 11.10 *Let us present the properties of $\varepsilon_\kappa^-(\xi)$ and $\varepsilon_\kappa^+(\xi)$:*

$$\varepsilon_\kappa^-(\xi_1) - \varepsilon_\kappa^-(\xi_2) \le (\xi_1 - \xi_2)(\xi_1 - \kappa)_- \quad \forall \xi_1, \xi_2 \in \mathbb{R}, \quad (11.28)$$
$$\varepsilon_\kappa^+(\xi_1) - \varepsilon_\kappa^+(\xi_2) \le (\xi_1 - \xi_2)(\xi_1 - \kappa)_+ \quad \forall \xi_1, \xi_2 \in \mathbb{R}. \quad (11.29)$$

Proof. Proof of this lemma can be carried out in the same manner as the proof of Lemma 8.13. □

Let $\kappa_\sharp \in \mathbb{R}$, such that $\kappa_\sharp \leq u_{i0} + x_2$ a.e. in Ω. Then we have $(u_{ip}^n + x_2 - \kappa_\sharp)_- \in W_0^{1,2}(\Omega), i = 1,2$ and thus we may test (11.13) with

$$\phi_i = 1/\omega_i[u_{ip}^n + x_2 - \kappa_\sharp]_-,$$

and sum the equation for $i = 1,2$. We get

$$\begin{aligned}\frac{1}{\omega_i}\sum_{i=1}^{2}\int_\Omega \frac{b_i(u_{ip}^n) - b_i(u_{ip}^{n-1})}{\tau}[u_{ip}^n + x_2 - \kappa_\sharp]_- \,\mathrm{d}x \\ + \frac{1}{\omega_i}\sum_{i=1}^{2}\int_\Omega k_i(\theta_{ip}^{n-1}, u_{ip}^{n-1})\left(\nabla u_{ip}^n + \boldsymbol{e}_z\right)\cdot\nabla[u_{ip}^n + x_2 - \kappa_\sharp]_- \,\mathrm{d}x \\ + \int_\Omega \alpha_\omega(u_{1p}^{n-1}, u_{2p}^{n-1})(u_{1p}^n - u_{2p}^n)[u_{1p}^n + x_2 - \kappa_\sharp]_- \,\mathrm{d}x \\ + \int_\Omega \alpha_\omega(u_{1p}^{n-1}, u_{2p}^{n-1})(u_{2p}^n - u_{1p}^n)[u_{2p}^n + x_2 - \kappa_\sharp]_- \,\mathrm{d}x = 0. \quad (11.30)\end{aligned}$$

Let us set $\tilde{\kappa}_\sharp = \max_{x_2\in\Omega}(\kappa_\sharp - x_2)$. We now use (8.38) on the first term in (11.30) and (11.28) for the third and fourth term in (11.30), further we slightly modify the elliptic term and we obtain (recall the definition of $\beta_{\tilde{\kappa}_\sharp}^-$ in (8.36))

$$\begin{aligned}\frac{1}{\omega_i}\sum_{i=1}^{2}\int_\Omega \frac{\beta_{\tilde{\kappa}_\sharp}^-(u_{ip}^n) - \beta_{\tilde{\kappa}_\sharp}^-(u_{ip}^{n-1})}{\tau}\,\mathrm{d}x \\ + \frac{1}{\omega_i}\sum_{i=1}^{2}\int_\Omega k_i(\theta_{ip}^{n-1}, u_{ip}^{n-1})\left|\nabla[u_{ip}^n + x_2 - \kappa_\sharp]_-\right|^2 \mathrm{d}x \\ + \int_\Omega \alpha_\omega(u_{1p}^{n-1}, u_{2p}^{n-1})\left(\varepsilon_{\tilde{\kappa}_\sharp}^-(u_{1p}^n) - \varepsilon_{\tilde{\kappa}_\sharp}^-(u_{2p}^n)\right)\mathrm{d}x \\ + \int_\Omega \alpha_\omega(u_{1p}^{n-1}, u_{2p}^{n-1})\left(\varepsilon_{\tilde{\kappa}_\sharp}^-(u_{2p}^n) - \varepsilon_{\tilde{\kappa}_\sharp}^-(u_{1p}^n)\right)\mathrm{d}x \leq 0. \quad (11.31)\end{aligned}$$

The sum of coupling terms equals zero, the elliptic terms are nonnegative because k is a nonnegative function. This allows us to repeat the procedure in (8.45)–(8.49) to show that

$$u_{ip}^n \geq \tilde{\kappa}_\sharp, \quad i = 1,2. \quad (11.32)$$

Further let $\kappa^\sharp \in \mathbb{R}$, such that $\kappa^\sharp \geq u_{i0} + x_2$ a.e. in Ω. Then we have $(u_{ip}^n + x_2 - \kappa^\sharp)_+ \in W_0^{1,2}(\Omega), i = 1,2$ and we are allowed to test (11.13) with

$$\phi_i = 1/\omega_i[u_{ip}^n + x_2 - \kappa^\sharp]_+.$$

Now we sum the equations for $i = 1,2$, set $\tilde{\kappa}^\sharp = \min_{x_2\in\Omega}(\kappa^\sharp - x_2)$ and repeat the procedure presented in (8.51)–(8.56) to show that

$$u_{ip}^n \leq \tilde{\kappa}^\sharp, \quad i = 1,2. \quad (11.33)$$

Taking together (11.32) and (11.33) we arrive at

$$\tilde{\kappa}_\sharp \le u_{ip}^n \le \tilde{\kappa}^\sharp \tag{11.34}$$

almost everywhere in Ω, $n = 1, 2, \cdots p$. Note that (11.34) becomes

$$\|\bar{u}_{ip}\|_{L^\infty(Q_T)} \le c. \tag{11.35}$$

The a-priori estimate (11.35) allows us to conclude that there exists $u \in L^\infty(Q_T)$ such that, letting $p \to +\infty$ (along a selected subsequence),

$$\bar{\boldsymbol{u}}_p \rightharpoonup \boldsymbol{u} \qquad \text{weakly star in } L^\infty(Q_T)^2. \tag{11.36}$$

L^∞-bound for $\bar{\theta}_{ip}$. In order to show L^∞-bound for $\bar{\theta}_{ip}$ we follow [29]. Let ℓ be an odd integer. First, we use $\phi_i = [\ell/(\ell+1)][\theta_{ip}^n]^{\ell+1}$ as a test function in (11.13) to get

$$\begin{aligned}\frac{\ell}{\ell+1}\int_\Omega \frac{b_i(u_{ip}^n) - b_i(u_{ip}^{n-1})}{\tau}[\theta_{ip}^n]^{\ell+1}\,\mathrm{d}x&\\ + \frac{\ell}{\ell+1}\int_\Omega k_i(\theta_{ip}^{n-1}, u_{ip}^{n-1})\left(\nabla u_{ip}^n + \boldsymbol{e}_z\right)\cdot\nabla[\theta_{ip}^n]^{\ell+1}\,\mathrm{d}x&\\ + \frac{\ell}{\ell+1}\int_\Omega \omega_i\alpha_\omega(u_1^{n-1}, u_2^{n-1})(u_1^n - u_2^n)[\theta_{ip}^n]^{\ell+1}\,\mathrm{d}x &= 0\end{aligned} \tag{11.37}$$

and similarly, we set $\psi_i = [\theta_{ip}^n]^\ell$ as a test function in (11.14) to obtain

$$\begin{aligned}\frac{1}{\tau}\int_\Omega \left[\left(b_i(u_{ip}^n) + \varrho_i\right)\theta_{ip}^n - \left(b_i(u_{ip}^{n-1}) + 1\right)\theta_{ip}^{n-1}\right][\theta_{ip}^n]^\ell\,\mathrm{d}x + \int_\Omega \lambda_i(\theta_{ip}^{n-1}, u_{ip}^{n-1})\nabla\theta_{ip}^n\cdot\nabla[\theta_{ip}^n]^\ell\mathrm{d}x&\\ + \int_\Omega \theta_{ip}^n k_i(\theta_{ip}^{n-1}, u_{ip}^{n-1})\left(\nabla u_{ip}^n + \boldsymbol{e}_z\right)\cdot\nabla[\theta_{ip}^n]^\ell\mathrm{d}x = \int_\Omega F_i(u_1^n, u_2^n, \theta_1^n, \theta_2^n)[\theta_{ip}^n]^\ell\,\mathrm{d}x.&\end{aligned} \tag{11.38}$$

Now we divide the equation (11.38) by positive constant ϱ_i, $i = 1, 2$ and, for simplicity, denote all obtained coefficients with the same symbols. Next, we subtract (11.37) from (11.38), we sum the equations for $i = 1, 2$ and arrive at

$$\begin{aligned}&\sum_{i=1}^2 \frac{1}{\tau}\int_\Omega \left[\left(b_i(u_{ip}^n) + 1\right)\theta_{ip}^n - \left(b_i(u_{ip}^{n-1}) + 1\right)\theta_{ip}^{n-1}\right][\theta_{ip}^n]^\ell\,\mathrm{d}x\\ &\qquad + \sum_{i=1}^2\int_\Omega \lambda_i(\theta_{ip}^{n-1}, u_{ip}^{n-1})\nabla\theta_{ip}^n\cdot\nabla[\theta_{ip}^n]^\ell\mathrm{d}x\\ &= \sum_{i=1}^2\int_\Omega F_i(u_1^n, u_2^n, \theta_1^n, \theta_2^n)[\theta_{ip}^n]^\ell\,\mathrm{d}x + \sum_{i=1}^2 \frac{\ell}{\ell+1}\frac{1}{\tau}\int_\Omega \left(b_i(u_{ip}^n) - b_i(u_{ip}^{n-1})\right)[\theta_{ip}^n]^{\ell+1}\,\mathrm{d}x\\ &\qquad + \sum_{i=1}^2 \frac{\ell}{\ell+1}\int_\Omega \omega_i\alpha_\omega(u_1^{n-1}, u_2^{n-1})(u_1^n - u_2^n)[\theta_{ip}^n]^{\ell+1}\,\mathrm{d}x.\end{aligned} \tag{11.39}$$

For the first term on the third line in (11.39) we can write (recall (IV))

$$\sum_{i=1}^{2}\int_\Omega F_i(u_1^n,u_2^n,\theta_1^n,\theta_2^n)[\theta_{ip}^n]^\ell\,\mathrm{d}x \le \sum_{i=1}^{2}\int_\Omega c\,(1+|\theta_1^n|+|\theta_2^n|)\,[\theta_{ip}^n]^\ell\,\mathrm{d}x. \tag{11.40}$$

Hence

$$\begin{aligned}\sum_{i=1}^{2}\int_\Omega F_i(u_1^n,u_2^n,\theta_1^n,\theta_2^n)[\theta_{ip}^n]^\ell\,\mathrm{d}x \le c\int_\Omega &\left([\theta_{1p}^n]^\ell+[\theta_{2p}^n]^\ell\right)\,\mathrm{d}x\\ +c\int_\Omega &\left([\theta_{1p}^n]^{\ell+1}+[\theta_{2p}^n]^{\ell+1}\right)\,\mathrm{d}x\\ +c\int_\Omega &\left(\theta_{1p}^n[\theta_{2p}^n]^\ell+\theta_{2p}^n[\theta_{1p}^n]^\ell\right)\,\mathrm{d}x.\end{aligned} \tag{11.41}$$

Now we use Young's inequality on the first term on the righthand side of (11.41) to get

$$c\int_\Omega \left([\theta_{1p}^n]^\ell+[\theta_{2p}^n]^\ell\right)\,\mathrm{d}x \le c\int_\Omega\left(\frac{1}{\ell+1}c^{\ell+1}+\frac{\ell}{\ell+1}[\theta_{1p}^n]^{\ell+1}+\frac{1}{\ell+1}c^{\ell+1}+\frac{\ell}{\ell+1}[\theta_{2p}^n]^{\ell+1}\right)\,\mathrm{d}x \tag{11.42}$$

hence, we obtain

$$c\int_\Omega \left([\theta_{1p}^n]^\ell+[\theta_{2p}^n]^\ell\right)\,\mathrm{d}x \le 2\frac{\mu(\Omega)}{\ell+1}c^{\ell+1}+\frac{\ell}{\ell+1}\int_\Omega\left([\theta_{1p}^n]^{\ell+1}+[\theta_{2p}^n]^{\ell+1}\right)\,\mathrm{d}x. \tag{11.43}$$

Similarly, we use Young's inequality on the third term on the righthand side of (11.41) to obtain

$$\int_\Omega\left(\theta_{1p}^n[\theta_{2p}^n]^\ell+\theta_{2p}^n[\theta_{1p}^n]^\ell\right)\,\mathrm{d}x \le c\int_\Omega\left([\theta_{1p}^n]^{\ell+1}+[\theta_{2p}^n]^{\ell+1}\right)\,\mathrm{d}x. \tag{11.44}$$

Hence, putting (11.43) and (11.44) in (11.41) we arrive at

$$\sum_{i=1}^{2}\int_\Omega F_i(u_1^n,u_2^n,\theta_1^n,\theta_2^n)[\theta_{ip}^n]^\ell\,\mathrm{d}x \le 2\frac{\mu(\Omega)}{\ell+1}c^{\ell+1}+c\int_\Omega\left([\theta_{1p}^n]^{\ell+1}+[\theta_{2p}^n]^{\ell+1}\right)\,\mathrm{d}x. \tag{11.45}$$

Further, for the coupling term on the fourth line of (11.39) we can write (recall (II))

$$\int_\Omega\frac{\ell}{\ell+1}\int_\Omega \omega_i\alpha_\omega(u_1^{n-1},u_2^{n-1})(u_1^n-u_2^n)[\theta_{ip}^n]^{\ell+1}\,\mathrm{d}x \le c\,\frac{\ell}{\ell+1}\int_\Omega[\theta_{ip}^n]^{\ell+1}\,\mathrm{d}x. \tag{11.46}$$

Hence, putting (11.45) and (11.46) in (11.39) and considering that the elliptic term in (11.39) is nonnegative we obtain

$$\sum_{i=1}^{2}\frac{1}{\tau}\int_\Omega \left[b_i(u_{ip}^n)\theta_{ip}^n - b_i(u_{ip}^{n-1})\theta_{ip}^{n-1}\right][\theta_{ip}^n]^\ell\,\mathrm{d}x + \sum_{i=1}^{2}\frac{1}{\tau}\int_\Omega\left[\theta_{ip}^n-\theta_{ip}^{n-1}\right][\theta_{ip}^n]^\ell\,\mathrm{d}x$$
$$-\sum_{i=1}^{2}\frac{\ell}{\ell+1}\frac{1}{\tau}\int_\Omega\left(b_i(u_{ip}^n)-b_i(u_{ip}^{n-1})\right)[\theta_{ip}^n]^{\ell+1}\,\mathrm{d}x$$
$$\le 2\frac{\mu(\Omega)}{\ell+1}c^{\ell+1} + c\int_\Omega\left([\theta_{1p}^n]^{\ell+1}+[\theta_{2p}^n]^{\ell+1}\right)\,\mathrm{d}x. \quad (11.47)$$

Now, we multiply (11.47) by τ, again use Young's inequality and after little lengthy computation we arrive at

$$\sum_{i=1}^{2}\int_\Omega[\theta_{ip}^n]^{\ell+1}\,\mathrm{d}x + \sum_{i=1}^{2}\frac{1}{\ell+1}\int_\Omega b(u_{ip}^n)[\theta_{ip}^n]^{\ell+1}\,\mathrm{d}x - \sum_{i=1}^{2}\frac{1}{\ell+1}\int_\Omega b(u_{ip}^{n-1})[\theta_{ip}^{n-1}]^{\ell+1}\,\mathrm{d}x$$
$$\le \sum_{i=1}^{2}\int_\Omega[\theta_{ip}^{n-1}][\theta_{ip}^n]^\ell\,\mathrm{d}x + \tau c_1\int_\Omega\left([\theta_{1p}^n]^{\ell+1}+[\theta_{2p}^n]^{\ell+1}\right)\,\mathrm{d}x + 4\tau\frac{\mu(\Omega)}{\ell+1}c^{\ell+1}. \quad (11.48)$$

Hence

$$\sum_{i=1}^{2}(1-2\tau c_1)\int_\Omega[\theta_{ip}^n]^{\ell+1}\,\mathrm{d}x + \sum_{i=1}^{2}\frac{1}{\ell+1}\int_\Omega b(u_{ip}^n)[\theta_{ip}^n]^{\ell+1}\,\mathrm{d}x$$
$$-\sum_{i=1}^{2}\frac{1}{\ell+1}\int_\Omega b(u_{ip}^{n-1})[\theta_{ip}^{n-1}]^{\ell+1}\,\mathrm{d}x$$
$$\le \sum_{i=1}^{2}\int_\Omega[\theta_{ip}^{n-1}][\theta_{ip}^n]^\ell\,\mathrm{d}x + 4\tau\frac{\mu(\Omega)}{\ell+1}c^{\ell+1}. \quad (11.49)$$

For sufficiently small τ, such that $(1-2\tau c_1)>0$, we can divide (11.49) by $(1-2\tau c_1)$

$$\sum_{i=1}^{2}\int_\Omega[\theta_{ip}^n]^{\ell+1}\,\mathrm{d}x + \sum_{i=1}^{2}\frac{1}{\ell+1}\frac{1}{1-2\tau c_1}\int_\Omega b(u_{ip}^n)[\theta_{ip}^n]^{\ell+1}\,\mathrm{d}x$$
$$-\sum_{i=1}^{2}\frac{1}{\ell+1}\frac{1}{1-2\tau c_1}\int_\Omega b(u_{ip}^{n-1})[\theta_{ip}^{n-1}]^{\ell+1}\,\mathrm{d}x$$
$$\le \sum_{i=1}^{2}\frac{1}{1-2\tau c_1}\int_\Omega[\theta_{ip}^{n-1}][\theta_{ip}^n]^\ell\,\mathrm{d}x + 4\tau\frac{\mu(\Omega)}{\ell+1}\frac{1}{1-2\tau c_1}c^{\ell+1}. \quad (11.50)$$

Let us mention, that we can write

$$\frac{1}{1-2\tau c_1} = 1 + \left(2c_1 + \frac{2c_1\tau 2c_1}{1-2\tau c_1}\right)\tau. \quad (11.51)$$

Now we denote

$$L := 2c_1, \quad \omega_p := \frac{2c_1\tau 2c_1}{1-2\tau c_1}. \tag{11.52}$$

Let us mention that $\omega_p \to 0$, for $\tau \to 0$. We employ notation (11.52) and again use Young's inequality on (11.50)

$$\begin{aligned}\sum_{i=1}^{2}\int_\Omega [\theta_{ip}^n]^{\ell+1}\,\mathrm{d}x &+ \sum_{i=1}^{2}\frac{1}{\ell+1}\frac{1}{1-2\tau c_1}\int_\Omega b(u_{ip}^n)[\theta_{ip}^n]^{\ell+1}\,\mathrm{d}x \\ &- \sum_{i=1}^{2}\frac{1}{\ell+1}\frac{1}{1-2\tau c_1}\int_\Omega b(u_{ip}^{n-1})[\theta_{ip}^{n-1}]^{\ell+1}\,\mathrm{d}x \\ \le [1+(L+\omega_p)\tau]^{\ell+1}\,&\frac{1}{\ell+1}\sum_{i=1}^{2}\int_\Omega [\theta_{ip}^{n-1}]^{\ell+1}\,\mathrm{d}x + \frac{\ell}{\ell+1}\sum_{i=1}^{2}\int_\Omega [\theta_{ip}^n]^{\ell+1}\,\mathrm{d}x \\ &+ 4\tau\frac{\mu(\Omega)}{\ell+1}\frac{1}{1-2\tau c_1}c^{\ell+1}. \end{aligned} \tag{11.53}$$

Now we denote

$$y_n := \sum_{i=1}^{2}\int_\Omega [\theta_{ip}^n]^{\ell+1}\,\mathrm{d}x, \tag{11.54}$$

$$Y_n := \sum_{i=1}^{2}\frac{1}{\ell+1}\frac{1}{1-2\tau c_1}\int_\Omega b(u_{ip}^n)[\theta_{ip}^n]^{\ell+1}\,\mathrm{d}x, \tag{11.55}$$

$$Y_{n-1} := \sum_{i=1}^{2}\frac{1}{\ell+1}\frac{1}{1-2\tau c_1}\int_\Omega b(u_{ip}^{n-1})[\theta_{ip}^{n-1}]^{\ell+1}\,\mathrm{d}x, \tag{11.56}$$

$$y_{n-1} := \sum_{i=1}^{2}\int_\Omega [\theta_{ip}^{n-1}]^{\ell+1}\,\mathrm{d}x. \tag{11.57}$$

With this notation we rewrite (11.53) as

$$y_n + Y_n \le [1+(L+\omega_p)\tau]^{\ell+1}\left[y_{n-1} + Y_{n-1} + 4\tau c^{\ell+1}\right]. \tag{11.58}$$

The recurrence relation (11.58) can be also rewritten as

$$y_n + Y_n \le [1+(L+\omega_p)\tau]^{(\ell+1)n}\left(y_0 + Y_0 + \tau\sum_{j=1}^{n}4c^{\ell+1}\right). \tag{11.59}$$

Because Y_n is nonnegative (recall that ℓ is an odd integer), we can write

$$y_n \le [1+(L+\omega_p)\tau]^{(\ell+1)n}\left(y_0 + Y_0 + \tau\sum_{j=1}^{n}4c^{\ell+1}\right). \tag{11.60}$$

We take the $\ell + 1$-th root

$$y_n^{\frac{1}{\ell+1}} \leq [1 + (L + \omega_p)\tau]^n \left(y_0^{\frac{1}{\ell+1}} + Y_0^{\frac{1}{\ell+1}} + \tau^{\frac{1}{\ell+1}} \sum_{j=1}^{n} 4^{\frac{1}{\ell+1}} c^{\ell+1} \right). \tag{11.61}$$

Recall, that $n = 1, \ldots, p$. Hence we can write

$$\left[1 + (L + \omega_p)\frac{T}{p}\right]^n \leq \left[1 + (L + \omega_p)\frac{T}{p}\right]^p \leq \left[1 + \frac{c}{p}\right]^p \to e^c, \tag{11.62}$$

for $p \to \infty$.

Now we put the estimate (11.62) in (11.61)

$$y_n^{\frac{1}{\ell+1}} \leq c_1 \left(y_0^{\frac{1}{\ell+1}} + Y_0^{\frac{1}{\ell+1}} + c_2 \right). \tag{11.63}$$

Taking into account (11.54), from (11.63) we have

$$\left(\int_\Omega [\theta_{1p^n}]^{\ell+1} \, \mathrm{d}x + \int_\Omega [\theta_{2p^n}]^{\ell+1} \, \mathrm{d}x \right)^{\frac{1}{\ell+1}} \leq c_1 \left(y_0^{\frac{1}{\ell+1}} + Y_0^{\frac{1}{\ell+1}} + c_2 \right). \tag{11.64}$$

Hence

$$\|\theta_{ip}^p\|_{L^{\ell+1}(\Omega))} \leq C. \tag{11.65}$$

Now letting $\ell \to +\infty$ we get

$$\|\theta_{ip}^n\|_{L^\infty(\Omega))} \leq C. \tag{11.66}$$

Let us mention that (11.66) becomes

$$\|\bar{\theta}_{ip}\|_{L^\infty(Q_T)} \leq C. \tag{11.67}$$

The a-priori estimate (11.67) allows us to conclude that there exist $\theta_i \in L^\infty(Q_T)$ such that, letting $p \to +\infty$ (along a selected subsequence),

$$\bar{\boldsymbol{\theta}}_p \rightharpoonup \boldsymbol{\theta} \qquad \text{weakly star in } L^\infty(Q_T)^2. \tag{11.68}$$

Energy estimate for $\bar{u}_{ip}$. To derive energy a-priori estimate for $\bar{u}_{ip}(t)$ we test the equation (11.24) with $\phi_i = \bar{u}_{ip}(t)/\omega_i$ and sum the equations for $i = 1, 2$.

The sum of coupling terms is equal to zero. Using usual estimates for parabolic and elliptic term we arrive at

$$\sum_{i=1}^{2} \frac{1}{\tau} \int_\Omega B(\bar{u}_{ip}(t)) - B(\bar{u}_{ip}(t - \tau)) \, \mathrm{d}x + \sum_{i=1}^{2} \|\bar{u}_{ip}(t)\|^2_{W_0^{1,2}(\Omega)} \leq c. \tag{11.69}$$

We integrate (11.69) with respect to time from 0 to s ($s = k\tau, k \in \mathbb{N}, 1 \le k \le p$). We arrive at

$$\sum_{i=1}^{2} -\int_\Omega B(\bar{u}_{ip}(0))\,\mathrm{d}x + \sum_{i=1}^{2}\int_\Omega B(\bar{u}_{ip}(k\tau))\,\mathrm{d}x + \sum_{i=1}^{2}\int_0^{k\tau} \|\bar{u}_{ip}(t)\|^2_{W_0^{1,2}(\Omega)}\,\mathrm{d}t \le k\tau c. \tag{11.70}$$

Hence

$$\sum_{i=1}^{2} \sup_{0\le t\le T}\int_\Omega B(\bar{u}_{ip}(t))\,\mathrm{d}x + \sum_{i=1}^{2}\int_0^{T} \|\bar{u}_{ip}(t)\|^2_{W_0^{1,2}}\,\mathrm{d}t \le Tc. \tag{11.71}$$

Let us note that (11.71) becomes

$$\|\bar{u}_{ip}\|_{L^2(I,W_0^{1,2}(\Omega))} \le Tc. \tag{11.72}$$

Hence

$$\bar{\boldsymbol{u}}_p \rightharpoonup \boldsymbol{u} \qquad \text{weakly in } L^2(I;W_0^{1,2}(\Omega)^2). \tag{11.73}$$

Energy estimate for $\bar{\theta}_{ip}$. In order to derive the energy estimate for $\bar{\theta}_{ip}$ we use $\psi_i = 2\theta_{ip}^n$ as a test function in (11.14), $\phi_i = (\theta_{ip}^n)^2$ as a test function in (11.13), combine the equations and sum for $i = 1, 2$ to get

$$\begin{aligned}
&\sum_{i=1}^{2}\int_\Omega \frac{\left(\theta_{ip}^n\right)^2\left(b_i(u_{ip}^n)+\varrho_i\right) - \left(\theta_{ip}^{n-1}\right)^2\left(b_i(u_{ip}^{n-1})+\varrho_i\right)}{\tau}\,\mathrm{d}x\\
&+\sum_{i=1}^{2}\int_\Omega \frac{1}{\tau}\left[\left(\theta_{ip}^n\right)-\left(\theta_{ip}^{n-1}\right)\right]^2\left(b_i(u_{ip}^{n-1})+\varrho_i\right)\,\mathrm{d}x + 2\sum_{i=1}^{2}\int_\Omega \lambda_i(\theta_{ip}^{n-1},u_{ip}^{n-1})\nabla\theta_{ip}^n\cdot\nabla\theta_{ip}^n\,\mathrm{d}x\\
&=\sum_{i=1}^{2}\int_\Omega 2F_i(u_{1p}^n,u_{2p}^n,\theta_{1p}^n,\theta_{2p}^n)\theta_{ip}^n\,\mathrm{d}x + \sum_{i=1}^{2}\int_\Omega \omega_i\alpha_\omega(u_{1p}^{n-1},u_{2p}^{n-1})[u_{1p}^n-u_{2p}^n](\theta_{ip}^n)^2\,\mathrm{d}x.
\end{aligned} \tag{11.74}$$

Recall that b is a positive function and ϱ is a positive constant, the second integral is nonnegative. Further, using Friedrich's inequality for the elliptic term, we obtain

$$2\sum_{i=1}^{2}\int_\Omega \lambda_i(\theta_{ip}^{n-1},u_{ip}^{n-1})\nabla\theta_{ip}^n\cdot\nabla\theta_{ip}^n\,\mathrm{d}x \ge \sum_{i=1}^{2} c\|\theta_{ip}^n\|^2_{W_0^{1,2}(\Omega)}. \tag{11.75}$$

Using (11.75) in (11.74), considering u_i are bouded functions we can write

$$\begin{aligned}
&\sum_{i=1}^{2}\int_\Omega \frac{\left(\theta_{ip}^n\right)^2\left(b_i(u_{ip}^n)+\varrho_i\right) - \left(\theta_{ip}^{n-1}\right)^2\left(b_i(u_{ip}^{n-1})+\varrho_i\right)}{\tau}\,\mathrm{d}x + \sum_{i=1}^{2} c\|\theta_{ip}^n\|^2_{W_0^{1,2}(\Omega)}\\
&\qquad\le \sum_{i=1}^{2}\int_\Omega 2F_i(\theta_{1p}^n,\theta_{2p}^n)\theta_{ip}^n\,\mathrm{d}x + \sum_{i=1}^{2}\int_\Omega c(\theta_{ip}^n)^2\,\mathrm{d}x.
\end{aligned} \tag{11.76}$$

Using (IV) we have

$$\sum_{i=1}^{2}\int_\Omega \left[\left(\theta_{ip}^n\right)^2\left(b_i(u_{ip}^n)+\varrho_i\right)-\left(\theta_{ip}^{n-1}\right)^2\left(b_i(u_{ip}^{n-1})+\varrho_i\right)\right]\,\mathrm{d}x+\sum_{i=1}^{2}c\tau\|\theta_{ip}^n\|^2_{W_0^{1,2}(\Omega)}$$
$$\le \tau\sum_{i=1}^{2}\int_\Omega(1+\theta_{1p}^n+\theta_{2p}^n)\theta_{ip}^n\,\mathrm{d}x+\tau\sum_{i=1}^{2}\int_\Omega c(\theta_{ip}^n)^2\,\mathrm{d}x. \quad (11.77)$$

Now, summing (11.77) for $n=1,\dots,k$, $1<k\le p$, we obtain

$$\sum_{i=1}^{2}\int_\Omega \left(\theta_{ip}^k\right)^2\left(b_i(u_{ip}^k)+\varrho_i\right)-\left(\theta_{ip}^{0}\right)^2\left(b_i(u_{ip}^{0})+\varrho_i\right)\,\mathrm{d}x+\sum_{n=1}^{k}\sum_{i=1}^{2}c\tau\|\theta_{ip}^n\|^2_{W_0^{1,2}(\Omega)}$$
$$\le \tau\sum_{n=1}^{k}\sum_{i=1}^{2}\int_\Omega(1+\theta_{1p}^n+\theta_{2p}^n)\theta_{ip}^n\,\mathrm{d}x+\tau\sum_{n=1}^{k}\sum_{i=1}^{2}\int_\Omega c(\theta_{ip}^n)^2\,\mathrm{d}x. \quad (11.78)$$

Using (I) we obtain

$$\sum_{i=1}^{2}\int_\Omega \left(\theta_{ip}^k\right)^2+\sum_{n=1}^{k}\sum_{i=1}^{2}c\tau\|\theta_{ip}^n\|^2_{W_0^{1,2}(\Omega)}$$
$$\le c+\tau\sum_{n=1}^{k}\sum_{i=1}^{2}\int_\Omega(1+\theta_{1p}^n+\theta_{2p}^n)\theta_{ip}^n\,\mathrm{d}x+c\tau\sum_{n=1}^{p}\sum_{i=1}^{2}\int_\Omega(\theta_{ip}^n)^2\,\mathrm{d}x. \quad (11.79)$$

Now, we rewrite the second term on the righthand side of the inequality (11.79) to get

$$\sum_{i=1}^{2}\int_\Omega \left(\theta_{ip}^k\right)^2+\sum_{n=1}^{k}\sum_{i=1}^{2}c\tau\|\theta_{ip}^n\|^2_{W_0^{1,2}(\Omega)}$$
$$\le c+\tau\sum_{n=1}^{k}\int_\Omega\left[\theta_{1p}^n+\theta_{2p}^n+2\theta_{1p}^n\theta_{2p}^n+\left(\theta_{1p}^n\right)^2+\left(\theta_{2p}^n\right)^2\right]\,\mathrm{d}x+c\tau\sum_{n=1}^{k}\sum_{i=1}^{2}\int_\Omega(\theta_{ip}^n)^2\,\mathrm{d}x. \quad (11.80)$$

We use Young's inequality with parameters ϵ

$$\sum_{i=1}^{2}\int_\Omega \left(\theta_{ip}^k\right)^2+\sum_{n=1}^{k}\sum_{i=1}^{2}c\tau\|\theta_{ip}^n\|^2_{W_0^{1,2}(\Omega)}$$
$$\le \tau\sum_{n=1}^{k}\int_\Omega(\epsilon_1+c_1(\epsilon)\left(\theta_{1p}^n\right)^2+\epsilon+c_2(\epsilon)\left(\theta_{2p}^n\right)^2+2(\epsilon)\left(\theta_{1p}^n\right)^2$$
$$+2c_3(\epsilon)\left(\theta_{2p}^n\right)^2+\left(\theta_{1p}^n\right)^2+\left(\theta_{2p}^n\right)^2)\,\mathrm{d}x$$
$$+c+c\tau\sum_{n=1}^{k}\sum_{i=1}^{2}\int_\Omega(\theta_{ip}^n)^2\,\mathrm{d}x. \quad (11.81)$$

Hence

$$\sum_{i=1}^{2}\int_{\Omega}\left(\theta_{ip}^{k}\right)^{2}\,\mathrm{d}x+\sum_{n=1}^{k}\sum_{i=1}^{2}c\tau\|\theta_{ip}^{n}\|_{W_0^{1,2}(\Omega)}^{2}\leq c+c\tau\sum_{n=1}^{k}\sum_{i=1}^{2}\int_{\Omega}(\theta_{ip}^{n})^{2}\,\mathrm{d}x. \tag{11.82}$$

This yields

$$\sum_{i=1}^{2}\int_{\Omega}\left(\theta_{ip}^{k}\right)^{2}\,\mathrm{d}x\leq c_1+c_2\tau\sum_{n=1}^{k}\sum_{i=1}^{2}\int_{\Omega}(\theta_{ip}^{n})^{2}\,\mathrm{d}x. \tag{11.83}$$

Now, using the discrete form of Gronwall's inequality D.5 we deduce

$$\sum_{i=1}^{2}\int_{\Omega}\left(\theta_{ip}^{k}\right)^{2}\,\mathrm{d}x\leq c. \tag{11.84}$$

Hence, the equation (11.84) yields

$$\sum_{i=1}^{2}\max_{n=1,\ldots,p}\int_{\Omega}\left|\theta_{ip}^{p}\right|^{2}\,\mathrm{d}x+\tau\sum_{i=1}^{2}\sum_{n=1}^{p}\int_{0}^{T}\|\theta_{ip}^{n}\|_{W_0^{1,2}}^{2}\,\mathrm{d}t\leq c. \tag{11.85}$$

Let us mention that (11.85) becomes

$$\|\bar{\theta}_{ip}\|_{L^2(I;W_0^{1,2}(\Omega))}\leq c. \tag{11.86}$$

Hence

$$\bar{\boldsymbol{\theta}}_p\rightharpoonup\boldsymbol{\theta}\qquad\text{weakly in } L^2(I;W_0^{1,2}(\Omega)^2). \tag{11.87}$$

Convergence almost everywhere of $\bar{u}_{ip}$, $\bar{\theta}_{ip}$. Apart from energy estimates and L^∞-bounds, we need to show convergence almost everywhere of the interpolant functions due to the nonlinear terms in the system.

Theorem 11.11 *(Convergence almost everywhere of $\bar{u}_{ip}$ and $\bar{\theta}_{ip}$) Let the assumptions* (I)–(V) *be satisfied, then*

$$\begin{aligned}\bar{u}_{ip}&\to u_i &&\textit{almost everywhere on } Q_T,\\ \bar{\theta}_{ip}&\to\theta_i &&\textit{almost everywhere on } Q_T.\end{aligned}$$

Proof. We procceed in the same way as in Section 8.4.4 and, using suitable test functions, we verify the assumptions of Lemma 8.15. Since the verification is technical and the main ideas have already been presented, we refer for more details to Section 8.4.4.

11.7 Passage to the limit for $p \to \infty$

In the preceeding sections, we have shown that letting $p \to +\infty$ (along a selected subsequence),

$$\begin{aligned}
\bar{u}_{ip} &\rightharpoonup u_i && \text{weakly in } L^2(I; W_0^{1,2}(\Omega)),\\
\bar{u}_{ip} &\rightharpoonup u_i && \text{weakly star in } L^\infty(Q_T),\\
\bar{\theta}_{ip} &\rightharpoonup \theta_i && \text{weakly in } L^2(I; W_0^{1,2}(\Omega)),\\
\bar{\theta}_{ip} &\rightharpoonup \theta_i && \text{weakly star in } L^\infty(Q_T).
\end{aligned}$$

Further, we have also

$$\begin{aligned}
\bar{u}_{ip} &\to u_i && \text{almost everywhere on } Q_T,\\
\bar{\theta}_{ip} &\to \theta_i && \text{almost everywhere on } Q_T.
\end{aligned}$$

Hence, the above established convergences are sufficient for taking the limit $p \to \infty$ in (11.24)–(11.25) (along a selected subsequence) to show that the pair $[\boldsymbol{u}, \boldsymbol{\theta}]$ is a weak solution of the system (11.1)–(11.6).

Part V

Further extensions

In this Section, we briefly introduce the model with general boundary conditions arising from the dual porosity approach (see Section 3.2) and a coupled diffusion-convection-dispersion model, including coupled transport of heat, moisture and dissolved species. In this text, we will formulate the problems in a variational sense and present the assumptions on parameters. For more information, we refer the reader to Appendices A and B, where we add our papers [6] and [7], which deal with these problems in detail. Let us also note, that in what follows, we adopt the notation from the mentioned papers.

12 Model with general boundary conditions

Strong formulation. Let Ω be a bounded domain in $\mathbb{R}^2$, $\Omega \in C^{0,1}$ and let Γ_D and Γ_N be open disjoint subsets of $\partial\Omega$ (not necessarily connected) such that $\Gamma_D \neq \emptyset$ and the $\partial\Omega \backslash (\Gamma_D \cup \Gamma_N)$ is a finite set. Let $T \in (0, \infty)$ be fixed, $I = (0, T)$ and $Q_T = \Omega \times I$ denotes the space-time cylinder, $\Gamma_{DT} = \Gamma_D \times I$ and $\Gamma_{NT} = \Gamma_N \times I$.

We shall study the following initial boundary value problem ($i = 1, 2$)

$$\partial_t b_i(u_i) = \nabla \cdot \boldsymbol{a}_i(\theta_i, u_i, \nabla u_i) + f_i(b_1(u_1), b_2(u_2)) \quad \text{in } Q_T, \tag{12.1}$$

$$\begin{aligned} \partial_t \left[b_i(u_i)\theta_i + \varrho_i \theta_i \right] = {} & \nabla \cdot (\lambda_i(\theta_i, u_i) \nabla \theta_i) \\ & + \nabla \cdot (\theta_i \boldsymbol{a}_i(\theta_i, u_i, \nabla u_i)) - h_i(\theta_1, \theta_2) \quad \text{in } Q_T, \end{aligned} \tag{12.2}$$

$$\boldsymbol{u} = \boldsymbol{u}_D \quad \text{on } \Gamma_{DT}, \tag{12.3}$$

$$\boldsymbol{\theta} = \boldsymbol{\theta}_D \quad \text{on } \Gamma_{DT}, \tag{12.4}$$

$$-\boldsymbol{a}_i(\theta_i, u_i, \nabla u_i) \cdot \boldsymbol{n} = -\gamma_i \quad \text{on } \Gamma_{NT}, \tag{12.5}$$

$$-\lambda_i(\theta_i, u_i) \nabla \theta_i \cdot \boldsymbol{n} = \alpha_i(\theta_i) - g_i \quad \text{on } \Gamma_{NT}, \tag{12.6}$$

$$\boldsymbol{u}(x, 0) = \boldsymbol{u}_0(x) \quad \text{in } \Omega, \tag{12.7}$$

$$\boldsymbol{\theta}(x, 0) = \boldsymbol{\theta}_0(x) \quad \text{in } \Omega. \tag{12.8}$$

The system (12.1)–(12.8) arises from the coupled water movement and heat transfer through the dual porous system following the Kirchhoff transformation. Here $u_i : Q_T \to \mathbb{R}$ and $\theta_i : Q_T \to \mathbb{R}$ are the unknown functions. $\boldsymbol{u} = (u_1, u_2)$ corresponds to the Kirchhoff transformation of the matric potential and $\boldsymbol{\theta} = (\theta_1, \theta_2)$ represents the temperature of the dual porous system. The vector function $\boldsymbol{a}_i : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ admits the structure $\boldsymbol{a}_i(r, s, \boldsymbol{z}) = a_i(r)\boldsymbol{z} + \boldsymbol{e}_i(r, s)$, $a_i : \mathbb{R} \to \mathbb{R}$, $\boldsymbol{e}_i : \mathbb{R}^2 \to \mathbb{R}^2$, $b_i : \mathbb{R} \to \mathbb{R}$, $\lambda_i : \mathbb{R}^2 \to \mathbb{R}$, $f_i : \mathbb{R}^2 \to \mathbb{R}$, $h_i : \mathbb{R}^2 \to \mathbb{R}$, $u_{iD} : Q_T \to \mathbb{R}$, $\theta_{iD} : Q_T \to \mathbb{R}$, $\gamma_i : \Gamma_{NT} \to \mathbb{R}$, $g_i : \Gamma_{NT} \to \mathbb{R}$, $\alpha_i : \mathbb{R} \to \mathbb{R}$, $u_{0i} : \Omega \to \mathbb{R}$ and $\theta_{0i} : \Omega \to \mathbb{R}$ are given functions, ϱ_i is a real positive constant and $\boldsymbol{n}$ is the outward unit normal vector.

Structure and data properties. We introduce our assumptions on functions in (12.1)–(12.8).

(12i) b_i is a positive continuous strictly monotone function such that

$$\begin{aligned} &0 < b_i(\xi) \le b^\sharp < +\infty && \forall \xi \in \mathbb{R} \quad (b^\sharp = \text{const}),\\ &(b_i(\xi_1) - b_i(\xi_2))\,(\xi_1 - \xi_2) > 0 && \forall \xi_1, \xi_2 \in \mathbb{R},\ \xi_1 \ne \xi_2. \end{aligned}$$

(12ii) a_i and λ_i are continuous functions satisfying

$$\begin{aligned} &0 < a_\sharp \le a_i(\xi) \le a^\sharp < +\infty && \forall \xi \in \mathbb{R} \quad (a_\sharp, a^\sharp = \text{const}),\\ &0 < \lambda_\sharp \le \lambda_i(\xi, \zeta) \le \lambda^\sharp < +\infty && \forall \xi, \zeta \in \mathbb{R} \quad (\lambda_\sharp, \lambda^\sharp = \text{const}). \end{aligned}$$

$\boldsymbol{e}_i : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable vector function, such that

$$|\boldsymbol{e}_i(\xi, \zeta)| \le e^\sharp < +\infty \qquad \forall \xi, \zeta \in \mathbb{R} \quad (e^\sharp = \text{const}).$$

(12iii) $f_i : \mathbb{R}^2 \to \mathbb{R}$ is continuous.

(12iv) $h_i : \mathbb{R}^2 \to \mathbb{R}$ $(i = 1, 2)$ admits the structure

$$h_1(r, s) = \varepsilon(r - s), \qquad h_2(r, s) = \varepsilon(s - r),$$

where ε is a positive constant.

(12v) $\alpha_i : \mathbb{R} \to \mathbb{R}$ admits the structure

$$\alpha_i(r) = c|r|^3 r - \sigma(r), \quad c > 0,$$

where σ is a continuous function satisfying the linear growth condition

$$|\sigma(r)| \le c(1 + |r|).$$

(12vi) Assume

$$\begin{aligned} &\boldsymbol{u}_0, \boldsymbol{\theta}_0 \in L^2(\Omega),\\ &\boldsymbol{u}_D, \boldsymbol{\theta}_D \in L^2(I; W^{1,2+\delta}(\Omega)) \cap W^{1,1}(I; L^\infty(\Omega)) \cap L^\infty(I; L^\infty(\Gamma_D)),\\ &\boldsymbol{\gamma}, \boldsymbol{g} \in C(\overline{Q_T})^2 \end{aligned}$$

with some $\delta > 0$.

Weak formulation. A weak solution of (12.1)–(12.8) is a pair $[\boldsymbol{u}, \boldsymbol{\theta}]$ such that

$$
\begin{aligned}
&\boldsymbol{u} \in \boldsymbol{u}_D + L^2(I; W^{1,2}_{\Gamma_D}(\Omega)^2),\\
&\boldsymbol{\theta} \in \boldsymbol{\theta}_D + L^2(I; W^{1,2}_{\Gamma_D}(\Omega)^2) \cap L^\infty(I; L^2(\Omega)^2),\\
&\alpha_i(\theta_i) \in L^{5/4}(I; L^{5/4}(\Gamma_N)),
\end{aligned}
$$

which satisfies $(i = 1, 2)$

$$
\begin{aligned}
&- \int_{Q_T} b_i(u_i)\partial_t\phi_i \,\mathrm{d}x\mathrm{d}t + \int_{Q_T} (a(\theta_i)\nabla u_i + \boldsymbol{e}_i(\theta_i, u_i)) \cdot \nabla\phi_i \,\mathrm{d}x\mathrm{d}t\\
&= \int_{Q_T} f_i(b_1(u_1), b_2(u_2))\phi_i \,\mathrm{d}x\mathrm{d}t + \int_\Omega b_i(u_{0i})\phi_i(0)\,\mathrm{d}x + \int_{\Gamma_{NT}} \gamma_i\phi_i \,\mathrm{d}S\mathrm{d}t
\end{aligned}
\tag{12.9}
$$

$\forall \phi_i \in C^\infty(\overline{Q}_T)$, $\phi_i(x, T) = 0$ $\forall x \in \Omega$ and $\phi_i = 0$ on Γ_D;

$$
\begin{aligned}
&- \int_{Q_T} (b_i(u_i)\theta_i + \varrho_i\theta_i)\,\partial_t\psi_i \,\mathrm{d}x\mathrm{d}t + \int_{Q_T} \lambda_i(\theta_i, u_i)\nabla\theta_i \cdot \nabla\psi_i \,\mathrm{d}x\mathrm{d}t\\
&+ \int_{Q_T} (\theta_i\,(a_i(\theta_i)\nabla u_i + \boldsymbol{e}_i(\theta_i, u_i))) \cdot \nabla\psi_i \,\mathrm{d}x\mathrm{d}t + \int_{Q_T} h_i(\theta_1, \theta_2)\psi_i \,\mathrm{d}x\mathrm{d}t\\
&+ \int_{\Gamma_{NT}} \alpha_i(\theta_i)\psi_i \,\mathrm{d}S\mathrm{d}t - \int_{\Gamma_{NT}} \theta_i\,\gamma_i\,\psi_i \,\mathrm{d}S\mathrm{d}t\\
&= \int_\Omega (b_i(u_{0i})\theta_{0i} + \varrho_i\theta_{0i})\,\psi_i(0)\,\mathrm{d}x + \int_{\Gamma_{NT}} g_i\psi_i \,\mathrm{d}S\mathrm{d}t
\end{aligned}
\tag{12.10}
$$

$\forall \psi_i \in C^\infty(\overline{Q}_T)$, $\psi_i(x, T) = 0$ $\forall x \in \Omega$ and $\psi_i = 0$ on Γ_D.

Theorem 12.1 *Let the assumptions* (12i)–(12vi) *be satisfied. Then there exists at least one weak solution of the system* (12.1)–(12.8).

13 Diffusion-convection-dispersion model

Strong formulation. Let Ω be a bounded domain in $\mathbb{R}^2$, $\Omega \in C^{0,1}$ and let Γ_D and Γ_N be open disjoint subsets of $\partial\Omega$ (not necessarily connected) such that $\Gamma_D \neq \emptyset$ and the $\partial\Omega \backslash (\Gamma_D \cup \Gamma_N)$ is a finite set. Let $T \in (0, \infty)$ be fixed, $I = (0, T)$ and $Q_T = \Omega \times I$ denotes the space-time cylinder, $\Gamma_{DT} = \Gamma_D \times I$ and $\Gamma_{NT} = \Gamma_N \times I$.

We present the initial boundary value problem in Q_T

$$\partial_t b(u) = \nabla \cdot [a(\theta)\nabla u], \tag{13.1}$$

$$\partial_t [b(u)w] = \nabla \cdot [b(u)D_w(u)\nabla w] + \nabla \cdot [wa(\theta)\nabla u], \tag{13.2}$$

$$\partial_t [b(u)\theta + \varrho\theta] = \nabla \cdot [\lambda(\theta, u)\nabla\theta] + \nabla \cdot [\theta a(\theta)\nabla u], \tag{13.3}$$

with the mixed-type boundary conditions

$$u = 0,\ w = 0,\ \theta = 0 \qquad \text{on } \Gamma_{DT}, \tag{13.4}$$

$$\nabla u \cdot \boldsymbol{n} = 0,\ \nabla w \cdot \boldsymbol{n} = 0,\ \nabla\theta \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma_{NT} \tag{13.5}$$

and the initial conditions

$$u(\cdot, 0) = u_0,\ w(\cdot, 0) = w_0,\ \theta(\cdot, 0) = \theta_0 \qquad \text{in } \Omega. \tag{13.6}$$

Here $u : Q_T \to \mathbb{R}$, $w : Q_T \to \mathbb{R}$ and $\theta : Q_T \to \mathbb{R}$ are the unknown functions. In particular, u corresponds to the Kirchhoff transformation of the matric potential [2], w represents concentration of dissolved species and θ represents the temperature of the porous system. Further, $a : \mathbb{R} \to \mathbb{R}$, $D_w : \mathbb{R} \to \mathbb{R}$, $b : \mathbb{R} \to \mathbb{R}$, $\lambda : \mathbb{R}^2 \to \mathbb{R}$, $u_0 : \Omega \to \mathbb{R}$, $w_0 : \Omega \to \mathbb{R}$, and $\theta_0 : \Omega \to \mathbb{R}$ are given functions, ϱ is a real positive constant and $\boldsymbol{n}$ is the outward unit normal vector.

Structure and data properties. Let us introduce the assumptions on functions in (13.1)–(13.6).

(13i) $b \in C^1(\mathbb{R})$, $0 < b'(\xi) < b_*$ and

$$0 < b(\xi) \leq b_2 < +\infty \quad \forall \xi \in \mathbb{R} \quad (b_2, b_* = \text{const}).$$

(13ii) a, $D_w \in C(\mathbb{R})$ and $\lambda \in C(\mathbb{R}^2)$ such that

$$\begin{aligned} &0 < a(\xi),\ 0 < D_w(\xi) && \forall \xi \in \mathbb{R}, \\ &0 < \lambda(\xi, \zeta) && \forall \xi, \zeta \in \mathbb{R}. \end{aligned}$$

(13iii) Assume

$$u_0,\ w_0,\ \theta_0 \in L^\infty(\Omega),$$

such that

$$-\infty < u_1 < u_0 < 0 \qquad \text{a.e. in } \Omega \quad (u_1 = \text{const}). \tag{13.7}$$

Weak formulation. A weak solution of (13.1)–(13.6) is a triplet $[u, w, \theta]$ such that

$$u \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega)),\; w \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega)) \cap L^\infty(Q_T),\; \theta \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega)) \cap L^\infty(Q_T),$$

which satisfies

$$-\int_{Q_T} b(u)\partial_t\phi \,\mathrm{d}x\mathrm{d}t + \int_{Q_T} a(\theta)\nabla u \cdot \nabla\phi \,\mathrm{d}x\mathrm{d}t = \int_\Omega b(u_0)\phi(\boldsymbol{x}, 0)\,\mathrm{d}x \qquad (13.8)$$

for any $\phi \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega)) \cap W^{1,1}(I; L^\infty(\Omega))$ with $\phi(\cdot, T) = 0$;

$$-\int_{Q_T} b(u)w\partial_t\eta \,\mathrm{d}x\mathrm{d}t + \int_{Q_T} b(u)D_w(u)\nabla w \cdot \nabla\eta \,\mathrm{d}x\mathrm{d}t$$
$$+\int_{Q_T} wa(\theta)\nabla u \cdot \nabla\eta \,\mathrm{d}x\mathrm{d}t = \int_\Omega b(u_0)w_0\eta(\boldsymbol{x}, 0)\,\mathrm{d}x \qquad (13.9)$$

for any $\eta \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega)) \cap W^{1,1}(I; L^\infty(\Omega))$ with $\eta(\cdot, T) = 0$;

$$-\int_{Q_T} [b(u)\theta + \varrho\theta]\partial_t\psi \,\mathrm{d}x\mathrm{d}t + \int_{Q_T} \lambda(\theta, u)\nabla\theta \cdot \nabla\psi \,\mathrm{d}x\mathrm{d}t$$
$$+\int_{Q_T} \theta a(\theta)\nabla u \cdot \nabla\psi \,\mathrm{d}x\mathrm{d}t = \int_\Omega [b(u_0)\theta_0 + \varrho\theta_0]\psi(\boldsymbol{x}, 0)\,\mathrm{d}x \qquad (13.10)$$

for any $\psi \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega)) \cap W^{1,1}(I; L^\infty(\Omega))$ with $\psi(\cdot, T) = 0$.

Theorem 13.1 *Let the assumptions* (13i)–(13iii) *be satisfied. Then there exists at least one weak solution of the system* (13.1)–(13.6).

Part VI

Appendix

A Paper 1

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Weak solutions of coupled dual porosity flows in fractured rock mass and structured porous media

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A B S T R A C T

This paper deals with a fully nonlinear degenerate parabolic system with natural (critical) growths and under non-linear boundary conditions. Such problems arise from the heat and water flow through a partially saturated fractured rock mass and structured porous media. Existence of a global weak solution of the problem (on an arbitrary interval of time) is proved by means of semidiscretization in time, deriving suitable a-priori estimates based on $W^{1,p}$-regularity of the approximate solution and by passing to the limit from discrete approximations.

1. Introduction

In this paper we deal with mathematical analysis of fully nonlinear degenerate parabolic system modeling coupled heat transport and preferential movement of water in dual structured porous media. Variably-saturated porous medium is treated as a multi-phase material. At the microscale the individual phases can be clearly identified, however, at the macroscale, where measurements are usually carried out, the only observable quantities correspond to the effective behaviour. Because the detailed description of the geometry of the porous space is seldom known in practice, the macroscale-level equations are sought as suitable averages of the microscale balance law, for example in the framework of the hybrid mixture theory, originally proposed in [18–20]. In this context, the porous medium is considered as continuum of independent overlapping phases. For each constituent its conservation equation is derived according to principles of continuum mechanics.

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1.1. Conservation of mass

In mixture theory, the derivation of the mass balance equation is based on mass conservation of α-phase inside the spatial domain Ω of interest. A general form of a mass balance law reads [34]

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathcal{B}} \varrho^{\alpha}\,\mathrm{d}x + \int\limits_{\partial\mathcal{B}} \varrho^{\alpha}\boldsymbol{v}_{\alpha}\cdot\boldsymbol{n}\,\mathrm{d}S = \int\limits_{\mathcal{B}} s_{\alpha}\,\mathrm{d}x \tag{1.1}$$

to be satisfied for any regular subdomain $\mathcal{B} \subset \overline{\mathcal{B}} \subset \Omega$. Here, $\varrho^{\alpha} = \Theta_{\alpha}\varrho_{\alpha}$ represents the phase averaged density, Θ_{α} [-] is the volume fraction of the α-phase, ϱ_{α} [$\mathrm{kg\,m^{-3}}$] stands for the intrinsic phase averaged density and s_{α} [$\mathrm{kg\,m^{-3}\,s^{-1}}$] is a production term. Further, $\boldsymbol{v}_{\alpha}$ [$\mathrm{m\,s^{-1}}$] is the velocity of α-phase and $\boldsymbol{n}$ represents an outward unit normal vector to the boundary $\partial\mathcal{B}$. Applying the divergence theorem to (1.1) and owing to the arbitrariness of the domain $\mathcal{B}$ one arrives at the local form of the balance law

$$\frac{\partial(\Theta_{\alpha}\varrho_{\alpha})}{\partial t} + \nabla\cdot(\Theta_{\alpha}\varrho_{\alpha}\boldsymbol{v}_{\alpha}) = s_{\alpha}. \tag{1.2}$$

1.2. Conservation of heat energy

The balance of heat energy for the α-phase can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathcal{B}} e_{\alpha}\,\mathrm{d}x + \int\limits_{\partial\mathcal{B}} (\boldsymbol{q}_T)_{\alpha}\cdot\boldsymbol{n}\,\mathrm{d}S = \int\limits_{\mathcal{B}} \mathcal{Q}_{\alpha}\,\mathrm{d}x + \int\limits_{\mathcal{B}} \mathcal{E}_{\alpha}\,\mathrm{d}x - \int\limits_{\mathcal{B}} H_{\alpha}s_{\alpha}\,\mathrm{d}x, \tag{1.3}$$

where e_{α} [$\mathrm{J\,m^{-3}}$] is the total internal energy of the α-phase in $\mathcal{B}$, $(\boldsymbol{q}_T)_{\alpha}$ [$\mathrm{W\,m^{-2}}$] is the heat flux, $\mathcal{Q}_{\alpha}$ stands for the volumetric heat source, $\mathcal{E}_{\alpha}$ represents the term expressing energy exchange with the other phases and the symbol H_{α} [$\mathrm{J\,kg^{-1}}$] stands for the specific enthalpy of the α-phase. Here we assume

$$e_{\alpha} = \varrho^{\alpha}C_{\alpha}T_{\alpha}, \tag{1.4}$$

where T_{α} [K] is the absolute temperature and C_{α} [$\mathrm{J\,kg^{-1}\,K^{-1}}$] represents the specific isobaric heat of the α-phase. Further, the heat flux vector $(\boldsymbol{q}_T)_{\alpha}$ includes the conductive flux $\boldsymbol{q}_{\alpha}$ and convection

$$(\boldsymbol{q}_T)_{\alpha} = \boldsymbol{q}_{\alpha} + \varrho^{\alpha}C_{\alpha}T_{\alpha}\boldsymbol{v}_{\alpha}. \tag{1.5}$$

Hence, applying the divergence theorem to (1.3) and using (1.4) and (1.5) one obtains the heat energy conservation equation for the α-phase in the differential form

$$\partial_t\left(\varrho^{\alpha}C_{\alpha}T_{\alpha}\right) + \nabla\cdot(\boldsymbol{q}_{\alpha} + \varrho^{\alpha}C_{\alpha}T_{\alpha}\boldsymbol{v}_{\alpha}) = \mathcal{Q}_{\alpha} + \mathcal{E}_{\alpha} - H_{\alpha}s_{\alpha}. \tag{1.6}$$

1.3. Single porosity model

In the simplest case, consider the flow of a single homogeneous fluid through a porous solid, such as variably saturated water flow in soils. The mass conservation equation for the α-phase (1.2) can be particularized to both the water phase ($\alpha = w$) and the solid phase ($\alpha = s$). The mass conservation equations for the water and solid phases, respectively, become (neglecting source terms)

$$\frac{\partial(\Theta_w\varrho_w)}{\partial t} + \nabla\cdot(\Theta_w\varrho_w\boldsymbol{v}_w) = 0 \tag{1.7}$$

and

$$\frac{\partial\left(\Theta_s \rho_s\right)}{\partial t}+\nabla \cdot\left(\Theta_s \varrho_s \boldsymbol{v}_s\right)=0.$$

Under local thermal equilibrium conditions between water and solid phases ($T = T_s = T_w$) and under the assumption that the solid phase is immobile, summing up the energy conservation equations (1.6) over water and solid phases one obtains (neglecting source terms)

$$\partial_t\left(C_w \varrho_w \Theta_w T+C_s \varrho_s T\right)+\nabla \cdot \boldsymbol{q}+\nabla \cdot\left(T C_w \varrho_w \Theta_m \boldsymbol{v}_w\right)=0. \tag{1.8}$$

Equations (1.7) and (1.8), which describe the conservations of mass of water and heat energy of porous media, respectively, may be used to model the coupled flow of water and heat in a porous medium. However, in most practical applications the structured nature of a porous medium in structured soils or fractured rock formations requires a more complicated approach to describe the water movement in the porous material. One commonly used approach of this type is referred to as the dual porosity model [15].

1.4. Dual porosity model

The dual porosity medium is composed of two distinct pore homogeneous systems with contrasted hydraulic properties, the network of fractures and the matrix pore system, respectively. The amount of water present at a certain matric potential h [m] of the porous medium is characterized by the water retention curve $\Theta = \Theta(h)$ [-]. In dual porosity type structured media two retention functions are taken into account, for the matrix $\Theta_m = \Theta_m(h_m)$ [-] and fractures $\Theta_f = \Theta_f(h_f)$ [-]. Water flow is considered for both, the fractures and the matrix pore system. The transfer of water across the fracture–matrix interface is described macroscopically using a first-order coupling term [37]. Water flow in the dual porosity medium is governed by the following system of equations [39]

$$\partial_t(\varrho_w \Theta_m)+\nabla \cdot \varrho_w \Theta_m \boldsymbol{v}_m+S_m(\Theta_m, \Theta_f)=0, \tag{1.9}$$

$$\partial_t(\varrho_w \Theta_f)+\nabla \cdot \varrho_w \Theta_f \boldsymbol{v}_f+S_f(\Theta_m, \Theta_f)=0. \tag{1.10}$$

The following system of equations expresses the first law of thermodynamics in the dual porous medium allowing for the heat transfer between fractures and the matrix pore system (that is, one no longer has local thermal equilibrium between matrix and fractures)

$$\partial_t\left(C_w \varrho_w \Theta_m T_m+C_{sm} \varrho_{sm} T_m\right)+\nabla \cdot \boldsymbol{q}_m+\nabla \cdot\left(T_m C_w \varrho_w \Theta_m \boldsymbol{v}_m\right)-\beta(T_f-T_m)=0, \tag{1.11}$$

$$\partial_t\left(C_w \varrho_w \Theta_f T_f+C_{sf} \varrho_{sf} T_f\right)+\nabla \cdot \boldsymbol{q}_f+\nabla \cdot\left(T_f C_w \varrho_w \Theta_f \boldsymbol{v}_f\right)-\beta(T_m-T_f)=0. \tag{1.12}$$

A critical aspect of using this approach lies in the determination of the appropriate coupling functions S_m and S_f and a value of β in the heat exchange terms [37]. In (1.9)–(1.12), the subscripts f and m, respectively, denote the subsystems of fractures (macropores) and matrix blocks (micropores), respectively. The primary unknowns in the model are the absolute temperature of matrix T_m [K], the absolute temperature of fractures T_f [K], the fracture matric potential h_f [m] and matrix matric potential (matrix pressure head) h_m [m] (single-valued functions of the time t and the spatial position $x \in \Omega$). Further, ϱ_w [kg m^{-3}] is the density of water, C_w [J kg^{-1} K^{-1}] represents the isobaric heat capacity of water, ϱ_{sm}, ϱ_{sf} [kg m^{-3}] and C_{sm}, C_{sf} [J kg^{-1} K^{-1}], respectively, are the mass densities and the isobaric heat capacities of solid microstructure corresponding to matrix and fractures, respectively.

1.5. Initial and boundary conditions

To complete the introduction of the model, let us specify the boundary and initial conditions. The boundary conditions may be of Neumann or Dirichlet type. The water flux across the boundary is quantified by the Neumann type boundary conditions

$$\Theta_m \boldsymbol{v}_m \cdot \boldsymbol{n} = \gamma_m, \qquad \Theta_f \boldsymbol{v}_f \cdot \boldsymbol{n} = \gamma_f,$$

where the couple (γ_m, γ_f) represents the liquid flux imposed on the boundary.

As for the heat flux, we consider the natural boundary condition given by

$$\boldsymbol{q}_m \cdot \boldsymbol{n} = \alpha_c (T_m - T_\infty) + e\sigma_{SB}(T_m^4 - T_\infty^4) + g_m,$$

in which the symbol α_c designates the heat transfer coefficient, e stands for the relative surface emissivity, σ_{SB} represents the Stefan–Boltzmann constant, T_∞ [K] is the temperature of the environment and g_m represents the heat flux imposed on the boundary. Analogously, corresponding Neumann boundary conditions are considered for fractures.

The Dirichlet boundary conditions are usually given by prescribed values of the matric potential and the temperature on the boundary

$$h_m = h_{Dm}, \quad h_f = h_{Df}, \quad T_m = T_{Dm}, \quad T_f = T_{Df}.$$

The initial conditions are set as follows:

$$h_m(\cdot, 0) = h_{0m}(\cdot), \quad h_f(\cdot, 0) = h_{0f}(\cdot), \quad T_m(\cdot, 0) = T_{0m}(\cdot), \quad T_f(\cdot, 0) = T_{0f}(\cdot).$$

Here, h_{0m}, h_{0f}, T_{0m} and T_{0f} represent the initial distributions of the primary unknowns, matric potentials and temperatures.

2. Constitutive relationships and hydraulic characteristics. Application of the Kirchhoff transformation

Physical models of coupled water flow and heat transport possess a common structure, derived from balance laws for mass of water and heat energy of the system. Further, we apply Darcy's constitutive law for the mass flux

$$\Theta \boldsymbol{v} = -K(\nabla h + \boldsymbol{e}_z), \tag{2.1}$$

where $\boldsymbol{e}_z$ stands for the vertical unit vector and K [m s^{-1}] represents the hydraulic permeability of the porous media. Similarly we assume the conductive heat flux $\boldsymbol{q}$ to be given by Fourier's law

$$\boldsymbol{q} = -\Lambda \nabla T \tag{2.2}$$

with the thermal conductivity function Λ [W m^{-1} K^{-1}]. Usually, under non-isothermal processes, given functions K and λ depend on the temperature and liquid water content and are measured experimentally.

Concerning retention curves of the fracture and matrix pore systems, respectively, we mention here the commonly used relation proposed by van Genuchten and Mualem (see, for instance, [39])

$$\Theta(h) = \Theta_r + (\Theta_s - \Theta_r)[1 + |\alpha h|^{n_1}]^{-n_2}, \tag{2.3}$$

where Θ_s is the soil saturated water content [-], Θ_r is the soil residual water content [-], α [m^{-1}], n_1 and n_2 are parameters.

The temperature–pressure head dependence of the hydraulic conductivity is given by [7,36]

$$K(T,h) = k_s\,\nu_0\,\frac{\kappa(h)}{\nu(T)}, \tag{2.4}$$

where k_s [m s^{-1}] is the saturated hydraulic conductivity at the reference temperature T_0 [K], κ [m s^{-1}] is the h-dependent relative hydraulic conductivity,

$$\kappa(h) = \sqrt{S(h)}\left(1-\left(1-S(h)^{1/n_2}\right)^{n_2}\right)^2 \tag{2.5}$$

for $h<0$ (unsaturated porous media), $S(h) = \frac{\Theta(h)-\Theta_r}{\Theta_s-\Theta_r}$. Here Θ_r and Θ_s are positive constants. Finally, ν [m s^{-2}] is the temperature dependent kinematic viscosity, $\nu_0 := \nu(T_0)$. In particular, material parameters in functions (2.3)–(2.5) need to be determined for the fracture and matrix pore systems, respectively.

Let us note that the system (1.9)–(1.10) with the constitutive relationship (2.1) and material data functions (2.3)–(2.5) is degenerate with degeneracies in both elliptic and parabolic parts. It is a common treatment of nonlinear problems to introduce the so called Kirchhoff transformation, which converts these degeneracies only to the parabolic term (see [2]). In particular, define the functions $\beta_i : \mathbb{R} \to \mathbb{R}^+$, $\zeta = \beta_i(\xi)$, $i = 1,2$, by

$$\beta_1(\xi) = \int_0^\xi \kappa_m(s)\mathrm{d}s, \quad \beta_2(\xi) = \int_0^\xi \kappa_f(s)\mathrm{d}s,$$

where κ_m and κ_f are the relative hydraulic conductivities (recall (2.5)) particularized for the matrix pore system and fractures, respectively.

Finally, in order to simplify mathematical formulations, let us introduce the following notation:

$$\begin{aligned}
b_1(u_1) &:= \Theta_m(\beta_1^{-1}(u_1)), & b_2(u_2) &:= \Theta_f(\beta_2^{-1}(u_2)),\\
a_1(T) &:= (k_s)_m\,(\nu_0)_m\,\frac{1}{\nu_m(T)}, & a_2(T) &:= (k_s)_f\,(\nu_0)_f\,\frac{1}{\nu_f(T)},\\
\varrho_1 &:= \frac{C_{sm}\varrho_{sm}}{C_w\varrho_w}, & \varrho_2 &:= \frac{C_{sf}\varrho_{sf}}{C_w\varrho_w},\\
\lambda_m(T,u_1) &:= \frac{\Lambda_m(T,\beta_1^{-1}(u_1))}{C_w\varrho_w}, & \lambda_f(T,u_2) &:= \frac{\Lambda_f(T,\beta_2^{-1}(u_2))}{C_w\varrho_w},\\
\boldsymbol{e}_1(T,u_1) &:= \frac{\boldsymbol{e}_z K_m(T,\beta_1^{-1}(u_1))}{C_w\varrho_w}, & \boldsymbol{e}_2(T,u_2) &:= \frac{\boldsymbol{e}_z K_f(T,\beta_2^{-1}(u_2))}{C_w\varrho_w}
\end{aligned}$$

and finally

$$\begin{aligned}
f_1(b_1(u_1),b_2(u_2)) &:= \frac{1}{\varrho_w} S_m(\Theta_m(\beta_1^{-1}(u_1)),\Theta_f(\beta_2^{-1}(u_2))),\\
f_2(b_1(u_1),b_2(u_2)) &:= \frac{1}{\varrho_w} S_f(\Theta_m(\beta_1^{-1}(u_1)),\Theta_f(\beta_2^{-1}(u_2))).
\end{aligned}$$

This formally leads to the system (3.1)–(3.2) introduced in the next section and qualitatively analyzed in the rest of the paper.

3. Strong formulation of the problem

Let Ω be a bounded domain in $\mathbb{R}^2$, $\Omega \in C^{0,1}$ and let Γ_D and Γ_N be open disjoint subsets of $\partial\Omega$ (not necessarily connected) such that $\Gamma_D \neq \emptyset$ and the $\partial\Omega \backslash (\Gamma_D \cup \Gamma_N)$ is a finite set. Let $T \in (0, \infty)$ be fixed throughout the paper, $I = (0, T)$ and $Q_T = \Omega \times I$ denotes the space–time cylinder, $\Gamma_{DT} = \Gamma_D \times I$ and $\Gamma_{NT} = \Gamma_N \times I$.

We shall study the following initial boundary value problem ($i = 1, 2$)

$$\partial_t b_i(u_i) = \nabla \cdot \boldsymbol{a}_i(\theta_i, u_i, \nabla u_i) + f_i(b_1(u_1), b_2(u_2)) \quad \text{in } Q_T, \tag{3.1}$$

$$\partial_t \left[b_i(u_i)\theta_i + \varrho_i\theta_i\right] = \nabla \cdot (\lambda_i(\theta_i, u_i)\nabla\theta_i) + \nabla \cdot (\theta_i \boldsymbol{a}_i(\theta_i, u_i, \nabla u_i)) - h_i(\theta_1, \theta_2) \quad \text{in } Q_T, \tag{3.2}$$

$$\boldsymbol{u} = \boldsymbol{u}_D \quad \text{on } \Gamma_{DT}, \tag{3.3}$$

$$\boldsymbol{\theta} = \boldsymbol{\theta}_D \quad \text{on } \Gamma_{DT}, \tag{3.4}$$

$$-\boldsymbol{a}_i(\theta_i, u_i, \nabla u_i) \cdot \boldsymbol{n} = -\gamma_i \quad \text{on } \Gamma_{NT}, \tag{3.5}$$

$$-\lambda_i(\theta_i, u_i)\nabla\theta_i \cdot \boldsymbol{n} = \alpha_i(\theta_i) - g_i \quad \text{on } \Gamma_{NT}, \tag{3.6}$$

$$\boldsymbol{u}(x, 0) = \boldsymbol{u}_0(x) \quad \text{in } \Omega, \tag{3.7}$$

$$\boldsymbol{\theta}(x, 0) = \boldsymbol{\theta}_0(x) \quad \text{in } \Omega. \tag{3.8}$$

The system (3.1)–(3.8) arises from the coupled water movement and heat transfer through the dual porous system following the Kirchhoff transformation. Here $u_i : Q_T \to \mathbb{R}$ and $\theta_i : Q_T \to \mathbb{R}$ are the unknown functions. $\boldsymbol{u} = (u_1, u_2)$ corresponds to the Kirchhoff transformation of the matric potential and $\boldsymbol{\theta} = (\theta_1, \theta_2)$ represents the temperature of the dual porous system. The vector function $\boldsymbol{a}_i : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ admits the structure

$$\boldsymbol{a}_i(r, s, \boldsymbol{z}) = a_i(r)\boldsymbol{z} + \boldsymbol{e}_i(r, s), \tag{3.9}$$

$a_i : \mathbb{R} \to \mathbb{R}$, $\boldsymbol{e}_i : \mathbb{R}^2 \to \mathbb{R}^2$, $b_i : \mathbb{R} \to \mathbb{R}$, $\lambda_i : \mathbb{R}^2 \to \mathbb{R}$, $f_i : \mathbb{R}^2 \to \mathbb{R}$, $h_i : \mathbb{R}^2 \to \mathbb{R}$, $u_{iD} : Q_T \to \mathbb{R}$, $\theta_{iD} : Q_T \to \mathbb{R}$, $\gamma_i : \Gamma_{NT} \to \mathbb{R}$, $g_i : \Gamma_{NT} \to \mathbb{R}$, $\alpha_i : \mathbb{R} \to \mathbb{R}$, $u_{0i} : \Omega \to \mathbb{R}$ and $\theta_{0i} : \Omega \to \mathbb{R}$ are given functions, ϱ_i is a real positive constant and $\boldsymbol{n}$ is the outward unit normal vector.

In this paper we study the existence of the solution to the system (3.1)–(3.9). In the last decades, a considerable effort has been invested into detailed analysis of parabolic systems arising from the coupled heat and mass flows in porous media. The related works in this context are, for instance, due to Vala [38], Li and Sun [27], Li et al. [29] and Li and Sun [28]. Most theoretical results on parabolic systems exclude the case of non-symmetrical parabolic parts [2,13,22]. Such systems are applicable e.g. in problems modeling degradation processes in wet concrete [30–32], motion of interacting populations of colloidal species [23,25,24], population dynamics [6], water movement in porous media with a dual porosity structure [15,16,10–12] etc. Although the approach in [38] admits non-symmetry in the parabolic term, it requires unrealistic symmetry in the elliptic part. In [8,21], the authors studied the existence, uniqueness and regularity of coupled quasilinear equations modeling evolution of fluid species influenced by thermal, electrical and diffusive forces. In [27,29,28], the authors studied a model of specific structure of a heat and mass transfer arising from textile industry and proved the global existence for one-dimensional problems in [27,29] and three-dimensional problems in [28]. In [40], the authors proved the global existence of positive/non-negative weak solutions of the fully nonlinear, degenerate and strongly coupled parabolic system modeling one-dimensional heat and sweat transport in porous textile media with a non-local thermal radiation and phase change. Giaquinta and Modica in [17] proved the local-in-time solvability of quasilinear diagonal parabolic systems with nonlinear boundary conditions (without assuming any growth condition), see also [41]. Recently, the existence of local-in-time strong solutions for coupled moisture and heat transfer in multi-layer porous structures governed by the doubly nonlinear system is proven in [4]. In the present paper we extend our previous existence result for coupled heat and mass flows in porous media [3] to more general coupled parabolic system in non-smooth domains and under highly nonlinear mixed boundary conditions.

The rest of this paper is organized as follows. In Section 4, we introduce basic notation and suitable function spaces and specify our assumptions on data and coefficient functions in the problem. In Section 5, we formulate the problem in the variational sense and state the main result, the global-in-time existence of the weak solution. The main result is proved by an approximation procedure in Section 6. First we formulate the semi-discrete scheme and prove the existence of its solution (Subsection 6.1). The crucial a-priori estimates of time interpolants of the solution are proved in Subsection 6.2. Finally, we conclude that the solutions of semi-discrete scheme converge and the limit is the solution of the original time-continuous problem (Subsection 6.3).

4. Preliminaries

4.1. Notations and some properties of Sobolev spaces

Vectors and vector functions are denoted by boldface letters. Throughout the paper, we will always use positive constants C, c, c_1, c_2, $\dots$, which are not specified and which may differ from line to line. Throughout this paper we suppose $s, q, s' \in [1,\infty]$, s' denotes the conjugate exponent to $s > 1$, $1/s + 1/s' = 1$. $L^s(\Omega)$ denotes the usual Lebesgue space equipped with the norm $\|\cdot\|_{L^s(\Omega)}$ and $W^{k,s}(\Omega)$, $k \geq 0$ (k need not to be an integer, see [26]), denotes the usual Sobolev–Slobodecki space with the norm $\|\cdot\|_{W^{k,s}(\Omega)}$. We define $W^{1,2}_{\Gamma_D}(\Omega) := \left\{\phi \in W^{1,2}(\Omega);\ \phi\big|_{\Gamma_D} = 0\right\}$. By E^* we denote the space of all continuous, linear forms on Banach space E and by $\langle\cdot,\cdot\rangle$ we denote the duality between E and E^*. By $L^s(I;E)$ we denote the Bochner space (see [1]). Therefore, $L^s(I;E)^* = L^{s'}(I;E^*)$.

Remark 4.1. (See [1,26,35].) There exists a continuous linear operator (trace operator) $\mathcal{R} : W^{1,p}(\Omega) \to L^1(\partial\Omega)$ such that, for any $\phi \in C^1(\overline{\Omega})$, we have $\mathcal{R}(\phi) = \phi\big|_{\partial\Omega}$. $\mathcal{R}$ remains continuous as the mapping (for $N = 2$ in our paper) $\phi \to \phi\big|_{\partial\Omega} : W^{1,p}(\Omega) \to L^q(\partial\Omega)$, where

$$q := \begin{cases} \frac{p}{2-p}, & \text{for } 1 \leq p < 2, \\ \text{an arbitrarily large real} & \text{for } p = 2, \\ +\infty & \text{for } p > 2). \end{cases}$$

Remark 4.2. Another useful result holds for a certain interpolation between the Sobolev and Lebesgue spaces, see [13, Remark 4]. For all η sufficiently small, say $0 < \eta \leq \eta_0$, η_0 being given, we have

$$\oint_{\partial\Omega} |\phi|^2 \mathrm{d}S \leq \eta \int_\Omega |\nabla\phi|^2 \,\mathrm{d}x + C(\eta) \int_\Omega |\phi|^2 \,\mathrm{d}x \quad \text{for all } \phi \in W^{1,2}(\Omega). \tag{4.1}$$

4.2. Structure and data properties

We start by introducing our assumptions on functions in (3.1)–(3.8).

(i) b_i is a positive continuous strictly monotone function such that

$$0 < b_i(\xi) \leq b^\sharp < +\infty \qquad \forall \xi \in \mathbb{R} \quad (b^\sharp = \text{const}),$$
$$(b_i(\xi_1) - b_i(\xi_2))\,(\xi_1 - \xi_2) > 0 \qquad \forall \xi_1, \xi_2 \in \mathbb{R},\ \xi_1 \neq \xi_2.$$

(ii) a_i and λ_i are continuous functions satisfying

$$0 < a_\sharp \le a_i(\xi) \le a^\sharp < +\infty \qquad \forall \xi \in \mathbb{R} \quad (a_\sharp, a^\sharp = \text{const}),$$
$$0 < \lambda_\sharp \le \lambda_i(\xi,\zeta) \le \lambda^\sharp < +\infty \qquad \forall \xi,\zeta \in \mathbb{R} \quad (\lambda_\sharp, \lambda^\sharp = \text{const}).$$

$\boldsymbol{e}_i : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable vector function, such that

$$|\boldsymbol{e}_i(\xi,\zeta)| \le e^\sharp < +\infty \qquad \forall \xi,\zeta \in \mathbb{R} \quad (e^\sharp = \text{const}).$$

(iii) $f_i : \mathbb{R}^2 \to \mathbb{R}$ is continuous.

(iv) $h_i : \mathbb{R}^2 \to \mathbb{R}$ $(i = 1, 2)$ admits the structure

$$h_1(r,s) = \varepsilon(r-s), \qquad h_2(r,s) = \varepsilon(s-r),$$

where ε is a positive constant.

(v) $\alpha_i : \mathbb{R} \to \mathbb{R}$ admits the structure

$$\alpha_i(r) = c|r|^3 r - \sigma(r), \quad c > 0,$$

where σ is a continuous function satisfying the linear growth condition

$$|\sigma(r)| \le c(1 + |r|).$$

(vi) (Boundary and initial data) Assume

$$\boldsymbol{u}_0, \boldsymbol{\theta}_0 \in L^2(\Omega),$$
$$\boldsymbol{u}_D, \boldsymbol{\theta}_D \in L^2(I; W^{1,2+\delta}(\Omega)) \cap W^{1,1}(I; L^\infty(\Omega)) \cap L^\infty(I; L^\infty(\Gamma_D)),$$
$$\boldsymbol{\gamma}, \boldsymbol{g} \in C(\overline{Q_T})^2$$

with some $\delta > 0$.

4.3. Auxiliary results

Remark 4.3. (See [2], Section 1.1.) Let us note that (i) implies that there is a (strictly) convex C^1-function $\Phi_i : \mathbb{R} \to \mathbb{R}$, $\Phi_i(0) = 0$, $\Phi_i'(0) = 0$, such that $b_i(z) - b_i(0) = \Phi_i'(z)$ $\forall z \in \mathbb{R}$. Introduce the Legendre transform

$$B_i(z) := \int_0^1 (b_i(z) - b_i(sz))z \,\mathrm{d}s = \int_0^z (b_i(z) - b_i(s)) \,\mathrm{d}s.$$

Let us present some properties of B_i [2]:

$$B_i(z) := \int_0^1 (b_i(z) - b_i(sz))z \,\mathrm{d}s \ge 0 \qquad \forall z \in \mathbb{R},$$
$$B_i(s) - B_i(r) \ge (b_i(s) - b_i(r))r \qquad \forall r, s \in \mathbb{R},$$
$$b_i(z)z - \Phi_i(z) + \Phi_i(0) = B_i(z) \le b_i(z)z \qquad \forall z \in \mathbb{R}.$$

Proposition 4.4. *(See [2], Lemma 1.5.) Suppose* (vi). *Let* $u_i \in u_{Di} + L^2(I; W^{1,2}_{\Gamma_D}(\Omega))$, *such that* $b_i(u_i) \in L^\infty(I; L^1(\Omega))$, $\partial_t b_i(u_i) \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega)^*)$, *and*

$$\int_0^T \langle \partial_t b_i(u_i), \phi \rangle \mathrm{d}t + \int_{Q_T} (b_i(u_i) - b_i(u_{0i}))\partial_t \phi \, \mathrm{d}x\mathrm{d}t = 0$$

for every test function $\phi \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega)) \cap W^{1,1}(I; L^\infty(\Omega))$ *with* $\phi(T) = 0$. *Then* $B_i(u_i) \in L^\infty(I; L^1(\Omega))$ *and for almost all* t *the following formula holds*

$$\int_\Omega B_i(u_i(t)) \, \mathrm{d}x - \int_\Omega B_i(u_{0i}) \, \mathrm{d}x = \int_0^t \langle \partial_t b_i(u_i), u_i - u_{Di} \rangle \mathrm{d}s$$
$$- \int_0^t \int_\Omega (b_i(u_i) - b_i(u_{0i}))\partial_t u_{Di} \, \mathrm{d}x\mathrm{d}s + \int_\Omega (b_i(u_i(t)) - b_i(u_{0i}))u_{Di}(t) \, \mathrm{d}x.$$

5. The main result

The aim of this paper is to prove the existence of a weak solution to the problem (3.1)–(3.9). First we formulate our problem in a variational sense.

Definition 5.1. A weak solution of (3.1)–(3.9) is a pair $[\boldsymbol{u}, \boldsymbol{\theta}]$ such that

$$\begin{aligned} &\boldsymbol{u} \in \boldsymbol{u}_D + L^2(I; W^{1,2}_{\Gamma_D}(\Omega)^2), \\ &\boldsymbol{\theta} \in \boldsymbol{\theta}_D + L^2(I; W^{1,2}_{\Gamma_D}(\Omega)^2) \cap L^\infty(I; L^2(\Omega)^2), \\ &\alpha_i(\theta_i) \in L^{5/4}(I; L^{5/4}(\Gamma_N)), \end{aligned}$$

which satisfies $(i = 1, 2)$

$$\begin{aligned} &- \int_{Q_T} b_i(u_i)\partial_t \phi_i \, \mathrm{d}x\mathrm{d}t + \int_{Q_T} (a(\theta_i)\nabla u_i + \boldsymbol{e}_i(\theta_i, u_i)) \cdot \nabla \phi_i \, \mathrm{d}x\mathrm{d}t \\ &= \int_{Q_T} f_i(b_1(u_1), b_2(u_2))\phi_i \, \mathrm{d}x\mathrm{d}t + \int_\Omega b_i(u_{0i})\phi_i(0) \, \mathrm{d}x + \int_{\Gamma_{NT}} \gamma_i \phi_i \, \mathrm{d}S\mathrm{d}t \end{aligned} \tag{5.1}$$

$\forall \phi_i \in C^\infty(\overline{Q}_T)$, $\phi_i(x, T) = 0 \ \forall x \in \Omega$ and $\phi_i = 0$ on Γ_D;

$$\begin{aligned} &- \int_{Q_T} (b_i(u_i)\theta_i + \varrho_i\theta_i) \, \partial_t \psi_i \, \mathrm{d}x\mathrm{d}t + \int_{Q_T} \lambda_i(\theta_i, u_i)\nabla\theta_i \cdot \nabla\psi_i \, \mathrm{d}x\mathrm{d}t \\ &+ \int_{Q_T} (\theta_i \, (a_i(\theta_i)\nabla u_i + \boldsymbol{e}_i(\theta_i, u_i))) \cdot \nabla\psi_i \, \mathrm{d}x\mathrm{d}t + \int_{Q_T} h_i(\theta_1, \theta_2)\psi_i \, \mathrm{d}x\mathrm{d}t \\ &+ \int_{\Gamma_{NT}} \alpha_i(\theta_i)\psi_i \, \mathrm{d}S\mathrm{d}t - \int_{\Gamma_{NT}} \theta_i \, \gamma_i \, \psi_i \, \mathrm{d}S\mathrm{d}t \\ &= \int_\Omega (b_i(u_{0i})\theta_{0i} + \varrho_i\theta_{0i}) \, \psi_i(0) \, \mathrm{d}x + \int_{\Gamma_{NT}} g_i\psi_i \, \mathrm{d}S\mathrm{d}t \end{aligned} \tag{5.2}$$

$\forall \psi_i \in C^\infty(\overline{Q}_T)$, $\psi_i(x, T) = 0 \ \forall x \in \Omega$ and $\psi_i = 0$ on Γ_D.

The main result of this paper reads as follows.

Theorem 5.2 *(Main result). Let the assumptions* (i)–(vi) *be satisfied. Then there exists at least one weak solution of the system* (3.1)–(3.9).

To prove the main result of the paper we use the method of semidiscretization in time by constructing temporal approximations and limiting procedure. The proof can be divided into three steps. In the first step we approximate our problem by means of a semi-implicit time discretization scheme (which preserve the pseudo-monotone structure of the discrete problem) and prove the existence and $W^{1,p}(\Omega)$-regularity of piecewise constant time interpolants of $\boldsymbol{u}$. In the second step we derive suitable a-priori estimates. Finally, in the third step we pass to the limit from discrete approximations.

6. Proof of the main result

6.1. Approximations

Let us fix $p \in \mathbb{N}$ and set $\tau := T/p$ be a time step. Further, let us consider

$$\left.\begin{array}{ll} q_{ip}^n(x) := \frac{1}{\tau}\int_{(n-1)\tau}^{n\tau} q_i(x,s)\mathrm{d}s, & n = 1,\dots,p, \\ g_{ip}^n(x) := \frac{1}{\tau}\int_{(n-1)\tau}^{n\tau} g_i(x,s)\mathrm{d}s, & n = 1,\dots,p, \\ {u_D}_{ip}^n(x) := \frac{1}{\tau}\int_{(n-1)\tau}^{n\tau} u_{Di}(x,s)\mathrm{d}s, & n = 1,\dots,p, \\ {\theta_D}_{ip}^n(x) := \frac{1}{\tau}\int_{(n-1)\tau}^{n\tau} \theta_{Di}(x,s)\mathrm{d}s, & n = 1,\dots,p, \\ u_{ip}^0(x) := u_{0i}(x), & \\ \theta_{ip}^0(x) := \theta_{0i}(x). & \end{array}\right\} \text{ a.e. on } \Omega.$$

We approximate our evolution problem by a semi-implicit time discretization scheme. Then we define, in each time step, $[\boldsymbol{u}_p^n, \boldsymbol{\theta}_p^n]$ as a solution of the following steady problem.

Problem 6.1. Find a pair $[\boldsymbol{u}_p^n, \boldsymbol{\theta}_p^n] \in [{\boldsymbol{u}_D}_p^n, {\boldsymbol{\theta}_D}_p^n] + W^{1,2}_{\Gamma_D}(\Omega)^2 \times W^{1,2}_{\Gamma_D}(\Omega)^2$, $n = 1,\dots,p$, such that

$$\int_\Omega \frac{b_i(u_{ip}^n) - b_i(u_{ip}^{n-1})}{\tau}\phi_i \,\mathrm{d}x + \int_\Omega \left(a_i(\theta_{ip}^{n-1})\nabla u_{ip}^n + \boldsymbol{e}_i(\theta_{ip}^{n-1}, u_{ip}^n)\right)\cdot\nabla\phi_i \,\mathrm{d}x$$
$$= \int_\Omega f_i(b_1(u_{1p}^n), b_2(u_{2p}^n))\phi_i \,\mathrm{d}x + \int_{\Gamma_N} \gamma_{ip}^n \phi_i \,\mathrm{d}S \tag{6.1}$$

$\forall \phi_i \in C^\infty(\overline{\Omega})$ and $\phi_i = 0$ on Γ_D;

$$\int_\Omega \frac{b_i(u_{ip}^n)\theta_{ip}^n - b_i(u_{ip}^{n-1})\theta_{ip}^{n-1}}{\tau}\psi_i \,\mathrm{d}x + \varrho_i \int_\Omega \frac{\theta_{ip}^n - \theta_{ip}^{n-1}}{\tau}\psi_i \,\mathrm{d}x$$
$$+ \int_\Omega \lambda_i(\theta_{ip}^{n-1}, u_{ip}^{n-1})\nabla\theta_{ip}^n \cdot \nabla\psi_i \mathrm{d}x + \int_\Omega \theta_{ip}^n \left(a_i(\theta_{ip}^{n-1})\nabla u_{ip}^n + \boldsymbol{e}_i(\theta_{ip}^{n-1}, u_{ip}^n)\right)\cdot\nabla\psi_i \mathrm{d}x$$
$$+ \int_\Omega h_i(\theta_{1p}^n, \theta_{2p}^n)\psi_i \,\mathrm{d}x + \int_{\Gamma_N} \alpha_i(\theta_{ip}^n)\psi_i \,\mathrm{d}S - \int_{\Gamma_N} \theta_{ip}^n \gamma_{ip}^n \psi_i \,\mathrm{d}S$$
$$= \int_{\Gamma_N} g_{ip}^n \psi_i \,\mathrm{d}S \tag{6.2}$$

$\forall \psi_i \in C^\infty(\overline{\Omega})$ and $\psi_i = 0$ on Γ_D.

Theorem 6.2 *(Existence of the solution to (6.1)). Let $[\boldsymbol{u}_p^{n-1}, \boldsymbol{\theta}_p^{n-1}] \in L^2(\Omega)^2$ be given and the assumptions* (i)–(vi) *be satisfied. Then there exists $\boldsymbol{u}_p^n \in \boldsymbol{u}_{D_p}^n + W^{1,2}_{\Gamma_D}(\Omega)^2$ the solution to the discrete problem (6.1).*

Proof. Let $[\boldsymbol{u}_p^{n-1}, \boldsymbol{\theta}_p^{n-1}] \in L^2(\Omega)^2$ be given. Let us write $\boldsymbol{u}_p^n = \tilde{\boldsymbol{u}}_p^n + \boldsymbol{u}_{D_p}^n$ with a new unknown function $\tilde{\boldsymbol{u}}_p^n \in W^{1,2}_{\Gamma_D}(\Omega)^2$. This amounts to solving the problem with the homogeneous Dirichlet boundary condition on Γ_D and shifted data

$$\tilde{\boldsymbol{e}}_i(x, \theta_{ip}^{n-1}, \tilde{u}_{ip}^n) := \boldsymbol{e}_i(\theta_{ip}^{n-1}, \tilde{u}_{ip}^n + u_{D_{ip}}^n(x)), \quad \tilde{b}_i(x, \tilde{u}_{ip}^n) := b_i(\tilde{u}_{ip}^n + u_{D_{ip}}^n(x))$$

almost everywhere on Ω. Define the functional $\boldsymbol{\mu} \in [W^{1,2}_{\Gamma_D}(\Omega)^2]^*$ by

$$\langle \boldsymbol{\mu}, \boldsymbol{\phi} \rangle = \frac{1}{\tau} \sum_{i=1}^{2} \int_\Omega b_i(u_{ip}^{n-1}) \phi_i \mathrm{d}x + \sum_{i=1}^{2} \int_{\Gamma_N} \gamma_{ip}^n \phi_i \mathrm{d}S - \sum_{i=1}^{2} \int_\Omega a_i(\theta_{ip}^{n-1}) \nabla u_{D_{ip}}^n \cdot \nabla \phi_i \mathrm{d}x$$

$\forall \phi_i \in W^{1,2}_{\Gamma_D}(\Omega)$. Further, define the operator $\mathcal{A} : W^{1,2}_{\Gamma_D}(\Omega)^2 \to [W^{1,2}_{\Gamma_D}(\Omega)^2]^*$ by the equation

$$\langle \mathcal{A}(\tilde{\boldsymbol{u}}_p^n), \boldsymbol{\phi} \rangle = \sum_{i=1}^{2} \int_\Omega \left(a_i(\theta_{ip}^{n-1}) \nabla \tilde{u}_{ip}^n + \tilde{\boldsymbol{e}}_i(x, \theta_{ip}^{n-1}, \tilde{u}_{ip}^n) \right) \cdot \nabla \phi_i \, \mathrm{d}x$$
$$+ \frac{1}{\tau} \sum_{i=1}^{2} \int_\Omega \tilde{b}_i(x, \tilde{u}_{ip}^n) \phi_i \, \mathrm{d}x - \sum_{i=1}^{2} \int_\Omega f_i(\tilde{b}_1(x, \tilde{u}_{1p}^n), \tilde{b}_2(x, \tilde{u}_{2p}^n)) \phi_i \, \mathrm{d}x$$

$\forall \phi_i \in W^{1,2}_{\Gamma_D}(\Omega)$. The operator equation $\mathcal{A}(\tilde{\boldsymbol{u}}_p^n) = \boldsymbol{\mu}$ has a solution if and only if $\boldsymbol{u}_p^n \in \boldsymbol{u}_{D_p}^n + \tilde{\boldsymbol{u}}_p^n$ solves (6.1). The operator $\mathcal{A}$ is monotone in the main part. Further, for any $\tilde{\boldsymbol{u}}_p^n \in W^{1,2}_{\Gamma_D}(\Omega)^2$ we have, taking into account (i)–(iii),

$$|\langle \mathcal{A}(\tilde{\boldsymbol{u}}_p^n), \boldsymbol{\phi} \rangle| \le \left(c_1 \|\tilde{\boldsymbol{u}}_p^n\|_{W^{1,2}_{\Gamma_D}(\Omega)^2} + c_2 \right) \|\boldsymbol{\phi}\|_{W^{1,2}_{\Gamma_D}(\Omega)^2}$$

$\forall \boldsymbol{\phi} \in W^{1,2}_{\Gamma_D}(\Omega)^2$. Therefore, we can write

$$\|\mathcal{A}(\tilde{\boldsymbol{u}}_p^n)\|_{[W^{1,2}_{\Gamma_D}(\Omega)^2]^*} = \sup_{\boldsymbol{\phi} \in W^{1,2}_{\Gamma_D}(\Omega)^2} \frac{|\langle \mathcal{A}(\tilde{\boldsymbol{u}}_p^n), \boldsymbol{\phi} \rangle|}{\|\boldsymbol{\phi}\|_{W^{1,2}_{\Gamma_D}(\Omega)^2}} \le c_1 \|\tilde{\boldsymbol{u}}_p^n\|_{W^{1,2}_{\Gamma_D}(\Omega)^2} + c_2.$$

Moreover, applying Young's inequality one derives in a standard way

$$\langle \mathcal{A}(\tilde{\boldsymbol{u}}_p^n), \tilde{\boldsymbol{u}}_p^n) \rangle \ge c_1 \|\tilde{\boldsymbol{u}}_p^n\|^2_{W^{1,2}_{\Gamma_D}(\Omega)^2} - c_2.$$

Now we conclude that the operator $\mathcal{A}$ is pseudomonotone and coercive (cf. [35, Lemma 2.31, Lemma 2.32], see also [33]). Hence $\mathcal{A} : W^{1,2}_{\Gamma_D}(\Omega)^2 \to [W^{1,2}_{\Gamma_D}(\Omega)^2]^*$ is surjective, see [5]. This completes the proof. □

Theorem 6.3 *($W^{1,s}$-regularity of the solution to (6.1)). Let $\boldsymbol{u}_p^n \in \boldsymbol{u}_{D_p}^n + W^{1,2}_{\Gamma_D}(\Omega)^2$ be the weak solution to the discrete problem (6.1). Then $\boldsymbol{u}_p^n \in W^{1,s}(\Omega)^2$ with some $s > 2$.*

Theorem 6.4. *(See [14, Theorem 4], [9].) Let Ω be a bounded connected open set with a Lipschitz continuous boundary of $\mathbb{R}^N$. Let Γ be a regular part of $\partial\Omega$ and $\widetilde{\Gamma} = \partial\Omega \setminus \Gamma$. Suppose $\widetilde{\Gamma}$ has a non-null $(N-1)$-dimensional measure. There is a real number s_0, $2^* \geq s_0 > 2$, such that, if u is the weak solution of (A represents a function from $L^\infty(\Omega)$ satisfying the ellipticity condition)*

$$\begin{cases} u \in W^{1,2}_{\Gamma_D}(\Omega), \\ \int_\Omega A(x)\nabla u(x) \cdot \nabla\varphi(x)\,\mathrm{d}\Omega = \langle f, \varphi\rangle_{W^{1,2}_{\Gamma_D}(\Omega)^*, W^{1,2}_{\Gamma_D}(\Omega)}, \quad \forall\varphi \in W^{1,2}_{\Gamma_D}(\Omega), \end{cases}$$

where $f \in W^{1,s'}_{\Gamma_D}(\Omega)^*$, $s' = s/(s-1)$, $s \in [2, s_0)$. Then u belongs to $W^{1,s}_{\Gamma_D}(\Omega)$ and there exists a real number $C(s)$ such that

$$\|u\|_{W^{1,s}_{\Gamma_D}(\Omega)} \leq C(s)\|f\|_{W^{1,s'}_{\Gamma_D}(\Omega)^*}.$$

Moreover, s_0 only depends on A and Ω and $C(s)$ on A, Ω, s, not on f.

Proof of Theorem 6.3. Let us note that, provided $[\boldsymbol{u}_p^{n-1}, \boldsymbol{\theta}_p^{n-1}] \in L^2(\Omega)^2$ and by virtue of (i)–(ii) and (vi), $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{u}_{D_p}^n$ and $\boldsymbol{\gamma}_p^n$ are smooth enough to guarantee $\boldsymbol{\mu} \in [W^{1,r'}_{\Gamma_D}(\Omega)^2]^*$, $r' = r/(r-1)$, with some $r > 0$. Rewrite the equation $\mathcal{A}(\tilde{\boldsymbol{u}}_p^n) = \boldsymbol{\mu}$ in the form (transferring the lower-order terms to the right hand side)

$$\sum_{i=1}^{2} \int_\Omega a_i(\theta_{ip}^{n-1})\nabla\tilde{u}_{ip}^n \cdot \nabla\phi_i\,\mathrm{d}x = -\sum_{i=1}^{2}\int_\Omega \tilde{\boldsymbol{e}}_i(x, \theta_{ip}^{n-1}, \tilde{u}_{ip}^n) \cdot \nabla\phi_i\,\mathrm{d}x$$
$$-\frac{1}{\tau}\sum_{i=1}^{2}\int_\Omega \tilde{b}_i(x, \tilde{u}_{ip}^n)\phi_i\,\mathrm{d}x + \sum_{i=1}^{2}\int_\Omega f_i(\tilde{b}_1(x, \tilde{u}_{1p}^n), \tilde{b}_2(x, \tilde{u}_{2p}^n))\phi_i\,\mathrm{d}x + \langle\boldsymbol{\mu}, \boldsymbol{\phi}\rangle.$$

Following the proof of Theorem 6.2 we have $\tilde{\boldsymbol{u}}_p^n \in W^{1,2}_{\Gamma_D}(\Omega)^2$. Since f_i is continuous and a_i, $\tilde{b}_i$ and $\tilde{\boldsymbol{e}}_i$ are bounded functions (essentially bounded functions in Ω as compound functions of the spatial variable $x \in \Omega$), we can directly apply Theorem 6.4 to conclude the proof. □

Theorem 6.5 *(Existence of the solution to (6.2)). Let $[\boldsymbol{u}_p^{n-1}, \boldsymbol{\theta}_p^{n-1}] \in L^2(\Omega)^2$ and $\boldsymbol{u}_p^n \in W^{1,s}(\Omega)^2$, with some $s > 2$, be the solution to (6.1) and the assumptions* (i)–(vi) *be satisfied. Let τ be sufficiently small. Then there exists the solution $\boldsymbol{\theta}_p^n \in \boldsymbol{\theta}_{D_p}^n + W^{1,2}_{\Gamma_D}(\Omega)^2$ to the discrete problem (6.2).*

Proof. We proceed in the same way as in the proof of Theorem 6.2. Let $\boldsymbol{u}_p^n \in W^{1,s}(\Omega)^2$ with some $s > 2$ be the solution to the discrete problem (6.1). Writing $\boldsymbol{\theta}_p^n = \tilde{\boldsymbol{\theta}}_p^n + \boldsymbol{\theta}_{D_p}^n$ amounts to solving the problem with a new unknown function $\tilde{\boldsymbol{\theta}}_p^n$ vanishing on Γ_D (in the sense of traces). Now define the functional $\boldsymbol{\mu} \in [W^{1,2}_{\Gamma_D}(\Omega)^2]^*$ by the equation

$$\langle\boldsymbol{\mu}, \boldsymbol{\psi}\rangle = \frac{1}{\tau}\sum_{i=1}^{2}\int_\Omega \left(b_i(u_{ip}^{n-1})\theta_{ip}^{n-1} - b_i(u_{ip}^n)\theta_{D_{ip}}^n\right)\psi_i\mathrm{d}x$$
$$+\frac{1}{\tau}\sum_{i=1}^{2}\int_\Omega \varrho_i(\theta_{ip}^{n-1} - \theta_{D_{ip}}^n)\psi_i\mathrm{d}x + \sum_{i=1}^{2}\int_{\Gamma_N} g_{ip}^n\psi_i\mathrm{d}S + \sum_{i=1}^{2}\int_{\Gamma_N}\theta_{D_{ip}}^n\gamma_{ip}^n\psi_i\mathrm{d}S$$
$$-\sum_{i=1}^{2}\int_\Omega \lambda_i(\theta_{ip}^{n-1}, u_{ip}^{n-1})\nabla\theta_{D_{ip}}^n \cdot \nabla\psi_i\mathrm{d}x$$

$$-\sum_{i=1}^{2}\int_{\Omega}{\theta_D}^n_{ip}\left(a_i(\theta^{n-1}_{ip})\nabla u^n_{ip}+\boldsymbol{e}_i(\theta^{n-1}_{ip},u^n_{ip})\right)\cdot\nabla\psi_i\mathrm{d}x$$
$$-\sum_{i=1}^{2}\int_{\Omega}h_i({\theta_D}^n_{1p},{\theta_D}^n_{2p})\psi_i\mathrm{d}x$$

$\forall\psi_i\in W^{1,2}_{\Gamma_D}(\Omega)$. The regularity of $\boldsymbol{u}^n_p$ and (i), (ii), (iv) and (vi) guarantee that all integrals are well defined. Further, define the operator $\mathcal{A}: W^{1,2}_{\Gamma_D}(\Omega)^2\to[W^{1,2}_{\Gamma_D}(\Omega)^2]^*$ by the equation

$$\begin{aligned}\langle\mathcal{A}(\tilde{\boldsymbol{\theta}}^n_p),\boldsymbol{\psi}\rangle=&\frac{1}{\tau}\sum_{i=1}^{2}\int_{\Omega}\left[b_i(u^n_{ip})+\varrho_i\right]\tilde{\theta}^n_{ip}\psi_i\,\mathrm{d}x\\&+\sum_{i=1}^{2}\int_{\Omega}\lambda_i(\theta^{n-1}_{ip},u^{n-1}_{ip})\nabla\tilde{\theta}^n_{ip}\cdot\nabla\psi_i\,\mathrm{d}x\\&+\sum_{i=1}^{2}\int_{\Omega}\tilde{\theta}^n_{ip}\left(a_i(\theta^{n-1}_{ip})\nabla u^n_{ip}+\boldsymbol{e}_i(\theta^{n-1}_{ip},u^n_{ip})\right)\cdot\nabla\psi_i\,\mathrm{d}x\\&+\sum_{i=1}^{2}\int_{\Gamma_N}\tilde{\alpha}_i(x,\tilde{\theta}^n_{ip})\psi_i\,\mathrm{d}S-\sum_{i=1}^{2}\int_{\Gamma_N}\tilde{\theta}^n_{ip}\,\gamma^n_{ip}\,\psi_i\,\mathrm{d}S\\&+\sum_{i=1}^{2}\int_{\Omega}h_i(\tilde{\theta}^n_{1p},\tilde{\theta}^n_{2p})\psi_i\,\mathrm{d}x\end{aligned}$$

$\forall\psi_i\in W^{1,2}_{\Gamma_D}(\Omega)$, where $\tilde{\alpha}_i(x,\tilde{\theta}^n_{ip}):=\alpha_i(\tilde{\theta}^n_{ip}+{\theta_D}^n_{ip})$. The operator $\mathcal{A}$ is monotone in the main part. Further, since $\boldsymbol{u}^n_p\in W^{1,s}(\Omega)^2$ with some $s>2$, we have for any given $\tilde{\boldsymbol{\theta}}^n_p\in W^{1,2}_{\Gamma_D}(\Omega)^2$ the estimate

$$|\langle\mathcal{A}(\tilde{\boldsymbol{\theta}}^n_p),\boldsymbol{\psi}\rangle|\le\left(c_1\|\tilde{\boldsymbol{\theta}}^n_p\|_{W^{1,2}_{\Gamma_D}(\Omega)^2}+c_2\right)\|\boldsymbol{\psi}\|_{W^{1,2}_{\Gamma_D}(\Omega)^2}$$

$\forall\boldsymbol{\psi}\in W^{1,2}_{\Gamma_D}(\Omega)^2$. Therefore we can write

$$\|\mathcal{A}(\tilde{\boldsymbol{\theta}}^n_p)\|_{[W^{1,2}_{\Gamma_D}(\Omega)^2]^*}=\sup_{\boldsymbol{\psi}\in W^{1,2}_{\Gamma_D}(\Omega)^2}\frac{|\langle\mathcal{A}(\tilde{\boldsymbol{\theta}}^n_p),\boldsymbol{\psi}\rangle|}{\|\boldsymbol{\psi}\|_{W^{1,2}_{\Gamma_D}(\Omega)^2}}\le c_1\|\tilde{\boldsymbol{\theta}}^n_p\|_{W^{1,2}_{\Gamma_D}(\Omega)^2}+c_2.$$

In order to show coercivity of $\mathcal{A}$ we use $\phi_i=(\tilde{\theta}^n_{ip})^2$ in (6.1) to obtain

$$\begin{aligned}&\int_{\Omega}\left(a_i(\theta^{n-1}_{ip})\nabla u^n_{ip}+\boldsymbol{e}_i(\theta^{n-1}_{ip},u^n_{ip})\right)\cdot\tilde{\theta}^n_{ip}\nabla\tilde{\theta}^n_{ip}\,\mathrm{d}x\\&=\frac{1}{2}\int_{\Omega}f_i(b_1(u^n_{1p}),b_2(u^n_{2p}))(\tilde{\theta}^n_{ip})^2\,\mathrm{d}x+\frac{1}{2}\int_{\Gamma_N}\gamma^n_{ip}(\tilde{\theta}^n_{ip})^2\,\mathrm{d}S\\&-\frac{1}{2}\int_{\Omega}\frac{b_i(u^n_{ip})-b_i(u^{n-1}_{ip})}{\tau}(\tilde{\theta}^n_{ip})^2\,\mathrm{d}x.\end{aligned}\tag{6.3}$$

Let us explicitly mention that, because of the regularity $\boldsymbol{u}_p^n \in W^{1,s}(\Omega)^2$ with some $s > 2$, all integrals in (6.3) make sense. On the other hand, by the definition of the operator $\mathcal{A}$ we can write

$$
\begin{aligned}
\langle \mathcal{A}(\tilde{\boldsymbol{\theta}}_p^n), \tilde{\boldsymbol{\theta}}_p^n \rangle = {} & \frac{1}{\tau} \sum_{i=1}^{2} \int_{\Omega} \left[b_i(u_{ip}^n) + \varrho_i \right] (\tilde{\theta}_{ip}^n)^2 \mathrm{d}x + \sum_{i=1}^{2} \int_{\Omega} \lambda_i(\theta_{ip}^{n-1}, u_{ip}^{n-1}) |\nabla \tilde{\theta}_{ip}^n|^2 \mathrm{d}x \\
& + \sum_{i=1}^{2} \int_{\Omega} \left(a_i(\theta_{ip}^{n-1}) \nabla u_{ip}^n + \boldsymbol{e}_i(\theta_{ip}^{n-1}, u_{ip}^n) \right) \cdot \tilde{\theta}_{ip}^n \nabla \tilde{\theta}_{ip}^n \, \mathrm{d}x \\
& + \sum_{i=1}^{2} \int_{\Gamma_N} \tilde{\alpha}_i(x, \tilde{\theta}_{ip}^n) \tilde{\theta}_{ip}^n \, \mathrm{d}S - \sum_{i=1}^{2} \int_{\Gamma_N} \gamma_{ip}^n \, (\tilde{\theta}_{ip}^n)^2 \, \mathrm{d}S \\
& + \sum_{i=1}^{2} \int_{\Omega} h_i(\tilde{\theta}_{1p}^n, \tilde{\theta}_{2p}^n) \tilde{\theta}_{ip}^n \, \mathrm{d}x.
\end{aligned} \tag{6.4}
$$

Exploiting (6.3) we can modify (6.4) to get

$$
\begin{aligned}
\langle \mathcal{A}(\tilde{\boldsymbol{\theta}}_p^n), \tilde{\boldsymbol{\theta}}_p^n \rangle = {} & \frac{1}{2\tau} \sum_{i=1}^{2} \int_{\Omega} \left[b_i(u_{ip}^n) + b_i(u_{ip}^{n-1}) + 2\varrho_i \right] (\tilde{\theta}_{ip}^n)^2 \, \mathrm{d}x \\
& + \sum_{i=1}^{2} \int_{\Omega} \lambda_i(\theta_{ip}^{n-1}, u_{ip}^{n-1}) |\nabla \tilde{\theta}_{ip}^n|^2 \, \mathrm{d}x \\
& + \frac{1}{2} \sum_{i=1}^{2} \int_{\Omega} f_i(b_1(u_{1p}^n), b_2(u_{2p}^n)) (\tilde{\theta}_{ip}^n)^2 \, \mathrm{d}x \\
& - \frac{1}{2} \sum_{i=1}^{2} \int_{\Gamma_N} \gamma_{ip}^n (\tilde{\theta}_{ip}^n)^2 \, \mathrm{d}S + \sum_{i=1}^{2} \int_{\Gamma_N} \tilde{\alpha}_i(x, \tilde{\theta}_{ip}^n) \tilde{\theta}_{ip}^n \, \mathrm{d}S \\
& + \sum_{i=1}^{2} \int_{\Omega} h_i(\tilde{\theta}_{1p}^n, \tilde{\theta}_{2p}^n) \tilde{\theta}_{ip}^n \, \mathrm{d}x.
\end{aligned} \tag{6.5}
$$

First two integrals on the right hand side in (6.5) are nonnegative. Remaining integrals can be estimated in the following way

$$
\frac{1}{2} \sum_{i=1}^{2} \int_{\Omega} f_i(b_1(u_{1p}^n), b_2(u_{2p}^n)) (\tilde{\theta}_{ip}^n)^2 \, \mathrm{d}x \geq k_1 \sum_{i=1}^{2} \int_{\Omega} (\tilde{\theta}_{ip}^n)^2 \, \mathrm{d}x, \tag{6.6}
$$

where k_1 represents some real constant. Further,

$$
\sum_{i=1}^{2} \int_{\Omega} h_i(\tilde{\theta}_{1p}^n, \tilde{\theta}_{2p}^n) \tilde{\theta}_{ip}^n \, \mathrm{d}x \geq 0. \tag{6.7}
$$

Boundary terms can be estimated using (4.1) to obtain

$$
-\frac{1}{2} \sum_{i=1}^{2} \int_{\Gamma_N} \gamma_{ip}^n (\tilde{\theta}_{ip}^n)^2 \, \mathrm{d}S \geq k_1 \sum_{i=1}^{2} \int_{\Gamma_N} (\tilde{\theta}_{ip}^n)^2 \, \mathrm{d}S
$$

$$\geq -C(\eta)\sum_{i=1}^{2}\int_{\Omega}(\tilde{\theta}_{ip}^{n})^{2}\,\mathrm{d}x - \eta\sum_{i=1}^{2}\int_{\Omega}|\nabla\tilde{\theta}_{ip}^{n}|^{2}\,\mathrm{d}x \tag{6.8}$$

and applying (v), (4.1) and Young's inequality with sufficiently small positive parameter ϵ we have

$$\begin{aligned}
&\sum_{i=1}^{2}\int_{\Gamma_N}\tilde{\alpha}_i(x,\tilde{\theta}_{ip}^{n})\tilde{\theta}_{ip}^{n}\,\mathrm{d}S\\
&=\sum_{i=1}^{2}\int_{\Gamma_N}c|\tilde{\theta}_{ip}^{n}+{\theta_D}_{ip}^{n}|^{3}(\tilde{\theta}_{ip}^{n}+{\theta_D}_{ip}^{n})\tilde{\theta}_{ip}^{n}-\sigma(\tilde{\theta}_{ip}^{n}+{\theta_D}_{ip}^{n})\tilde{\theta}_{ip}^{n}\,\mathrm{d}S\\
&=\sum_{i=1}^{2}\int_{\Gamma_N}c|\tilde{\theta}_{ip}^{n}+{\theta_D}_{ip}^{n}|^{5}-c|\tilde{\theta}_{ip}^{n}+{\theta_D}_{ip}^{n}|^{3}(\tilde{\theta}_{ip}^{n}+{\theta_D}_{ip}^{n}){\theta_D}_{ip}^{n}\,\mathrm{d}S-\sum_{i=1}^{2}\int_{\Gamma_N}\sigma(\tilde{\theta}_{ip}^{n}+{\theta_D}_{ip}^{n})\tilde{\theta}_{ip}^{n}\,\mathrm{d}S\\
&\geq\sum_{i=1}^{2}\int_{\Gamma_N}c_1|\tilde{\theta}_{ip}^{n}+{\theta_D}_{ip}^{n}|^{5}-\epsilon|\tilde{\theta}_{ip}^{n}+{\theta_D}_{ip}^{n}|^{5}-C(\epsilon)|{\theta_D}_{ip}^{n}|^{5}\,\mathrm{d}S-c_2\sum_{i=1}^{2}\int_{\Gamma_N}(1+|\tilde{\theta}_{ip}^{n}+{\theta_D}_{ip}^{n}|)|\tilde{\theta}_{ip}^{n}|\,\mathrm{d}S\\
&\geq k_1\sum_{i=1}^{2}\int_{\Gamma_N}|\tilde{\theta}_{ip}^{n}|^{2}\,\mathrm{d}S-c_1\\
&\geq -C(\eta)\sum_{i=1}^{2}\int_{\Omega}(\tilde{\theta}_{ip}^{n})^{2}\,\mathrm{d}x-\eta\sum_{i=1}^{2}\int_{\Omega}|\nabla\tilde{\theta}_{ip}^{n}|^{2}\,\mathrm{d}x-c_1,
\end{aligned}\tag{6.9}$$

where k_1 represents some real constant and η stands for sufficiently small positive number, cf. (4.1). Hence, combining (6.5)–(6.9), for τ sufficiently small, we can write

$$\langle\mathcal{A}(\tilde{\boldsymbol{\theta}}_p^{n}),\tilde{\boldsymbol{\theta}}_p^{n}\rangle\geq c_1\|\tilde{\boldsymbol{\theta}}_p^{n}\|^{2}_{W^{1,2}_{\Gamma_D}(\Omega)^2}-c_2.$$

With the same arguments as in the proof of Theorem 6.2 we conclude that the operator $\mathcal{A}: W^{1,2}_{\Gamma_D}(\Omega)^2\to[W^{1,2}_{\Gamma_D}(\Omega)^2]^*$ is pseudomonotone and coercive and hence, surjective. The abstract equation $\mathcal{A}(\tilde{\boldsymbol{\theta}}_p^{n})=\boldsymbol{\mu}$ has a solution if and only if the function $\boldsymbol{\theta}_p^{n}={\boldsymbol{\theta}_D}_p^{n}+\tilde{\boldsymbol{\theta}}_p^{n}\in W^{1,2}_{\Gamma_D}(\Omega)^2$ is the solution to the variational equation (6.2). This completes the proof. □

6.2. A-priori estimates

Here we prove some uniform estimates (with respect to p) for the time interpolants of the solution. We define the piecewise linear time interpolants ($n=1,2,\dots,p$)

$$\left.\begin{aligned}
\hat{\phi}_{ip}(t)&=\phi_{ip}^{n-1}+\tfrac{t-(n-1)\tau}{\tau}(\phi_{ip}^{n}-\phi_{ip}^{n-1})\\
\hat{b}_{ip}(t)&=b_i(u_{ip}^{n-1})+\tfrac{t-(n-1)\tau}{\tau}(b_i(u_{ip}^{n})-b_i(u_{ip}^{n-1}))\\
\hat{B}_{ip}(t)&=b_i(u_{ip}^{n-1})\theta_{ip}^{n-1}+\tfrac{t-(n-1)\tau}{\tau}(b_i(u_{ip}^{n})\theta_{ip}^{n}-b_i(u_{ip}^{n-1})\theta_{ip}^{n-1})
\end{aligned}\right\}$$

for $t\in((n-1)\tau,n\tau]$ and the piecewise constant interpolants $\bar{\phi}_{ip}(t)=\phi_{ip}^{n}$ for $t\in((n-1)\tau,n\tau]$ and, in addition, we extend $\bar{\phi}_{ip}$ for $t\leq 0$ by $\bar{\phi}_{ip}(t)=\phi_{i0}$ for $t\in(-\tau,0]$. For a function φ we often use the simplified notation $\varphi:=\varphi(t)$, $\varphi_\tau(t):=\varphi(t-\tau)$, $\partial_t^{-\tau}\varphi(t):=\frac{\varphi(t)-\varphi(t-\tau)}{\tau}$, $\partial_t^{\tau}\varphi(t):=\frac{\varphi(t+\tau)-\varphi(t)}{\tau}$. Then, following (6.1) and (6.2), the piecewise constant time interpolants $\bar{\boldsymbol{u}}_p\in L^\infty(I;W^{1,s}(\Omega)^2)$ (with some $s>2$), and $\bar{\boldsymbol{\theta}}_p\in L^\infty(I;W^{1,2}(\Omega)^2)$ satisfy the equations

$$\int_\Omega \partial_t^{-\tau} b_i(\bar u_{ip}(t))\phi_i \,\mathrm{d}x + \int_\Omega \big(a_i(\bar\theta_{ip}(t-\tau))\nabla \bar u_{ip}(t) + \boldsymbol{e}_i(\bar\theta_{ip}(t-\tau), \bar u_{ip}(t))\big)\cdot\nabla\phi_i\,\mathrm{d}x$$
$$= \int_\Omega f_i(b_1(\bar u_{1p}(t)), b_2(\bar u_{2p}(t)))\phi_i\,\mathrm{d}x + \int_{\Gamma_N} \bar\gamma_i(t)\phi_i\,\mathrm{d}S \tag{6.10}$$

$\forall\phi_i \in C^\infty(\overline{\Omega})$ and $\phi_i = 0$ on Γ_D and

$$\int_\Omega \partial_t^{-\tau}\left(b_i(\bar u_{ip}(t))\bar\theta_{ip}(t) + \varrho_i\bar\theta_{ip}(t)\right)\psi_i\,\mathrm{d}x + \int_\Omega \lambda_i(\bar\theta_{ip}(t-\tau), \bar u_{ip}(t-\tau))\nabla\bar\theta_{ip}(t)\cdot\nabla\psi_i\,\mathrm{d}x$$
$$+ \int_\Omega \bar\theta_{ip}(t)\left(a_i(\bar\theta_{ip}(t-\tau))\nabla\bar u_{ip}(t) + \boldsymbol{e}_i(\bar\theta_{ip}(t-\tau), \bar u_{ip}(t))\right)\cdot\nabla\psi_i\,\mathrm{d}x$$
$$+ \int_{\Gamma_N} \alpha_i(\bar\theta_{ip}(t))\psi_i\,\mathrm{d}S - \int_{\Gamma_N}\bar\theta_{ip}(t)\bar\gamma_{ip}(t)\psi_i\,\mathrm{d}S + \int_\Omega h_i(\bar\theta_{1p}(t), \bar\theta_{2p}(t))\psi_i\,\mathrm{d}x$$
$$= \int_{\Gamma_N} \bar g_{ip}(t)\psi_i\,\mathrm{d}S \tag{6.11}$$

$\forall\psi_i \in C^\infty(\overline{\Omega})$ and $\psi_i = 0$ on Γ_D.

We test with $\phi_i = \bar u_{ip}(t) - \overline{u_D}_{ip}(t)$ and integrate (6.10) over t from 0 to s. For the parabolic term we can write

$$\int_0^s\int_\Omega \partial_t^{-\tau} b_i(\bar u_{ip}(t))\left(\bar u_{ip}(t) - \overline{u_D}_{ip}(t)\right)\mathrm{d}x\mathrm{d}t$$
$$= \int_0^s\int_\Omega \partial_t^{-\tau} b_i(\bar u_{ip}(t))\bar u_{ip}(t)\,\mathrm{d}x\mathrm{d}t + \int_0^s\int_\Omega \left(b_i(\bar u_{ip}(t)) - b_i(\bar u_{ip}^0)\right)\partial_t^{\tau}\overline{u_D}_{ip}(t)\,\mathrm{d}x\mathrm{d}t$$
$$+ \frac{1}{\tau}\int_{s-\tau}^s\int_\Omega \left(b_i(\bar u_{ip}^0) - b_i(\bar u_{ip}(t))\right)\overline{u_D}_{ip}(t+\tau)\,\mathrm{d}x\mathrm{d}t$$
$$\geq \frac{1}{\tau}\int_{s-\tau}^s\int_\Omega B_i(\bar u_{ip}(t)) - B_i(\bar u_{ip}^0(t))\,\mathrm{d}x\mathrm{d}t + \int_0^s\int_\Omega \left(b_i(\bar u_{ip}(t)) - b_i(\bar u_{ip}^0)\right)\partial_t^{\tau}\overline{u_D}_{ip}(t)\,\mathrm{d}x\mathrm{d}t$$
$$+ \frac{1}{\tau}\int_{s-\tau}^s\int_\Omega \left(b_i(\bar u_{ip}^0) - b_i(\bar u_{ip}(t))\right)\overline{u_D}_{ip}(t+\tau)\,\mathrm{d}x\mathrm{d}t. \tag{6.12}$$

Further, adding (6.10) over $i = 1, 2$, using (6.12), applying the usual estimates for the elliptic part and, finally, using Gronwall's lemma, we obtain the a-priori estimate

$$\sum_{i=1}^2 \sup_{0\le t\le T}\int_\Omega B_i(\bar u_{ip}(t))\mathrm{d}x + \sum_{i=1}^2\int_0^T\int_\Omega |\nabla\bar u_{ip}(t)|^2\mathrm{d}x\mathrm{d}t \le c. \tag{6.13}$$

As a consequence of the preceding a-priori estimate (6.13) we see that there exists a function $\boldsymbol{u} \in L^2(I; W^{1,2}(\Omega)^2)$ such that, along a selected subsequence (letting $p \to \infty$), we have

$$\bar{\boldsymbol{u}}_p(t) \rightharpoonup \boldsymbol{u} \qquad \text{weakly in } L^2(I; W^{1,2}(\Omega)^2). \tag{6.14}$$

In order to show that $\bar{\boldsymbol{u}}_p$ converges to $\boldsymbol{u}$ almost everywhere on Q_T we follow [2]. Let $k \in \mathbb{N}$ and use

$$\phi_i(t) = \partial_t^{k\tau} \left(\bar{u}_{ip}(s) - \overline{u_D}_{ip}(s) \right)$$

for $j\tau \le t \le (j+k)\tau$ with $(j-1)\tau \le s \le j\tau$ and $1 \le j \le \frac{T}{\tau} - k$, as a test function in (6.10). For the parabolic term, we can write

$$\begin{aligned}
&\int_{j\tau}^{(j+k)\tau} \int_\Omega \partial_t^{-\tau} b_i(\bar{u}_{ip}(t))\, \partial_t^{k\tau} \left(\bar{u}_{ip}(t) - \overline{u_D}_{ip}(t) \right) \mathrm{d}x\mathrm{d}t \\
&= \frac{1}{k\tau^2} \int_{(j-1)\tau}^{j\tau} \int_\Omega \left(b_i(\bar{u}_{ip}(t+k\tau)) - b_i(\bar{u}_{ip}(t)) \right) \left(\bar{u}_{ip}(t+k\tau) - \bar{u}_{ip}(t) \right) \mathrm{d}x\mathrm{d}t \\
&- \frac{1}{k\tau^2} \int_{(j-1)\tau}^{j\tau} \int_\Omega \left(b_i(\bar{u}_{ip}(t+k\tau)) - b_i(\bar{u}_{ip}(t)) \right) \left(\overline{u_D}_{ip}(t+k\tau) - \overline{u_D}_{ip}(t) \right) \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Hence, summing over $j = 1, \ldots, p-k$ we get the estimate

$$\begin{aligned}
&\sum_{j=1}^{p-k} \int_{j\tau}^{(j+k)\tau} \int_\Omega \partial_t^{-\tau} b_i(\bar{u}_{ip}(t))\, \partial_t^{k\tau} \left(\bar{u}_{ip}(t) - \overline{u_D}_{ip}(t) \right) \mathrm{d}x\mathrm{d}t \\
&\quad \ge \frac{1}{k\tau^2} \int_0^{T-k\tau} \int_\Omega \left(b_i(\bar{u}_{ip}(t+k\tau)) - b_i(\bar{u}_{ip}(t)) \right) \left(\bar{u}_{ip}(t+k\tau) - \bar{u}_{ip}(t) \right) \mathrm{d}x\mathrm{d}t \\
&\quad - \frac{c}{\tau} \int_{(j-1)\tau}^{j\tau} \int_\Omega |\partial_t^{k\tau} \overline{u_D}_{ip}(t)| \,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{6.15}$$

Similarly, for the elliptic term, after a little lengthy but straightforward computation we obtain

$$\begin{aligned}
&\sum_{j=1}^{p-k} \int_{j\tau}^{(j+k)\tau} \int_\Omega \left(a_i(\bar{\theta}_{ip\tau} \nabla \bar{u}_{ip} + \boldsymbol{e}_i(\bar{\theta}_{ip\tau}, \bar{u}_{ip}) \right) \cdot \nabla \partial_t^{k\tau} \left(\bar{u}_{ip} - \overline{u_D}_{ip} \right) \mathrm{d}x\mathrm{d}t \\
&= \sum_{\ell=1}^{k} \sum_{j=1}^{p-k} \int_{(j+\ell-1)\tau}^{(j+\ell)\tau} \int_\Omega \left(a_i(\bar{\theta}_{ip\tau}) \nabla \bar{u}_{ip} + \boldsymbol{e}_i(\bar{\theta}_{ip\tau}, \bar{u}_{ip}) \right) \cdot \nabla \partial_t^{k\tau} \left(\bar{u}_{ip} - \overline{u_D}_{ip} \right) \mathrm{d}x\mathrm{d}t \\
&= \sum_{\ell=1}^{k} \int_{\ell\tau}^{T-k\tau+\ell\tau} \int_\Omega \left(a_i(\bar{\theta}_{ip\tau}(t)) \nabla \bar{u}_{ip}(t) + \boldsymbol{e}_i(\bar{\theta}_{ip\tau}(t), \bar{u}_{ip}(t)) \right) \cdot \nabla \partial_t^{k\tau} \bar{u}_{ip}(t-\ell\tau) \,\mathrm{d}x\mathrm{d}t \\
&- \sum_{\ell=1}^{k} \int_{\ell\tau}^{T-k\tau+\ell\tau} \int_\Omega \left(a_i(\bar{\theta}_{ip\tau}(t)) \nabla \bar{u}_{ip}(t) + \boldsymbol{e}_i(\bar{\theta}_{ip\tau}(t), \bar{u}_{ip}(t)) \right) \cdot \nabla \partial_t^{k\tau} \overline{u_D}_{ip}(t-\ell\tau) \,\mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$\leq \frac{c_1}{\tau} \int\limits_{Q_T} |a_i(\bar{\theta}_{ip\tau}) \nabla \bar{u}_{ip} + \boldsymbol{e}_i(\bar{\theta}_{ip\tau}, \bar{u}_{ip})|^2 \, \mathrm{d}x\mathrm{d}t + \frac{c_2}{\tau} \int\limits_{Q_T} |\nabla \bar{u}_{ip}|^2 + |\nabla \overline{u_D}_{ip}|^2 \, \mathrm{d}x\mathrm{d}t$$

$$\leq \frac{C}{\tau}. \tag{6.16}$$

Similarly, for the last terms we arrive at the estimate

$$\sum_{j=1}^{p-k} \int\limits_{j\tau}^{(j+k)\tau} \int\limits_{\Omega} f_i(b_1(\bar{u}_{1p}), b_2(\bar{u}_{2p})) \partial_t^{k\tau} \left(\bar{u}_{ip} - \overline{u_D}_{ip}\right) \, \mathrm{d}x\mathrm{d}t + \sum_{j=1}^{p-k} \int\limits_{j\tau}^{(j+k)\tau} \int\limits_{\Gamma_N} \bar{\gamma}_i \partial_t^{k\tau} \left(\bar{u}_{ip} - \overline{u_D}_{ip}\right) \, \mathrm{d}S\mathrm{d}t \leq \frac{C}{\tau}. \tag{6.17}$$

Combining (6.15)–(6.17) and using (6.13) we obtain

$$\sum_{i=1}^{2} \int\limits_{0}^{T-k\tau} \left(b_i(\bar{u}_{ip}(s+k\tau)) - b_i(\bar{u}_{ip}(s))\right) \left(\bar{u}_{ip}(s+k\tau) - \bar{u}_{ip}(s)\right) \mathrm{d}s \leq ck\tau.$$

Using the compactness argument one can show in the same way as in [2, Lemma 1.9] and [13, Eqs. (2.10)–(2.12)]

$$b_i(\bar{u}_{ip}) \to b_i(u_i) \text{ in } L^1(Q_T) \tag{6.18}$$

and almost everywhere on Q_T. Since b_i is strictly monotone, it follows from (6.18) that [22, Proposition 3.35]

$$\bar{\boldsymbol{u}}_p \to \boldsymbol{u} \qquad \text{almost everywhere on } Q_T. \tag{6.19}$$

Now we use $\psi_i(t) = 2(\bar{\theta}_{ip}(t) - \overline{\theta_D}_{ip}(t))$ as a test function in (6.11) to obtain

$$\begin{aligned}
&\int\limits_{\Omega} \partial_t^{-\tau} b_i(\bar{u}_{ip}(t)) 2\bar{\theta}_{ip}(t)^2 \, \mathrm{d}x - \int\limits_{\Omega} \partial_t^{-\tau} b_i(\bar{u}_{ip}(t)) 2\bar{\theta}_{ip}(t) \overline{\theta_D}_{ip}(t) \, \mathrm{d}x \\
&+ \int\limits_{\Omega} \partial_t^{-\tau} \bar{\theta}_{ip}(t) 2(\bar{\theta}_{ip}(t) - \overline{\theta_D}_{ip}(t)) \left(b_i(\bar{u}_{ip}(t-\tau)) + \varrho_i\right) \mathrm{d}x \\
&+ 2 \int\limits_{\Omega} \lambda_i(\bar{\theta}_{ip}(t-\tau), \bar{u}_{ip}(t-\tau)) \nabla \bar{\theta}_{ip}(t) \cdot \nabla(\bar{\theta}_{ip}(t) - \overline{\theta_D}_{ip}(t)) \, \mathrm{d}x \\
&+ \int\limits_{\Omega} \left(a_i(\bar{\theta}_{ip}(t-\tau)) \nabla \bar{u}_{ip}(t) + \boldsymbol{e}_i(\bar{\theta}_{ip}(t-\tau), \bar{u}_{ip}(t))\right) \cdot 2\bar{\theta}_{ip}(t) \nabla \bar{\theta}_{ip}(t) \, \mathrm{d}x \\
&- \int\limits_{\Omega} \left(a_i(\bar{\theta}_{ip}(t-\tau)) \nabla \bar{u}_{ip}(t) + \boldsymbol{e}_i(\bar{\theta}_{ip}(t-\tau), \bar{u}_{ip}(t))\right) \cdot 2\bar{\theta}_{ip}(t) \nabla \overline{\theta_D}_{ip} \, \mathrm{d}x \\
&+ 2 \int\limits_{\Gamma_N} \alpha_i(\bar{\theta}_{ip}(t))(\bar{\theta}_{ip}(t) - \overline{\theta_D}_{ip}(t)) \, \mathrm{d}S - 2 \int\limits_{\Gamma_N} \bar{\theta}_{ip}(t) \bar{\gamma}_{ip}(t)(\bar{\theta}_{ip}(t) - \overline{\theta_D}_{ip}(t)) \, \mathrm{d}S \\
&+ 2 \int\limits_{\Omega} h_i(\bar{\theta}_{1p}(t), \bar{\theta}_{2p}(t))(\bar{\theta}_{ip}(t) - \overline{\theta_D}_{ip}(t)) \, \mathrm{d}x \\
&= 2 \int\limits_{\Gamma_N} \bar{g}_{ip}(t)(\bar{\theta}_{ip}(t) - \overline{\theta_D}_{ip}(t)) \, \mathrm{d}S.
\end{aligned} \tag{6.20}$$

One is allowed to use $\phi_i(t) = \bar{\theta}_{ip}(t)^2 - \overline{\theta_D}_{ip}(t)^2$ as a test function in (6.10) to obtain

$$
\begin{aligned}
&\int_\Omega \partial_t^{-\tau} b_i(\bar{u}_{ip}(t))\bar{\theta}_{ip}(t)^2 \,\mathrm{d}x \\
&+ \int_\Omega \big(a_i(\bar{\theta}_{ip}(t-\tau))\nabla \bar{u}_{ip}(t) + \boldsymbol{e}_i(\bar{\theta}_{ip}(t-\tau), \bar{u}_{ip}(t))\big) \cdot \nabla \bar{\theta}_{ip}(t)^2 \,\mathrm{d}x \\
&= \int_\Omega \partial_t^{-\tau} b_i(\bar{u}_{ip}(t))\overline{\theta_D}_{ip}(t)^2 \,\mathrm{d}x \\
&+ \int_\Omega \big(a_i(\bar{\theta}_{ip}(t-\tau))\nabla \bar{u}_{ip}(t) + \boldsymbol{e}_i(\bar{\theta}_{ip}(t-\tau), \bar{u}_{ip}(t))\big) \cdot \nabla \overline{\theta_D}_{ip}(t)^2 \,\mathrm{d}x \\
&+ \int_{\Gamma_N} \bar{\gamma}_i(t) \left(\bar{\theta}_{ip}(t)^2 - \overline{\theta_D}_{ip}(t)^2\right) \,\mathrm{d}S + \int_\Omega f_i(b_1(\bar{u}_{1p}), b_2(\bar{u}_{2p})) \left(\bar{\theta}_{ip}(t)^2 - \overline{\theta_D}_{ip}(t)^2\right) \,\mathrm{d}x.
\end{aligned}
\tag{6.21}
$$

Combining (6.20) and (6.21) we deduce

$$
\begin{aligned}
&\int_\Omega \partial_t^{-\tau} \left[\left(\bar{\theta}_{ip}(t) - \overline{\theta_D}_{ip}(t)\right)^2 (b_i(\bar{u}_{ip}(t)) + \varrho_i)\right] \mathrm{d}x \\
&+ \int_\Omega \partial_t^{-\tau} \overline{\theta_D}_{ip}(t) 2(\bar{\theta}_{ip}(t) - \overline{\theta_D}_{ip}(t)) \left(b_i(\bar{u}_{ip}(t-\tau)) + \varrho_i\right) \mathrm{d}x \\
&+ \int_\Omega \frac{1}{\tau} \left[\left(\bar{\theta}_{ip}(t) - \overline{\theta_D}_{ip}(t)\right) - \left(\bar{\theta}_{ip}(t-\tau) - \overline{\theta_D}_{ip}(t-\tau)\right)\right]^2 \left(b_i(\bar{u}_{ip}(t-\tau)) + \varrho_i\right) \mathrm{d}x \\
&+ 2\int_\Omega \lambda_i(\bar{\theta}_{ip}(t-\tau), \bar{u}_{ip}(t-\tau)) \nabla \bar{\theta}_{ip}(t) \cdot \nabla(\bar{\theta}_{ip}(t) - \overline{\theta_D}_{ip}(t)) \,\mathrm{d}x \\
&+ \int_\Omega \big(a_i(\bar{\theta}_{ip}(t-\tau))\nabla \bar{u}_{ip}(t) + \boldsymbol{e}_i(\bar{\theta}_{ip}(t-\tau), \bar{u}_{ip}(t))\big) \cdot 2\overline{\theta_D}_{ip}(t) \nabla \overline{\theta_D}_{ip}(t) \,\mathrm{d}x \\
&- \int_\Omega \big(a_i(\bar{\theta}_{ip}(t-\tau))\nabla \bar{u}_{ip}(t) + \boldsymbol{e}_i(\bar{\theta}_{ip}(t-\tau), \bar{u}_{ip}(t))\big) \cdot 2\bar{\theta}_{ip}(t) \nabla \overline{\theta_D}_{ip}(t) \,\mathrm{d}x \\
&+ 2\int_{\Gamma_N} \alpha_i(\bar{\theta}_{ip}(t))(\bar{\theta}_{ip}(t) - \overline{\theta_D}_{ip}(t)) \,\mathrm{d}S - 2\int_{\Gamma_N} \bar{\theta}_{ip}(t)\bar{\gamma}_{ip}(t)(\bar{\theta}_{ip}(t) - \overline{\theta_D}_{ip}(t)) \,\mathrm{d}S \\
&+ 2\int_\Omega h_i(\bar{\theta}_{1p}(t), \bar{\theta}_{2p}(t))(\bar{\theta}_{ip}(t) - \overline{\theta_D}_{ip}(t)) \,\mathrm{d}x \\
&+ \int_\Omega f_i(b_1(\bar{u}_{1p}), b_2(\bar{u}_{2p})) \left(\bar{\theta}_{ip}(t)^2 - \overline{\theta_D}_{ip}(t)^2\right) \mathrm{d}x \\
&= 2\int_{\Gamma_N} \bar{g}_{ip}(t)(\bar{\theta}_{ip}(t) - \overline{\theta_D}_{ip}(t)) \,\mathrm{d}S - \int_{\Gamma_N} \bar{\gamma}_i(t) \left(\bar{\theta}_{ip}(t)^2 - \overline{\theta_D}_{ip}(t)^2\right) \mathrm{d}S.
\end{aligned}
\tag{6.22}
$$

Adding (6.22) over $i = 1, 2$, integrating with respect to time t and using Gronwall's argument we obtain the a-priori estimate

$$\sum_{i=1}^{2} \sup_{0\le t\le T} \int_\Omega |\bar\theta_{ip}(t)|^2 \mathrm{d}x + \sum_{i=1}^{2} \int_0^T \|\bar\theta_{ip}(t)\|^2_{W^{1,2}(\Omega)} \mathrm{d}t + \sum_{i=1}^{2} \int_0^T \|\bar\theta_{ip}(t)\|^5_{L^5(\Gamma_N)} \mathrm{d}t \le c. \tag{6.23}$$

Let us mention that (6.23) becomes

$$\|\bar{\boldsymbol\theta}_p\|_{L^2(I;W^{1,2}(\Omega)^2)} \le c, \tag{6.24}$$

$$\|\bar{\boldsymbol\theta}_p\|_{L^\infty(I;L^2(\Omega)^2)} \le c, \tag{6.25}$$

$$\|\bar{\boldsymbol\theta}_p\|_{L^5(I;L^5(\Gamma_N)^2)} \le c. \tag{6.26}$$

The a-priori estimate (6.24) allows us to conclude that there exists $\boldsymbol\theta \in L^2(I;W^{1,2}(\Omega)^2)$ such that, letting $p \to +\infty$ (along a selected subsequence),

$$\bar{\boldsymbol\theta}_p \rightharpoonup \boldsymbol\theta \qquad \text{weakly in } L^2(I;W^{1,2}(\Omega)^2). \tag{6.27}$$

Now our aim is to show the a-priori bound $\|\partial_t(\hat B_{ip} + \varrho_i\hat\theta_{ip})\|_{L^{5/4}(I;W^{1,5}_{\Gamma_D}(\Omega)^*)} \le c$ that can be deduced directly from equation (6.11) exploiting the uniform bounds (6.13) and (6.23). Assume $\psi_i \in L^5(I;W^{1,5}_{\Gamma_D}(\Omega))$ and integrate (6.11) over I to obtain

$$\begin{aligned}
&\int_{Q_T} \partial_t^{-\tau} \left(b_i(\bar u_{ip}(t))\bar\theta_{ip}(t) + \varrho_i\bar\theta_{ip}(t)\right) \psi_i \,\mathrm{d}x\mathrm{d}t \\
&= \int_{\Gamma_{NT}} \bar g_{ip}(t)\psi_i \,\mathrm{d}S\mathrm{d}t - \int_{Q_T} \lambda_i(\bar\theta_{ip}(t-\tau), \bar u_{ip}(t-\tau))\nabla\bar\theta_{ip}(t)\cdot\nabla\psi_i \,\mathrm{d}x\mathrm{d}t \\
&- \int_{Q_T} \bar\theta_{ip}(t)\left(a_i(\bar\theta_{ip}(t-\tau))\nabla\bar u_{ip}(t) + \boldsymbol e_i(\bar\theta_{ip}(t-\tau), \bar u_{ip}(t))\right)\cdot\nabla\psi_i \,\mathrm{d}x\mathrm{d}t \\
&- \int_{\Gamma_{NT}} \alpha_i(\bar\theta_{ip}(t))\psi_i \,\mathrm{d}S\mathrm{d}t + \int_{\Gamma_{NT}} \bar\theta_{ip}(t)\bar\gamma_{ip}(t)\psi_i \,\mathrm{d}S\mathrm{d}t \\
&- \int_{Q_T} h_i(\bar\theta_{1p}(t), \bar\theta_{2p}(t))\psi_i \,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{6.28}$$

By means of a simple interpolation argument (see [3, eqs. (5.38) and (5.39)] for the details) we have

$$L^2(I;W^{1,2}(\Omega)) \cap L^\infty(I;L^2(\Omega)) \hookrightarrow L^{10/3}(Q_T). \tag{6.29}$$

Using (6.13), (6.24), (6.25) and (6.29) we get

$$\begin{aligned}
&\|\bar\theta_{ip}\left[a_i(\bar\theta_{ip\tau})\nabla\bar u_{ip} + \boldsymbol e_i(\bar\theta_{ip\tau}, \bar u_{ip})\right]\|_{L^{5/4}(Q_T)^2} \\
&\le \|\bar\theta_{ip}\|_{L^{10/3}(Q_T)}\left(\|a_i(\bar\theta_{ip\tau})\nabla\bar u_{ip}\|_{L^2(Q_T)^2} + \|\boldsymbol e_i(\bar\theta_{ip\tau}, \bar u_{ip})\|_{L^2(Q_T)^2}\right) \le c.
\end{aligned} \tag{6.30}$$

The latter relation yields the uniform bound of the “critical” convective term in equation (6.28) in the sense

$$\begin{aligned}
&\left|\int_{Q_T} \bar\theta_{ip}\left[a_i(\bar\theta_{ip\tau})\nabla\bar u_{ip} + \boldsymbol e_i(\bar\theta_{ip\tau}, \bar u_{ip})\right]\cdot\nabla\psi_i \,\mathrm{d}x\mathrm{d}t\right| \\
&\le c\|\bar\theta_{ip}\left[a_i(\bar\theta_{ip\tau})\nabla\bar u_{ip} + \boldsymbol e_i(\bar\theta_{ip\tau}, \bar u_{ip})\right]\|_{L^{5/4}(Q_T)^2}\|\psi_i\|_{L^5(I;W^{1,5}_{\Gamma_D}(\Omega))} \\
&\le c\|\psi_i\|_{L^5(I;W^{1,5}_{\Gamma_D}(\Omega))}.
\end{aligned} \tag{6.31}$$

The nonlinear boundary term can be handled as follows

$$\begin{aligned}\Big|\int\limits_{\Gamma_{NT}} \alpha_i(\bar{\theta}_{ip})\psi_i \,\mathrm{d}S\mathrm{d}t\Big| &\le \Bigg(\int\limits_{\Gamma_{NT}} |\alpha_i(\bar{\theta}_{ip})|^{5/4}\,\mathrm{d}S\mathrm{d}t\Bigg)^{4/5}\Bigg(\int\limits_{\Gamma_{NT}} |\psi_i|^5\,\mathrm{d}S\mathrm{d}t\Bigg)^{1/5}\\ &\le c\|\bar{\theta}_{ip}\|_{L^5(I;L^5(\Gamma_N))}\|\psi_i\|_{L^5(I;W^{1,5}_{\Gamma_D}(\Omega))}\\ &\le c\|\psi_i\|_{L^5(I;W^{1,5}_{\Gamma_D}(\Omega))}. \end{aligned} \tag{6.32}$$

The other terms on the right hand side of (6.28) can be handled in a more straightforward way. Moreover, it is easy to see that

$$\int\limits_{Q_T} \partial_t^{-\tau}\left(b_i(\bar{u}_{ip}(t))\bar{\theta}_{ip}(t) + \varrho_i\bar{\theta}_{ip}(t)\right)\psi_i\,\mathrm{d}x\mathrm{d}t = \int\limits_{Q_T} \partial_t(\hat{B}_{ip}(t) + \varrho_i\hat{\theta}_{ip}(t))\psi_i\,\mathrm{d}x\mathrm{d}t \tag{6.33}$$

for all $\psi_i \in L^5(I;W^{1,5}_{\Gamma_D}(\Omega))$. Finally, equation (6.28) combined with (6.33) and estimates (6.24), (6.26), (6.31) and (6.32) gives rise to the desired bound

$$\|\partial_t(\hat{B}_{ip} + \varrho_i\hat{\theta}_{ip})\|_{L^{5/4}(I;W^{1,5}_{\Gamma_D}(\Omega)^*)} \le c. \tag{6.34}$$

Further, we can write

$$\|\hat{B}_{ip} + \varrho_i\hat{\theta}_{ip}\|_{L^{5/4}(I;W^{1,5/4}(\Omega))} \le c.$$

Since

$$W^{1,5/4}(\Omega) \hookrightarrow\hookrightarrow W^{1-\beta,5/4}(\Omega) \hookrightarrow W^{1,5}_{\Gamma_D}(\Omega)^*,$$

where β is a small positive real number, the Aubin–Lions lemma yields the existence of $\chi_i \in L^{5/4}(I;W^{1-\beta,5/4}(\Omega))$ such that (modulo a subsequence)

$$\hat{B}_{ip} + \varrho_i\hat{\theta}_{ip} \to \chi_i \qquad \text{strongly in } L^{5/4}(I;W^{1-\beta,5/4}(\Omega))$$

and almost everywhere on Q_T and therefore also we have

$$b_i(\bar{u}_{ip})\bar{\theta}_{ip} + \varrho_i\bar{\theta}_{ip} \to \chi_i \qquad \text{strongly in } L^{5/4}(I;W^{1-\beta,5/4}(\Omega)).$$

Since $\bar{u}_{ip}$ converges almost everywhere on Q_T to u_i, we conclude

$$\bar{\boldsymbol{\theta}}_p \to \boldsymbol{\theta} \qquad \text{almost everywhere on } Q_T. \tag{6.35}$$

Hence, $b_i(\bar{u}_{ip})\bar{\theta}_{ip} + \varrho_i\bar{\theta}_{ip}$ converges almost everywhere on Q_T to $b_i(u_i)\theta_i + \varrho_i\theta_i$ and

$$\chi_i = b_i(u_i)\theta_i + \varrho_i\theta_i.$$

Now, taking into account (6.34), we get

$$\partial_t^{-\tau}\left(b_i(\bar{u}_{ip}(t))\bar{\theta}_{ip}(t) + \varrho_i\bar{\theta}_{ip}(t)\right) \rightharpoonup \partial_t(b_i(u_i)\theta_i + \varrho_i\theta_i) \quad \text{weakly in } L^{5/4}(I;W^{1,5}_{\Gamma_D}(\Omega)^*). \tag{6.36}$$

Finally, [13, Lemma 3], together with (6.23), (6.26) and (6.35), yields

$$\bar{\boldsymbol{\theta}}_p \to \boldsymbol{\theta} \qquad \text{almost everywhere on } \Gamma_{NT}, \tag{6.37}$$

$$\bar{\boldsymbol{\theta}}_p \rightharpoonup \boldsymbol{\theta} \qquad \text{weakly in } L^5(I; L^5(\Gamma_N)^2). \tag{6.38}$$

6.3. Passage to the limit for $p \to \infty$

The above established convergences (6.14), (6.19) and (6.27), (6.35), (6.36), (6.37) and (6.38) are sufficient for taking the limit $p \to \infty$ in (6.10) and (6.11) (along a selected subsequence) to get the weak solution of the system (3.1)–(3.9) in the sense of Definition 5.1. This completes the proof of the main result stated by Theorem 5.2.

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GLOBAL WEAK SOLUTIONS TO DEGENERATE COUPLED DIFFUSION-CONVECTION-DISPERSION PROCESSES AND HEAT TRANSPORT IN POROUS MEDIA

MICHAL BENEŠ, LUKÁŠ KRUPIČKA

Abstract. In this contribution we prove the existence of weak solutions to degenerate parabolic systems arising from the coupled moisture movement, transport of dissolved species and heat transfer through partially saturated porous materials. Physically motivated mixed Dirichlet-Neumann boundary conditions and initial conditions are considered. Existence of a global weak solution of the problem is proved by means of semidiscretization in time and by passing to the limit from discrete approximations. Degeneration occurs in the nonlinear transport coefficients which are not assumed to be bounded below and above by positive constants. Degeneracies in all transport coefficients are overcome by proving suitable a priori L^∞-estimates for the approximations of primary unknowns of the system.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$, $\Omega \in C^{0,1}$ and let Γ_D and Γ_N be open disjoint subsets of $\partial\Omega$ (not necessarily connected) such that $\Gamma_D \neq \emptyset$ and the $\partial\Omega \backslash (\Gamma_D \cup \Gamma_N)$ is a finite set. Let $T \in (0,\infty)$ be fixed throughout the paper, $I = (0,T)$ and $Q_T = \Omega \times I$ denotes the space-time cylinder, $\Gamma_{DT} = \Gamma_D \times I$ and $\Gamma_{NT} = \Gamma_N \times I$.

We shall study the following initial boundary value problem in Q_T,

$$\partial_t b(u) = \nabla \cdot [a(\theta)\nabla u], \tag{1.1}$$

$$\partial_t [b(u)w] = \nabla \cdot [b(u)D_w(u)\nabla w] + \nabla \cdot [wa(\theta)\nabla u], \tag{1.2}$$

$$\partial_t [b(u)\theta + \varrho\theta] = \nabla \cdot [\lambda(\theta, u)\nabla\theta] + \nabla \cdot [\theta a(\theta)\nabla u], \tag{1.3}$$

with the mixed-type boundary conditions

$$u = 0, \quad w = 0, \quad \theta = 0 \quad \text{on } \Gamma_{DT}, \tag{1.4}$$

$$\nabla u \cdot \mathbf{n} = 0, \quad \nabla w \cdot \mathbf{n} = 0, \quad \nabla\theta \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_{NT} \tag{1.5}$$

and the initial conditions

$$u(\cdot, 0) = u_0, \quad w(\cdot, 0) = w_0, \quad \theta(\cdot, 0) = \theta_0 \quad \text{in } \Omega. \tag{1.6}$$

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System 1.1–1.6 arises from the coupled moisture movement, transport of dissolved species and heat transfer through the porous system [4, 20]. For simplicity, the gravity terms and external sources are not included since they do not affect the analysis. For specific applications we refer the reader to e.g. [19]. Here $u : Q_T \to \mathbb{R}$, $w : Q_T \to \mathbb{R}$ and $\theta : Q_T \to \mathbb{R}$ are the unknown functions. In particular, u corresponds to the Kirchhoff transformation of the matric potential [2], w represents concentration of dissolved species and θ represents the temperature of the porous system. Further, $a : \mathbb{R} \to \mathbb{R}$, $D_w : \mathbb{R} \to \mathbb{R}$, $b : \mathbb{R} \to \mathbb{R}$, $\lambda : \mathbb{R}^2 \to \mathbb{R}$, $u_0 : \Omega \to \mathbb{R}$, $w_0 : \Omega \to \mathbb{R}$, and $\theta_0 : \Omega \to \mathbb{R}$ are given functions, ϱ is a real positive constant and $\mathbf{n}$ is the outward unit normal vector. In this paper we study the existence of the weak solution to (1.1)–(1.6).

Nowadays, description of heat, moisture or soluble/non-soluble contaminant transport in concrete, soil or rock porous matrix is frequently based on time dependent models. Coupled transport processes (diffusion processes, heat conduction, moister flow, contaminant transport or coupled flows through porous media) are typically associated with systems of strongly nonlinear degenerate parabolic partial differential equations of type (written in terms of operators A, Ψ, F)

$$\partial_t \Psi(\mathbf{u}) - \nabla \cdot A(\mathbf{u}, \nabla \mathbf{u}) = F(\mathbf{u}), \tag{1.7}$$

where $\mathbf{u}$ stands for the unknown vector of state variables. There is no complete theory for such general problems. However, some particular results assuming special structure of operators A and Ψ and growth conditions on F can be found in the literature, see [22].

Most theoretical results on parabolic systems exclude the case of non-symmetrical parabolic parts [2, 8, 13].

Giaquinta and Modica [10] proved the local-in-time solvability of quasilinear diagonal parabolic systems with nonlinear boundary conditions (without assuming any growth condition), see also [23].

The existence of weak solutions to more general non-diagonal systems like (1.7) subject to mixed boundary conditions has been proven in [2]. The authors proved an existence result assuming the operator Ψ to be only (weak) monotone and subgradient. This result has been extended in [8], where the authors presented the local existence of the weak solutions for the system with nonlinear Neumann boundary conditions and under more general growth conditions on nonlinearities in $\mathbf{u}$. These results, however, are not applicable if Ψ does not take the subgradient structure, which is typical of coupled transport models in porous media. Thus, the analysis needs to exploit the specific structure of such problems.

The existence of a local-in-time strong solution for moisture and heat transfer in multi-layer porous structures modelling by the doubly nonlinear parabolic system is proven in [5]. In [21], the author proved the existence of the solution to the purely diffusive hygro-thermal model allowing non-symmetrical operators Ψ, but requiring non-realistic symmetry in the elliptic part. In [7, 12], the authors studied the existence, uniqueness and regularity of coupled quasilinear equations modeling evolution of fluid species influenced by thermal, electrical and diffusive forces. In [15, 16, 17], the authors studied a model of specific structure of a heat and mass transfer arising from textile industry and proved the global existence for one-dimensional problems in [15, 16] and three-dimensional problems in [17].

In the present paper we extend our previous existence result for coupled heat and mass flows in porous media [6] to more general problem (including the convection-dispersion equation) modeling coupled moisture, solute and heat transport in porous media. This leads to a fully nonlinear degenerate parabolic system with natural (critical) growths and degeneracies in all transport coefficients.

The rest of this paper is organized as follows. In Section 2, we introduce basic notation and suitable function spaces and specify our assumptions on data and coefficient functions in the problem. In Section 3, we formulate the problem in the variational sense and state the main result, the global-in-time existence of the weak solution. The main result is proved by an approximation procedure in Section 4. First we formulate the semi-discrete scheme and prove the existence of its solution. The crucial a priori estimates and uniform boundness of time interpolants are proved in part 4.2. Finally, we conclude that the solutions of semi-discrete scheme converge and the limit is the solution of the original problem (Subsection 4.3).

Remark 1.1. The present analysis can be straightforwardly extended to a setting with nonhomogeneous boundary conditions (see [6] for details). Here we work with homogeneous boundary conditions, ignoring the gravity terms and excluding external sources to simplify the presentation and avoid unnecessary technicalities in the existence result.

2. Preliminaries

2.1. **Notation and some properties of Sobolev spaces.** Vectors and vector functions are denoted by boldface letters. Throughout the paper, we will always use positive constants C, c, c_1, c_2, ..., which are not specified and which may differ from line to line. Throughout this paper we suppose $s, q, s' \in [1,\infty]$, s' denotes the conjugate exponent to $s > 1$, $1/s + 1/s' = 1$. $L^s(\Omega)$ denotes the usual Lebesgue space equipped with the norm $\|\cdot\|_{L^s(\Omega)}$ and $W^{k,s}(\Omega)$, $k \geq 0$ (k need not to be an integer, see [14]), denotes the usual Sobolev-Slobodecki space with the norm $\|\cdot\|_{W^{k,s}(\Omega)}$. We define

$$W^{1,2}_{\Gamma_D}(\Omega) := \{v \in W^{1,2}(\Omega) : v\big|_{\Gamma_D} = 0\}.$$

By E^* we denote the space of all continuous, linear forms on Banach space E and by $\langle \cdot, \cdot \rangle$ we denote the duality between E and E^*. By $L^s(I;E)$ we denote the Bochner space (see [1]). Therefore, $L^s(I;E)^* = L^{s'}(I;E^*)$.

2.2. **Structure and data properties.** We start by introducing our assumptions on functions in (1.1)–(1.6).

(i) $b \in C^1(\mathbb{R})$, $0 < b'(\xi) < b_*$ and

$$0 < b(\xi) \leq b_2 < +\infty \quad \forall \xi \in \mathbb{R} \quad (b_2, b_* = \text{const}).$$

(ii) a, $D_w \in C(\mathbb{R})$ and $\lambda \in C(\mathbb{R}^2)$ such that

$$0 < a(\xi), \quad 0 < D_w(\xi) \quad \forall \xi \in \mathbb{R},$$
$$0 < \lambda(\xi,\zeta) \quad \forall \xi, \zeta \in \mathbb{R}.$$

(iii) (Initial data) Assume $u_0, w_0, \theta_0 \in L^\infty(\Omega)$, such that

$$-\infty < u_1 < u_0 < 0 \qquad \text{a.e. in } \Omega \quad (u_1 = \text{const}). \tag{2.1}$$

2.3. **Auxiliary results.**

Remark 2.1 ([2, Section 1.1]). Let us note that (i) implies that there is a (strictly) convex C^1-function $\Phi : \mathbb{R} \to \mathbb{R}$, $\Phi(0) = 0$, $\Phi'(0) = 0$, such that $b(z) - b(0) = \Phi'(z)$ for all $z \in \mathbb{R}$. Introduce the Legendre transform

$$B(z) := \int_0^1 (b(z) - b(sz))z \,\mathrm{d}s = \int_0^z (b(z) - b(s)) \,\mathrm{d}s.$$

Let us present some properties of B [2]:

$$\begin{gathered} B(z) := \int_0^1 (b(z) - b(sz))z \,\mathrm{d}s \geq 0 \quad \forall z \in \mathbb{R}, \\ B(s) - B(r) \geq (b(s) - b(r))r \quad \forall r, s \in \mathbb{R}, \\ b(z)z - \Phi(z) + \Phi(0) = B(z) \leq b(z)z \quad \forall z \in \mathbb{R}. \end{gathered}$$

3. Main result

The aim of this paper is to prove the existence of a weak solution to problem (1.1)–(1.6). First we formulate our problem in a variational sense.

Definition 3.1. A weak solution of (1.1)–(1.6) is a triplet $[u, w, \theta]$ such that

$$\begin{gathered} u \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega)), \quad w \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega)) \cap L^\infty(Q_T), \\ \theta \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega)) \cap L^\infty(Q_T), \end{gathered}$$

which satisfies

$$-\int_{Q_T} b(u)\partial_t \phi \,\mathrm{d}x\mathrm{d}t + \int_{Q_T} a(\theta)\nabla u \cdot \nabla \phi \,\mathrm{d}x\mathrm{d}t = \int_\Omega b(u_0)\phi(\mathbf{x}, 0) \,\mathrm{d}x \tag{3.1}$$

for any $\phi \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega)) \cap W^{1,1}(I; L^\infty(\Omega))$ with $\phi(\cdot, T) = 0$;

$$\begin{aligned} &-\int_{Q_T} b(u)w\partial_t \eta \,\mathrm{d}x\mathrm{d}t + \int_{Q_T} b(u)D_w(u)\nabla w \cdot \nabla \eta \,\mathrm{d}x\mathrm{d}t \\ &+ \int_{Q_T} wa(\theta)\nabla u \cdot \nabla \eta \,\mathrm{d}x\mathrm{d}t \\ &= \int_\Omega b(u_0)w_0\eta(\mathbf{x}, 0) \,\mathrm{d}x \end{aligned} \tag{3.2}$$

for any $\eta \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega)) \cap W^{1,1}(I; L^\infty(\Omega))$ with $\eta(\cdot, T) = 0$;

$$\begin{aligned} &-\int_{Q_T} [b(u)\theta + \varrho\theta]\partial_t \psi \,\mathrm{d}x\mathrm{d}t + \int_{Q_T} \lambda(\theta, u)\nabla \theta \cdot \nabla \psi \,\mathrm{d}x\mathrm{d}t \\ &+ \int_{Q_T} \theta a(\theta)\nabla u \cdot \nabla \psi \,\mathrm{d}x\mathrm{d}t \\ &= \int_\Omega [b(u_0)\theta_0 + \varrho\theta_0]\psi(\mathbf{x}, 0) \,\mathrm{d}x \end{aligned} \tag{3.3}$$

for any $\psi \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega)) \cap W^{1,1}(I; L^\infty(\Omega))$ with $\psi(\cdot, T) = 0$.

The main result of this paper reads as follows.

Theorem 3.2. *Let assumptions* (i)–(iii) *be satisfied. Then there exists at least one weak solution of the system* (1.1)–(1.6).

To prove the main result of the paper we use the method of semidiscretization in time by constructing temporal approximations and limiting procedure. The proof can be divided into three steps. In the first step we approximate our problem by means of a semi-implicit time discretization scheme (which preserve the pseudo-monotone structure of the discrete problem) and prove the existence and $W^{1,s}(\Omega)$-regularity (with some $s > 2$) of temporal approximations. In the second step we construct piecewise constant time interpolants and derive suitable a priori estimates. The key point is to establish L^∞-estimates to overcome degeneracies in transport coefficients. Finally, in the third step we pass to the limit from discrete approximations.

4. Proof of the main result

4.1. **Approximations.** Let us fix $p \in \mathbb{N}$ and set $\tau := T/p$ (a time step). Further, let us consider $u_p^0 := u_0$, $w_p^0 := w_0$, $\theta_p^0 := \theta_0$ a.e. on Ω. We approximate our evolution problem by a semi-implicit time discretization scheme. Then we define, in each time step $n = 1, \dots, p$, a triplet $[u_p^n, w_p^n, \theta_p^n]$ as a solution of the following recurrence steady problem.

For a given triplet $[u_p^{n-1}, w_p^{n-1}, \theta_p^{n-1}]$, $n = 1, \dots, p$, $u_p^{n-1} \in L^\infty(\Omega)$, $w_p^{n-1} \in L^\infty(\Omega)$, $\theta_p^{n-1} \in L^\infty(\Omega)$, find $[u_p^n, w_p^n, \theta_p^n]$, such that $u_p^n \in W^{1,s}_{\Gamma_D}(\Omega)$, $w_p^n \in W^{1,s}_{\Gamma_D}(\Omega)$, $\theta_p^n \in W^{1,s}_{\Gamma_D}(\Omega)$ with some $s > 2$ and

$$\int_\Omega \frac{b(u_p^n) - b(u_p^{n-1})}{\tau}\phi \,\mathrm{d}x + \int_\Omega a(\theta_p^{n-1})\nabla u_p^n \cdot \nabla\phi \,\mathrm{d}x = 0 \tag{4.1}$$

for any $\phi \in W^{1,2}_{\Gamma_D}(\Omega)$;

$$\begin{aligned} &\int_\Omega \frac{b(u_p^n)w_p^n - b(u_p^{n-1})w_p^{n-1}}{\tau}\eta \,\mathrm{d}x \\ &+ \int_\Omega b(u_p^{n-1})D_w(u_p^{n-1})\nabla w_p^n \cdot \nabla\eta \,\mathrm{d}x + \int_\Omega w_p^n a(\theta_p^{n-1})\nabla u_p^n \cdot \nabla\eta \,\mathrm{d}x = 0 \end{aligned} \tag{4.2}$$

for any $\eta \in W^{1,2}_{\Gamma_D}(\Omega)$;

$$\begin{aligned} &\int_\Omega \frac{b(u_p^n)\theta_p^n - b(u_p^{n-1})\theta_p^{n-1}}{\tau}\psi \,\mathrm{d}x + \int_\Omega \varrho\frac{\theta_p^n - \theta_p^{n-1}}{\tau}\psi \,\mathrm{d}x \\ &+ \int_\Omega \lambda(\theta_p^{n-1}, u_p^{n-1})\nabla\theta_p^n \cdot \nabla\psi \mathrm{d}\Omega + \int_\Omega \theta_p^n a(\theta_p^{n-1})\nabla u_p^n \cdot \nabla\psi \mathrm{d}\Omega = 0 \end{aligned} \tag{4.3}$$

for any $\psi \in W^{1,2}_{\Gamma_D}(\Omega)$.

Next we show the existence of the solution to (4.1)–(4.3).

Theorem 4.1. *Let $u_p^{n-1} \in L^\infty(\Omega)$, $w_p^{n-1} \in L^\infty(\Omega)$, $\theta_p^{n-1} \in L^\infty(\Omega)$ be given and the assumptions* (i)–(iii) *be satisfied. Then there exists $[u_p^n, w_p^n, \theta_p^n]$, such that $u_p^n \in W^{1,s}_{\Gamma_D}(\Omega)$, $w_p^n \in W^{1,s}_{\Gamma_D}(\Omega)$ and $\theta_p^n \in W^{1,s}_{\Gamma_D}(\Omega)$ with some $s > 2$ satisfying* (4.1)–(4.3).

Proof. The proof rests on the $W^{1,p}$-regularity of elliptic problems presented in [9, 11] and the embedding $W^{1,s}_{\Gamma_D}(\Omega) \subset L^\infty(\Omega)$ if $s > 2$ (recall that Ω is a bounded domain in $\mathbb{R}^2$).

The existence of $u_p^n \in W^{1,s}_{\Gamma_D}(\Omega)$ with some $s > 2$ and $\theta_p^n \in W^{1,2}_{\Gamma_D}(\Omega)$, solutions to problems (4.1) and (4.3), respectively, is proven in [6]. The existence of $w_p^n \in W^{1,2}_{\Gamma_D}(\Omega)$, the solution to (4.2), can be handled in the same way.

Now, with $w_p^n \in W^{1,2}_{\Gamma_D}(\Omega)$ in hand, rewrite the equation (4.2) in the form (transferring the lower-order terms to the right hand side)

$$\begin{aligned}&\int_\Omega b(u_p^{n-1}) D_w(u_p^{n-1}) \nabla w_p^n \cdot \nabla \eta \,\mathrm{d}x \\ &= -\int_\Omega \frac{b(u_p^n) w_p^n - b(u_p^{n-1}) w_p^{n-1}}{\tau} \eta \,\mathrm{d}x - \int_\Omega w_p^n a(\theta_p^{n-1}) \nabla u_p^n \cdot \nabla \eta \,\mathrm{d}x.\end{aligned}$$

Since $u_p^{n-1} \in L^\infty(\Omega)$, $u_p^n \in W^{1,s}_{\Gamma_D}(\Omega)$ with some $s > 2$, $w_p^{n-1} \in L^\infty(\Omega)$, $\theta_p^{n-1} \in L^\infty(\Omega)$, both integrals on the right hand side make sense for any $\eta \in W^{1,r'}_{\Gamma_D}(\Omega)$, $r' = r/(r-1)$ with some $r > 2$. Now we are able to apply [9, Theorem 4] to obtain $w_p^n \in W^{1,s}_{\Gamma_D}(\Omega)$ with some $s > 2$. Analysis similar to the above implies that $\theta_p^n \in W^{1,s}_{\Gamma_D}(\Omega)$ with some $s > 2$. □

4.2. **A priori estimates.** In this part we prove some uniform estimates (with respect to p) for the time interpolants of the solution. In the following estimates, many different constants will appear. For simplicity of notation, C represents generic constants which may change their numerical value from one formula to another but do not depend on p and the functions under consideration.

4.2.1. *Construction of temporal interpolants.* With the sequences u_p^n, w_p^n, θ_p^n constructed in Section 4.1, we define the piecewise constant interpolants $\bar{\phi}_p(t) = \phi_p^n$ for $t \in ((n-1)\tau, n\tau]$ and, in addition, we extend $\bar{\phi}_p$ for $t \le 0$ by $\bar{\phi}_p(t) = \phi_0$ for $t \in (-\tau, 0]$. For a function φ we often use the simplified notation $\varphi := \varphi(t)$, $\varphi_\tau(t) := \varphi(t-\tau)$, $\partial_t^{-\tau}\varphi(t) := \frac{\varphi(t)-\varphi(t-\tau)}{\tau}$, $\partial_t^{\tau}\varphi(t) := \frac{\varphi(t+\tau)-\varphi(t)}{\tau}$. Then, following (4.1)–(4.3), the piecewise constant time interpolants $\bar{u}_p \in L^\infty(I; W^{1,s}_{\Gamma_D}(\Omega))$, $\bar{w}_p \in L^\infty(I; W^{1,s}_{\Gamma_D}(\Omega))$ and $\bar{\theta}_p \in L^\infty(I; W^{1,s}_{\Gamma_D}(\Omega))$ (with some $s > 2$) satisfy the equations

$$\int_\Omega \partial_t^{-\tau} b(\bar{u}_p(t))\phi \,\mathrm{d}x + \int_\Omega a(\bar{\theta}_p(t-\tau)) \nabla \bar{u}_p(t) \cdot \nabla \phi \,\mathrm{d}x = 0 \tag{4.4}$$

for any $\phi \in W^{1,2}_{\Gamma_D}(\Omega)$,

$$\begin{aligned}&\int_\Omega \partial_t^{-\tau}[b(\bar{u}_p(t))\bar{w}_p(t)]\eta \,\mathrm{d}x + \int_\Omega b(\bar{u}_p(t-\tau)) D_w(\bar{u}_p(t-\tau)) \nabla \bar{w}_p(t) \cdot \nabla \eta \,\mathrm{d}x \\ &+ \int_\Omega \bar{w}_p(t) a(\bar{\theta}_p(t-\tau)) \nabla \bar{u}_p(t) \cdot \nabla \eta \,\mathrm{d}x = 0\end{aligned} \tag{4.5}$$

for any $\eta \in W^{1,2}_{\Gamma_D}(\Omega)$ and

$$\begin{aligned}&\int_\Omega \partial_t^{-\tau} \left[b(\bar{u}_p(t))\bar{\theta}_p(t) + \varrho \bar{\theta}_p(t) \right] \psi \,\mathrm{d}x \\ &+ \int_\Omega \lambda(\bar{\theta}_p(t-\tau), \bar{u}_p(t-\tau)) \nabla \bar{\theta}_p(t) \cdot \nabla \psi \,\mathrm{d}x \\ &+ \int_\Omega \bar{\theta}_p(t) a(\bar{\theta}_p(t-\tau)) \nabla \bar{u}_p(t) \cdot \nabla \psi \,\mathrm{d}x = 0\end{aligned} \tag{4.6}$$

for any $\psi \in W^{1,2}_{\Gamma_D}(\Omega)$.

4.2.2. *L^∞-bound for $\bar{u}_p$, $\bar{w}_p$ and $\bar{\theta}_p$.* First we prove the L^∞-estimate for $\bar{u}_p$. Let us set

$$\phi := [b(\bar{u}_p) - b(u_1)]_- = \begin{cases} b(\bar{u}_p) - b(u_1), & \bar{u}_p < u_1, \\ 0, & \bar{u}_p \geq u_1, \end{cases} \tag{4.7}$$

as a test function in (4.4). Note that ϕ vanishes on Γ_D. It is a simple matter to derive

$$\frac{1}{2}\int_\Omega [b(\bar{u}_p(t)) - b(u_1)]_-^2 \,\mathrm{d}x + \int_{Q_t} a(\bar{\theta}_p(s-\tau)) b'(\bar{u}_p(s)) |\nabla \bar{u}_p(s)|^2 \chi_{\{\bar{u}_p < u_1\}} \mathrm{d}x\mathrm{d}s \leq 0$$

for almost every $t \in I$. Hence we conclude that the set $\{x \in \Omega : \bar{u}_p(x,t) < u_1\}$ has a measure zero for almost every $t \in I$.

Now setting

$$\phi = [b(\bar{u}_p) - b(0)]_+ = \begin{cases} b(\bar{u}_p) - b(0), & \bar{u}_p > 0, \\ 0, & \bar{u}_p \leq 0, \end{cases} \tag{4.8}$$

we obtain, using similar arguments,

$$\frac{1}{2}\int_\Omega [b(\bar{u}_p) - b(0)]_+^2 \,\mathrm{d}x = 0 \quad \text{for almost every } t \in I.$$

Hence the set $\{x \in \Omega : \bar{u}_p(x,t) > 0\}$ has a measure zero for almost every $t \in I$. Finally, combining the previous arguments, we deduce

$$\|\bar{u}_p\|_{L^\infty(Q_T)} \leq C, \tag{4.9}$$

where C does not depend on p.

Now we prove a similar estimate for $\bar{w}_p$. Let ℓ be an odd integer. Using $\phi = [\ell/(\ell+1)](\bar{w}_p)^{\ell+1}$ as a test function in (4.4) and $\eta = (\bar{w}_p)^\ell$ in (4.5) and combining both equations we obtain

$$\begin{aligned}
&\frac{1}{\tau}\frac{1}{\ell+1}\int_\Omega b(\bar{u}_p(s))[\bar{w}_p(s)]^{\ell+1}\,\mathrm{d}x \\
&- \frac{1}{\tau}\frac{1}{\ell+1}\int_\Omega b(\bar{u}_p(s-\tau))[\bar{w}_p(s-\tau)]^{\ell+1}\,\mathrm{d}x \\
&+ \frac{1}{\tau}\frac{1}{\ell+1}\int_\Omega b(\bar{u}_p(s-\tau))[\bar{w}_p(s-\tau)]^{\ell+1}\,\mathrm{d}x \\
&+ \frac{1}{\tau}\frac{\ell}{\ell+1}\int_\Omega b(\bar{u}_p(s-\tau))[\bar{w}_p(s)]^{\ell+1}\,\mathrm{d}x \\
&- \frac{1}{\tau}\int_\Omega b(\bar{u}_p(s-\tau))\bar{w}_p(s-\tau)[\bar{w}_p(s)]^{\ell}\,\mathrm{d}x \\
&+ \int_\Omega \ell[\bar{w}_p(s)]^{\ell-1} b(\bar{u}_p(s-\tau)) D_w(\bar{u}_p(s-\tau)) \nabla\bar{w}_p(s)\cdot\nabla\bar{w}_p(s)\,\mathrm{d}x = 0.
\end{aligned} \tag{4.10}$$

Applying the Young's inequality we can write for the term in the third line

$$\begin{aligned}
&\frac{1}{\tau}\int_\Omega b(\bar u_p(s-\tau))\bar w_p(s-\tau)[\bar w_p(s)]^\ell\,\mathrm{d}x\\
&\le \frac{1}{\tau}\frac{1}{\ell+1}\int_\Omega b(\bar u_p(s-\tau))[\bar w_p(s-\tau)]^{\ell+1}\,\mathrm{d}x\\
&+\frac{1}{\tau}\frac{\ell}{\ell+1}\int_\Omega b(\bar u_p(s-\tau))[\bar w_p(s)]^{\ell+1}\,\mathrm{d}x.
\end{aligned}\tag{4.11}$$

Combining (4.10) and (4.11) we deduce

$$\begin{aligned}
&\frac{1}{\tau}\frac{1}{\ell+1}\int_\Omega b(\bar u_p(s))[\bar w_p(s)]^{\ell+1}\,\mathrm{d}x\\
&-\frac{1}{\tau}\frac{1}{\ell+1}\int_\Omega b(\bar u_p(s-\tau))[\bar w_p(s-\tau)]^{\ell+1}\,\mathrm{d}x\\
&+\int_\Omega \ell[\bar w_p(s)]^{\ell-1}b(\bar u_p(s-\tau))D_w(\bar u_p(s-\tau))\nabla\bar w_p(s)\cdot\nabla\bar w_p(s)\,\mathrm{d}x\le 0.
\end{aligned}\tag{4.12}$$

Now, integrating (4.12) over s from 0 to t we obtain

$$\begin{aligned}
&\int_\Omega(\bar w_p(t))^{\ell+1}b(\bar u_p(t))\mathrm{d}x\\
&+\int_{\Omega_t}(\ell+1)\ell[\bar w_p(s)]^{\ell-1}b(\bar u_p(s-\tau))D_w(\bar u_p(s-\tau))|\nabla\bar w_p(s)|^2\mathrm{d}x\mathrm{d}s\\
&\le\int_\Omega(w_0)^{\ell+1}b\left(u_0\right)\mathrm{d}x.
\end{aligned}\tag{4.13}$$

Note that the second integral in (4.13) is nonnegative (ℓ is supposed to be the odd integer). Moreover, from (4.13) and (4.9) it follows that

$$\|\bar w_p\|_{L^\infty(0,T;L^{\ell+1}(\Omega))}\le C,\tag{4.14}$$

where the constant C is independent of ℓ and p. Now, let $\ell\to+\infty$ in (4.14), we obtain

$$\|\bar w_p\|_{L^\infty(Q_T)}\le C.\tag{4.15}$$

In the same manner we arrive at the estimate for $\bar\theta_p$, i.e.

$$\|\bar\theta_p\|_{L^\infty(Q_T)}\le C.\tag{4.16}$$

4.2.3. *Energy estimates for $\bar u_p$, $\bar w_p$ and $\bar\theta_p$.* We test (4.4) with $\phi=\bar u_p(t)$ and integrate (4.4) over t from 0 to s. For the parabolic term we can write

$$\int_0^s\int_\Omega\partial_t^{-\tau}b(\bar u_p(t))\bar u_p(t)\,\mathrm{d}x\mathrm{d}t\ge\frac{1}{\tau}\int_{s-\tau}^s\int_\Omega B(\bar u_p(t))-B(u_0)\,\mathrm{d}x\mathrm{d}t.\tag{4.17}$$

Further, using (4.9) and (4.17), applying the usual estimates for the elliptic part (see also [2]), we obtain the a priori estimate

$$\sup_{0\le t\le T}\int_\Omega B(\bar u_p(t))\mathrm{d}x+\int_0^T\int_\Omega|\nabla\bar u_p(t)|^2\mathrm{d}x\mathrm{d}t\le C.\tag{4.18}$$

Now it follows that there exists a function $u\in L^2(I;W^{1,2}_{\Gamma_D}(\Omega))$ such that, along a selected subsequence (letting $p\to\infty$), we have $\bar u_p(t)\rightharpoonup u$ weakly in $L^2(I;W^{1,2}_{\Gamma_D}(\Omega))$.

Now we prove similar result for $\bar{w}_p(t)$. Using $\eta(t) = 2\bar{w}_p(t)$ as a test function in (4.5) we obtain

$$\begin{aligned}
&\int_\Omega \partial_t^{-\tau} b(\bar{u}_p(t)) 2\bar{w}_p(t)^2 \,\mathrm{d}x + \int_\Omega \partial_t^{-\tau} \bar{w}_p(t) 2\bar{w}_p(t) b(\bar{u}_p(t-\tau)) \,\mathrm{d}x \\
&+ 2\int_\Omega b(\bar{u}_p(t-\tau)) D_w(\bar{u}_p(t-\tau)) \nabla \bar{w}_p(t) \cdot \nabla \bar{w}_p(t) \,\mathrm{d}x \\
&+ \int_\Omega a(\bar{\theta}_p(t-\tau)) \nabla \bar{u}_p(t) \cdot 2\bar{w}_p(t) \nabla \bar{w}_p(t) \,\mathrm{d}x = 0.
\end{aligned} \tag{4.19}$$

One is allowed to use $\phi(t) = \bar{w}_p(t)^2$ as a test function in (4.4) to obtain

$$\int_\Omega [\partial_t^{-\tau} b(\bar{u}_p(t))] \bar{w}_p(t)^2 \,\mathrm{d}x + \int_\Omega a(\bar{\theta}_p(t-\tau)) \nabla \bar{u}_p(t) \cdot \nabla \bar{w}_p(t)^2 \,\mathrm{d}x = 0. \tag{4.20}$$

Combining (4.19) and (4.20) we deduce

$$\begin{aligned}
&\int_\Omega \partial_t^{-\tau} \left[\bar{w}_p(t)^2 b(\bar{u}_p(t)) \right] \,\mathrm{d}x + \int_\Omega \frac{1}{\tau} \left[\bar{w}_p(t) - \bar{w}_p(t-\tau) \right]^2 b(\bar{u}_p(t-\tau)) \,\mathrm{d}x \\
&+ 2\int_\Omega b(\bar{u}_p(t-\tau)) D_w(\bar{u}_p(t-\tau)) \nabla \bar{w}_p(t) \cdot \nabla \bar{w}_p(t) \,\mathrm{d}x = 0.
\end{aligned} \tag{4.21}$$

In view of (4.9) we have

$$b(\bar{u}_p(t)),\ b(\bar{u}_p(t-\tau)),\ D_w(\bar{u}_p(t-\tau)) > C \qquad \text{in } \Omega \times (-\tau, T). \tag{4.22}$$

Recall that C does not depend on p. Now, integrating (4.21) with respect to time t we obtain

$$\sup_{0 \le t \le T} \int_\Omega |\bar{w}_p(t)|^2 \mathrm{d}\Omega + \int_0^T \|\bar{w}_p(t)\|^2_{W^{1,2}_{\Gamma_D}(\Omega)} \mathrm{d}\Omega \le C.$$

From this we can write

$$\|\bar{w}_p\|_{L^2(I; W^{1,2}_{\Gamma_D}(\Omega))} \le C. \tag{4.23}$$

Similarly, we use $\psi(t) = 2\bar{\theta}_p(t)$ as a test function in (4.6) to obtain

$$\begin{aligned}
&\int_\Omega \partial_t^{-\tau} b(\bar{u}_p(t)) 2\bar{\theta}_p(t)^2 \,\mathrm{d}x + \int_\Omega \partial_t^{-\tau} \bar{\theta}_p(t) 2\bar{\theta}_p(t) b(\bar{u}_p(t-\tau)) \,\mathrm{d}x \\
&+ 2\int_\Omega \lambda(\bar{\theta}_p(t-\tau), \bar{u}_p(t-\tau)) \nabla \bar{\theta}_p(t) \cdot \nabla \bar{\theta}_p(t) \,\mathrm{d}x \\
&+ \int_\Omega a(\bar{\theta}_p(t-\tau)) \nabla \bar{u}_p(t) \cdot 2\bar{\theta}_p(t) \nabla \bar{\theta}_p(t) \,\mathrm{d}x \le 0.
\end{aligned} \tag{4.24}$$

Using $\phi(t) = \bar{\theta}_p(t)^2$ as a test function in (4.4) we obtain

$$\int_\Omega \partial_t^{-\tau} b(\bar{u}_p(t)) \bar{\theta}_p(t)^2 \,\mathrm{d}x + \int_\Omega a(\bar{\theta}_p(t-\tau)) \nabla \bar{u}_p(t) \cdot \nabla \bar{\theta}_p(t)^2 \,\mathrm{d}x = 0. \tag{4.25}$$

Combining (4.24) and (4.25) we deduce

$$\begin{aligned}
&\int_\Omega \partial_t^{-\tau} \left[\left(\bar{\theta}_p(t) \right)^2 b(\bar{u}_p(t)) \right] \,\mathrm{d}x + \int_\Omega \frac{1}{\tau} \left[\bar{\theta}_p(t) - \bar{\theta}_p(t-\tau) \right]^2 b(\bar{u}_p(t-\tau)) \,\mathrm{d}x \\
&+ 2\int_\Omega \lambda(\bar{\theta}_p(t-\tau), \bar{u}_p(t-\tau)) \nabla \bar{\theta}_p(t) \cdot \nabla \bar{\theta}_p(t) \,\mathrm{d}x \le 0.
\end{aligned} \tag{4.26}$$

Integrating (4.26) with respect to time t we obtain the a priori estimate (using (4.9) and (4.16))

$$\sup_{0\le t\le T}\int_\Omega |\bar\theta_p(t)|^2 \mathrm{d}x + \int_0^T \|\bar\theta_p(t)\|^2_{W^{1,2}_{\Gamma_D}(\Omega)} \mathrm{d}t \le C. \tag{4.27}$$

From this we have

$$\|\bar\theta_p\|_{L^2(I;W^{1,2}_{\Gamma_D}(\Omega))} \le C. \tag{4.28}$$

4.2.4. *Further estimates.* To show that $\bar u_p$ converges to u almost everywhere on Q_T we follow [2]. Let $k\in\mathbb{N}$ and use

$$\phi(t) = \partial_t^{k\tau}\bar u_p(s)$$

for $j\tau \le t \le (j+k)\tau$ with $(j-1)\tau \le s\le j\tau$ and $1\le j\le \frac{T}{\tau}-k$, as a test function in (4.4). For the parabolic term, we can write

$$\begin{aligned}
&\int_{j\tau}^{(j+k)\tau}\int_\Omega \partial_t^{-\tau} b(\bar u_p(t))\,\partial_t^{k\tau}\bar u_p(t)\,\mathrm{d}x\mathrm{d}t\\
&= \frac{1}{k\tau^2}\int_{(j-1)\tau}^{j\tau}\int_\Omega \left(b(\bar u_p(t+k\tau)) - b(\bar u_p(t))\right)\left(\bar u_p(t+k\tau)-\bar u_p(t)\right)\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Hence, summing over $j=1,\dots,p-k$ we obtain the estimate

$$\begin{aligned}
&\sum_{j=1}^{p-k}\int_{j\tau}^{(j+k)\tau}\int_\Omega \partial_t^{-\tau} b(\bar u_p(t))\partial_t^{k\tau}\bar u_p(t)\,\mathrm{d}x\mathrm{d}t\\
&\ge \frac{1}{k\tau^2}\int_0^{T-k\tau}\int_\Omega \left(b(\bar u_p(t+k\tau)) - b(\bar u_p(t))\right)\left(\bar u_p(t+k\tau)-\bar u_p(t)\right)\,\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{4.29}$$

Similarly, for the elliptic term, after a little lengthy but straightforward computation we obtain

$$\begin{aligned}
&\sum_{j=1}^{p-k}\int_{j\tau}^{(j+k)\tau}\int_\Omega a(\bar\theta_p(t-\tau))\nabla\bar u_p\cdot\nabla\partial_t^{k\tau}\bar u_p\,\mathrm{d}x\mathrm{d}t\\
&=\sum_{\ell=1}^{k}\sum_{j=1}^{p-k}\int_{(j+\ell-1)\tau}^{(j+\ell)\tau}\int_\Omega \big(a(\bar\theta_p(t-\tau))\nabla\bar u_p\big)\cdot\nabla\partial_t^{k\tau}\bar u_p\mathrm{d}x\mathrm{d}t\\
&=\sum_{\ell=1}^{k}\int_{\ell\tau}^{T-k\tau+\ell\tau}\int_\Omega a(\bar\theta_p(t-\tau))\nabla\bar u_p(t)\cdot\nabla\partial_t^{k\tau}\bar u_p(t-\ell\tau)\,\mathrm{d}x\mathrm{d}t\\
&\le \frac{c_1}{\tau}\int_{Q_T}|a(\bar\theta_p(t-\tau))\nabla\bar u_p|^2\,\mathrm{d}x\mathrm{d}t + \frac{c_2}{\tau}\int_{Q_T}|\nabla\bar u_p|^2\,\mathrm{d}x\mathrm{d}t\\
&\le \frac{C}{\tau}.
\end{aligned}\tag{4.30}$$

Combining (4.29)–(4.30) and using (4.18) we obtain

$$\int_0^{T-k\tau}\left(b(\bar u_p(s+k\tau)) - b(\bar u_p(s))\right)\left(\bar u_p(s+k\tau)-\bar u_p(s)\right)\mathrm{d}s \le Ck\tau. \tag{4.31}$$

Using the compactness argument one can show in the same way as in [2, Lemma 1.9] and [8, Eqs. (2.10)–(2.12)]

$$b(\bar u_p)\to b(u)\quad \text{in } L^1(Q_T) \tag{4.32}$$

and almost everywhere on Q_T. Since b is strictly monotone, it follows from (4.32) that [13, Proposition 3.35]

$$\bar{u}_p \to u \quad \text{almost everywhere on } Q_T. \tag{4.33}$$

Further, in much the same way as in (4.31), we arrive at

$$\int_0^{T-k\tau} |b(\bar{u}_p(s+k\tau))\bar{w}_p(s+k\tau) - b(\bar{u}_p(s))\bar{w}_p(s)|^2 \mathrm{d}s \le Ck\tau. \tag{4.34}$$

From this we conclude, using (4.15), that

$$\int_0^{T-k\tau} |\bar{w}_p(s+k\tau) - \bar{w}_p(s)|^2 \mathrm{d}s \le Ck\tau. \tag{4.35}$$

Finally, in a similar way, using (4.16), we arrive at

$$\int_0^{T-k\tau} |\bar{\theta}_p(s+k\tau) - \bar{\theta}_p(s)|^2 \mathrm{d}s \le Ck\tau. \tag{4.36}$$

4.3. **Passage to the limit.** The a priori estimates (4.15), (4.16), (4.18), (4.23), (4.28), (4.31), (4.35), (4.36) allow us to conclude that there exist $u \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega))$, $w \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega)) \cap L^\infty(Q_T)$ and $\theta \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega)) \cap L^\infty(Q_T)$ such that, letting $p \to +\infty$ (along a selected subsequence),

$$\begin{gathered}
\bar{u}_p \rightharpoonup u \quad \text{weakly in } L^2(I; W^{1,2}_{\Gamma_D}(\Omega)),\\
\bar{u}_p \to u \quad \text{almost everywhere on } Q_T,\\
\bar{w}_p \rightharpoonup w \quad \text{weakly in } L^2(I; W^{1,2}_{\Gamma_D}(\Omega)),\\
\bar{w}_p \rightharpoonup w \quad \text{weakly star in } L^\infty(Q_T),\\
\bar{w}_p \to w \quad \text{almost everywhere on } Q_T,\\
\bar{\theta}_p \rightharpoonup \theta \quad \text{weakly in } L^2(I; W^{1,2}_{\Gamma_D}(\Omega)),\\
\bar{\theta}_p \rightharpoonup \theta \quad \text{weakly star in } L^\infty(Q_T),\\
\bar{\theta}_p \to \theta \quad \text{almost everywhere on } Q_T.
\end{gathered}$$

The above established convergences are sufficient for taking the limit $p \to \infty$ in (4.4)–(4.6) (along a selected subsequence) to get the weak solution of the system (1.1)–(1.6) in the sense of Definition 3.1. This completes the proof of the main result stated in Theorem 3.2.

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C Function spaces

Here we introduce some function spaces which have been used throughout the text.

Definition C.1 (Lebesgue space $L^p(\Omega), p \in [1;\infty)$ [36]) *Let $p \in [1;\infty)$, let Ω be a measurable subset of Eucledian N-space $\mathbb{R}^N$. We denote by $L^p(\Omega)$ the set of all measurable functions f defined almost everywhere on Ω and such that the Lebesgue integral*

$$\int_\Omega |f(x)|^p \,\mathrm{d}x$$

is finite.

Definition C.2 (Norm in Lebesgue space $L^p(\Omega), p \in [1;\infty)$[36]) *Let $p \in [1;\infty)$, let Ω be a measurable subset of Eucledian N-space $\mathbb{R}^N$, let f be a measurable function defined almost everywhere in Ω. We denote by $\|f\|_{L^p(\Omega)}$ norm in Lebesgue space $L^p(\Omega)$ such that*

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f(x)|^p \,\mathrm{d}x\right)^{1/p}.$$

Definition C.3 (Lebesgue space $L^\infty(\Omega)$ [36]) *Let Ω be a measurable subset of Eucledian N-space $\mathbb{R}^N$. We denote by $L^\infty(\Omega)$ the set of all measurable functions f defined almost everywhere on Ω, such that there exists a constant $K > 0$ with the property*

$$|f(x)| \leq K.$$

Definition C.4 (Sobolev space $W^{1,p}(\Omega); p \in [1;\infty)$) *Let Ω be a measurable subset of Eucledian N-space $\mathbb{R}^N$. We denote by $W^{1,p}(\Omega)$ the set of all measurable functions f defined almost everywhere on Ω, such that*

$$f \in L^p(\Omega)$$

and

$$\frac{\partial f}{\partial x_i} \in L^p(\Omega) \quad i = 1, 2, ..N.$$

Definition C.5 (Sobolev space $W_0^{1,p}(\Omega); p \in [1;\infty)$) *Let Ω be a measurable subset of Eucledian N-space $\mathbb{R}^N$ with boundary $\partial\Omega$. We denote by $W_0^{1,p}(\Omega)$ the set of all measurable functions f defined almost everywhere on Ω, such that*

$$f \in W^{1,p}(\Omega)$$

and

$$f|_{\partial\Omega} = 0.$$

Definition C.6 (Sobolev space $W_D^{1,p}(\Omega); p \in [1;\infty)$) *Let Ω be a measurable subset of Eucledian N-space $\mathbb{R}^N$ with boundary $\partial\Omega \subseteq \Gamma_D$. We denote by $W_D^{1,p}(\Omega)$ the set of all measurable functions f defined almost everywhere on Ω, such that*

$$f \in W^{1,p}(\Omega)$$

and

$$f|_{\Gamma_D} = 0.$$

Definition C.7 (Norm in Sobolev space $W^{1,p}(\Omega); p \in [1;\infty)$) *Let Ω be a measurable subset of Eucledian N-space $\mathbb{R}^N$ and f a measurable function defined almost everywhere on Ω. We define the norm in Sobolev space $W^{1,p}(\Omega); p \in [1;\infty)$ such that*

$$\|f\|_{W^{1,p}(\Omega)} := \left(\int_\Omega \left[f^p(x) + \sum_{i=1}^N \left(\frac{\partial f(x)}{\partial x_i}\right)^p\right] \mathrm{d}x\right)^{1/p}.$$

D Important inequalities

In this section we introduce some well known inequalities which have been used in the text.

Lemma D.1 (Hölder's inequality ([16], Sec. B.2)) *Let $1 < p,\ q < +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $f(x) \in L^p(\Omega)$, $g(x) \in L^q(\Omega)$. The following inequality holds*

$$\int_\Omega f(x)g(x)\,\mathrm{d}x \leq \left(\int_\Omega f(x)^p\,\mathrm{d}x\right)^{1/p} \left(\int_\Omega g(x)^q\,\mathrm{d}x\right)^{1/q}.$$

Lemma D.2 (Friedrichs' inequality ([51], Theorem 30.3)) *Let Ω be a domain with a lipschitz boundary Γ, further Γ_1 is a part of the boundary Γ with nonnegative measure, then there exists a constant $c > 0$, which depends on the domain and the part of the boundary Γ_1 so that for all functions $f(x) \in W^{1,2}(\Omega)$ holds*

$$\|f(x)\|^2_{W^{1,2}(G)} \leq c\left(\int_\Omega (\nabla f(x))^2\,\mathrm{d}x + \int_{\Gamma 1} f(x)^2 \mathrm{d}S\right).$$

Lemma D.3 (Young's inequality ([16], Sec. B.2)) *Let $1 < p, q < +\infty$ a $\frac{1}{p} + \frac{1}{q} = 1$. Then the following inequality holds*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

for all $a, b > 0$. Further let $\alpha \neq 0$. We can write

$$\alpha\, a\, \frac{b}{\alpha} \leq \frac{(\alpha\, a)^p}{p} + \left(\frac{b}{\alpha}\right)^q \frac{1}{q}.$$

Further
$$ab \leq \frac{\alpha^p}{p}\, a^p + \frac{1}{q\alpha^p}\, b^q.$$
Let us denote $\eta = \frac{\alpha^p}{p}$, $c(\eta) = \frac{1}{q\alpha^p}$ *and we can write*
$$ab \leq \eta\, a^p + c(\eta)\, b^{\,q}.$$

Lemma D.4 (Gronwall's inequality in integral form ([16], Appendix B, paragraph k.)) *Let* $\xi(t)$ *be a nonnegative, summable function on* $[0, T]$ *which satisfies*
$$\xi(t) \leq C_1 \int_0^t \xi(s) ds,$$
for a.e $0 \leq t \leq T$, *then*
$$\xi(t) = 0$$
almost everywhere.

Lemma D.5 (Gronwall's inequality in discrete form ([52], Chapter 1)) *Let* $y_l \leq C + \tau \sum_{k=1}^{l-1}(a_k y_k + b_k)$ *for any* $l \geq 0$. *The discrete form of the Gronwall's inequality reads as follows*
$$y_l \leq c + \tau \sum_{k=1}^{l}(a y_k + b_k). \tag{D.1}$$
We will often use $a_k = a$ *constant, and the condition*
$$y_l \leq C + \tau \sum_{k=1}^{l}(a y_k + b_k),$$
from which can be derived $y_l \leq (1 - a\tau)^{-1} \left(c + \tau b_0 + \tau \sum_{k=1}^{l-1}(a y_k + b_{k+1})\right)$, *so that* (D.1) *gives*
$$y_l \leq \frac{e^{\tau l a/(1-a\tau)}}{1 - a\tau} \left(c + \tau \sum_{k=1}^{l} b_k\right) \quad \text{if} \quad \tau < \frac{1}{a}.$$

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