Czech Technical University in Prague

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Doctoral Thesis

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Algebraic and measure-theoretic properties of the structures close to Boolean algebras

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Study programme:	P2612 Electrical Engineering and Informatics
Specialization:	3901V021 Mathematical Engineering

Prague, July 2019

I would like to thank my supervisor, Prof. RNDr. Pavel Pták, DrSc., for his guidance throughout my study and many helpful comments with writing this thesis.

I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

Prague, 25th July 2019 Michal Hroch

Anotace

Abstrakt: Téma této disertační práce je studium konečně aditivních měr na strukturách blízkých Booleovým algebrám. Ve formulaci přijaté v teoretické fyzice jde o vyšetřování stavů na kvantových logikách. Přínos této disertační práce je obsažen v pěti přiložených článcích. V prvním z těchto článků se autor zabývá stavy s hodnotami v obecném tělese charakteristiky nula. Cílem je zobecnit klasickou Hornovu-Tarského větu o rozšiřování stavů (HT). Autor částečně uspěl v některých speciálních případech, ale obecně se ukázalo, že přirozená reformulace věty HT pro tělesa neplatí (kromě znaménkové formulace HT, kde se podařilo původní reálně hodnotovou HT zobecnit na tělesovou formulaci). V druhém článku autor vyšetřuje ortomodulární svazy, které dovolují zavedení symetrické diference. Tyto struktury, které jsou v současné době intenzivně studovány, byly v tomtu článku obohaceny o vhodný pojem stavu. Vznikla pak otázka, kdy se dá daný stav rozšířit na větší logiku (případně i takovou, která není množinově reprezentovatelná). Autor ukázal, že stavová rozšíření jsou možná pokud je stav definován na Booleově algebře. V obecnosti ani logiky velice blízké Booleovským stavová rozšíření nedovolují. Ve třetím článku autor dále analyzoval množinově reprezentovatelné logiky. Jako hlavní výsledky lze jmenovat kritérium pro rozšiřování stavů na Gudderových logikách a příspěvek k jistým fyzikálně motivovaným otázkám pro klasickou hustotní logiku (určitý nový pohled na Banachovy limity). Ve čtvrtém článku se autor zabývá pravděpodobnostně motivovaným pojmem Jauchova-Pironova stavu. Autor nalezl nutnou a postačující podmínku pro rozšiřování stavů definovaných na Booleově algebře a zachovávající při rozšíření Jauchovu-Pironovu vlastnost. Aplikací tohoto výsledku je dokázána věta o rozšiřování Jauchových-Pironových stavů na projektorovou logiku $L(H)$. V pátém článku je zavedeno jisté nekonečné rozšíření Gudderovy logiky. V poněkud překvapivém kontrastu s konečnou Gudderovou logikou je dokázáno, že tyto zobecněné logiky dovolují rozšiřování stavů na potenční množinu.

Klíčová slova: ortomodulární částečně uspořádaná množina, symetrická diference, konečně aditivní míra, stav, Booleova algebra, Hornovo-Tarského kritérium o rozšiřování stavů, kompaktnost stavového prostoru, Tichonovova věta

Annotation

Abstract: The theme of the thesis is, in general terms, a study of finitely additive measures on the structures similar to Boolean algebras. Formulating this theme in the language of theoretical physics, it is an investigation of states on (algebraic) quantum logic. The contribution of the thesis is based on five papers included in the thesis. In the first paper, the author considers field-valued states for the fields of characteristics zero. The intention is to generalize the classic Horn-Tarski state extension theorem (HT). The author partially succeded in special cases but in general it was shown that a natural formulation of HT cannot be obtained (except for the signed form of HT where the original real-valued HT result allows for the field-valued formulation). In the second paper, the author investigated the orthomodular lattices that can be endowed with a symmetric difference. This orthomodular lattices, recently intensely studied, was enriched in this paper with an appropriate notion of state. Then a question was asked when, given a state, one can extend it over a bigger (possibly non-set-representable) logic. The author showed that this state extension is possible when the "domain" logic is a Boolean algebra, but in general even "almost Boolean" logics do not allow for a state extension. In the third paper, the author further analyzed the set-representable logics. Main results obtained are a state extension criterion for Gudder's logics and a clarification of certain physically motivated questions on the classic density logic (a certain new view of Banach limits). The fourth paper deals with the quantum-probabilistically justified notion of Jauch-Piron states. The author found a necessary and sufficient condition for the extension of states, as Jauch-Piron states, defined on Boolean algebras. This result, as an application, establishes a Jauch-Piron state extension over the projection logic $L(H)$. The fifth paper brings an infinite generalization of Gudder's logics and shows, in a slightly surprising contrast to Gudder's logics, that the states on these generalized logics allow for the extension over the power set.
Keywords: orthomodular poset, symmetric difference, finitely additive measure, state, Boolean algebra, Horn-Tarski extension criterion, compactness of the state space, Tychonoff's theorem

Contents

Introduction

Quantum structures and related topics have been attractive areas for many researchers in mathematics and physics for a longer time. After the celebrated paper by G. Birkhoff and J. von Neumann [1], there has been an intense research in algebraic and measure-theoretic problems of "quantum logics". Typically, the quantum logics are identified with orthomodular posets and the states on them are identified with probability measures. Considering both the algebraic and measure-theoretic (resp. functionally-analytic) lines, there is a series of monographs and fundamental papers on the subject (see A. Horn and A. Tarski (1948) [7], A. Dvurečenskij [3], G. Mackey (1963) [12], P. Mittelstaedt (1970) [13], R. J. Greechie (1971) [4], S. Gudder (1979) [9], G. Kalmbach (1983) [11], M. Navara, P. Pták and V. Rogalewicz (1988) [10], I. Pitowski (1989) [14], P. Pták and S. Pulmannová (1991) [15], L. Bunce and J. Wright (1992) [2], H. Weber (1994) [17], S. S. Holland, Jr. (1995) [5], M. Rèdei (1996) [16], J. Hamhalter (2003) [6], Handbook of Quantum Logic (2007) [8], etc.). The present state of the art of algebraic theories offers a large area of interesting problems on the intrinsic and state-space properties of the structures dealt with. Technically, the study mostly concerns several types of orthomodular posets and several types of probability measures on them. In this account, we want to contribute to the questions on the extension of states on (typically set-representable) quantum logics. The logics pursued lie near the so called "standard logics", meaning they lie intrinsically near Boolean algebras.

The questions pursued in this thesis are motivated by theoretical physics (the mathematical foundation of quantum theories) but they are sometimes brought in by a mere mathematical curiosity. The research reflexes the state of the art of this part of the theory of ordered structures. More specifically, the results obtained extend on and complement some of the recent findings.

A center theme of this thesis are problems on the extensions of states, sometimes related to subadditivity with respect to the symmetric difference. Though the results may have a link with questions of theoretical physics, it is believed that the results obtained in this thesis will defend themselves as regards the criticism of pure mathematicians (the specialists in orthomodular structures and in ordered sets at large, and the specialists in generalized measure theory).

The thesis is founded on five published papers, Chapters 1–5. Let us comment on the contents of the papers.

In the first part, Chapter 1, *A note on field valued measures*, one contributes to the question of extending field-valued states (a generalized form of the Horn-Tarski theorem). For technical reasons, we restrict ourselves to the fields of characteristic 0 (thus, we assume that no finite sum of 1 is 0). We first find out a link between the Farkas lemma and the famous Horn-Tarski extension result (Theorem 1.1). We then show that if the field in question contains rational numbers as a topologically dense subfield, we infer that the condition for the extension of non-negative field valued assignment is equivalent of a weak form of Farkas lemma (Theorem 1.3). In trying to obtain a full generalization of Horn-Tarski extension result, we ended up in a rather negative conclusion—we have been able to find a counterexample to the field valued form of the Horn-Tarski result (Proposition 1.4). We then pass to discussing signed measures and in this area we establish that a full extension of Horn-Tarski result is possible (Theorem 1.5).

In the second part, Chapter 2, *States on orthocomplemented difference posets (Extensions)*, one adds to the nowadays developing theory of orthomodular posets with a symmetric difference (the ODPs)—we show that a state on a Boolean subalgebra B of a general state-rich ODP, Q, extends over Q (Theorem 2.2). By a counterexample we show that this result cannot be obtained when B is an "almost Boolean" ODP (= when B is pseudocomplemented). This counterexample is established in a rather elaborate construction of Theorem 2.4 that uses an insight borrowed from other publications.

In the third part, Chapter 3, *Concrete quantum logics, Δ-logics, states and Δ-states*, which is perhaps the most technically involved section of this account, we investigate several previously considered set-representable orthomodular posets and their symmetric-difference closures. We first subject the states studied to certain natural properties and discuss the possibilities of their extensions. It is, for instance, decided about extensibility of states on the divisibility logics (Theorem 3.2). We then consider the classic density logics and proved several new properties, e. g., we find that the Δ-closure of the density logic is the set of all subsets which relates to the celebrated Banach construction (Theorem 3.3). We also study almost disjoint subsets of Ω of the cardinality equal to continuum and observe that the corresponding factor is pseudocomplemented. Finally, this chapter contains an answer to a published open question on the archetypical even-coeven logics (Theorem 3.1).

In the fourth part, Chapter 4, *Jauch-Piron states on quantum logics*, we take up the question of when a state on a Boolean algebra can be extended, as a Jauch-Piron state, over a bigger logic. In the main result we found a sufficient condition for such an extension (Theorem 4.2). As a consequence, such an extension is showed for the projection logic $L(H)$ (Theorem 4.3). In the appendix to this chapter, we looked for a formally analogous extension theorem in set-representable logics. However, we found that if we require all Dirac states Jauch-Piron, then the logic automatically becomes the Boolean algebra (Theorem 4.4).

In the last part, Chapter 5, *Quantum logics defined by divisibility conditions*, we consider a specific set-representable logics as a generalization of Gudders logics. Suppose that p is a prime number and S is a countable set. Let us consider the collection Div_p^S of all subsets of S whose cardinalities are multiples of p, and the complements of such sets. We proved that a state on Div_p^S can be extended to a state on the Boolean algebra of all subsets of S. Further, we show that each pure state on Div_p^S has to be two-valued. This result may be of interest when viewed incomparism with previous studies (it is known that if S is finite and Div_p^S reduces to Gudder's logic, then neither of the results are true) and, also, when the results are viewed within the interpretation in theoretical physics, it is shown that state space of a non-standard (non-Boolean) logic embeds into the state space of a Boolean algebra.

In concluding the introduction let us mention that the author presented the results of his doctoral thesis on several academic occasions. In order to name the important ones related to the thesis, they are: The thirteenth Biennial IQSA Conference Quantum Structures, July 2016, Leicester (UK); The intermediate IQSA Quantum Structures Workshop, July 2017, Nijmegen (NED); Lecture at the seminar of the Mathematical Institute of Slovac Academy of Sciences, October 2017, Bratislava (SVK); The fourteenth Biennial IQSA Conference Quantum Structures, July 2018, Kazan (RUS); Workshop STOCHASTIKA, February 2019, Kohútka (CZE).

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1 A note on field-valued measures

The paper enclosed appeared in Mathematica Slovaca 67 (2017), 1295–1300, ISSN 0139-9918—IF 0,314. The contribution of Michal Hroch to the contents of the paper is 33% (agreed with the coauthor).

A note on field-valued measures

Anna De Simone, Michal Hroch and Pavel Pták [1]

Abstract We consider the Horn-Tarski condition for the extension of (signed) measures (resp., non-negative measures) in the setup of field-valued assignments. For a finite collection $\mathcal{C}$ of subsets of Ω, we find that the extension from $\mathcal{C}$ over the collection $\exp\Omega$ of all subsets of Ω is implied by, and indeed equivalent to, a certain type of Frobenius theorem (resp. a certain type of Farkas lemma). This links classical notions of linear algebra with a generalized version of Horn-Tarski condition on extensions of measures. We also observe that for a general (infinite) $\mathcal{C}$ the Horn-Tarski condition guarantees the extension of signed measures (here the standard Zorn lemma applies). However, we find out that the extensions for non-negative ordered-field-valued measures are generally not available.

Keywords Boolean algebra, field-valued measure, Horn-Tarski condition.

Mathematics Subject Classifications 03G05, 12E99, 15A06, 06E99 and 28A60

Dedicated to Professor Paolo de Lucia with gratefulness, and thanks

[1]The research of first author was partially supported by the PRIN research project "Metodi logici per il trattamento dell'informazione" (2010) of MIUR (Italian Ministry of University and Research). The second author was supported by the Grant Agency of the Czech Technical University in Prague, grant No. SGS15/193/OHK3/3T/13

The Horn-Tarski condition (H-T condition, see [1] and [9]) characterizes the situations in which real-valued assignments allow for extensions as measures. This condition implies, for instance, the classical result that each bounded non-negative measure on a Boolean collection $\mathcal{C}$, $\mathcal{C} \subset \exp \Omega$, extends over the entire $\exp \Omega$ of all subsets of Ω. Recently the H-T condition has been applied in certain questions of quantum logics (for basics on quantum logics, see e.g. [7]), in both the finite $\mathcal{C}$ (see [4] and [6]) and the infinite $\mathcal{C}$ (see [5]). Being partially inspired by the latter findings and a potential application in a quantum measurement and elsewhere, we consider field-valued measures and a field versions of H-T conditions. We then show the results indicated in the abstract. It should be noted that the ordered fields and the fields that contain rational numbers can be very complicated (for instance, the so called hyper-real fields obtained as factors of rings of continuous functions—see e.g. [10], [12] and [13]), so our contribution can be viewed as a generalization of the hitherto known results (see [1] and [6]).

Let us introduce the notions we shall deal with in the sequel. Let F be a field of characteristic 0 (i.e., let $F = (F, +, \cdot, 0, 1)$ be a commutative ring in which each non-zero element has an inverse and no finite sum of 1 is 0). Obviously, F contains the field of rational numbers. *Throughout the paper, let us reserve F for a field of characteristic 0.*

The following definition generalizes the classical notion of measure.

Definition 1.1. Let Ω be a set and let $\mathcal{B}$ be a Boolean algebra of subsets of Ω. Then a mapping $m : \mathcal{B} \to F$ is said to be an F-valued signed measure on $\mathcal{B}$ if
(i) $m(\emptyset) = 0$,
(ii) if $A, B \in \mathcal{B}$ and $A \cap B = \emptyset$, then $m(A \cup B) = m(A) + m(B)$.

Though the definition of F-valued measure concerns only the (additive) group part of F, in certain questions about a possible extension of F-valued assignment $p : \mathcal{C} \to F$, $\mathcal{C} \subset \exp \Omega$, to an F-valued measure on a Boolean algebra, the field structure of F may play a role. An important observation is that if one manages to extend p over the Boolean algebra generated by $\mathcal{C}$, the rest in extending to $\exp \Omega$ follows from the theorem that says that each group-valued measure extends over $\exp \Omega$ (in [2], the authors present the proof for the group R of real numbers but they correctly comment that the same technique enables the proof for a general group; for general measure-theoretic considerations on group-valued measures, see e.g. [3]).

Let us first consider the following type of an assignment $p : \mathcal{C} \to F$.

Definition 1.2. Let Ω be a set and let $\mathcal{C}$ be a collection of subsets of Ω. Let $p : \mathcal{C} \to F$ be a mapping with the following property (let us recall that $\chi_S : \Omega \to \{0, 1\}$ denotes the characteristic function of S, $S \subset \Omega$, where $\chi_S(x) = 1$ exactly when $x \in S$): If $\sum_{i=1}^{k} \chi_{B_i} = \sum_{j=1}^{l} \chi_{C_j}$ for some $B_1, B_2, \ldots, B_k, C_1, C_2, \ldots, C_l$ in $\mathcal{C}$ then $\sum_{i=1}^{k} p(B_i) = \sum_{j=1}^{l} p(C_j)$. Then we say that $p : \mathcal{C} \to F$ fulfils the Horn-Tarski signed condition (H-T signed condition).

We shall ask when $p : \mathcal{C} \to F$ that fulfils the H-T signed condition allows for an extension over a Boolean algebra as a signed measure. Let us first observe that if $\mathcal{C}$ happens to be Boolean, then $p : \mathcal{C} \to F$ has the H-T signed condition exactly when p is a signed measure. Thus, the H-T signed condition is a *conditio sine qua non* for the meaningfulness of the effort of extending p over a Boolean algebra.

In this first part we will restrict ourselves to a finite $\mathcal{C}$. In that case there is a rather interesting link with elements of linear algebra. Let us introduce the following notion (let us agree to call an element c of F an integer if c is a finite sum of 1 or -1). With a usual abuse of notation, we will denote by the symbol 0 both the zero element in a field F and the zero vector of F^n, considering the context this should not lead to a misunderstanding. Further, we will use the symbol $\leq$ equally in F and in F^n, in the latter case we will understand it coordinatewisely. Also, we write $v > w$ when $v \geq w$ and $v \neq w$.

Definition 1.3. Let A be an $m \times n$ matrix with the entries 0 or 1 ($0, 1 \in F$). Let $Ax = b$ be a linear system with the right-hand vector $b = (b_1, b_2, \ldots, b_m)^T$. This system $Ax = b$ is said to be the 0-1 F-Frobenius if the following implication holds true. If $A^T r = 0$ for a vector $r = (r_1, r_2, \ldots, r_m)^T$ and all r_i, $i \leq m$, are integers in F, then $b^T r = 0$.

Our first result reads as follows (naturally, we say that two propositions are equivalent if one of them directly provides a proof of the other).

Theorem 1.1. *The following propositions are equivalent:*
Prop 1: Each 0-1 F-Frobenius system $Ax = b$ has a solution.
Prop 2: Let $\mathcal{C}$ be a finite collection of subsets of Ω. If $p : \mathcal{C} \to F$ fulfils the H-T signed condition then there is an extension of p over $\exp \Omega$ as a signed measure.

Proof. Let us first show that Prop 1 implies Prop 2. We may (and shall) assume that Ω is finite (this translation to the "skeleton" of $\mathcal{C}$ and p is easy and done in detail in [6]). Thus, let us assume that $\Omega = \{1, 2, \ldots, u\}$ and $\mathcal{C} = \{A_1, A_2, \ldots, A_t\} \subset \exp \Omega$. Let us denote by A the $t \times u$ matrix whose i-th row consists of the characteristic functions of A_i. Set $b = (p(A_1), p(A_2), \ldots, p(A_t))^T$. We are to show that the system $Ax = b$ has a solution provided $p : \mathcal{C} \to F$ fulfils the H-T signed condition. In view of Prop 1, it suffices to check that $b^T r = 0$ is valid for an integer vector $r = (r_1, r_2, \ldots, r_t)^T$ provided $A^T r = 0$. Let us separate the positive and negative coordinates, $r = r^+ - r^-$, where $r^+ = (q_1, \ldots, q_t)$ (resp., $r^- = (q_1, \ldots, q_t)$) with $q_i = r_i$ for $r_i > 0$ (resp., $q_i = -r_i$ for $r_i < 0$) and $q_i = 0$ otherwise. Then the equality $A^T r = 0$ can be rewritten as $\sum_{i=1}^t r_i^+ \cdot \chi_{A_i} = \sum_{j=1}^t r_j^- \cdot \chi_{A_j}$. So we obtain the H-T signed condition when we repeat A_i and A_j according to the multiplicities given by r_i^+ and r_j^-. Then we have $\sum_{i=1}^t r_i^+ \cdot p(A_i) = \sum_{j=1}^t r_j^- \cdot p(A_j)$ but this means the equality $(p(A_1), p(A_2), \ldots, p(A_t))(r_1, r_2, \ldots, r_t)^T = 0$. This gives us $b^T r = 0$ and we have derived the implication Prop 1 implies Prop 2.

Let us prove that Prop 2 implies Prop 1. Let us start with a linear system $Ax = b$, where A is a $t \times r$ matrix with the entries 0 or 1. Let us suppose that this system is 0-1 F-Frobenius. Write $b = (b_1, b_2, \ldots, b_t)^T$ and set $\Omega = \{1, 2, \ldots, r\}$. Associate to each row of A a subset Ω that corresponds to the characteristic function of the i-th row. Denote these subsets by A_i, $i \leq t$, and consider the assignment $p : \mathcal{C} \to F$ by having $\mathcal{C} = \{A_1, A_2, \ldots, A_t\}$ and setting $p(A_i) = b_i$. Take two collections $B_1, B_2, \ldots, B_k, C_1, C_2, \ldots, C_l$ in $\mathcal{C}$. Then define $c_h = \operatorname{card}\{i : B_i = A_h\}$ and $d_h = \operatorname{card}\{j : C_j = A_h\}$, $h \leq t$. Thus, the equality $\sum_{i=1}^k \chi_{B_i} = \sum_{j=1}^l \chi_{C_j}$ is equivalent with $A^T c = A^T d$, $c = (c_1, c_2, \ldots, c_t)^T$, $d = (d_1, d_2, \ldots, d_t)^T$, and $b^T c = b^T d$ is equivalent with $\sum_{i=1}^k p(B_i) = \sum_{j=1}^l p(C_j)$. This means that the equality $\sum_{i=1}^k \chi_{B_i} = \sum_{j=1}^l \chi_{C_j}$ implies $\sum_{i=1}^k p(B_i) = \sum_{j=1}^l p(C_j)$. Hence, since the system $Ax = b$ is 0-1 F-Frobenius, the mapping p fulfils the H-T signed condition and therefore there is an extension of p over $\exp \Omega$ that gives us the solution of $Ax = b$. We see that Prop 2 implies Prop 1. □

Though we were essentially interested in extending $p : \mathcal{C} \to F$ for a general (infinite) $\mathcal{C}$, and this problem will be addressed later on with a solution technique of well-ordering, we found it worthwhile formulating the above equivalence to see an explicit relation with fundamentals of linear algebra (and the similar approach will be made use of in the sequel in ordered fields). Obviously, both statements of Prop 1 and Prop 2 in Theorem 1.1 are true and therefore the extension is available in this case. Indeed, consider the statement of Prop 1. Denote by L the linear space generated by the columns of A and by $\bar{L}$ the space generated by $L \cup b$. Since the complement $L^{\perp}$ of L equals to the complement $(L \cup b)^{\perp}$ of $L \cup b$ (this is true since there is a basis of $L^{\perp}$ consisting of vectors with integer coordinates), we see that $\dim(L) = \dim(L \cup b)$ and the system has a solution due to the Frobenius theorem.

In the next part we will take up the extensions of non-negative $p : \mathcal{C} \to F$, $\mathcal{C}$ finite, over the Boolean algebra B generated by $\mathcal{C}$ (as will be seen later, the further extension over $\exp \Omega$ may be even more difficult). Let F be an ordered field (F is endowed with a linear ordering and this ordering cooperates well with the field operations, see e.g. [12]). As known, each ordered field is of characteristic 0 and therefore it contains the field Q of rational numbers. Thus, we can stick to our convention about denoting F both the field and ordered field case.

Let B be a Boolean algebra. An assignment $p : B \to F$ is said to be a *non-negative* measure if p is a signed measure with $p(a) \geq 0$ for all $a \in B$. We want to derive an equivalence analogous to Theorem 1.1. However, our effort here is met with a weaker success.

We shall need two definitions. Let us first introduce a "non-negative" version of Horn-Tarski condition.

Definition 1.4. Let F be an ordered field. Let $\mathcal{C}$ be a collection of subsets of Ω. Let $p : \mathcal{C} \to F$ be a mapping such that $p(A) \geq 0$ for any $A \in \mathcal{C}$. Let us suppose that the following implication holds true: If $\sum_{i=1}^{k} \chi_{B_i} \leq \sum_{j=1}^{l} \chi_{C_j}$ for some sets $B_1, B_2, \ldots, B_k, C_1, C_2, \ldots, C_l$ in $\mathcal{C}$, then $\sum_{i=1}^{k} p(B_i) \leq \sum_{j=1}^{l} p(C_j)$. Then we say that $p : \mathcal{C} \to F$ fulfils the Horn-Tarski condition (the H-T condition).

An adequate version of the 0-1 F-Frobenius condition (called now the 0-1 F-Farkas condition) in linear systems reads as follows.

Definition 1.5. Let F be an ordered field. Let A be an $m \times n$ matrix with the entries 0 and 1 ($0, 1 \in F$). Consider the system $Ax = b$, $b = (b_1, b_2, \ldots, b_m)^T$. Then we say that the system $Ax = b$ fulfils the 0-1 F-Farkas condition, if the following implication holds true: If $A^T r \geq 0$ for a vector r with all coordinates integer, then $b^T r \geq 0$.

We can again spell out the following equivalence.

Theorem 1.2. *Let F be an ordered field. The following two propositions are equivalent.*

Prop $1_{\leq}$*: Let $Ax = b$ be a system that fulfils the 0-1 F-Farkas condition. Then the system $Ax = b$ has a non-negative solution.*

Prop $2_{\leq}$*: Let $\mathcal{C}$ be a finite collection of subsets of Ω. Let $p : \mathcal{C} \to F$ be a mapping that fulfils the H-T condition. Then p can be extended over the Boolean algebra generated by $\mathcal{C}$ as a non-negative measure.*

The proof verbatim follows the procedure of Theorem 1.1, we only have to exchange the sign $=$ with the sign $\leq$ (see also [6], passing from the field R of reals to a general ordered field F is routine). However, in this case this equivalence does not seem to imply the measure extension – the statement of Prop $1_{\leq}$ seems difficult to be proved and therefore it is not seen how to prove Prop $2_{\leq}$. We only have the following partial result.

Theorem 1.3. *Let F be an ordered field. If the field Q of rational numbers is dense in F when F is considered with the order-topology given by the ordering of F, then each system $Ax = b$ which fulfils the 0-1 F-Farkas condition has a non-negative solution. Thus, for an ordered field F with the above property related to Q, if $\mathcal{C}$ is finite and $p : \mathcal{C} \to F$ fulfils the H-T condition, then there is an extension of p over the Boolean algebra generated by $\mathcal{C}$ as a non-negative measure.*

Proof. Recall first the general version of the Farkas lemma. Let F be an ordered field. Let A be an $m \times n$ matrix with entries in F and let us consider the linear system $Ax = b$ (in F). Then the following statement holds true: The system $Ax = b$ has a non-negative solution $x = (x_1, x_2, \ldots, x_n)^T$ if and only if the inequality $A^T v \geq 0$ ($v = (v_1, v_2, \ldots, v_m)^T$) implies $b^T v \geq 0$ (this result can be proved without any topological considerations as demonstrated in [8]). It suffices to show that under our assumptions the system is subject to the (full) F-Farkas condition. It means that it suffices to show that if there is a vector v with $A^T v \geq 0$ and $b^T v < 0$, then there is a column vector w with rational coordinates and, also, $A^T w \geq 0$ and $b^T w < 0$ (passing from rational to integer is then a matter of multiplying with a natural scalar). If $A^T v > 0$, then such a rational vector v can be easily found in view of the density of Q in F. Suppose that some coordinates of $A^T v$ are 0 (this also covers the case when $A^T v = 0$). Without a lost of generality, let us assume that $(A^T v)_1 = (A^T v)_2 = \ldots = (A^T v)_k = 0$ and $(A^T v)_l > 0$ for $l > k$. Since the entries of A are either 0 or 1, the orthogonal complement of the k first rows has a basis consisting of vectors with integer coordinates. Due to the density of Q in F again, we can easily find such a vector w with rational coefficients that $(A^T w)_1 = (A^T w)_2 = \ldots = (A^T w)_k = 0$ and $(A^T w)_l > 0$ for $l > k$. This completes the proof. $\square$

As we see, if F densely contains the rational numbers, which could occur for many ordered fields different from the field R of real numbers (see [10], [12]), the measure extension from $\mathcal{C}$ over a Boolean algebra exists. If a proper interval in F is compact, then F is isomorphic with R and the extension can be required over the entire $\exp \Omega$ (see [1] and [6]). However, such an extension is not possible in general.

Proposition 1.4. *Let us consider the (ordered) set $Q_{\langle 0,1)}$ of rational numbers of the interval $\langle 0, 1)$. Let B be the Boolean algebra generated by all half-open intervals of the type $\langle a, b)$, $0 \leq a < b \leq 1$. Then there is a non-negative measure $m : B \to Q$ which cannot be extended as a non-negative measure over* $\exp\langle 0, 1)$.

Proof. It is easy to see that B consists of finite disjoint unions of generating sets (see, e.g., [11]). Thus, if $A \in B$ then A is a disjoint union of the form $A = \langle a_1, b_1) \cup \langle a_2, b_2) \cup \ldots \cup \langle a_n, b_n)$. Consider the restriction of the Lebesgue measure on B and denote this measure by m. Then $m : B \to Q$ can be viewed as a non-negative Q-valued measure. This measure cannot be extended over $\exp\langle 0, 1)$ as a Q-valued non-negative measure. Indeed, take an interval $\langle 0, z)$ where z is an irrational number, $z < 1$. If $\bar{m} : \exp\langle 0, 1) \to Q$ were a non-negative extension of m, then $\bar{m}(\langle 0, z)) \geq q_1$ for each rational number $q_1 < z$ and $\bar{m}(\langle 0, z)) \leq q_2$ for each rational number $q_2 > z$ ($\bar{m}$ has to be order-preserving on the sets of $\exp\langle 0, 1)$). But this is absurd. $\square$

In the rest we want to consider an arbitrary (possibly infinite) $\mathcal{C}$ and an arbitrary F (however, F is still subject to our original requirement about F being of characteristic 0). As the previous example shows, we have to resign on hoping to extend non-negative F-measures to non-negative F-measures. In the case of signed measures, however, we find that the methods applied for real valued measures (see [1] and [9]) can effectively be used in this general case as well.

Theorem 1.5. *Let F be a field (of characteristic 0). Let $\mathcal{C}$ be a collection of subsets of a set Ω and let $p : \mathcal{C} \to F$ be a mapping that fulfils the H-T signed condition. Then p can be extended over* $\exp \Omega$ *as a signed measure.*

We shall need two auxiliary results. Since they are proved by the known methods (see [1]), we only indicate their proofs. Let us recall that F has to contain the field Q of rational numbers.

Proposition 1.6. *Let $\mathcal{C}$ be a collection of subsets of a set Ω. Let $L(\mathcal{C}) = \{f : \Omega \to R, f = \sum_{i=1}^{n} r_i \cdot \chi_{A_i}, A_1, A_2, \ldots, A_n$ belong to $\mathcal{C}$, $r_1, r_2, \ldots, r_n$ are rational numbers. So $L(\mathcal{C})$ is the linear space over the field Q the basis of which consists of χ_{A_i}, $A_i \in \mathcal{C}$. Let $p : \mathcal{C} \to F$ be a mapping. Let us define another mapping $\ell : L(\mathcal{C}) \to F$ by putting $\ell(\sum_{i=1}^{n} r_i \cdot \chi_{A_i}) = \sum_{i=1}^{n} r_i \cdot p(A_i)$. Then ℓ is a linear mapping exactly when $p : \mathcal{C} \to F$ fulfils the H-T signed condition.*

Proof. The result is easily seen as soon as one verifies that the definition of ℓ is correct. This amounts to establishing that if $\sum_{i=1}^{k} r_i \cdot \chi_{A_i} = \sum_{j=1}^{t} s_j \cdot \chi_{B_j}$ then $\sum_{i=1}^{k} r_i \cdot p(A_i) = \sum_{j=1}^{t} s_j \cdot p(B_j)$. Multiplying with a suitable integer we obtain integers c_i, d_j such that $\sum_{i=1}^{k} c_i \cdot \chi_{A_i} = \sum_{j=1}^{t} d_j \cdot \chi_{B_j}$. Considering the number of occurrence of A_i and B_j, we can again rewrite the latter equality in the form $\sum_{i=1}^{u} \chi_{D_i} = \sum_{j=1}^{v} \chi_{E_j}$. But then $\sum_{i=1}^{u} p(D_i) = \sum_{j=1}^{v} p(E_j)$ and, by going back in an analogous way, we obtain $\sum_{i=1}^{k} r_i \cdot p(A_i) = \sum_{j=1}^{t} s_j \cdot p(B_j)$. This is what we wanted to show. □

Proposition 1.7. *Let F be a field that contains Q and let $\mathcal{C}$ be a collection of subsets of Ω. Let $p : \mathcal{C} \to F$ be a mapping that fulfils the H-T signed condition. Let $A \subset \Omega$ be a set that belongs to the Boolean algebra generated by $\mathcal{C}$. Then there is a mapping $\bar{p} : \mathcal{C} \cup \{A\} \to F$ that extends p and fulfils the H-T signed condition.*

Proof. Consider the linear mapping $\ell : L(\mathcal{C}) \to F$ defined in Proposition 1.6. Let us set $\bar{p}(A) = d$, where d is determined as follows. If $\chi_A \in L(\mathcal{C})$, then we define $d = \bar{p}(A) = \ell(\chi_A)$. If $\chi_A \notin L(\mathcal{C})$, then we choose an arbitrary $d \in F$ and put $d = \bar{p}(A)$. We claim that if we set $\bar{p}(C) = p(C)$ for any $C \in \mathcal{C}$ and $\bar{p}(A) = d$, we obtain a mapping $\bar{p} : \mathcal{C} \cup \{A\} \to F$ that fulfils the H-T signed condition. We only have to show (Proposition 1.6) that the corresponding mapping $\ell : L(\mathcal{C} \cup \{A\}) \to F$ is linear. It reduces to showing that ℓ is correctly defined. In the case of $A \in \mathcal{C}$, this is evident. Suppose that $A \notin \mathcal{C}$. Then $L(\mathcal{C} \cup \{A\}) = \{f + r \cdot \chi_A,$ where $f \in L(\mathcal{C}), r \in Q\}$. On the set $L(\mathcal{C} \cup \{A\})$ we have defined ℓ as follows: $\ell(f + r \cdot \chi_A) = \ell(f) + r \cdot d$. Let us see that ℓ is correctly defined, the rest is obvious. If $f_1 + r_1 \cdot \chi_A = f_2 + r_2 \cdot \chi_A$ then $f_1 - f_2 = (r_2 - r_1) \cdot \chi_A$. But $\chi_A \notin L(\mathcal{C})$ and therefore $f_1 - f_2 = 0$. This verifies the correctness of ℓ and the proof is complete. □

Let us return to the proof of Theorem 1.5. Consider all collections $\mathcal{E}$ of subsets of Ω that satisfy two following conditions: 1. $\mathcal{C} \subset \mathcal{E}$ and $\mathcal{E}$ is a subset of the Boolean algebra generated by $\mathcal{C}$, 2. there is a mapping $\bar{p} : \mathcal{E} \to F$ which extends $p : \mathcal{C} \to F$ and fulfils the H-T condition. Let us denote by $\mathfrak{E}$ this collection of all such $\mathcal{E}$. Let us order $\mathfrak{E}$ by inclusion. Then $\mathfrak{E}$ has to have a maximal element (Zorn's lemma). Denote this maximal element by $\bar{\mathcal{E}}$ and consider the corresponding $\bar{p} : \bar{\mathcal{E}} \to F$. By Proposition 1.7, $\bar{\mathcal{E}}$ must be the Boolean algebra generated by $\mathcal{C}$. Since $\bar{p}$ is defined on a Boolean algebra and fulfils the H-T signed condition, $\bar{p}$ must be an F-valued signed measure. This measure can be extended over $\exp \Omega$ by the theorem on the group-valued signed measure extension [2].

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2 States on orthocomplemented difference posets (Extensions)

The paper enclosed appeared in Letters in Mathematical Physics 106 (2016), 1131–1137, ISSN 0377-9017—IF 1,671. The contribution of Michal Hroch to the contents of the paper is 50% (agreed with the coauthor).

States On Orthocomplemented Difference Posets (Extensions)

Michal Hroch[2] and Pavel Pták

Abstract We continue the investigation of orthocomplemented posets that are endowed with a symmetric difference (ODPs). The ODPs are orthomodular and therefore can be viewed as "enriched" quantum logics. In this note we introduce states on ODPs. We derive their basic properties and study the possibility of extending them over larger ODPs. We show that there are extensions of states from Boolean algebras over unital ODPs. Since unital ODPs do not in general have to be set-representable, this result can be applied to a rather large class of ODPs. We then ask the same question after replacing Boolean algebras with "nearly Boolean" ODPs (the pseudocomplemented ODPs). Making use of a few results on ODPs, some known and some new, we construct a pseudocomplemented ODP, P, and a state on P that does not allow for extensions over larger ODPs.

Keywords Orthocomplemented poset with a symmetric difference (ODP), Boolean algebra, states on ODPs.

Mathematics Subject Classifications 06C15, 03G12, 81B10

[2]The author was supported by the Grant Agency of the Czech Technical University in Prague, grant No. SGS15/193/OHK3/3T/13

In this note we deal with the orthocomplemented posets with the symmetric difference (ODPs) as introduced in [7]. Since the ODPs are found to be orthomodular, they can be viewed as some types of quantum logics (see e.g. [1], [4], [5] and [13] for properties of quantum logics). We introduce states on ODPs and find their basic properties. Then we contribute to the question on the extensions of states. In a sense, we complement and extend on the results of the papers [2], [3], [6], [7], [8] and [9].

Let us first recall the structure which we are going to be interested in [7].

Definition 2.1. Let $P = (X, \leq, ^{\perp}, 0, 1, \triangle)$, where $(X, \leq, ^{\perp}, 0, 1)$ is an orthocomplemented poset and $\triangle$ is a binary operation. Let us say that P is said to be an *orthocomplemented difference poset* (abbr. , an ODP) if P satisfies the following properties:

(D1) $x\triangle(y\triangle z) = (x\triangle y)\triangle z$,

(D2) $x\triangle 1 = x^{\perp}, 1\triangle x = x^{\perp}$,

(D3) $x \leq z$ and $y \leq z$ implies $x\triangle y \leq z$.

For the convenience of the reader, let us shortly review some basic properties of the operations of an ODP. They can be easily verified and have been checked in detail in [7]. Suppose that P is an ODP and $x, y \in P$. Then (1) $x\triangle 0 = x$, (2) $x\triangle x = 0$, (3) $0\triangle x = x$, (4) $x\triangle y = y\triangle x$, (5) $x\triangle y^{\perp} = x^{\perp}\triangle y = (x\triangle y)^{\perp}$, (6) $x^{\perp}\triangle y^{\perp} = x\triangle y$, (7) $x\triangle y = 0$ exactly when $x = y$.

For the purpose of expressing our results in a transparent form, let us review basic notions on the morphisms in ODPs. Let P, Q be some ODPs and let $f : P \to Q$ be an ODP morphism (in a standard manner, f is an ODP morphism if f respects all the operations involved in the definition of ODP). If f is injective and f^{-1} is again a morphism, we call f an embedding. Of course, if $f : P \to R$ is an embedding, we can view P as a subset of R and call P a subODP of R. Further, if f is an embedding and if f is onto then we call f an isomorphism.

Each Boolean algebra is an ODP, of course, but obviously there are many others (see e.g. [7]).

Example 1. Let $\Omega = \{1, 2, \ldots, 2k-1, 2k\}$ be a set $k \in N$. Let Ω_{even} be the collection of all subsets of Ω consisting of an even number of elements. Then this Ω_{even} understood with the inclusion ordering and with the standard set–theoretic symmetric difference is an ODP.

Example 2. The projection logic $L(R^2)$ can be viewed as an ODP (one of the proofs of that result uses the Gödel coding, see [8]). It is an open problem (yet unsolved but with a conjecturably positive answer [9]) that $L(R^3)$ could be embedded in an ODP.

The following notion is crucial in this paper.

Definition 2.2. Let P be an ODP. Let $s : P \to [0,1]$ be a mapping that is subject to the following conditions ($a, b \in P$):
(S1) $s(1) = 1$,
(S2) if $a \leq b^\perp$, then $s(a \vee b) = s(a) + s(b)$,
(S3) $s(a \triangle b) \leq s(a) + s(b)$.

As noted before, an ODP is automatically an orthomodular poset. Thus, the first conditions S1, S2 of the ODP states are nothing but the standard requirement for a state in the theory of quantum logics. Since the operation $\triangle$ models in a sense the logical connective of "exclusive or", it seems natural (or at least not in a conflict with quantum considerations) to require the third condition for quantum mechanical events a, b in a state s.

The following result brings some basic properties of the state space of an ODP.

Proposition 2.1. *Let L be an ODP and let $\mathcal{S}(L)$ be the set of all states on L. If $\mathcal{S}(L)$ is viewed as a subset of $[0,1]^L$, then $\mathcal{S}(L)$ is convex and compact.*

Proof. Suppose that $s_1, s_2 \in \mathcal{S}(L)$. Consider the mapping $v = \alpha s_1 + (1-\alpha)s_2$, $\alpha \in [0,1]$. Then v obviously satisfies the conditions (S1) and (S2) of the definition of a state. Let us verify the condition (S3). Suppose that $a, b \in L$ and compute $v(a \triangle b)$. We obtain
$v(a\triangle b) = \alpha s_1(a\triangle b) + (1-\alpha)s_2(a\triangle b) \leq \alpha\,(s_1(a) + s_1(b)) + (1-\alpha)(s_2(a) + s_2(b)) = \alpha s_1(a) + (1-\alpha)s_2(a) + \alpha s_1(b) + (1-\alpha)s_2(b) = v(a) + v(b)$.
Thus, $\mathcal{S}(L)$ is closed under the formation of convex combinations. The compactness of $\mathcal{S}(L)$ follows from the Tychonoff theorem – $\mathcal{S}(L)$ is closed under the formation of pointwise limits and therefore $\mathcal{S}(L)$ is a closed subspace of the (compact) space $[0,1]^L$. □

Let us recall that L is said to be *unital* if for each $a \in L, a \neq 0$ there is a state $s \in \mathcal{S}(L)$ such that $s(a) = 1$. Our next result reads as follows (the compactness argument of [12] is adopted for the situation of ODPs).

Theorem 2.2. *Let L be a unital ODP. Let us suppose that B is a Boolean subalgebra of L and let s be a state on B. Then s can be extended over L as a state, i.e. there is a state, t, on L such that $t(a) = s(a)$ for any $a \in B$. If s is two-valued and L is unital with respect to two-valued states, then the state t can be required two-valued, too.*

Proof. We may (and shall) understand B as a subset of L. Let us consider the set $\mathcal{P}$ of all partitions of B. Recall that $P = \{p_1, p_2, \ldots, p_n\}$ is said to be a partition of B, $P \in \mathcal{P}$, where $p_i \in B,\ i \leq n$ and the following conditions are satisfied (the symbols $\vee$ and $\wedge$ mean the supremum and infimum) : $\vee_{i=1}^{n} p_i = 1$, and $p_i \wedge p_j = 0$ whenever $i \neq j$. The set $\mathcal{P}$ is (upon an obvious identification) a partially ordered set with the refinement relation $\prec$: For two partitions $P = \{p_1, p_2, \ldots, p_n\}$ and $Q = \{q_1, q_2, \ldots, q_m\}$, we write $P \prec Q$ if for any $p_i,\ i \leq n$, there is a $q_j,\ j \leq m$ such that $p_i \leq q_j$. Moreover, the ordering $\prec$ is directed. Indeed, if $P = \{p_1, p_2, \ldots, p_n\}$ and $Q = \{q_1, q_2, \ldots, q_m\}$, then the partition $P \wedge Q$ that consist of all nonempty intersections $\{p_i \wedge q_j \mid i \leq n,\ j \leq m\}$ is a lower bound of P and Q. Consider a state $s \in \mathcal{S}(B)$. For each partition $P = \{p_1, p_2, \ldots, p_n\}$, let us write $\mathcal{S}_P = \{t \in \mathcal{S}(L) \mid s(p_i) = t(p_i) \text{ for all } i \leq n\}$. Obviously, $\mathcal{S}_P$ is a closed subset of $\mathcal{S}(L)$. Moreover, $\mathcal{S}_P \neq \emptyset$. To see that, if $p_i \in P$ and $p_i \neq 0$, then there is a state $s_i \in \mathcal{S}(L)$ such that $s_i(p_i) = 1$ (this state s_i is guaranteed by the unitality of L). Since $\sum_{i=1}^{n} s(p_i) = 1$, we can make use of the convexity of $\mathcal{S}(L)$ and define $s_P = \sum_{i=1}^{n} s(p_i) \cdot s_i$. Then $s_P \in \mathcal{S}_P$ and therefore $\mathcal{S}_P \neq \emptyset$ for each partition P. In the rest we utilized the compactness of $\mathcal{S}(L)$. Going over all partitions of B, we obtain a centered family $\mathcal{S}_P$, $P \in \mathcal{P}$ of closed subsets of $\mathcal{S}(L)$. As a consequence of the compactness of $\mathcal{S}(L)$, $\bigcap\limits_{P \in \mathcal{P}} \mathcal{S}_P \neq \emptyset$, and if we take a state $t \in \bigcap\limits_{P \in \mathcal{P}} \mathcal{S}_P$, we easily see that $t(a) = s(a)$ for any $a \in B$. Obviously, if s is two-valued then t is two-valued, too. The proof is complete. $\square$

Remark. If we identify the hidden variables of a quantum system (meaning of an ODP in our case) with a two-valued state [5], the final "two-valued" statement of Theorem 2.2 can be rephrased as follows: Choosing a classic subsystem B of a "reasonable" quantum system L, all two-valued states on B are traces of hidden variables of L. Without taking any risk in intellectualizing about this statement, the hidden variable hypothesis might be linked with another rather bizarre circumstance—the abundance of two-valued states on B is equivalent with the Axiom of Choice in the Set Theory.

Let us return to Theorem 2.2 at large. It should be noted that if L is set-representable, this result of Theorem 2.2 has been obtained in [2]. However, there is a huge class of unital ODPs that are not set-representable and can play an important role in quantum theories (like e.g. the projection logic $L(R^3)$ does in the usual quantum logic theory). The set-representable as well as non-set-representable ODPs have been largely studied in [7], though state questions have not been considered. In fact, the mere horizontal sum technique in ODPs provides interesting examples of non-set-representable ODPs. To make this note self-contained and allow the reader to see that the construction product is indeed a unital ODP (and so are indeed all non-set-representable ODPs constructed in [7]), let us exhibit a simple example.

Example 3. There is a finite ODP, P, which is unital and not set-representable.

Proof. Let B be a Boolean algebra of all subsets of the set $\{1,2,3,4,5\}$, thus $B = \exp\{1,2,3,4,5\}$. Let us consider the following three subalgebras of B:

$B_1 = \{\emptyset, \{1,2\}, \{3\}, \{4,5\}, \{3,4,5\}, \{1,2,4,5\}, \{1,2,3\}, \{1,2,3,4,5\}\}$

$B_2 = \{\emptyset, \{1,5\}, \{2\}, \{3,4\}, \{2,3,4\}, \{1,3,4,5\}, \{1,2,5\}, \{1,2,3,4,5\}\}$

$B_3 = \{\emptyset, \{1,3\}, \{2,4\}, \{5\}, \{2,4,5\}, \{1,3,5\}, \{1,2,3,4\}, \{1,2,3,4,5\}\}$.

Let us denote by P the (orthomodular) horizontal sum of the algebras B_1, B_2, B_3. Since B_1, B_2, B_3 are mutually disjoint except for $\emptyset$ and $\{1,2,3,4,5\}$, we can set $x \triangle_P y = x \triangle_B y$. It is easy to see that P is an ODP. Also, it is evident, that the state space of B agrees with the state space of P, hence P is unital. On the other hand, it can be proved that P is not set-representable. For that, let us make use of Theorem 4.7 of [7] which asserts that, in a certain analogy with the Stone representation of Boolean algebras, an ODP is set-representable exactly when for two non-comparable elements $x, y \in P$ there is a two-valued evaluation s (in our terminology, a two-valued state s) on P such that $s(x) = 1$ and $s(y) = 0$. Set $x = \{3\}$ and $y = \{2,3,4\}$. Since $x \in B_1$ and $y \in B_2$, the elements x, y are obviously non-comparable. Assume that there is a two-valued state s on P such that $s(\{3\}) = 1$ and $s(\{2,3,4\}) = 0$. Then $s(\{3\}^\perp) = s(\{1,2,4,5\}) = 0$ and therefore $s(\{1,2\}) = s(\{3,4\}) = s(\{4,5\}) = s(\{2\}) = 0$. Since $\{1,2,3,4\} = \{1,2\} \triangle \{3,4\}$ and $\{2,4,5\} = \{4,5\} \triangle \{2\}$, we see that $s(\{5\}^\perp) = s(\{1,2\} \triangle \{3,4\}) = 0$, and analogously, $s(\{1,3\}^\perp) = s(\{2,4,5\}) = 0$. Since $\{1,3\} \subset \{5\}^\perp = \{1,2,3,4\}$, we conclude that $s(\{1,3\}) = 0$, but $s(\{2,4,5\}) = 0$, too, and this is a contradiction. $\square$

Let us address a potential generalization of Theorem 2.2—let us ask the same question for pseudocomplemented ODPs. Unfortunately, we are in for a negative answer. Let us recall that an ODP is said to be *pseudocomplemented* if the following implication holds true: If $a \wedge b = 0$ then $a \leq b^{\perp}$. It should be noted that this class of pseudocomplemented ODPs is of specific interest with quantum theories because each pseudocomplemented ODP is set-representable [7] and the relation of compatibility manifests itself "algebraically". The elements a, b are compatible precisely when $a \wedge b$ exists. Observe also that a pseudocomplemented ODP is Boolean whenever it is finite or it is a lattice. So the pseudocomplemented ODPs are rather close to Boolean algebras.

We shall need two auxiliary results. The first result is based on an idea of [2] and we will express it in the form we shall need later.

Example 4. Let $\Omega = \{1, \ldots, 10\}$ and let us consider the following subsets $A, B, C, D \ : A = \{1, 2, 5, 8\}, B = \{1, 3, 6, 9\}, C = \{1, 2, 3, 4\}, D = \{1, 5, 6, 7\}$ of Ω. Let P be the set-representable ODP of subsets of Ω that is generated by A, B, C and D. Then there is a (two-valued) state on P that does not allow for an extension over $\exp \Omega$.

Proof. Let us set $s(\Omega) = 1, s(A) = 0, s(B) = 1, s(C) = 1, s(D) = 1, s(A \triangle B) = 1, s(A \triangle C) = 1, s(A \triangle D) = 1, s(B \triangle C) = 0, s(B \triangle D) = 0, s(C \triangle D) = 0,$ $s(A \triangle B \triangle C) = 0, s(A \triangle B \triangle D) = 0, s(A \triangle C \triangle C) = 0, s(B \triangle C \triangle D) = 1$ and $s(A \triangle B \triangle C \triangle D) = 1$. We have defined a two-valued state on P. This state does not allow for an extension over $\exp \Omega$. To see that, one first takes into account that each state t on $\exp \Omega$ is a convex combination of Dirac states (i.e., the convex combination of states that are concentrated in singletons). But it easily follows from the construction of the sets A, B, C, D that there is no such t that would extend s. Indeed, let us suppose that t is such an extension. Then $t(\{1, 3, 6, 9\}) = t(\{1, 2, 3, 4\}) = 1$ and therefore $t(\{1, 3\}) = 1$. But $t(\{1\}) = t(\{1, 2, 5, 8\}) = 0$ and if follows that $t(\{3\}) = 1$, and this is absurd. □

The second result might also be of a certain interest in its own right.

Proposition 2.3. *Each set-representable ODP can be embedded in a pseudocomplemented ODP.*

Proof. Suppose that P is a set-representable ODP and suppose that the underlying set of P is X. Let us take a mapping f, $f : Y \to X$ such that f maps Y

onto X and such that $\operatorname{card} f^{-1}(x) > \operatorname{card} X$ for any $x \in X$. When we take the full preimage of P with respect to f^{-1}, we obviously obtain an ODP which is isomorphic with P (the preimages preserve the operations of intersection and union as well as the operation $\triangle$ of complementation). Let us denote by Q the ODP we have obtained, $Q \subset \exp Y$. Let us denote by R the smallest (set-representable) ODP that is generated by Q and all finite and co-finite subsets of Y. Then R is a pseudocomplemented ODP and the proof is complete. □

Theorem 2.4. *There is a unital ODP, R, and a pseudocomplemented ODP, P, such that P is a subODP of R and such that we can construct a state s on P that does not allow for an extension over R.*

Proof. Let us apply the construction of Prop. 2.3 by taking for P the ODP on $\Omega = \{1, 2, 3, 4, 5\}$ constructed in Example 3. Then the preimage Q under f^{-1} (Prop. 2.3) is isomorphic with P and therefore Q has the same state-space. We have constructed a state s on P that cannot be extended over $\exp \Omega$. Let us denote by t the state on R (R constructed in Prop. 2.3) obtained by lifting the state s with f^{-1} and with additionally defining $t(F) = 0$ for each finite subset of Y. Then t cannot be extended over $\exp Y$ and this completes the proof. □

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3 Concrete quantum logics, $\triangle$-logics, states and $\triangle$-states

The paper enclosed appeared International Journal of Theoretical Physics 56 (2017), 3852–3859, ISSN 0020-7748—IF 0,968. The contribution of Michal Hroch to the contents of the paper is 50% (agreed with the coauthor).

Concrete quantum logics and $\triangle$-logics, states and $\triangle$-states

Michal Hroch and Pavel Pták[3]

Abstract By a concrete quantum logic (in short, by a logic) we mean the orthomodular poset that is set-representable. If $L = (\Omega, \mathcal{L})$ is a logic and $\mathcal{L}$ is closed under the formation of symmetric difference, Δ, we call L a Δ-logic. In the first part we situate the known results on logics and states to the context of Δ-logics and Δ-states (the Δ-states are the states that are subadditive with respect to the symmetric difference). Moreover, we observe that the rather prominent logic $\mathcal{E}_\Omega^{\text{even}}$ of all even-coeven subsets of the countable set Ω possesses only Δ-states. Then we show when a state on the logics given by the divisibility relation allows for an extension as a state. In the next paragraph we consider the so called density logic and its Δ-closure. We find that the Δ-closure coincides with the power set. Then we investigate other properties of the density logic and its factor.

Keywords concrete quantum logic, symmetric difference, Δ-logic, state, density logic, Banach limit.

Mathematics Subject Classifications 03G12, 06E99 , 28E99, 81P10

[3]The authors were supported by the Grant Agency of the Czech Technical University in Prague, grant No. SGS15/193/OHK3/3T/13

The (concrete) logics and Δ-logics have been investigated by several authors ([1], [2], [4], [3], [5], [7], [14], [15], [18], [20], [23], [24]). In this note, we extend on this investigation.

Let us review the basic notions as we shall use them in the sequel (by $\exp\Omega$ we mean a collection of all subsets of Ω).

Definition 3.1. A concrete quantum logic (abbr., a logic) is a pair $(\Omega, \mathcal{L})$ where Ω is a set and $\mathcal{L}$, $\mathcal{L} \subset \exp\Omega$ is such a collection of sets that is subject to the following conditions:
1. $\Omega \in \mathcal{L}$,
2. if $A \in \mathcal{L}$, then $\Omega \backslash A \in \mathcal{L}$,
3. if $A, B \in \mathcal{L}$ and $A \cap B = \emptyset$, then $A \cup B \in \mathcal{L}$.

A logic is said to be a Δ-logic if it is closed under the formation of the symmetric difference: If $A, B \in \mathcal{L}$, then $A \,\Delta\, B = (A \backslash B) \cup (B \backslash A) \in \mathcal{L}$.

Let us observe that if $L = (\Omega, \mathcal{L})$ is a logic then the Δ-logic generated in Ω by $\mathcal{L}$ consists of all elements, D, of the type $D = A_1 \,\Delta\, A_2 \,\Delta \ldots \Delta\, A_n$, where $A_i \in \mathcal{L}$. Let us denote by $(\Omega, \Delta\,\mathcal{L})$ the Δ-logic generated by $\mathcal{L}$ in Ω. Obviously, if a collection $\mathcal{K}$ is closed under the formation of the symmetric difference and $\Omega \in \mathcal{K}$, then $\mathcal{K}$ is a logic.

The previous research revealed a large variety of concrete logics, including Boolean algebras, of course. It is easily seen that $(\Omega, \mathcal{L})$ is a Boolean algebra exactly when $A \cap B \in \mathcal{L}$ for any pair $A, B \in \mathcal{L}$. (It may be noted that some authors—including the inventor of Δ-logics P. G. Ovchinnikov [18]—preferred the expression "symmetric logic" to Δ-logic, we feel that Δ-logic is more suggestive and short.)

Definition 3.2. Let $L = (\Omega, \mathcal{L})$ be a logic. A mapping $s : \mathcal{L} \to [0, 1]$ is said to be a state on L (or, alternatively, s is said to be a state on $\mathcal{L}$ if we do not need to refer to Ω) if
1. $s(\Omega) = 1$,
2. if $A, B \in \mathcal{L}$ and $A \cap B = \emptyset$, then $s(A \cup B) = s(A) + s(B)$.

If $(\Omega, \mathcal{L})$ is a Δ-logic and s is a state on $\mathcal{L}$, then s is called a Δ-state if $s(A \,\Delta\, B) \leq s(A) + s(B)$ for any $A, B \in \mathcal{L}$.

In the first part of the paper we ask when a state on $(\Omega, \mathcal{L})$ can be extended over $(\Omega, \Delta\,\mathcal{L})$ as a Δ-state. This question is related to "discrete integration" as pursued e.g. in [11], [13], [16]—if a state on $(\Omega, \mathcal{L})$ allows for an extension over $(\Omega, \Delta\,\mathcal{L})$ as a Δ-state, the corresponding integral is Δ-subadditive. In the ideal case when the state s extends over $\exp\Omega$, the corresponding integral is additive. The degree of additivity of the integral can be a significant matter in e.g. coarse-grained measurement or in economic theories (see [4], [12]).

The first instance to be taken up is the situation when the logic $(\Omega, \mathcal{L})$ is already a Δ-logic. A most natural question then reads as follows: When a state on $(\Omega, \mathcal{L})$ is automatically a Δ-state? In [1] the authors asked this question. They showed that if $(\Omega, \mathcal{L})$ is a non-Boolean logic and $\mathcal{L}$ is finite then there is always a state on $(\Omega, \mathcal{L})$ that is not a Δ-state. In a certain contrast, they proved that if $\mathcal{L} = \mathcal{E}_\Omega^{\mathrm{even}}$ is a logic of all even-coeven subsets of Ω and Ω is *uncountable*, then each state on $(\Omega, \mathcal{E}_\Omega^{\mathrm{even}})$ is automatically a Δ-state. The authors of [1] omit the case of Ω countable. In the following theorem we take care of this case.

Theorem 3.1. *Let Ω be an (infinite) countable set and let $\mathcal{E}_\Omega^{\mathrm{even}}$ be the quantum logic of all even-coeven subsets of Ω. Let s be a state on $\mathcal{E}_\Omega^{\mathrm{even}}$. Then s is a Δ-state.*

Proof. The proof makes use of the insight taken from [1] plus a few new observations. Let $A, B \in \mathcal{E}_\Omega^{\mathrm{even}}$. We have to show that $s(A\,\Delta\,B) \le s(A) + s(B)$. First, if $A \cap B \in \mathcal{E}_\Omega^{\mathrm{even}}$, then both $A\backslash B$ and $B\backslash A$ belong to $\mathcal{E}_\Omega^{\mathrm{even}}$ and the inequality is obvious: $s(A\,\Delta\,B) = s((A\backslash B) \cup (B\backslash A)) = s(A\backslash B) + s(B\backslash A) \le s(A) + s(B)$. Suppose therefore that $A \cap B \notin \mathcal{E}_\Omega^{\mathrm{even}}$. Then neither of the sets $A\backslash B$ and $B\backslash A$ belong to $\mathcal{E}_\Omega^{\mathrm{even}}$. Let us discuss the situation by cases. Suppose first that both A and B are infinite. Thus, $A = \Omega\backslash\{a_1, a_2, \ldots, a_{2k}\}$ and $B = \Omega\backslash\{b_1, b_2, \ldots, b_{2l}\}$, where k, l are positive integers. Then $A\backslash B = \{b_1, b_2, \ldots, b_{2l}\}\backslash\{a_1, a_2, \ldots, a_{2k}\}$ and $B\backslash A = \{a_1, a_2, \ldots, a_{2k}\}\backslash\{b_1, b_2, \ldots, b_{2l}\}$. By our assumption, both $A\backslash B$ and $B\backslash A$ are of odd cardinalities. Then there is an $x \in A$ and a $y \in B$ such that both $(A\backslash B) \cup \{x\}$ and $(B\backslash A) \cup \{y\}$ belong to $\mathcal{E}_\Omega^{\mathrm{even}}$. If $x \ne y$, which is easy to satisfy, then $s(A\,\Delta\,B) = s(((A\backslash B) \cup \{x\}) \cup ((B\backslash A) \cup \{y\})) \le s((A\backslash B) \cup \{x\}) + s((B\backslash A) \cup \{y\}) \le s(A) + s(B)$. Secondly, suppose that A is infinite and B is finite. Then $A = \Omega\backslash\{a_1, a_2, \ldots, a_{2k}\}$ and $B = \{b_1, b_2, \ldots, b_{2l}\}$. It means that $A\backslash B = \Omega\backslash(\{a_1, a_2, \ldots, a_{2k}\} \cup \{b_1, b_2, \ldots, b_{2l}\})$ and $B\backslash A = \{a_1, a_2, \ldots, a_{2k}\} \cap \{b_1, b_2, \ldots, b_{2l}\}$. Suppose that the cardinality of $A \cap B$ is greater than or equal to 3. Then we can easily find two distinct points $x, y \in A \cap B$ such that $((A\backslash B) \cup \{x\}) \in \mathcal{E}_\Omega^{\mathrm{even}}$ and $((B\backslash A) \cup \{y\}) \in \mathcal{E}_\Omega^{\mathrm{even}}$. Then $s(A\,\Delta\,B) = s((A\backslash B) \cup (B\backslash A)) \le$

$s(((A\backslash B)\cup\{x\})\cup((B\backslash A)\cup\{y\})) = s((A\backslash B)\cup\{x\})+s((B\backslash A)\cup\{y\}) \leq s(A)+s(B)$. Suppose therefore that $A\cap B$ is a singleton. Write $A\cap B = \{c\}$. We are going to show that for any ε, $\varepsilon > 0$, we have the inequality $s(A\,\Delta\,B) \leq s(A) + s(B) + \varepsilon$. This implies that $s(A\,\Delta\,B) \leq s(A) + s(B)$. Since $A\backslash B$ is infinite, there is an infinite number of disjoint two-point sets in $A\backslash B$. Among these two-point sets there must be one, say $\{u, v\}$, such that $s(\{u, v\}) \leq \varepsilon$ (otherwise we have a contradiction with the additivity of s). Then we have the following inequalities: $s(A\,\Delta\,B) = s((A\backslash B) \cup (B\backslash A)) = s((A\backslash\{c\}) \cup (B\backslash\{c\})) = s((A\backslash\{c, u\}) \cup ((B\backslash\{c\}) \cup \{u\})) = s((A\backslash\{c, u\})) + s((B\backslash\{c\}) \cup \{u\})) \leq s(A) + s(B \cup \{u, v\}) = s(A) + s(B) + s(\{u, v\}) = s(A) + s(B) + \varepsilon$. Finally, suppose that both A and B are finite. So $A = \{a_1, a_2, \ldots, a_{2k}\}$ and $B = \{b_1, b_2, \ldots, b_{2l}\}$. If the cardinality of $A \cap B$ is greater than or equal to 3, we can again find two distinct points $x, y \in A \cap B$ such that $((A\backslash B) \cup \{x\}) \in \mathcal{E}_\Omega^{\text{even}}$ and $((B\backslash A) \cup \{y\}) \in \mathcal{E}_\Omega^{\text{even}}$. As before, $s(A\,\Delta\,B) = s((A\backslash B) \cup (B\backslash A)) \leq s(((A\backslash B) \cup \{x\}) \cup ((B\backslash A) \cup \{y\})) = s((A\backslash B) \cup \{x\}) + s((B\backslash A) \cup \{y\}) \leq s(A) + s(B)$. The only case that remains is when $A \cap B$ is a singleton. Then we will show that for any ε, $\varepsilon > 0$, we have the inequality $s(A\,\Delta\,B) \leq s(A) + s(B) + 2\varepsilon$. Consider again infinitely many two-point sets in $\Omega\backslash(A \cup B)$. If an ε, $\varepsilon > 0$ is given, there must be a two-point set $\{u, v\}$, $\{u, v\} \subset \Omega\backslash(A \cup B)$ with $s(\{u, v\}) \leq \varepsilon$. Consider the sets $A \cup \{u, v\}$ and $B \cup \{u, v\}$. Then $(A \cup \{u, v\}) \cap (B \cup \{u, v\})$ has 3 elements and, moreover, $(A \cup \{u, v\})\,\Delta(B \cup \{u, v\}) = A\,\Delta\,B$. So we obtain $s(A\,\Delta\,B) = s((A \cup \{u, v\})\,\Delta(B \cup \{u, v\})) \leq s(A \cup \{u, v\}) + s(B \cup \{u, v\}) \leq s(A) + \varepsilon + s(B) + \varepsilon \leq s(A) + s(B) + 2\varepsilon$. The proof is complete. □

The result above supports the conjecture that a state on $\mathcal{E}_\Omega^{\text{even}}$ extends over the Boolean algebra of finite-cofinite sets. This question seems to be open so far.

It should be noted that in [23] the author shows that there is a Δ-logic $(\Omega, \mathcal{L})$ on which each state is subadditive (a state on $(\Omega, \mathcal{L})$ is said to be subadditive if for any $A, B \in \mathcal{L}$ there is a $C \in \mathcal{L}$ such that $A \cup B \subset C$ and $s(C) \leq s(A) + s(B)$). Since each subadditive state is a Δ-state, this example somewhat strengthens the uncountable example of [1] and, in addition, it enjoys several other algebraic properties (for instance, it is pseudocomplemented). The example of [23] does require the set Ω uncountable.

Another conceptually important example is the case of the divisibility logics. Suppose that $n = mk$ with numbers $m, n, k \in N$ and $k \geq 2$. Let $\Omega = \{1, 2, \ldots, n\}$ and let us denote by Div_k the logic of all subsets of Ω whose number of elements is divisible by k. Thus, the cardinality of Div_k is $\sum_{i=0}^{m}\binom{n}{ik}$. Consider the logic (Ω, Div_k). If $m = 2$, then problem trivializes—there is always a state on (Ω, Div_k) that cannot be extended over $(\Omega, \Delta\, Div_k)$ (see [3], [21]). A rather interesting situation occurs when $k \geq 3$ and $m \geq 3$. Since the case of k even reduces to the situation covered by $k = 2$ (in this case $\Delta\, Div_k = Div_2$), let us assume that k is odd. In this case $\Delta\, Div_k = \exp\Omega$ and we therefore ask whether a state on (Ω, Div_k) extends over $\exp\Omega$. This question was investigated in a nice paper [21] with the answer that there are always states that do not extend over $\exp\Omega$ as states but that each state on (Ω, Div_k) always allows for an extension over $\exp\Omega$ as a *signed* state. In other words, there is always a set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of real (not necessarily non-negative) numbers with $\sum_{i=1}^{n}\alpha_1 = 1$ and with the property that the combination of the Dirac states on $\exp\Omega$ with the coefficients α_i, $(i \leq n)$, gives us the original state s when restricted to (Ω, Div_k). We would like to contribute to this result by formulating—in the line of Farkas lemma (see [8], [9])—a necessary condition for a state to be extended as a state (we could strengthen it to obtain a necessary and sufficient condition but the formulation is then perhaps less satisfactory and less elegant). To do that, we have to express the result of [21] in more detail and fix some terminology.

Let $n = mk$, when $k \geq 3$, $m \geq 3$ and k is odd. Let us assign to the logic (Ω, Div_k) a matrix, $P(Div_k)$, in the following manner. The matrix $P(Div_k)$ is an $(n-1)\times(n-1)$ matrix such that the rows of $P(Div_k)$ are the following vectors (each vector contains k many of 1's and $(n-1-k)$ many of 0's):

$$\begin{aligned}
r_1 &= (1,1,\ldots,1,0,0,\ldots,0,0,0),\\
r_2 &= (0,1,1,\ldots,1,1,0,\ldots,0,0),\\
&\vdots\\
r_{n-k-1} &= (0,\ldots,0,1,1,\ldots,1,1,1,0),\\
r_{n-k} &= (0,0,\ldots,0,1,1,\ldots,1,1,1),\\
r_{n-k+1} &= (1,0,0,\ldots,0,1,1,\ldots,1,1),\\
r_{n-k+2} &= (1,1,0,0,\ldots,0,1,1,\ldots,1),\\
&\vdots\\
r_{n-2} &= (1,1,\ldots,1,0,0,\ldots,0,1,1),\\
r_{n-1} &= (1,1,1,\ldots,1,0,0,\ldots,0,1).
\end{aligned}$$

Let us assign to each vector of the row the set A_i $(i \leq n-1)$ of Div_k which "copies the coordinates" (for instance, to the vector $r_1 = (1,\ldots,1,0,\ldots,0)$ we assign $A_1 = \{1,2,\ldots,k\}$, to the vector $r_2 = (0,1,\ldots,1,0,\ldots,0)$ we assign $A_2 =$

$\{2, 3, \ldots, k+1\}$, etc.). Write $s_i = s(A_i)$. The author of [21] shows that the system $P(Div_k) \cdot x^T = s_i$ has a precisely one solution ($\det(P(Div_k)) = k$). In fact, by the Cramer rule we obtain $x_i = \frac{\det(P_i)}{k}$, $i \leq n-1$, where P_i is the i-th Cramer matrix associated to $P(Div_k)$. The coordinates of the vector determine the values of the extension of s over $\exp \Omega$. This follows from the fact proved in [21] that the sets A_i are generators of Div_k. It further shows that there is always an extension of s over $\exp \Omega$ as a signed state. A necessary condition for a non-negative extension is given by the following version of Farkas lemma (by an additional condition, we can even arrive to a characterization).

Theorem 3.2. *Let $n = mk$, where $k \geq 3$, $m \geq 3$ and k is odd. Write $\Omega = \{1, 2, \ldots, n\}$ and consider the logic (Ω, Div_k). Then*
1) $(\Omega, \Delta\, Div_k) = (\Omega, \exp \Omega)$,
2) if s is a state on (Ω, Div_k) and $P(Div_k)$ is the matrix associated to (Ω, Div_k), then the validity of the following implication is a necessary condition for s to be extended over $\exp \Omega$ as a state: If $(P(Div_k))^T \cdot p^T \geq 0$ for a vector p with all coordinates integer, then $(s_1, s_2, \ldots, s_{n-1}) \cdot p^T \geq 0$,
3) suppose that the implication in the condition 2) above is valid and suppose that $s(\{1, 2, \ldots, k-1, n\}) - \sum_{i=1}^{k-1} \frac{\det(P_i)}{k} \geq 0$. Then s extends over $\exp \Omega$ as a state.

Proof. Since $P(Div_k)$ is a matrix with the entries 0 and 1 only, we can apply the variant of Farkas lemma proved in [6]. The condition 3) guarantees that we can find the non-negative extension for the singleton $\{n\}$, too. □

Let us note that the system of linear equations considered above may indeed have a "properly signed" solution (thus, there is a state on (Ω, Div_k) that cannot be extended over $\exp \Omega$ as a state. Take, for instance, $n = 9$, $k = 3$ and $m = 3$. Thus $\Omega = \{1, 2, \ldots, 9\}$. Consider the evaluation $e : \Omega \to R$ such that $e(1) = -\frac{1}{7}$, $e(2) = e(3) = \ldots = e(9) = \frac{1}{7}$. This evaluation uniquely determines a (non-negative) state on (Ω, Div_k) by setting $s(A) = \sum_{a \in A} e(a)$. The state s cannot be extended over $\exp \Omega$ as a state. This can be verified directly or it suffices to take, in our condition 2), the vector $p = (3, -2, 0, 2, 0, -2, 3, -1)$.

It should be noted, in connection with the theme of our paper, that an analogous question about extensions of states has been asked and investigated in [12] for so called coarse-grained logics and fully answered in [19] (for a further extension on this type of research, see [4] and [5]). Recalling briefly the definition, if we again write $n = mk$ and $\Omega = \{1, 2, \ldots, n\}$, then the coarse-grained logic is the one generated by consecutive k-tuples in Ω understood $\mod k$. Hence the generating sets are $\{1, 2, \ldots, k\}$, $\{2, 3, \ldots, k+1\}$, ..., $\{n-k+1, n-k+2, \ldots, n\}$, $\{n-k+2, \ldots, n, 1\}$, ..., $\{n, 1, 2, \ldots, k-1\}$. So the number of generators is n (this number could be lowered but this is not a matter of our interest in this paper). In a rather interesting manner, the nature of the extension problem differs considerably from the previous situation. If $m \geq 3$ and $(\Omega, \mathcal{L})$ is a coarse-grained logic on Ω, then a state on $(\Omega, \mathcal{L})$ *always* allows for an extension over $\exp \Omega$ as a state, and therefore the state always allows for an extension over $(\Omega, \Delta \mathcal{L})$ as a state. (In order to expose the structural difference of the two situations, let us again consider the example of the previous paragraph given by the evaluation $e : \Omega \to R$ such that $e(1) = -\frac{1}{7}$, $e(2) = e(3) = \ldots = e(9) = \frac{1}{7}$. If understood as a state of (Ω, Div_3), it cannot be extended over $\exp \Omega$ as a state. However, if understood as a state on the coarse-grained logic $(\Omega, \mathcal{L})$, $k = 3$, it does allow for an extension as a state (indeed, it suffices to take $s(\{1\}) = \frac{1}{7}$, $s(\{4\}) = s(\{7\}) = \frac{3}{7}$, $s(\{2\}) = s(\{3\}) = s(\{5\}) = s(\{6\}) = s(\{8\}) = s(\{9\}) = 0$).

Let us introduce the final area of questions which we want to take up (and contribute to) in this paper. Let $N = \{1, 2, \ldots, n, \ldots\}$ be the set of all natural numbers and let $\mathcal{L}$ be the collection of all subsets A, $A \subset N$ such that $\lim_{n\to\infty} \frac{\mathrm{card}(A\cap\{1,2,\ldots,n\})}{n}$ exists. Put $\Omega = N$ and let us consider $(\Omega, \mathcal{L})$. Let us call $(\Omega, \mathcal{L})$ a *d-logic* (the letter d indicates "density" as sometimes referred to in the literature). This classical structure of number theory and analysis has apparently not been considered from the point of view of quantum logics (in the paper [26] this example was mentioned without any further discussion). Let us formulate and prove certain properties of $(\Omega, \mathcal{L})$ for a potential further investigation within quantum logics.

Theorem 3.3. *Let $(\Omega, \mathcal{L})$ be a d-logic. Thus, $\Omega = N$ and $\mathcal{L}$ consists of all subsets of Ω that are determined by the limit condition introduced in the paragraph above. Then*

1) $(\Omega, \mathcal{L})$ is a (concrete) quantum logic,

2) if we write, for any $A \in \mathcal{L}$, $s(A) = \lim_{n\to\infty} \frac{\mathrm{card}(A\cap\{1,2,\ldots,n\})}{n}$, then s is a state on $\mathcal{L}$,

3) $\mathcal{L}$ is not a lattice (and therefore $\mathcal{L}$ is not Boolean). In fact, any couple A, $B \in \mathcal{L}$ such that $A \cap B \notin \mathcal{L}$ and $A \cap B$ is infinite does not have an infimum,

4) $\Delta \mathcal{L} = \exp \Omega$. More explicitly, for each A, $A \subset \exp \Omega$ there are sets $B, C \in \mathcal{L}$ such that $s(B) = s(C) = \frac{1}{2}$ and $A = B \Delta C$,

5) the state s can be extended over $\exp \Omega$ *as a state,*
6) there is a family of $2^{\aleph_0}$ *almost disjoint subsets of* Ω, $\{A_\alpha, \alpha < 2^{\aleph_0}\}$, *such that* $s(A_\alpha) = 0$ *for each* $\alpha, \alpha < 2^{\aleph_0}$. *A consequence: Let us consider the quantum logic* $\mathcal{K} = \mathcal{L}/\mathcal{F}$ *obtained as the factor of* $(\Omega, \mathcal{L})$ *with respect to the ideal* $\mathcal{F}$ *of all finite sets. Then* $\mathcal{K}$ *has* $2^{\aleph_0}$ *elements and* $\mathcal{K}$ *is atomless. Moreover, this factor logic* $\mathcal{K}$ *is pseudocomplemented (i.e., the elements* A, $B \in \mathcal{K}$ *are compatible exactly when* $A \wedge B$ *exists).*

Proof. The statements 1) and 2) can be proved by routine verifications. Let us consider the statement 3). It is easy to check that for any couple referred to in statement 3) the infimum does not exists (the logic $\mathcal{L}$ contains all finite sets). What remains to show is that such a couple exists at all. Indeed, it suffices to take for A the set of all odd numbers and to construct the set B as follows. First we put into the set B the elements 2 and 3, then precisely all even numbers from the segment $(2^k + 1)$ up to $(\frac{3}{2}2^k)$ and precisely all odd numbers from the segment $(\frac{3}{2}2^k + 1)$ up to $(2^{k+1} + 1)$ for all natural k, $k \geq 2$. It is easy to see that $s(A) = s(B) = \frac{1}{2}$ and that the sequence $d_n = \frac{\text{card}((B \cap C) \cap \{1,2,\ldots,n\})}{n}$ has the values $\frac{1}{4}$ and $\frac{1}{6}$ for its cluster points (thus, $\lim_{n\to\infty} d_n$ does not exist and hence $B \cap C \notin \mathcal{L}$).

Let us take up the proof of statement 4). The formal expression of B and C would be rather difficult and cumbersome, we will indicate the construction idea which is sufficiently intuitive. Let us consider A expressed as a union of subsets, $A = \bigcup_{i=1}^{\infty} I_i$, where each I_i $(i \in N)$ is a segment of consecutive points. Also, let us express the set $\Omega \backslash A$ as a union of subsets, $\Omega \backslash A = \bigcup_{i=1}^{\infty} H_i$ $(i \in N)$, where each H_i is a segment of consecutive points. In our argument, let us refer to an I_i as an "island" in A and to an H_i as a "hole" in A. If either of I_i of H_i is equal to Ω up to a finite set, then the proof is easy. Suppose therefore that both the families I_i and H_i $(i \in N)$ are infinite and each I_i and H_i is a finite set. We can consider I_i and H_i with its order inherited from $N(= \Omega)$. Call this order the natural order of I_i and H_i. Let us construct the sets B and C. Firstly, consider those islands I_i which consist of an even number of elements. In this case the set B to be constructed contains precisely the odd elements in the I_i considered in the natural order and, analogously, the set C to be constructed contains precisely the even elements in the I_i. Secondly, consider the holes H_i which consist of an even number of elements. Then we put the same points into both sets B and C, and these sets will consist precisely of the odd elements in the H_i. It remains to take up the odd-elements sets I_i and H_i. Then the situation is slightly more complicated. Let us first consider the set of all the odd-elements islands I_i. If the latter set is empty, we do not have anything to do. Otherwise, there is the first i ($i \in N$), some i_1, such that I_{i_1} is the first odd-elements island. Further, we construct the set B from all the odd-ordered elements of I_{i_1} and the set C from all the even-ordered elements of I_{i_1}. Then all odd-elements islands $I_{i_{2k+1}}$ will be treated equally. Next, we are to take the points from the islands of the type $I_{i_{2k}}$. In this case we distribute the odd-ordered points of $I_{i_{2k}}$ into the set C and the even-ordered elements to the set B. Finally, let us consider the set of all the odd-elements holes H_i. If the latter set is empty, we do not have anything to do. Otherwise there is the first i ($i \in N$), some i_1, such that H_{i_1} is the first odd-elements hole. Then we construct both the sets B and C from the odd-ordered elements of H_{i_1}. Then all holes $H_{i_{2k+1}}$ will be treated equally. Further, we are to take points from the holes of the type $H_{i_{2k}}$. In this case we construct the both sets B and C from the even-ordered points of $H_{i_{2k}}$. By the construction of the sets B and C, it is not difficult to check that $s(B) = s(C) = \frac{1}{2}$.

The statement 5) can be proved by the classical result on the Banach limits (see e.g. [27], p. 41).

In order to show the statement 6), let us first see that there is a collection of $2^{\aleph_0}$ almost disjoint subsets of Ω. An easy proof of this known result can be obtain as follows (see also [10]). Identify Ω with the set of all rational numbers Q. For each irrational number $r \in R$, let us choose a sequence $(q_n^r)_{n\in N}$ of rational numbers that converges to r. Consider the family of the previously constructed sequences $S_r = \{(q_n^r), n \in N\}$. Let us take the collection $\mathcal{S} = \{S_r, r \text{ is an irrational number}\}$. Then this collection is an almost disjoint family of subsets of Q with the cardinality $2^{\aleph_0}$. Going back to Ω, we have the required almost disjoint collection. Continuing our argument let us first observe that each infinite subset of Ω contains a subset M with $s(M) = 0$. For each S_r choose such a set M_r. Write $\mathcal{M} = \{M_r, r \text{ is an irrational number}\}$. Since the sets of $\mathcal{S}$ are pairwisely eventually almost disjoint, so are the sets of $\mathcal{M}$. This proves the first part of statement 6). To complete the proof, we only need to observe that $\mathcal{F}$ consists of central elements of $(\Omega, \mathcal{L})$ and hence the factor $\mathcal{L}/\mathcal{F}$ gives us a quantum logic [25]. One only takes into account that the pseudocomplementedness ($a \leq b' \iff a \wedge b = 0$) can be equivalently expressed by the equivalence ($a \wedge b$ exists $\iff$ a is compatible with b, see e.g. [17]). The rest is easy. □

The results above indicate certain potential for the interpretation of the d-logic in the realm of quantum logics. In concluding our paper, let us for instance note a link of the d-logic with the projection logic $L(H)$. Let us take an orthonormal basis, $\mathcal{E} = \{v_i, i \in N\}$, in H. Then $\mathcal{E}$ understood as elements of $L(H)$ generates a Boolean subalgebra $B_\mathcal{E}$ of $L(H)$. Obviously, $B_\mathcal{E}$ is Boolean isomorphic to $\exp N$. Consider a state t on $L(H)$. Let us call it an *$\mathcal{E}$-d-state* if the restriction of t on $B_\mathcal{E}$ is a Banach extension of the state s on the d-logic understood as being underlied by the set $\mathcal{E}$. Observe that the $\mathcal{E}$-d-states exist. Indeed, a Banach extension of s considered as a state on $\exp \mathcal{E}$ can be extended over the entire $L(H)$ [22]. It may be interesting to see what the size of the closure of the convex hull, $\overline{\mathrm{conv}(T)}$, comes to ($T$ is the set of all $\mathcal{E}$-d-states for all choices of orthonormal bases $\mathcal{E}$). How smaller this $\overline{\mathrm{conv}(T)}$ is than the entire state space of $L(H)$?

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4 Jauch-Piron states on quantum logics

The paper enclosed will appear in International Journal of Algebra and Its Applications (DOI: 10.1142/S0219498820500176). The contribution of Michal Hroch to the contents of the paper is 50% (agreed with the coauthor).

Jauch-Piron states on quantum logics

Michal Hroch[4] and Pavel Pták[5]

Abstract. We show in this note that if B is a Boolean subalgebra of the lattice quantum logic L, then each state on B can be extended over L as a Jauch-Piron state provided L is Jauch-Piron unital with respect to B (i.e., for each non-zero $b \in B$, there is a Jauch-Piron state s on L such that $s(b) = 1$). We then discuss this result for the case of L being the Hilbert space logic $L(H)$ and L being a set-representable logic.

Keywords: Boolean algebra, orthomodular lattice, quantum logic, Jauch-Piron states, extensions of states

AMS Classifications: 06C15, 03G12, 81B10

Notions and results

Let L be a lattice quantum logic (i.e., let $L = (L, 0, 1, \wedge, \vee, ')$ be an *orthomodular lattice*, see [4] and [6]). Let us reserve the letter L for a lattice quantum logic and call L simply a logic. By a *state* on L we mean a mapping $s : L \to [0, 1]$ such that $s(1) = 1$, and $s(a \vee b) = s(a) + s(b)$ provided $a \leq b'$. Thus, a state on L is a (normalized finitely additive) measure. Let us denote by $\mathcal{S}(L)$ the set of all states on L. For a systematic treatment of states on a logic, see e.g. [10] and [13].

If we identify L with a set of events of a quantum experiment and if we identify s with a state of L, it may seem desirable, in view of quantum phenomena, to assume that s satisfies the following requirement: If $s(a) = s(b) = 1$, then

[4]The author was supported by the Grant Agency of the Czech Technical University in Prague, grant No. SGS18/131/OHK3/2T/13.

[5]This work was supported by the project OPVVV CAAS CZ.02.1.01/0.0/0.0/16_019/0000778.

$s(a \wedge b) = 1$. Translating into probabilistic language, this means the requirement that if a and b are "almost sure events" within the state s, then so is the conjunction $a \wedge b$. The state with the latter property is said to be a *Jauch-Piron state*. It is called in this way after the names of the physicists who pointed out the importance of this property (see [1], [5], [15], etc.).

Each state on a Boolean algebra (= each state on the event structure of a classical experiment) is Jauch-Piron, of course. The situation in general is rather more complicated. Let us observe, in view of the formulation of our results, that each logic can be embedded into a logic that has no Jauch-Piron states at all or, quite curiously, that has exactly one Jauch-Piron state (both statements can be easily derived from [3] and [10]). Consequently, each logic can be embedded in a logic whose Jauch-Piron states form a simplex. It seems conjecturable (see [16]) that each logic can be embedded in a logic whose Jauch-Piron states form a given compact convex set. But this may constitute a rather hard problem. As a rule, apart from Jauch-Piron states a logic usually contains the states that are not Jauch-Piron. So does, for instance, the prominent logic $L(H)$ of all projectors in a separable Hilbert space (see [5], Prop 8.2.2 and Prop. 10.1.3).

Let B be a Boolean sublogic of a logic L (thus, $B \subset L$ and B forms a Boolean algebra with respect to the operations inherited from L). We show that if s is a state on a B and if L has an abundance of Jauch-Piron states with respect to B then s allows for an extension over L as a Jauch-Piron state. By its mathematical character, this result belongs to the (non-commutative) measure theory. In the possible interpretation in theoretical physics, the result says that a state on a classical event system allows, under natural conditions, a "desirable" extension over a quantum event system.

The method of the proof is not entirely new, it makes use of an appropriate modification of the technique of the paper [12]. However, since the application of our result to the Hilbert space logic $L(H)$ is slightly surprising, we think it worthwhile publishing a detailed proof.

Let us first formulate a proposition concerning the Jauch-Piron states.

Proposition 4.1. *Let L be a lattice logic and let $\mathcal{S}_{JP}(L)$ be the set of all Jauch-Piron states on L. Then $\mathcal{S}_{JP}(L)$, when viewed as an affine and a topological subspace of $[0,1]^L$, forms a convex and compact set.*

Proof. If s_1, s_2 are Jauch-Piron states and if $s = ts_1 + (1-t)s_2$, where $t \in [0,1]$, then s is again a Jauch-Piron state. Indeed, if $s(a) = s(b) = 1$ and if $0 < t < 1$, then $s_1(a) = s_1(b) = s_2(a) = s_2(b) = 1$. As a result, $s_1(a \wedge b) = 1$ and $s_2(a \wedge b) = 1$. Thus, $s(a \wedge b) = ts_1(a \wedge b) + (1-t)s_2(a \wedge b) = 1$.

Further, suppose that s_α converges pointwisely to s, where α goes over a directed set. Then if any s_α is Jauch-Piron, we see that s is obviously Jauch-Piron, too. We infer that the set $\mathcal{S}_{\mathrm{JP}}(L)$ of all Jauch-Piron states on L is a close subspace of the topological product $[0,1]^L$ and therefore (Tychonoff's theorem) the set $\mathcal{S}_{\mathrm{JP}}(L)$ is compact. □

Theorem 4.2. *Let B be a Boolean subalgebra of a quantum logic L. Let s be a state on B. Let us suppose that for each $b \in B$, $b \neq 0$, there is a Jauch-Piron state $\tilde{s}$ on L such that $\tilde{s}(b) = 1$. Then there is a state t on L such that t is Jauch-Piron and t restricted to B coincides with s.*

Proof. Let us consider the collection $\mathcal{P}$ of all partitions of B. By a partition of B we mean a collection $P = \{p_1, \ldots, p_n\}$ such that $p_i \wedge p_j = 0$ in B provided $i \neq j$, and $\vee_{i=1}^n p_i = 1$ in B. The set $\mathcal{P}$ can be viewed as a directed partially ordered set. Indeed, $\mathcal{P}$ is naturally ordered by the refinement relation $\preceq$, where $P = \{p_i, \ldots, p_n\}$ is less or equal than $Q = \{q_i, \ldots, q_m\}$ (in symbols $P \preceq Q$) if for any p_i, $i \leq n$, there is q_j, $j \leq m$, with $p_i \leq q_j$. The ordering $\preceq$ on $\mathcal{P}$ is obviously directed—if $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_i, \ldots, q_m\}$, then the partition $R = \{p_i \wedge q_j, i \leq n,\ j \leq m\}$ is a lower bound of both P and Q.

Let us consider the given state $s \in \mathcal{S}(B)$. For each partition $P = \{p_1, \ldots, p_n\}$ of B let us set $\mathcal{S}_P = \{v \in \mathcal{S}_{\mathrm{JP}}(L),\ v(p_i) = s(p_i)\}$. Considering the topology of $\mathcal{S}_{\mathrm{JP}}(L)$ and making use of Proposition 4.1, this set $\mathcal{S}_P$ is compact. By our assumption, $\mathcal{S}_P \neq \emptyset$ for any $P \in \mathcal{P}$. Indeed, if $p_i \neq 0$ then the assumption of Theorem 4.2 guarantees that there is a Jauch-Piron state $\tilde{s}_i$ on L such that $\tilde{s}_i(p_i) = 1$. Obviously $\sum_{i=1}^n s(p_i) = 1$ and the convexity property of Jauch-Piron states gives us that $s_P = \sum_{i=1}^n s(p_i) \cdot \tilde{s}_i$ is a Jauch-Piron state on L. Thus, $s_P \in \mathcal{S}_P$ and therefore $\mathcal{S}_P \neq \emptyset$. We will end up the proof by using the compactness of the set of all Jauch-Piron states on L. Since $\mathcal{P}$ is directed and each $\mathcal{S}_P$ is non-void for each $P \in \mathcal{P}$, we conclude that $\bigcap_{P \in \mathcal{P}} S_P \neq \emptyset$. Take a state $t \in \bigcap_{P \in \mathcal{P}} S_P$. By the construction, $t(b) = s(b)$ for each $b \in B$. The proof is complete. □

The following consequence of the previous theorem seems to be unknown. To a certain extent, it has been a motivation for our study. Let us denote by $L(H)$ the quantum logic of all closed subspaces in a separable Hilbert space H. As recalled above, $L(H)$ has Jauch-Piron states and, also, it has states that are not Jauch-Piron. (It may be noted that the σ-additive setup of the question studied has a completely different character—all σ-additive states on $L(H)$ are Jauch-Piron, see e. g. [14].)

Theorem 4.3. *Let B a Boolean sublogic of $L(H)$. Let s be a state on B. Then s can be extended over $L(H)$ as a Jauch-Piron state.*

Proof. If $b \in B,\ b \neq 0$, then b can be viewed as a non-zero element of $L(H)$. Thus, b is a non-zero closed subspace of H. If we take a unit vector $u \in B$ and consider the "Gleason" standard vector state s_u on $L(H)$, $s_u(A)$ being the scalar product $\langle u, u(A)\rangle$ where $u(A)$ is the projection of u into A, then s_u is a Jauch-Piron state on $L(H)$ and the result follows from the previous theorem. □

It may be noted that Theorem 4.3 can be easily generalized from $L = L(H)$ to $L = L_1 \times L_2$, where L_1 is a Boolean algebra and L_2 is a finite product of Hilbert space logics, $L_2 = L(H_1) \times L(H_2) \times \ldots \times L(H_n)$.

Another conceptually important class of quantum logics are the set-representable quantum logics as a certain antithesis of $L(H)$ (see [4] and [13]). A set-representable quantum logic can be identified with a collection Δ of subsets of a set Ω such that 1. Δ is a lattice with respect to the ordering given by inclusion, and 2. Δ contains Ω and Δ is closed under the formation of finite disjoint unions in Ω (see e.g. [11]). These logics form a large class—they form a variety of algebras (see [2], [8], etc.). Since each Boolean algebra is set-representable (Stone's theorem), it seems conceivable to think of a result analogous to Theorem 4.3 after we replace Gleason's states of $L(H)$ with Dirac's states of set-representable logics (recall that a state s on the logic (Ω, Δ) is said to be *Dirac* if there is a point $p \in \Omega$ such that $s(A) = 1$ provided $p \in A$, otherwise $s(B) = 0,\ A, B \in \Delta$). However, the situation is quite different as the following result shows—we in fact arrive at a characterization of Boolean algebras among set-representable logics. The proof can be obtained as a consequence of results of [7], here we present a short direct proof.

Theorem 4.4. *Let L be a lattice set-representable logic. If each Dirac state on L is Jauch-Piron, then L is a Boolean algebra.*

Proof. Suppose that L is a collection, Δ, of subsets of Ω. In order to show that $L = \Delta$ is Boolean, it is sufficient to verify that $A \cap B \in \Delta$ for any couple $A, B \in \Delta$. Since $A \wedge B$ exist, we infer that the complement $(A \wedge B)'$ belongs to

Δ and we have the equality $B \wedge (A \cap (A \wedge B)') = \emptyset$ (the set $A \cap (A \wedge B)'$ belongs to Δ, too, as easily follows from the orthomodularity of Δ). But this means that $B \cap (A \cap (A \wedge B)') = \emptyset$. Indeed, if $(A \wedge B)$ is a proper subset of $(A \cap B)$, we see that there is a point p such that $p \in (A \cap B) \setminus (A \wedge B)$. As a consequence, we obtain for the Dirac state s given by p that $s(B) = 1$ and $s(A \cap (A \wedge B)') = 1$. Since s is supposed to be Jauch-Piron, we infer that $s(B \wedge (A \cap (A \wedge B)') = 1$ which is absurd. Thus, we have $B \cap (A \cap (A \wedge B)') = (B \cap A) \cap (A \wedge B)' = \emptyset$ and therefore $A \wedge B = A \cap B$. This completes the proof. □

Let us note in concluding this paper that Theorem 4.4 obviously fails in quantum logics L that are not lattices—it is even possible that all states are Jauch-Piron without L being Boolean (see [9]).

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5 Quantum logics defined by divisibility conditions

The paper enclosed will appear in International Journal of Theoretical Physics (DOI: 10.1007/s10773-018-3977-y). The contribution of Michal Hroch to the contents of the paper is 33% (agreed with the coauthor).

Quantum logics defined by divisibility conditions

Michal Hroch, Mirko Navara, and Pavel Pták[6]

Abstract. Let p be a prime number and let S be a countable set. Let us consider the collection Div_p^S of all subsets of S whose cardinalities are multiples of p and the complements of such sets. Then the collection Div_p^S constitutes a (set-representable) quantum logic (i.e., Div_p^S is an orthomodular poset). We show in this note that each state on Div_p^S can be extended over the Boolean algebra $\exp S$ of all subsets of S. We also prove that all pure states on Div_p^S are two-valued. (If we lend to a main result a possible interpretation in terms of quantum entities, the logics Div_p^S have higher degree of noncompatibility but somewhat classical states.)

Keywords: set-representable quantum logic, state, extensions of states

AMS Classifications: 06C15, 03G12, 06E99 , 28E15, 81P10

Notions and results

Let Div_p^S be the quantum logic defined in the abstract. Thus, Div_p^S is a subset of $\exp S$. Standardly, a mapping $s\colon Div_p^S \to [0,1]$ is said to be a *state* if (i) $s(S) = 1$, and (ii) if $A, B \in Div_p^S$ and $A \cap B = \emptyset$, then $s(A \cup B) = s(A) + s(B)$. If s satisfies only the condition (ii) and if s is allowed to attain any real value, then s is called a *signed measure*. We are going to show that each state s on Div_p^S can be extended over the Boolean algebra $\exp S$. (Let us note that a certain inspiration for our

[6]The first author was supported by the Grant Agency of the Czech Technical University in Prague, grant No. SGS18/131/OHK3/2T/13. The second and the third author were supported by the project OPVVV CAAS CZ.02.1.01/0.0/0.0/16_019/0000778.

consideration comes from the paper [1] where the authors pursued the logic Div_2^P for P uncountable. A related study has also been carried on in [2] and [4].)

The proof of our result makes use of the following propositions.

Proposition 5.1. *Let p be a prime number and K be a set whose cardinality is $p \cdot k$ for some $k \in \mathbb{N},\ k \geq 3$. Let us denote by Div_p^K the collection of all subsets of K whose cardinalities are multiples of p. Let s be a state on Div_p^K. Then there exists a unique signed measure t on $\exp K$ such that $s(A) = t(A)$ for each set $A \in Div_p^K$.*

The proof of Prop. 5.1 can be found in [7]. It should be noted (see also [7]) that t can actually attain negative values.

Proposition 5.2. *Let p be a prime number. Let S be a countable set and let Div_p^S be the corresponding quantum logic defined in the abstract. Form a disjoint covering of S with the sets $L_i, i \in \mathbb{N}$, such that $\operatorname{card}(L_i) = 3p$ for each $i \in \mathbb{N}$. Thus, $S = \bigcup_{i=1}^{\infty} L_i$ and $L_i \cap L_j = \emptyset$ provided $i \neq j$. Let s be a state on Div_p^S and let an ε, $\varepsilon > 0$, be given. Then there is an n_0, $n_0 \in \mathbb{N}$, such that $s(L_i) \leq \varepsilon$ for each $i \geq n_0$.*

Proof. If it were not the case, we would have infinitely many disjoint sets L_i with $s(L_i) > \varepsilon$ and this is impossible. □

Theorem 5.3. *Let p be a prime number. Let S be a countable set and let s be a state on Div_p^S. Then s can be extended to a state on the Boolean algebra $\exp S$ of all subsets of S.*

Proof. Obviously, it is only sufficient to extend s over the Boolean algebra generated by Div_p^S, the rest (i.e., the extension over the entire $\exp S$) follows from the classical result (see e.g. [3]). The Boolean algebra generated by Div_p^S is clearly the algebra of finite and cofinite subsets of S. Hence we have to extend s over the Boolean algebra of all finite-cofinite subsets of S. Let us do it by induction (we in fact find this extension unique). Let us cover S by the $3p$-element sets $L_i, i \in \mathbb{N}$, as done in Prop. 5.2. Consider L_1. If $s(L_1) = 0$ we assign 0 to all points of L_1. If $s(L_1) \neq 0$, then by Prop. 5.1 the state s restricted to subsets of L_1 is a positive multiple of a state and it can therefore be extended to a signed measure t on $\exp L_1$. We claim that t is non-negative. Suppose on the contrary that there is a point q, $q \in L_1$, such that $t(q) = u < 0$. Take a positive number

v such that $v < |u|$. Then there is an i_0, $i_0 > 1$, such that $s(L_{i_0}) < v$. Consider the set $L_1 \cup L_{i_0}$ and take into consideration Prop. 5.1 (i.e., observe that the state s restricted to the union $L_1 \cup L_{i_0}$ must be defined by a signed measure on $\exp(L_1 \cup L_{i_0})$). It is easily seen that by exchanging a suitable point of L_{i_0} with q we obtain a set M such that $s(M) < 0$. This is impossible. As a consequence, t must be a non-negative measure and therefore t presents an extension of s over $\exp L_1$. Suppose that we have obtained the extension over $\exp(\bigcup_{i=1}^{m} L_i)$ and we want to extend it over $\exp(\bigcup_{i=1}^{m+1} L_i)$ to some $\tilde{t}$. By Prop. 5.1 again, we obtain the signed measure on $\exp(\bigcup_{i=1}^{m+1} L_i)$ which has to agree with the so far defined signed measure on $\exp(\bigcup_{i=1}^{m} L_i)$. If there was a point in L_{m+1} for which the function $\tilde{t}$ is negative, we would use Prop. 5.1 in a similar manner like to the case concerning L_1 and would obtain a contradiction. As a result of our induction procedure, we can define the desired extension over $\exp(\bigcup_{i=1}^{\infty} L_i)$ and the proof is complete. □

One of the implications of Theorem 5.3 is the following result considered individually in [1] and [4].

Lemma 5.4. *Let p be a prime number. Let S be a countable set and let s be a state on Div_p^S. Let $A, B \in Div_p^S$ and let Δ denote the operation of symmetric difference for subsets of S. Let $A \Delta B \in Div_p^S$. Then $s(A \Delta B) \leq s(A) + s(B)$.*

Proof. Since s can be extended over $\exp S$, the inequality reduces to the well known inequality valid for states on Boolean algebras. □

It should be noted that analogous results are in force for S of an arbitrary cardinality—we can easily show that the state on Div_p^S must live on a countable subset of S.

A state is called *pure* if it cannot be expressed as a non-trivial convex combination of different states. (Here "non-trivial" means "with coefficients different from 0 and 1".) It is called *two-valued* if it attains only the values 0 and 1. Obviously, two-valued states are pure in any quantum logic (and vice versa for a Boolean logic). It is known (see [5] and [6]) that for a finite K, the logic Div_p^K possesses pure states that are not two-valued. We show that this is not the case for Div_p^S.

Theorem 5.5. *Let p be a prime number and S be a countable set. Let s be a state on Div_p^S. If s is a pure state then s is two-valued. A consequence: The state space on Div_p^S is the closure of the convex hull of all two-valued states.*

Proof. Suppose that s is a state on Div_p^S which is not two-valued, i.e., suppose that there exists a set $M \in Div_p^S$ such that $0 < s(M) < 1$. We may assume that M is finite (otherwise we pass to the complement). We shall show that the state s is not pure.

According to Theorem 5.3, s can be extended to a state t on $\exp S$. As $s(M) = t(M) = \sum_{m \in M} t(\{m\})$, there must be some $q \in M$ such that $c = t(\{q\})$ satisfies $0 < c \leq s(M) < 1$. Let us express t as a non-trivial convex combination,

$$t = c\,u + (1-c)\,v\,,$$

of the "Dirac" state u concentrated in q (determined uniquely by $u(\{q\}) = 1$) and the state v that is defined by the formula

$$v(A) = \frac{1}{1-c}\,t(A \setminus \{q\})$$

for all $A \in \exp S$. The states u and v defined on $\exp S$ are distinct. The state s is a non-trivial convex combination (with the coefficients $c, 1-c$) of the restrictions of u and v to Div_p^S. It remains to show that these restrictions are not identical.

As in Prop. 5.2, we can find a set Q, $q \notin Q$, with $p-1$ elements such that $v(Q) < c$. Then $T = Q \cup \{q\} \in Div_p^S$ and therefore $u(T) \geq c$ whereas $v(T) < c$. The proof is complete. The "consequence" statement follows from the Krein–Milman theorem. □

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Participation of all authors is equivalent and authors are ordered alphabetically.