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**Faculty of Electrical Engineering
Department of Cybernetics**

Bachelor's Thesis

Graph Theory and Quantum Structures

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BACHELOR'S THESIS ASSIGNMENT

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II. Bachelor's thesis details

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Graph Theory and Quantum Structures

Bachelor's thesis title in Czech:

Teorie grafů a kvantové struktury

Guidelines:

Summarize recent results of graph theory related to the bounds of number of edges of hypergraphs with given girth [3]. Apply them to questions from the event structures appearing in quantum theory. In particular, try to improve results concerning the existence of quantum structures with "small" state spaces which are "small" in terms of number of atoms, elements, or blocks [1, 2, 4].

Bibliography / sources:

- [1] Navara, M.: An orthomodular lattice admitting no group-valued measure. Proc. Amer. Math. Soc. 122 (1994), 7–12.
- [2] Weber, H.: There are orthomodular lattices without non-trivial group valued states; a computer-based construction. J. Math. Anal. Appl. 183, 89–94 (1994).
- [3] Verstraëte, J.: Extremal problems for cycles in graphs. In: A. Beveridge et al. (eds.), Recent Trends in Combinatorics, The IMA Volumes in Mathematics and its Applications 159, DOI 10.1007/978-3-319-24298-9_4.
- [4] Navara, M.: Small quantum structures with small state spaces. Internat. J. Theoret. Phys. 47 (2008), No. 1, 36–43.

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Acknowledgement / Declaration

I would like to express my sincere gratitude to my advisor, prof. Ing. Mirko Navara, DrSc., for allowing me to do mathematics, and for his patient guidance throughout the work.

I would also like to thank my family, friends, and Hanka for their support.

I declare that the presented work was developed independently and that I have listed all sources of information used within it in accordance with the methodical instructions for observing the ethical principles in the preparation of university theses.

Prague, 23. 5. 2019

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Abstrakt / Abstract

S příchodem kvantové teorie klasická logika přestala být dostatečnou pro popis fyzikálních událostí, protože distributivní zákon výrokové logiky vede ke sporům v kvantové teorii. To vedlo k vytvoření kvantových logik. Ortomodulární svazy jsou algebraické struktury, které splňují vlastnosti kvantové logiky.

V této práci studujeme ortomodulární svazy, a také obecnější kvantové struktury bez netriviálních měr z libovolné komutativní aditivní grupy. Ukážeme, jak kvantové struktury souvisí s hypergrafy, a použijeme grafově teoretické metody pro vytvoření omezení velikostí studovaných kvantových struktur.

Prezentujeme nové výsledky jak z teorie grafů, kde jsme našli horní odhad počtu hran při daném počtu vrcholů pro určitý typ hypergrafů, tak z teorie kvantových struktur, kde jsme značně zjednodušili příklady a argumenty pro existenci kvantových struktur bez měr z libovolné komutativní aditivní grupy.

Klíčová slova: kvantové struktury, ortomodulární svaz, ortomodulární poset, ortoalgebra, hypergraf.

Překlad titulu: Teorie grafů a kvantové struktury

With arise of the quantum theory, the classical logic was no longer sufficient for describing physical events because the distributive law of propositional logic leads to contradictions in quantum mechanics. This led to the introduction of quantum logics. An algebraic structure capturing properties of quantum logic operations is called orthomodular lattice.

In the thesis, we study orthomodular lattices and also more general quantum structures without non-trivial measures from any additive commutative group. We show how quantum structures are related to hypergraphs and we use graph theory for deriving constraints on the size of such quantum structures.

We present our new results in both, the graph-theory, and the theory of quantum structures. In graph theory we derived a new upper bound on the number of edges for a given number of vertices for a certain type of hypergraphs. In the theory of quantum structures, we significantly simplified the arguments for the existence of quantum structures with no non-trivial group-valued measures. We also constructed small such quantum structures.

Keywords: quantum structure; orthomodular lattice; orthomodular poset; orthoalgebra; hypergraph.

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Chapter 1

Introduction

1.1 Motivation

In 1932 John von Neumann in [1] noted that projections on a Hilbert space describe well propositions about observables in quantum physics. Later, in 1936, Garrett Birkhoff and John von Neumann in [2] expressed the flaws of classical logic when applied to quantum mechanics, and proposed properties of quantum logic. Those flaws were mainly caused by the Heisenberg's Uncertainty Principle, i.e., the more accurately is known the position of a particle, the less known is its momentum, and by the Principle of Non-commutativity of Observations, i.e., most pairs of observations cannot be measured simultaneously.

1.1.1 Heisenberg's Uncertainty Principle

Heisenberg's Uncertainty Principle states the relation between uncertainty in the momentum and the position of a particle. Loosely speaking, it can be formulated as follows:

$$\Delta x \Delta p \geq \hbar/2,$$

where Δx and Δp are uncertainty in position and momentum respectively, and \hbar is the reduced Planck constant. The precise formulation involves the standard deviation of momentum and position.

We present a simple example of why this principle violates the distributive law of propositional logic. The example is not exact, but we hope it gives the right intuition behind the principle. As additional background mathematics and physics is needed for the correct understanding of the phenomenon, we refer the reader to [2-3]. The latter source is a review paper.

Example 1.1. For simplicity, we use a system of units where the reduced Planck constant is 6. Let us introduce logical variables for the following propositions in some units:

- A = the particle's position is in the interval $[0, 1]$.
- B = the particle's momentum is in the interval $[-2, 0]$.
- C = the particle's momentum is in the interval $[0, 2]$.

Let the uncertainty of position or momentum be the length of the interval from which we know the respective property takes value. For example, the uncertainty of the momentum in proposition B and C is 2. The uncertainty of the position in proposition A is 1.

Let us also define logical operations between such propositions.

- $B \vee C$ = the particle's momentum is in the interval $[-2, 2]$.

1.2 Related Work

With the arise of the theory of quantum structures, there developed constructions of such structures admitting few or no states. The first example of an orthomodular lattice without states was made by R. Greechie in 1971 in [10].

This example of a stateless orthomodular lattice directly led to a proof that the state space of an orthomodular lattice can be an arbitrary compact convex set in [11].

R. Mayet simplified this example of a stateless-orthomodular lattice, which can be found in [12]. In [13], there are arguments why this construction could be the smallest possible (smallest in the sense of the number of atoms). So far, the example is still the smallest known.

For the next 20 years, there was an open question of whether there is an orthomodular lattice without non-trivial group-valued measures. This was positively answered independently by M. Navara in [14] and H. Weber in [15]. Since then, there is no result directly addressing constructions of quantum structures with no non-trivial group-valued measures.

1.3 Goals of Thesis

Here we shall describe our goals of the thesis.

- Do a survey in extremal graph theory. Find bounds on the number of edges, given a number of vertices for hypergraphs representing quantum structures.
- Modify these bounds to create lower bound on the number of atoms of quantum structures which do not admit any non-trivial group-valued measure.
- Construct small quantum structures admitting no non-trivial group-valued measure. This serves as an upper bound on the minimal number of atoms of such quantum structures.
- Construct a quantum structure with no non-trivial group-valued measures such that the argument for the non-existence of the measures is simple.

Chapter 2

Hypergraphs

Hypergraphs are a generalization of graphs, but unlike in graphs, here an edge may contain an arbitrary number of vertices, and not just 2. Hypergraphs are commonly used for visualizing quantum structures, as hypergraphs capture their structure well.

Here we define all the needed terms and state needed lemmas, so that in the following chapters, we can directly prove theorems.

2.1 Definitions

Definition 2.1. A *hypergraph* is a pair (V, E) , where V is a finite set of vertices and E is a set of edges. Every edge $e \in E$ is a subset of V . The reader is referred to [16–17] for more details.

This definition was widely used by Erdős and Hungarian mathematicians in general. There are also different formal definitions of hypergraphs, as in [16, 18], a hypergraph H is a subset of the powerset of some finite set. The members of H are called edges. The union of all edges is called the set of vertices. This definition is perhaps used more often.

These definitions are not equivalent, as the latter does not admit isolated vertices, i.e., vertices which do not belong to any edge of the hypergraphs. Although we are not interested in isolated vertices. We stick to the former definition, as with the latter, the existence of some hypergraphs has to be assumed.

Definition 2.2.

- A hypergraph (V, E) is *r-uniform*, or also an *r-graph*, if every edge $e \in E$ contains exactly r vertices.
- A *graph* is a 2-uniform hypergraph.
- Hypergraph (V, E) is *simple* if there are no two distinct edges $e_1, e_2 \in E$ such that $e_1 \subseteq e_2$.
- Let (V, E) be a hypergraph and $v \in V$ be its vertex. The number of edges $e \in E$ containing v is the *degree* of v , written $d(v)$.
- Let (V, E) be a hypergraph. Vertices $u, v \in V$ are called *neighbors* if there is an edge $e \in E$ such that $u, v \in e$.

Hypergraphs are used for the representation of quantum structures. Such hypergraphs have to fulfill some conditions in order to represent a certain quantum structure. An important property of a hypergraph for these purposes is the length of its shortest cycle.

Definition 2.3. Let (V, E) be a hypergraph. An alternating sequence of vertices and edges: $v_1, e_1, v_2, e_2, \dots, v_n, e_n$ is called a *cycle*, if the following conditions are fulfilled:

- $n \geq 2$.
- Vertices $v_1, v_2, \dots, v_n \in V$ are pairwise distinct.
- Edges $e_1, e_2, \dots, e_n \in E$ are pairwise distinct.
- $v_i, v_{i+1} \in e_i$ for every $i \in \{1, 2, \dots, n-1\}$ and $v_n, v_1 \in e_n$.

The *length* of a cycle is the number of vertices it contains.

There are also other definitions of cycles. For more details see [19].

Definition 2.4. The *girth* of a hypergraph is the length of its shortest cycle.

We will draw hypergraphs as dots connected by smooth curves. Dots represent vertices and maximal smooth curves are edges. See Figure 2.1 for illustration. Every edge of this hypergraph contains three vertices, the hypergraph is 3-uniform. Its girth is 5.

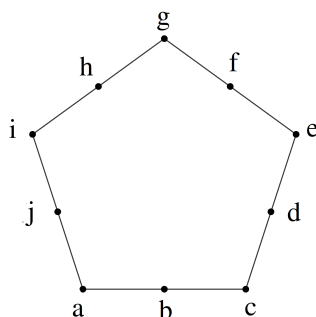


Figure 2.1. A hypergraph
 $(\{a, b, \dots, j\}, \{\{a, b, c\}, \{c, d, e\}, \{e, f, g\}, \{g, h, i\}, \{i, j, a\}\})$.

As we study measures on quantum structures, we will mainly focus on those, for which a specific hypergraph representation exists. We will define it in a later chapter. It turns out that the measures on quantum structures are in some sense equivalent to measures on hypergraphs. For now, we define measures on hypergraphs and show all the connections later. As we are interested in group-valued measures, we define it using groups.

Definition 2.5. Let $H = (V, E)$ be a hypergraph and $\mathbf{1} \notin V$ be a special element. A *group-valued measure* (abbr. *measure*) on H is a mapping $m : (V \cup \{\mathbf{1}\}) \rightarrow X$, where $(X, +)$ is a commutative, additive group, see Definition 3.1, such that every edge $e \in E$ satisfies the following condition:

- $\sum_{v \in e} m(v) = m(\mathbf{1})$.

We now introduce notation, which we will use later.

- If $(X, +) = (\mathbb{R}, +)$, i.e., X is the set of real numbers and $+$ is the standard addition, we call that measure a *real-valued measure*.
- If m is a real-valued measure, $m(\mathbf{1}) = 1$, and for every vertex $v \in V : m(v) \geq 0$, we call such a measure a *probability measure*.
- m is called a *constant measure* if for all vertices $u, v \in V$, $m(u) = m(v)$.
- m is called a *trivial measure* if for all vertices $v \in V$, $m(v) = 0$.

Example 2.1. Two examples of measures on a hypergraph from Figure 2.1. Both of them are real-valued. $m_1(\mathbf{1}) = 2$ and $m_2(\mathbf{1}) = 1$.

Chapter 3

Groups

The motivation for the introduction of groups is that we are interested in quantum structures with no non-trivial group-valued measures. With this in mind, for our purpose, it is sufficient to state basic definitions and theorems about solving systems of linear equations over a group.

3.1 Definitions

Definition 3.1. A *group* is a pair (G, \circ) , where G is a non-empty set and a group operation $\circ : G^2 \rightarrow G$ satisfying the following for all elements $a, b, c \in G$:

- $a \circ (b \circ c) = (a \circ b) \circ c$ (associativity)
- There is an element $e \in G$ such that for every element $a \in G$: $a \circ e = e \circ a = a$. This element is called an *identity element*.
- For every element $a \in G$ there is an element $b \in G$ such that $a \circ b = b \circ a = e$. b is called an *inverse element* of a .

Furthermore, if the operation \circ is commutative, we call such group a *commutative group* or also an *Abelian group*.

- For all $a, b \in G$: $a \circ b = b \circ a$. (commutativity)

It is an easy consequence that in every group there is a unique identity element and every element of a group has the unique inverse.

If the group operation should be considered as an addition in some sense, the group is called an *additive group*. The group operation is often denoted as $+$, the identity element is 0 and the inverse of an element x is $-x$. A summation of an element with itself will be abbreviated to multiplying by an integer. For example, we write $3a$ instead of $a + a + a$ and $-2a$ instead of $-a - a$.

Let (G, \circ) be a group, we call underlying set a group. I.e., we call G a group. This is a common practice for algebraic structures built upon a set. We will use it for all later defined algebraic structures.

Example 3.1. Here we bring some examples of groups.

- A pair $(\{0\}, \circ)$, where $0 \circ 0 = 0$, is a group. As all group properties are trivially satisfied, the group is called trivial.
- A pair $(\mathbb{R}, +)$, where \mathbb{R} is the set of real numbers and $+$ is the standard addition. Associativity is fulfilled, as addition is associative. The identity element is 0 and the inverse element of an element $x \in \mathbb{R}$ is $-x$, so this is also a group.
- A pair (\mathbb{R}_+, \cdot) , where \mathbb{R}_+ is the set of positive real numbers and \cdot is the standard multiplication, which is associative. The identity element is 1 and the inverse element of an element $x \in \mathbb{R}_+$ is $1/x$, hence it is a group.

Proof. Let M be the matrix of H . It has more columns than rows, hence its null space is non-trivial. Therefore there exists a non-trivial solution to the system of linear equations $Mx = \mathbf{0}$, where x is real-valued. \square

Theorem 3.3. Let M be the matrix of a hypergraph. If the system of linear equations $Mx = \mathbf{0}$ has a non-trivial solution in a group,¹ then it has a non-trivial solution in \mathbb{Z}_p for some prime p .

Proof. Can be found in [14–15]. \square

Corollary 3.4. Let M be the matrix of a hypergraph H . If M can be transformed using a division-less Gauss-elimination to the identity matrix, then H does not admit any non-trivial group-valued measure.

Theorem 3.5. Let M be the matrix of a hypergraph H , assuming that M is a square matrix. Then H does not admit any non-trivial group-valued measures, iff $\det(M) = \pm 1$.

Proof. If $\det(M) = \pm 1$, then M has the inverse. Elements of M^{-1} are quotients of minors of M , which are integers, and $\det(M)$, which is ± 1 . Hence M^{-1} is a matrix with integer entries. Multiplying a matrix by its inverse can be seen as applying the steps of Gaussian elimination needed to get an identity matrix. Since M^{-1} consists only of integers, the elimination was division-less.

Let the determinant be 0. Then the matrix has non-trivial null space and the system has a non-trivial real-valued solution.

Let the determinant be different from 0, ± 1 . Then it is divisible by a prime p and the matrix of the system has not the full rank when considered as a mapping of a vector space over the group \mathbb{Z}_p . Hence there is a non-trivial solution to the system of equations. \square

¹ Which means that x is a vector consisting of elements of a group and not all of them are 0.

Chapter 4

Orthoalgebras

Orthoalgebras are the first quantum structures we will address.

Firstly, in Section 4.1 we define OAs, measures on them and prove some of their important properties so that we can represent OAs as hypergraphs. Definitions here are based on those in [7–8]. In Section 4.2 we review the results of extremal graph theory related to orthoalgebras and finally we adapt those results to constructions of OAs with no probability, or group-valued, measures.

4.1 Definitions

Definition 4.1. An *orthoalgebra* is a quadruple $(L, \oplus, \mathbf{0}, \mathbf{1})$, where L is a set, $\mathbf{0}$ and $\mathbf{1}$ are its distinct elements, and a partial¹ operation $\oplus : L^2 \rightarrow L$, which satisfies the following properties for all $a, b, c \in L$:

- If $a \oplus b$ is defined, then so is $b \oplus a$ and $a \oplus b = b \oplus a$. (commutativity law)
- If $b \oplus c$ and $a \oplus (b \oplus c)$ are defined, then so are $a \oplus b$ and $(a \oplus b) \oplus c$ and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$. (associativity law)
- For every element $a \in L$, there exists a unique element $b \in L$ such that $a \oplus b$ is defined and $a \oplus b = \mathbf{1}$. (orthocomplementation law)
- If $a \oplus a$ is defined, then $a = \mathbf{0}$.

Let us assume L is finite, as we deal only with finite structures.

If $a \oplus b = \mathbf{1}$, then b is the orthocomplement of a , denoted by $a' = b$, clearly $b' = a$.

The associativity law allows us not to write brackets in expressions, as every valid bracketing is equivalent. We use them only when it clarifies the expression.

Let us note some properties of OAs.

4.1.1 Basic Properties

Lemma 4.1. Let L be an OA, then for all elements $a, b \in L$ the following holds.

- (i) $a'' = a$.
- (ii) If $a \oplus \mathbf{1}$ is defined, then $a = \mathbf{0}$.
- (iii) $\mathbf{1}' = \mathbf{0}$.
- (iv) $a \oplus \mathbf{0}$ is defined and $a \oplus \mathbf{0} = a$.
- (v) If $a \oplus b = \mathbf{0}$, then $a = b = \mathbf{0}$.
- (vi) If $a \oplus b = a$, then $b = \mathbf{0}$.

¹ Partial operation means that it is not necessarily defined for the whole domain.

Proof.

- (i) The unique orthocomplement of a' is both, a and a'' , thus $a'' = a$.
- (ii) Let $a \oplus \mathbf{1}$ be defined, then

$$\mathbf{1} = (a \oplus \mathbf{1}) \oplus (a \oplus \mathbf{1})',$$

$$\mathbf{1} = a \oplus (a \oplus \mathbf{1} \oplus (a \oplus \mathbf{1})') \oplus (a \oplus \mathbf{1})',$$

hence $a \oplus a$ is defined and $a = \mathbf{0}$.

- (iii) An easy corollary of (ii).
- (iv) $\mathbf{1} = \mathbf{1} \oplus \mathbf{1}' = (a \oplus a') \oplus \mathbf{0} = a \oplus (a' \oplus \mathbf{0})$, hence $a' = a' \oplus \mathbf{0}$.
- (v) Let $a \oplus b = \mathbf{0}$, then $a \oplus b \oplus \mathbf{1}$ is defined. According to (ii), $\mathbf{0} \oplus \mathbf{1}$ is defined. Hence $a \oplus \mathbf{1}$ and $b \oplus \mathbf{1}$ are defined, then $a = b = \mathbf{0}$.
- (vi) $\mathbf{1} = a' \oplus a = a' \oplus a \oplus b = \mathbf{1} \oplus b$, hence $b = \mathbf{0}$.

□

4.1.2 Measures on Orthoalgebras

Let us recall that a *group* is an abbreviation for an additive commutative group.

Definition 4.2. Let L be an OA. A *group-valued measure* (abbr. *measure*) on L is a mapping $m : L \rightarrow X$, where $(X, +)$ is a group, if the following holds:

- For all elements $a, b \in L$: If $a \oplus b$ is defined, then $m(a \oplus b) = m(a) + m(b)$.

If $(X, +)$ is $(\mathbb{R}, +)$, for every element $e \in L$: $m(e) \geq 0$, and $m(\mathbf{1}) = 1$, then m is a probability measure and is called a *state on OA*. If there is no measure which is a probability measure, then the OA is called *stateless*.

If for all elements $e \in L$: $m(e)$ equals the identity element of X , the measure is called *trivial*.

Corollary 4.2. Let L be an OA and m be a measure on it. Then $m(\mathbf{0}) = 0$.

4.1.3 Atoms and Blocks

Definition 4.3. Let L be an OA. An element $e \in L, e \neq \mathbf{0}$, is called an *atom* if there are no two elements $a, b \in L, a, b \neq \mathbf{0}$ such that $e = a \oplus b$.

Theorem 4.3. Let L be a finite OA and m be a measure on L . Then m is determined by its values on atoms.

Proof. If $L = \{\mathbf{0}, \mathbf{1}\}$, then the only atom is $\mathbf{1}$ and the theorem holds.

Otherwise, we proceed by contradiction. Assume to the contrary that there is an element $e_1 \in L$ such that $m(e_1)$ is undetermined, given values of measure on atoms. We call such elements as *undetermined* elements. An undetermined element is not an atom. Hence, according to the definition of an atom, it can be written as $e_1 = e_2 \oplus f_2$ for some $e_2, f_2 \in L, e_2, f_2 \neq \mathbf{0}$. At least one of these elements, WLoG e_2 , is undetermined, otherwise, $m(e_1) = m(e_2) + m(f_2)$ would be known. By repeated application of this idea, we can expand these elements in the following way:

$$e_1 = e_2 \oplus f_2,$$

$$e_2 = e_3 \oplus f_3,$$

If a hypergraph $H = (V, E)$ represents an OA, then H is simple. There are also no two edges $e_1, e_2 \in E$ such that $|e_1| = 1 + |e_1 \cap e_2|$. For more details see the following theorem.

Theorem 4.8. Let L be an OA and b_1, b_2 be its blocks. Then $|b_1| \neq 1 + |b_1 \cap b_2|$. *Proof.* We will demonstrate this on an example.

We proceed by contradiction. Assume to the contrary that there is an OA with blocks $\{a, b, e\}$ and $\{a, b, c, d\}$. See the hypergraph in Figure 4.1 for illustration. Then $e = (a \oplus b)' = (c \oplus d)$, hence e is not an atom of L . A contradiction.

There is also a simpler case where the blocks have the same cardinality. The argument here is different. In short, for the contrary assume different blocks $\{a, b, c\}$ and $\{a, b, d\}$, then $c = (a \oplus b)' = d$, hence the blocks are not different.

The same arguments apply when the cardinalities of blocks are larger. □

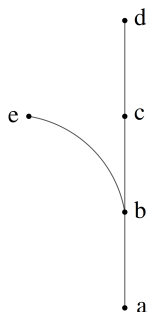


Figure 4.1. A hypergraph $(\{a, b, c, d, e\}, \{\{a, b, c, d\}, \{a, b, e\}\})$, which does not represent any OA.

See Figure 4.2 for an example of a hypergraph representing an OA.

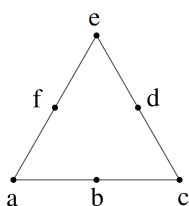


Figure 4.2. A hypergraph $(\{a, b, c, d, e, f\}, \{\{a, b, c\}, \{c, d, e\}, \{e, f, a\}\})$, which represents an OA. The OA is called the *Wright triangle*.

These conditions are necessary but not sufficient. As far as we know, the set of conditions, which is both, necessary and sufficient, is not known yet. In the next section, we introduce conditions on hypergraphs, which are sufficient for the existence of an OA represented by such hypergraphs.

We show a hypergraph which does not violate our conditions, but does not represent any OA.

Example 4.1. There is no OA represented by a hypergraph in Figure 4.3.

Theorem 4.9. Let $H = (V, E)$ be a Greechie diagram. Let $n = |V|$ and $m = |E|$, then

$$m \leq \frac{n(n-1)}{6}.$$

Proof. Let E_2 be a set of neighbor pairs from H , i.e., $\{u, v\} \in E_2$ iff u and v are neighbors in H . Let $G = (V, E_2)$. Every edge $e \in E$ adds at least 3 edges into G . As G has at most $\frac{n(n-1)}{2}$ edges, H has at most $\frac{n(n-1)}{2 \cdot 3} = \frac{n(n-1)}{6}$ edges. \square

We will use such hypergraphs for which the inequality turns into equality later. See Figures 4.6 and 4.7 for illustration.

Corollary 4.10. If equality holds, then every two distinct vertices are neighbors and H is a 3-graph.

If equality is attained, the hypergraph becomes equivalent to a *Steiner triple system*. Steiner triple systems were extensively studied in [20–22]. A Steiner triple system on n vertices is abbreviated to $STS(n)$. An $STS(n)$ does not need to be unique up to isomorphism.

Theorem 4.11. The equality in 4.9 can be attained iff $n \equiv 1 \pmod{6}$ or $n \equiv 3 \pmod{6}$ and H is a 3-graph.

Proof. We show that the equality cannot hold when $n \not\equiv 1 \pmod{6}$ and $n \not\equiv 3 \pmod{6}$. Then we show construction of such a hypergraph for one of the remaining cases. The construction is similar to that in [22].

Let us assume that H has $\frac{n(n-1)}{6}$ edges.

- If $n \equiv 2 \pmod{6}$ or $n \equiv 5 \pmod{6}$, then the assumed number of edges is not an integer, hence $n \not\equiv 2 \pmod{6}$ and $n \not\equiv 5 \pmod{6}$.
- Take an arbitrary vertex $v \in V$. As every edge containing v contains two neighbors of v , there is an even number of neighbors of v . Since v is a neighbor of all vertices, there is an odd number of vertices, hence $n \not\equiv 0 \pmod{6}$ and $n \not\equiv 4 \pmod{6}$.

Hence if $n \not\equiv 1 \pmod{6}$ and $n \not\equiv 3 \pmod{6}$, the equality cannot hold. Let us show a construction where $n = 6k + 3$, for an integer k .

- Let $a_0, a_1, \dots, a_{6k+2} \in V$ be vertices of H . Then we can arrange them in a table as follows:

a_0	a_1	\dots	a_{2k}
a_{2k+1}	a_{2k+2}	\dots	a_{4k+1}
a_{4k+2}	a_{4k+3}	\dots	a_{6k+2}

Table 4.1. Arrangement of vertices of $STS(6k + 3)$

For every column, there is an edge containing all its vertices. For example, $\{a_0, a_{2k+1}, a_{4k+2}\} \in E$. There is also an edge containing two vertices from one row and one vertex from the following one. The following row for the last one is the first row. Let a_x, a_y be vertices from one row and a_z from the following one, then $\{a_x, a_y, a_z\}$ is an edge iff

$$x + y \equiv 2z \pmod{2k + 1}.$$

This edge cannot contain two vertices from one column. Assume on the contrary that two vertices, WLoG x and z , are in the same column, i.e., $x = z$

It is an easy adjustment of the proof if we consider an edge with 4 vertices, which will have six 2-element subsets. It leads to the following:

$$m \leq \frac{n(n-1)}{6} - 1.$$

Then

$$n + 1 \leq m \leq \frac{n(n-1)}{6} - 1,$$

$$n \geq 9.$$

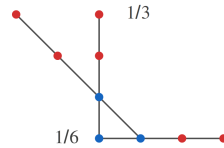
Clearly, if $|e| \geq 5$, then the bound on n is higher. □

Theorem 4.14. Let $H = (V, E)$ be a Greechie diagram, then there is a probability measure on H if $|V| \leq 9$.

Proof. It is sufficient to show that there is no such hypergraph with no probability measure and 9 vertices, so we restrict ourselves to this case. We will do so by constructing a probability measure for all hypergraphs satisfying assumptions.

A necessary condition for H to have no probability measure is to have at least one edge with 4 or more vertices. We assume this restriction and break the proof into numerous cases. In the proof, we use figures capturing the probability measures. Vertices of the same color have the same measure in a figure.

- There are three edges $e, f, g \in E$ with 4 or more vertices. Then there is the following probability measure:



If there are other edges in the hypergraph, then they contain three red vertices, $1/3 + 1/3 + 1/3 = 1$.

- There are precisely two edges $e, f \in E$ with 4 or more vertices.
 - If $e \cap f = \emptyset$, then there is at most one vertex v not contained in these edges. Hence if there is any edge in E distinct from e and f , it contains precisely one vertex from e , one vertex from f , and v . A measure which assigns $1/|e|$ and $1/|f|$ to all vertices in e and f respectively and possibly $1 - 1/|e| - 1/|f|$ to v , if v exists, is a probability measure.
 - Otherwise, $e \cap f = \{v\}, v \in V$.
 - If $|e| + |f| \geq 9$, then there is at most one vertex $u \in V$ not contained in e and f , hence there is no edge beside e, f containing v . Therefore if there is another edge in E , then it consists of u , an element of $e \setminus \{v\}$, and an element of $f \setminus \{v\}$. Then there is an analogous probability measure to the previous case, which assigns 0 to v .
 - Otherwise, $|e| = |f| = 4$. There are two vertices $u, w \in V$ not contained in e and f . If there is no edge containing v , we can say the measure of v is 0 and the measure of all other elements is $1/3$. Otherwise, there is an edge $\{u, v, w\} \in E$. Then there is the following probability measure:

4.2.3 Stateless Orthoalgebras

There exists a Greechie diagram with 10 vertices for a stateless OA. The first example of a stateless OA with 10 atoms is in [7]. The Greechie diagram for their example is in Figure 4.4.

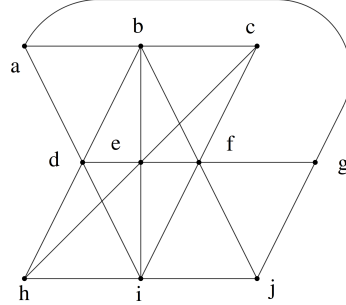


Figure 4.4. Greechie diagram for a stateless OA.

Proposition 4.15. The orthoalgebra represented in Figure 4.4 admits no probability measure.

Proof. We proceed by contradiction. Assume to the contrary that there is a probability measure m .

$$\text{Let } S = \sum_{v \in V} m(v).$$

■ There are the following families of blocks:

- (i) $\{a, b, c\}, \{d, e, f, g\}, \{h, i, j\}$, hence $S = 3$.
- (ii) $\{a, g, j\}, \{h, d, b\}, \{i, f, c\}$, hence $S - m(e) = 3$ and $m(e) = 0$.
- (iii) $\{a, d, i\}, \{b, f, j\}, \{h, e, c\}$, hence $S - m(g) = 3$ and $m(g) = 0$.

■ Now take the following blocks:

- (i) $\{b, e, i\}, \{h, e, c\}, \{a, g, j\}$, hence $S + m(e) - m(f) - m(d) = 3$ and $m(f) = m(d) = 0$.

Therefore $m(d) + m(e) + m(f) + m(g) = 0$, but it should equal 1.

□

Proposition 4.16. The orthoalgebra represented in Figure 4.4 admits a non-trivial group-valued measure.

Proof. There is a \mathbb{Z}_2 -valued measure m . The sum of measures of vertices over an edge is 0.

$$m(a) = m(c) = m(d) = m(f) = m(h) = m(j) = 1,$$

and

$$m(b) = m(e) = m(i) = m(g) = 0.$$

□

4.2.4 Orthoalgebras With No Group-Valued Measures

If the orthoalgebra has no group-valued measures and is representable by a Greechie diagram, it cannot be arbitrarily small. In the previous sections, we showed that it has at least 10 atoms. We found an example with 10 atoms, see Figure 4.5 and already published it in [23].

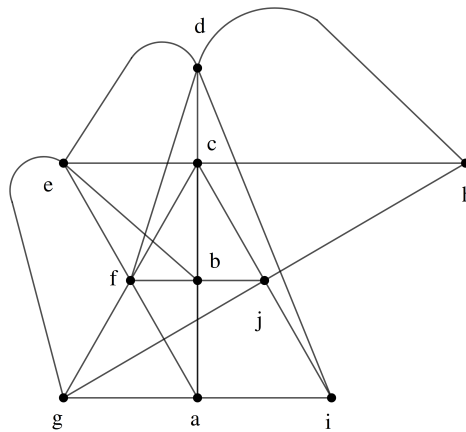


Figure 4.5. Greechie diagram for an OA admitting no group valued measures.

Proposition 4.17. The orthoalgebra represented in Figure 4.5 does not admit any non-trivial group-valued measures.

Proof. The OA is represented by a hypergraph H . The matrix M of H is captured in Table 4.2.

a	b	c	d	e	f	g	h	i	j	U
1	1	1	1	0	0	0	0	0	0	-1
1	0	0	0	1	1	0	0	0	0	-1
1	0	0	0	0	0	1	0	1	0	-1
0	1	0	0	1	0	1	0	0	0	-1
0	1	0	0	0	1	0	0	0	1	-1
0	0	1	0	1	0	0	1	0	0	-1
0	0	1	0	0	1	1	0	0	0	-1
0	0	1	0	0	0	0	0	1	1	-1
0	0	0	1	1	0	0	0	1	0	-1
0	0	0	1	0	1	0	1	0	0	-1
0	0	0	0	0	0	1	1	0	1	-1

Table 4.2. Matrix of a hypergraph representing OA with no group-valued measures.

The determinant of M is 1, and according to Theorem 3.5, the only group-valued measure is trivial. □

We shall also provide an example of an OA with no group-valued measures such that the argument is simple and does not require advanced results from the group theory.

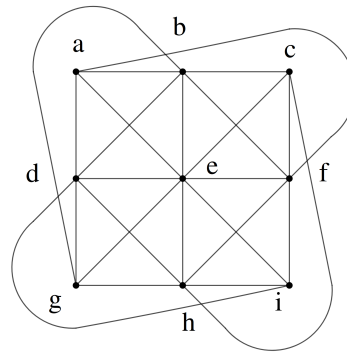


Figure 4.6. Greechie diagram for an OA. The hypergraph is an STS(9).

Lemma 4.18. If the OA captured in Figure 4.6 admits a non-constant group-valued measure m , then the domain of m is \mathbb{Z}_3 .

Proof. Take arbitrary two vertices, e.g., b, i , and sum up measures over all vertices incident with them.

$$\text{Let } S = \sum_{v \in V} m(v).$$

- $[m(a) + m(b) + m(c)] + [m(b) + m(e) + m(h)] + [m(b) + m(d) + m(i)] + [m(b) + m(f) + m(g)] = S + 3m(b)$
- $[m(c) + m(f) + m(i)] + [m(g) + m(h) + m(i)] + [m(a) + m(e) + m(i)] + [m(b) + m(d) + m(i)] = S + 3m(i)$

Hence $S + 3m(b) = S + 3m(i)$ and $3(m(b) - m(i)) = 0$, so either $3 = 0$ or $m(b) = m(i)$. The other cases are solved similarly. \square

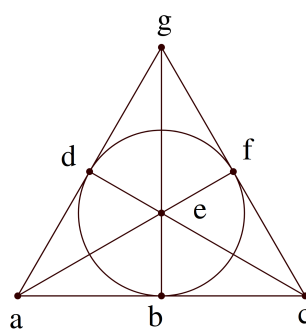


Figure 4.7. Greechie diagram for an OA. The hypergraph is an STS(7) and is known as a Fano plane.

Lemma 4.19. If the OA captured in Figure 4.7 admits a non-constant group-valued measure m , then the domain of m is \mathbb{Z}_2 .

The proof is analogical to the previous one and is therefore omitted.

Chapter 5

Orthomodular Posets

Now we move to the more concrete algebra capturing quantum logic. Here the definition is more involved. We show orthomodular posets as a special family of OAs. Hence we will not need to define everything again. For more details about OMPs see [6]. Small OMPs were also studied in [24–25].

5.1 Definitions

As the definition of an OMP is quite complicated, we split it into several parts. Firstly we define what a partially ordered set (abbr. *poset*) is, then we move to the definition of an orthocomplemented poset, and finally, we define an orthomodular poset.

Definition 5.1. A *partially ordered set* is a pair (P, \leq) , where P is a set and \leq is a binary relation over P , which is reflexive, antisymmetric, and transitive. That is, for all elements $a, b, c \in P$ the following holds:

- $a \leq a$. (reflexivity)
- If $a \leq b$ and $b \leq a$, then $a = b$. (antisymmetry)
- If $a \leq b$ and $b \leq c$, then $a \leq c$. (transitivity)

Let $a \vee b$ be the supremum, i.e., the least element which is greater than or equal to both, a and b . Similarly, let $a \wedge b$ be the infimum, the greatest element among the elements which are less than or equal to both, a and b . The supremum and infimum do not need to exist, and if they do, they are unique.

Example 5.1. Here we bring some examples of posets.

- A pair (\mathbb{R}, \leq) , where \mathbb{R} is the set of real numbers and \leq is the standard less-than-or-equal relation, is a poset.
- A pair (P, \leq) , where P is an arbitrary subset of positive integers and $a \leq b$ iff a divides b , is a poset. See an example in Figure 5.1.

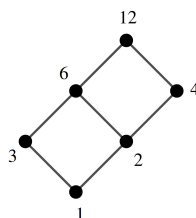


Figure 5.1. Hasse diagram of a poset. There is a non-decreasing path from an element a to an element b iff $a \leq b$. A trivial path is also a non-decreasing one.

Definition 5.2. An *orthocomplemented poset* is a poset (P, \leq) with the greatest element $\mathbf{1}$ and least element $\mathbf{0}$, together with a unary operation $' : P \rightarrow P$, called orthocomplementation, if it satisfies the following for all $a, b \in L$.

Definition 5.5. Let $H = (V, E)$ be a hypergraph, and the following conditions hold:

- For all edges $e \in E$, $|e| \geq 3$ (every edge contains at least 3 vertices).
- The girth of H is at least 4.

Then H is called a *Greechie diagram for OMP*. In this section we write just a Greechie diagram. It represents an OMP P in the sense that V represents the set of atoms of P , and E represents the set of blocks of P .

Again, we will use Greechie diagrams for representing OMPs when possible. For an example of a Greechie diagram of an OMP, see Figure 5.2.

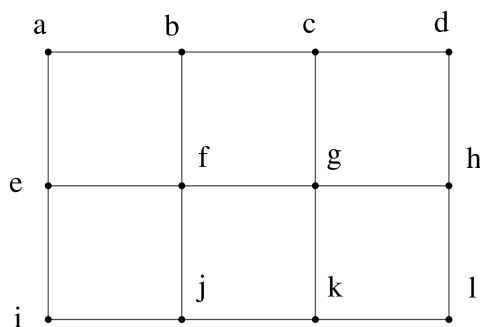


Figure 5.2. Greechie diagram for a stateless OMP.

Proposition 5.4. The OMP represented in Figure 5.2 does not admit any probability measure. The example was given in [10].

Proof. We proceed by contradiction. Assume to the contrary that there is a probability measure. Then if we sum up measures of atoms over the horizontal edges, we get that sum of measures of all vertices is 3. On the other hand, if we do the summation over the vertical edges, we get that the sum of all vertices is 4. A contradiction. \square

Proposition 5.5. The OMP represented in Figure 5.2 admits a non-trivial group-valued measure.

Proof. There is a \mathbb{Z}_3 -valued measure m . The sum of measures of vertices over an edge is 0.

$$m(a) = m(f) = 1.$$

$$m(b) = m(e) = -1.$$

$$m(c) = m(d) = m(g) = m(h) = m(i) = m(j) = m(k) = m(l) = 0.$$

\square

5.4 Hypergraphs of Girth 4

Here we are interested in Greechie diagrams of OMPs. Firstly we show that just mentioned example of a stateless OMP is, in fact, the smallest possible. Then, as we are searching for small OMPs with no group-valued measures, we construct a bound on the maximal number of edges for a Greechie diagram for OMPs, given the number of vertices. To our best knowledge, no such bound has

The bound is not asymptotically tight. In [27], it is shown that for any positive constant c , there is a number of vertices n such that $m \leq cn^2$. On the other hand, this is not evident if we consider only hypergraphs with not more vertices than edges. See Figure 5.3, for more details. We computed the “best upper-bound” by generating of all 3-uniform hypergraphs of girth at least 4.

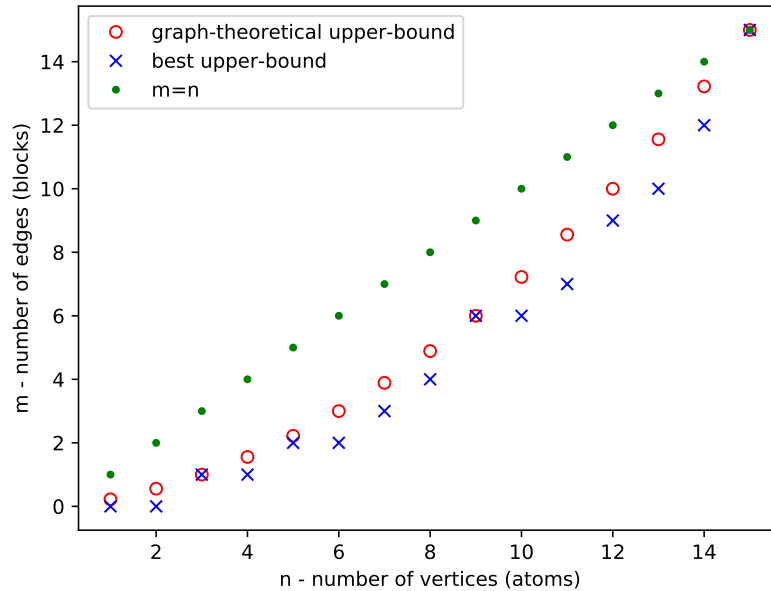


Figure 5.3. Relation between the graph-theoretical upper bound, and the real maximum of the number of edges, given a number of edges for a 3-uniform hypergraph of girth at least 4.

Corollary 5.8. According to graph theory, a Greechie diagram representing an OMP needs to have at least 17 vertices, to have more edges than vertices.

Corollary 5.9. According to computer-aided generating of hypergraphs, a Greechie diagram needs to have at least 19 vertices, to have more edges than vertices.

There has to be an edge with 4 or more edges in a hypergraph to forbid probability measures, then, according to computer-aided search, the Greechie diagram of an OMP with no non-trivial group-valued measures has at least 20 vertices. We found such an example with 21 vertices. There is hope that it is the smallest possible, as no matrix of a hypergraph with 20 vertices, which we found, has full rank.

Theorem 5.10. A hypergraph represented by its matrix in Table 5.1 has no non-trivial group-valued measure.

Proof. The determinant of the matrix is 1, and according to Theorem 3.5, the only group-valued measure is trivial. \square

Chapter 6

Orthomodular Lattices

Orthomodular lattices are the most specific algebras for capturing properties of quantum logic, among those on which we focused. OMLs are special cases of OMPs, so we use definitions from previous chapters. OMLs were studied in [26, 28–29].

6.1 Definitions

Definition 6.1. A *lattice* is a poset for which every two elements have the supremum and infimum.

Definition 6.2. An *orthomodular lattice* is an orthomodular poset which is a lattice.

A measure on an OML L is also a measure on the OMP L .

Example 6.1. The OMP from Figure 5.2 is not an OML.

Proof. There are three common upper-bounds for a and f , namely b' , e' , and $\mathbf{1}$. Elements b' and e' are less than or equal to $\mathbf{1}$ and they are not comparable. There is no supremum for a and f . Hence the OMP is not a lattice. \square

OMLs are special cases of OMPs. Hence we shall define Greechie diagrams for OMLs, as we showed that not every OMP representable by a Greechie diagram is an OML.

Proposition 6.1. Let $H = (V, E)$ be a hypergraph, and the following conditions hold:

- For every edge $e \in E$, $|e| \geq 3$ (every edge contains at least 3 vertices).
- The girth of H is at least 5.

Then H represents an OML L , in the sense that V represents the set of atoms of L , and E represents the set of blocks of L .

Not for all OMLs a Greechie diagram exists. In [30, 29], it is shown how a hypergraph representing OML can even have cycles of length 2.

6.2 Hypergraphs of Girth 5

Again, we look for hypergraphs of girth at least 5 with more edges than vertices. But at first, we show an OML with no probability measure.

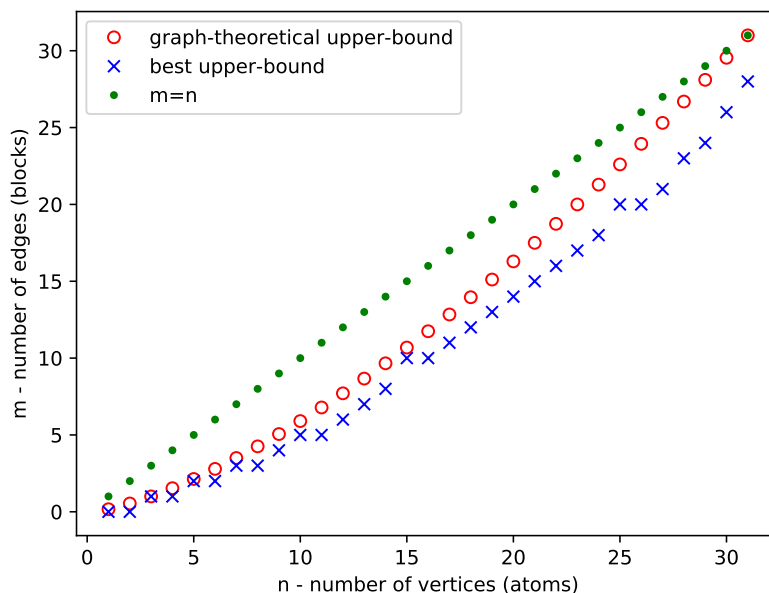


Figure 6.2. Relation between the graph-theoretical upper bound, and the real maximum of the number of edges, given a number of edges for a 3-uniform hypergraph of girth at least 5.

Corollary 6.3. According to graph-theory, a Greechie diagram needs to have at least 34 vertices to have more edges than vertices.

Corollary 6.4. According to the generation of hypergraphs representing OMLs in [31], there is no Greechie diagram for OML with at most 36 vertices having more edges than vertices.

We shall construct an OML with no group-valued measures.

Theorem 6.5. A Greechie diagram for OML with vertices $\{1, 2, \dots, 67\}$, and edges:

$$\begin{aligned} & \{(1, 5, 65), (1, 9, 21), (1, 45, 56), (2, 3, 18), (2, 23, 64), (2, 28, 40), \\ & (3, 7, 17), (3, 38, 63), (4, 5, 6), (4, 8, 24), (4, 46, 62), (5, 13, 37), \\ & (5, 51, 52), (6, 7, 60), (6, 15, 31), (6, 20, 35), (7, 11, 32), (7, 44, 67), \\ & (8, 11, 12), (8, 27, 29), (9, 12, 18), (9, 14, 20), (10, 20, 28), (10, 26, 32), \\ & (10, 46, 63), (11, 19, 25), (12, 42, 49), (13, 17, 22), (13, 42, 58), (14, 19, 58), \\ & (14, 23, 36), (15, 26, 29), (15, 39, 43), (16, 19, 65), (16, 27, 35), (16, 38, 39), \\ & (17, 21, 29), (19, 31, 59), (21, 46, 53), (22, 24, 28), (22, 45, 48), (23, 49, 52), \\ & (24, 36, 38), (25, 28, 50), (25, 30, 47), (27, 40, 57), (30, 43, 61), (31, 34, 48), \\ & (32, 41, 52), (33, 50, 54), (33, 57, 63), (33, 58, 60), (34, 37, 44), (34, 49, 50), \\ & (35, 42, 47), (39, 41, 54), (40, 41, 59), (40, 44, 56), (42, 56, 62), (43, 49, 55), \\ & (45, 55, 60), (46, 55, 59), (47, 63, 66), (48, 51, 53), (51, 57, 61), (53, 54, 67), \\ & (61, 62, 64), (64, 65, 66, 67)\} \end{aligned}$$

does not admit any non-trivial group-valued measures.

Proof. Construct the matrix of the hypergraph. Its determinant is 1, according to Theorem 3.5 it does not admit any non-trivial group-valued measure. \square

Chapter 7

Contribution

We reviewed results of extremal graph theory, related to hypergraphs of girth 3, 4, 5. In the case of girth 3 and 5, we adjusted the bound on the maximal number of edges, given the number of vertices, as we added to consideration that there is at least one edge with 4 or more vertices. For the hypergraphs of girth 4, there was no bound, so we created our own.

We showed that all these three bounds are quite tight for a small number of vertices. Such hypergraphs with a small number of vertices are interesting for the theory of quantum structures. This corresponds to the lower bounds on the number of atoms of quantum structures, for which a Greechie diagram exists.

From the other side, we constructed quantum structures with a small number of atoms admitting no non-trivial group-valued measures; this is an upper bound on the minimal size of such quantum structures.

To our best knowledge, there is no work addressing the minimal number of atoms of quantum structures with no non-trivial group-valued measures.

The upper bound was 73 atoms for all quantum structures. Here we significantly optimized the constructions.

- Orthoalgebras
 - Lower bound - 10 atoms.
 - Upper bound - 10 atoms.
- Orthomodular posets
 - Lower bound - 20 atoms.
 - Upper bound - 21 atoms.
- Orthomodular lattices
 - Lower bound - 36 atoms.
 - Upper bound - 67 atoms.

We showed that the smallest known examples of stateless orthoalgebra and orthomodular poset are the smallest possible.

We also developed new methods of generating quantum structures, which are much more general than the current known. We moved this to Appendix B.

We present a sufficient condition for a hypergraph to not have a non-trivial group-valued measure. We check if the determinant of a matrix is ± 1 or not. The previous works needed much more complicated arguments for this.

We also showed the construction of an orthoalgebra with no non-trivial group-valued measures such that the arguments for the nonexistence of those measures are simple. It does not require advanced theorems from group theory. In contrast to all current known examples of quantum structures with no non-trivial group-valued measures, it does not require a computer to check if there are group-valued measures or not.



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Appendix A

The Root-Mean Square-Arithmetic Mean Inequality

The inequality between root-mean square, which is sometimes called quadratic mean, and arithmetic mean (abbr. RMS-AM inequality) can be widely used in extremal graph theory. We show it as a special case of Cauchy-Schwarz inequality. It is also a special case of Jensen inequality, as Cauchy-Schwarz inequality is itself a special case of Jensen inequality.

Theorem A.1. (Root-Mean Square-Arithmetic Mean inequality)

Let x_1, x_2, \dots, x_n be real numbers. Then

$$\sqrt{\frac{x_1^2 + x_2^2 + \dots + x_n^2}{n}} \geq \frac{x_1 + x_2 + \dots + x_n}{n}.$$

Proof. If the right hand side of the inequality is negative we are done. Assume it is not, then we equivalently rewrite the inequality using summation notation.

$$n(x_1^2 + x_2^2 + \dots + x_n^2) \geq (x_1 + x_2 + \dots + x_n)^2$$

$$n \sum_{i=1}^n x_i^2 \geq \left(\sum_{i=1}^n x_i \right)^2$$

As both of these inequalities are equivalent to the stated one, it is sufficient to prove the latter. Now consider the inequality $(a - b)^2 \geq 0$, so $a^2 + b^2 \geq 2ab$, which clearly holds. If we sum up these inequalities for $a, b \in \{x_1, x_2, \dots, x_n\}$, the resulting inequality will also hold.

$$\sum_{i=1}^n \sum_{j=1}^n (x_i^2 + x_j^2) \geq \sum_{i=1}^n \sum_{j=1}^n 2x_i x_j$$

$$\sum_{i=1}^n \sum_{j=1}^n (x_i^2 + x_j^2) = \sum_{i=1}^n \sum_{j=1}^n x_i^2 + \sum_{i=1}^n \sum_{j=1}^n x_j^2 = \sum_{i=1}^n n x_i^2 + n \sum_{j=1}^n x_j^2 = 2n \sum_{i=1}^n x_i^2 \geq$$

$$\geq \sum_{i=1}^n \sum_{j=1}^n 2x_i x_j = 2 \sum_{i=1}^n x_i \sum_{j=1}^n x_j = 2 \left(\sum_{i=1}^n x_i \right)^2$$

$$n \sum_{i=1}^n x_i^2 \geq \left(\sum_{i=1}^n x_i \right)^2$$

□

Corollary A.2. RMS-AM Inequality becomes an equality iff

$$x_1 = x_2 = \dots = x_n \geq 0.$$

A.1 RMS-AM Inequality as a Special Case of Cauchy-Schwarz Inequality.

Let $u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n$ be real numbers, Then Cauchy-Schwarz inequality can be stated as follows:

$$(u_1^2 + u_2^2 + \dots + u_n^2)(v_1^2 + v_2^2 + \dots + v_n^2) \geq (|u_1v_1| + |u_2v_2| + \dots + |u_nv_n|)^2.$$

Let $u_1 = u_2 = \dots = u_n = 1$ and $v_1 = x_1, v_2 = x_2, \dots, v_n = x_n$, then:

$$n(x_1^2 + x_2^2 + \dots + x_n^2) \geq (x_1 + x_2 + \dots + x_n)^2,$$

which is equivalent to RMS-AM inequality.

A.2 RMS-AM Inequality as a Special Case of Jensen Inequality

The finite form of Jensen inequality generalizes the statement that a secant line of a convex function is above the graph.

Let f be a convex function, x_1, x_2, \dots, x_n be in its domain and

$$\lambda_1, \lambda_2, \dots, \lambda_n \geq 0,$$

$$\sum_{i=1}^n \lambda_i = 1.$$

Then the following inequality is known as Jensen inequality.

$$\sum_{i=1}^n \lambda_i f(x_i) \geq f\left(\sum_{i=1}^n \lambda_i x_i\right)$$

Now let $f(x) = x^2$, which is a convex function, and $\lambda_1 = \lambda_2 = \dots = \lambda_n = \frac{1}{n}$, then we rewrite the Jensen inequality as follows.

$$\frac{\sum_{i=1}^n x_i^2}{n} \geq \left(\frac{\sum_{i=1}^n x_i}{n}\right)^2$$

$$n \sum_{i=1}^n x_i^2 \geq \left(\sum_{i=1}^n x_i\right)^2$$

The latter is again equivalent to RMS-AM inequality.

As Jensen inequality is closely related to probability theory, we will also add a probability-theoretical explanation of RMS-AM inequality.

Let x_1, x_2, \dots, x_n be real numbers. Let X be a random variable attaining values x_1, x_2, \dots, x_n , each with probability $\frac{1}{n}$. The expected value of such random variable is:

$$\mathbb{E}X = \frac{(\sum_{i=1}^n x_i)}{n}.$$

Appendix B

Implementation

Here we briefly describe our programs used for generating quantum structures. They are on the enclosed CD, together with a `README.txt` file.

B.1 `orthoalgebra.cpp`

The program generates all Greechie diagrams of orthoalgebras with the given number of vertices. As the number of different hypergraphs is relatively small, it is possible to enumerate all of them in a reasonable time.

There are 744 624 Greechie diagrams of orthoalgebras with 10 vertices, an edge with at least 4 vertices, and at least 11 edges such that no edge could be added to the hypergraph. This takes about 2 seconds¹. To generate them and about 2 minutes to save them².

The speed of implementation is caused by choosing appropriate data structures. We use an unsigned integer for the representation of an edge. The binary code of an integer is for our purpose an indicator function. For example, an edge represented by the binary integer 1010 contains vertices 1 and 3. This allows us to store them efficiently in an array. Hence the algorithm is cache-friendly.

For two edges (unsigned integers), we can calculate the number of common vertices as follows:

```
// count the number of bits
unsigned weight(unsigned x){
    unsigned count;
    for(count=0; x; count++)
        x &= x-1;
    return count;
}

unsigned a = ...;
unsigned b = ...;
unsigned common_vertices = weight(a&b);
```

This is efficient as the number of common vertices is generally small.

The algorithm itself is a simple recursive DFS. In a function call either no edge can be added and the hypergraph is processed, or an edge is added to the hypergraph, the domain of edges is restricted and a recursive call follows.

B.2 `gen.cpp`

In the previous case, there was a small number of hypergraphs to generate. Hence it was sufficient not to address graph isomorphism. Here, in the case of

¹ On an Intel Core i5 – 8250U 1.6GHz

² It is 180 MB of plain text

OMLs and OMPs, there are significantly more hypergraphs representing them, even if they have a small number of vertices, for example, 30. Hence ignoring isomorphisms in an exhaustive generation is not possible here. The problem of graph isomorphism is hard. There is no algorithm running in polynomial time, which tests general graph-isomorphism. Isomorphism-free generation of hypergraphs of OMLs was done in [31]. However, they did not generate any hypergraph with more edges than vertices.

According to the preceding arguments, there is no hope for a systematic exploration of hypergraphs representing OMLs, which are large enough to have no non-trivial group-valued measure.

We decided to do a random search with no back-tracking, as we want to explore as diverse parts of space of hypergraphs as possible.

Our algorithm works as follows, although our implementation slightly differs:

1. Create a hypergraph H with n vertices and one edge, with 4 vertices. Declare the minimal length of a cycle.
2. Declare $d = 0$, where d is the maximal allowed degree of a vertex in H .
3. `new_edge := true`.
4. If `new_edge = false`, terminate and process the constructed hypergraph.
5. `d := d+1`.
6. `new_edge = false`.
7. Create a set of all possible edges with 3 vertices.
8. If S is empty, go to step 4.
9. Remove a random edge $e \in S$ from S . If the degree of all vertices of e is less than d , and the addition of e does not create a cycle of a smaller length than the minimal length of cycle, add it to H . `added_edge := true`
10. Go to step 8.



Appendix C

Contents of CD

- Thesis.pdf
- gen.cpp
- orthoalgebra.cpp
- README.txt