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Two-dimensional universal algebra

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Declaration of originality I hereby declare on my honour that this thesis contains, to my best knowledge and belief, no material previously published by any other person except where due acknowledgement has been made.

Matěj Dostál

Annotation This thesis studies categorical universal algebra. Many results from classical categorical universal algebra are generalised to the setting of enriched categories, or to the setting of higher categories. Among other results, we show a Morita equivalence theorem for many-sorted algebras, Birkhoff's variety theorem for algebras over categories, a formal pseudoadjoint functor theorem, and we describe wreaths for pseudomonads.

Keywords categorical universal algebra, enriched algebraic theory, pseudomonad

Anotace Tato práce se zabývá kategoriální universální algebrou. Zobecňuje mnoho výsledků klasické kategoriální universální algebry pro obohacené kategorie, popřípadě pro vícerozměrné kategorie. Mimo jiné patří mezi výsledky práce věta o Moritově ekvivalenci pro vícesortové algebry, Birkhoffova věta pro algebry nad kategoriemi, formální věta o pseudoadjunktu, a popis věnečků pro pseudomonády.

Klíčová slova kategoriální universální algebra, obohoacená algebraická teorie, pseudomonáda

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Contents

Preface

The topic of this text is the study of higher-dimensional categorical universal algebra algebra. We study two-dimensional algebraic theories and monads (and their properties), and we are interested in generalising the results of categorical universal algebra outside the setting of ordinary category theory.

The study of ordinary categorical universal algebra is well-established and it essentially started with Lawvere's thesis [61] in the 1960s, where he introduced the notion of an algebraic theory for the first time. His notion corresponds to the universal-algebraic notion of an equational theory that dates back to Birkhoff's paper [16], which established universal algebra as a field of mathematics. For an excellent introduction to the theory of algebraic theories, we refer to a recent textbook by Adámek, Rosický and Vitale [5] that covers the main results concerning algebraic theories in the setting of ordinary category theory.

Moving from ordinary category theory, we wish to study generalisations of the notions of Lawvere's algebraic theories and of monads using the theory of enriched categories. Instead of ordinary categories, we consider categories enriched in a suitable monoidal category $\mathscr{V}$, say the category Pos of all posets and monotone maps, or the category Cat of all small categories and functors. We are then able to study interesting structures of algebraic nature that are equipped with an underlying poset or an underlying category, instead of a set. The fact that the underlying "object" of an algebra has an additional structure of categorical nature explains why we speak of "higher-dimensional" categorical algebra.

The study of categorical universal algebra using the methods of enriched category theory is not new either: the study of cocompletions of enriched categories by Kelly [41], and the recent investigations of enriched algebraic theories of Lack and Rosický [57], recover large portions of the theory of ordinary algebraic theories in a very general setting. Still, it is possible to find interesting parts of the ordinary theory that have not yet been generalised in a satisfactory manner; and perhaps even more importantly, we want to persuade the reader that even in the cases where there is a general result for enriched algebraic theories, it is worthwhile to study its ramifications for concrete enrichments.

Going even further, some of the results that are definitely of algebraic flavour cannot be stated with the methods of enriched category theory. We have to work in a yet weaker setting of 2-dimensional categories, pseudofunctors and pseudonatural transformations (see, e.g., [35]) instead of categories enriched in Cat. The "pseudo" prefix here indicates that many of the requirements present as axioms in enriched category theory are relaxed to fit the examples we have in mind.

We give a more detailed account of the contents of the text below.

Structure of the text.

Chapter 1 In the first chapter we introduce basic notions of (categorical) universal algebra and give a motivation for its study.

Chapter 2 We turn to generalising some of the basic notions of Chapter 1 to the setting of enriched categories. The essentials of cocompletions of categories are reviewed and their connection to categorical universal algebra is hinted at.

Chapter 3 *Morita equivalence* detects the situation in which two different theories give rise to the same class of models. In this chapter we first quickly generalise the notion of an *algebraic theory* to the context of enriched category theory, then we generalise the result of Adámek, Sobral and Sousa concerning Morita equivalence of many-sorted algebraic theories. The result is parametric in the choice of enrichment and in the choice of the class of colimits in the theory. We apply the result to recover some known Morita equivalence theorems and significantly extend them.

Chapter 4 Every *theory map* between two theories gives rise to a well-behaved *algebraic functor* between the categories of algebras for the respective theories. An example of this phenomenon is the "quotient" of the theory of groups to the theory of commutative groups, that gives rise to the inclusion of the category of commutative groups in the category of all groups. In fact, there is a dual correspondence between (certain) theories and algebraic categories that is usually dubbed *Gabriel-Ulmer duality*. We give a very general account of the duality in this chapter.

Chapter 5 *Sifted colimits* are those colimits that commute with finite products in sets. They play a major role in categorical universal algebra. For example, varieties of algebras are precisely the free cocompletions under sifted colimits of algebraic theories. In this chapter we give an elementary characterisation of sifted weights for the enrichment in categories. We also provide a number of examples of sifted weights using our elementary criterion.

Chapter 6 *Birkhoff's variety theorem* from universal algebra characterises equational subcategories of varieties. We give an analogue of Birkhoff's theorem in the setting of enrichment in categories. For a suitable notion of an equational subcategory we characterise these subcategories by their closure properties in the ambient algebraic category.

Chapter 7 When studying weak categorical notions such as pseudoadjunctions and pseudomonads formally, it is conveninent to consider them as constructions *in a* **Gray**-*category* instead of in a general tricategory. **Gray**-categories are a strict kind of tricategories and known coherence results state that one does not lose any generality working with **Gray**-categories, rather than in tricategories. We introduce **Gray**-categorical notions and review the theory behind giving *presentations* of **Gray**-categories.

Chapter 8 Using the theory of Chapter 7, we give presentations of important **Gray**-categories: the *free pseudoadjunction* **Gray**-category **psa**, the *free pseudomonad* **Gray**-category **psm**, and their *KZ-variants*.

Chapter 9 The formal adjoint functor theorem states that a right adjoint is equivalently a certain absolute left Kan extension. Naively reformulating this result from ordinary category theory, we would presume that a pseudoadjunction is equivalently a certain *absolute left Kan pseudoextension*, given the right notion of a Kan pseudoextension. Such a notion already appeared in literature, and we show that using this notion the expected formal pseudoadjoint functor theorem indeed holds.

Chapter 10 We turn to study the properties of KZ-pseudoadjunctions, KZ-pseudomonads and their properties. We show that KZ-pseudomonads are *property-like* in a certain technical sense, loosely meaning that pseudoalgebras for these pseudomonads are objects “with additional properties” instead of objects “with additional structure”.

Chapter 11 Wreaths are a generalisation of the notion of a distributive law. We introduce the notion of a wreath in a **Gray**-category and give an elementary description of wreaths in this setting.

Original results contained in the text. The original results of the author are collected (and properly cited) in the following list. Other results that are contained in the text but not in the list are either not original, or easy/folklore.

1. Chapter 3 contains a refinement for many-sorted algebraic theories of the basic Morita-type theorem. This chapter closely follows the exposition of [31].

2. Chapter 4 contains a proof of Gabriel-Ulmer duality for Ψ-theories and Ψ-algebraic categories enriched in a suitable monoidal category $\mathscr{V}$. The result has not yet been published elsewhere.

3. Chapter 5 contains a coend characterisation of flat weights for a sound class of weights in a general enrichment. In the special case of the enrichment in categories and for the sound class of weights for finite coproducts, this yields an elementary characterisation of sifted weights. The results of this chapter have appeared in [30].

4. Chapter 6 contains a proof of a two-dimensional Birkhoff theorem. The results of this chapter are contained in [28].

5. Chapter 8 contains presentations of **Gray**-categories **psa**, **psm**, **kza**, **kza** detecting pseudoadjunctions, pseudomonads, KZ-pseudoadjunctions and KZ-pseudomonads, respectively. These presentations are spelled out in detail.

6. Chapter 9 contains a proof of formal pseudoadjoint functor theorem. This is a novel result contained in the preprint [29].

7. Chapter 10 contains a characterisation of KZ-pseudoadjunctions and KZ-pseudomonads that expands the characterisation of KZ-pseudomonads already present in [67].

8. Chapter 11 contains a description of wreaths of 2-functors around pseudomonads. This description is new and has not appeared anywhere in the literature.

Part I

Enriched categorical universal algebra

Chapter 1

Introduction to categorical universal algebra

The intention of this chapter is to introduce *categorical universal algebra* and to motivate this subject by showing its connections to “ordinary” universal algebra. We give the basic definitions pertaining to categorical algebra, and then reformulate and generalise them in the setting of enriched category theory.

The topic of the remaining chapters of Part I of this thesis is “categorical universal algebra in the enriched setting”. To give a very quick taste of the difference between classical universal algebra and enriched universal algebra, consider

- The example of *natural numbers.* Natural numbers can be thought of as a *set* $\mathbb{N}$ endowed with *algebraic structure*: certain *operations* on this set, such as addition or multiplication, that are subject to laws or *axioms*: for example, for any two natural numbers a, b the commutativity condition

  $$a \times b = b \times a$$

  holds.

- The example of *sets themselves.* Consider (putting foundational issues aside) the collection Set of all sets and mappings. This huge collection is also endowed with algebraic structure in some sense. Similarly to the case of natural numbers, we can take two elements of the collection Set and perform some *operations* on them. The *cartesian product* of two sets is a well-known operation on sets, taking sets A and B and producing the set

  $$A \times B = \{(a, b) \mid a \in A,\ b \in B\}$$

  of ordered pairs of elements from A and B. In fact, the operation × extends to pairs $f : A \to A'$ and $g : B \to B'$ of mappings, producing the mapping

  $$\begin{aligned} f \times g : A \times B &\to A' \times B' \\ (a, b) &\mapsto (f(a), g(b)) \end{aligned}$$

  and thus being “functorial”. This operation is also subject to axioms. For example, for any two sets, their cartesian product is *almost commutative.* That is, the equality

  $$A \times B = B \times A$$

does *not* hold. However, the sets $A \times B$ and $B \times A$ are in an abstract sense *almost the same*: there is a natural isomorphism (bijection) map

$$s : A \times B \to B \times A$$

that for each pair (a, b) returns the pair (b, a).

The need for study of the "higher-dimensional" algebraic structures in the sense of the last example arises very often when we turn from studying concrete examples of algebras and instead begin to study *classes* of algebras and relations between such classes of algebras. In this chapter we give a basic overview of universal algebra in the sense of the first example, and in subsequent chapters we study the theory of categorical universal algebra in the enriched setting, motivated by the abundance of the examples of the second kind.

Structure of the chapter.

1. In Section 1.1 we will give the most basic definitions from universal algebra and show examples from computer science that enlighten the importance of many-sorted algebras.

2. Section 1.2 introduces the notion of an *algebraic theory*. This is the abstract structure generalising the usual notion of an equational theory from universal algebra. We will argue that the level of generality of the definition allows interesting variations of the notion of an algebra with almost no overhead in the difficulty of presentation.

3. Then we shall cover the notion of a *sifted colimit* in Section 1.3. Sifted colimits are those colimits that commute with finite products, and they satisfy many useful properties in categories of algebras for an algebraic theory.

4. Morita equivalence studies the situation when two different equational theories yield the same category of algebras. We will deal with the basics of the theory in Section 1.4.

5. Section 1.5 defines the notion of an equationally defined subcategory of an algebraic category, and states Birkhoff's HSP theorem in categorical language.

6. We shall sometimes use the *monad* approach for studying universal algebraic phenomena. Monads give an alternative abstract definition of the notion of a *theory*. We quickly introduce them in Section 1.6.

All the notions developed and all the results stated in this chapter are well-known and standard. The exposition in this chapter is quick and it serves the role of settling the notation rather than being completely self-contained.

1.1 Universal algebra

Universal algebra was founded by Garrett Birkhoff in 1935 in his seminal paper [16]. The unifying strength of universal algebra quickly established it as an important field of algebra that conceptually explains the similarities between various classes of abstract algebras.

We shall give a short exposition of the basic notions of universal algebra using some well-known structures from computer science. In anticipation of further generalisations, our presentation of basic notions of universal algebra differs slightly from those given in standard universal algebra textbooks (e.g. [26]). Namely, we intend to use category theory notation from the very outset, since it makes the generalisation easy to grasp.

An example of an algebraic structure abundant in both pure mathematics and theoretical computer science is that of a monoid.

Definition 1.1.1. A *monoid* is a set A together with an identity element $e \in A$ and a binary operation $* : A \times A \to A$, subject to the unit and associativity axioms

$$e * a = a, \qquad a * e = a, \qquad (a * b) * c = a * (b * c)$$

holding for all elements $a, b, c \in A$.

Example 1.1.2. Let us fix a finite set $S = \{s_1, \dots, s_n\}$, thought of as an alphabet with symbols $s_1, \dots, s_n$. If we denote by S^* the set of all (finite) words in the alphabet S, we can form a *free monoid* $FS = (S^*, *, \epsilon)$ by defining the operation

$$* : S^* \times S^* \to S^*$$

as concatenation of strings, and denoting the empty word by ϵ. The axioms of a monoid are satisfied since concatenation is associative and ϵ plays the role of unit with respect to concatenation.

Observe that the free monoid $F1$ on a one-element set 1 is isomorphic to the monoid $(\mathbb{N}, +, 0)$ of natural numbers with addition.

The example of monoids shows three main ingredients of an abstract algebra. It is a *set* that comes equipped with some *operations* of a given *signature*, and the operations satisfy certain prescribed *axioms*. Firstly we will introduce the notion of an algebra for a general signature.

Definition 1.1.3. A *signature* Σ is a set of operation symbols together with an arity function

$$\mathsf{ar} : \Sigma \to \mathbb{N}$$

that assigns to each operation symbol σ its arity $\mathsf{ar}(\sigma)$.

An *algebra* for a signature Σ is a set A together with with an n-ary operation $\sigma^A : A^n \to A$ for every operation $\sigma \in \Sigma$ such that $\mathsf{ar}(\sigma) = n$. We often write 1 (the one-element set) instead of A^0.

A *homomorphism* of Σ-algebras from A to B is a function $f : A \to B$ that preserves the operations: for every n-ary operation $\sigma \in \Sigma$, the square

$$\begin{tikzcd}
A^n \arrow[r, "f^n"] \arrow[d, "\sigma^A"'] & B^n \arrow[d, "\sigma^B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

commutes.

The introduction of axioms for algebras requires the notion of a Σ-*term*. Without being too technical, we will settle with saying that a Σ-term is a well-constructed algebraic expression (a syntactic tree) constructed from variables and operation symbols from Σ. A Σ-*equation* is then a formal equality $l \approx r$ between two Σ-terms l and r, and a Σ-algebra A *satisfies* the equation $l \approx r$ if the algebraic expressions l and r, when interpreted in A, yield the same output for every valuation of the variables in A.

Example 1.1.4. Using the notions defined above, a monoid is an algebra for the signature

$$\Sigma = \{*, e\}$$

with $\mathsf{ar}(*) = 2$ and $\mathsf{ar}(e) = 0$, satisfying the formal equalities

$$e * x \approx x, \qquad x * e \approx x, \qquad x * (y * z) \approx (x * y) * z.$$

The algebraic structure of a monoid is one-sorted, since it involves essentially only one set: namely the “carrier set” of the algebra. Many applications of algebra in computer science (see, e.g., [81]) naturally demand a many-sorted approach to universal algebra, as introduced in [17]. An elementary example of a many-sorted algebra is a directed graph, since every graph consists of two sorts of data: the vertices, and the edges.

Example 1.1.5. A *directed graph* is a pair (E, V) of sets together with two maps $s : E \to V$ and $t : E \to V$, being the source and target maps, respectively. In a diagram, we have a parallel pair

$$\begin{tikzcd} E \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & V \end{tikzcd}$$

of sets and mappings.

A slightly more involved example of a many-sorted algebra is a stack. In this example the two sorts of data considered are the alphabet sort and the stack sort. Unlike in the example of a directed graph, we need operations with arities of mixed sorts: the operation of pushing a symbol on top of a stack requires a symbol (that has the alphabet sort), the stack involved (of the stack sort), and the operation returns a new stack. Moreover, we need to specify axioms guaranteeing the stack behaviour of the algebra.

Example 1.1.6 ([81])**.** A *stack* is a pair (A, S) of sets, together with a nullary operation $\mathrm{empty} : 1 \to S$, a unary operation $\mathrm{pop} : S \to S$, and a binary operation $\mathrm{push} : A \times S \to S$, subject to axioms

$$\mathrm{pop}(\mathrm{empty}) = \mathrm{empty}, \qquad \mathrm{pop}(\mathrm{push}(a, s)) = s.$$

It is possible to give a formal definition of a many-sorted signature, many-sorted algebra and a homomorphism of many-sorted algebras in the same spirit as in Definition 1.1.3. These definitions are, however, more involved than in the one-sorted case. We shall introduce the notion of an algebraic theory in the following section, showing how this approach yields the expected definitions for free.

1.2 Algebraic theories

The categorical notion of a one-sorted algebraic theory was introduced by W. Lawvere in his PhD thesis [61]. We are going to introduce the notion of an algebraic theory and its variants. We use the standard notation and basic notions of category theory; we refer to the standard textbooks [10] and [64].

The set $\mathbb{N}$ will now be considered as a category (and denoted by $\mathbb{N}$ again). The category $\mathbb{N}$ has natural numbers as objects, and the only morphisms in $\mathbb{N}$ are those that guarantee the existence of all the finite coproducts of the form $n = 1 + \cdots + 1 = n \bullet 1$.

Definition 1.2.1 (Lawvere [61]). A *Lawvere theory* $\mathscr{T}$ is a category with finite coproducts, together with a finite-coproduct-preserving functor $T : \mathbb{N} \to \mathscr{T}$ that is bijective on objects.

Given a Lawvere theory $\mathscr{T}$, we denote by $\mathsf{Alg}(\mathscr{T})$ the full subcategory of $[\mathscr{T}^{op}, \mathsf{Set}]$ spanned by all finite-product-preserving presheaves. Every finite-product-preserving presheaf $A : \mathscr{T}^{op} \to \mathsf{Set}$ is called an *algebra* for the theory $\mathscr{T}$, a *homomorphism* between two algebras A and B is a natural transformation $f : A \to B$ between the respective presheaves, and the category $\mathsf{Alg}(\mathscr{T})$ is called the category of algebras for $\mathscr{T}$. Any category equivalent to $\mathsf{Alg}(\mathscr{T})$ for some theory $\mathscr{T}$ is called an *algebraic category*.

Example 1.2.2. Define $\mathscr{T}$ to be the category that has as objects the free monoids Fn on n generators for every natural number n, and as morphisms all the monoid homomorphisms between them. This category has finite coproducts, because the isomorphism $F(m) + F(n) \cong F(m + n)$ holds for all pairs m, n of natural numbers. If we equip $\mathscr{T}$ with the obvious finite-coproduct-preserving functor $T : \mathbb{N} \to \mathscr{T}$ that maps a natural number n to the free monoid Fn, we observe that $\mathscr{T}$ is a Lawvere theory.

It is easy to prove that the category $\mathsf{Alg}(\mathscr{T})$ of algebras for the Lawvere theory $\mathscr{T}$ is equivalent to the category Mon of all monoids and monoid homomorphisms.

The introduced formalism of a theory as of a certain category with finite coproducts admits a quick and natural generalisation.

To obtain a notion of an S-sorted theory, we replace $\mathbb{N}$ with $\mathscr{S}^*$. Here, $\mathscr{S}^*$ has the set of all strings S^* over an alphabet S as the set of objects, and the morphisms in $\mathscr{S}^*$ are such that any word $vw \in S^*$ is a coproduct $v + w$ in $\mathscr{S}^*$.

Remark 1.2.3. Let us slightly reformulate the above comments. Given a set S, we can form the discrete category $\mathscr{S}$ consisting of the object set S, and the only morphisms in $\mathscr{S}$ being the identity morphisms for each object $s \in S$. The above introduced category $\mathscr{S}^*$ is obtained by "freely adjoining" finite coproducts to the discrete category $\mathscr{S}$. In more precise terms, $\mathscr{S}^*$ is the *free cocompletion* of $\mathscr{S}$ under finite coproducts.

Definition 1.2.4 (Bénabou [14]). Let S be any set. An *S-sorted theory* $\mathscr{T}$ is a category with finite coproducts, together with a finite-coproduct-preserving functor $T : \mathscr{S}^* \to \mathscr{T}$ that is bijective on objects.

Given an S-sorted theory $\mathscr{T}$, the definition of an algebra, homomorphism and the category of algebras is the same as in the case when $\mathscr{T}$ is a Lawvere theory.

Note that by the above definition, the notion of a Lawvere theory is a special instance of the notion of an S-sorted theory for S being a one-element set 1. This follows from observing that $\mathbb{N} \simeq \mathbf{1}^*$, where $\mathbf{1}$ denotes the one-morphism category.

Example 1.2.5. The category Graph of directed graphs and their homomorphisms is equivalent to the presheaf category $[\mathscr{D}^{op}, \mathsf{Set}]$ for the category $\mathscr{D}$ defined by the following diagram:

$$\begin{tikzcd} e & v \arrow[l, bend right, "s"'] \arrow[l, bend left, "t"] \end{tikzcd}$$

We may form a many-sorted algebraic theory $\mathscr{T}$ from $\mathscr{D}$ by freely adjoining finite coproducts to the category $\mathscr{D}$. Then the presheaf category $[\mathscr{D}^{op}, \mathsf{Set}]$ is equivalent to the category $\mathsf{Alg}(\mathscr{T})$ of algebras for the theory $\mathscr{T}$.

The construction of an algebraic theory for the example of stacks is slightly more involved, since the specification of the stack data type contains operations of with signatures of mixed sorts. We, however, indicate the basic idea:

1. The two sorts (stack and alphabet) are denoted by s and a. The set of objects of the theory consists is the set S^* of the words in the alphabet $S = \{s, a\}$.

2. We consider the category $\mathscr{S}^*$ that endows S^* with a finite coproduct structure, and "freely adjoin" the "rewrite rules"

   $$\text{pop} : s \to s, \quad \text{push} : s \to as, \quad \text{empty} : s \to \epsilon$$

   as morphisms in $\mathscr{S}^*$. Loosely said, to freely adjoin these morphisms means that we consider the above three morphisms as *rewrite rules* on the words in the alphabet S, and add also all "derived" rewrite rules that can be formed from the generating rewrite rules. For example, the theory for stacks contains the morphism $\text{pop} \cdot \text{pop} \cdot \text{push} : s \to s \to s \to as$.

The conceptual ease in introducing many-sorted algebraic theories is one of the advantages of the categorical approach to universal algebra. Moreover, we can define a more general notion of an algebraic theory that does not refer to any sorting whatsoever.

Definition 1.2.6. An *algebraic theory* is any category $\mathscr{T}$ with finite coproducts.

Remark 1.2.7.

1. The definition of an algebraic theory subsumes both the one-sorted and many-sorted definition, and allows much of the theory to be developed without any regard to sorts. This conceptual simplification often leads to simpler proofs, see e.g. [5].

2. Some authors define the notion of an algebraic theory dually by taking it to have finite *products* instead of coproducts. While this approach has the advantage of being closer to the classical universal algebraic approach, the notion of theories with coproducts usually gives notationally nicer proofs.

Another advantage of the categorical approach to universal algebra is that we can very easily consider models of an algebraic theory in an ambient category different from Set. Given an algebraic theory for a given type of an algebraic structure, we can e.g. retrieve the ordered or topological variants of the algebraic structure.

Example 1.2.8. Given an algebraic theory $\mathscr{T}$ and a category $\mathscr{X}$ with finite products, an *$\mathscr{X}$-model* of $\mathscr{T}$ is a finite-product-preserving functor $\mathscr{T}^{op} \to \mathscr{X}$. The full subcategory of the functor category $[\mathscr{T}^{op}, \mathscr{X}]$ on all finite-product-preserving functors is called the category of $\mathscr{X}$-models of $\mathscr{T}$.

Denote by Pos the category of posets and monotone maps, and by Top the category of topological spaces and continuous mappings. If $\mathscr{T}$ is the Lawvere theory of monoids, the Pos-models of $\mathscr{T}$ are exactly the partially ordered monoids, and the Top-models of $\mathscr{T}$ are exactly the topological monoids.

1.3 Commutativity of limits and colimits

Computing *limits* of algebras is easy; they are computed essentially on the level of their underlying sets due to the fact that limits commute with limits. Computation of *colimits* of algebras is usually much harder. However, there is an important class of colimits of algebras that can be computed easily as well. More specifically, colimits that commute with finite products in Set are very well behaved in algebraic categories: they can also be computed on the level of the underlying sets of the involved algebras. The importance of such colimits has already been noted by Lawvere in [61].

Thus a very important aspect of the categorical approach to universal algebra is the study of commutativity of limits and colimits.

Definition 1.3.1. Given two small categories $\mathscr{D}$ and $\mathscr{C}$, we say that *$\mathscr{D}$-colimits commute with $\mathscr{C}$-limits* in Set, if for any diagram $D : \mathscr{D} \times \mathscr{C} \to \mathsf{Set}$ the canonical morphism

$$\mathsf{can} : \operatorname*{colim}_{\mathscr{D}} \lim_{\mathscr{C}} D(d,c) \to \lim_{\mathscr{C}} \operatorname*{colim}_{\mathscr{D}} D(d,c)$$

is an isomorphism. Given two classes Φ and Ψ of small categories, we say that Ψ-colimits commute with Φ-limits if for every $\mathscr{D}$ in Ψ and $\mathscr{C}$ in Φ it holds that $\mathscr{D}$-colimits commute with $\mathscr{C}$-limits in Set.

In practice, one often fixes the class Ψ of schemes for limits and *defines* the appropriate class Ψ^+ of schemes for colimits by the commutativity condition.

The case that is most useful in the setting of algebraic categories is the case where we set Ψ to be the class of finite discrete categories in the above definition.

Example 1.3.2. We say that $\mathscr{D}$ is a *sifted* category if $\mathscr{D}$-colimits commute with finite products in Set. A colimit is called sifted whenever the domain of its diagram is sifted.

Similarly, we say that a category $\mathscr{D}$ is *filtered* if $\mathscr{D}$-colimits commute with finite limits in Set. A colimit is called filtered whenever its diagram is filtered.

The basic example of a sifted colimit is a reflexive coequaliser.

Example 1.3.3. Denote by $\mathscr{D}$ the category

$$\begin{tikzcd} a \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & b \end{tikzcd}$$

and by $\mathscr{D}_r$ the category

$$\begin{tikzcd} a \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & b \arrow[l, "r" description] \end{tikzcd}$$

where the equalities

$$s \cdot r = 1_b, \qquad t \cdot r = 1_b$$

hold. We call the colimit of a diagram $D : \mathscr{D} \to \mathscr{X}$ a *coequaliser* of D, and the colimit of a diagram $D_r : \mathscr{D}_r \to \mathscr{X}$ in X is called a *reflexive coequaliser* of D_r.

We will see that the category $\mathscr{D}_r$ is sifted, and therefore reflexive coequalisers are sifted colimits. This is not the case for ordinary coequalisers, as the following example shows: Let $D_1 : \mathscr{D} \to \mathsf{Set}$ be the diagram corresponding to a discrete graph on one vertex and $D_2 : \mathscr{D} \to \mathsf{Set}$ be the one-arrow graph. Then

$$\operatorname{colim}(D_1 \times D_2) \cong 2, \qquad \operatorname{colim} D_1 \times \operatorname{colim} D_2 \cong 1,$$

so the canonical morphism

$$\operatorname{colim}(D_1 \times D_2) \to \operatorname{colim} D_1 \times \operatorname{colim} D_2$$

cannot be an isomorphism.

There are many facets of the importance of sifted colimits in categorical universal algebra. For example, they play a central role in the characterisation of algebraic categories. We will note some easy facts concerning sifted colimits that are very handy in practical computation with algebras.

Remark 1.3.4. The following two observations are direct corollaries of the definition of a sifted colimit and are proved e.g. in [5].

1. Any algebraic category $\mathsf{Alg}(\mathscr{T})$ is closed in $[\mathscr{T}^{op}, \mathsf{Set}]$ under sifted colimits. Let $\mathscr{D}$ be a sifted category, $D : \mathscr{D} \to \mathsf{Alg}(\mathscr{T})$ a diagram landing in an algebraic category for a Lawvere theory $\mathscr{T}$. Sifted colimits of algebras are computed on the level of underlying sets. For example, if

$$\begin{tikzcd} Da \arrow[r, bend left, "Ds"] \arrow[r, bend right, "Dt"'] & Db \arrow[l, "Dr" description] \end{tikzcd}$$

is a reflexive pair of algebras, the reflexive coequaliser C of the above reflexive pair can be computed by computing the reflexive coequaliser $C(1)$ of the pair

$$\begin{tikzcd} Da(1) \arrow[r, bend left, "Ds(1)"] \arrow[r, bend right, "Dt(1)"'] & Db(1) \arrow[l, "Dr(1)" description] \end{tikzcd}$$

and then observing that $C(n) \cong (C(1))^n$ has to hold by siftedness of $\mathscr{D}$.

2. Since sifted colimits commute with finite products in the category Set, they do so in any presheaf category $[\mathscr{T}^{op}, \mathsf{Set}]$, because there they are computed pointwise. Any algebraic category $\mathsf{Alg}(\mathscr{T})$ is closed under limits and sifted colimits in $[\mathscr{T}^{op}, \mathsf{Set}]$, and thus sifted colimits commute with finite products in $\mathsf{Alg}(\mathscr{T})$ as well.

Since sifted colimits enjoy many nice properties, it is very handy to have a usable characterisation of sifted diagrams. The following characterisation goes back to [36].

Proposition 1.3.5 (Characterisation of sifted categories.)**.** *A small category $\mathscr{D}$ is sifted if and only if it is non-empty and for every pair d_1, d_2 of objects from $\mathscr{D}$ the category of cospans on d_1 and d_2 is connected.*

The proof, along with other observations about sifted colimits, is contained for example in Chapter 2 of [5]. We also give another argument in Example 5.1.5.

Remark 1.3.6. We recall that a cospan in a category $\mathscr{D}$ is a diagram of the form

$$\begin{tikzcd}
& d & \\
d_1 \arrow[ur, "f_1"] & & d_2 \arrow[ul, "f_2"']
\end{tikzcd}$$

for some morphisms $f_1 : d_1 \to d$ and $f_2 : d_2 \to d$ in $\mathscr{D}$. The category of cospans in $\mathscr{D}$ on d_1 and d_2 has diagrams of the above shape as objects, and given two cospans

$$\begin{tikzcd}
d & d' \\
d_1 \arrow[u, "f_1"] \arrow[ur, "f_1'" near start] & d_2 \arrow[ul, "f_2"' near start] \arrow[u, "f_2'"']
\end{tikzcd}$$

the morphism $f : d \to d'$ is a morphism of the two cospans if the diagrams

$$\begin{tikzcd}
d \arrow[r, "f"] & d' & & d \arrow[r, "f"] & d' \\
& & \text{and} & & \\
d_1 \arrow[uu, "f_1"] \arrow[uur, "f_1'"'] & & & & d_2 \arrow[uul, "f_2"] \arrow[uu, "f_2'"']
\end{tikzcd}$$

commute. Then the connectedness of the category of cospans on d_1 and d_2 requires the category of cospans to be non-empty and for any pair (f_1, f_2) and (f_1', f_2') of cospans on d_1 and d_2 there has to be a zig-zag of cospan morphisms from (f_1, f_2) to (f_1', f_2').

1.4 Morita equivalence

Sometimes it is possible to describe some kind of an algebraic structure in two different, yet equivalent ways.

Example 1.4.1 ([5]). Directed reflexive graphs can be seen as either two-sorted or one-sorted algebras. This is a simple example of two "theories" having equivalent categories of models. Let $\mathscr{D}$ be the category

$$\begin{tikzcd} e \arrow[r, "r" description] & v \arrow[l, bend right, "s"'] \arrow[l, bend left, "t"] \end{tikzcd}$$

with the following equations satisfied:

$$r \cdot s = 1_v, \qquad r \cdot t = 1_v.$$

The presheaf category $[\mathscr{D}^{op}, \mathsf{Set}]$ of "models" of $\mathscr{D}$ is the category RGraph of *reflexive directed graphs*. That is, the models are graphs that have for each vertex a distinguished loop on that vertex. Now we can use the argument that each vertex is essentially definable by its distinguished loop. Let $\mathscr{D}'$ be the subcategory

$$\begin{tikzcd} e \arrow[r, bend left, "s \cdot r"] \arrow[r, bend right, "t \cdot r"'] & e \end{tikzcd}$$

of $\mathscr{D}$. Both morphisms $s \cdot r$ and $t \cdot r$ are *idempotent*: they satisfy the equations

$$s \cdot r \cdot s \cdot r = s \cdot r, \qquad t \cdot r \cdot t \cdot r = t \cdot r.$$

Given a model $A : (\mathscr{D}')^{op} \to \mathsf{Set}$ of $\mathscr{D}'$, we can construct a directed reflexive graph from it by taking $A(e)$ to be the set of edges of A and the image of $A(e)$ under $A(s \cdot r)$ as the set of vertices. Showing the equivalence $[\mathscr{D}^{op}, \mathsf{Set}] \simeq [(\mathscr{D}')^{op}, \mathsf{Set}]$ is then easy.

The above example gives the gist of the basic motivation for studying Morita theory. Given an algebraic theory $\mathscr{T}$, we study its category $\mathsf{Alg}(\mathscr{T})$ of algebras and ask whether there is any other algebraic theory $\mathscr{T}'$ such that the categories $\mathsf{Alg}(\mathscr{T})$ and $\mathsf{Alg}(\mathscr{T}')$ are equivalent. A similar question asks for a characterisation of all algebraic theories $\mathscr{T}'$ that have, up to an equivalence, the same category of algebras as a given algebraic theory $\mathscr{T}$.

Definition 1.4.2. We say that two algebraic theories $\mathscr{T}$ and $\mathscr{T}'$ are *Morita equivalent* if their categories of algebras are equivalent as categories; that is, if

$$\mathsf{Alg}(\mathscr{T}) \simeq \mathsf{Alg}(\mathscr{T}')$$

holds.

The original motivation for the theory of Morita equivalence comes from module theory. There the question was the following: given two rings R and S, when are the categories ${}_R\mathsf{Mod}$ and ${}_S\mathsf{Mod}$ of left modules over R and S categorically equivalent? This problem was solved in the paper [70] by Morita.

Dukarm in [32] provided a characterisation of Morita equivalent Lawvere theories, and this characterisation was generalised for the case of S-sorted theories in Adámek, Sobral and Sousa [7]. We will state the more general result of [7].

Theorem 1.4.3. *Two S-sorted theories $\mathscr{T}$ and $\mathscr{T}'$ are Morita equivalent if and only if $\mathscr{T}' \simeq \mathscr{T}_u$ for some pseudoinvertible idempotent u in $\mathscr{T}$.*

The notion of a pseudoinvertible idempotent used in the previous theorem is quite involved; here we shall only comment that it is a generalisation of the phenomenon that occurred in Example 1.4.1, where we constructed a new "theory" $\mathscr{D}'$ from $\mathscr{D}$ using certain idempotents present in $\mathscr{D}$. We will give a proper generalisation of the notion of a pseudoinvertible idempotent in Chapter 3, and it will coincide with the usual notion of a pseudoinvertible idempotent for ordinary algebraic theories.

1.5 Birkhoff's variety theorem

Birkhoff's variety theorem is a celebrated result from [16] that characterises equationally defined subcategories of a category of algebras. In short, let Σ be a (one-sorted) signature in the sense of Section 1.1 and let us denote by $\Sigma\text{-}\mathsf{Alg}$ the category of all Σ-algebras and their homomorphisms. Given a set E of Σ-equations, there is a full subcategory $\mathscr{A}$ of $\Sigma\text{-}\mathsf{Alg}$ of all Σ-algebras that satisfy every equation in the set E. Every such subcategory has nice closure properties: it is closed under forming products of algebras, regular quotients (homomorphic images) of algebras, and subalgebras. The surprising fact is that any full subcategory $\mathscr{A}$ of $\Sigma\text{-}\mathsf{Alg}$ that satisfies these closure properties is essentially an equationally defined subcategory of $\Sigma\text{-}\mathsf{Alg}$.

We shall introduce the notions needed for formally stating the Birkhoff theorem. To be able to speak about equationally defined subcategories, we first need a definition of an equation in an algebraic theory.

Definition 1.5.1. Given an algebraic theory $\mathscr{T}$, an *equation* $l \approx r$ is a pair $l, r : s \to t$ of morphisms in $\mathscr{T}$. An algebra $A : \mathscr{T}^{op} \to \mathsf{Set}$ *satisfies* the equation $l \approx r$ if $A(l) = A(r)$ holds.

Example 1.5.2. If $\mathscr{T}$ is the one-sorted algebraic theory for monoids, the elements of $\mathscr{T}(F1, Fn)$ correspond to words in an n-element alphabet. If we take 1 to be the one-element set $\{x\}$ and 2 to be the set $\{a, b\}$, then the two morphisms $l : F1 \to F2$ and $r : F1 \to F2$ defined by

$$l(x) = ab, \qquad r(x) = ba$$

constitute an equation in $\mathscr{T}$. Monoids $A : \mathscr{T}^{op} \to \mathsf{Set}$ satisfying this equation are precisely the commutative ones.

Definition 1.5.3. Let E be a set of equations in an algebraic theory $\mathscr{T}$. We say that the full subcategory $\mathscr{A}$ of $\mathsf{Alg}(\mathscr{T})$ spanned by algebras satisfying the equations from E is the *variety* generated by E.

Example 1.5.4. The category CMon of commutative monoids and their homomorphisms is the variety in Mon generated by the equation $l \approx r$ from Example 1.5.2.

Varieties have nice closure properties in the category $\Sigma\text{-}\mathsf{Alg}$ of Σ-algebras.

Remark 1.5.5. Any variety $\mathscr{A}$ in $\Sigma\text{-}\mathsf{Alg}$ is closed in $\Sigma\text{-}\mathsf{Alg}$ under

1. regular quotients,

2. subalgebras,

3. products,

4. filtered colimits.

Recall that an object B in $\mathscr{A}$ is a *regular quotient* of an object A if there is a regular epimorphism $e : A \to B$ in $\mathscr{A}$, that is, if there is a pair $x_1 : X \to A$ and $x_2 : X \to A$ of morphisms such that in the following diagram

$$\begin{tikzcd} X \arrow[r, bend left, "x_1"] \arrow[r, bend right, "x_2"'] & A \arrow[r, "e"] & B \end{tikzcd}$$

the morphism e is a coequaliser of x_1 and x_2. In algebraic categories, having a regular quotient B of A corresponds to saying that B is (isomorphic to) the quotient algebra A/θ of A generated by the kernel congruence θ of e.

For our needs, we say that in any algebraic category, an algebra A is a *subalgebra* of B if there exists a mono $m : A \to B$, that is, a morphism that has the right cancellation property:

$$m \cdot x = m \cdot y \text{ implies } x = y$$

for any pair of morphisms x and y. This definition coincides (up to an isomorphism of algebras) with the usual notion of a subalgebra from universal algebra.

Since the (categorical) product of two algebras again coincides (up to an isomorphism of algebras) with the notion of a cartesian product of algebras, the last interesting closure property is the closure under filtered colimits. Recall from Example 1.3.2 that a category $\mathscr{D}$ is filtered if $\mathscr{D}$-colimits commute with finite limits in Set.

With these definitions, we can now state the Birkhoff theorem. Observe that, in contrast to the original Birkhoff theorem from [16], the closedness requirements include closure under filtered colimits.

Theorem 1.5.6 ([5]). *Let $\mathscr{A}$ be a full subcategory of $\mathsf{Alg}(\mathscr{T})$ for some algebraic theory $\mathscr{T}$. The category $\mathscr{A}$ is (equivalent to) a variety if and only if it is closed in $\mathsf{Alg}(\mathscr{T})$ under*

1. *regular quotients,*

2. *subalgebras,*

3. *products,*

4. *filtered colimits.*

The closure properties 1.–3. from the above theorem are dubbed "HSP" conditions. The additional requirement for closure under filtered colimits *cannot* be in general excluded. The reason this closure does not appear in [16] is that it is not needed in the special case of a *one-sorted* theory. It is another of the successes of categorical algebra that the need for this closure property, often mistakenly omitted, has been pointed out in [6].

Remark 1.5.7. Let E be a set of equations in an algebraic theory $\mathscr{T}$. For any such set we can define a new set $\mathsf{Con}(E)$ of all consequences of E. The set $\mathsf{Con}(E)$ contains all equations $l' \approx r'$ such that any algebra A in $\mathsf{Alg}(\mathscr{T})$ satisfying all equations $l \approx r$ from E also satisfies $l' \approx r'$. This set $\mathsf{Con}(E)$ constitutes a *congruence* on the category $\mathscr{T}$. That is, there is a quotient category $\mathscr{T}/\mathsf{Con}(E)$ defined as having the same objects as $\mathscr{T}$, and for two objects t and t', the hom-set $\mathscr{T}/\mathsf{Con}(E)(t, t')$ is the quotient of the hom-set $\mathscr{T}(t, t')$ by the equivalence relation

$$\{(l, r) \mid l : t \to t', r : t \to t' \text{ and } l \approx r \in \mathsf{Con}(E)\}.$$

The fact that $\mathsf{Con}(E)$ is a congruence precisely states that the identities and composition in $\mathscr{T}/\mathsf{Con}(E)$ can be defined on the equivalence classes of morphisms in the usual manner. Moreover, there is a “theory map”

$$e : \mathscr{T} \to \mathscr{T}/\mathsf{Con}(E),$$

i.e., a functor preserving finite coproducts (the structure of the theory). Algebraic theories and theory maps form a category, and $e : \mathscr{T} \to \mathscr{T}/\mathsf{Con}(E)$ then is a regular quotient of algebraic theories. The variety defined by the set E of equations is equivalent to the algebraic category $\mathsf{Alg}(\mathscr{T}/\mathsf{Con}(E))$.

With this observation we can rephrase Theorem 1.5.6 as follows: *There is a one-to-one correspondence between regular quotients $e : \mathscr{T} \to \mathscr{T}'$ of algebraic theories and their morphisms, and full subcategories $\mathscr{A}$ of $\mathsf{Alg}(\mathscr{T})$ that are closed in $\mathsf{Alg}(\mathscr{T})$ under regular quotients, subalgebras, products, and filtered colimits.*

1.6 Monads

Sometimes it is more convenient to use a different formalism capturing the notion of an algebraic theory. We shall sometimes use the formalism of *monads*. As a first approximation, a monad can be thought of as the abstraction of the collection of all terms of an algebraic theory, together with abstract rules concerning their substitution.

Example 1.6.1. Consider the example of monoids. Given a set X of variables, the set of monoid terms in the set X of variables is

$$TX = X^*,$$

i.e., the words in the alphabet X. As we can form the set X^* of all words for any alphabet X, this gives us an endofunctor

$$T : \mathsf{Set} \to \mathsf{Set}$$

of sets, acting on a mapping $f : X \to Y$ to give a mapping $Tf = f^* : TX \to TY$ that maps a word from alphabet X to a word from alphabet Y by “element-wise translation using f”.

Since any letter in an alphabet X can be thought of as a one-letter word, we get a *unit* mapping

$$\eta_X : X \to TX$$

for every alphabet X. Given a “word of words in alphabet X”, i.e., the set TTX, we have a “flattening” mapping

$$\mu_X : TTX \to TX$$

that concatenates all the words into a single one: for example,

$$\mu_X((xy)(yzx)(x)) = xyyzzxx.$$

The data in the above example satisfy many intuitively obvious laws, which we will abstract into axioms in the following definition.

Definition 1.6.2. A *monad* (T, η, μ) on a category $\mathscr{X}$ is an endofunctor $T : \mathscr{X} \to \mathscr{X}$ together with two natural transformations $\eta : 1_{\mathscr{X}} \to T$ and $\mu : TT \to T$, subject to the following axioms:

1. The *unit* triangles

$$\begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, "1_T"'] & TT \arrow[d, "\mu"] \\ & T \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} TT \arrow[d, "\mu"'] & T \arrow[l, "T\eta"'] \arrow[dl, "1_T"] \\ T & \end{tikzcd}$$

   commute, and

2. the *associativity* square

$$\begin{tikzcd} TTT \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & TT \arrow[d, "\mu"] \\ TT \arrow[r, "\mu"'] & T \end{tikzcd}$$

   commutes.

Definition 1.6.3. An *algebra* for a monad (T, η, μ) on $\mathscr{X}$ is a morphism $a : TX \to X$ in $\mathscr{X}$ such that the diagrams

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "1_X"'] & TX \arrow[d, "a"] \\ & X \end{tikzcd}$$

and

$$\begin{tikzcd} TTX \arrow[r, "Ta"] \arrow[d, "\mu_X"'] & TX \arrow[d, "a"] \\ TX \arrow[r, "a"'] & X \end{tikzcd}$$

commute. Given two algebras $a : TX \to X$ and $b : TY \to Y$ for a monad (T, η, μ), a morphism $h : X \to Y$ in $\mathscr{X}$ is a *homomorphism* from a to b if the square

$$\begin{tikzcd} TX \arrow[r, "Th"] \arrow[d, "a"'] & TY \arrow[d, "b"] \\ X \arrow[r, "h"'] & Y \end{tikzcd}$$

commutes.

Example 1.6.4. It is now straightforward to check that a monoid, as introduced in Definition 1.1.1, gives rise to an algebra for the monoid monad (T, η, μ) introduced in Example 1.6.1, and that conversely any algebra $a : TX \to X$ for this monad is a monoid, with the multiplication operation $x * y$ being defined as $a(xy)$, and the unit e being defined as $a(\epsilon)$. Analogously, homomorphisms of monoids in the sense of universal algebra are exactly the homomorphisms of the respective algebras for the monoid monad.

We shall use the formalism of monads e.g. in our treatment of Birkhoff's theorem in Chapter 6.

Chapter 2

Preliminary notions

As we concern ourselves with categorical universal algebra in the setting of *enriched* categories in this thesis, it is vital that we give a short overview of some of the basic notions that will be used throughout the thesis. Since the theory of enriched categories is very rich and broad, we shall not attempt to give a full account of it. Instead, we introduce its basic notions more for the need of establishing the notation for the rest of the text; and the choice of the topics covered in this chapter hints at what will follow in the rest of the text.

Thus we give prominent examples of the categories in which we can enrich, review the basics of limits and colimits (and colimit cocompletions) in the enriched setting, introduce the basic algebraic notions and discuss some of the technical definitions that arise when we study algebraic phenomena in this level of generality.

Structure of the chapter.

1. We recall some of the basic notions of enriched category theory in Section 2.1. We shortly discuss cocompletions of enriched categories under weighted colimits, and commutativity of limits and colimits.

2. In Section 2.2 we give definitions of a Ψ-theory and Ψ-algebra parametric in the choice of a class Ψ of weights, and discuss the connection with cocompletions.

2.1 Colimits in enriched categories, cocompletions

In this section we quickly recall the notions of a limit and colimit for enriched categories, and cocompletions of categories. For a deeper exposition of the enriched notions we refer to [41].

Enriched categories

Assumption 2.1.1. From now on, whenever we speak of categories enriched in a category $\mathscr{V}$, this $\mathscr{V}$ is assumed to be a complete and cocomplete symmetric monoidal closed category $\mathscr{V} = (\mathscr{V}_0, \otimes, I, [-,-])$. ($I$ being the unit of the tensor and $[-,-]$ being the internal hom.)

Notation 2.1.2. We shall denote by $\mathscr{V}$-CAT the 2-category of all $\mathscr{V}$-categories, $\mathscr{V}$-functors and $\mathscr{V}$-natural transformations.

To avoid heavy notation concerning $\mathscr{V}$-categories, $\mathscr{V}$-functors and $\mathscr{V}$-natural transformations, we use the usual convention present, e.g., in [41].

Notation 2.1.3. Whenever the base category $\mathscr{V}$ is fixed, we stop writing the prefix "$\mathscr{V}$-" in $\mathscr{V}$-category, $\mathscr{V}$-functor, etc. Instead, we speak simply of a category, a functor, etc. When it is necessary, we distinguish a $\mathscr{V}$-category and a Set-category by dubbing the latter one *ordinary*.

Example 2.1.4. In various parts of the thesis we use many examples of the categories $\mathscr{V}$ in which we enrich.

1. As in Chapter 1, we denote by Set the category of all sets and mappings, the tensor operation being the cartesian product, and the unit $I = \mathbf{1}$ being "the" one-element set.

2. We denote by Pos the category of all posets and monotone maps equipped with the cartesian product as tensor, and the one-element poset as unit. The category of all preorders and monotone maps will be denoted by Pre.

3. We use Cat for the category of small categories and functors, with the cartesian tensor and with the one-morphism category $I = \mathbf{1}$ as unit.

4. From the previous examples we can construct the categories $\mathsf{Set}_\bullet$ of *pointed* sets and point-preserving maps, $\mathsf{Pos}_\bullet$ of pointed posets and point-preserving monotone maps, and $\mathsf{Cat}_\bullet$ of pointed categories and point-preserving functors.

5. We denote by Ab the category of abelian groups and group homomorphisms, the tensor $\otimes$ being the tensor product of groups, and the unit I being the additive group $\mathbb{Z}$ of integers.

6. Let $\mathscr{V}_0$ be a complete lattice equipped with a monotone, commutative and associative *tensor* operation $\otimes$ with unit e. If for all $v \in \mathscr{V}_0$ the monotone map $- \otimes v : \mathscr{V}_0 \to \mathscr{V}_0$ has a right adjoint $[v, -] : \mathscr{V}_0 \to \mathscr{V}_0$, we say that $\mathscr{V} = (\mathscr{V}_0, \otimes, e)$ is a *quantale*. For example:

   - The 2-element boolean algebra $\mathbf{2}$ forms a quantale with the tensor being the conjunction operation $\wedge$.
   - The real half-line $[0, \infty]$ with the natural ordering $\geqslant$ and addition as tensor forms a quantale. (For the addition operation, $\infty + x = x + \infty = \infty$ holds for all $x \in [0, \infty]$.)

7. In Chapter 7 we shall work with the monoidal category **Gray** of 2-categories and 2-functors equipped with the *Gray* tensor product. As the Gray product is quite involved, we defer its description for the later chapters.

Sometimes we will need to work with the underlying *ordinary* category of a $\mathscr{V}$-category.

Notation 2.1.5. Given a $\mathscr{V}$-category $\mathscr{K}$, we denote its *underlying ordinary category* by $\mathscr{K}_0$. Analogously, given a $\mathscr{V}$-functor $H : \mathscr{K} \to \mathscr{L}$, we denote by $H_0 : \mathscr{K}_0 \to \mathscr{L}_0$ its *underlying ordinary functor.*

Notation 2.1.6. We will denote by $\mathbf{1}$ the *unit category* with one object $*$ and the hom $\mathbf{1}(*, *) = I$.

Weighted limits and colimits When working with enriched categories, it is necessary to introduce *weighted* limits and colimits, as they are the appropriate generalisation of ordinary (conical) limits and colimits in ordinary category theory.

Notation 2.1.7. Given a small[1] category $\mathscr{D}$, we denote by $[\mathscr{D}^{op}, \mathscr{V}]$ the category of presheaves on $\mathscr{D}$. The objects of $[\mathscr{D}^{op}, \mathscr{V}]$ are functors $\varphi : \mathscr{D}^{op} \to \mathscr{V}$, called *weights*, and the hom $[\mathscr{D}^{op}, \mathscr{V}](\varphi, \psi)$ is computed by the *end*

$$\int_d [\varphi(d), \psi(d)],$$

an instance of a *weighted limit*; these are introduced below.

Definition 2.1.8 (Hat and tilde conjugates). Given a diagram $D : \mathscr{D} \to \mathscr{K}$ (with $\mathscr{D}$ a small category), we define its *tilde-conjugate* $\widetilde{D} : \mathscr{K} \to [\mathscr{D}^{op}, \mathscr{V}]$ by the assignment

$$X \mapsto \mathscr{K}(D-, X)$$

for every X in $\mathscr{K}$. The action on morphisms is defined as expected. The *hat-conjugate* $\widehat{D}$ of the diagram D is the functor $\widehat{D} : \mathscr{K} \to [\mathscr{D}, \mathscr{V}]^{op}$ defined by the assignment

$$X \mapsto \mathscr{K}(X, D-)$$

on objects of $\mathscr{K}$.

Definition 2.1.9 (Weighted limits and colimits). A *colimit of* $D : \mathscr{D} \to \mathscr{K}$ *weighted by* $\varphi : \mathscr{D}^{op} \to \mathscr{V}$ is an object $\varphi \star D$ together with an isomorphism

$$\mathscr{K}(\varphi \star D, X) \cong [\mathscr{D}^{op}, \mathscr{V}](\varphi, \widetilde{D}X)$$

that is natural in X.

A *limit of* $D : \mathscr{D}^{op} \to \mathscr{K}$ *weighted by* $\varphi : \mathscr{D}^{op} \to \mathscr{V}$ is an object $\{\varphi, D\}$ together with an isomorphism

$$\mathscr{K}(X, \{\varphi, D\}) \cong [\mathscr{D}^{op}, \mathscr{V}]^{op}(\widehat{D}X, \varphi)$$

natural in X.

Remark 2.1.10. Note our use of a diagram $D : \mathscr{D}^{op} \to \mathscr{K}$ instead of $D : \mathscr{D} \to \mathscr{K}$ in the definition of a weighted *limit*. Using the opposite category of $\mathscr{D}$ as the domain category of a limit diagram enables us to weigh both colimits and limits by *presheaves* of the form $\varphi : \mathscr{D}^{op} \to \mathscr{V}$. This convention is useful when dealing with notions pertaining to *classes of weights*. Of course, the "alternative" definition of a weighted limit of $D : \mathscr{D} \to \mathscr{K}$ and $\varphi : \mathscr{D} \to \mathscr{V}$ as an object $\{\varphi, D\}$ together with an isomorphism

$$\mathscr{K}(X, \{\varphi, D\}) \cong [\mathscr{D}, \mathscr{V}]^{op}(\widehat{D}X, \varphi)$$

natural in X is equivalent to ours.

---

[1] I.e., a category having a *set* of objects.

Example 2.1.11. We give an example of colimits and limits weighted by particularly nice weights.

1. Given a diagram $D : \mathbf{1} \to \mathscr{K}$ in $\mathscr{K}$ with the domain being the unit category, the diagram is determined by the image of the unique object of $\mathbf{1}$ under D. Say we have a diagram $D : \mathbf{1} \to \mathscr{K}$ with $D(*) = Z$ and a weight $\varphi : \mathscr{D}^{op} \to \mathscr{V}$ with $\varphi(*) = A$. The weighted colimit $\varphi * D$ in $\mathscr{K}$ is called a *tensor of Z by A* and is denoted by $A \bullet Z$. In this case the universal property defining the colimit reduces to an isomorphism
$$\mathscr{K}(A \bullet Z, X) \cong \mathscr{V}(A, \mathscr{K}(Z, X))$$
natural in X.

2. The limit notion dual to the one above is called a *cotensor*. Given a diagram $D : \mathbf{1}^{op} \to \mathscr{K}$ in $\mathscr{K}$ and a weight $\varphi : \mathbf{1}^{op} \to \mathscr{V}$ with $D(*) = Z$ and $\varphi(*) = B$, a *cotensor of Z by B* in $\mathscr{K}$ is the weighted limit $\{\varphi, D\}$, denoted by $B \pitchfork Z$ (or Z^B) such that there is an isomorphism
$$\mathscr{K}(X, B \pitchfork Z) \cong \mathscr{V}(B, \mathscr{K}(X, Z))$$
natural in X.

Definition 2.1.12. Given a class Φ of weights, a category $\mathscr{K}$ is called Φ-*cocomplete* if for any $\varphi : \mathscr{D}^{op} \to \mathscr{V}$ from Φ and any diagram $D : \mathscr{D} \to \mathscr{K}$ the colimit $\varphi \star D$ exists in $\mathscr{K}$. The category $\mathscr{K}$ is Φ-*complete*, if $\mathscr{K}^{op}$ is Φ-cocomplete, i.e., if for any weight $\varphi : \mathscr{D}^{op} \to \mathscr{V}$ from Φ and any diagram $D : \mathscr{D}^{op} \to \mathscr{K}$ the limit $\{\varphi, D\}$ exists in $\mathscr{K}$.

A functor $F : \mathscr{K} \to \mathscr{L}$ is Φ-*cocontinuous* if for any colimit $\varphi \star D$ weighted by φ in Φ the colimit $\varphi \star (F \cdot D)$ exists, and the canonical morphism
$$\varphi \star (F \cdot D) \to F(\varphi \star D)$$
is an isomorphism. Likewise, $F : \mathscr{K} \to \mathscr{L}$ is Φ-*continuous* if for any *limit* $\{\varphi, D\}$ weighted by φ in Φ the limit $\{\varphi, F \cdot D\}$ exists, and the canonical morphism
$$F\{\varphi, D\} \to \{\varphi, F \cdot D\}$$
is an isomorphism.

Free cocompletions of categories There is a deep interplay between the theory of free cocompletions of categories and between the theory of (generalised) algebraic theories. We refer again to [41] for a comprehensive account of cocompletions of categories and give an outline of the basics here.

Remark 2.1.13. Recall that in Notation 2.1.7 we posited that for any *small* category $\mathscr{D}$ there exists a category $[\mathscr{D}^{op}, \mathscr{V}]$ of presheaves on $\mathscr{D}$. Considering now a not necessarily small category $\mathscr{K}$, dealing with "category" $[\mathscr{K}^{op}, \mathscr{V}]$ runs into size issues: the homs of $[\mathscr{K}^{op}, \mathscr{V}]$ may not be objects of $\mathscr{V}$. However, this problem may be overcome by considering the *legitimate* category $\mathcal{P}(\mathscr{K})$ of *small* presheaves instead of $[\mathscr{K}^{op}, \mathscr{V}]$.

Definition 2.1.14 ([27])**.** We say that a presheaf $F : \mathscr{K}^{op} \to \mathscr{V}$ is *small* if F is of the form $\mathrm{Lan}_{J^{op}}\varphi$ for some weight $\varphi : \mathscr{D}^{op} \to \mathscr{V}$ and for some $J : \mathscr{D} \to \mathscr{K}$. (Recall that a weight has a small domain).

Remark 2.1.15. From the above definition we can quickly see that any weight is a small preshaf. Moreover, given any category $\mathscr{K}$ and an object X in $\mathscr{K}$, the representable functor $\mathscr{K}(-,X) : \mathscr{K}^{op} \to \mathscr{V}$ is also a small presheaf.

Notation 2.1.16. We denote by $\mathcal{P}(\mathscr{K})$ the category of small presheaves; we think of $\mathcal{P}(\mathscr{K})$ as of a full subcategory of $[\mathscr{K}^{op}, \mathscr{V}]$, even though the latter one is in fact *illegitimate* due to size issues. We will use the notation

$$Y_{\mathscr{K}} : \mathscr{K} \to \mathcal{P}(\mathscr{K})$$

for the "Yoneda embedding" defined by the assignment

$$X \mapsto \mathscr{K}(-,X).$$

We abuse the notation slightly by denoting in the same way the restricted embedding into the category of small presheaves and the proper Yoneda embedding $Y : \mathscr{K} \to [\mathscr{K}^{op}, \mathscr{V}]$.

Remark 2.1.17. In fact, the embedding

$$Y_{\mathscr{K}} : \mathscr{K} \to \mathcal{P}(\mathscr{K})$$

exhibits $\mathcal{P}(\mathscr{K})$ as a *free cocompletion of* $\mathscr{K}$ *under all colimits*. That is, $\mathcal{P}(\mathscr{K})$ is cocomplete and the embedding $Y_{\mathscr{K}} : \mathscr{K} \to \mathcal{P}(\mathscr{K})$ has the universal property such that given any cocomplete category $\mathscr{L}$ and any $F : \mathscr{K} \to \mathscr{L}$, there exists, up to isomorphism, a unique cocontinuous functor $F^\sharp : \mathcal{P}(\mathscr{K}) \to \mathscr{L}$ such that the diagram

$$\begin{tikzcd}
\mathscr{K} \arrow[r, "Y_{\mathscr{K}}"] \arrow[dr, "F"'] & \mathcal{P}(\mathscr{K}) \arrow[d, dotted, "F^\sharp"] \\
 & \mathscr{L}
\end{tikzcd}$$

commutes up to isomorphism.

Example 2.1.18. For a *small* category $\mathscr{D}$, the free cocompletion of $\mathscr{D}$ under all colimits is the presheaf category $\mathcal{P}(\mathscr{D}) = [\mathscr{D}^{op}, \mathscr{V}]$, with the unit $Y : \mathscr{D} \to [\mathscr{D}^{op}, \mathscr{V}]$ being the Yoneda embedding.

Given a class Φ of weights, we may use the Yoneda embedding $Y_{\mathscr{K}} : \mathscr{K} \to \mathcal{P}(\mathscr{K})$ to introduce free cocompletion of $\mathscr{K}$ under colimits weighted by weights φ in Φ.

Notation 2.1.19 (Free cocompletions). The *free cocompletion*

$$Z^{\Phi}_{\mathscr{K}} : \mathscr{K} \to \Phi(\mathscr{K})$$

of the category $\mathscr{K}$ under Φ-colimits exists for any class of weights and any category (see [41]), and it is given as the factorisation

$$\begin{tikzcd}
\mathscr{K} \arrow[r, "Z^{\Phi}_{\mathscr{K}}"] \arrow[dr, "Y_{\mathscr{K}}"'] & \Phi(\mathscr{K}) \arrow[d, "W^{\Phi}_{\mathscr{K}}"] \\
 & \mathcal{P}(\mathscr{K})
\end{tikzcd}$$

of the Yoneda embedding given by the *closure of $\mathscr{K}$ in $\mathcal{P}(\mathscr{K})$ under Φ-colimits.* Since it is given by a closure, $\Phi(\mathscr{K})$ can be thought of as a full subcategory of $\mathcal{P}(\mathscr{K})$ via the fully faithful inclusion $W^{\Phi}_{\mathscr{K}} : \Phi(\mathscr{K}) \to \mathcal{P}(\mathscr{K})$.

The *free completion* of $\mathscr{K}$ under Φ-limits is then given by

$$(Z^{\Phi}_{\mathscr{K}^{op}})^{op} : \mathscr{K} \to (\Phi(\mathscr{K}^{op}))^{op}$$

using the free cocompletion construction.

Remark 2.1.20. Consider now the *2-category*

$$\Phi\text{-}\mathsf{COCTS}$$

of all Φ-cocomplete categories, all Φ-cocontinuous functors and all natural transformations. There is an obvious forgetful 2-functor

$$U_{\Phi} : \Phi\text{-}\mathsf{COCTS} \to \mathscr{V}\text{-}\mathsf{CAT}$$

Since the free cocompletion of a category under a class of colimits is determined only up to equivalence of categories, we cannot claim that this 2-functor has a strict left (2-)adjoint. However, there is a *pseudofunctor*[2]

$$F_{\Phi} : \mathscr{V}\text{-}\mathsf{CAT} \to \Phi\text{-}\mathsf{COCTS}$$

given by the free cocompletion procedure that gives rise to an equivalence

$$\Phi\text{-}\mathsf{COCTS}(F_{\Phi}(\mathscr{K}), \mathscr{L}) \simeq \mathscr{V}\text{-}\mathsf{CAT}(\mathscr{K}, U_{\Phi}(\mathscr{L}))$$

of categories for any $\mathscr{K}$ and a Φ-cocomplete $\mathscr{L}$. Without delving too deep into the technical issues, we comment that these data give rise to a *pseudoadjunction*

$$F_{\Phi} \dashv U_{\Phi} : \Phi\text{-}\mathsf{COCTS} \to \mathscr{V}\text{-}\mathsf{CAT}$$

with unit $Z^{\Phi}_{\mathscr{K}} : \mathscr{K} \to \Phi(\mathscr{K})$ for $\mathscr{K}$ in $\mathscr{V}$-CAT.

Classes of weights We now introduce some technical notions concerning classes of weights, and some interesting classes of weights.

Definition 2.1.21. A class Φ of weights is *locally small* (see [48]) if the free cocompletion $\Phi(\mathscr{D})$ of $\mathscr{D}$ under Φ-colimits is a small category for every small category $\mathscr{D}$.

When dealing with cocompletions, it will be sometimes easier for us to consider classes of weights that are *saturated*:

Definition 2.1.22 ([8])**.** The *saturation* Φ^* of Φ is the largest class of weights such that the 2-categories Φ-COCTS and Φ^*-COCTS coincide. In case $\Phi^* = \Phi$, the class Φ is called *saturated*.

---

[2] A *pseudofunctor* is a "functor up to isomorphism". As we will not delve into the technical details, we omit the precise definitions here. See Chapter 7 or Chapter 7 of Volume 1 of [19].

In more elementary terms, the saturation Φ^* of Φ is the largest class of weights such that whenever a category is Φ-cocomplete it is also Φ^*-cocomplete, and whenever a functor is Φ-cocontinuous it is also Φ^*-cocontinuous.

Remark 2.1.23. Saturation of classes of weights induces a closure operator

$$\Phi \mapsto \Phi^*$$

on the ordered collection of all classes of weights, since the conditions

- $\Phi \subseteq \Phi^*$,
- whenever $\Phi \subseteq \Psi$ holds, then $\Phi^* \subseteq \Psi^*$ holds,
- $\Phi^{**} = \Phi^*$

all hold for any pair Φ and Ψ of classes of weights.

For any class Φ of weights and any category $\mathscr{K}$ it is easy to prove that $\Phi(\mathscr{K}) = \Phi^*(\mathscr{K})$; thus cocompletion-wise it does not matter if we deal with a class Φ of weights or with its saturation. Moreover, considering a saturated class Φ of weights from the very beginning allows for a nice description of the Φ-cocompletion procedure for $\mathscr{K}$.

Notation 2.1.24. Let $\mathscr{D}$ be a small category and Φ a class of weights. We denote by

$$\Phi[\mathscr{D}]$$

the full subcategory of $[\mathscr{D}^{op}, \mathscr{V}]$ spanned by weights in Φ.

Remark 2.1.25 ([48])**.** For saturated classes Φ of weights, the Φ-cocompletion of a category $\mathscr{K}$ can be done in *one step*, i.e., the closure of $\mathscr{K}$ under Φ-colimits in $\mathcal{P}(\mathscr{K})$ consists of adding objects of the form $\psi * YD$ for a weight $\psi : \mathscr{D}^{op} \to \mathscr{V}$ in $\Phi[\mathscr{D}]$ and for some diagram $D : \mathscr{D} \to \mathscr{K}$.

For not necessarily saturated classes Φ, adding the objects of the form $\psi * YD$ is the first (nontrivial) step $\Phi_1(\mathscr{K})$ of the transfinite cocompletion process for the given category $\mathscr{K}$. That is, we have a factorisation

$$\begin{tikzcd} \mathscr{K} \arrow[r, "Z_1"] & \Phi_1(\mathscr{K}) \arrow[r, "V_1"] & \Phi(\mathscr{K}) \end{tikzcd}$$

of the unit $Z^{\Phi}_{\mathscr{K}} : \mathscr{K} \to \Phi(\mathscr{K})$. Saturatedness of Φ then implies the equality $\Phi_1(\mathscr{K}) = \Phi(\mathscr{K})$.

Example 2.1.26. Let us consider the case of ordinary categories ($\mathscr{V} = \mathsf{Set}$) and look at some of the most important classes of weights.

1. There is the trivial empty class $\Psi = \varnothing$ of weights. It is clearly locally small.
2. The class of *all* weights will be denoted by $\mathcal{P}$. This notation aligns with the fact that for a category $\mathscr{K}$ its free cocompletion under all (small) colimits is denoted by $\mathcal{P}(\mathscr{K})$, see Notation 2.1.16. This class is not locally small in general.
3. Let Ψ be the saturated class of weights such that a Ψ-cocomplete category is one with finite coproducts. We shall denote this class by Π. This is a locally small class of weights, since it is small [41].
4. The saturated class Ψ of weights such that a Ψ-cocomplete category is one with *finite colimits* is also locally small.

Commutativity of limits and colimits Recall from Section 1.3 that there is an important connection between *finite products* and *sifted colimits* that is important in categorical universal algebra. In the ordinary setting algebraic theories are categories $\mathscr{T}$ with *finite coproducts* and algebras are presheaves $\mathscr{T}^{op} \to \mathsf{Set}$ preserving *finite products*. It is true that the algebraic category $\mathsf{Alg}(\mathscr{T})$ is cocomplete, but the most important and best-behaved colimits are *sifted colimits*, defined as those that commute with finite products in the category Set. The importance of commutativity of limits and colimits for categorical universal algebra extends to the enriched context and to Ψ-theories.

Definition 2.1.27. Given classes Φ and Ψ of weights, we say that *Φ-colimits commute with Ψ-limits in $\mathscr{V}$* if for any $\varphi : \mathscr{D}^{op} \to \mathscr{V}$ in Φ the functor

$$\varphi * (-) : [\mathscr{D}, \mathscr{V}] \to \mathscr{V}$$

preserves Ψ-limits.

With a class of weights there are two important classes of weights obtained by a commutativity condition.

Definition 2.1.28. Consider two classes Φ and Ψ of weights.

1. We denote by Ψ^+ the class of all weights φ such that φ-colimits commute with Ψ-limits in $\mathscr{V}$. We name this class the class of *Ψ-flat* weights.

2. We denote by Φ^- the class of *Φ-presentable* weights; those ψ such that Φ-colimits commute with ψ-limits in $\mathscr{V}$.

Flat and presentable weights are connected via a Galois connection.

Remark 2.1.29. Let Φ and Ψ be classes of weights.

1. It follows straightforwardly from the definitions of flat and presentable weights that the inclusion $\Phi \subseteq \Psi^+$ holds precisely when the inclusion $\Psi \subseteq \Phi^-$ holds. This means that the two assignments $\Psi \mapsto \Psi^+$ and $\Phi \mapsto \Phi^-$ constitute a Galois connection on the ordered collection of all classes of weights.

2. The equalities
$$\Psi^+(\mathscr{D}) = \Psi^+[\mathscr{D}], \qquad \Phi^-(\mathscr{D}) = \Phi^-[\mathscr{D}]$$
hold for every small $\mathscr{D}$; and thus the classes Ψ^+ and Φ^- are saturated.

We will be especially interested in Ψ-flat weights for various classes Ψ.

Example 2.1.30. Consider again $\mathscr{V} = \mathsf{Set}$.

1. Take the empty class $\Psi = \varnothing$ of weights. For this class of weights, the class Ψ^+ of Ψ-flat weights is the class $\mathcal{P}$ of all weights: every weight is $\varnothing$-flat.

2. Consider the (saturated) class Π of weights for finite coproducts. This is a locally small class of weights, since it is small [41]. The weights in Ψ^+ are precisely the weights for *sifted* colimits [60]. We will study weights for sifted colimits in Chapter 5.

3. The class of Ψ-flat weights for the class Ψ of weights for finite colimits consists precisely of the weights for *filtered* colimits. The free cocompletion of $\mathscr{K}$ under filtered colimits is usually denoted by $\mathsf{Ind}(\mathscr{K})$ [3].

4. Take the class $\mathcal{P}$ of all weights. The class $\mathcal{P}^+$ of all $\mathcal{P}$-flat weights is denoted by $\mathcal{Q}$; we shall consider it again when dealing with *small-projective* weights in Definition 3.1.1.

2.2 Enriched algebraic theories

In this section we give a reasonably general definition of a theory that will cover enough interesting examples as instances of the definition. We also comment on *soundness*, a technical condition of classes of weights.

Definition 2.2.1 (Theories and algebras). A small Ψ-cocomplete category $\mathscr{T}$ is called a Ψ*-theory*. A Ψ-cocontinuous functor between Ψ-theories is called a Ψ*-theory morphism*.

Given a Ψ-theory $\mathscr{T}$, a $\mathscr{T}$*-algebra* is a Ψ-continuous presheaf $A : \mathscr{T}^{op} \to \mathscr{V}$. The full subcategory of the presheaf category $[\mathscr{T}^{op}, \mathscr{V}]$ spanned by all $\mathscr{T}$-algebras is denoted by $\Psi\text{-}\mathsf{Alg}(\mathscr{T})$.

Remark 2.2.2. Some authors prefer the *limit* definition of a theory: a Ψ-theory $\mathscr{T}$ then is a small Ψ-*complete* category, and a $\mathscr{T}$-algebra is a Ψ-*continuous* functor $A : \mathscr{T} \to \mathscr{V}$. This approach can be seen e.g. in [5]. We stick to the colimit definition of a theory. The reason is that the standard notation is biased w.r.t. colimits and free cocompletions rather than limits and free completions. Of course, it is possible, although notationally uncomfortable, to rephrase all the results with the dual definition of a theory.

Example 2.2.3. The definition of a Ψ-theory covers many important concepts of ordinary category theory (i.e., the case when $\mathscr{V} = \mathsf{Set}$) as examples.

1. By taking the empty class $\Psi = \varnothing$ of weights, any small category $\mathscr{T}$ is a Ψ-theory. The category of algebras for an $\varnothing$-theory $\mathscr{T}$ is the presheaf category $[\mathscr{T}^{op}, \mathsf{Set}]$.

2. For the case of Ψ being the class Π, a Π-theory is a category $\mathscr{T}$ with finite coproducts. This is our Definition 1.2.6 of an algebraic theory, and the dual of the definition of an algebraic theory from [5]. The algebras for $\mathscr{T}$ are finite-product-preserving presheaves in $[\mathscr{T}^{op}, \mathsf{Set}]$.

3. Taking the class Ψ of weights for finite colimits, a Ψ-theory is the (dual of the) notion of an *essentially algebraic theory* [3]. The algebras for such a theory constitute a locally finitely presentable category.

Remark 2.2.4. Since Ψ^+-colimits commute with Ψ-limits, the category $\Psi\text{-}\mathsf{Alg}(\mathscr{T})$ of algebras is closed in $[\mathscr{T}^{op}, \mathscr{V}]$ under Ψ^+-colimits for any Ψ-theory $\mathscr{T}$. In other words, there is always an inclusion

$$\Psi^+(\mathscr{T}) \subseteq \Psi\text{-}\mathsf{Alg}(\mathscr{T}).$$

We shall restrict our attention only to Ψ-theories for those classes Ψ of weights that satisfy a technical notion called *soundness*. A class Ψ of weights is sound if the requirement from Definition 2.1.28 for a weight φ to be Ψ-flat can be weakened:

Definition 2.2.5 (Soundness [1]). A class Ψ of weights is *sound* if a weight $\varphi : \mathscr{D}^{op} \to \mathscr{V}$ is in the class Ψ^+ of Ψ-flat weights whenever the functor

$$\varphi * (-) : [\mathscr{D}, \mathscr{V}] \to \mathscr{V}$$

preserves Ψ-limits *of representables*.

Remark 2.2.6. Soundness of a class Ψ of weights has pleasant consequences for studying Ψ-theories and their algebras. Consider the class Ψ to be sound. Given a Ψ-theory $\mathscr{T}$, the category of $\mathscr{T}$-algebras is a free Ψ-flat cocompletion of $\mathscr{T}$:

$$\Psi\text{-}\mathsf{Alg}(\mathscr{T}) = \Psi^+(\mathscr{T}). \tag{2.1}$$

First, we know that the inclusion $\Psi^+(\mathscr{T}) \subseteq \Psi\text{-}\mathsf{Alg}(\mathscr{T})$ holds always. Secondly, use that Ψ^+ is always saturated (see Remark 2.1.29); for any $\mathscr{T}$-algebra $\varphi : \mathscr{T}^{op} \to \mathscr{V}$ the functor $\varphi * (-)$ preserves Ψ-limits of representables, and thus φ is Ψ-flat by soundness of Ψ. It also follows from [48] that $\Psi^+(\mathscr{T})$ is complete and cocomplete.

Definition 2.2.5 is the *abstract formulation* of soundness from Remark 2.7 of [1]. In contrast to the *concrete* formulation of soundness (see Definition 2.2 of [1]), the abstract formulation generalises well to the enriched setting. For more details about soundness in the ordinary setting, see [1].

Example 2.2.7. We will list some examples of sound classes of weights.

1. The empty class $\varnothing$ of weights is sound. Indeed, every weight $\varphi : \mathscr{D}^{op} \to \mathscr{V}$ is in the class $\mathcal{P} = \varnothing^+$, and also every such weight φ preserves $\varnothing$-limits of representables, since this condition is void.

2. The class Π of weights for finite coproducts is sound [55]. For the case of $\mathscr{V} = \mathsf{Set}$ this implies, by (2.1), that the category of algebras $\Pi\text{-}\mathsf{Alg}(\mathscr{T})$ for a theory $\mathscr{T}$ is the free cocompletion $\mathsf{Sind}(\mathscr{T})$ of $\mathscr{T}$ under sifted colimits, see [5].

3. Similarly, the class Ψ of weights for finite colimits is sound as well. If in the ordinary case ($\mathscr{V} = \mathsf{Set}$) we denote the class by $\Psi = \mathsf{Lex}$, we get by (2.1) the well-known result
$$\mathsf{Lex}\text{-}\mathsf{Alg}(\mathscr{T}) = \mathsf{Ind}(\mathscr{T}),$$
stating that the locally finitely presentable category $\mathsf{Lex}(\mathscr{T}^{op}, \mathsf{Set})$ of finite-limit-preserving presheaves $\mathscr{T}^{op} \to \mathsf{Set}$ (which coincides with the category $\mathsf{Lex}\text{-}\mathsf{Alg}(\mathscr{T})$ of algebras for the Lex-theory $\mathscr{T}$) is precisely the free cocompletion of $\mathscr{T}$ under filtered colimits, see [3].

To sum up, whenever Ψ is sound the categories of algebras $\Psi\text{-}\mathsf{Alg}(\mathscr{T})$ for a Ψ-theory $\mathscr{T}$ are given by a free cocompletion

$$Z_{\mathscr{T}}^{\Phi^+} : \mathscr{T} \to \Psi^+(\mathscr{T})$$

under Ψ-flat colimits. This fact plays an important role in the development of the theory in the following chapters, and particularly when we deal with generalised Gabriel-Ulmer duality in Chapter 4.

Chapter 3

Enriched Morita equivalence

The study of the following problem has a long and fruitful history: Given two *theories*, when do they have the same *models*? In the context of general algebra, Morita was the first to successfully solve this question in [70] for the case of theories being rings, and models of the theory being modules over the chosen ring. More precisely: two rings R and S are called *Morita equivalent*, when the respective categories ${}_R\mathsf{Mod}$ and ${}_S\mathsf{Mod}$ of left modules are equivalent as categories.

Of course, the above question makes sense in a large variety of situations: consider, for example, two monoids M and N and their respective categories $\mathsf{Act}(M)$ and $\mathsf{Act}(N)$ of monoid actions. Again, the monoids M and N are called Morita equivalent if there is an equivalence of categories $\mathsf{Act}(M) \simeq \mathsf{Act}(N)$. This is the non-additive version of the problem of rings, and it has been studied independently by Banaschewski [9] and Knauer [49]. Perhaps more surprisingly, the characterisation of the most general situation occurring in universal algebra (given two algebraic theories, when do they give rise to equivalent categories of algebras?) is quite similar to the case of rings or monoids. Such results are due to Dukarm [32] (for the case of algebraic theories being Lawvere theories), and due to Adámek, Sobral and Sousa [7] (for the case of many-sorted algebraic theories).

For each of the above examples the theories can be seen as categories $\mathscr{T}$ (enriched in a suitable $\mathscr{V}$), possibly with an additional colimit structure given by a class Ψ of weights, and the models (or algebras) are the functors from $\mathscr{T}$ to $\mathscr{V}$ preserving the additional colimit structure: i.e., we deal with Ψ-theories and Ψ-algebras in the sense of Chapter 2. Rings are one-object categories enriched in the category Ab of abelian groups, modules are additive functors from $\mathscr{T}^{op}$ to Ab. Monoids are ordinary one-object categories, the category of actions over a given monoid is again the category of functors from the monoid into sets. Algebraic theories are ordinary categories with finite coproducts, algebras are presheaves preserving finite products.

In characterising Morita equivalence, one notion keeps reoccurring: namely the notion of a *pseudoinvertible idempotent*. With the proper definition of a pseudoinvertible idempotent, all of the Morita equivalence results can be stated as follows: two theories $\mathscr{T}'$ and $\mathscr{T}$, having the same sorts, are Morita equivalent if and only if $\mathscr{T}'$ is an idempotent modification of $\mathscr{T}$, given some choice of a pseudoinvertible idempotent in $\mathscr{T}$. In short, this is the main result of this chapter.

We thus first prove this very general Morita equivalence result, and then we show that the mentioned examples can be recovered very quickly. Moreover, we show some variants of the standard results in other enrichments, as they can be proved almost for free. For example, we get the characterisation of Morita equivalent partially ordered monoids, pointed categories, or algebraic theories that are 2-dimensional (i.e., enriched in categories).

Structure of the chapter

- In Section 3.1 we introduce the notion of a *Cauchy completion* that is central to the general theory of Morita equivalence, and we recall a basic enriched Morita-type result.
- After having defined all the required notions, Section 3.2 provides us with the core result of this chapter. We state what an $\mathscr{S}$-sorted theory is, what an *idempotent modification* of an $\mathscr{S}$-sorted theory is in general, and what it means for an idempotent to be *pseudoinvertible*. Then we prove that two $\mathscr{S}$-sorted theories are Morita equivalent precisely when one is an idempotent modification of the other, provided the idempotent is pseudoinvertible.
- The result of Section 3.2 is applied in Section 3.3 by looking at specific examples arising from the general theory. We characterise Morita equivalent monoids, partially ordered monoids, monoids enriched in categories, $\mathscr{S}$-sorted categories in various enrichments, and we observe that the case of enrichment in abelian groups gives the classical result of Morita. Then we show how (enriched) algebraic theories fit into the introduced framework by recovering the results of Dukarm and Adámek, Sobral, Sousa, and proving their enriched variants.

The results of this chapter have been published in [31] by J. Velebil and the author. The wording of the chapter is a slight modification of the text of the paper.

3.1 Cauchy completeness and basic Morita result

In order to introduce the basics of Morita theory, we will need to use the notion of Φ-*presentability* of an object in a category. This notion generalises those of a finitely presentable or perfectly presentable algebra from [5].

Definition 3.1.1. Given a class of weights Φ and a Φ-cocomplete category $\mathscr{K}$, we say that an object X in $\mathscr{K}$ is Φ-*presentable* if the functor

$$\mathscr{K}(X,-) : \mathscr{K} \to \mathscr{V}$$

is Φ-cocontinuous. In particular, if Φ is the class $\mathcal{P}$ of all weights, we say that a $\mathcal{P}$-presentable object X is *small-projective*.

Notation 3.1.2. We denote by $\mathscr{K}_\Phi$ the full subcategory of $\mathscr{K}$ spanned by all Φ-presentable objects.

Example 3.1.3. Given a class Ψ of weights and a Ψ-theory $\mathscr{T}$, we will be in particular interested in Ψ^+-presentable objects of the category $\Psi\text{-}\mathsf{Alg}(\mathscr{T})$ of algebras for $\mathscr{T}$. Let $\mathscr{V} = \mathsf{Set}$.

1. When $\mathscr{T}$ is a $\varnothing$-theory, the $\varnothing^+$-presentable algebras in $\varnothing\text{-}\mathsf{Alg}(\mathscr{T})$ are precisely the small-projectives in $\varnothing\text{-}\mathsf{Alg}(\mathscr{T}) = [\mathscr{T}^{op}, \mathsf{Set}]$, since we know from Example 2.1.30 that $\varnothing^+ = \mathcal{P}$. Small-projectives are called *absolutely presentable* in [5].

2. Let $\mathscr{T}$ be a Lex-theory, i.e., a small category with finite colimits (recall Example 2.2.7). The Lex^+-presentable (or Ind-presentable) algebras in $\Psi\text{-}\mathsf{Alg}(\mathscr{T})$ are exactly the *finitely presentable* presheaves in $[\mathscr{T}^{op}, \mathsf{Set}]$.

3. Similarly, when $\mathscr{T}$ is an Π-theory, the Π^+-presentable algebras in $\Psi\text{-}\mathsf{Alg}(\mathscr{T})$ are exactly the *perfectly presentable* presheaves in $[\mathscr{T}^{op}, \mathsf{Set}]$, see [5].

Now we can recall some known results of the theory of Morita equivalence. We will formulate a *basic Morita theorem* (see Theorem 3.1.9 below) that we will build upon later. Firstly, we introduce the class of all small-projective weights and study its properties.

Notation 3.1.4. The class of *all* small-projective weights is denoted by $\mathcal{Q}$. The free cocompletion of $\mathscr{K}$ under small-projective weights is then $Z^{\mathcal{Q}}_{\mathscr{K}} : \mathscr{K} \to \mathcal{Q}(\mathscr{K})$, or shortly $\mathcal{Q}(\mathscr{K})$. We call it the *Cauchy completion* of $\mathscr{K}$.

Remark 3.1.5. We shall often use that the free cocompletion of a category under colimits of small-projective weights is the same as its free completion under limits of small-projective weights. That is, given a category $\mathscr{K}$, there is an equivalence

$$\mathcal{Q}(\mathscr{K}) \simeq (\mathcal{Q}(\mathscr{K}^{op}))^{op}.$$

This equivalence is proved in Proposition 7.4 of [48], and it explains why we can say that $\mathcal{Q}(\mathscr{K})$ is the Cauchy *completion* of a category $\mathscr{K}$.

Some authors prefer the names *Karoubi envelope* or *idempotent completion* for what we call Cauchy completion. We use the terminology of [20].

Example 3.1.6 (Examples of Cauchy completions). The class $\mathcal{Q}$ of small-projective weights is always saturated: from Example 2.1.30 we know that $\mathcal{Q} = \mathcal{P}^+$, and the class Ψ^+ is always saturated for any class Ψ of weights (see also [48]). Therefore, the Cauchy completion of a given small category $\mathscr{D}$ can be described as the subcategory of the category of presheaves $[\mathscr{D}^{op}, \mathscr{V}]$ spanned by small-projective weights. In the case of the enrichments in $\mathscr{V} = \mathsf{Set}$, $\mathsf{Set}_\bullet$, Pos, or Cat, a weight $\varphi : \mathscr{D}^{op} \to \mathscr{V}$ is small-projective if and only if it is a retract of a representable functor. The proof of this fact is standard and the case of $\mathscr{V} = \mathsf{Set}$ is shown e.g. in [41]. The other cases are easy reformulations of the ordinary result.

In this example we show a more explicit description of the Cauchy completion of a small category in the following cases:

1. $\mathscr{V} = \mathsf{Set}$: Given a small category $\mathscr{D}$, its Cauchy completion $\mathcal{Q}(\mathscr{D})$ has as objects the idempotents of $\mathscr{D}$, and given two idempotents $u : d \to d$ and $v : d' \to d'$, the morphisms $f : u \to v$ in $\mathcal{Q}(\mathscr{D})$ are the morphisms $f : d \to d'$ in $\mathscr{D}$ that make the diagram

$$\begin{tikzcd} d \arrow[r, "f"] \arrow[d, "u"'] \arrow[rd, "f"] & d' \arrow[d, "v"] \\ d \arrow[r, "f"'] & d' \end{tikzcd}$$

commute.

2. $\mathscr{V} = \mathsf{Pos}$: We can describe the Cauchy completion of $\mathscr{D}$ again using idempotents. In Pos-enrichment we only have to take care of the two-dimensional aspect. Given two morphisms $f : u \to v$ and $f' : u \to v$ in $\mathcal{Q}(\mathscr{D})$, we define $f \leqslant f'$ if and only if the inequality $f \leqslant f'$ holds in $\mathscr{D}$ for the morphisms $f : d \to d'$ and $f' : d \to d'$.

3. $\mathscr{V} = \mathsf{Cat}$: Given a Cat-enriched category $\mathscr{D}$, its Cauchy completion is the Cauchy completion of the ordinary underlying category $\mathscr{D}_0$ (i.e., the ordinary category obtained by discarding the 2-cells of $\mathscr{D}$), with the two-cells inherited from $\mathscr{D}$: the 2-cells β from $f : u \to v$ to $f' : u \to v$ in $\mathcal{Q}(\mathscr{D})(u, v)$ are precisely those of the form $v * \alpha * u$ for some 2-cell α from f to f' in $\mathscr{D}(d, d')$ (where the product $*$ denotes the Godement product).

4. $\mathscr{V} = \mathsf{Set}_\bullet$: The Cauchy completion of $\mathscr{D}$ in $\mathsf{Set}_\bullet$-enrichment is the same as in the ordinary case, we need only to specify the distinguished point in $\mathcal{Q}(\mathscr{D})(u, v)$ for every pair $u : d \to d$ and $v : d' \to d'$ of objects in $\mathcal{Q}(\mathscr{D})$. If the distinguished point in $\mathscr{D}(d, d')$ is called p, then the distinguished point in $\mathcal{Q}(\mathscr{D})(u, v)$ is $v \cdot p \cdot u$. In fact, a completely analogous statement holds for categories enriched in the categories $\mathsf{Pos}_\bullet$ and $\mathsf{Cat}_\bullet$ of pointed posets and categories, respectively.

5. $\mathscr{V} = \mathsf{Ab}$: If $\mathscr{D}$ is a one-object category, it can equivalently be seen as a ring with a unit. The situation is substantially different from the previous case: a weight $\varphi : \mathscr{D}^{op} \to \mathscr{V}$ is small-projective if and only if it is a finitely generated projective left $\mathscr{D}$-module [20]. Then $\mathcal{Q}(\mathscr{D})$ is the category of finitely generated projective left $\mathscr{D}$-modules. Such modules are precisely the retracts of finitely generated free modules.

In all the above examples, the Cauchy completion of a small category is again small. This is not the case for every enrichment. Let CL be the monoidal category of complete lattices with sup-preserving functions and the usual tensor product. The Cauchy completion of a small CL-category need not be small [40]. This is due to the fact that CL is not a locally finitely presentable category: by results of [40], the class $\mathcal{Q}$ is locally small for any $\mathscr{V}$ whose underlying category is locally finitely presentable.

Assumption 3.1.7. In the rest of this chapter we use the following two assumptions.

1. The class $\mathcal{Q}$ is locally small.

2. The class Ψ is a fixed locally small sound class of weights.

We can now introduce the main concept used in this chapter: the notion of Morita equivalent Ψ-theories.

Definition 3.1.8. Let $\mathscr{T}$ and $\mathscr{T}'$ be two Ψ-theories. We call $\mathscr{T}$ and $\mathscr{T}'$ *Morita equivalent*, if there is an equivalence

$$\Psi\text{-}\mathsf{Alg}(\mathscr{T}) \simeq \Psi\text{-}\mathsf{Alg}(\mathscr{T}')$$

of their categories of algebras.

There is a very general result characterising Morita equivalent Ψ-theories from [48], which uses the Cauchy completions of the respective theories.

Theorem 3.1.9 (Basic Morita theorem). *For any two Ψ-theories $\mathscr{T}$ and $\mathscr{T}'$, we have*

$$\mathscr{T} \text{ is Morita equivalent to } \mathscr{T}' \text{ iff the categories } \mathcal{Q}(\mathscr{T}) \text{ and } \mathcal{Q}(\mathscr{T}') \text{ are equivalent.}$$

Proof. Once we prove that $\mathcal{Q} \subseteq \Psi^+$, the result follows immediately from Proposition 7.7 of [48]. But $\mathcal{P}^+ \subseteq \Psi^+$ holds, since $\Psi \subseteq \mathcal{P}$ does and $\mathcal{P}^+ = \mathcal{Q}$ (see Example 2.1.30 or Remark 6.21 of [48]). ∎

Remark 3.1.10. It is possible to add a third equivalent condition to the above theorem: that the categories

$$[\mathscr{T}^{op}, \mathscr{V}] \text{ and } [\mathscr{T}'^{op}, \mathscr{V}]$$

are equivalent. Indeed, this again follows quickly from Proposition 7.7 of [48], since $\mathcal{Q} \subseteq \mathcal{P}$ and $\mathcal{P}(\mathscr{T}) = [\mathscr{T}^{op}, \mathscr{V}]$.

3.2 Morita theorem for $\mathscr{S}$-sorted theories

We shall sharpen the basic Morita theorem 3.1.9 for the case of *many-sorted* theories. We will obtain a characterisation result for Morita equivalent theories that is similar in spirit to those contained in [3, 7]: two many-sorted theories are Morita equivalent if one is a certain *idempotent modification* of the other.

Remark 3.2.1. Recall that $\mathscr{S}$ is called *discrete* if its homs are defined as

$$\mathscr{S}(s, s') = \begin{cases} I & \text{if } s = s' \\ \perp & \text{otherwise,} \end{cases}$$

where $\perp$ denotes the initial object of $\mathscr{V}$ and I denotes the unit of the monoidal structure on $\mathscr{V}$ (recall Assumption 2.1.1).

Definition 3.2.2. Suppose $\mathscr{S}$ is a small discrete category. A Ψ-theory $\mathscr{T}$ is called *$\mathscr{S}$-sorted*, if there is a functor

$$T : \Psi(\mathscr{S}) \to \mathscr{T}$$

that is both identity on objects and a morphism of Ψ-theories.

If $T : \Psi(\mathscr{S}) \to \mathscr{T}$ is an $\mathscr{S}$-sorted Ψ-theory, then composition with T yields a faithful functor of the form

$$\Psi\text{-}\mathsf{Alg}(\mathscr{T}) \to \Psi\text{-}\mathsf{Alg}(\Psi(\mathscr{S})).$$

Due to the definition of Ψ-algebras and since $\Psi(\mathscr{S})$ is a free cocompletion of $\mathscr{S}$ under Ψ-colimits, the above functor is, up to equivalence, a "forgetful functor" of the form

$$\Psi\text{-}\mathsf{Alg}(\mathscr{T}) \to [\mathscr{S}^{op}, \mathscr{V}].$$

This explains our terminology: Ψ-algebras for an $\mathscr{S}$-sorted theory "live" over $[\mathscr{S}^{op}, \mathscr{V}]$, i.e., over "$\mathscr{S}$-sorted $\mathscr{V}$".

Example 3.2.3. The definition of an $\mathscr{S}$-sorted theory covers some important examples.

1. Let $\mathscr{V}$ be arbitrary, and Ψ be the empty class of weights. Then to give an $\mathscr{S}$-sorted Ψ-theory $T : \mathscr{S} \to \mathscr{T}$ is to give $\mathscr{T}$ having the same objects as $\mathscr{S}$. If $\mathscr{S}$ has only one object, the $\mathscr{S}$-sorted theories are then monoids in the case of $\mathscr{V} = \mathsf{Set}$, ordered monoids in the case of $\mathscr{V} = \mathsf{Pos}$, and rings when $\mathscr{V} = \mathsf{Ab}$.

2. Let $\mathscr{V} = \mathsf{Set}$ and consider the class Π of weights. The $\mathscr{S}$-sorted Π-theories are the (duals of) *S-sorted Lawvere theories* from [5].

For any Ψ-theory $\mathscr{T}$ we can construct its respective "canonical theory" $\mathcal{Q}(\mathscr{T})$. This generalises the concept of the canonical theory of a Lawvere theory $\mathscr{T}$, see Chapter 8 of [5]. For a Lawvere theory $\mathscr{T}$, its canonical theory is given by the idempotent completion of $\mathscr{T}$ (which coincides with $\mathcal{Q}(\mathscr{T})$, see Example 3.1.6 above).

Proposition 3.2.4. *For any Ψ-theory $\mathscr{T}$, the category $\mathcal{Q}(\mathscr{T})$ is a Ψ-theory. The unit $Z^{\mathcal{Q}}_{\mathscr{T}} : \mathscr{T} \to \mathcal{Q}(\mathscr{T})$ of the free $\mathcal{Q}$-cocompletion of $\mathscr{T}$ is a morphism of Ψ-theories.*

Proof. We have to prove that 1. the category $\mathcal{Q}(\mathscr{T})$ has Ψ-colimits and that 2. the functor $Z^{\mathcal{Q}}_{\mathscr{T}} : \mathscr{T} \to \mathcal{Q}(\mathscr{T})$ preserves them.

1. By soundness of Ψ we know that the equality $\Psi\text{-}\mathsf{Alg}(\mathscr{T}) = \Psi^+(\mathscr{T})$ holds, see (2.1).

   We will show that the category $\Psi^+(\mathscr{T})$ is cocomplete: By using Theorem 8.11 of [48], we see that for any small category $\mathscr{D}$, the closure of $\Psi^*(\mathscr{D})$ in $[\mathscr{D}^{op}, \mathscr{V}]$ under Ψ^+-colimits is all of $[\mathscr{D}^{op}, \mathscr{V}]$, By Proposition 8.8 of [48], each object of $[\mathscr{T}^{op}, \mathscr{V}]$ has a reflection in $\Psi^*\text{-}\mathsf{Alg}(\mathscr{T}) = \Psi\text{-}\mathsf{Alg}(\mathscr{T}) = \Psi^+(\mathscr{T})$ and thus $\Psi^+(\mathscr{T})$ is indeed cocomplete.

   Because $\mathcal{Q} \subseteq \Psi^+$ holds (see the proof of Theorem 3.1.9 above), the category $\mathcal{Q}(\mathscr{T})$ is precisely the category of all Ψ^+-presentable objects of the category $\Psi^+(\mathscr{T})$, see Proposition 7.5 of [48]. Thus $\mathcal{Q}(\mathscr{T})$ has Ψ-colimits and it is small since $\mathcal{Q}$ is locally small by Assumption 3.1.7 above.

2. Consider the factorisation

$$\begin{tikzcd}
\mathscr{T} \arrow[r, "Z^{\mathcal{Q}}_{\mathscr{T}}"'] \arrow[rr, bend left, "Z^{\Psi^+}_{\mathscr{T}}"] & \mathcal{Q}(\mathscr{T}) \arrow[r, "H"'] & \Psi^+(\mathscr{T})
\end{tikzcd}$$

   The inclusion $Z^{\Psi^+}_{\mathscr{T}} : \mathscr{T} \to \Psi^+(\mathscr{T})$ preserves Ψ-colimits (use the equality $\Psi\text{-}\mathsf{Alg}(\mathscr{T}) = \Psi^+(\mathscr{T})$ again and Corollary 8.5 of [48]). Since H preserves and reflects Ψ-colimits, the functor $Z^{\mathcal{Q}}_{\mathscr{T}}$ preserves Ψ-colimits.

■

The following definition is a generalisation of an *idempotent modification* of a theory $\mathscr{T}$ from [7]. As we will see in Section 3.3, the definition in fact covers many constructions that appear in various characterisation theorems of Morita equivalence in different contexts.

Definition 3.2.5. Suppose $T : \Psi(\mathscr{S}) \to \mathscr{T}$ is an $\mathscr{S}$-sorted Ψ-theory and suppose $u : \mathscr{S} \to \mathcal{Q}(\mathscr{T})$ is any functor.

1. The closure under Ψ-colimits in $\mathcal{Q}(\mathscr{T})$ of the full subcategory of $\mathcal{Q}(\mathscr{T})$ spanned by objects of the form $u(s)$ will be denoted by $\mathscr{T}_u$. The Ψ-theory $\mathscr{T}_u$ is called the *u-modification* of $\mathscr{T}$.

2. The functor u is called *pseudoinvertible* if the closure of $\mathscr{T}_u$ under $\mathcal{Q}$-colimits in $\mathcal{Q}(\mathscr{T})$ is all of $\mathcal{Q}(\mathscr{T})$.

Example 3.2.6 (The canonical pseudoinvertible functor for a theory). Suppose $T : \Psi(\mathscr{S}) \to \mathscr{T}$ is an $\mathscr{S}$-sorted Ψ-theory. Consider the composite

$$\begin{tikzcd} c \equiv \mathscr{S} \arrow[r, "Z^{\Psi}_{\mathscr{S}}"] & \Psi(\mathscr{S}) \arrow[r, "T"] & \mathscr{T} \arrow[r, "Z^{\mathcal{Q}}_{\mathscr{T}}"] & \mathcal{Q}(\mathscr{T}) \end{tikzcd}$$

We claim that c is pseudoinvertible. Moreover, the categories $\mathscr{T}_c$ and $\mathscr{T}$ are equal.

Recall from Proposition 3.2.4 that $\mathcal{Q}(\mathscr{T})$ is a Ψ-theory and consider the essentially unique extension $c^\sharp : \Psi(\mathscr{S}) \to \mathcal{Q}(\mathscr{T})$ of c to a morphism of Ψ-theories. This extension $c^\sharp$ is (isomorphic to) the composite $Z^{\mathcal{Q}}_{\mathscr{T}} \cdot T$, since the latter functor preserves Ψ-colimits. Moreover, the categories $\mathscr{T}_c$ and $\mathscr{T}$ are the same, since $\mathscr{T}$ is closed in $\mathcal{Q}(\mathscr{T})$ under Ψ-colimits.

The following lemma establishes the main ingredient in the first part of the characterisation of Morita equivalent $\mathscr{S}$-sorted theories: a pseudoinvertible idempotent u in a theory $\mathscr{T}$ gives rise to an idempotent modification of $\mathscr{T}$ that is Morita equivalent to $\mathscr{T}$.

Lemma 3.2.7. *For every pseudoinvertible $u : \mathscr{S} \to \mathcal{Q}(\mathscr{T})$, the Ψ-theories $\mathscr{T}$ and $\mathscr{T}_u$ are Morita equivalent.*

Proof. Denote by $E : \mathscr{T}_u \to \mathcal{Q}(\mathscr{T})$ the full inclusion from the definition of $\mathscr{T}_u$. It suffices to prove that $E : \mathscr{T}_u \to \mathcal{Q}(\mathscr{T})$ is a free cocompletion of $\mathscr{T}_u$ under $\mathcal{Q}$-colimits. Indeed, then $\mathcal{Q}(\mathscr{T}_u)$ and $\mathcal{Q}(\mathscr{T})$ would be equivalent as categories and the claim would follow from the basic Morita theorem 3.1.9.

To finish the proof, observe that the following four conditions are satisfied:

1. E is fully faithful. This is trivial.

2. $\mathcal{Q}(\mathscr{T})$ has $\mathcal{Q}$-colimits. Again, this is trivial.

3. The closure of $\mathscr{T}_u$ in $\mathcal{Q}(\mathscr{T})$ under $\mathcal{Q}$-colimits is all of $\mathcal{Q}(\mathscr{T})$. This is a restatement of pseudoinvertibility of u.

4. Every object a of $\mathscr{T}_u$ is $\mathcal{Q}$-presentable in $\mathcal{Q}(\mathscr{T})$. Indeed, the functor $\mathcal{Q}(\mathscr{T})(Ea, -) : \mathcal{Q}(\mathscr{T}) \to \mathscr{V}$ preserves $\mathcal{Q}$-colimits, since $\mathcal{Q}$-colimits are preserved by any functor, see [78].

By Proposition 4.2 of [48], the above four conditions prove precisely that $E : \mathscr{T}_u \to \mathcal{Q}(\mathscr{T})$ is a free cocompletion of $\mathscr{T}_u$ under $\mathcal{Q}$-colimits. ∎

Now we are ready for the characterisation: Every theory $\mathscr{T}'$ Morita equivalent to $\mathscr{T}$ is essentially an idempotent modification of $\mathscr{T}$.

Theorem 3.2.8 (The Morita theorem for many-sorted theories). *Suppose $T : \Psi(\mathscr{S}) \to \mathscr{T}$ and $T' : \Psi(\mathscr{S}) \to \mathscr{T}'$ are $\mathscr{S}$-sorted Ψ-theories. Then the following conditions are equivalent:*

1. *Theories $\mathscr{T}$ and $\mathscr{T}'$ are Morita equivalent.*

2. *There is a pseudoinvertible $u : \mathscr{S} \to \mathcal{Q}(\mathscr{T})$ such that the categories $\mathscr{T}_u$ and $\mathscr{T}'$ are equivalent.*

Proof. 1. implies 2. Due to the basic Morita theorem, one can choose an adjoint equivalence

$$L \dashv R : \mathcal{Q}(\mathscr{T}) \to \mathcal{Q}(\mathscr{T}').$$

Define $u : \mathscr{S} \to \mathcal{Q}(\mathscr{T})$ to be the composite

$$u \equiv \mathscr{S} \xrightarrow{c} \mathcal{Q}(\mathscr{T}') \xrightarrow{L} \mathcal{Q}(\mathscr{T})$$

That is, u is defined as the canonical pseudoinvertible functor c for the theory $\mathscr{T}'$ (see Example 3.2.6), composed with L. The functor u is pseudoinvertible, since c is and L is an equivalence of categories. By Example 3.2.6 we know that $\mathscr{T}'_c = \mathscr{T}'$. Since L is an equivalence of categories, it preserves all colimits; thus the image of $\mathscr{T}'$ under L in $\mathcal{Q}(\mathscr{T})$ is the u-modification $\mathscr{T}_u$ of $\mathscr{T}$.

Again using that $L \dashv R$ is an adjoint equivalence, the image of $\mathscr{T}_u$ under R in $\mathcal{Q}(\mathscr{T}')$ is the $R \cdot u$-modification $\mathscr{T}'_{R \cdot u}$ of $\mathscr{T}'$. The composite $R \cdot u$ is naturally isomorphic to the canonical pseudoinvertible functor c for $\mathscr{T}'$, and thus the categories $\mathscr{T}'$ and $\mathscr{T}'_{R \cdot u}$ are equivalent as categories by construction. This establishes the existence of an equivalence of categories between $\mathscr{T}'$ and $\mathscr{T}_u$.

2. implies 1. Choose an equivalence $\mathscr{T}_u \simeq \mathscr{T}'$. Then the categories $\mathcal{Q}(\mathscr{T}_u)$ and $\mathcal{Q}(\mathscr{T}')$ are equivalent. Furthermore, by Lemma 3.2.7, the categories $\mathcal{Q}(\mathscr{T}_u)$ and $\mathcal{Q}(\mathscr{T})$ are equivalent. Use the basic Morita theorem 3.1.9 to conclude the proof. ■

3.3 Examples

In this section we are going to apply our general result in various contexts to show its unifying nature. Namely, we can vary classes Ψ of weights (working thus with various notions of theories) and we can vary the base category $\mathscr{V}$. We show both the one-sorted and many-sorted case wherever this distinction is applicable.

However, the freedom of the choice of Ψ can be somewhat limited in some enrichments. For example, when $\mathscr{V} = \mathsf{Set}$ it does not make much sense to consider classes Ψ that contain weights for coequalisers. In fact, in this case any Ψ-theory $\mathscr{T}$ has coequalisers and is therefore idempotent complete [5]. Thus we have an equivalence $\mathcal{Q}(\mathscr{T}) \simeq \mathscr{T}$ of categories. Given two Ψ-theories $\mathscr{T}$ and $\mathscr{T}'$ that are Morita equivalent, we have a chain of equivalences

$$\mathscr{T} \simeq \mathcal{Q}(\mathscr{T}) \simeq \mathcal{Q}(\mathscr{T}') \simeq \mathscr{T}'.$$

The above argument can be used for a general $\mathscr{V}$ and any class Ψ of weights such that $\mathcal{Q} \subseteq \Psi$ holds. Thus we conclude that for such a Ψ, two Ψ-theories $\mathscr{T}'$ and $\mathscr{T}$ are Morita equivalent if and only if $\mathscr{T}$ and $\mathscr{T}'$ are equivalent as categories.

3.3.1 The case of the empty class

Let $\mathscr{V}$ be arbitrary and let Ψ be the empty class of weights. Thus a Ψ-theory is a small category and $\Psi\text{-}\mathsf{Alg}(\mathscr{T}) = [\mathscr{T}^{op}, \mathscr{V}]$.

We show how the choice of an empty class of weights affects the notion of a theory and of Morita equivalence in various enrichments.

Example 3.3.1. Let $\mathscr{V} = \mathsf{Cat}$ and let $\mathscr{S}$ be a discrete category on one object, say s. An $\mathscr{S}$-sorted Ψ-theory $\mathscr{T}$ is therefore a Cat-enriched monoid. Any functor $u : \mathscr{S} \to \mathcal{Q}(\mathscr{T})$ chooses an object $u(s) \in \mathrm{ob}(\mathcal{Q}(\mathscr{T}))$. This object corresponds to an idempotent in $\mathscr{T}$, which we are going to denote by $u : s \to s$ for notational simplicity.

The u-modification $\mathscr{T}_u$ of $\mathscr{T}$ is the Cat-monoid $\mathcal{Q}(\mathscr{T})(u,u)$ of morphisms $f : s \to s$ from $\mathscr{T}$ that satisfy the equalities $u \cdot f = f = f \cdot u$:

$$\begin{tikzcd}
s \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, "f"] & s \arrow[d, "u"] \\
s \arrow[r, "f"'] & s
\end{tikzcd}$$

Equivalently, each such morphism f has to satisfy the equality $u \cdot f \cdot u = f$. Let f and f' be two morphisms in $\mathcal{Q}(\mathscr{T})(u,u)$. The 2-cells $\alpha : f \to f'$ in $\mathscr{T}_u$ are exactly the 2-cells $\alpha : f \to f'$ in $\mathscr{T}$ for which $u * \alpha * u = \alpha$.

The functor $u : \mathscr{S} \to \mathcal{Q}(\mathscr{T})$ is pseudoinvertible if and only if any object p from $\mathcal{Q}(\mathscr{T})$ is a retract of $u : s \to s$ in the category $\mathcal{Q}(\mathscr{T})$.

We shall now show that for u to be pseudoinvertible it is enough that 1_s is a retract of u. From this it will follow that any p is a retract of u: Suppose we have

$$\begin{tikzcd}
s \arrow[r, "m"] \arrow[rr, bend left, "1_s"] \arrow[d, "1_s"'] \arrow[dr, "m"] & s \arrow[r, "e"] \arrow[d, "u"] \arrow[dr, "e"] & s \arrow[d, "1_s"] \\
s \arrow[r, "m"'] \arrow[rr, bend right, "1_s"'] & s \arrow[r, "e"'] & s
\end{tikzcd}$$

with $e \cdot m = 1_s$. The diagram

$$\begin{tikzcd}
s \arrow[r, "p"] \arrow[rr, bend left, "p"] \arrow[d, "p"'] \arrow[dr, "p"] & s \arrow[r, "p"] \arrow[d, "1_s"] \arrow[dr, "p"] & s \arrow[d, "p"] \\
s \arrow[r, "p"'] \arrow[rr, bend right, "p"'] & s \arrow[r, "p"'] & s
\end{tikzcd}$$

shows that p is a retract of 1_s, since $p : s \to s$ is the identity morphism in the hom-set $\mathcal{Q}(\mathscr{T})(p,p)$. This allows us to conclude that p indeed is a retract of u by inspecting the diagram

$$\begin{tikzcd}
s \arrow[r, "p"] \arrow[d, "p"'] \arrow[dr, "p"] & s \arrow[r, "m"] \arrow[d, "1_s"] \arrow[dr, "m"] & s \arrow[r, "e"] \arrow[d, "u"] \arrow[dr, "e"] & s \arrow[r, "p"] \arrow[d, "1_s"] \arrow[dr, "p"] & s \arrow[d, "p"] \\
s \arrow[r, "p"'] & s \arrow[r, "m"'] & s \arrow[r, "e"'] & s \arrow[r, "p"'] & s
\end{tikzcd}$$

and observing that $p \cdot e \cdot m \cdot p = p$.

We have therefore shown that the only Cat-monoids Morita equivalent to $\mathscr{T}$ are those of the form $\mathscr{T}_u$ for a pseudoinvertible idempotent u of $\mathscr{T}$. In detail, given an idempotent $u : s \to s$ from $\mathscr{T}$ satisfying the equalities

$$e \cdot m = 1 \quad u \cdot m = m \quad e \cdot u = u$$

for some morphism m and e in $\mathscr{T}$, we have a monoid $\mathscr{T}_u$ consisting of the morphisms $f : s \to s$ satisfying the equalities

$$u \cdot f = f = f \cdot u$$

and equipped by the 2-cells $u * \alpha * u : f \to f'$ derived from the 2-cells $\alpha : f \to f'$ from $\mathscr{T}$. Such monoids are Morita equivalent to $\mathscr{T}$, and they are the only ones Morita equivalent to $\mathscr{T}$.

Example 3.3.2 (Morita equivalence of 2-categories). Slightly generalising Example 3.3.1, let us consider the case of $\mathscr{S}$ possibly having more than one object. An $\mathscr{S}$-sorted theory $\mathscr{T}$ is any category with objects being precisely the sorts from $\mathscr{S}$. A functor $u : \mathscr{S} \to \mathscr{Q}(\mathscr{T})$ then chooses an object $u(s)$ from $\mathscr{Q}(\mathscr{T})$ for every sort $s \in \mathrm{ob}(\mathscr{S})$. This amounts to choosing an idempotent $u(s) : t_s \to t_s$ from $\mathscr{T}$ for every sort s.

The u-modification $\mathscr{T}_u$ of the theory $\mathscr{T}$ can be described either as a full subcategory of $\mathscr{Q}(\mathscr{T})$ spanned by the objects of the form $u(s)$ for $s \in \mathrm{ob}(\mathscr{S})$, or more concretely as follows: The objects of $\mathscr{T}_u$ are the objects t_s from $\mathscr{T}$ that are (co)domains of some idempotent $u(s) : t_s \to t_s$. The morphisms in $\mathscr{T}_u(t_s, t_{s'})$ are the morphisms $f : t_s \to t_{s'}$ from $\mathscr{T}$ for which the diagram

$$\begin{tikzcd}
t_s \arrow[r, "f"] \arrow[d, "u(s)"'] \arrow[dr, "f"] & t_{s'} \arrow[d, "u(s')"] \\
t_s \arrow[r, "f"'] & t_{s'}
\end{tikzcd}$$

commutes in $\mathscr{T}$. And the 2-cells between $f : t_s \to t_{s'}$ and $f' : t_s \to t_{s'}$ are the 2-cells $u(s') * \alpha * u(s) : f \to f'$ for every 2-cell $\alpha : f \to f'$ in $\mathscr{T}(t_s, t_{s'})$.

The functor $u : \mathscr{S} \to \mathscr{Q}(\mathscr{T})$, that chooses the idempotents, is pseudoinvertible if and only if there is an equivalence $\mathscr{Q}(\mathscr{T}_u) \simeq \mathscr{Q}(\mathscr{T})$. As in the case of $\mathscr{S}$ having one object, every object $p : t \to t$ in $\mathscr{Q}(\mathscr{T})$ has to be a retract of some $u(s) : t_s \to t_s$. By the same argument as in Example 3.3.1, it is enough to require that every identity morphism $1_t : t \to t$ from $\mathscr{T}$ is a retract of some $u(s) : t_s \to t_s$ in $\mathscr{Q}(\mathscr{T})$. This is true because every object $p : t \to t$ is a retract of $1_t : t \to t$ in $\mathscr{Q}(\mathscr{T})$. Thus u is pseudoinvertible if and only if for every sort t there is an idempotent $u(s)$ and morphisms $m_t : t \to t_s$ and $e_t : t_s \to t$ such that the diagram

$$\begin{tikzcd}
t_s \arrow[r, "u(s)"] & t_s \arrow[d, "e_t"] \\
t \arrow[u, "m_t"] \arrow[r, "1_t"'] & t
\end{tikzcd}$$

commutes in $\mathscr{T}$. The Morita theorem 3.2.8 then says that the only $\mathscr{S}$-sorted theories Morita equivalent to $\mathscr{T}$ are those of the form $\mathscr{T}_u$ as described above. We have therefore generalised the characterisation of Morita equivalent Cat-monoids and we have described Morita equivalent Cat-*categories* over a fixed set of objects.

Example 3.3.3. The results from the previous examples transfer directly to the case of enrichments in Pos and Set.

For the enrichment in $\mathscr{V} = \mathsf{Pos}$ and the empty class Ψ of weights, an $\mathscr{S}$-sorted theory $\mathscr{T}$ is a small (Pos-)category with the set $\mathrm{ob}(\mathscr{S})$ of objects. Recall from Example 3.1.6 that the objects in $\mathcal{Q}(\mathscr{T})$ are the idempotents of $\mathscr{T}$, a morphism $f : k \to k'$ in $\mathscr{T}$ is a morphism from u to v in $\mathcal{Q}(\mathscr{T})$ if

$$v \cdot f = f = f \cdot u,$$

and $f \leqslant f'$ holds in $\mathcal{Q}(\mathscr{T})(u, v)$ holds if and only if it holds in $\mathscr{T}$.

Given an $\mathscr{S}$-sorted theory $\mathscr{T}$, the functor $u : \mathscr{S} \to \mathcal{Q}(\mathscr{T})$ is a choice of idempotents $u(s) : t_s \to t_s$ from $\mathscr{T}$ for each sort $s \in \mathrm{ob}(\mathscr{S})$. The u-modification $\mathscr{T}_u$ of the theory $\mathscr{T}$ has the set $\mathrm{ob}(\mathscr{T}_s) = \{t_s \mid s \in \mathrm{ob}(\mathscr{S})\}$ of objects, where t_s is the domain of $u(s)$ for each s, and a morphism $f : t_s \to t_{s'}$ from $\mathscr{T}(t_s, t_{s'})$ belongs to $\mathscr{T}_u(t_s, t_{s'})$ if and only if

$$u(s') \cdot f = f = f \cdot u(s)$$

holds in $\mathscr{T}$. To say that u is pseudoinvertible is to say that $\mathcal{Q}(\mathscr{T}_u) \simeq \mathcal{Q}(\mathscr{T})$ holds, and this in turn means that any object $p : t \to t$ is a retract of some $u(s) : t_s \to t_s$ in $\mathcal{Q}(\mathscr{T})$. Thus for each sort s the equality

$$\begin{tikzcd}
t_s \arrow[r, "u(s)"] & t_s \arrow[d, "e_t"] \\
t \arrow[u, "m_t"] \arrow[r, "1_t"'] & t
\end{tikzcd}$$

has to hold for some morphisms $m_t : t \to t_s$ and $e_t : t_s \to t$ in $\mathscr{T}$, and this condition is sufficient for $u : \mathscr{S} \to \mathcal{Q}(\mathscr{T})$ to be pseudoinvertible by the same reasoning as in Example 3.3.2.

We have now generalised one of the results of [52] that discusses Morita equivalence of partially ordered monoids. If we consider the one-object category $\mathscr{S}$ of sorts, an $\mathscr{S}$-sorted theory $\mathscr{T}$ is a partially ordered monoid. Translating the above characterisation to the usual algebraic language, we get that an idempotent u of a partially ordered monoid $(N, \cdot, 1, \leqslant)$ is pseudoinvertible if and only if there are elements $m, e \in N$ such that $e \cdot u \cdot m = 1$. The Morita theorem 3.2.8 then says that a partially ordered monoid M is Morita equivalent to N if and only if $M \cong uNu$ for some pseudoinvertible idempotent u in N, where uNu is the partially ordered monoid with the underlying set $\{u \cdot n \cdot u \mid n \in N\}$, multiplication operation defined as in N, and unit u.

The above arguments from the case of $\mathscr{V} = \mathsf{Pos}$ carry unchanged to the case of $\mathscr{V} = \mathsf{Set}$ by ignoring the 2-dimensional aspect: thus for $\mathscr{V} = \mathsf{Set}$, we get that all monoids Morita equivalent to a given monoid $(N, \cdot, 1)$ are isomorphic to a monoid of the form $(uNu, \cdot, u)$ for an idempotent $u \in N$, where $uNu = \{u \cdot n \cdot u \mid n \in N\}$, the operation $\cdot$ is the same as in N, and there are elements $m, e \in N$ such that

$$e \cdot u \cdot m = 1$$

holds. We have thus reproved the result of [9, 49] characterising Morita equivalent ordinary monoids.

Example 3.3.4. In the case of the enrichment $\mathscr{V} = \mathsf{Ab}$, the situation is as follows: If $\mathscr{S}$ is a one-object category, then an $\mathscr{S}$-sorted Ψ-theory $\mathscr{T}$ is a ring with a unit. An object in $\mathscr{Q}(\mathscr{T})$ is a retract of a finitely generated projective $\mathscr{T}$-module. Any $\mathscr{T}$-module M in $\mathscr{Q}(\mathscr{T})$ yields a ring $\mathscr{Q}(\mathscr{T})(M, M)$ of endomorphisms. This ring is pseudoinvertible if $\mathscr{T}$, considered as a module over itself, is a retract of a finite coproduct $\coprod_l M$. By this we recapture the original result of Morita from [70]: Let $\mathscr{T}$ and $\mathscr{T}'$ be two rings. Denote by $\mathscr{T}^{[k]}$ the ring of all $k \times k$ matrices over $\mathscr{T}$. For any idempotent $u \in \mathscr{T}$, denote by $\mathscr{T}_u$ the ring of elements $r \in \mathscr{T}$ such that $ru = r = ur$, with neutral element u and multiplication defined as in $\mathscr{T}$. Two rings $\mathscr{T}$ and $\mathscr{T}'$ are Morita equivalent if and only if $\mathscr{T}'$ is isomorphic to the idempotent modification $\mathscr{T}_u^{[k]}$ of the matrix ring $\mathscr{T}^{[k]}$ for some $k > 0$ and a pseudoinvertible idempotent matrix u, that is, a matrix such that $e \cdot u \cdot m = 1$ for some $k \times k$ matrices e and m.

3.3.2 The case of finite coproducts

We shall now consider the class Π of weights for finite coproducts, see Example 2.1.26. We know that Π is locally small and that it is sound by the results of [45, 55]. Recall that in $\mathscr{V} = \mathsf{Set}$, the free cocompletion $\Pi(\mathscr{K})$ of $\mathscr{K}$ under is constructed by adding formal finite coproducts to $\mathscr{K}$: The objects of $\Pi(\mathscr{K})$ are finite words $w = k_0 \dots k_{n-1}$ over the alphabet $\mathrm{ob}(\mathscr{K})$. Given two objects $v = z_0 \dots z_{m-1}$ and $w = k_0 \dots k_{n-1}$ from $\Pi(\mathscr{K})$, the morphisms $\Pi(\mathscr{K})(v, w)$ are tuples (f, α) with $f : m \to n$ a function and $\alpha = (\alpha_j)_{j<m}$ being a choice of morphisms $\alpha_j : z_j \to k_{f(j)}$. The identities and composition in $\Pi(\mathscr{K})$ are defined as expected. In the case of $\mathscr{V} = \mathsf{Pos}$ and $\mathscr{V} = \mathsf{Cat}$, we also need to describe the 2-cells of $\Pi(\mathscr{K})$. As we will be computing free cocompletions only for discrete categories $\mathscr{S}$, we shall not need to compute this 2-dimensional aspect, as the only 2-cells in $\Pi(\mathscr{S})$ will be the trivial ones.

Specialising the enrichments we obtain the following examples:

Example 3.3.5. In the case of $\mathscr{V} = \mathsf{Cat}$ and the class Π of weights, a Π-theory is a small category $\mathscr{T}$ with finite coproducts. The objects of an $\mathscr{S}$-sorted theory $\mathscr{T}$ are words over the alphabet $\mathrm{ob}(\mathscr{S})$ equipped with the usual injection morphisms. We then specialise to two cases:

1. In case that $\mathscr{S}$ is the unit category $\mathbf{1}$ (i.e., one object, one morphism, one 2-cell), an $\mathscr{S}$-sorted theory $\mathscr{T}$ is a 2-Lawvere theory, i.e. the Cat-enriched version of the notion of a Lawvere theory [61].

   A functor $u : \mathscr{S} \to \mathscr{Q}(\mathscr{T})$ chooses one idempotent of the form $u(s) : t \to t$ from $\mathscr{T}$ (where $t = n \bullet s$ for some natural number n).

   For a nontrivial choice of idempotents u (meaning that for $u(s) : t \to t$ the object t is not initial in $\mathscr{T}$), the u-modification $\mathscr{T}_u$ has finite coproducts $n \bullet t$ of t as objects. Then a morphism $f : m \bullet t \to n \bullet t$ is present in $\mathscr{T}_u(m \bullet t, n \bullet t)$ if and only if

$$\begin{tikzcd}
m \bullet t \arrow[r, "f"] \arrow[d, "m \bullet u(s)"'] \arrow[dr, "f"] & n \bullet t \arrow[d, "n \bullet u(s)"] \\
m \bullet t \arrow[r, "f"'] & n \bullet t
\end{tikzcd}$$

commutes. The identity morphism for $m \bullet t$ is $m \bullet u(s) : m \bullet t \to m \bullet t$. Given two morphisms $f : m \bullet t \to n \bullet t$ and $f : m \bullet t \to n \bullet t$ and $f : m \bullet t \to n \bullet t$ and $f' : m \bullet t \to n \bullet t$ in $\mathscr{T}_u(m \bullet t, n \bullet t)$, a 2-cell $\beta : f \to f'$ from $\mathscr{T}(m \bullet t, n \bullet t)$ is in $\mathscr{T}_u(m \bullet t, n \bullet t)$ if and only if it is of the form $(n \bullet u(s)) * \alpha * (m \bullet u(s))$ for some 2-cell $\alpha : f \to f'$ from $\mathscr{T}(m \bullet t, n \bullet t)$.

We shall now state the requirements for $u : \mathscr{S} \to \mathcal{Q}(\mathscr{T})$ to be pseudoinvertible. By definition, the equivalence $\mathcal{Q}(\mathscr{T}_u) \simeq \mathcal{Q}(\mathscr{T})$ must hold. Any object $v : r \to r$ from $\mathcal{Q}(\mathscr{T})$ thus has to be a retract of some object $m \bullet u(s) : m \bullet t \to m \bullet t$ from $\mathscr{T}_u$. Since every $v : r \to r$ is a retract of $1_r : r \to r$ in $\mathcal{Q}(\mathscr{T})$, it is enough to show that every object $1_r : r \to r$ from $\mathcal{Q}(\mathscr{T})$ is a retract of some such object $m \bullet u(s)$. Further, $r = n \bullet s$ for some natural number n, and thus we only need to check whether $1_s : s \to s$ is a retract of some $m \bullet u(s) : m \bullet t \to m \bullet t$. In elementary terms, this says that there have to be two morphisms $m : s \to m \bullet t$ and $e : m \bullet t \to s$ such that

$$\begin{tikzcd}
m \bullet t \arrow[r, "m \bullet u(s)"] & m \bullet t \arrow[d, "e"] \\
s \arrow[u, "m"] \arrow[r, "1_s"'] & s
\end{tikzcd} \tag{3.1}$$

commutes. Therefore the only $\mathscr{S}$-sorted theories Morita equivalent to $\mathscr{T}$ are those of the form $\mathscr{T}_u$ with u being an idempotent of $\mathscr{T}$ satisfying the pseudoinvertibility condition of the diagram (3.1).

2. The second case is that of $\mathrm{ob}(\mathscr{S})$ containing (in general) more than one element. Then the notion of an $\mathscr{S}$-sorted theory $\mathscr{T}$ corresponds to a category $\mathscr{T}$ with the set $\mathrm{ob}(\mathscr{T})$ of objects consisting of finite words over the alphabet $\mathrm{ob}(\mathscr{S})$ of sorts, and every word $w = s \dots s'$ being the coproduct $s + \dots + s'$ of sorts.

   A functor $u : \mathscr{S} \to \mathcal{Q}(\mathscr{T})$ is a choice of an idempotent $u(s) : t_s \to t_s$ in $\mathscr{T}$ for each sort $s \in \mathrm{ob}(\mathscr{S})$.

   The u-modification of $\mathscr{T}$ is defined as a closure under finite coproducts of the subcategory of $\mathcal{Q}(\mathscr{T})$ spanned by the objects $u(s)$ for some $s \in \mathrm{ob}(\mathscr{S})$.

   That is, the objects of $\mathscr{T}_u$ are of the form

$$u(t_s) + \dots + u(t_{s'}) : t_s + \dots + t_{s'} \to t_s + \dots + t_{s'}$$

   for some n-tuple $s, \dots, s'$ of sorts from $\mathscr{S}$, and the morphisms

$$f : u(t_s) + \dots + u(t_{s'}) \to u(t_q) + \dots + u(t_{q'})$$

   are precisely the morphisms

$$f : t_s + \dots + t_{s'} \to t_q + \dots + t_{q'}$$

   for which the following diagram

$$\begin{tikzcd}
t_s + \dots + t_{s'} \arrow[r, "f"] \arrow[d, "u(t_s)+\dots+u(t_{s'})"'] \arrow[dr, "f"] & t_q + \dots + t_{q'} \arrow[d, "u(t_q)+\dots+u(t_{q'})"] \\
t_s + \dots + t_{s'} \arrow[r, "f"'] & t_q + \dots + t_{q'}
\end{tikzcd}$$

commutes in $\mathscr{T}$. A 2-cell $\beta : f \to f'$ from $\mathscr{T}$ is in $\mathscr{T}_u$ if and only if it is of the form

$$\beta = (u(t_q) + \cdots + u(t_{q'})) * \alpha * (u(t_s) + \cdots + u(t_{s'}))$$

for some 2-cell $\alpha : f \to f'$ from $\mathscr{T}$.

Finding out when $u : \mathscr{S} \to \mathcal{Q}(\mathscr{T})$ is pseudoinvertible is similar to the one-sorted case. Any object $v : r \to r$ from $\mathcal{Q}(\mathscr{T})$ has to be a retract of an object $p : t_s + \cdots + t_{s'} \to t_s + \cdots + t_{s'}$ from $\mathscr{T}_u$. Equivalently, any object $1_r : r \to r$ from $\mathcal{Q}(\mathscr{T})$ has to be a retract of an object $p : t_s + \cdots + t_{s'} \to t_s + \cdots + t_{s'}$ from $\mathscr{T}_u$. This implies that any object $1_{t_0} : t_0 \to t_0$ with $t_0 \in \mathrm{ob}(\mathscr{S})$ has to be a retract of some p from $\mathscr{T}_u$. Thus for every sort $t_0 \in \mathrm{ob}(\mathscr{S})$ there has to be an object $p : t_s + \cdots + t_{s'} \to t_s + \cdots + t_{s'}$ from $\mathscr{T}_u$ and two morphisms $m_{t_0} : t_0 \to t_s + \cdots + t_{s'}$ and $e_{t_0} : t_s + \cdots + t_{s'} \to t_0$ such that

$$\begin{tikzcd}
t_s + \cdots + t_{s'} \arrow[r, "p"] & t_s + \cdots + t_{s'} \arrow[d, "e_{t_0}"] \\
t_0 \arrow[u, "m_{t_0}"] \arrow[r, "1_{t_0}"'] & t_0
\end{tikzcd} \tag{3.2}$$

commutes in $\mathscr{T}$. Moreover, this is a sufficient condition for pseudoinvertibility of u, since a retract of a finite coproduct of identity morphisms coincides with a finite coproduct of retracts of identity morphisms. We have therefore characterised the $\mathscr{S}$-sorted theories Morita equivalent to $\mathscr{T}$ as the theories $\mathscr{T}_u$ for which the pseudoinvertibility condition (3.2) holds.

Example 3.3.6. The cases of $\mathscr{V} = \mathsf{Pos}$ and $\mathscr{V} = \mathsf{Set}$ are again simplifications of the Cat-enriched case. We are going to spell out the details in the many-sorted case. Then an $\mathscr{S}$-sorted theory $\mathscr{T}$ is a category equipped with a finite-coproduct-preserving functor $\Psi(\mathscr{S}) \to \mathscr{T}$ that is bijective on objects. The objects of $\mathscr{T}$ can therefore be interpreted as finite words $w = s \dots s'$ over the alphabet $\mathrm{ob}(\mathscr{S})$, with every w being the coproduct $s + \cdots + s'$. When $\mathscr{V} = \mathsf{Set}$, the notion of an $\mathscr{S}$-sorted theory $\mathscr{T}$ corresponds to the standard notion of a many-sorted algebraic theory over the set of sorts $\mathrm{ob}(\mathscr{S})$ (as can be seen e.g. in [7]).

As in the Cat-enrichment case, a choice of idempotents $u : \mathscr{S} \to \mathcal{Q}(\mathscr{T})$ is a choice of an idempotent $u(s) : t_s \to t_s$ in $\mathscr{T}$ for each sort $s \in \mathrm{ob}(\mathscr{S})$.

The u-modification $\mathscr{T}_u$ of $\mathscr{T}$ is defined as the closure under finite coproducts of the subcategory of $\mathcal{Q}(\mathscr{T})$ spanned by the objects $u(s)$ for some $s \in \mathrm{ob}(\mathscr{S})$. The construction of $\mathscr{T}_u$ proceeds as in Example 3.3.5. For $\mathscr{V} = \mathsf{Set}$, this construction gives exactly the notion of an idempotent modification of a theory $\mathscr{T}$ from [7]. For $\mathscr{V} = \mathsf{Pos}$, we put the inequality $f \leqslant g$ between two morphisms $f : u(t_s) + \cdots + u(t_{s'}) \to u(t_q) + \cdots + u(t_{q'})$ and $f' : u(t_s) + \cdots + u(t_{s'}) \to u(t_q) + \cdots + u(t_{q'})$ if and only if there is an inequality $f \leqslant g$ between the underlying morphisms $f : t_s + \cdots + t_{s'} \to t_q + \cdots + t_{q'}$ and $f' : t_s + \cdots + t_{s'} \to t_q + \cdots + t_{q'}$ in $\mathscr{T}$.

The choice of idempotents $u : \mathscr{S} \to \mathcal{Q}(\mathscr{T})$ is pseudoinvertible if for every sort $t_0 \in \mathrm{ob}(\mathscr{S})$ there is an idempotent $p : t_s + \cdots + t_{s'} \to t_s + \cdots + t_{s'}$ from $\mathscr{T}$ and two morphisms

$m_{t_0} : t_0 \to t_s + \cdots + t_{s'}$ and $e_{t_0} : t_s + \cdots + t_{s'} \to t_0$ such that

$$\begin{tikzcd}
t_s + \cdots + t_{s'} \arrow[r, "p"] & t_s + \cdots + t_{s'} \arrow[d, "e_{t_0}"] \\
t_0 \arrow[u, "m_{t_0}"] \arrow[r, "1_{t_0}"'] & t_0
\end{tikzcd}$$

commutes. Our notion of pseudoinvertibility therefore captures the notion of pseudoinvertibility as defined in [7], and for the one-sorted case, the pseudoinvertibility condition from [5].

Thus we get the characterisation of $\mathscr{S}$-sorted theories $\mathscr{T}' \simeq \mathscr{T}_u$ Morita equivalent to the theory $\mathscr{T}$ as it is present in [7], and its Pos-enriched variant.

Remark 3.3.7. Let us note that the technique and results of this section stay unchanged if we change the enrichment from $\mathscr{V} = \mathsf{Set}$ to the enrichment in pointed sets ($\mathscr{V} = \mathsf{Set}_\bullet$). Even more generally, the characterisation of Morita equivalent categories and Π-theories stays unchanged for the enrichment in *pointed posets* or *pointed categories*.

Chapter 4

Gabriel-Ulmer duality

Gabriel-Ulmer duality states that locally presentable categories are dually equivalent to essentially algebraic theories. A similar result can be obtained for algebraic theories: Cauchy complete algebraic theories are dually equivalent to algebraic categories. In the context of ordinary categories, a general version of the theorem of Gabriel and Ulmer was proved by Centazzo in [25]. We state and prove the generalisation of this result in the setting of enriched categories.

Structure of the chapter

- We first recall in Section 4.1 the notions used in this chapter (theories, algebras etc.) to make the chapter essentially self-contained. We then prove some elementary facts about algebraic functors.
- In Section 4.2 we state and prove the duality theorem.

Various forms of Gabriel-Ulmer duality have appeared in the literature. It seems that the main result of this chapter might be considered folklore. However, since the proof of the result has (to our best knowledge) *not* appeared in the literature in such a general form, and since the proof is very slick, we find it worthwhile to give our presentation of it.

4.1 Preliminaries

In this section we introduce and recall the important notions that arise in the statement and proof of Gabriel-Ulmer duality.

Finitely presentable objects and monoidal structure A category $\mathscr{V}$ is *locally finitely presentable as a monoidal category* if $\mathscr{V}_0$ is *locally finitely presentable* (recall from Example 2.2.3), and if finitely presentable objects (recall Example 3.1.3) of $\mathscr{V}_0$ are closed under the tensor of $\mathscr{V}$ and the monoidal unit I is a finitely presentable object.

To be able to state the duality, we shall also need a way to form duals of 2-categories.

Definition 4.1.1. Given a 2-category $\mathscr{K}$, we define its *horizontal dual* $\mathscr{K}^{op}$ by putting

$$\mathscr{K}^{op}(X, Y) = \mathscr{K}(Y, X)$$

and its *vertical dual* $\mathscr{K}^{co}$ by putting

$$\mathscr{K}^{co}(X,Y) = (\mathscr{K}(X,Y))^{op}.$$

Composition and identities in $\mathscr{K}^{op}$ and $\mathscr{K}^{co}$ are defiend in a straightforward way.

Assumption 4.1.2. In the rest of this chapter, whenever we are given a class Ψ of weights, we assume it to be locally small and sound.

Algebraic categories

Definition 4.1.3. Given Ψ-theories $\mathscr{T}$ and $\mathscr{T}'$, the functor $M : \mathscr{T} \to \mathscr{T}'$ is called a *Ψ-theory morphism* if it is Ψ-cocontinuous.

We denote by

$$\Psi\text{-}\mathsf{Th}$$

the 2-category of Ψ-theories, Ψ-theory morphisms and all natural transformations. Thus Ψ-Th is a locally fully faithful sub-2-category of the 2-category $\mathscr{V}$-Cat of all small $\mathscr{V}$-categories, $\mathscr{V}$-functors and $\mathscr{V}$-natural transformations, and there is a full inclusion

$$\Psi\text{-}\mathsf{Th} \to \Psi\text{-}\mathsf{COCTS}$$

into the 2-category of Ψ-cocomplete categories, Ψ-cocontinuous functors and all ($\mathscr{V}$-)natural transformations.

Recall that given a Ψ-theory $\mathscr{T}$, all $\mathscr{T}$-algebras span a full subcategory Ψ-$\mathsf{Alg}(\mathscr{T})$ of the presheaf category $[\mathscr{T}^{op}, \mathscr{V}]$, and we denote the inclusion by

$$W_{\mathscr{T}} : \Psi\text{-}\mathsf{Alg}(\mathscr{T}) \to [\mathscr{T}^{op}, \mathscr{V}].$$

Any category equivalent to the category Ψ-$\mathsf{Alg}(\mathscr{T})$ of $\mathscr{T}$-algebras for some Ψ-theory $\mathscr{T}$ is called *Ψ-algebraic*. A functor $H : \mathscr{K} \to \mathscr{K}'$ between two Ψ-algebraic categories is called *Ψ-algebraic* if it preserves limits and Ψ^+-colimits.

Remark 4.1.4. Recall from Remark 2.2.6 that the category Ψ-$\mathsf{Alg}(\mathscr{T})$ is (equivalent to) the free cocompletion $\Psi^+(\mathscr{T})$ of $\mathscr{T}$ under Ψ^+-colimits for any Ψ-theory $\mathscr{T}$. We thus have the factorisation

$$\mathscr{T} \xrightarrow{Z_{\mathscr{T}}} \Psi\text{-}\mathsf{Alg}(\mathscr{T}) \xrightarrow{W_{\mathscr{T}}} [\mathscr{T}^{op}, \mathscr{V}]$$

of the Yoneda embedding $\mathscr{T} \to [\mathscr{T}^{op}, \mathscr{V}]$.

Any Ψ-theory morphism $M : \mathscr{T} \to \mathscr{T}'$ gives rise to a Ψ-algebraic functor

$$\Psi\text{-}\mathsf{Alg}(M) : \Psi\text{-}\mathsf{Alg}(\mathscr{T}') \to \Psi\text{-}\mathsf{Alg}(\mathscr{T})$$

defined as the restriction

$$\begin{tikzcd}
{[(\mathscr{T}')^{op}, \mathscr{V}]} \arrow[r, "{[M^{op}, \mathscr{V}]}"] & {[\mathscr{T}^{op}, \mathscr{V}]} \\
\Psi\text{-}\mathsf{Alg}(\mathscr{T}') \arrow[u, "W_{\mathscr{T}'}"] \arrow[r, dotted, "\Psi\text{-}\mathsf{Alg}(M)"'] & \Psi\text{-}\mathsf{Alg}(\mathscr{T}) \arrow[u, "W_{\mathscr{T}}"']
\end{tikzcd}$$

of the precomposition functor $[M^{op}, \mathscr{V}]$, as we show now.

Algebraic functors and their left adjoints

Lemma 4.1.5. *Given a Ψ-theory morphism $M : \mathscr{T} \to \mathscr{T}'$, the functor* $\Psi\text{-}\mathsf{Alg}(M) : \Psi\text{-}\mathsf{Alg}(\mathscr{T}') \to \Psi\text{-}\mathsf{Alg}(\mathscr{T})$ *is algebraic: it preserves limits and Ψ^+-colimits.*

Proof. Let us first observe that in the diagram

$$\begin{tikzcd}
{[(\mathscr{T}')^{op}, \mathscr{V}]} \arrow[r, "{[M^{op}, \mathscr{V}]}"] & {[\mathscr{T}^{op}, \mathscr{V}]} \\
\Psi\text{-}\mathsf{Alg}(\mathscr{T}') \arrow[u, "W_{\mathscr{T}'}"] \arrow[r, dotted, "\Psi\text{-}\mathsf{Alg}(M)"'] & \Psi\text{-}\mathsf{Alg}(\mathscr{T}) \arrow[u, "W_{\mathscr{T}}"']
\end{tikzcd}$$

the functors $W_{\mathscr{T}}$ and $W_{\mathscr{T}'}$ are in fact the functors $\widetilde{Z}_{\mathscr{T}}$ and $\widetilde{Z}_{\mathscr{T}'}$, respectively. Both $\widetilde{Z}_{\mathscr{T}}$ and $\widetilde{Z}_{\mathscr{T}'}$ are fully faithful since $Z_{\mathscr{T}}$ and $Z_{\mathscr{T}'}$ are dense. Moreover, $\Psi\text{-}\mathsf{Alg}(\mathscr{T})$ is closed in $[\mathscr{T}^{op}, \mathscr{V}]$ under limits and Ψ^+-colimits, and so is $\Psi\text{-}\mathsf{Alg}(\mathscr{T}')$ in $[\mathscr{T}'^{op}, \mathscr{V}]$. That $\Psi\text{-}\mathsf{Alg}(M)$ preserves limits and Ψ^+-colimits then follows from the fact that $[M^{op}, \mathscr{V}]$ preserves both limits and colimits: it has both a right and a left adjoint (given by right and left Kan extensions, respectively). ■

Algebraic functors that arise from a theory morphism have a left adjoint given by left Kan extension.

Lemma 4.1.6. *Given a Ψ-theory morphism $M : \mathscr{T} \to \mathscr{T}'$, the functor* $\Psi\text{-}\mathsf{Alg}(M) : \Psi\text{-}\mathsf{Alg}(\mathscr{T}') \to \Psi\text{-}\mathsf{Alg}(\mathscr{T})$ *has a left adjoint.*

Proof. Let us take a functor $L = \mathrm{Lan}_{Z_{\mathscr{T}}}(Z_{\mathscr{T}'}M)$ which makes the square

$$\begin{tikzcd}
\Psi\text{-}\mathsf{Alg}(\mathscr{T}') & \Psi\text{-}\mathsf{Alg}(\mathscr{T}) \arrow[l, "\mathrm{Lan}_{Z_{\mathscr{T}}}(Z_{\mathscr{T}'}M)"'] \\
\mathscr{T}' \arrow[u, "Z_{\mathscr{T}'}"] \arrow[ur, phantom, "\cong"] & \mathscr{T} \arrow[l, "M"] \arrow[u, "Z_{\mathscr{T}}"']
\end{tikzcd}$$

commute up to isomorphism. On objects L is defined as

$$LA = \Psi\text{-}\mathsf{Alg}(\mathscr{T})(Z_{\mathscr{T}}-, A) * Z_{\mathscr{T}'}M,$$

and we thus get the following series of isomorphisms:

$$\begin{aligned}
\Psi\text{-}\mathsf{Alg}(\mathscr{T}')(LA, B) &\cong \{\widetilde{Z}_{\mathscr{T}}(A), \Psi\text{-}\mathsf{Alg}(\mathscr{T})(Z_{\mathscr{T}'}M-, B)\} \\
&\cong [\mathscr{T}^{op}, \mathscr{V}](\widetilde{Z}_{\mathscr{T}}(A), \widetilde{Z}_{\mathscr{T}'}(B)M^{op}) \\
&\cong [\mathscr{T}^{op}, \mathscr{V}](\widetilde{Z}_{\mathscr{T}}(A), \widetilde{Z}_{\mathscr{T}}(\Psi\text{-}\mathsf{Alg}(M)B)) \\
&\cong \Psi\text{-}\mathsf{Alg}(\mathscr{T})(A, \Psi\text{-}\mathsf{Alg}(M)B)
\end{aligned}$$

This shows that $L = \mathrm{Lan}_{Z_{\mathscr{T}}}(Z_{\mathscr{T}'}M)$ indeed is a left adjoint to $\Psi\text{-}\mathsf{Alg}(M)$. ■

In fact, *every* algebraic functor has a left adjoint. This fact will be important in showing that every algebraic functor is essentially of the form $\Psi\text{-}\mathsf{Alg}(M)$ for some theory morphism M. First we prove an auxiliary lemma.

Lemma 4.1.7. *Suppose a functor* $F : \Psi^+(\mathscr{T}) \to \mathscr{V}$ *preserves limits and* Ψ^+*-colimits. Such* F *is representable.*

Proof. This result follows quickly from a general representability theorem: see that by Theorem 4.82 of [41] and by the fact that $F = \mathrm{Lan}_{Z_\mathscr{T}} F Z_\mathscr{T}$ is representable if and only if the (small) limit $\{FZ_\mathscr{T}, Z_\mathscr{T}\}$ exists and if F preserves this limit. But $\{FZ_\mathscr{T}, Z_\mathscr{T}\}$ exists by completeness of $\Psi^+(\mathscr{T})$ (recall Remark 2.2.6) and F preserves this limit since it preserves all limits. ∎

Lemma 4.1.8. *Any algebraic functor* $G : \Psi\text{-}\mathsf{Alg}(\mathscr{T}') \to \Psi\text{-}\mathsf{Alg}(\mathscr{T})$ *has a left adjoint.*

Proof. We first note that we can equivalently consider an algebraic functor $G : \Psi^+(\mathscr{T}') \to \Psi^+(\mathscr{T})$. This functor has a left adjoint if $\Psi^+(\mathscr{T})(A, G-) : \Psi^+(\mathscr{T}) \to \mathscr{V}$ is representable for every A in $\Psi^+(\mathscr{T})$.

For Ψ-presentable objects A in $\Psi^+(\mathscr{T})$ we see that $\Psi^+(\mathscr{T})(A, G-)$ preserves limits and Ψ^+-colimits, since both $\Psi^+(\mathscr{T})(A, -)$ and G do. In fact, $\Psi^+(\mathscr{T})(A, G-)$ preserves limits and Ψ^+-colimits for any A in $\Psi^+(\mathscr{T})$, as any such A is a Ψ^+-colimit of Ψ-presentable objects. Since any functor $F : \Psi^+(\mathscr{T}) \to \mathscr{V}$ that preserves limits and Ψ^+-colimits is representable by Lemma 4.1.7, the proof is finished. ∎

Taking all Ψ-algebraic categories, we can form a (non-full) sub-2-category

$$\Psi\text{-}\mathsf{ALG}$$

of the 2-category $\mathscr{V}\text{-}\mathsf{CAT}$ of all $\mathscr{V}$-categories. The morphisms in $\Psi\text{-}\mathsf{ALG}$ consist of all Ψ-algebraic functors, and the inclusion $\Psi\text{-}\mathsf{ALG} \hookrightarrow \mathscr{V}\text{-}\mathsf{CAT}$ is locally fully faithful (that is, any natural transformation between two Ψ-algebraic functors is a 2-cell in $\Psi\text{-}\mathsf{ALG}$).

Definition 4.1.9. Given a class Ψ of weights, the assignment

$$\mathscr{T} \mapsto \Psi\text{-}\mathsf{Alg}(\mathscr{T})$$

mapping a Ψ-theory $\mathscr{T}$ to its category of algebras can be extended to a 2-functor of the form

$$\Psi\text{-}\mathsf{Alg}(-) : (\Psi\text{-}\mathsf{Th})^{coop} \to \Psi\text{-}\mathsf{ALG}.$$

A Ψ-theory morphism $M : \mathscr{T} \to \mathscr{T}'$ is mapped to the algebraic functor $\Psi\text{-}\mathsf{Alg}(M) : \Psi\text{-}\mathsf{Alg}(\mathscr{T}') \to \Psi\text{-}\mathsf{Alg}(\mathscr{T})$. Given two Ψ-theory morphisms $M, M' : \mathscr{T} \to \mathscr{T}'$, a natural transformation $\alpha : M \to M'$ is mapped to the natural transformation $\Psi\text{-}\mathsf{Alg}(\alpha) : \Psi\text{-}\mathsf{Alg}(M') \to \Psi\text{-}\mathsf{Alg}(M)$ whose component at an algebra $A : (\mathscr{T}')^{op} \to \mathscr{V}$ is $A(\alpha)^{op} : A \cdot (M')^{op} \Rightarrow A \cdot M^{op}$.

Remark 4.1.10. Let us observe that $\Psi\text{-}\mathsf{Alg}(-)$ is indeed a 2-functor. Given two Ψ-theory morphisms $M : \mathscr{T} \to \mathscr{T}'$ and $N : \mathscr{T}' \to \mathscr{T}''$, the composition on 1-cells is preserved by

$\Psi\text{-}\mathsf{Alg}(-)$ strictly, as it is seen from the diagram

$$\begin{tikzcd}
{[(\mathscr{T}'')^{op}, \mathscr{V}]} \arrow[r, "{[N^{op},\mathscr{V}]}"] & {[(\mathscr{T}')^{op}, \mathscr{V}]} \arrow[r, "{[M^{op},\mathscr{V}]}"] & {[\mathscr{T}^{op}, \mathscr{V}]} \\
\Psi\text{-}\mathsf{Alg}(\mathscr{T}'') \arrow[u, "W_{\mathscr{T}''}"] \arrow[r, dotted, "\Psi\text{-}\mathsf{Alg}(N)"'] & \Psi\text{-}\mathsf{Alg}(\mathscr{T}') \arrow[u, "W_{\mathscr{T}'}"] \arrow[r, dotted, "\Psi\text{-}\mathsf{Alg}(M)"'] & \Psi\text{-}\mathsf{Alg}(\mathscr{T}), \arrow[u, "W_{\mathscr{T}}"']
\end{tikzcd}$$

whose outer rectangle corresponds to

$$\begin{tikzcd}
{[(\mathscr{T}'')^{op}, \mathscr{V}]} \arrow[r, "{[(M\cdot N)^{op},\mathscr{V}]}"] & {[\mathscr{T}^{op}, \mathscr{V}]} \\
\Psi\text{-}\mathsf{Alg}(\mathscr{T}'') \arrow[u, "W_{\mathscr{T}'}"] \arrow[r, "\Psi\text{-}\mathsf{Alg}(M\cdot N)"'] & \Psi\text{-}\mathsf{Alg}(\mathscr{T}). \arrow[u, "W_{\mathscr{T}}"']
\end{tikzcd}$$

Likewise, for the 2-cells the 2-functoriality of $\Psi\text{-}\mathsf{Alg}(-)$ follows quickly, since componentwise $\Psi\text{-}\mathsf{Alg}(-)$ acts as postcomposition.

Definition 4.1.11. We say that a Ψ-theory is *Cauchy complete* if it is $\mathcal{Q}$-cocomplete. We denote the full sub-2-category of the 2-category $\Psi\text{-}\mathsf{Th}$ spanned by all Cauchy complete Ψ-theories by $\Psi\text{-}\mathsf{Th}_{cc}$.

Remark 4.1.12. In the context of ordinary categories and algebraic categories, Cauchy complete algebraic theories are called *canonical theories*. See, for example, Chapter 8 of [5] and Proposition 3.2.4. As in the case of Gabriel-Ulmer duality for algebraic theories and algebraic categories in the ordinary setting, we need to restrict ourselves to "canonical theories" only in order to obtain the duality result.

4.2 Gabriel-Ulmer duality

For ordinary categories, duality results consist of showing that two categories $\mathscr{K}$ and $\mathscr{L}$ of interest are dually equivalent, that is, there is an equivalence

$$\mathscr{K}^{op} \to \mathscr{L}$$

of categories. We shall need to be more careful in our formulation of Gabriel-Ulmer duality, as the "categories" (2-categories in fact) $\Psi\text{-}\mathsf{Th}_{cc}$ and $\Psi\text{-}\mathsf{ALG}$ are *not* equivalent in this sense. The theory morphisms in $\Psi\text{-}\mathsf{Th}_{cc}$ and algebraic functors in $\Psi\text{-}\mathsf{ALG}$ are in correspondence only up to equivalence; hence we need to use an adequately weakened notion of an equivalence.

Definition 4.2.1 ([77]). Let $\mathscr{K}$ and $\mathscr{L}$ be 2-categories. A 2-functor $T : \mathscr{K} \to \mathscr{L}$ is a *biequivalence* if T is

1. essentially surjective, i.e., for each object Y in $\mathscr{L}$ there exists an object X in $\mathscr{K}$ such that TX is *equivalent* to Y,

2. locally an equivalence, i.e., the action
$$T_{X,X'} : \mathscr{K}(X,X') \to \mathscr{L}(TX,TX')$$
is an *equivalence* of categories for each pair X, X' of objects in $\mathscr{K}$.

Now we can state the main result of this chapter.

Theorem 4.2.2 (Gabriel-Ulmer duality). *The 2-functor*
$$\Psi\text{-}\mathsf{Alg}(-) : (\Psi\text{-}\mathsf{Th}_{cc})^{coop} \to \Psi\text{-}\mathsf{ALG}$$
is a biequivalence of 2-categories.

Proof. We will proceed in two parts:

1. We will prove that Ψ-$\mathsf{Alg}(-)$ is essentially surjective on objects. That is, we will show that each Ψ-algebraic category $\mathscr{K}$ is essentially a category of Ψ-algebras for a Cauchy complete Ψ-theory $\mathscr{T}$.
2. We will prove that for each pair $\mathscr{T}$ and $\mathscr{T}'$ of Cauchy complete Ψ-theories the action
$$(\Psi\text{-}\mathsf{Th}_{cc}(\mathscr{T},\mathscr{T}'))^{op} \to \Psi\text{-}\mathsf{ALG}(\Psi\text{-}\mathsf{Alg}(\mathscr{T}'), \Psi\text{-}\mathsf{Alg}(\mathscr{T}))$$
of the 2-functor Ψ-$\mathsf{Alg}(-)$ is an equivalence of categories.

Ad 1.: We need to show that given a Ψ-algebraic category $\mathscr{K}$, there exists a Cauchy complete Ψ-theory $\mathscr{T}$ such that $\Psi\text{-}\mathsf{Alg}(\mathscr{T}) \simeq \mathscr{K}$ holds.

We shall show here that $\mathscr{T} := \mathscr{K}_{\Psi^+}$ (the full subcategory of $\mathscr{K}$ spanned by Ψ^+-presentable objects) is a Cauchy complete Ψ-theory, and that $\Psi\text{-}\mathsf{Alg}(\mathscr{K}_{\Psi^+}) \simeq \mathscr{K}$ holds. (The proof is also contained in [48].)

First, $\mathscr{K}_{\Psi^+}$ has Ψ-colimits, as it is closed in $\mathscr{K}$ under Ψ-colimits: consider a diagram $D : \mathscr{D} \to \mathscr{K}$ that factorises through $\mathscr{K}_{\Psi^+}$, and a Ψ-weight $\psi : \mathscr{D}^{op} \to \mathscr{V}$. Using the natural isomorphism $\mathscr{K}(\psi * D, X) \cong [\mathscr{D}^{op}, \mathscr{V}](\psi, \mathscr{K}(D-, X))$, we see that $\psi * D$ is Ψ^+-presentable since the functors $\mathscr{K}(Dd, -)$ and $[\mathscr{D}^{op}, \mathscr{V}](\psi, -) = \{\psi, -\}$ preserve Ψ^+-colimits, and thus $\mathscr{K}(\psi * D, -)$ preserves them as well.

Secondly, $\mathscr{K}_{\Psi^+}$ is Cauchy complete. From the inclusion $\mathcal{Q} \subseteq \Psi^+$ we see that a similar argument as above can be used to prove that $\mathscr{K}_{\Psi^+}$ is closed in $\mathscr{K}$ under $\mathcal{Q}$-colimits. Indeed, given a diagram $D : \mathscr{D} \to \mathscr{K}_{\Psi^+} \to \mathscr{K}$ and a weight $\psi : \mathscr{D}^{op} \to \mathscr{V}$ from $\mathcal{Q}$, the functor $\mathscr{K}(Dd, -)$ preserves Ψ^+-colimits for any d from $\mathscr{D}$, and $[\mathscr{D}^{op}, \mathscr{V}](\psi, -)$ preserves Ψ^+-colimits, as it in fact preserves *all* colimits. Thus $\psi * D$ is Ψ^+-presentable as we needed.

Thirdly, from $\mathcal{Q} \subseteq \Psi^+$ we infer that
$$\mathcal{Q}(\mathscr{T}) \simeq \Psi^+(\mathscr{T})_{\Psi^+} \simeq \mathscr{K}_{\Psi^+},$$
and from this it follows that
$$\Psi\text{-}\mathsf{Alg}(\mathscr{T}) \simeq \Psi^+(\mathscr{T}) \simeq \Psi^+(\mathcal{Q}(\mathscr{T})) \simeq \Psi\text{-}\mathsf{Alg}(\mathscr{K}_{\Psi^+}).$$

Ad 2.: Let us fix two theories $\mathscr{T}$ and $\mathscr{T}'$. The (ordinary) functor
$$(\Psi\text{-}\mathsf{Th}_{cc}(\mathscr{T},\mathscr{T}'))^{op} \to \Psi\text{-}\mathsf{ALG}(\Psi\text{-}\mathsf{Alg}(\mathscr{T}'), \Psi\text{-}\mathsf{Alg}(\mathscr{T}))$$

is an equivalence precisely when it is fully faithful and essentially surjective.

For essential surjectivity we take an algebraic functor $G : \Psi\text{-}\mathsf{Alg}(\mathscr{T}') \to \Psi\text{-}\mathsf{Alg}(\mathscr{T})$ and construct an appropriate theory morphism $M : \mathscr{T} \to \mathscr{T}'$. Since G has a left adjoint (let us denote it by $L : \Psi\text{-}\mathsf{Alg}(\mathscr{T}) \to \Psi\text{-}\mathsf{Alg}(\mathscr{T}')$), we have the composite dotted functor as in the following diagram:

$$\begin{tikzcd}
\Psi\text{-}\mathsf{Alg}(\mathscr{T}) \arrow[r, "L"] & \Psi\text{-}\mathsf{Alg}(\mathscr{T}') \\
\mathscr{T} \arrow[u, "Z_{\mathscr{T}}"] \arrow[ur, dotted] & \mathscr{T}' \arrow[u, "Z_{\mathscr{T}'}"']
\end{tikzcd}$$

Since left adjoints preserve colimits (and map Ψ^+-presentable objects to Ψ^+-presentable objects), the dotted functor can be factored through $Z_{\mathscr{T}'}$ to the sought functor M:

$$\begin{tikzcd}
\Psi\text{-}\mathsf{Alg}(\mathscr{T}) \arrow[r, "L"] & \Psi\text{-}\mathsf{Alg}(\mathscr{T}') \\
\mathscr{T} \arrow[u, "Z_{\mathscr{T}}"] \arrow[r, dotted, "M"'] & \mathscr{T}' \arrow[u, "Z_{\mathscr{T}'}"']
\end{tikzcd}$$

By construction L is left adjoint to $\Psi\text{-}\mathsf{Alg}(M)$, and thus $\Psi\text{-}\mathsf{Alg}(M)$ is isomorphic to G.

To show that

$$(\Psi\text{-}\mathsf{Th}_{cc}(\mathscr{T}, \mathscr{T}'))^{op} \to \Psi\text{-}\mathsf{ALG}(\Psi\text{-}\mathsf{Alg}(\mathscr{T}'), \Psi\text{-}\mathsf{Alg}(\mathscr{T}))$$

is fully faithful, see that natural transformations

$$\begin{tikzcd}
\mathscr{T} \arrow[r, bend left, "M"] \arrow[r, bend right, "M'"'] \arrow[r, phantom, "\Downarrow"] & \mathscr{T}'
\end{tikzcd}$$

correspond precisely to natural transformations

$$\begin{tikzcd}
\Psi^+(\mathscr{T}) \arrow[r, bend left, "\Psi^+(M)"] \arrow[r, bend right, "\Psi^+(M')"'] \arrow[r, phantom, "\Downarrow"] & \Psi^+(\mathscr{T}')
\end{tikzcd} \tag{4.1}$$

as the cocompletion pseudoadjunction

$$\begin{tikzcd}
\Psi^+\text{-}\mathsf{Cocts} \arrow[d, bend left] \arrow[d, phantom, "\dashv"] \\
\mathscr{V}\text{-}\mathsf{Cat} \arrow[u, bend left]
\end{tikzcd}$$

has a locally fully faithful forgetful right adjoint. Now since $\Psi^+(M) \dashv \Psi\text{-}\mathsf{Alg}(M)$ and $\Psi^+(M') \dashv \Psi\text{-}\mathsf{Alg}(M)$ both hold, we can use the units and counits of these adjunctions (say η^M, μ^M and $\eta^{M'}$ and $\mu^{M'}$ respectively) to form mates

$$\begin{tikzcd}
 & & \Psi^+(\mathscr{T}) & & & \\
\Psi^+(\mathscr{T}) \arrow[urr, bend left, "1"] \arrow[urr, phantom, "\swarrow\eta^{M}" description] \arrow[rr, bend left, "\Psi^+(M)"] \arrow[rr, bend right, "\Psi^+(M')"'] \arrow[rr, phantom, "\Downarrow"] & & \Psi^+(\mathscr{T}') \arrow[u, "\Psi\text{-}\mathsf{Alg}(M)"'] & = & \Psi\text{-}\mathsf{Alg}(\mathscr{T}') \arrow[r, bend left, "\Psi\text{-}\mathsf{Alg}(M')"] \arrow[r, bend right, "\Psi\text{-}\mathsf{Alg}(M)"'] \arrow[r, phantom, "\Downarrow"] & \Psi\text{-}\mathsf{Alg}(\mathscr{T}) \\
\Psi^+(\mathscr{T}') \arrow[u, "\Psi\text{-}\mathsf{Alg}(M')"] \arrow[urr, bend right, "1"'] \arrow[urr, phantom, "\searrow\varepsilon^{M'}" description] & & & & &
\end{tikzcd}$$

that are in one to one correspondence to the natural transformations of the form in diagram (4.1). ■

Example 4.2.3. Instantiating our general result, we obtain the following known dualities.

1. For $\mathscr{V} = \mathsf{Set}$ and Ψ being the class of weights for finite colimits, the general result recovers the original result of Gabriel and Ulmer from [36] that essentially algebraic theories are dually equivalent to locally presentable categories.

2. For $\mathscr{V} = \mathsf{Set}$ and Ψ being the class of weights for finite coproducts, we recover the duality of canonical algebraic theories and algebraic categories given in [2] and [25].

3. For $\Psi = \varnothing$ being the empty class of weights, we obtain the "Morita duality": the duality of presheaf categories and small Cauchy complete categories. That is, functors $[\mathscr{T}', \mathscr{V}] \to [\mathscr{T}, \mathscr{V}]$ preserving limits (and trivially $\mathscr{Q}$-colimits) essentially correspond to (trivially $\mathscr{Q}$-cocontinuous) functors $\mathscr{T} \to \mathscr{T}'$.

4. For $\mathscr{V} = \mathsf{Set}$ and Ψ being weights for *conical* colimits, we obtain the results of Centazzo [25]. We will study conical weights more in Chapter 5.

Varying the enrichment, we obtain the obvious variations of the above results for *ordered* algebraic categories, *ordered* locally presentable categories, etc.

Remark 4.2.4. We can also vary the classes of weights: consider two classes Ψ and Ψ' of weights such that $\Psi \subseteq \Psi'$ holds. This inclusion gives rise to an inclusion 2-functor

$$i_{\Psi',\Psi} : \Psi'\text{-}\mathsf{Th}_{cc} \to \Psi\text{-}\mathsf{Th}_{cc}$$

which has a left *pseudoadjoint* given by a *Ψ-conservative Ψ'-cocompletion.* More in detail, we have for every Ψ-theory $\mathscr{T}$ the restricted Yoneda embedding

$$Z^{\Psi}_{\mathscr{T}} : \mathscr{T} \to \Psi\text{-}\mathsf{Alg}(\mathscr{T})$$

and this embedding can be factorised: Consider the closure $\overline{\mathscr{T}}$ of $\mathscr{T}$ in $\Psi\text{-}\mathsf{Alg}(\mathscr{T})$ under Ψ'-colimits. We get a diagram

$$\begin{tikzcd}
\mathscr{T} \arrow[r, "Z^{\Psi,\Psi'}_{\mathscr{T}}"] \arrow[dr, "Z^{\Psi}_{\mathscr{T}}"'] \arrow[dr, phantom, bend left, "\cong"] & \overline{\mathscr{T}} \arrow[d] \\
& \Psi\text{-}\mathsf{Alg}(\mathscr{T})
\end{tikzcd}$$

in which $\overline{\mathscr{T}}$ is a Ψ'-theory (since Ψ' is assumed to be locally small), the functor

$$Z^{\Psi,\Psi'}_{\mathscr{T}} : \mathscr{T} \to \overline{\mathscr{T}}$$

preserves Ψ-colimits and is the $\mathscr{T}$-component of the *unit* of the pseudoadjunction

$$\begin{tikzcd}
\Psi'\text{-}\mathsf{Th}_{cc} \arrow[d, bend left, "i_{\Psi',\Psi}"] \arrow[d, phantom, "\dashv"] \\
\Psi\text{-}\mathsf{Th}_{cc}. \arrow[u, bend left, "\overline{(-)}"]
\end{tikzcd}$$

Moreover, this pseudoadjunction induces via the biequivalences $\Psi\text{-}\mathsf{Alg}(-) : (\Psi\text{-}\mathsf{Th}_{cc})^{coop} \to \Psi\text{-}\mathsf{ALG}$ and $\Psi'\text{-}\mathsf{Alg}(-) : (\Psi'\text{-}\mathsf{Th}_{cc})^{coop} \to \Psi'\text{-}\mathsf{ALG}$ the right-hand side pseudoadjunction in the following diagram:

$$\begin{tikzcd}
(\Psi'\text{-}\mathsf{Th}_{cc})^{coop} \arrow[r, dotted, "\simeq"] \arrow[d, bend left, "i^{coop}_{\Psi',\Psi}"] \arrow[d, phantom, "\vdash"] & \Psi'\text{-}\mathsf{ALG} \arrow[d, bend left] \arrow[d, phantom, "\vdash"] \\
(\Psi\text{-}\mathsf{Th}_{cc})^{coop} \arrow[r, dotted, "\simeq"'] \arrow[u, bend left, "\overline{(-)}^{coop}"] & \Psi\text{-}\mathsf{ALG} \arrow[u, bend left]
\end{tikzcd}$$

The right pseudoadjoint

$$\Psi'\text{-}\mathsf{ALG} \to \Psi\text{-}\mathsf{ALG}$$

has the expected behaviour on objects: given a Ψ'-algebraic category $\Psi'\text{-}\mathsf{Alg}(\mathscr{T})$ for a Ψ'-theory $\mathscr{T}$, we may consider this category as $\Psi\text{-}\mathsf{Alg}(\mathscr{T})$, since $\mathscr{T}$ is trivially also a Ψ-theory, and any Ψ'-algebra $\mathscr{T}^{op} \to \mathscr{V}$ preserves Ψ'-limits, thus Ψ-limits as well.

The left pseudoadjoint

$$\Psi\text{-}\mathsf{ALG} \to \Psi'\text{-}\mathsf{ALG}$$

gives for a Ψ-algebraic category $\Psi\text{-}\mathsf{Alg}(\mathscr{T})$ the Ψ'-algebraic category $\Psi'\text{-}\mathsf{Alg}(\overline{\mathscr{T}})$ with $\overline{\mathscr{T}}$ being the Ψ'-theory obtained from $\mathscr{T}$ by “conservatively adjoining Ψ'-colimits”.

Example 4.2.5. Consider the case of ordinary categories $\mathscr{V} = \mathsf{Set}$ and the classes $\Psi = \Pi$ and $\Psi' = \mathsf{Lex}$ of weights. Taking an algebraic Cauchy complete (Π-)theory $\mathscr{T}$, we may consider $\mathscr{T}$ as a subcategory of $\Pi\text{-}\mathsf{Alg}(\mathscr{T})$ and form the closure under finite colimits. This gives us the category $\overline{\mathscr{T}}$ of finitely presentable objects of $\Pi\text{-}\mathsf{Alg}(\mathscr{T})$ (see [5]), and $\overline{\mathscr{T}}$ is an essentially algebraic theory, having all finite colimits. Moreover, the categories $\mathsf{Lex}\text{-}\mathsf{Alg}(\overline{\mathscr{T}})$ and $\Pi\text{-}\mathsf{Alg}(\mathscr{T})$ are equivalent, see Corollary 5.12 in [5].

Chapter 5

Sifted weights

The classical theory of locally presentable [36] and accessible categories [59], [65] (see also the more recent [3]) hinges a lot upon the interplay of two classes of categories: the class of λ-small categories for limits and the class of λ-filtered categories for colimits, where λ is a fixed regular cardinal. The precise nature of the interplay is that

> λ-small limits commute with λ-filtered colimits in the category of all sets and mappings.

The idea of [1] was to develop a more general theory of locally presentable and accessible categories based on the fact that one has a fixed class $\mathbb{D}$ of small categories that replaces the class of λ-small categories. The corresponding class of colimits, called $\mathbb{D}$-*filtered*, is then *defined* by the requirement that

> $\mathbb{D}$-limits commute with $\mathbb{D}$-filtered colimits in the category of sets and mappings.

It has been showed in [1] that a great deal of the classical theory can be developed for the concept of $\mathbb{D}$-filteredness, provided that the class $\mathbb{D}$ satisfies a side condition that is called *soundness* in [1].

For example, the class $\mathbb{D}$ consisiting of finite discrete categories is sound. The corresponding $\mathbb{D}$-filtered colimits turn out to be precisely the *sifted* colimits of [60]. Free cocompletions of small categories under sifted colimits generalise the notion of a variety, as shown in [4]. In fact, the notion of a sifted colimit turned out to be a cornerstone notion in the categorical treatment of universal algebra, see [5].

The approach of [1] can be generalised further, as we hinted at in Definition 2.1.27.

That is, we can study commutativity of *weighted* limits and colimits in a general enrichment, and look at "well-behaved" classes Ψ of weights that give rise to a nice notion of a Ψ-theory. These are precisely the *sound* classes of weights as we defined them in Definition 2.2.5.

The "ordinary" definition of soudness from [1] and our notion of soundness are closely tied: they coincide when we consider ordinary categories and *conical* weights. We shall then pass from categories to categories enriched in Cat and characterise *sifted weights* in this environment. This characterisation is very much in the style of the characterisation of sifted colimits for ordinary categories, and it allows for a "calculus" for detecting whether a weight is sifted.

Structure of the chapter

- In Section 5.1 we recall some necessary basic notions and then show that
  1. a weight is Π-flat precisely when its category of elements is sifted in the ordinary sense of [5],
  2. our definition of soundness coincides with the definition of soundness from [1].
- In Section 5.2 we give the elementary characterisation of sifted weights for $\mathscr{V} = \mathsf{Cat}$ and show the usage of this characterisation on an example of an interesting weight by proving its siftedness.
- Having studied sifted weights in the enrichment $\mathscr{V} = \mathsf{Cat}$, we get some observations on sifted weights in the enrichment $\mathscr{V} = \mathsf{Pre}$ almost for free in Section 5.3.

The results contained in this chapter appeared in the manuscript [30] by J. Velebil and the author. The wording of the chapter is a slight modification of the text of the manuscript.

5.1 Preliminaries

Notation 5.1.1. In this chapter we denote the identity morphism on an object A by $id_A : A \to A$. As we shall be working with categories whose objects are natural numbers, this notation prevents possible confusion. For the same reason the terminal object in a cartesian closed category $\mathscr{V}$ will be denoted by $\top$.

In the setting of ordinary categories, we most often deal with *conical* limits and colimits. The notion of a conical limit or conical colimit can be introduced for any enrichment in $\mathscr{V}$, provided that $\mathscr{V}$ is cartesian closed.

Example 5.1.2. Suppose $\mathscr{V}$ is cartesian closed. By $\mathsf{const}_\top : \mathscr{D}^{op} \to \mathscr{V}$ we denote the weight that is constantly the terminal object $\top$. Such weights will be called *conical*. Any class $\mathbb{D}$ of small categories induces a class

$$\Psi_{\mathbb{D}}$$

of conical weights $\mathsf{const}_\top : \mathscr{D}^{op} \to \mathscr{V}$ with $\mathscr{D}^{op}$ in $\mathbb{D}$.

1. Suppose $\mathscr{V} = \mathsf{Set}$. Then to say that a small category $\mathscr{D}$ is $\mathbb{D}$-*filtered* in the sense of [1] is to say that the conical weight $\mathsf{const}_\top : \mathscr{D}^{op} \to \mathsf{Set}$ is $\Psi_{\mathbb{D}}$-flat. Indeed: (co)limits of diagrams weighted by conical weights yield the usual notions defined by (co)cones.

2. Suppose $\mathscr{V}$ is arbitrary (but still cartesian closed). The class $\Psi_{\mathbb{D}}$ for $\mathbb{D}$ consisting of all finite discrete categories will be denoted by Π^{cone}. The corresponding class of Π^{cone}-flat weights will be called the class of *sifted* weights. Compare with Example 2.1.30: we have extended the definition of sifted weights to enrichments other than $\mathscr{V} = \mathsf{Set}$.

Remark 5.1.3. Recall from Definition 2.2.5 that a class Ψ of weights is called *sound* if a weight $\varphi : \mathscr{D}^{op} \to \mathscr{V}$ is Ψ-flat whenever the functor

$$\varphi * (-) : [\mathscr{D}, \mathscr{V}] \to \mathscr{V}$$

preserves Ψ-limits of representables.

Recall (see, e.g., [64]) that in the enrichment $\mathscr{V} = \mathsf{Set}$, every $\varphi : \mathscr{D}^{op} \to \mathsf{Set}$ has a *category of elements* $\mathsf{elts}(\varphi)$: the objects are pairs (x, d) with $x \in \varphi(d)$ and a morphism from (x, d) to (x', d') is a morphism $t : d \to d'$ in $\mathscr{D}$ such that $\varphi t(x') = x$ holds.

Recall also from 2.1.25 that for a class Ψ of weights we denote by $\Psi_1(\mathscr{D})$ the first step of the free Ψ-cocompletion step for the category $\mathscr{D}$.

The following easy result shows that the "testing weights" for Ψ-flatness can be taken in a special form:

Proposition 5.1.4. *For a class Ψ the following are equivalent:*

1. *Ψ is sound.*
2. *The weight $\varphi : \mathscr{D}^{op} \to \mathscr{V}$ is Ψ-flat, whenever $\varphi * (-)$ preserves $\Psi_1(\mathscr{D})$-limits of representables, i.e., whenever the canonical morphism*

$$\mathsf{can} : \varphi * \{\psi, Y-\} \to \{\psi, \varphi\} \tag{5.1}$$

is an isomorphism, for every $\psi : \mathscr{D}^{op} \to \mathscr{V}$ in $\Psi_1(\mathscr{D})$.

Proof. Let $Y : \mathscr{D}^{op} \to [\mathscr{D}, \mathscr{V}]$ be the Yoneda embedding. Definition 2.2.5 requires the canonical morphism

$$\varphi * \{\psi, YT^{op}-\} \to \{\psi, \varphi \cdot T^{op}\}$$

to be an isomorphism, for every $\psi : \mathscr{G}^{op} \to \mathscr{V}$ in Ψ and every $T : \mathscr{G} \to \mathscr{D}$.

The weight $\mathrm{Lan}_{T^{op}}\psi : \mathscr{D}^{op} \to \mathscr{V}$ is in $\Psi_1(\mathscr{D})$ and every weight in $\Psi_1(\mathscr{D})$ has this form, for some $\psi : \mathscr{G}^{op} \to \mathscr{V}$ in Ψ and some $T : \mathscr{G} \to \mathscr{D}$.

Since there are isomorphisms

$$\{\psi, YT^{op}\} \cong \{\mathrm{Lan}_{T^{op}}\psi, Y\}, \quad \{\psi, \varphi \cdot T^{op}\} \cong \{\mathrm{Lan}_{T^{op}}\psi, \varphi\}$$

the equivalence of 1. and 2. follows. ■

The canonical morphism in (5.1) can be rewritten using coends and Yoneda Lemma as the morphism

$$\mathsf{can} : \int^{d} [\mathscr{D}^{op}, \mathscr{V}](Yd, \varphi) \otimes [\mathscr{D}^{op}, \mathscr{V}](\psi, Yd) \to [\mathscr{D}^{op}, \mathscr{V}](\psi, \varphi) \tag{5.2}$$

that is given by composition in $[\mathscr{D}^{op}, \mathscr{V}]$. We illustrate now on two well-known classes that this coend description yields precisely the "classical" description of flatness by means of the category of cocones.

Example 5.1.5 (Sifted weights and flat weights for $\mathscr{V} = \mathsf{Set}$). Suppose $\mathscr{V} = \mathsf{Set}$.

1. The class Π^{cone} is a sound class of weights. The category $\Pi_1^{\mathrm{cone}}(\mathscr{D})$ is spanned by finite coproducts of representables in $[\mathscr{D}^{op}, \mathsf{Set}]$. Hence a general testing weight $\psi : \mathscr{D}^{op} \to \mathsf{Set}$ for Π-flatness by Proposition 5.1.4 has the form $\coprod_{i \in I} Y d_i$ where I is a finite set.

   We show now that (5.2) yields the well-known characterisation of *sifted* weights.

   Indeed, given a general weight $\varphi : \mathscr{D}^{op} \to \mathsf{Set}$, the mapping can has the form

$$\mathsf{can} : \int^{d} \varphi(d) \times \prod_{i \in I} \mathscr{D}(d_i, d) \to \prod_{i \in I} \varphi d_i, \quad [(x, (t_i))] \mapsto (\varphi t_i(x))$$

   Hence can is a bijection if and only if the following two conditions hold:

   (a) The mapping can is surjective, i.e., for every element of $\prod_{i \in I} \varphi(d_i)$, i.e., for every I-tuple (x_i) of elements of φ there is a d, an element $x \in \varphi(d)$ and an I-tuple $t_i : d_i \to d$ of morphisms in $\mathscr{D}$ such that $\varphi t_i(x) = x_i$.

   Briefly: on every I-tuple of objects of $\mathsf{elts}(\varphi)$ there is a cocone.

   (b) The mapping can is injective, i.e., for any pair $(x, (t_i))$, $(x', (t_i'))$ such that $\varphi t_i(x) = \varphi t_i'(x')$ holds for all i, i.e., for any two cocones of the same I-tuple of objects of $\mathsf{elts}(\varphi)$ there is a zig-zag in $\mathscr{D}$ that connects these cocones in $\mathsf{elts}(\varphi)$.

   To summarise: a weight $\varphi : \mathscr{D}^{op} \to \mathsf{Set}$ is Π^{cone}-flat if and only if its category of elements is sifted, i.e. every finite family of elements has a cocone and every two cocones for the same finite family are connected by a zig-zag. (Recall from Proposition 1.3.5 and Remark 1.3.6 an equivalent characterisation of sifted categories.)

2. Let Ψ be the sound class of finite (conical) colimits, i.e., let $\Psi = \Psi_{\mathbb{D}}$ for the class $\mathbb{D}$ of finite categories.

   The category $\Psi_1(\mathscr{D})$ is spanned by finite colimits of representable functors in $[\mathscr{D}^{op}, \mathsf{Set}]$. Thus, a general testing weight $\psi : \mathscr{D}^{op} \to \mathsf{Set}$ for Ψ-flatness has the form $\psi = \operatorname{colim} YC$ for a diagram $C : \mathscr{C} \to \mathscr{D}$ with $\mathscr{C}$ finite.

   Given a general weight $\varphi : \mathscr{D}^{op} \to \mathsf{Set}$, the mapping can is a bijection if and only if two conditions hold:

   (a) The mapping can is surjective, i.e., every finite diagram in $\mathsf{elts}(\varphi)$ has a cocone.

   (b) The mapping can is injective, i.e., any two cocones for the same finite diagram in $\mathsf{elts}(\varphi)$ are connected by a zig-zag in $\mathsf{elts}(\varphi)$.

   The above two conditions together state that the category of cocones of finite diagrams in $\mathsf{elts}(\varphi)$ is nonempty and connected. This means that the category $\mathsf{elts}(\varphi)$ is filtered. As expected, Ψ-flat weights are precisely the flat ones.

We prove now that our definition of soundness from Definition 2.2.5 coincides with the definition from [1]. The definition from [1] is condition 2 of the following proposition.

Proposition 5.1.6. *Suppose $\mathscr{V} = \mathsf{Set}$. For a class $\mathbb{D}$ of small categories, the following conditions are equivalent:*

1. *The class $\Psi_{\mathbb{D}}$ of conical weights $\mathsf{const}_{\top} : \mathscr{D}^{op} \to \mathsf{Set}$ with $\mathscr{D}^{op}$ in $\mathbb{D}$ is sound.*

2. *A category $\mathscr{D}$ is $\mathbb{D}$-filtered whenever the category of cocones for any functor $T : \mathscr{G} \to \mathscr{D}$ with $\mathscr{G}^{op}$ in $\mathbb{D}$ is nonempty and connected.*

Proof. We will use the canonical morphism (5.2). Observe first that $[\mathscr{D}^{op}, \mathsf{Set}](\psi, \mathsf{const}_\top)$ is a one-element set for any small category $\mathscr{D}$ and any $\psi : \mathscr{D}^{op} \to \mathsf{Set}$, since $\mathsf{const}_\top$ is a terminal object in $[\mathscr{D}^{op}, \mathsf{Set}]$.

By Proposition 5.1.4 any testing weight $\psi : \mathscr{D}^{op} \to \mathsf{Set}$ for $\Psi_{\mathbb{D}}$-flatness of $\mathsf{const}_\top : \mathscr{D}^{op} \to \mathsf{Set}$ has the form $\mathrm{Lan}_{T^{op}}\mathsf{const}_\top$ for some $T : \mathscr{G} \to \mathscr{D}$, where $\mathsf{const}_\top : \mathscr{G}^{op} \to \mathsf{Set}$ is in $\Psi_{\mathbb{D}}$. The left-hand side of (5.2) therefore has the form

$$\begin{aligned}\int^d [\mathscr{D}^{op}, \mathsf{Set}](\mathrm{Lan}_{T^{op}}\mathsf{const}_\top, Yd) &\cong \int^d [\mathscr{G}^{op}, \mathsf{Set}](\mathsf{const}_\top, Yd \cdot T^{op}) \\ &\cong \int^d [\mathscr{G}^{op}, \mathsf{Set}](\mathsf{const}_\top, \mathscr{D}(T-, d))\end{aligned}$$

Observe that the category of elements of $[\mathscr{G}^{op}, \mathsf{Set}](\mathsf{const}_\top, \mathscr{D}(T-, d))$ is precisely the category of cocones for T that have d as a vertex.

Thus (5.2) is a bijection if and only if

$$\int^d [\mathscr{G}^{op}, \mathsf{Set}](\mathsf{const}_\top, \mathscr{D}(T-, d)) \cong \top$$

holds. From this, the equivalence of 1. and 2. follows immediately. ■

We shall now analyse the isomorphism (5.2) in more detail for the enrichment in Cat. We then turn the analysis into a useful elementary criterion of *siftedness* of weights enriched in Cat.

Suppose that $\mathscr{V} = \mathsf{Cat}$. Let $\psi, \varphi : \mathscr{D}^{op} \to \mathsf{Cat}$ be any weights. Then the coend

$$\int^d [\mathscr{D}^{op}, \mathsf{Cat}](Yd, \varphi) \times [\mathscr{D}^{op}, \mathsf{Cat}](\psi, Yd) \tag{5.3}$$

is a category that can be computed as a coequaliser in Cat of the parallel pair

$$\begin{tikzcd} \coprod\limits_{d,d'} [\mathscr{D}^{op}, \mathsf{Cat}](Yd', \varphi) \times \mathscr{D}(d, d') \times [\mathscr{D}^{op}, \mathsf{Cat}](\psi, Yd) \arrow[d, shift right=2, "L"'] \arrow[d, shift left=2, "R"] \\ \coprod\limits_{d} [\mathscr{D}^{op}, \mathsf{Cat}](Yd, \varphi) \times [\mathscr{D}^{op}, \mathsf{Cat}](\psi, Yd) \end{tikzcd} \tag{5.4}$$

of functors

$$\begin{aligned} L &: (\widehat{x} : Yd' \to \varphi, f : d \to d', \tau : \psi \to Yd) \mapsto (\widehat{x} \cdot Yf : Yd \to \varphi, \tau : \psi \to Yd) \\ R &: (\widehat{x} : Yd' \to \varphi, f : d \to d', \tau : \psi \to Yd) \mapsto (\widehat{x} : Yd' \to \varphi, Yf \cdot \tau : \psi \to Yd') \end{aligned}$$

Thus the coend (5.3) has the following description (see, e.g., [61]):

1. The objects are equivalence classes
$$[(\widehat{x}, \tau)]_\sim$$
where $\widehat{x} : Yd \to \varphi$ and $\tau : \psi \to Yd$ are natural transformations. The equivalence is generated by
$$(\widehat{x}, Yf \cdot \tau) \sim (\widehat{x} \cdot Yf, \tau)$$
for all $\widehat{x} : Yd \to \varphi$, $\tau : \psi \to Yd'$ $f : d' \to d$ in $\mathscr{D}$.

2. The morphisms are equivalence classes
$$[((u_1, v_1), \ldots, (u_n, v_n))]_\approx$$
of finite sequences $((u_1, v_1), \ldots, (u_n, v_n))$ such that every pair (u_i, v_i) is a morphism in the category $\coprod_d [\mathscr{D}^{op}, \mathsf{Cat}](Yd, \varphi) \times [\mathscr{D}^{op}, \mathsf{Cat}](\psi, Yd)$ and
$$\operatorname{cod}(u_1, v_1) \sim \operatorname{dom}(u_2, v_2), \quad \ldots, \quad \operatorname{cod}(u_{n-1}, v_{n-1}) \sim \operatorname{dom}(u_n, v_n)$$
The equivalence relation $\approx$ is generated from the following two conditions
$$(u * Yw, v) \approx (u, Yw * v), \quad ((u_1, v_1), (u_2, v_2)) \approx (u_2 \cdot u_1, v_2 \cdot v_1)$$
by reflexivity, symmetry, transitivity and composition (concatenation).

It will be useful to work with the following graphical representation. The sequence $((u_1, v_1), \ldots, (u_n, v_n))$ as above is going to be depicted as

$$\begin{tikzcd}
\varphi \arrow[r, equal] & \varphi \arrow[r, equal] & \varphi & \cdots & \varphi \arrow[r, equal] & \varphi \arrow[r, equal] & \varphi \\
d \arrow[u, "\widehat{x}"] \arrow[r, squiggly, no head] & d_1 \arrow[u, "\widehat{x}_1"] \arrow[r, equal] \arrow[ur, phantom, "\overset{u_1}{\Rightarrow}"] & d_1 \arrow[u, "\widehat{x}'_1"'] & \cdots & d_{n-1} \arrow[u, "\widehat{x}_n"] \arrow[r, equal] \arrow[ur, phantom, "\overset{u_n}{\Rightarrow}"] & d_{n-1} \arrow[u, "\widehat{x}'_n"'] \arrow[r, squiggly, no head] & d' \arrow[u, "\widehat{x}'"'] \\
\psi \arrow[u, "\tau"] \arrow[r, equal] & \psi \arrow[u, "\tau_1"] \arrow[r, equal] \arrow[ur, phantom, "\overset{v_1}{\Rightarrow}"] & \psi \arrow[u, "\tau'_1"'] & \cdots & \psi \arrow[u, "\tau_n"] \arrow[r, equal] \arrow[ur, phantom, "\overset{v_n}{\Rightarrow}"] & \psi \arrow[u, "\tau'_n"'] \arrow[r, equal] & \psi \arrow[u, "\tau'"']
\end{tikzcd} \tag{5.5}$$

The above picture is called a *hammock* from $(\widehat{x}, \tau)$ to $(\widehat{x}', \tau')$. The wiggly arrow in the above hammock, for example from $(\widehat{x}, \tau)$ to $(\widehat{x}_1, \tau_1)$, represents a zig-zag connecting d and d_1 in $\mathscr{D}$ that witnesses the equivalence $(\widehat{x}, \tau) \sim (\widehat{x}_1, \tau_1)$.

The whole hammock (5.5) gets evaluated to the composite modification
$$(u_n * v_n) \cdot (u_{n-1} * v_{n-1}) \cdot \cdots \cdot (u_1 * v_1) : \widehat{x} \cdot \tau \to \widehat{x}' \cdot \tau'$$
in $[\mathscr{D}^{op}, \mathsf{Cat}]$. Up to the equivalence $\approx$, this is how the evaluation functor can works.

The functor can is an isomorphism of categories if and only if it is bijective on objects and fully faithful. Hence, the following two conditions have to hold:

1. *The 1-dimensional aspect.* To give $\alpha : \psi \to \varphi$ is to give a unique $[(\widehat{x}, \tau)]_\sim$ such that $\widehat{x} \cdot \tau = \alpha$ holds.

2. *The 2-dimensional aspect.* To give a modification $\Xi : \alpha \to \alpha'$ is to give a unique equivalence class $[((u_1, v_1), \ldots, (u_n, v_u))]_\approx$ such that Ξ is the composite $(u_n * v_n) \cdot \cdots \cdot (u_1 * v_1)$.

5.2 Siftedness for enrichment in categories

Let us now fix the class Π^{cone} of (conical) weights for finite coproducts for this section, and study sifted weights in the enrichment $\mathscr{V} = \mathsf{Cat}$.

It is proved in [45] that the class $\Pi_1^{\mathrm{cone}}(\mathscr{D})$ of testing weights for siftedness of a weight $\varphi : \mathscr{D}^{op} \to \mathsf{Cat}$ can be reduced further to the empty coproduct $\mathsf{const}_0 : \mathscr{D}^{op} \to \mathsf{Cat}$ of representables and to binary coproducts $\mathscr{D}(-, d_1) + \mathscr{D}(-, d_2) : \mathscr{D}^{op} \to \mathsf{Cat}$. Hence, using (5.2), the following result holds:

Lemma 5.2.1. *A weight $\varphi : \mathscr{D}^{op} \to \mathsf{Cat}$ is sifted if and only if the following two conditions hold:*

1. *The unique functor from $\int^d \varphi(d)$ to the one-morphism category $\mathbf{1}$ is an isomorphism.*
2. *For any d_1, d_2 in $\mathscr{D}$, the canonical morphism*
$$\mathsf{can} : \int^d \varphi(d) \times \mathscr{D}(d_1, d) \times \mathscr{D}(d_2, d) \to \varphi(d_1) \times \varphi(d_2)$$
is an isomorphism.

Remark 5.2.2. By analogy to the case $\mathscr{V} = \mathsf{Set}$, we may call the first condition above *connectedness* of the weight $\varphi : \mathscr{D}^{op} \to \mathsf{Cat}$ and the second condition expresses that the diagonal 2-functor $\Delta : \mathscr{D} \to \mathscr{D} \times \mathscr{D}$ is *cofinal* in the sense that the 2-cell

$$\begin{tikzcd}
\mathscr{D}^{op} \arrow[rr, "\Delta^{op}"] \arrow[dr, "\varphi"'] & {} \arrow[d, phantom, "\overset{\delta}{\Rightarrow}"] & \mathscr{D}^{op} \times \mathscr{D}^{op} \arrow[dl, "(d_1,d_2)\mapsto\varphi(d_1)\times\varphi(d_2)"] \\
& \mathsf{Cat} &
\end{tikzcd}$$

where $\delta_d : \varphi(d) \to \varphi(d) \times \varphi(d)$ is the diagonal functor, is a left Kan extension. Indeed, it suffices to consider the isomorphism

$$\int^d \varphi(d) \times \mathscr{D}(d_1, d) \times \mathscr{D}(d_2, d) \cong \int^d \varphi(d) \times (\mathscr{D}^{op} \times \mathscr{D}^{op})(\Delta^{op}(d), (d_1, d_2))$$

We apply the criteria of Lemma 5.2.1, together with the analysis of (5.2) using hammocks, for giving elementary proofs of siftedness of various weights.

Example 5.2.3 (A weight that is not sifted). We start with an example of a weight $\varphi : \mathscr{D}^{op} \to \mathsf{Cat}$ that is not sifted, although the 'underlying' ordinary functor

$$\mathscr{D}_0^{op} \xrightarrow{\varphi_0} \mathsf{Cat}_0 \xrightarrow{\mathrm{ob}} \mathsf{Set},$$

where ob denotes the forgetful "object" functor, is sifted.

Consider the one-morphism category $\mathscr{S}$ with the only object s. Denote by $\mathscr{D}^{op}$ the free completion of $\mathscr{S}$ under finite products. It follows immediately that the only 2-cells in $\mathscr{D}^{op}$ are identities.

Let $\chi : \mathscr{D}^{op} \to \mathsf{Cat}$ be the product-preserving functor defined by $\chi(s) = \mathbf{2}$, where $\mathbf{2}$ is the two-element chain, considered as a category. We define φ to be the following modification of χ: where $\chi(s^n) = \mathbf{2}^n$, we let $\varphi(s^n) = 2^n$ for every $n > 1$. The structure on 2^n is that of an almost discrete preorder with the only nontrivial inequality being $(0,\dots,0) \leqslant (1,\dots,1)$. The action of φ on morphisms is defined as for χ. Of course, φ does *not* preserve products, but the composite

$$\mathscr{D}_0^{op} \xrightarrow{\varphi_0} \mathsf{Cat}_0 \xrightarrow{\mathrm{ob}} \mathsf{Set}$$

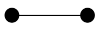
does; in fact, it is not hard to see that this ordinary functor constitutes an algebra for the ordinary algebraic theory $\mathscr{D}_0$ and thus it is a sifted weight by [5].

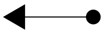
It is enough now to find pairs (x_1, x_2) and (y_1, y_2) from $\varphi(s) \times \varphi(s)$ such that $(x_1, x_2) \leqslant (y_1, y_2)$ holds but there is no hammock to witness this inequality. Consider $(x_1, x_2) = (0, 1)$ and $(y_1, y_2) = (1, 1)$. Firstly, we make use of the fact that there are no nontrivial 2-cells in $\mathscr{D}^{op}$. This implies that the 'lax' parts of the hammock consist only of inequalities between the elements of $\varphi(s^n) = 2^n$ for some s^n. But these are precisely the diagonal inequalities $(0,\dots,0) \leqslant (1,\dots,1)$. Together with the fact that the only morphisms of the form $s^n \to s$ in $\mathscr{D}^{op}$ are the product projections, it is easy to see that there is no way how any hammock could evaluate its right-hand side to $(1, 1)$ and its left-hand side to $(0, 1)$.

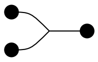
Remark 5.2.4. Siftedness of the composite $\mathrm{ob} \cdot \varphi_0 : \mathscr{D}_0^{op} \to \mathsf{Set}$ establishes precisely the 1-dimensional aspect of siftedness: the functor can is bijective on objects iff $\mathsf{ob} \cdot \varphi_0$ is sifted. From this it immediately follows that a weight $\varphi : \mathscr{D}^{op} \to \mathsf{Cat}$ with $\mathscr{D}$ locally discrete (i.e., with only the identity 2-cells) and such that every $\varphi(d)$ is a discrete category is sifted if and only if the composite $\mathrm{ob} \cdot \varphi_0 : \mathscr{D}_0^{op} \to \mathsf{Set}$ is sifted in the ordinary sense.

The 2-dimensional aspect of siftedness of $\varphi : \mathscr{D}^{op} \to \mathsf{Cat}$ has to be verified in general. Example 5.2.3 exhibits such a situation when $\mathscr{D}$ is locally discrete and Example 5.2.6 shows a conical weight $\mathsf{const}_{\top} : \mathscr{D}^{op} \to \mathsf{Cat}$ that is not sifted although the underlying ordinary category $\mathscr{D}_0$ is sifted in the ordinary sense.

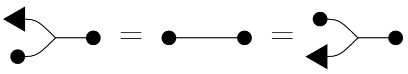
Example 5.2.5 (Siftedness for weights based on the simplicial category). Recall from, e.g., [64], that the simplicial category Δ has finite ordinals as objects and monotone maps as morphisms. It can be proved rather easily that the morphisms of Δ can be obtained from $id_1 : 1 \to 1$, $\eta : 0 \to 1$ and $\mu : 2 \to 1$ by ordinal sums subject to monad axioms. Hence we will draw the morphisms of Δ as string diagrams that are generated from the following strings



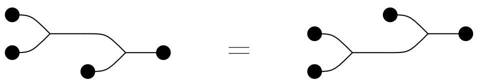




that represent $id_1 : 1 \to 1$, $\eta : 0 \to 1$ and $\mu : 2 \to 1$, respectively, by vertical concatenation that is subject to the unit axioms



and the associativity axiom



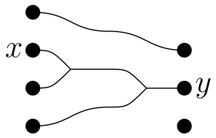
We show that both the conical weight on Δ and the weight given by inclusion of Δ into Cat are sifted weights. In fact, from our reasoning it will be clear that the same holds of almost any *truncation* Δ_n of Δ. The truncated category Δ_n is just the full subcategory of Δ spanned by finite ordinals up to n.

1. It is easy to show that $\mathsf{const}_\top : \Delta \to \mathsf{Set}$ is an ordinary sifted weight, and therefore even the conical weight $\mathsf{const}_\top : \Delta \to \mathsf{Cat}$ is sifted due to the fact that there are no non-trivial 2-cells in Δ, see Remark 5.2.4. Every truncation $\Delta^{(n)}$ (for $n \geqslant 1$) of the simplicial category Δ gives rise to a conical sifted weight as well.

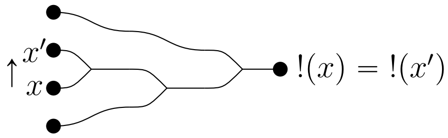
2. Suppose the weight $\varphi : \Delta \to \mathsf{Cat}$ is given by inclusion. Here

$$\Delta_0 \xrightarrow{\varphi_0} \mathsf{Cat}_0 \xrightarrow{\text{ob}} \mathsf{Set}$$

is an (ordinary) representable weight $\Delta_0(1, -) : \Delta_0 \to \mathsf{Set}$. For each object n of Δ the category $\varphi(n)$ is the free linearly ordered category on an n-element chain. We will show an elementary proof that φ is a sifted weight. First of all, let us check that the coend $\int^n \varphi(n)$ is isomorphic to the terminal object $\top$, i.e., the one-morphism category $\mathbf{1}$. Of course, the category $\int^n \varphi(n)$ has precisely one object: given any two objects $x \in \varphi(n)$ and $y \in \varphi(m)$, they are equivalent by $\sim$ if there exists a string diagram $\sigma : \varphi(n) \to \varphi(m)$ such that x gets mapped to y by σ. A diagram like this always exists; we illustrate this on an example situation with $n = 4$ and $m = 3$:



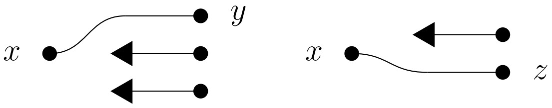
Now given any morphism $f : x \to x'$ in $\varphi(n)$, we show that $f \approx id_*$, where id_* is the identity morphism on the only object $*$ of $\varphi(1)$. This is again immediate when using the string diagrams: consider the only string diagram $! : \varphi(n) \to \varphi(1)$. It maps all morphisms in $\varphi(n)$ to the identity morphism, see for example the diagram below.



So the category $\int^n \varphi(n)$ indeed has only one morphism. Now we show the isomorphism

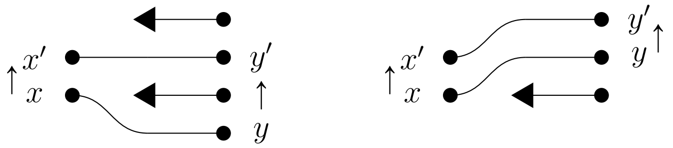
$$\int^n \varphi(n) \times \Delta(n, n_1) \times \Delta(n, n_2) \cong \varphi(n_1) \times \varphi(n_2)$$

by showing that the canonical morphism is bijective on objects and fully faithful. On objects, the canonical morphism takes an object $x \in \varphi(n)$, two string diagrams $\sigma : \varphi(n) \to \varphi(n_1)$ and $\tau : \varphi(n) \to \varphi(n_2)$, and computes the pair $(\sigma(x), \tau(x))$. It is immediate that for any pair (y, z) in $\varphi(n_1) \times \varphi(n_2)$ there exists a tuple (x, σ, τ) that is mapped to (y, z). More is true: we can always choose $x = * \in \varphi(1)$ and the string diagrams σ, τ are the obvious diagrams choosing y and z, respectively.



This proves that can is bijective on objects. In order to prove that can is full, we will show that given any pair of morphisms $g : y \to y'$ and $h : z \to z'$ in $\varphi(m)$ and

$\varphi(p)$ respectively, there is a morphism $f : x \to x'$ in $\varphi(n)$ and two string diagrams sending the morphism f to g and h, respectively. But there is again a canonical such $f : x \to x'$ in $\varphi(2)$ with the obvious inclusions, as is shown in the example diagram below.



Thus we have proved fullness *and* faithfulness of the canonical functor can. The weight φ is sifted.

We have actually proved that any truncation $\varphi^{(n)} : \Delta^{(n)} \to$ Cat of the inclusion weight is also sifted for $n \geqslant 1$.

The 2-dimensional aspect of siftedness is crucial for Cat-enriched weights even in the case of conical weights, as we show in the following easy example.

Example 5.2.6 (A conical weight that is not sifted). Consider the diagram scheme for reflexive coequalisers satisfying $\delta_0 \cdot \sigma = \delta_1 \cdot \sigma = id_1$, and adjoin freely a 2-cell α to it:

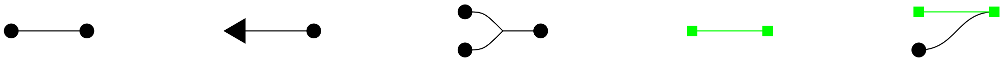
$$\begin{tikzcd}
2 \arrow[r, bend left=60, "\delta_0", ""{name=U, below}] \arrow[r, bend right=60, "\delta_1"', ""{name=D, above}] & 1 \arrow[l, "\sigma"'{name=M}]
\arrow[from=U, to=M, phantom, "\|"]
\arrow[from=M, to=D, Rightarrow, "\alpha"]
\end{tikzcd}$$

The resulting 2-category $\mathscr{D}$, when considered as a conical weight, is *not* sifted (the 2-cell is not reflexive), although the underlying ordinary category $\mathscr{D}_0$ is sifted in the ordinary sense (see, e.g., Chapter 3 of [5]).

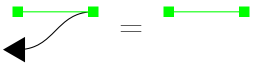
Example 5.2.7 (Siftedness for the weight for Kleisli objects). The weight $\varphi : \mathscr{D}^{op} \to$ Cat such that φ-colimits yield Kleisli objects is described in [62]. We will recall the definition of the weight φ and prove that it is sifted. That the weight φ is sifted is known from Proposition 8.43 in [21]: in this example we show an elementary proof of this fact.

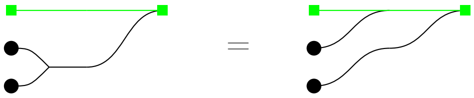
The 2-category $\mathscr{D}$ is the suspension $\Sigma\Delta$ of the simplicial category Δ. This means that $\mathscr{D}$ has a unique object, say d_0, and that the hom-category $\mathscr{D}(d_0, d_0)$ is the category Δ. Morphisms in $\mathscr{D}$ are finite ordinals, and the 2-cells are 'monad-like' string diagrams as described in Example 5.2.5.

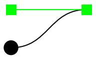
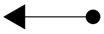
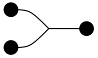
The category $\varphi(d_0)$ is defined as follows: the objects are finite *non-zero* ordinals, that is, objects of the form $n + 1$ for some natural number n. Every object $n + 1$ is understood as a $(n + 1)$-element chain with a distinguished top element. The morphisms in $\varphi(d_0)$ are precisely the monotone maps that preserve the distinguished top element. This definition of $\varphi(d_0)$ again allows a pictorial description in terms of string diagrams. The morphisms in $\varphi(d_0)$ are string diagrams generated by the basic diagrams



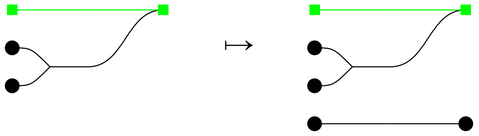
subject to monad axioms and the two axioms

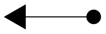


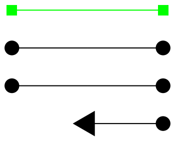


that express the fact that the diagram  is an algebra for the monad given by the unit  and multiplication  .

The 2-functor $\varphi : \mathscr{D}^{op} \to \mathsf{Cat}$ is defined on the morphisms and 2-cells of $\mathscr{D}^{op}$ by concatenation: for a given morphism $n : e_0 \to e_0$, the functor $\varphi(n) : \varphi(e_0) \to \varphi(e_0)$ maps an object $m+1 \in \varphi(e_0)$ to the object $m+n+1$. A string diagram s in $\varphi(e_0)$ is mapped to the diagram $\varphi(n)(s)$, defined as the diagram s concatenated with n identity strings. We show an example of this assignment for $n = 1$:



Likewise, given a 2-cell $\theta : m \to n$ in $\mathscr{D}$, the natural transformation $\varphi(\theta)$ is defined componentwise: for an object $m+1$ in $\varphi(e_0)$, the morphism $\varphi(\theta)_{m+1}$ is the concatenation of the identity diagram on $m+1$ with the diagram θ. For example, given the diagram  as θ and $m = 2$, the component $\varphi(\theta)_3$ is the following string diagram in $\varphi(d_0)$:

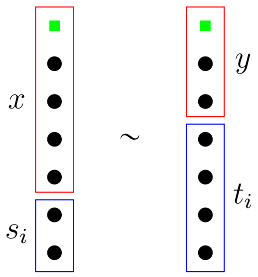


Now to prove that φ is a sifted weight, we need to verify that there are canonical isomorphisms

$$\int^{d} \varphi(d) \cong \mathbf{1}, \qquad \int^{d} \varphi(d) \times \mathscr{D}(d_0, d) \times \mathscr{D}(d_0, d) \cong \varphi(d_0) \times \varphi(d_0) \tag{5.6}$$

proving that $\varphi * (-)$ preserves nullary and binary products. We first analyse parts of a general hammock (5.5) for the weight φ with the testing weight $\psi = \coprod_{i \in I} \mathscr{D}(-, d_i)$. The left-hand side rectangle on the diagram below

$$\begin{tikzcd}
\varphi \arrow[r, equal] & \varphi \\
e_0 \arrow[u, "x"] \arrow[r, "f"] & e_0 \arrow[u, "y"'] \\
\psi \arrow[u, "s_i"] \arrow[r, equal] & \psi \arrow[u, "t_i"']
\end{tikzcd}$$



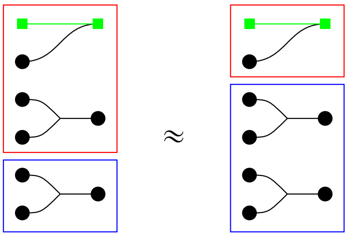
represents the information that for each $i \in I$ and the morphisms given in the diagram we have that equalities $f + s_i = t_i$ and $x = y + f$ hold in natural numbers. This situation is depicted on the right-hand side of the above diagram. In general, the tuples (x, s_i) and

(y, t_i) are related by the equivalence relation $\sim$ if and only if $x + s_i = y + t_i$ holds for all $i \in I$.

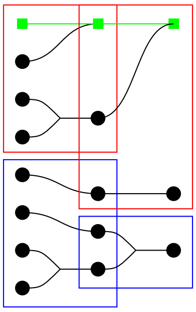
The rectangle of the form

$$\begin{tikzcd}
\varphi \arrow[r, equal] & \varphi \\
e_0 \arrow[u, "x"] \arrow[r, "f"] \arrow[ru, phantom, "\overset{u}{\Rightarrow}"] & e_0 \arrow[u, "y"'] \\
\psi \arrow[u, "s_i"] \arrow[r, equal] \arrow[ru, phantom, "\overset{v_i}{\Rightarrow}"] & \psi \arrow[u, "t_i"']
\end{tikzcd}$$

is represented by the concatenation of two string diagrams u and v_i for each $i \in I$.



The above diagram is an example of string diagrams that are equivalent: the 'sliding' of the division between the string diagrams generates the equivalence relation $\approx$. Observe moreover that morphisms in the coend are n-tuples of composable string diagrams. Any such n-tuple is equivalent to a 1-tuple, but the fact that we are allowed to vertically 'decompose' any string diagram to n parts is important in the proof of siftedness for φ. In the following diagram



we can see such a decomposition of a string diagram into a 2-tuple of shorter string diagrams.

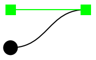
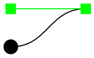
With the complete description of the weight φ and of the hammocks, we can conclude that we have the canonical isomorphisms in (5.6):

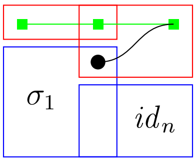
1. The weight φ satisfies the isomorphism

$$\int^d \varphi(d) \cong \mathbf{1}.$$

Indeed, the coend $\int^d \varphi(d)$ has precisely one object: any pair $n+1$ and $m+1$ of objects in $\varphi(d_0)$ is related by a hammock of length 2:

$$\begin{tikzcd}
\varphi \arrow[r, equal] & \varphi \arrow[r, equal] & \varphi \\
d_0 \arrow[u, "n+1"] \arrow[r, "n"'] & d_0 \arrow[u, "1"] & d_0 \arrow[u, "m+1"'] \arrow[l, "m"]
\end{tikzcd}$$

To show that $\int^d \varphi(d)$ has a unique morphism, we will prove that any string diagram $\sigma : m+1 \to n+1$ is congruent by the equivalence relation $\approx$ to an identity string diagram $id_k : k \to k$ for some natural number k. We have to distinguish two cases. If the diagram σ does not contain  as a subdiagram, then it is trivially a concatenation of two string diagrams $\sigma_0 = id_1 : 1 \to 1$ and $\sigma_1 : m \to n$, and therefore $\sigma \approx id_1$ holds. If σ contains , then it is necessary to factor it into a composition of two diagrams (denote the red part of the diagram by ω):



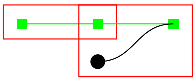
This decomposition is unique. Take the identity morphism id_n and decompose it in the same way into a concatenation of ω with (τ_1, id_n). By the first case we have that $\tau_1 \approx \sigma_1$, and the equivalence $id_n \approx id_n$ is trivial. This decomposition thus witnesses the equivalence $\sigma \approx id_n$.

2. The second isomorphism

$$\int^d \varphi(d) \times \mathscr{D}(d_0, d) \times \mathscr{D}(d_0, d) \cong \varphi(d_0) \times \varphi(d_0)$$

is proved similarly to the case above. Given two objects $m+1$ and $n+1$ from $\varphi(d_0)$, there is a triple $(1, m, n)$ that gets mapped exactly to $(m+1, n+1)$ by the canonical functor. For any other triple (k, m', n') that is mapped to $(m+1, n+1)$ we have the equalities $k + m' = m + 1$ and $k + n' = n + 1$. Therefore $(1, m, n) \sim (k, m', n')$ holds and the canonical functor is bijective on objects.

To prove that the canonical functor is full, we show that for any two string diagrams $\sigma : m+1 \to n+1$ and $\tau : p+1 \to q+1$ there is a triple (ω, α, β) getting mapped to (σ, τ). But again, as in the case of the first isomorphism, take ω to be the diagram



and factor the diagrams σ and τ into pairs $\alpha = (\sigma_1, id_n)$ and $\beta = (\tau_1, id_q)$ in a way that $\omega * \alpha = \sigma$ and $\omega * \beta = \tau$, where $*$ denotes the horizontal composition. Faithfulness of the canonical functor then comes easily from the fact that the morphisms in the coend have the above mentioned 'normal form'.

The proof of siftedness is complete.

5.3 Siftedness for enrichment in preorders

The enrichment in the category Pre of preorders and monotone maps is in many aspects similar to the enrichment in Cat, but the computations are much simpler. In fact, we will be able to give a full characterisation of sifted *conical* weights $\mathsf{const}_\top : \mathscr{D}^{op} \to \mathsf{Pre}$, see Example 5.3.1.

The crucial coend

$$\int^d \varphi(d) \times [\mathscr{D}^{op}, \mathsf{Pre}](\psi, Yd)$$

is computed as a coequaliser in Pre of two monotone maps L and R that are defined in the same way as for $\mathscr{V} = \mathsf{Cat}$, see (5.4). Moreover, the coequaliser of L and R can be computed in two steps. First we compute the coequaliser on the level of underlying sets. This yields a set of equivalence classes of the form $[(\widehat{x}, \tau)]_\sim$ with respect to the equivalence $\sim$ generated by L and R. The set of equivalence classes is then equipped with a least preorder $\sqsubseteq$ satisfying the following condition:

$$\text{If } (\widehat{x}, \tau) \leqslant (\widehat{y}, \sigma), \text{ then } [(\widehat{x}, \tau)]_\sim \sqsubseteq [(\widehat{y}, \sigma)]_\sim.$$

where $\leqslant$ denotes the preorder of the coproduct $\coprod_d \varphi(d) \times [\mathscr{D}^{op}, \mathsf{Pre}](\psi, Yd)$.

Below, we will also use hammocks for the enrichment in Pre. These are pictures like (5.5) but the 2-cells u_i, v_i are replaced by mere inequality signs.

We show now that for *conical* weights $\varphi : \mathscr{D}^{op} \to \mathsf{Pre}$ the 2-dimensional aspect of siftedness is vacuous. That this is not true for *general* weights $\varphi : \mathscr{D}^{op} \to \mathsf{Pre}$ is demonstrated by the weight of Example 5.2.3: all categories there are in fact enriched in Pre.

Example 5.3.1 (Sifted conical weights). The reasoning is similar to Example 5.1.5 above. Elements of $\Pi_1^{\mathrm{cone}}(\mathscr{D})$ are finite coproducts $\coprod_{i \in I} Yd_i$ of representables in $[\mathscr{D}^{op}, \mathsf{Pre}]$. By Yoneda Lemma, every $\tau : \psi \to Yd$ can be identified with a cocone $t_i : d_i \to d$. Then the requirement that for any two natural transformations $\tau : \psi \to Yd$ and $\sigma : \psi \to Yd$ the equivalence $\tau \sim \sigma$ has to hold, corresponds to the fact that the cocones $t_i : d_i \to d$ and $s_i : d_i \to d$ (corresponding to τ and σ respectively) have to be connected by a zig-zag. The 2-dimensional aspect of siftedness is vacuous in this case.

Thus a weight $\mathsf{const}_\top : \mathscr{D}^{op} \to \mathsf{Pre}$ is sifted if and only if the *ordinary* functor

$$\mathscr{D}_0^{op} \xrightarrow{(\mathsf{const}_\top)_0} \mathsf{Pre}_0 \xrightarrow{\mathrm{ob}} \mathsf{Set}$$

(with ob being the forgetful functor) is sifted in the *ordinary* sense.

Example 5.3.2 (Sifted weights in general). Consider a general weight $\varphi : \mathscr{D}^{op} \to \mathsf{Pre}$. To establish the isomorphism

$$\mathsf{can} : \int^d \varphi(d) \times \prod_{i \in I} \mathscr{D}(d_i, d) \to \prod_{i \in I} \varphi(d_i),$$

of preorders we need the monotone map can to be bijective and order-reflecting. As we noticed earlier, the coend is computed as a coequaliser in Set equipped with a freely generated preorder. More precisely, there are two conditions for a weight to be sifted:

1. To obtain bijectivity on objects of the can mapping we demand that

$$\mathscr{D}_0^{op} \xrightarrow{\varphi_0} \mathsf{Pre}_0 \xrightarrow{\mathrm{ob}} \mathsf{Set}$$

be an ordinary sifted weight.

2. Order-reflectivity of can means that given any two tuples $(x_i) \leqslant (x'_i)$ from $\prod_{i\in I}\varphi(d_i)$ we can form a hammock

$$\begin{tikzcd}
\varphi \arrow[r, equal] & \varphi \arrow[r, equal] & \varphi & \cdots & \varphi \arrow[r, equal] & \varphi \arrow[r, equal] & \varphi \\
d \arrow[u, "\widehat{x}"] \arrow[r, squiggly, no head] & d_1 \arrow[u, "\widehat{x}_1"] \arrow[r, equal] \arrow[ur, phantom, "\leqslant"] & d_1 \arrow[u, "\widehat{x}'_1"'] & \cdots & d_{n-1} \arrow[u, "\widehat{x}_n"] \arrow[r, equal] \arrow[ur, phantom, "\leqslant"] & d_{n-1} \arrow[u, "\widehat{x}'_n"'] \arrow[r, squiggly, no head] & d' \arrow[u, "\widehat{x}'"'] \\
(d_i) \arrow[u, "\tau"] \arrow[r, equal] & (d_i) \arrow[u, "\tau_1"] \arrow[r, equal] \arrow[ur, phantom, "\leqslant"] & (d_i) \arrow[u, "\tau'_1"'] & \cdots & (d_i) \arrow[u, "\tau_n"] \arrow[r, equal] \arrow[ur, phantom, "\leqslant"] & (d_i) \arrow[u, "\tau'_n"'] \arrow[r, equal] & (d_i) \arrow[u, "\tau'"']
\end{tikzcd}$$

such that its left-hand vertical side evaluates to (x_i), and its right-hand vertical side evaluates to (x'_i).

Remark 5.3.3. Observe that the characterisations of sifted weights for enrichments in Cat and Pre are strongly related. This is because the computations of coequalisers are essentially the same.

In fact, the requirements for a weight $\varphi : \mathscr{D}^{op} \to \mathsf{Pre}$ to be sifted (as enriched in Pre) are exactly the requirements of siftedness for the weight $\varphi' : \mathscr{D}^{op} \to \mathsf{Cat}$, with φ' being the weight φ considered as enriched in Cat.

The situation is rather different when considering sifted weights for the enrichment in the category Pos of all posets and monotone maps. The computation of a coequaliser in Pos runs in two steps: one computes the coequaliser in preorders *and then* performs the poset-reflection. It is the second step that brings in additional identifications and makes the characterisation of siftedness quite complex.

Chapter 6

Two-dimensional Birkhoff theorem

In this chapter we will state and give a proof of a 2-dimensional analogue of the Birkhoff theorem from universal algebra. Recall from Theorem 1.5.6 that in the ordinary setting, Birkhoff's theorem characterises equational subcategories of algebraic categories. An algebraic category can be viewed as a category $\mathsf{Alg}(T)$ of algebras for a *strongly finitary monad* T on Set. (A monad is strongly finitary if its underlying functor is strongly finitary, i.e., if it preserves sifted colimits [5].) A full subcategory $\mathscr{A}$ of $\mathsf{Alg}(T)$ is said to be an *equational subcategory* of $\mathsf{Alg}(T)$ if it is (equivalent to) the category $\mathsf{Alg}(T')$ of algebras for a strongly finitary monad T', where T' is constructed by "adding new equations" to the monad T. More precisely, we ask T' to be a quotient of T, meaning that there is a monad morphism $e : T \to T'$ that is moreover a regular epimorphism. The resulting algebraic functor

$$\mathsf{Alg}(e) : \mathsf{Alg}(T') \to \mathsf{Alg}(T)$$

then exhibits $\mathsf{Alg}(T')$ as an equational subcategory of $\mathsf{Alg}(T)$. Every such subcategory $\mathsf{Alg}(T') \to \mathsf{Alg}(T)$ has nice closure properties with respect to to the inclusion into $\mathsf{Alg}(T)$. The content of Birkhoff's theorem is that equational subcategories can be characterised by these closure properties (see, e.g., [81]). In essence, this theorem holds since algebraic categories are well-behaved with respect to quotients (regular epis) — they are *exact categories* [5].

Taking inspiration from the ordinary case, we want to give a characterisation of equational subcategories of algebraic categories in the enriched setting. Namely, we shall mainly work with categories enriched in the symmetric monoidal closed category $\mathscr{V} = \mathsf{Cat}$ and we will accordingly use the enriched notions of a functor, natural transformation, etc.

Analogously to the ordinary case, in defining the notion of an equational subcategory of $\mathsf{Alg}(T)$ the idea is again to consider "quotients" $e : T \to T'$ of strongly finitary 2-monads. Any subcategory $\mathsf{Alg}(e) : \mathsf{Alg}(T') \to \mathsf{Alg}(T)$ exhibited by a quotient $e : T \to T'$ is an equational subcategory of $\mathsf{Alg}(T)$.

Unlike to the $\mathscr{V} = \mathsf{Set}$ case, it is not immediately clear that some well-behaved notion of a quotient of strongly finitary 2-monads should exist. In $\mathscr{V} = \mathsf{Set}$, the quotients come as the "epi part" of the (regular epi, mono) factorisation system, and they are computed as certain colimits, the coequalisers. The solution in $\mathscr{V} = \mathsf{Cat}$ is to mimic this approach. Thus we should study factorisation systems on Cat (and the respective notions of a quotient), and find out which factorisation systems "lift up" from the category Cat to Cat-enriched algebraic categories. That is, we want to find factorisation systems that render the algebraic categories over Cat *exact* in some suitably generalised sense. This would allow us to talk about quotients of strongly finitary 2-monads while preserving the good behaviour of quotients as in Cat.

Recent advances in the theory of 2-dimensional exactness (see [22]) show that there are at least three notions of a quotient coming from three factorisation systems $(\mathcal{E}, \mathcal{M})$ on Cat, for which algebraic categories over Cat are exact:

1. (surjective on objects, injective on objects and fully faithful),

2. (bijective on objects, fully faithful),

3. (bijective on objects and full, faithful).

(For the $\mathcal{E}$ parts of the above systems, we will use the standard abbreviations, namely s.o. for surjective on objects, b.o. for bijective on objects, and b.o. full for bijective on objects and full.) We show that the 2-category $\mathsf{Mnd}_{\mathsf{sf}}(\mathsf{Cat})$ of strongly finitary 2-monads over Cat is exact in the sense of [22] with respect to all the three factorisation systems above as well.

We focus on the factorisation system (b.o. full, faithful). Unlike the other two systems, it corresponds to a meaningful notion of an equational subcategory, and it allows us to prove the 2-dimensional Birkhoff theorem by arguments very similar to those used in the proof of the ordinary Birkhoff theorem. For this factorisation system, the exactness of $\mathsf{Mnd}_{\mathsf{sf}}(\mathsf{Cat})$ implies that a monad morphism $e : T \to T'$ is a quotient if and only if $e_C : TC \to T'C$ is a b.o. full functor in Cat for every category C. We shall often use this "pointwise" nature of quotient monad morphisms.

The main result of the chapter characterises equational subcategories of algebraic categories as those that are closed under products, quotients, subalgebras and sifted colimits. This is a characterisation in the spirit of the ordinary Birkhoff theorem. In the universal algebraic formulations, only the first three closure properties are demanded, and are dubbed "HSP" conditions. However, it was found out in [6] that even in the ordinary case, the property of being closed under *filtered* colimits is necessary when dealing with infinitely-sorted algebras. It is thus not surprising that the additional requirement of closedness under *sifted* colimits might be needed in the 2-dimensional case: the *finitary* and *strongly finitary* 2-monads no longer coincide in Cat (see Remark 6.2.4 for a distinguishing example), and we are dealing with the strongly finitary ones. The choice of working with strongly finitary 2-monads is fairly natural, since the 2-category $\mathsf{Mnd}_{\mathsf{sf}}(\mathsf{Cat})$ is equivalent to the 2-category Law of Cat-enriched one-sorted algebraic theories (also dubbed *Lawvere 2-theories*) [57].

Structure of the chapter.

1. We summarise the relevant parts of the theory of enriched factorisation systems from [22] in Section 6.1. We also recall some basic notions of the theory of category-enriched monads (2-monads).

2. In Section 6.2 we prove the 2-dimensional Birkhoff's theorem (Theorem 6.2.3).

The results of this chapter were published in the paper [28] by the author. The wording of the chapter is a slight modification of the text of the paper.

6.1 Kernels and quotients in 2-categories

We shall make heavy use of factorisation systems in discussing and proving the Birkhoff theorem. The study of factorisation systems in general 2-categories is more involved than in the ordinary case. Following the exposition in [22], we first recall the definitions of enriched orthogonality and enriched factorisation systems in a general $\mathscr{V}$-category for a symmetric monoidal closed base category $\mathscr{V}$. Then we introduce *kernel-quotient systems* that generalise the correspondence between regular epimorphisms and kernels in exact categories, and we use this notion to introduce the (b.o. full, faithful) factorisation system on Cat. This factorisation system lifts up to a large class of algebraic categories, as is shown in Theorem 6.1.8. As an important corollary we show in Proposition 6.1.10 that the 2-category of strongly finitary monads on Cat inherits the (b.o. full, faithful) factorisation system, allowing us to study quotients of monads.

Definition 6.1.1. A morphism $f : A \to B$ in a $\mathscr{V}$-category $\mathscr{C}$ is *$\mathscr{V}$-orthogonal* to $g : C \to D$ (denoted by $f \perp g$) if the diagram

$$\begin{tikzcd}
\mathscr{C}(B,C) \arrow[r, "\mathscr{C}(B,g)"] \arrow[d, "\mathscr{C}(f,C)"'] & \mathscr{C}(B,D) \arrow[d, "\mathscr{C}(f,D)"] \\
\mathscr{C}(A,C) \arrow[r, "\mathscr{C}(A,g)"'] & \mathscr{C}(A,D)
\end{tikzcd}$$

is a pullback in $\mathscr{V}$. Given a class $\mathcal{G}$ of morphisms of $\mathscr{C}$, we define two classes of morphisms $\mathscr{V}$-orthogonal to those in $\mathcal{G}$:

- $\mathcal{G}^{\downarrow} := \{m \mid \forall g \in \mathcal{G} : g \perp m\}$,
- $\mathcal{G}^{\uparrow} := \{e \mid \forall g \in \mathcal{G} : e \perp g\}$.

Given an object C of $\mathscr{C}$, the morphism $f : A \to B$ is orthogonal to C if f is orthogonal to 1_C, i.e., if the precomposition map

$$\mathscr{C}(f,C) : \mathscr{C}(B,C) \to \mathscr{C}(A,C)$$

is invertible (i.e., an isomorphism). We denote this fact by $f \perp C$.

Let $\mathcal{E}$ and $\mathcal{M}$ be two classes of morphisms of $\mathscr{C}$. We say that $(\mathcal{E}, \mathcal{M})$ is a *$\mathscr{V}$-factorisation system* if

1. $\mathcal{M} = \mathcal{E}^{\downarrow}$,
2. $\mathcal{E} = \mathcal{M}^{\uparrow}$, and
3. every morphism f in $\mathscr{C}$ can be factorised as the composition $m \cdot e$ of a morphism m in $\mathcal{M}$ and a morphism e in $\mathcal{E}$.

Example 6.1.2. We examine when two morphisms $f : A \to B$ and $g : C \to D$ are orthogonal in $\mathscr{C}$ for the case of $\mathscr{V} = \mathsf{Cat}$. Firstly, the morphisms have to satisfy the usual diagonal fill-in property

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "x"'] & B \arrow[d, "y"] \arrow[dl, dotted, "\exists ! d"'] \\
C \arrow[r, "g"'] & D
\end{tikzcd}$$

for every pair $x : A \to C$ and $y : B \to D$ of morphisms in $\mathscr{C}$. Let us denote by $d : A \to D$ the diagonal fill-in for x and y, and denote by $d' : A \to D$ the diagonal fill-in for x' and y'. The second requirement on f and g to be orthogonal is that they satisfy the diagonal *2-cell* property: for every pair $\alpha : x \Rightarrow x'$ and $\beta : y \Rightarrow y'$ of 2-cells such that

$$\begin{tikzcd}
A \arrow[d, bend right=50, "x"'] \arrow[d, bend left=50, "x'"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] & \\
C \arrow[r, "g"'] & D
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[d, bend right=50, "y"'] \arrow[d, bend left=50, "y'"] \arrow[d, phantom, "\overset{\beta}{\Rightarrow}"] \\
 & D
\end{tikzcd}$$

there has to exist a *unique* 2-cell $\delta : d \Rightarrow d'$ such that the equalities

$$\begin{tikzcd}
A \arrow[d, bend right=50, "x"'] \arrow[d, bend left=50, "x'"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \\
C
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[dl, bend right=30, "d"'] \arrow[dl, bend left=30, "d'"] \arrow[dl, phantom, "\overset{\delta}{\Rightarrow}"] \\
C &
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[d, bend right=50, "y"'] \arrow[d, bend left=50, "y'"] \arrow[d, phantom, "\overset{\beta}{\Rightarrow}"] \\
D
\end{tikzcd}
=
\begin{tikzcd}
 & B \arrow[dl, bend right=30, "d"'] \arrow[dl, bend left=30, "d'"] \arrow[dl, phantom, "\overset{\delta}{\Rightarrow}"] \\
C \arrow[r, "g"'] & D
\end{tikzcd}$$

hold.

Similarly, a morphism $f : A \to B$ in $\mathscr{C}$ is orthogonal to an object C of $\mathscr{C}$ if

1. for every $g : A \to C$ there exists a unique morphism $h : B \to C$ such that $h \cdot f = g$, and
2. for every 2-cell $\alpha : g \Rightarrow g'$ there exists a unique 2-cell $\beta : h \Rightarrow h'$ such that $\beta * f = \alpha$ holds.

We now recall from [22] the notion of a *kernel-quotient system*. This notion generalises the notions of a kernel and its induced quotient, and allows treating factorisation systems in enriched categories parametric in the choice of the shape of "kernel data". Importantly, this approach covers the motivating ordinary (regular epi, mono) factorisation system on Set as well as the three factorisation systems on Cat that are mentioned in the introduction to this chapter.

In the following, we will restrict ourselves to $\mathscr{V}$ being a locally finitely presentable category as a monoidal closed category in the sense of [43], as we will need to impose a finiteness condition on the kernel-quotient system.

Let us denote by $\mathbf{2}$ the free $\mathscr{V}$-category on a morphism $1 \to 0$. We let $\mathscr{F}$ be a finitely presentable $\mathscr{V}$-category containing $\mathbf{2}$ as a full subcategory. Then there is the obvious inclusion $J : \mathbf{2} \to \mathscr{F}$ and the inclusion $I : \mathscr{K} \to \mathscr{F}$ of the full subcategory $\mathscr{K}$ of $\mathscr{F}$ spanned by all objects of $\mathscr{F}$ except 0. We call the data (J, I) a *kernel-quotient system*, and the role of $\mathscr{K}$ is, informally, to give the "shape" of the kernels. Given a complete and cocomplete $\mathscr{V}$-category $\mathscr{C}$, there is a chain of adjunctions as in the following diagram:

$$\begin{tikzcd}
{[\mathbf{2},\mathscr{C}]} \arrow[r, bend right, "\mathrm{Ran}_J"'] \arrow[r, phantom, "\perp"] & {[\mathscr{F},\mathscr{C}]} \arrow[l, bend right, "{[J,\mathscr{C}]}"'] \arrow[r, bend right, "{[I,\mathscr{C}]}"'] \arrow[r, phantom, "\perp"] & {[\mathscr{K},\mathscr{C}]} \arrow[l, bend right, "\mathrm{Lan}_I"']
\end{tikzcd}$$

We denote the composite adjunction by

$$\begin{tikzcd}
{[\mathbf{2},\mathscr{C}]} \arrow[r, bend right, "K"'] \arrow[r, phantom, "\perp"] & {[\mathscr{K},\mathscr{C}]} \arrow[l, bend right, "Q"']
\end{tikzcd}$$

and call it the *kernel-quotient adjunction* for $\mathscr{F}$.

In [22] the authors give a weaker definition of kernel-quotient adjunction to capture the cases where $\mathscr{C}$ is not complete and cocomplete. We do not need to introduce this weaker notion, as the 2-categories $\mathscr{C}$ in our examples always satisfy the completeness conditions.

Definition 6.1.3. Given a complete and cocomplete $\mathscr{V}$-category $\mathscr{C}$ together with the kernel-quotient adjunction for $\mathscr{F}$, we say that an object X in $[\mathscr{K},\mathscr{C}]$ is an *$\mathscr{F}$-kernel* if it is in the essential image of K. Any arrow $f : A \to B$ in $\mathscr{C}$ is called an *$\mathscr{F}$-quotient map* if it is in the essential image of Q, and it is called *$\mathscr{F}$-monic* if the morphism $K(1_A, f) : K(1_A) \to K(f)$ is an isomorphism.

Example 6.1.4. The motivating example of a kernel-quotient adjunction in the ordinary setting ($\mathscr{V} = \mathsf{Set}$) is given by taking the category $\mathscr{F}$ to be of the shape

$$\begin{tikzcd}
2 \arrow[r, bend left] \arrow[r, bend right] & 1 \arrow[r] & 0
\end{tikzcd}$$

with J and I being the obvious embeddings. Here the adjunction $Q \dashv K$ acts as follows. The functor Q sends a parallel pair $X = (f, g)$ to a coequaliser QX of the parallel pair (f, g). A morphism $f : A \to B$ in $\mathscr{C}$, thus an object in $[\mathbf{2},\mathscr{C}]$, is sent by K to the kernel pair $Kf = (k_1, k_2)$ of f. The $\mathscr{F}$-monic morphisms are precisely the monomorphisms in this example.

The kernel-quotient system in the previous example allows factoring every morphism in $\mathscr{C}$ as a regular epimorphism followed by a (not necessarily monomorphic) morphism. Since both the functors $I : \mathscr{K} \to \mathscr{F}$ and $J : \mathbf{2} \to \mathscr{F}$ are injective on objects and fully faithful, the functors Ran_J and Lan_I can *always* be taken as strict sections of the functors $[J,\mathscr{C}]$ and $[I,\mathscr{C}]$, respectively. Then the kernel-quotient adjunction $Q \dashv K$ may be taken to commute with the evaluation functors $[\mathbf{2},\mathscr{C}] \to \mathscr{C}$ and $[\mathscr{K},\mathscr{C}] \to \mathscr{C}$ that evaluate at the object 1. This results in the counit ε of $Q \dashv K$ having the following form for all objects f in $[\mathbf{2},\mathscr{C}]$:

$$\varepsilon_f : QKf \to f \qquad = \qquad \begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "QKf"'] & A \arrow[d, "f"] \\
C \arrow[r, "\varepsilon_f"'] & B
\end{tikzcd}$$

Thus we have a factorisation

$$f = \varepsilon_f \cdot QKf,$$

and QKf is a regular epi, being a coequaliser of the parallel pair Kf. If the morphism ε_f is a mono for every f, we obtain a factorisation system (regular epi, mono) on $\mathscr{C}$.

The above construction of the morphism ε_f is analogous in the case of enrichment in a general $\mathscr{V}$. We say that *$\mathscr{F}$-kernel-quotient factorisations in $\mathscr{C}$ converge immediately* whenever ε_f is $\mathscr{F}$-monic for each morphism f in $\mathscr{C}$. Whenever $\mathscr{F}$-kernel-quotient factorisations converge immediately in $\mathscr{C}$, we obtain a $\mathscr{V}$-factorisation system ($\mathscr{F}$-quotient, $\mathscr{F}$-monic) on $\mathscr{C}$ (by Proposition 4 of [22]).

Example 6.1.5. Given a 2-category $\mathscr{F}_{\mathsf{bof}}$ generated by

$$\begin{tikzcd}
2 \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\alpha \Downarrow\Downarrow \beta"] & 1 \arrow[r, "w"] & 0
\end{tikzcd}$$

subject to the identity $w * \alpha = w * \beta$, we obtain the following kernel-quotient system. The $\mathscr{F}_{\mathsf{bof}}$-kernel (or *equikernel*) of a morphism $f : A \to B$ is given by the following data:

$$\begin{tikzcd}
E \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\alpha \Downarrow\Downarrow \beta"] & A \arrow[r, "f"] & B
\end{tikzcd}$$

In Cat, the category E has as objects the parallel morphisms $p, q : a \to b$ from A for which the equality $f(p) = f(q)$ holds in B. The morphisms between objects $p_1, q_1 : a_1 \to b_1$ and $p_2, q_2 : a_2 \to b_2$ in E are the pairs (m, n) of morphisms $m : a_1 \to a_2$ and $n : b_1 \to b_2$, satisfying the equalities $n \cdot p_1 = p_2 \cdot m$ and $n \cdot q_1 = q_2 \cdot m$. The functors s and t then act as "source" and "target" functors. That is, given $p, q : a \to b$ as an object in E, we have that $s(p, q) = a$ and $t(p, q) = b$. The action of s and t on morphisms is as expected: using the above notation, $s(m, n) = m : a_1 \to a_2$ and $t(m, n) = n : b_1 \to b_2$. The natural transformations α and β then act as "morphism projections", i.e., $\alpha(p, q) = p : a \to b$ and $\beta(p, q) = q : a \to b$.

Given kernel-data X in $[\mathscr{K}, \mathscr{C}]$, its $\mathscr{F}_{\mathsf{bof}}$-quotient QX is its *coequifier*, i.e., a universal morphism $c : X1 \to C$ satisfying $c * X\alpha = c * X\beta$ (see [44] or Section 5.3 in [22]). A morphism in $\mathscr{C}$ is $\mathscr{F}_{\mathsf{bof}}$-monic precisely when it is representably faithful (i.e., faithful when $\mathscr{C} =$ Cat). As the coequifier morphisms are always bijective on objects and full in Cat, this hints that the $\mathscr{F}_{\mathsf{bof}}$ kernel-quotient system gives rise to the (b.o. full, faithful) factorisation system on Cat. This is indeed the case. In detail, given a functor $f : A \to B$, we can form its equikernel E and factorise f into two functors $e : A \to A/E$ and $m : A/E \to B$. The category A/E is the congruence category of A having the same objects as A, with the congruence on morphisms of A generated by the pairs $p, q : a \to b$ that are objects of E. Defining e as the canonical functor that assigns to each morphism of A its equivalence class in A/E, it is obviously bijective on objects and full. The functor m assigns to each object a of A its image $f(a)$, and to the equivalence class morphism $[p : a \to b]$ the image $f(p) : f(a) \to f(b)$. It follows immediately from the definition of the equikernel that m is well-defined and faithful.

To summarise, for $\mathscr{C} =$ Cat the kernel-quotient factorisations for $\mathscr{F}_{\mathsf{bof}}$ converge immediately, and they give rise to the (b.o. full, faithful) factorisation system.

The main focus of [22] is to study the generalised notions of regularity and exactness, parametric in the choice of a kernel-quotient system $\mathscr{F}$. This yields a theory of $\mathscr{F}$-regularity and $\mathscr{F}$-exactness. We do not need to introduce the theory of $\mathscr{F}$-exactness in detail. In fact, we use the results of [22] only to "lift" the (b.o. full, faithful) factorisation system of Example 6.1.5 on Cat to any algebraic category $\mathsf{Alg}(T)$ for a strongly finitary 2-monad T on Cat.

Remark 6.1.6. We say that a diagram

$$\begin{tikzcd} E \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\alpha \Downarrow\Downarrow \beta"] & A \end{tikzcd} \tag{6.1}$$

of kernel data is *reflexive* if there exists a morphism $i_A : A \to E$ as in the diagram

$$\begin{tikzcd} E \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\alpha \Downarrow\Downarrow \beta"] & A \arrow[l, bend left=60, "i_A"] \end{tikzcd}$$

that satisfies the reflexivity equalities

$$s \cdot i_A = t \cdot i_A = 1_A,$$
$$\alpha * i_A = \beta * i_A = 1.$$

In Cat, the equikernel (6.1) of any functor $f : A \to B$ is indeed reflexive. Recalling the description of E from Example 6.1.5, we see that the assignment

$$a \mapsto (1_a, 1_a),$$
$$m : a \to b \mapsto (m, m)$$

defines a morphism $i_A : A \to E$ that satisfies the reflexivity equalities.

It follows from the above remark that each b.o. full functor is the coequifier of a reflexive diagram: its equikernel. This observation is important because coequifiers of reflexive diagrams (reflexive coequifiers) are examples of *sifted* colimits. In the ordinary setting, sifted colimits are those colimits that commute with finite products in the category of sets, recall Example 1.3.2 or see [5]. In particular, Theorem 2.15 of [5] contains a useful characterisation of *diagrams* for sifted colimits. A diagram $\mathscr{D}$ is sifted if and only if it is connected and the diagonal $\Delta : \mathscr{D} \to \mathscr{D} \times \mathscr{D}$ is cofinal. See also Proposition 1.3.5 above. In the case of enrichment in Cat, sifted colimits are again those colimits that commute with finite products, now in the category Cat, and it is possible to characterise them in a manner similar to the ordinary characterisation, as we have shown in Chapter 5. A weight $\varphi : \mathscr{D}^{op} \to \mathsf{Cat}$ is sifted if and only if

1. φ is *connected*, meaning that the unique 2-functor $\int^d \varphi d \to \mathbf{1}$ is an isomorphism, and

2. the diagonal 2-functor $\Delta : \mathscr{D} \to \mathscr{D} \times \mathscr{D}$ is *cofinal*, meaning that the 2-cell

$$\begin{tikzcd}
\mathscr{D}^{op} \arrow[rr, "\Delta^{op}"] \arrow[dr, "\varphi"'] & \arrow[d, phantom, "\overset{\delta}{\Rightarrow}"] & \mathscr{D}^{op} \times \mathscr{D}^{op} \arrow[dl, "(d_1,d_2) \mapsto \varphi(d_1) \times \varphi(d_2)"] \\
& \mathsf{Cat} &
\end{tikzcd} \tag{6.2}$$

is a left Kan extension, where $\delta_d : \varphi d \to \varphi d \times \varphi d$ is the diagonal functor.

This characterisation is contained in Remark 5.2.2.

Remark 6.1.7. A 2-functor $T : \mathscr{C} \to \mathscr{C}$ is called *strongly finitary* if it preserves sifted colimits [45]. Using Remark 6.1.6 we see that every strongly finitary endo-2-functor $T : \mathsf{Cat} \to \mathsf{Cat}$ preserves b.o. full functors, as they are coequifiers for some reflexive diagram in Cat. We will use this fact very often in the following sections.

Let us denote by $\mathsf{Cat}_{\mathsf{sf}}$ the 2-category of natural numbers $n = \{0, 1, \dots, n-1\}$ and functions between them. There is an inclusion $\iota : \mathsf{Cat}_{\mathsf{sf}} \to \mathsf{Cat}$ that represents n as the discrete category with the object set n, and maps a function $f : m \to n$ to the corresponding functor with object assignment f. Theorem 8.31 of [21] states that Cat is the free cocompletion of $\mathsf{Cat}_{\mathsf{sf}}$ under sifted colimits. This observation is useful as it shows that any category $\mathscr{C}$ is a sifted colimit of finite discrete categories. Indeed, every discrete category is a filtered colimit of its finite discrete subcategories, and every category is a sifted colimit (a special codescent object) of discrete categories (see, e.g., Chapter 1 of [21]). Moreover, strongly finitary 2-functors $T : \mathsf{Cat} \to \mathsf{Cat}$ correspond (up to isomorphism) to 2-functors $T \cdot \iota : \mathsf{Cat}_{\mathsf{sf}} \to \mathsf{Cat}$, as the following diagram

$$\begin{tikzcd}
\mathsf{Cat}_{\mathsf{sf}} \arrow[rr, "\iota"] \arrow[dr, "T \cdot \iota"'] & \arrow[d, phantom, "\overset{\lambda}{\Rightarrow}"] & \mathsf{Cat} \arrow[dl, "T"] \\
& \mathsf{Cat} &
\end{tikzcd}$$

is a left Kan extension. This correspondence is stated and proved in Corollary 8.45 of [21]. Via this correspondence we may identify the 2-category $\mathsf{StrFin}(\mathsf{Cat})$ of strongly finitary endo-2-functors of Cat with the (2-functor) 2-category $[\mathsf{Cat}_{\mathsf{sf}}, \mathsf{Cat}]$. We will use this identification in the proof of Proposition 6.1.10.

The factorisation system given by $\mathscr{F}_{\mathsf{bof}}$ lifts from Cat to the categories of algebras for a strongly finitary 2-monad T. We will introduce the notion of an algebraic category and then state the "lifting theorem" for $\mathscr{F}_{\mathsf{bof}}$.

For a 2-monad T on a 2-category $\mathscr{X}$, we denote the 2-category of T-algebras and their *strict* homomorphisms by $\mathsf{Alg}(T)$. Recall that a morphism $a : TA \to A$ is a T-algebra if it satisfies the axioms

$$\begin{tikzcd}
A \arrow[r, "\eta^T_A"] \arrow[dr, "1_A"'] & TA \arrow[d, "a"] \\
& A
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
TTA \arrow[r, "Ta"] \arrow[d, "\mu^T_A"'] & TA \arrow[d, "a"] \\
TA \arrow[r, "a"'] & A,
\end{tikzcd}$$

and a morphism $h : A \to B$ is a strict homomorphism between T-algebras (A, a) and (B, b) if it makes the usual diagram

$$\begin{tikzcd}
TA \arrow[r, "Th"] \arrow[d, "a"'] & TB \arrow[d, "b"] \\
A \arrow[r, "h"'] & B
\end{tikzcd}$$

commute in $\mathscr{X}$. Let us recall the 2-dimensional structure of $\mathsf{Alg}(T)$. Given two T-algebras $a : TA \to A$ and $b : TB \to B$, and two homomorphisms $h, h' : A \to B$ between (A, a) and (B, b), the 2-cells $\alpha : h' \Rightarrow h$ between the homomorphisms h' and h are exactly those 2-cells $\alpha : h' \Rightarrow h$ in $\mathscr{X}$ that moreover satisfy the following equality:

$$\begin{tikzcd}
TA \arrow[r, bend left, "Th'"] \arrow[r, bend right, "Th"'] \arrow[r, phantom, "\Downarrow T\alpha"] & TB \arrow[d, "b"] & = & TA \arrow[d, "a"'] & \\
& B & & A \arrow[r, bend left, "h'"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow \alpha"] & B
\end{tikzcd}$$

For us, the algebraic 2-category $\mathsf{Alg}(T)$ is therefore what other authors commonly denote by $\mathsf{Alg}_s(T)$, see [56]. As we do not deal with the weaker kinds of morphisms, we will talk simply of homomorphisms instead of strict homomorphisms in what follows. We call the 2-categories equivalent to the 2-categories of the form $\mathsf{Alg}(T)$ *algebraic*.

Theorem 6.1.8. *Let T be a strongly finitary 2-monad on $[X, \mathsf{Cat}]$ (with X an arbitrary set). Then the $\mathscr{F}_{\mathsf{bof}}$ kernel-quotient factorisations converge immediately in the 2-category $\mathsf{Alg}(T)$ of T-algebras. These factorisations give rise to a factorisation system: the quotient morphisms are precisely those morphisms whose underlying morphisms are pointwise bijective on objects and full, and the monic morphisms are precisely those whose underlying morphisms are pointwise faithful.*

Proof. Observe that the forgetful 2-functor $U : \mathsf{Alg}(T) \to [X, \mathsf{Cat}]$ creates limits and sifted colimits. In particular, U creates equikernels and coequifiers of equikernels, since the equikernel is a reflexive pair in the sense of Remark 6.1.6, and therefore sifted. The factorisation of any morphism $h : (A, a) \to (B, b)$ in $\mathsf{Alg}(T)$ is thus computed as in $[X, \mathsf{Cat}]$, and there the $\mathscr{F}_{\mathsf{bof}}$ factorisations converge immediately. The $\mathscr{F}_{\mathsf{bof}}$ factorisations thus converge immediately in $\mathsf{Alg}(T)$. Moreover, any $\mathscr{F}_{\mathsf{bof}}$-quotient morphism in $[X, \mathsf{Cat}]$ is pointwise bijective on objects and full, as it is a coequifier and these are computed pointwise in $[X, \mathsf{Cat}]$. ■

In the context of categories of algebras, the lifted factorisation system gives rise to the notions of a quotient algebra and a subalgebra. Let T be a strongly finitary 2-monad T on Cat, and take an algebra (A, a) from $\mathsf{Alg}(T)$. We say that (B, b) is a *subalgebra* of (A, a) if there is a homomorphism $m : (B, b) \rightarrowtail (A, a)$ with m faithful, as in the left-hand side of the diagram (6.3). By a *quotient algebra* of (A, a) we mean a T-algebra (B, b)

together with a b.o. full morphism $h : A \twoheadrightarrow B$ in Cat that is a homomorphism, as in the right-hand side of the diagram (6.3).

$$\begin{tikzcd}
TB \arrow[r, "Tm"] \arrow[d, "b"'] & TA \arrow[d, "a"] & & TA \arrow[r, two heads, "Th"] \arrow[d, "a"'] & TB \arrow[d, "b"] \\
B \arrow[r, tail, "m"'] & A & & A \arrow[r, two heads, "h"'] & B
\end{tikzcd} \tag{6.3}$$

Let us remark that in the above diagram concerning quotient algebras, the morphism $Th : TA \twoheadrightarrow TB$ is indeed b.o. full by Remark 6.1.7 since h is b.o. full and T is strongly finitary.

Remark 6.1.9. Denote by N the discrete 2-category with natural numbers as objects. We have an obvious inclusion $J : N \to \mathsf{Cat}_{\mathsf{sf}}$ that is an identity on objects (recall the description of $\mathsf{Cat}_{\mathsf{sf}}$ from Remark 6.1.7), and it induces a 2-functor

$$V = [J, \mathsf{Cat}] : [\mathsf{Cat}_{\mathsf{sf}}, \mathsf{Cat}] \to [N, \mathsf{Cat}]$$

given by precomposition with J. Then let us denote by W the underlying 2-functor

$$W : \mathsf{Mnd}_{\mathsf{sf}}(\mathsf{Cat}) \to [\mathsf{Cat}, \mathsf{Cat}] \xrightarrow{[\iota, \mathsf{Cat}]} [\mathsf{Cat}_{\mathsf{sf}}, \mathsf{Cat}]$$

mapping a strongly finitary 2-monad (T, μ, η) on Cat to its underlying endo-2-functor T and restricting it to the 2-functor $T \cdot \iota : \mathsf{Cat}_{\mathsf{sf}} \to \mathsf{Cat}$. An argument from [53] shows that there is a chain

$$\begin{tikzcd}
\mathsf{Mnd}_{\mathsf{sf}}(\mathsf{Cat}) \arrow[d, bend left, "W"] \\
{[\mathsf{Cat}_{\mathsf{sf}}, \mathsf{Cat}]} \arrow[u, bend left, "H"] \arrow[u, phantom, "\dashv"] \arrow[d, bend left, "V"] \\
{[N, \mathsf{Cat}]} \arrow[u, bend left, "G"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

of adjunctions with the composite adjunction

$$\begin{tikzcd}
\mathsf{Mnd}_{\mathsf{sf}}(\mathsf{Cat}) \arrow[d, bend left, "U"] \\
{[N, \mathsf{Cat}]} \arrow[u, bend left, "F"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

being monadic. Thus $\mathsf{Mnd}_{\mathsf{sf}}(\mathsf{Cat})$ is equivalent to the 2-category $[N, \mathsf{Cat}]^M$ of algebras for the 2-monad $M = UF$. The 2-category $[N, \mathsf{Cat}]^M$ is a locally finitely presentable category (in the 2-dimensional sense of [43]), and so it is complete and cocomplete. We will show that the right adjoint U preserves sifted colimits, and therefore M is strongly finitary. Then, using Theorem 6.1.8, we will be able to conclude that $\mathsf{Mnd}_{\mathsf{sf}}(\mathsf{Cat})$ admits the (b.o. full, faithful) factorisation system of Example 6.1.5.

Proposition 6.1.10. *The 2-monad $M = UF$ given by the adjunction*

$$\begin{tikzcd}
\mathsf{Mnd}_{\mathsf{sf}}(\mathsf{Cat}) \arrow[d, bend left, "U"] \arrow[d, phantom, "\dashv"] \\
{[N, \mathsf{Cat}]} \arrow[u, bend left, "F"]
\end{tikzcd}$$

is strongly finitary.

Proof. In the notation of the previous remark, U is the composite of right adjoints W and V. The 2-functor V, being defined as a precomposition with J (recall Remark 6.1.9), has itself a right adjoint and is therefore strongly finitary. To deduce that U preserves sifted colimits, and that M is thus strongly finitary, it is enough to show that W preserves sifted colimits. The argument can be taken almost verbatim from Section 4 of [47], where the authors show a similar result for *finitary* monads. In the following we shall identify the 2-category $[\mathsf{Cat}_{\mathsf{sf}}, \mathsf{Cat}]$ with the (2-equivalent) 2-category $\mathsf{StrFin}(\mathsf{Cat})$ of strongly finitary endo-2-functors of Cat as in Remark 6.1.7. Take a weight $\varphi : \mathscr{D}^{op} \to \mathsf{Cat}$ for a sifted colimit (i.e., a sifted weight), and a diagram $D : \mathscr{D} \to \mathsf{Mnd}_{\mathsf{sf}}(\mathsf{Cat})$ sending d to a strongly finitary 2-monad $(T_d, \mu^{T_d}, \eta^{T_d})$. Denote the weighted colimit object $\varphi * WD$ in $[\mathsf{Cat}_{\mathsf{sf}}, \mathsf{Cat}]$ by T. For every strongly finitary $S : \mathsf{Cat} \to \mathsf{Cat}$, both $- \cdot S$ and $S \cdot -$ are again strongly finitary, the first by having a right adjoint, and the second one since colimits in $[\mathsf{Cat}_{\mathsf{sf}}, \mathsf{Cat}]$ are computed pointwise. Therefore the weighted colimit $(\varphi \times \varphi) * D'$ of the diagram $D' : \mathscr{D} \times \mathscr{D} \to [\mathsf{Cat}_{\mathsf{sf}}, \mathsf{Cat}]$ sending (d, d') to $T_d \cdot T_{d'}$ weighted by $\varphi \times \varphi : \mathscr{D}^{op} \times \mathscr{D}^{op} \to \mathsf{Cat}$ is the 2-functor TT. Since the diagonal 2-functor $\Delta : \mathscr{D} \to \mathscr{D} \times \mathscr{D}$ is cofinal with respect to the weight φ (recall diagram (6.2)), it follows that the weighted colimit $\varphi * D'\Delta$ is also the 2-functor TT. This in turn induces a multiplication $\mu : TT \to T$, and similarly we get the unit $\eta : Id \to T$. Thus T carries a monad structure, and it follows that W preserves sifted colimits. ■

Consider now a quotient $e : T \twoheadrightarrow T'$ of monads T and T' in $\mathsf{Mnd}_{\mathsf{sf}}(\mathsf{Cat})$. From Theorem 6.1.8 it follows that e is pointwise b.o. full. That is, the functor $e_n : Tn \twoheadrightarrow T'n$ is b.o. full for every finite discrete category n. Of course, for strongly finitary monads we may state an even stronger pointwise property of quotient monad maps: given a quotient $e : T \twoheadrightarrow T'$, its component $e_C : TC \to T'C$ is b.o. full for each *category* C. This is true since each category is a sifted colimit of finite discrete categories, and since T and T' preserve sifted colimits, see Remark 6.1.7.

Using the above observations, we shall see that quotients of monads correspond to equational subcategories of algebraic categories.

Remark 6.1.11. Let us give an algebraic meaning to the fact that a quotient $e : T \twoheadrightarrow T'$ of strongly finitary 2-monads on Cat implies that every $e_n : Tn \twoheadrightarrow T'n$ is b.o. full in Cat. Viewing the objects of Tn as n-ary terms, bijectivity on objects of e_n means that the quotient e does not postulate any new equations between terms. On the other hand, fullness of e_n means that $T'n$ is obtained from Tn by only identifying morphisms in Tn. On the level of algebras, this imposes equations between *morphisms* of the underlying category of an algebra. We will make the notion of an equation precise in Section 6.2.

Example 6.1.12. Let $\mathscr{A}$ be the 2-category of categories C equipped with the following algebraic structure, subject to no axioms:

1. one nullary operation I and one binary operation $\otimes$,
2. natural transformations: the associator α with components $\alpha_{a,b,c} : (a \otimes b) \otimes c \to a \otimes (b \otimes c)$ and the "back-associator" α' with components $\alpha'_{a,b,c} : a \otimes (b \otimes c) \to (a \otimes b) \otimes c$,
3. natural transformations: the left and right unitors λ and ρ with the components $\lambda_a : I \otimes a \to a$ and $\rho_a : a \otimes I \to a$, together with "back-unitors" λ' and ρ' with the components $\lambda'_a : a \to I \otimes a$ and $\rho'_a : a \to a \otimes I$.

In particular, the "back-associators" and "back-unitors" are not forced to be the inverses of associators and unitors. The morphisms in $\mathscr{A}$ are those functors that preserve the algebraic structure "on the nose", and the 2-cells are monoidal natural transformations between those functors. We can obtain from $\mathscr{A}$ a full subcategory $\mathscr{B}$ spanned by "monoidal categories without coherence": that is, consider only those categories C from $\mathscr{A}$ whose associator and unitors are in fact natural *isomorphisms*, with their corresponding inverse transformations being the "back-transformations". In an informal sense, $\mathscr{B}$ is an equational subcategory of $\mathscr{A}$ defined by the equations

$$\alpha \cdot \alpha' = \alpha' \cdot \alpha = 1, \quad \lambda \cdot \lambda' = 1, \quad \lambda' \cdot \lambda = 1, \quad \rho \cdot \rho' = 1, \quad \rho' \cdot \rho = 1.$$

Let MonCat be the 2-category of monoidal categories, strict monoidal functors and monoidal natural transformations between those functors. Informally again, MonCat can be obtained as an equational subcategory of $\mathscr{B}$ by considering those categories from $\mathscr{B}$ that satisfy the usual triangle and pentagon identities.

The 2-category $\mathscr{A}$ can be easily seen to be the 2-category $\mathsf{Alg}(R)$ of algebras for a strongly finitary 2-monad R on Cat. The results of Section 6.2 will show that there is a chain

$$R \twoheadrightarrow S \twoheadrightarrow T$$

of quotients of strongly finitary 2-monads R, S and T for which we have the correspondences

$$\mathscr{A} \simeq \mathsf{Alg}(R), \quad \mathscr{B} \simeq \mathsf{Alg}(S), \quad \mathsf{MonCat} \simeq \mathsf{Alg}(T).$$

Moreover, the monad morphism quotients induce the inclusions

$$\mathsf{Alg}(T) \to \mathsf{Alg}(S) \to \mathsf{Alg}(R)$$

that correspond to the inclusions of equational subcategories $\mathsf{MonCat} \subseteq \mathscr{B} \subseteq \mathscr{A}$. The theory developed in Section 6.2 will make these correspondences precise.

We will end the present section with a remark stating that b.o. full morphisms are epimorphisms with respect to morphisms and 2-cells. These properties will allow us to prove a Cat-enriched Birkhoff theorem in the following section, with the proof being very much in the spirit of the proof for ordinary Birkhoff theorem. Specifically, these properties will be crucial in proving that quotients of monads induce 2-dimensionally fully faithful algebraic functors (as defined in Definition 6.2.1).

Remark 6.1.13. Given a b.o. full $h : C \to A$ in Cat, the functor

$$\mathsf{Cat}(h, B) : \mathsf{Cat}(A, B) \to \mathsf{Cat}(C, B)$$

is injective on objects and fully faithful for every B. The injectivity on objects of $\mathsf{Cat}(h, B)$ corresponds to h being an epimorphism in Cat, faithfulness of $\mathsf{Cat}(h, B)$ states that h is an epimorphism with respect to 2-cells, and fullness of $\mathsf{Cat}(h, B)$ corresponds to a factorisation property of h w.r.t. 2-cells.

Consider the following diagram

$$\begin{tikzcd}
C \arrow[r, "h"] \arrow[d, "h"'] \arrow[dr, phantom, "\beta \Swarrow"] & A \arrow[d, "f"] \\
A \arrow[r, "g"'] & B
\end{tikzcd}$$

in Cat with h being b.o. full. Denote by

$$\begin{tikzcd}
K \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\gamma \Downarrow\Downarrow \delta"] & C \arrow[r, "h"] & A
\end{tikzcd}$$

the kernel-quotient pair of h. The morphism h is a coequifier of the kernel diagram since Cat is $\mathscr{F}_{\mathsf{bof}}$-exact. Both the composites $f \cdot h$ and $g \cdot h$ also coequify the kernel diagram. By the 2-dimensional universal property of coequifiers the equality

$$\begin{tikzcd}
C \arrow[r, "h"] \arrow[d, "h"'] \arrow[dr, phantom, "\beta \Swarrow"] & A \arrow[d, "f"] \\
A \arrow[r, "g"'] & B
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, "h"] & A \arrow[d, bend right=50, "g"'] \arrow[d, bend left=50, "f"] \arrow[d, phantom, "\overset{\alpha}{\Leftarrow}"] \\
& B
\end{tikzcd}$$

holds for a unique 2-cell α. This observation equivalently says that $\mathsf{Cat}(h, B)$ is fully faithful.

6.2 Birkhoff theorem for the kernel-quotient system $\mathscr{F}_{\mathsf{bof}}$

In this section we first recall basic definitions concerning subcategories and equivalence of categories in the Cat-enriched setting. After a short review of the properties of algebraic categories and algebraic functors we state and prove the Birkhoff theorem for the $\mathscr{F}_{\mathsf{bof}}$ kernel-quotient system.

Definition 6.2.1. A 2-functor $T : \mathscr{C} \to \mathscr{D}$ between 2-categories $\mathscr{C}$ and $\mathscr{D}$ is called *fully faithful* if for any pair A, B of objects of $\mathscr{C}$ the action $T_{A,B} : \mathscr{C}(A, B) \to \mathscr{D}(TA, TB)$ is an *isomorphism* of categories. We say that T exhibits $\mathscr{C}$ as a *full subcategory* of $\mathscr{D}$. When $\mathscr{C}$ is moreover closed in $\mathscr{D}$ under isomorphisms, we call $\mathscr{C}$ a *replete* full subcategory of $\mathscr{D}$. The 2-category $\mathscr{C}$ is closed in $\mathscr{D}$ under isomorphisms if for any object A in $\mathscr{C}$ and any isomorphism $i : TA \to D$ in $\mathscr{D}$ there exists an isomorphism $j : A \to B$ in $\mathscr{C}$ such that $Tj = i$.

The 2-categories $\mathscr{C}$ and $\mathscr{D}$ are *equivalent* if there is a fully faithful 2-functor $T : \mathscr{C} \to \mathscr{D}$ that is *essentially surjective*, that is, for any object D of $\mathscr{D}$ there exists an object A of $\mathscr{C}$ with TA being isomorphic to D, denoted by $TA \cong D$.

Remark 6.2.2. Recall that an algebraic 2-category is a 2-category that is equivalent to a 2-category $\mathsf{Alg}(T)$ for some 2-monad T on $\mathscr{C}$. We will look at some important properties of algebraic categories and algebraic functors (functors arising from a monad morphism):

1. Consider two 2-monads T and T' on $\mathscr{C}$, and a monad morphism $e : T \to T'$. This monad morphism gives rise to an algebraic 2-functor $\mathsf{Alg}(e) : \mathsf{Alg}(T') \to \mathsf{Alg}(T)$ between the 2-categories $\mathsf{Alg}(T')$ and $\mathsf{Alg}(T)$ of algebras for T' and T. On objects, $\mathsf{Alg}(e)$ acts as follows:

$$\begin{tikzcd} T'A \arrow[dd, "a'"'] & & TA \arrow[d, "e_A"] \\ & \mapsto & T'A \arrow[d, "a'"] \\ A & & A \end{tikzcd}$$

On morphisms and 2-cells $\mathsf{Alg}(e)$ acts as an identity. A homomorphism $h : (A, a') \to (B, b')$ between two T'-algebras $a' : T'A \to A$ and $b' : T'B \to B$ gets mapped to a homomorphism $h : (A, a' \cdot e_A) \to (B, b' \cdot e_B)$ of the corresponding T-algebras. Indeed, the outer rectangle in the diagram

$$\begin{tikzcd} TA \arrow[r, "Th"] \arrow[d, "e_A"'] & TB \arrow[d, "e_B"] \\ T'A \arrow[r, "T'h"] \arrow[d, "a'"'] & T'B \arrow[d, "b'"] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

clearly commutes. The same reasoning applies for the 2-cells $\alpha : h \Rightarrow h'$ between two homomorphisms $h : (A, a') \to (B, b')$ and $h' : (A, a') \to (B, b')$.

The action of $\mathsf{Alg}(e)$ on morphisms and 2-cells is thus faithful for any $e : T \to T'$.

2. The algebraic 2-category $\mathsf{Alg}(T)$ for a strongly finitary monad T on Cat is cowellpowered with respect to quotient algebras. Indeed, for every small category A there is, up to isomorphism, only a set of b.o. full functors of the form $h : A \to B$ in Cat. Thus for a T-algebra (A, a) there is, up to isomorphism, only a set of quotients $h : (A, a) \to (B, b)$ in $\mathsf{Alg}(T)$.

3. Given an algebraic 2-category $\mathsf{Alg}(T)$ for a strongly finitary 2-monad T on Cat, it is a standard observation that the underlying 2-functor $U : \mathsf{Alg}(T) \to \mathsf{Cat}$ creates 2-limits. See Theorem 6.8 of [15] for a proof that U preserves these limits and observe that it can be easily modified to show that U in fact creates these limits. Since

T is strongly finitary, the 2-functor U also creates sifted colimits. In particular, U creates reflexive coequifiers. That is, given a reflexive diagram

$$\begin{tikzcd}
(K,k) \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] \arrow[rr, phantom, "\Downarrow\Downarrow"] & & (A,a) \arrow[ll, bend left=70, "h"]
\end{tikzcd}$$

in $\mathsf{Alg}(T)$, and the coequifier of the U-image of the above diagram

$$\begin{tikzcd}
K \arrow[r, bend left, "Uf"] \arrow[r, bend right, "Ug"'] \arrow[r, phantom, "\Downarrow\Downarrow"] & A \arrow[l, bend left=70, "Uh"] \arrow[r, "\gamma"] & C
\end{tikzcd}$$

there exists a unique algebra (C, c) such that γ is a homomorphism between (A, a) and (C, c).

We now turn to the proof of the Birkhoff theorem. The proof is a 2-dimensional variant of the classical proof of Birkhoff's theorem in the setting of ordinary categories. The interested reader can compare the structure of the present proof with the proof of Theorem 3.3.6 in [66].

Theorem 6.2.3. *Let T be a strongly finitary 2-monad on* Cat *and let $\mathscr{A}$ be a full subcategory* $\mathsf{Alg}(T)$ *of the category of algebras for the 2-monad T. Then the following are equivalent:*

1. *There is a strongly finitary 2-monad T' and a b.o. full monad morphism $e : T \twoheadrightarrow T'$ such that the comparison 2-functor $\mathscr{A} \to \mathsf{Alg}(T')$ is an equivalence.*
2. *The category $\mathscr{A}$ is closed in $\mathsf{Alg}(T)$ under sifted colimits, 2-products, quotient algebras, and subalgebras.*

Proof. We first prove the implication 1. ⇒ 2. in the following manner:

(a) Given the monad morphism $e : T \twoheadrightarrow T'$, we get a 2-functor $\mathsf{Alg}(e) : \mathsf{Alg}(T') \to \mathsf{Alg}(T)$ that we show to be fully faithful.

(b) We show that $\mathsf{Alg}(e)$ preserves sifted colimits and 2-limits.

(c) Finally we show that $\mathsf{Alg}(T')$ is closed in $\mathsf{Alg}(T)$ under subalgebras and quotient algebras.

Ad (a): The action of $\mathsf{Alg}(e)$ on morphisms and 2-cells is faithful by point 1. of Remark 6.2.2. We prove that $\mathsf{Alg}(e)$ is indeed fully faithful by showing that its action on morphisms and 2-cells is full. The fullness on morphisms comes from observing that given any diagram of the form

$$\begin{tikzcd}
TA \arrow[r, "Th"] \arrow[d, two heads, "e_A"'] & TB \arrow[d, two heads, "e_B"] \\
T'A \arrow[r, "T'h"] \arrow[d, "a'"'] & T'B \arrow[d, "b'"] \\
A \arrow[r, "h"'] & B
\end{tikzcd}$$

such that the outer rectangle commutes, the lower square commutes since $e_A : TA \twoheadrightarrow T'A$ is b.o. full, and thus epi. Similarly, given a 2-cell $\alpha : h \Rightarrow h'$ in $\mathsf{Alg}(T)$, it is also a 2-cell in $\mathsf{Alg}(T')$ by 2-naturality of e (saying that $e_B * T\alpha = \alpha * e_A$), and since e_A is an epimorphism on 2-cells by Remark 6.1.13. The algebraic 2-functor $\mathsf{Alg}(e)$ is therefore indeed fully faithful.

Ad (b): Let us denote by $U^T : \mathsf{Alg}(T) \to \mathsf{Cat}$ and by $U^{T'} : \mathsf{Alg}(T') \to \mathsf{Cat}$ the underlying 2-functors of $\mathsf{Alg}(T)$ and $\mathsf{Alg}(T')$. Then $U^{T'} = U^T \cdot \mathsf{Alg}(e)$. The 2-functor $U^{T'}$ preserves 2-limits and sifted colimits and U creates them. Therefore $\mathsf{Alg}(e)$ preserves 2-limits and sifted colimits.

Ad (c): Now we show that the 2-category $\mathsf{Alg}(T')$ is closed in $\mathsf{Alg}(T)$ under subalgebras and quotient algebras. To this end, consider a T'-algebra (A, a') and its image $(A, a) = (A, a' \cdot e_A)$ under $\mathsf{Alg}(e)$. Given any subalgebra (B, b) of (A, a) as in the diagram

$$\begin{tikzcd}
TB \arrow[r, "Tm"] \arrow[dd, "b"'] & TA \arrow[d, "e_A"] \\
 & T'A \arrow[d, "a"] \\
B \arrow[r, tail, "m"'] & A,
\end{tikzcd}$$

we can use the naturality of e

$$\begin{tikzcd}
TB \arrow[rr, "Tm"] \arrow[dr, two heads, "e_B"'] \arrow[dd, "b"'] & & TA \arrow[d, two heads, "e_A"] \\
 & T'B \arrow[r, "T'm"'] \arrow[dl, dotted, "b'"] & T'A \arrow[d, "a'"] \\
B \arrow[rr, tail, "m"'] & & A
\end{tikzcd}$$

and define b' as the unique diagonal fill-in with respect to e_B and m in the above diagram. (Recall that (b.o. full, faithful) is a factorisation system on Cat.) This $b' : T'B \to B$ is a T'-algebra. We inspect the following diagrams to see that (B, b') satisfies both algebra axioms.

$$\begin{tikzcd}
 & TB \arrow[d, two heads, "e_B"] \arrow[dd, bend left=60, "b"] \\
B \arrow[ur, "\eta^T_B"] \arrow[r, "\eta^{T'}_B"] \arrow[dr, "1_B"'] & T'B \arrow[d, "b'"] \\
 & B
\end{tikzcd}$$

$$\begin{tikzcd}
TTB \arrow[r, two heads, "Te_B"] \arrow[dd, "\mu^T_B"'] & TT'B \arrow[r, "Tb'"] \arrow[d, two heads, "e_{T'B}"'] & TB \arrow[d, two heads, "e_B"] \\
 & T'T'B \arrow[r, "T'b'"] \arrow[d, "\mu^{T'}_B"'] & T'B \arrow[d, "b'"] \\
TB \arrow[r, "e_B"'] \arrow[rr, bend right=40, "b"'] & T'B \arrow[r, "b'"] & B
\end{tikzcd}$$

Consider the left-hand diagram. The upper triangle commutes by the unit axiom of the monad morphism e, and the outer triangle commutes since (B, b) is a T-algebra. Thus the lower triangle commutes by virtue of e_B being an epimorphism. In the right-hand diagram, the outer square commutes since (B, b) is a T-algebra. The upper rectangle is an instance of a monad morphism axiom, and the lower left square commutes by naturality of e. The morphism Te_B is b.o. full, as e_B is and T preserves b.o. full morphisms by Remark 6.1.7. Thus the composite morphism $e_{T'B} \cdot Te_B$ is b.o. full as well. By the cancellation property of b.o. full morphisms we obtain the commutativity of the square

$$\begin{tikzcd}
TT'B \arrow[r, "T'b'"] \arrow[d, "\mu^{T'}_B"'] & T'B \arrow[d, "b'"] \\
T'B \arrow[r, "b'"'] & B,
\end{tikzcd}$$

and this proves that (B, b') is a T'-algebra. In conclusion, $\mathsf{Alg}(T')$ is indeed closed in $\mathsf{Alg}(T)$ under subalgebras.

The closedness of $\mathsf{Alg}(T')$ under quotient algebras in $\mathsf{Alg}(T)$ follows from closedness under limits and sifted colimits. Whenever we are given a T'-algebra (A, a') and a quotient homomorphism $h : (A, a) = (A, a' \cdot e_A) \twoheadrightarrow (B, b)$ of T-algebras as in

$$\begin{tikzcd}
TA \arrow[r, two heads, "Th"] \arrow[d, two heads, "e_A"'] & TB \arrow[dd, "b"] \\
T'A \arrow[d, "a'"'] & \\
A \arrow[r, two heads, "h"'] & B,
\end{tikzcd}$$

the kernel (K, k) of h lies in $\mathsf{Alg}(T')$. This is true since (K, k) is easily seen to be a subalgebra of the cotensor algebra $(A, a)^{\bullet \rightrightarrows \bullet}$ (where $\bullet \rightrightarrows \bullet$ denotes the obvious category), and $(A, a)^{\bullet \rightrightarrows \bullet}$ is in turn a subalgebra of the product algebra $(A, a)^2$. Since the kernel (K, k) is reflexive and as $\mathsf{Alg}(T')$ is closed in $\mathsf{Alg}(T)$ under sifted colimits, it follows that (B, b) lies in $\mathsf{Alg}(T')$.

The second part of the proof is the implication 2. ⇒ 1. Given a strongly finitary 2-monad T and a full subcategory

$$J : \mathscr{A} \to \mathsf{Alg}(T)$$

of $\mathsf{Alg}(T)$ that is closed under sifted colimits, 2-products, quotient algebras and subalgebras, we need to find a strongly finitary 2-monad T' such that there is a monad morphism $T \twoheadrightarrow T'$ and the comparison $\mathscr{A} \to \mathsf{Alg}(T')$ is an equivalence. Observe that $\mathscr{A}$ is a replete subcategory of $\mathsf{Alg}(T)$ as this follows from closedness under unary products.

We will proceed as follows:

(a) We will form an *ordinary* left adjoint to J by using Freyd's adjoint functor theorem [64].

(b) We will show that that J preserves cotensors with **2** and that the ordinary adjunction is thus enriched in Cat, using Proposition 3.1 of [18].

(c) We will construct a monad morphism $T \twoheadrightarrow T'$ from the above adjunction and show the equivalence $\mathscr{A} \simeq \mathsf{Alg}(T')$.

Ad (a): We will show that $\mathscr{A}$ has and J preserves ordinary limits. Since J is fully faithful, it suffices to prove that $\mathscr{A}$ is closed in $\mathsf{Alg}(T)$ under ordinary limits. By assumption, $\mathscr{A}$ is closed in $\mathsf{Alg}(T)$ under 2-products. It is therefore closed under ordinary products as well, since 2-products and ordinary products coincide in Cat. We need to show that it is closed also under equalisers. To this end, consider an equaliser diagram

$$\begin{tikzcd}
(A,a) \arrow[r, tail] & JX \arrow[r, shift left, "Js"] \arrow[r, shift right, "Jt"'] & JY
\end{tikzcd}$$

in $\mathsf{Alg}(T)$. Equalisers in $\mathsf{Alg}(T)$ are computed on the level of underlying categories, which implies that $A \rightarrowtail UJX$ is faithful. Thus (A,a) is a subalgebra of JX. Since the 2-category $\mathscr{A}$ is closed under subalgebras in $\mathsf{Alg}(T)$, we proved that it is closed under equalisers as well.

To establish the existence of a left adjoint for J, we now only need to find an ordinary solution set for every object (A,a) of $\mathsf{Alg}(T)$. We claim that the solution set is the set $\{h_i : (A,a) \twoheadrightarrow JX_i \mid i \in I\}$ of all the (representatives of the) quotients of (A,a) that lie in $\mathscr{A}$. This is indeed a set due to the nature of b.o. fullness, recall point 2. of Remark 6.2.2. Given any morphism $f : (A,a) \to JY$, we can factorise it to obtain a triangle

$$\begin{tikzcd}
(A,a) \arrow[rr, "f"] \arrow[dr, two heads, "h"'] & & JY \\
& (B,b) \arrow[ur, tail] &
\end{tikzcd}$$

and moreover, since (B,b) is a subalgebra of JY, we have that $(B,b) \cong JX$ holds for some X from $\mathscr{A}$, and the solution set condition is satisfied. The unit of the adjunction is constructed as follows: we take the product $\prod_{i \in I} JX_i$ of all the codomains of the quotients in the solution set, and factorise the mediating morphism $(h_i) : (A,a) \to \prod_{i \in I} JX_i$ as in the following diagram.

$$\begin{tikzcd}
(A,a) \arrow[rr, "(h_i)"] \arrow[dr, two heads, "\eta_{(A,a)}"'] & & \prod_{i \in I} JX_i \\
& JL(A,a) \arrow[ur, tail] &
\end{tikzcd}$$

Note that $\eta_{(A,a)}$ is b.o. full for every algebra (A,a).

Ad (b): Take a T-algebra (A,a) that belongs to $\mathscr{A}$ and form its cotensor $(A,a)^{\mathbf{2}}$. By means of the inclusion functor $2 \to \mathbf{2}$, we have a canonical homomorphism $(A,a)^{\mathbf{2}} \to (A,a)^2$ whose underlying functor is faithful, and thus renders $(A,a)^{\mathbf{2}}$ as a subalgebra of a product of algebras contained in $\mathscr{A}$. By the closure properties imposed on $\mathscr{A}$, we have that $\mathscr{A}$ is closed in $\mathsf{Alg}(T)$ under forming cotensors with $\mathbf{2}$ as well.

Ad (c): We can now define the 2-monad T' and the monad morphism $\varphi : T \to T'$ for which we will show the equivalence $\mathscr{A} \simeq \mathsf{Alg}(T')$. Let us first settle the notation and write $(L \dashv J, \eta, \varepsilon)$ for the adjunction $L \dashv J : \mathscr{A} \to \mathsf{Alg}(T)$, denote by $(F^T, U^T, \eta^T, \varepsilon^T)$ the adjunction $F^T \dashv U^T : \mathsf{Alg}(T) \to \mathsf{Cat}$, and let $\mu^T : TT \to T$ be the multiplication of the 2-monad T.

This allows us to define the 2-functor $T' := U^T JLF^T$ which is the underlying endofunctor of a 2-monad $(T', \eta^{T'}, \mu^{T'})$ with the unit $\eta^{T'}$ and the composition $\mu^{T'}$ defined by the assignments

$$\eta^{T'} := U^T \eta F^T \cdot \eta^T, \qquad \mu^{T'} := U^T J\varepsilon LF^T \cdot U^T JL\varepsilon^T JLF^T.$$

Then there is a corresponding monad morphism $e = U^T \eta F^T : T \twoheadrightarrow T'$. The proof that e is indeed a monad morphism is standard and proceeds exactly as in the non-enriched case. Moreover, e is a quotient, since

1. $\eta_{(A,a)}$ is a quotient for each algebra (A, a), and
2. U^T preserves quotients since T does.

Let us denote by

$$\begin{tikzcd}
\mathscr{A} \arrow[rr, dotted, "K"] \arrow[dr, "U^T J"'] & & \mathsf{Alg}(T') \arrow[dl, "U^{T'}"] \\
& \mathsf{Cat} &
\end{tikzcd}$$

the ordinary comparison functor. We will apply the ordinary Beck's theorem to infer that K is an ordinary equivalence. Since $\mathscr{A}$ has and $U^T J$ preserves sifted colimits, $\mathscr{A}$ has and $U^T J$ preserves coequalisers of reflexive pairs. Moreover, since U^T reflects isomorphisms and J is fully faithful, the composite functor $U^T J$ also reflects isomorphisms. Therefore $K : \mathscr{A} \to \mathsf{Alg}(T')$ is indeed an equivalence in the ordinary sense.

We will now show that on objects, the inclusion $J : \mathscr{A} \to \mathsf{Alg}(T)$ factorises, up to isomorphism, as in the following triangle:

$$\begin{tikzcd}
& \mathsf{Alg}(T') \arrow[dd, "\mathsf{Alg}(e)"] \\
\mathscr{A} \arrow[ur, "K"] \arrow[dr, "J"'] & \\
& \mathsf{Alg}(T).
\end{tikzcd}$$

Indeed, for any object A of $\mathscr{A}$ the equality

$$KA = (U^T JA, U^T J\varepsilon_A \cdot U^T JL\varepsilon^T_{JA})$$

holds. The algebra KA gets mapped by the functor $\mathsf{Alg}(e)$ to an algebra with a structure map

$$\begin{aligned}
U^T J\varepsilon_A \cdot U^T JL\varepsilon^T_{JA} \cdot e_{U^T JA} &= U^T J\varepsilon_A \cdot U^T JL\varepsilon^T_{JA} \cdot U^T \eta_{F^T U^T JA} \\
&= U^T J\varepsilon_A \cdot U^T \eta_{JA} \cdot U^T \varepsilon^T_{JA} \\
&= U^T \varepsilon^T_{JA},
\end{aligned}$$

where the first equality holds by the definition of e, the second one follows from naturality of η, and the third one comes from the triangle identity of $L \dashv J$. But $(U^T JA, U^T \varepsilon^T_{JA})$ is isomorphic to JA, as $(U^T JA, U^T \varepsilon^T_{JA})$ is the image of JA under the trivial comparison functor

$$I : \mathsf{Alg}(T) \to \mathsf{Alg}(T).$$

Both J and $\mathsf{Alg}(e)$ are fully faithful in Cat-enriched sense: the 2-functor J is such by assumption and $\mathsf{Alg}(e)$ was proved to be fully faithful for a quotient monad morphism e in the first part of the proof. We can conclude that the ordinary equivalence $K : \mathscr{A} \to \mathsf{Alg}(T')$ is enriched in Cat, thus finishing the proof. ■

Remark 6.2.4. A point that needs to be discussed is that we demand $\mathscr{A}$ to be closed under sifted colimits in $\mathsf{Alg}(T)$ in the characterisation of equational subcategories of $\mathsf{Alg}(T)$. It is true that in the original Birkhoff theorem there is no need to demand closedness under any class of colimits whatsoever. However, even in the ordinary case of $\mathscr{V} = \mathsf{Set}$, closedness under filtered colimits (or directed unions) is essential in the case of many-sorted universal algebra, see [6]. In the case of $\mathscr{V} = \mathsf{Cat}$, at least the requirement for closedness under filtered colimits is arguably expectable. The reason why our version of the Birkhoff theorem asks for an even stronger closure property, i.e., closedness under *sifted* colimits, is the following. While finitary and strongly finitary monads on Set coincide (every finitary monad is strongly finitary), this is not the case for 2-monads on Cat: a finitary 2-monad need not be strongly finitary. For example, the 2-monad T that gives rise to the 2-category $\mathsf{Alg}(T)$ of categories $\mathscr{C}$ equipped with one "arrow-ary" operation $\mathscr{C}^{\mathbf{2}} \to \mathscr{C}$ is finitary, but T fails to preserve sifted colimits in general. Since we are dealing with *strongly finitary* 2-monads on Cat, being closed under *sifted* colimits is the corresponding closure property.

Remark 6.2.5. In our setting, the property of being closed under sifted colimits is equivalent to being closed under *conical* filtered colimits and under *codescent objects of strict reflexive coherence data* by Remark 8.44 of [21]. For our purposes, the only two important points concerning codescent objects are that

1. they are the colimit objects for a certain *sifted* diagram, and
2. in the categories $\mathsf{Alg}(T)$ for a strongly finitary 2-monad T on Cat, the *universal cocone* over such a diagram consists of a single *bijective on objects homomorphism.*

This allows us to state the conditions of our Birkhoff theorem in an alternative way. In $\mathsf{Alg}(T)$, define an algebra (B, b) to be a *(b.o.)-quotient* of (A, a) if there is a homomorphism $h : (A, a) \to (B, b)$ that is bijective on objects. Since every b.o. full functor is b.o., we may strengthen the property of being closed under quotient algebras to the property of being closed under (b.o.)-quotient algebras, and replace the requirement for closedness under sifted colimits by closedness under filtered colimits.

It remains to argue that a full subcategory $\mathscr{A}$ of $\mathsf{Alg}(T)$ closed under 2-limits and sifted colimits is closed under (b.o.)-quotients. Given a (b.o.)-quotient $h : (A, a) \to (B, b)$ with (A, a) contained in $\mathscr{A}$, it follows by the results of [22] (see Section 5.1 in particular) that h is the quotient of the kernel

$$\begin{tikzcd} h|h|h \arrow[r, bend left, "p"] \arrow[r, "m"] \arrow[r, bend right, "q"'] & h|h \arrow[r, bend left, "d"] \arrow[r, bend right, "c"'] & (A,a), \arrow[l, "i"'] \end{tikzcd}$$

where the component $h|h$ is a certain subalgebra of $(A,a)^2$, and the component $h|h|h$ is a pullback of c and d. Without loss of generality, the above kernel can be considered reflexive, and thus (B,b) belongs to $\mathscr{A}$, being a sifted colimit of algebras contained in $\mathscr{A}$.

The following alternative statement of Birkhoff theorem is a direct corollary of the above remark, and it may be more useful in practice for detecting equational subcategories of algebraic categories.

Corollary 6.2.6. *The full subcategory $\mathscr{A}$ of* $\mathsf{Alg}(T)$ *is an equational subcategory of* $\mathsf{Alg}(T)$ *if and only if it is closed in* $\mathsf{Alg}(T)$ *under 2-products, (b.o.)-quotient algebras, subalgebras and filtered colimits.*

In the ordinary setting, full algebraic subcategories induced by a quotient monad morphism can be characterised as a special kind of orthogonal subcategories. Without substantial changes to the reasoning, the same characterisation can be obtained for the case of $\mathscr{V} = \mathsf{Cat}$, as is shown below.

Given a 2-category $\mathscr{X}$ and a set $S = \{f_i : X_i \to Y_i \mid i \in I\}$ of morphisms of $\mathscr{X}$, we will denote by $S^{\perp}$ the full subcategory $J : \mathscr{Y} \to \mathscr{X}$ spanned by the objects Y that are orthogonal to all morphisms in S.

Corollary 6.2.7. *The equational subcategories*

$$J : \mathscr{A} \to \mathsf{Alg}(T)$$

of the 2-category $\mathsf{Alg}(T)$ *of algebras for a strongly finitary 2-monad T are precisely the orthogonal subcategories of* $\mathsf{Alg}(T)$ *of the form*

$$\mathscr{A} = \{f : F^T n \twoheadrightarrow (C,c) \mid f \in I\}^{\perp} = I^{\perp}$$

for some set I of quotient morphisms in $\mathsf{Alg}(T)$*. Moreover, each morphism in I has as its domain a free algebra on a finite discrete category.*

Proof. To see that one direction of this statement holds, observe that $\mathscr{A}$ is closed under subobjects in $\mathsf{Alg}(T)$: Given an algebra (B,b) in $\mathscr{A}$ and its subalgebra (A,a), we have for any $g : F^T n \to (A,a)$ a situation

$$\begin{tikzcd} F^T n \arrow[r, two heads, "f"] \arrow[d, "g"'] & (C,c) \arrow[dl, dotted] \arrow[d, dotted] \\ (A,a) \arrow[r, tail] & (B,b) \end{tikzcd}$$

where the unique morphism $(C,c) \to (B,b)$ exists since $f \perp (B,b)$, and the unique diagonal exists by the diagonal property of the factorisation system. The universal property of 2-products establishes that $\mathscr{A}$ is closed in $\mathsf{Alg}(T)$ under 2-products. To see that $\mathscr{A}$ is closed in $\mathsf{Alg}(T)$ under sifted colimits we show that $\mathsf{Alg}(T)(F^T n, -)$ preserves sifted colimits. This is the case, since $\mathsf{Alg}(T)(F^T n, -) \cong \mathsf{Cat}(n, U^T -)$ and both $\mathsf{Cat}(n, -)$ and U^T preserve sifted colimits. To show that $\mathscr{A}$ is closed in $\mathsf{Alg}(T)$ under quotients, observe first that $\mathsf{Alg}(T)(F^T n, -)$ preserves quotient maps since both U^T and $\mathsf{Cat}(n, -)$ are easily seen to preserve quotient maps. This property implies that $F^T n$ is projective with respect to quotients, and that the factorisation granted by projectivity is unique. Consider a quotient $h : (A, a) \twoheadrightarrow (B, b)$ with (A, a) in $\mathscr{A}$. To prove that (B, b) is in $\mathscr{A}$, observe that for any morphism $g : F^T n \to (B, b)$ there is a unique morphism $p : F^T n \to (A, a)$:

$$\begin{tikzcd}
F^T n \arrow[d, dotted, "p"'] \arrow[dr, "g"] & \\
(A,a) \arrow[r, two heads, "h"'] & (B,b)
\end{tikzcd}$$

Since (A, a) is orthogonal to f, we obtain a triangle

$$\begin{tikzcd}
F^T n \arrow[r, "f"] \arrow[d, "p"'] & (C,c) \arrow[dl, dotted, "o"] \\
(A,a) &
\end{tikzcd}$$

The composite $h \cdot o$ then proves that $f \perp (B, b)$. Indeed, given any other factorisation $g = i \cdot f$, the equality $i = h \cdot o$ holds since f is epi.

In the opposite direction, recall that reflective subcategories are *always* orthogonality classes. In our case we have that

$$\mathscr{A} = \{\eta_{(A,a)} : (A, a) \twoheadrightarrow JL(A, a) \mid (A, a) \in \mathsf{Alg}(T)\}^{\perp}.$$

We need to take a subset of the above class of morphisms such that the codomain of each morphism is a free algebra on a finite discrete category. For this, we first use that every algebra (A, a) is a sifted colimit of free algebras on finite discrete categories. Indeed, consider the full subcategory $\mathscr{G} \to \mathsf{Alg}(T)$ spanned by algebras of the form $F^T n$ for a natural number n. By Proposition 4.2 of [48], $\mathsf{Alg}(T)$ is a free cocompletion of $\mathscr{G}$ under sifted colimits; the only interesting property to check being that the closure of $\mathscr{G}$ in $\mathsf{Alg}(T)$ under sifted colimits is the whole of $\mathsf{Alg}(T)$. Observe that a free algebra $F^T X$ on a discrete category X is a filtered colimit of free algebras on finite discrete categories, a free algebra $F^T C$ on a category C is a sifted colimit (codescent object) of free algebras on discrete categories, and any algebra (A, a) is a reflexive coequaliser of free algebras $F^T A$ on A. The result follows from this reasoning.

Secondly, if an object is orthogonal to a given set of arrows, it is orthogonal to their colimit in the category of arrows as well. Since JL preserves sifted colimits, we get that

$$\{\eta_{(A,a)} : (A, a) \twoheadrightarrow JL(A, a) \mid (A, a) \in \mathsf{Alg}(T)\}^{\perp}$$

is equal to the subcategory

$$\{\eta_{F^T n} : F^T n \twoheadrightarrow JLF^T n \mid n \in \mathsf{Cat}, n \text{ finite discrete}\}^{\perp},$$

as we needed. ■

The above result may be reformulated to resemble the original universal algebraic formulation of Birkhoff's theorem even more. Taking again

$$\mathscr{A} = \{f : F^T n \twoheadrightarrow (C, c) \mid f \in I\}^{\perp},$$

we know that any morphism $f : F^T n \twoheadrightarrow (C, c)$ as above is the coequifier of its kernel:

$$\begin{tikzcd}(K,k) \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\gamma \Downarrow\Downarrow \delta"] & F^T n \arrow[r, two heads, "f"] & (C,c)\end{tikzcd}$$

Given an algebra (A, a), it is orthogonal to f precisely when each morphism $g : F^T n \to (A, a)$ coequifies the 2-cells γ and δ. Now consider the underlying discrete category K_0 of the category K by means of the b.o. inclusion functor $i : K_0 \to U^T(K, k)$. Transposing this functor, we get a homomorphism $i^\sharp : F^T K_0 \to (K, k)$ defined as the composite

$$\begin{tikzcd}F^T K_0 \arrow[rr, "i^\sharp"] \arrow[dr, "F^T i"'] & & (K,k) \\ & F^T U^T (K,k) \arrow[ur, "\varepsilon^T_{(K,k)}"'] & \end{tikzcd}$$

of two homomorphisms that are surjective on objects. The morphism $\varepsilon^T_{(K,k)}$ is surjective on objects since its underlying functor is a split epi k, and $F^T i$ is in fact b.o., because $T = U^T F^T$ as a strongly finitary monad preserves b.o. functors. A given morphism $g : F^T n \to (A, a)$ therefore coequifies γ and δ if and only if it coequifies the whiskered 2-cells $\gamma * i^\sharp$ and $\delta * i^\sharp$:

$$\begin{tikzcd}F^T K_0 \arrow[r, "i^\sharp"] & (K,k) \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\gamma \Downarrow\Downarrow \delta"] & F^T n\end{tikzcd}$$

As a left adjoint, F^T preserves coproducts, and thus

$$F^T K_0 \cong F^T\Big(\coprod_{\mathrm{ob}(K_0)} \mathbf{1}\Big) \cong \coprod_{\mathrm{ob}(K_0)} F^T \mathbf{1}$$

holds. This allows us to reduce the pair γ and δ of 2-cells into $\mathrm{ob}(K_0)$-many pairs γ_c and δ_c of 2-cells

$$\begin{tikzcd}F^T \mathbf{1} \arrow[r, bend left, "s_c"] \arrow[r, bend right, "t_c"'] \arrow[r, phantom, "\gamma_c \Downarrow\Downarrow \delta_c"] & F^T n\end{tikzcd}$$

such that a morphism $g : F^T n \to (A, a)$ coequifies γ and δ precisely when it coequifies all the pairs γ_c and δ_c. In fact, let us call each such a pair (γ_c, δ_c) an *equation in* T and observe that it corresponds precisely to a pair of morphisms in $U^T F^T n$. Let us now say that an algebra (A, a) from $\mathsf{Alg}(T)$ *satisfies the equation* $\gamma_c = \delta_c$ if every morphism $g : F^T n \to (A, a)$ coequifies γ_c and δ_c. We have just proved the following "universal-algebraic" version of Birkhoff's theorem.

Corollary 6.2.8. *For a full subcategory* $\mathscr{A}$ *of* $\mathsf{Alg}(T)$ *for a strongly finitary monad* T *on* Cat*, the following are equivalent:*

1. $\mathscr{A}$ *is closed under 2-products, subalgebras, quotient algebras and sifted colimits.*
2. *There is a set* $E = \{\gamma_i = \delta_i \mid i \in I\}$ *of equations in* T *such that* $\mathscr{A}$ *consists of algebras of* $\mathsf{Alg}(T)$ *that satisfy* E*.*

Part II

Formal categorical universal algebra

Chapter 7

Gray-categories and their presentations

In ordinary universal algebra, the notion of *freeness* can be given a precise meaning via the categorical notion of an *adjunction*. For example, consider the free monoid construction

$$S \mapsto F(S) = (S^*, *, \epsilon)$$

which gives, for a set S, the monoid FS given by the set of words S^* over alphabet S with concatenation as multiplication and the empty word as the unit (as in Example 1.1.2). This construction gives rise to an adjunction

$$\begin{tikzcd}
\mathsf{Mon} \arrow[d, bend left, "U"] \arrow[d, phantom, "\dashv"] \\
\mathsf{Set} \arrow[u, bend left, "F"]
\end{tikzcd}$$

with U being the underlying set functor, and the components of the unit $\eta_S : S \to S^*$ being the inclusion of the alphabet S into the set S^* of words over S. Equivalently, this construction satisfies a universal property: given a monoid $(X, *, e)$ and mapping $f : S \to X$, there exists a unique way to lift f to a monoid homomorphism $f^\sharp : FS \to (X, *, e)$ such that the diagram

$$\begin{tikzcd}
S \arrow[r, "\eta_S"] \arrow[dr, "f"'] & FS \arrow[d, "Uf^\sharp"] \\
& X
\end{tikzcd}$$

commutes.

The notion of freeness reoccurs "one level of abstraction up" when we consider limits and colimits in categories, and form *free (co)completions* of categories. Let us recall from Remark 2.1.20 that given a class Φ of weights and a category $\mathscr{K}$, there exists the *free cocompletion* $\Phi(\mathscr{K})$ *of* $\mathscr{K}$ *under the class of* Φ*-colimits* with the *unit*

$$Z^\Phi_{\mathscr{K}} : \mathscr{K} \to \Phi(\mathscr{K}).$$

This cocompletion *almost* satisfies a universal property similar to the property of the free monoid above. That is, given a Φ-cocomplete category $\mathscr{L}$ and any $F : \mathscr{K} \to \mathscr{L}$, there exists, *up to isomorphism*, a unique Φ-cocontinuous functor $\mathscr{F}^\sharp : \Phi(\mathscr{K}) \to \mathscr{L}$ such that the diagram

$$\begin{tikzcd}
\mathscr{K} \arrow[r, "\eta^{\Phi}_{\mathscr{K}}"] \arrow[dr, "F"'] & \Phi(\mathscr{K}) \arrow[d, "F^\sharp"] \\
& \mathscr{L}
\end{tikzcd}$$

commutes up to isomorphism. Since the free Φ-cocompletion of a category is determined only up to equivalence of categories, we cannot claim that there is a "free cocompletion (2)-functor"

$$F_\Phi : \mathscr{V}\text{-}\mathsf{CAT} \to \Phi\text{-}\mathsf{COCTS}$$

which would be a left adjoint to the forgetful (2)-functor

$$U_\Phi : \Phi\text{-}\mathsf{COCTS} \to \mathscr{V}\text{-}\mathsf{CAT}$$

mapping a Φ-cocomplete category to itself. However, F_Φ is a *pseudofunctor* and F_Φ and U_Φ form a *pseudoadjunction*

$$\begin{tikzcd}
\Phi\text{-}\mathsf{COCTS} \arrow[d, bend left, "U_\Phi"] \arrow[d, phantom, "\dashv"] \\
\mathscr{V}\text{-}\mathsf{CAT} \arrow[u, bend left, "F_\Phi"]
\end{tikzcd}$$

with *unit* η^Φ, where the notions of a pseudofunctor and pseudoadjuction are the appropriate generalisations of the "strict" notions of a functor and an adjunction. These notions are appropriate in the sense that they abstract the example of free cocompletions, describing precisely what the notion of freeness "up to equivalence of categories" should mean.

The above example of free cocompletions leads us to the field of higher category theory. The setting for the study of pseudoadjunctions leads us to 2-categories, pseudofunctors and pseudonatural transformations, all being *bicategorical* notions. However, working in such a general setting yields substantial technical difficulties, since even the definitions of the "pseudo versions" of basic categorical notions are quite involved. This observation leads us to the study of **Gray**-categories, that is, categories enriched in the category $\mathscr{V} =$ **Gray** of 2-categories and 2-functors, equipped with **Gray**-tensor product [39]. Without **Gray**-categories we would have to study the collection of 2-categories, pseudofunctors, pseudonatural transformations and modifications as a *tricategorical* structure. Such an approach is considerably difficult. **Gray**-categories allow us to study the collection of 2-categories, pseudofunctors, pseudonatural transformations and modifications abstractly in the context of enriched category theory. This motivation leads the topic of the present chapter. We introduce **Gray**-categories and show a way to give *presentations* of **Gray**-categories that will allow us to consider, for example, a pseudoadjunction P in a **Gray**-category $\mathbf{K}$ as a certain **Gray**-*functor* with codomain $\mathbf{K}$.

Purpose of the chapter. Since this chapter does not contain any new results of the author, it may be safely skipped by the reader who is not interested in the technical details of the theory behind presentations of **Gray**-categories. In fact, after reading Section 7.1

on the basics of **Gray**-categories, the rest of this chapter can be summarised as giving reasons for the claim

It is possible to give presentations of **Gray***-categories in the same spirit as we can present ordinary categories, or 2-categories.*

The reader who is content to believe this claim can move on to the next chapter which gives concrete examples of presentations of **Gray**-categories after reading Section 7.1.

Structure of the chapter.

- We introduce **Gray**-categories in Section 7.1 and shortly discuss their importance as “well-behaved” tricategories.
- In Section 7.2 we will introduce the notion of a *presentation* of an algebra, and show that there is an adjunction between the category of algebras of a given type and the category of presentations of the given type.
- In Section 7.3 we shall comment on the problems of presenting 2-categories and note why such presentations are important. We sketch the approach of Street, namely of his *computads*.
- To be able to give presentations of **Gray**-categories, we need to review some technically involved notions. Section 7.4 introduces *globular operads* via Kelly's *clubs*.
- Globular operads underlie the notion of a *globular computad*, a generalisation of Street's computads. We treat globular computads Section 7.5.

This chapter serves as an overview of **Gray**-categories and their presentations. We do not claim authorship of any of the results in this chapter.

7.1 Gray-categorical background

The collection of 2-categories, pseudofunctors, pseudonatural transformations and modifications[1] organises itself into a 3-dimensional categorical structure; we shall see that it *does not* form a (strict) 3-category. Rather, these data organise themselves into a *tricategory*. Working in the environment of a tricategory poses substantial technical difficulties. We shall introduce **Gray**-categories [38, 39] that are more pleasant to work with. An important example of a **Gray**-category is the collection of 2-categories, 2-functors, pseudonatural transformations and modifications. A coherence result states that every tricategory is *triequivalent* to a **Gray**-category. Thus we lose no extra generality when working with **Gray**-categories rather than with tricategories.

Example 7.1.1 (Difficulties in the composition of pseudofunctors and pseudonatural transformations [72]). The collection of of 2-categories, pseudofunctors and pseudonatural transformations does not form a *2-category*. Observe that the problem arises when dealing with the middle-four interchange law, as can be seen e.g. in Example 7.2 of [72]:

---

[1] The prefix *pseudo* roughly means that the equalities in the definitions of a functor and a natural transformation are replaced by coherent isomorphisms. See for example Chapter 7 of Volume 1 of [19].

Let us take the data

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\sigma \Downarrow"] & \mathscr{B} \end{tikzcd}$$

where σ is a pseudonatural transformation (i.e., a collection $\sigma_X : FX \to GX$ of morphisms of $\mathscr{A}$ such that there is a "coherent" isomorphism

$$\begin{tikzcd} FX \arrow[r, "\sigma_X"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\swarrow \sigma_f"] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\sigma_Y"'] & GY \end{tikzcd}$$

for any morphism $f : X \to Y$ in $\mathscr{A}$), and take a morphism $f : X \to Y$ *in* the 2-category $\mathscr{A}$. The morphism f determines a pseudonatural transformation

$$\begin{tikzcd} \mathbf{1} \arrow[r, bend left, "\ulcorner X \urcorner"] \arrow[r, bend right, "\ulcorner Y \urcorner"'] \arrow[r, phantom, "\ulcorner f \urcorner \Downarrow"] & \mathscr{A}. \end{tikzcd}$$

If the middle-four interchange law held, then σ would be *2-natural* with respect to f. This would hold for all f, yielding that σ is necessarily 2-natural, and thus ending in a contradiction.

Remark 7.1.2 (Interchange law up to isomorphism [39]). Let us inspect the problems with "middle-four interchange law" for 2-categories, pseudofunctors, pseudonatural transformations and modifications a bit further. Let us organise these data into a collection PSD; for 2-categories $\mathscr{A}$ and $\mathscr{B}$, we get that the collection $\mathsf{PSD}(\mathscr{A}, \mathscr{B})$ of pseudofunctors, pseudonatural transformations and modifications forms a 2-category. Consider the composition *assignment* (not claiming that it is a *2-functor itself*)

$$c_{\mathscr{A},\mathscr{B},\mathscr{C}} : \mathsf{PSD}(\mathscr{B}, \mathscr{C}) \times \mathsf{PSD}(\mathscr{A}, \mathscr{B}) \to \mathsf{PSD}(\mathscr{A}, \mathscr{C}).$$

Given the data

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\sigma \Downarrow"] & \mathscr{B} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\tau \Downarrow"] & \mathscr{C} \end{tikzcd}$$

both $c_{\mathscr{A},\mathscr{B},\mathscr{C}}(H, -) = H(-)$ and $c_{\mathscr{A},\mathscr{B},\mathscr{C}}(-, F) = (-)F$ are 2-functors; and both $H\sigma$ and τF are pseudonatural transformations. However, the square

$$\begin{tikzcd} HF \arrow[r, Rightarrow, "\tau F"] \arrow[d, Rightarrow, "H\sigma"'] & KF \arrow[d, Rightarrow, "K\sigma"] \\ HG \arrow[r, Rightarrow, "\tau G"'] & KG \end{tikzcd} \tag{7.1}$$

does not commute (as it should, were $c_{\mathscr{A},\mathscr{B},\mathscr{C}}$ a 2-functor): it commutes only up to an isomorphism

$$\begin{tikzcd}
HF \arrow[r, Rightarrow, "\tau F"] \arrow[d, Rightarrow, "H\sigma"'] \arrow[dr, phantom, "\Lleftarrow \tau_\sigma"] & KF \arrow[d, Rightarrow, "K\sigma"] \\
HG \arrow[r, Rightarrow, "\tau G"'] & KG
\end{tikzcd} \tag{7.2}$$

(i.e., the pseudonatural transformations are "not 'natural" in the sense that the middle-four interchange law holds only up to a coherent isomorphism modification). This shows that if we wanted $\mathsf{PSD}(\mathscr{A},\mathscr{B})$ to be an internal hom of PSD, the monoidal product of PSD cannot be chosen as the cartesian one.

The rather unfortunate situation in the above remark can be dealt with. We shall work with **Gray**-*categories*: categories enriched in the category $\mathscr{V} = \mathbf{Gray}$, which consists of 2-categories, 2-functors, pseudonatural transformations and modifications, and has a non-cartesian monoidal product (**Gray**-tensor product). We shall introduce **Gray**-categories explicitly in elementary terms.

Gray-categories A **Gray**-*category* $\mathbf{K}$ with objects $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, has a hom-2-category $\mathbf{K}(\mathscr{A},\mathscr{B})$ for each pair $\mathscr{A}$, $\mathscr{B}$ of objects, the unit 2-functor $u_{\mathscr{A}} : \mathscr{I} \to \mathbf{K}(\mathscr{A},\mathscr{A})$ sends the unique i-cell ($i = 0, 1, 2$) to the identity $(i + 1)$-cell of $\mathscr{A}$ (e.g., the unique object $*$ of $\mathscr{I}$ gets sent to $1_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}$), and the *composition*

$$\mathbf{K}(\mathscr{B},\mathscr{C}) \otimes \mathbf{K}(\mathscr{A},\mathscr{B}) \to \mathbf{K}(\mathscr{A},\mathscr{C})$$

is essentially the composition *cubical* functor

$$\mathbf{K}(\mathscr{B},\mathscr{C}) \times \mathbf{K}(\mathscr{A},\mathscr{B}) \to \mathbf{K}(\mathscr{A},\mathscr{C})$$

yielding, for any F in $\mathbf{K}(\mathscr{A},\mathscr{B})$ and any G in $\mathbf{K}(\mathscr{B},\mathscr{C})$, two 2-functors

$$\begin{aligned}(-)F &: \mathbf{K}(\mathscr{B},\mathscr{C}) \to \mathbf{K}(\mathscr{A},\mathscr{C})\\ G(-) &: \mathbf{K}(\mathscr{A},\mathscr{B}) \to \mathbf{K}(\mathscr{A},\mathscr{C})\end{aligned}$$

("precomposition" and "postcomposition"). Here, $(-)F$ is acting on the data

$$\begin{tikzcd}
\mathscr{B} \arrow[rr, bend left, "H"] \arrow[rr, bend right, "K"'] \arrow[rr, phantom, "\alpha \Downarrow \overset{\tau}{\Rrightarrow} \Downarrow \beta"] & & \mathscr{C}
\end{tikzcd}$$

to give

$$\begin{tikzcd}
\mathscr{B} \arrow[rr, bend left, "HF"] \arrow[rr, bend right, "KF"'] \arrow[rr, phantom, "\alpha F \Downarrow \overset{\tau F}{\Rrightarrow} \Downarrow \beta F"] & & \mathscr{C}
\end{tikzcd}$$

and $G(-)$ is acting on

$$\begin{tikzcd}[column sep=huge]
\mathscr{A} \arrow[r, bend left=40, "H"] \arrow[r, bend right=40, "K"'] \arrow[r, phantom, "\alpha \Downarrow \overset{\tau}{\Rrightarrow} \Downarrow \beta"] & \mathscr{B}
\end{tikzcd}$$

to give

$$\begin{tikzcd}[column sep=huge]
\mathscr{A} \arrow[r, bend left=40, "GH"] \arrow[r, bend right=40, "GK"'] \arrow[r, phantom, "G\alpha \Downarrow \overset{G\tau}{\Rrightarrow} \Downarrow G\beta"] & \mathscr{B}.
\end{tikzcd}$$

The 2-functors $(-)F$ and $G(-)$ are subject to the equality

$$(G)F = G(F) = GF$$

and for every pair

$$\begin{tikzcd}[column sep=large]
\mathscr{A} \arrow[r, bend left=40, "F"] \arrow[r, bend right=40, "F'"'] \arrow[r, phantom, "\alpha \Downarrow"] & \mathscr{B} & \mathscr{B} \arrow[r, bend left=40, "G"] \arrow[r, bend right=40, "G'"'] \arrow[r, phantom, "\beta \Downarrow"] & \mathscr{C}
\end{tikzcd}$$

there is an isomorphism

$$\begin{tikzcd}[column sep=large, row sep=large]
GF \arrow[r, Rightarrow, "\beta F"] \arrow[d, Rightarrow, "G\alpha"'] \arrow[dr, phantom, "\Lleftarrow \beta_\alpha"] & G'F \arrow[d, Rightarrow, "G'\alpha"] \\
GF' \arrow[r, Rightarrow, "\beta F'"'] & G'F'
\end{tikzcd}$$

subject to the cubical functor axioms:

1. Composition axioms

$$\begin{tikzcd}[column sep=large, row sep=large]
GF \arrow[r, Rightarrow, "\beta F"] \arrow[d, Rightarrow, "G\alpha"'] \arrow[dr, phantom, "\Lleftarrow \beta_\alpha"] & G'F \arrow[d, Rightarrow, "G'\alpha"] \\
GF' \arrow[r, Rightarrow, "\beta F'"'] \arrow[d, Rightarrow, "G\gamma"'] \arrow[dr, phantom, "\Lleftarrow \beta_\gamma"] & G'F' \arrow[d, Rightarrow, "G'\gamma"] \\
GF'' \arrow[r, Rightarrow, "\beta F''"'] & G'F''
\end{tikzcd}
=
\begin{tikzcd}[column sep=large, row sep=large]
GF \arrow[r, Rightarrow, "\beta F"] \arrow[d, Rightarrow, "G\alpha"'] & G'F \arrow[d, Rightarrow, "G'\alpha"] \\
GF' \arrow[r, phantom, "\Lleftarrow \beta_{\gamma\cdot\alpha}"] \arrow[d, Rightarrow, "G\gamma"'] & G'F' \arrow[d, Rightarrow, "G'\gamma"] \\
GF'' \arrow[r, Rightarrow, "\beta F''"'] & G'F''
\end{tikzcd}$$

and

$$\begin{tikzcd}
GF \arrow[r, Rightarrow, "\beta F"] \arrow[d, Rightarrow, "G\alpha"'] \arrow[dr, phantom, "\Lleftarrow \beta_\alpha"] & G'F \arrow[r, Rightarrow, "\delta F"] \arrow[d, Rightarrow, "G'\alpha"] \arrow[dr, phantom, "\Lleftarrow \delta_\alpha"] & G''F \arrow[d, Rightarrow, "G''\alpha"] \\
GF' \arrow[r, Rightarrow, "\beta F'"'] & G'F' \arrow[r, Rightarrow, "\delta F'"'] & G''F'
\end{tikzcd}$$

$$=$$

$$\begin{tikzcd}
GF \arrow[r, Rightarrow, "\beta F"] \arrow[d, Rightarrow, "G\alpha"'] & G'F \arrow[r, Rightarrow, "\delta F"] \arrow[d, phantom, "\Lleftarrow (\delta \cdot \beta)_\alpha"] & G''F \arrow[d, Rightarrow, "G''\alpha"] \\
GF' \arrow[r, Rightarrow, "\beta F'"'] & G'F' \arrow[r, Rightarrow, "\delta F'"'] & G''F'.
\end{tikzcd}$$

2. “Modification” axioms

$$\begin{tikzcd}
GF \arrow[r, Rightarrow, "\beta F"] \arrow[d, Rightarrow, "G\alpha"'] \arrow[dr, phantom, "\Lleftarrow \beta_\alpha"] & G'F \arrow[d, Rightarrow, "G'\alpha"'] \arrow[d, Rightarrow, bend left=90, looseness=2, "G'\alpha'"] \arrow[d, phantom, shift left=6ex, "\overset{G's}{\Lleftarrow}"] \\
GF' \arrow[r, Rightarrow, "\beta F'"'] & G'F'
\end{tikzcd}
=
\begin{tikzcd}
GF \arrow[r, Rightarrow, "\beta F"] \arrow[d, Rightarrow, "G\alpha'"] \arrow[d, Rightarrow, bend right=90, looseness=2, "G\alpha"'] \arrow[d, phantom, shift right=6ex, "\overset{Gs}{\Lleftarrow}"] \arrow[dr, phantom, "\Lleftarrow \beta_{\alpha'}"] & G'F \arrow[d, Rightarrow, "G'\alpha'"] \\
GF' \arrow[r, Rightarrow, "\beta F'"'] & G'F'
\end{tikzcd}$$

for any 3-cell $s : \alpha' \Rrightarrow \alpha$ and

$$\begin{tikzcd}
GF \arrow[r, Rightarrow, "\beta F"'] \arrow[r, Rightarrow, bend left=90, looseness=1.5, "\beta' F"] \arrow[r, phantom, shift left=4ex, "tF \Downarrow"] \arrow[d, Rightarrow, "G\alpha"'] \arrow[dr, phantom, "\Lleftarrow \beta_\alpha"] & G'F \arrow[d, Rightarrow, "G'\alpha"] \\
GF' \arrow[r, Rightarrow, "\beta F'"'] & G'F'
\end{tikzcd}
=
\begin{tikzcd}
GF \arrow[r, Rightarrow, "\beta' F"] \arrow[d, Rightarrow, "G\alpha"'] \arrow[dr, phantom, "\Lleftarrow {\beta'}_\alpha"] & G'F \arrow[d, Rightarrow, "G'\alpha"] \\
GF' \arrow[r, Rightarrow, "\beta' F'"] \arrow[r, Rightarrow, bend right=90, looseness=1.5, "\beta F"'] \arrow[r, phantom, shift right=4ex, "tF' \Downarrow"] & G'F'
\end{tikzcd}$$

for any 3-cell $t : \beta' \Rrightarrow \beta$.

Given any triple

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\alpha \Downarrow"] & \mathscr{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{B} \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\beta \Downarrow"] & \mathscr{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{C} \arrow[r, bend left, "H"] \arrow[r, bend right, "H'"'] \arrow[r, phantom, "\gamma \Downarrow"] & \mathscr{D}
\end{tikzcd}$$

the associativity equalities

$$\gamma_{(\beta F)} = (\gamma_\beta)F, \quad \gamma_{(G\alpha)} = (\gamma G)_\alpha, \quad H(\beta_\alpha) = (H\beta)_\alpha$$

hold, allowing us to relax the notation when working with the invertible 3-cells β_α. Finally, the unit equalities

$$1F = F, \quad G1 = G$$

hold, where 1 can stand for the identity 1-cell, 2-cell or 3-cell on $\mathscr{B}$.

Example 7.1.3. Let us consider the *category* $(2\text{-}\mathsf{Cat})_0$ of all 2-categories and all 2-functors. As in Example 7.1.2, when we equip this category with the 2-dimensional structure consisting of pseudonatural transformations and the 3-dimensional structure given by modifications, the resulting collection does *not* form a 3-category by the same argument. If we denoted the collection by **Gray**, then we would need to have a composition 2-functor

$$c_{\mathscr{A},\mathscr{B},\mathscr{C}} : \mathbf{Gray}(\mathscr{B},\mathscr{C}) \times \mathbf{Gray}(\mathscr{A},\mathscr{B}) \to \mathbf{Gray}(\mathscr{A},\mathscr{C})$$

which is impossible by Remark 7.1.2. However, we *do have* a cubical functor

$$c_{\mathscr{A},\mathscr{B},\mathscr{C}} : \mathbf{Gray}(\mathscr{B},\mathscr{C}) \times \mathbf{Gray}(\mathscr{A},\mathscr{B}) \to \mathbf{Gray}(\mathscr{A},\mathscr{C})$$

where for every situation

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\sigma \Downarrow"] & \mathscr{B} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\tau \Downarrow"] & \mathscr{C}
\end{tikzcd}$$

the relevant isomorphism is given by the 2-cell

$$\begin{tikzcd}
HF \arrow[r, Rightarrow, "\tau F"] \arrow[d, Rightarrow, "H\sigma"'] \arrow[dr, phantom, "\Lleftarrow \tau_\sigma"] & KF \arrow[d, Rightarrow, "K\sigma"] \\
HG \arrow[r, Rightarrow, "\tau G"'] & KG
\end{tikzcd}$$

and this 2-cell satisfies the cubical axioms stated above.

Our interest in **Gray**-categories stems from the fact that they are the maximally strict tricategories such that we lose no generality working with **Gray**-categories instead of general tricategories.[2] This is not true of 3-categories:

Proposition 7.1.4 ([72], Example 7.4). *Not every tricategory is triequivalent to a 3-category.*

---

[2] Roughly speaking, a tricategory relates to a 3-category similarly to how a bicategory relates to a 2-category: various types of composition are associative and unital only up to coherent isomorphisms. See [38] for the precise definition of a tricategory.

The above fact gives the basic motivation for introducing **Gray**-categories. The reason that Proposition 7.1.4 holds is that to give a one-object, one-morphism tricategory is to give a braided monoidal category, while giving a one-object, one-morphism *3-category* is giving a *strict* monoidal category with a *strict* symmetry, and these in general need not be equivalent. A concrete example is, e.g., Example 7.4 of [72]: the symmetric monoidal closed category Set with finite products. The symmetry map $X \times Y \to Y \times X$ is almost never the identity, and thus the symmetry is not strict.

In contrast to Proposition 7.1.4, the following coherence result holds:

Theorem 7.1.5 (Theorem 8.1 of [37]). *Every tricategory is triequivalent to a* **Gray**-*category.*

This result allows us to work in the setting of **Gray**-categories instead of general tricategories. For example, proving a fact about pseudomonads in a general **Gray**-category proves that the fact holds in the tricategory PSD. (See, e.g., the paper [67] developing the formal theory of KZ-monads.)

We shall now recall the notions of a **Gray**-functor, **Gray**-natural transformation, and the presheaf **Gray**-category.

Gray-functors A **Gray**-*functor* $\mathbf{F} : \mathbf{K} \to \mathbf{L}$ consists of

1. an object assignment
$$\mathscr{A} \mapsto \mathbf{F}\mathscr{A}$$
2. and the action on hom-2-categories, i.e., a 2-functor
$$\mathbf{F}_{\mathscr{A},\mathscr{B}} : \mathbf{K}(\mathscr{A},\mathscr{B}) \to \mathbf{L}(\mathbf{F}\mathscr{A},\mathbf{F}\mathscr{B})$$
defined by the assignment
$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left=50, "H"] \arrow[r, bend right=50, "K"'] \arrow[r, phantom, "\alpha \Downarrow \overset{\tau}{\Rrightarrow} \Downarrow \beta"] & \mathscr{B} \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \mathbf{F}\mathscr{A} \arrow[r, bend left=50, "\mathbf{F}H"] \arrow[r, bend right=50, "\mathbf{F}K"'] \arrow[r, phantom, "\mathbf{F}\alpha \Downarrow \overset{\mathbf{F}\tau}{\Rrightarrow} \Downarrow \mathbf{F}\beta"] & \mathbf{F}\mathscr{B} \end{tikzcd}$$
This assignment is subject to the following axioms:
   (a) For each $\mathscr{A}$ of $\mathbf{K}$ the assignment $\mathbf{F}$ maps identity cells to identity cells.
   (b) For all objects $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, a 1-cell $F : \mathscr{A} \to \mathscr{B}$, 1-cells $H, K : \mathscr{B} \to \mathscr{C}$, 2-cells $\alpha, \beta : H \Rightarrow K$ and a 3-cell $\tau : \alpha \Rrightarrow \beta$, we demand the equalities
$$\mathbf{F}(HF) = (\mathbf{F}H)(\mathbf{F}F), \qquad \mathbf{F}(KF) = (\mathbf{F}K)(\mathbf{F}F)$$
on the level of 1-cells. Furthermore, we demand the equalities
$$\mathbf{F}(\alpha F) = (\mathbf{F}\alpha)(\mathbf{F}F), \qquad \mathbf{F}(\beta F) = (\mathbf{F}\beta)(\mathbf{F}F)$$
on the level of 2-cells, and
$$\mathbf{F}(\tau F) = (\mathbf{F}\tau)(\mathbf{F}F)$$
on the level of 3-cells; these are the axioms given by "precomposition with F". Analogous axioms are given for postcomposition.

Gray-natural transformations A **Gray**-*natural transformation*

$$\begin{tikzcd} \mathbf{K} \arrow[r, bend left, "\mathbf{F}"] \arrow[r, bend right, "\mathbf{G}"'] \arrow[r, phantom, "\mathbf{t}\Downarrow"] & \mathbf{L} \end{tikzcd}$$

is a collection of 1-cells $\mathbf{t}_{\mathscr{A}} : \mathbf{F}\mathscr{A} \to \mathbf{G}\mathscr{A}$ indexed by objects of $\mathbf{K}$ such that given a 1-cell, 2-cell or 3-cell x in $\mathbf{K}$ with 0-cell domain $\mathscr{A}$ and 0-cell codomain $\mathscr{B}$, the equation

$$\begin{tikzcd} \mathbf{F}\mathscr{A} \arrow[rr, bend left, "\mathbf{F}H"] \arrow[rr, bend right, "\mathbf{F}K"'] \arrow[rr, phantom, "\overset{\mathbf{F}\tau}{\mathbf{F}\alpha \Downarrow \Rrightarrow \Downarrow \mathbf{F}\beta}"] & & \mathbf{F}\mathscr{B} \arrow[d, "\mathbf{t}_{\mathscr{B}}"] & = & \mathbf{F}\mathscr{A} \arrow[d, "\mathbf{t}_{\mathscr{A}}"] & & \\ & & \mathbf{G}\mathscr{B} & & \mathbf{G}\mathscr{A} \arrow[rr, bend left, "\mathbf{G}H"] \arrow[rr, bend right, "\mathbf{G}K"'] \arrow[rr, phantom, "\overset{\mathbf{G}\tau}{\mathbf{G}\alpha \Downarrow \Rrightarrow \Downarrow \mathbf{G}\beta}"] & & \mathbf{G}\mathscr{B} \end{tikzcd}$$

holds. With these definitions, we can form for every pair of **Gray**-categories $\mathbf{K}$ and $\mathbf{L}$ and **Gray**-functors $\mathbf{F}, \mathbf{G} : \mathbf{K} \to \mathbf{L}$ the hom-2-category $[\mathbf{K}, \mathbf{L}](\mathbf{F}, \mathbf{G})$.

Definition 7.1.6. Given **Gray**-categories $\mathbf{K}$ and $\mathbf{L}$ and **Gray**-functors $\mathbf{F}, \mathbf{G} : \mathbf{K} \to \mathbf{L}$ we define the hom-2-category $[\mathbf{K}, \mathbf{L}](\mathbf{F}, \mathbf{G})$ as follows:

1. The objects are the **Gray**-natural transformations from $\mathbf{F}$ to $\mathbf{G}$; i.e., an object is a collection $\mathbf{t}_{\mathscr{A}} : \mathbf{F}\mathscr{A} \to \mathbf{G}\mathscr{A}$ of 1-cells in $\mathbf{L}$ indexed by objects of $\mathbf{K}$.

2. The 1-cells $\alpha : \mathbf{s} \to \mathbf{t}$ consist of a collection of 2-cells

$$\alpha_{\mathscr{A}} : \mathbf{s}_{\mathscr{A}} \Rightarrow \mathbf{t}_{\mathscr{A}}$$

indexed by the objects of $\mathbf{K}$, such that for any 1-cell $f : \mathscr{A} \to \mathscr{B}$ in $\mathbf{K}$ the equality

$$\begin{tikzcd} \mathbf{F}\mathscr{A} \arrow[r, bend left, "\mathbf{s}_{\mathscr{A}}"] \arrow[r, bend right, "\mathbf{t}_{\mathscr{A}}"'] \arrow[r, phantom, "\alpha_A \Downarrow"] & \mathbf{G}\mathscr{A} \arrow[d, "\mathbf{G}f"] & = & \mathbf{F}\mathscr{A} \arrow[d, "\mathbf{F}f"] & \\ & \mathbf{G}\mathscr{B} & & \mathbf{F}\mathscr{B} \arrow[r, bend left, "\mathbf{s}_{\mathscr{B}}"] \arrow[r, bend right, "\mathbf{t}_{\mathscr{B}}"'] \arrow[r, phantom, "\alpha_B \Downarrow"] & \mathbf{G}\mathscr{B} \end{tikzcd}$$

holds. (We say that α is a *modification* between $\mathbf{s}$ and $\mathbf{t}$.)

3. The 2-cells $\rho : \alpha \Rightarrow \beta$ consist of a collection of 3-cells $\rho_{\mathscr{A}} : \alpha_{\mathscr{A}} \Rrightarrow \beta_{\mathscr{A}}$ indexed by the objects of $\mathbf{K}$ satisfying for each 1-cell $f : \mathscr{A} \to \mathscr{B}$ in $\mathbf{K}$ the equation

$$\begin{tikzcd} \mathbf{F}\mathscr{A} \arrow[r, bend left, "\mathbf{s}_{\mathscr{A}}"] \arrow[r, bend right, "\mathbf{t}_{\mathscr{A}}"'] \arrow[r, phantom, "\overset{\rho_{\mathscr{A}}}{\alpha_{\mathscr{A}} \Downarrow \Rrightarrow \Downarrow \beta_{\mathscr{A}}}"] & \mathbf{G}\mathscr{A} \arrow[d, "\mathbf{G}f"] & = & \mathbf{F}\mathscr{A} \arrow[d, "\mathbf{F}f"] & \\ & \mathbf{G}\mathscr{B} & & \mathbf{F}\mathscr{B} \arrow[r, bend left, "\mathbf{s}_{\mathscr{B}}"] \arrow[r, bend right, "\mathbf{t}_{\mathscr{B}}"'] \arrow[r, phantom, "\overset{\rho_{\mathscr{B}}}{\alpha_{\mathscr{B}} \Downarrow \Rrightarrow \Downarrow \beta_{\mathscr{B}}}"] & \mathbf{G}\mathscr{B} \end{tikzcd}$$

(We say that ρ is a *perturbation* between α and β.)

The units and composition are defined componentwise.

In fact, given the above definition of the hom-2-categories, $[\mathbf{K}, \mathbf{L}]$ forms a **Gray**-category itself.

Definition 7.1.7. The composition cubical functor

$$[\mathbf{K}, \mathbf{L}](\mathbf{G}, \mathbf{H}) \times [\mathbf{K}, \mathbf{L}](\mathbf{F}, \mathbf{G}) \to [\mathbf{K}, \mathbf{L}](\mathbf{F}, \mathbf{H})$$

is defined as follows:

1. The pair $(\mathbf{t}, \mathbf{s})$ is sent to the composite defined pointwise as $\mathbf{t}_{\mathscr{A}}\mathbf{s}_{\mathscr{A}} : \mathbf{F}\mathscr{A} \to \mathbf{G}\mathscr{A} \to \mathbf{H}\mathscr{A}$ for each object $\mathscr{A}$ of $\mathbf{K}$.
2. The pair (β, α) given by $\beta : \mathbf{t} \to \mathbf{t}'$ and $\alpha : \mathbf{s} \to \mathbf{s}'$ is sent to the collection of 2-cells

$$\begin{tikzcd}
\mathbf{t}_{\mathscr{A}}\mathbf{s}_{\mathscr{A}} \arrow[r, "\beta_{\mathscr{A}}\mathbf{s}_{\mathscr{A}}"] \arrow[d, "\mathbf{t}_{\mathscr{A}}\alpha_{\mathscr{A}}"'] \arrow[dr, phantom, "\swArrow \beta_{\mathscr{A}\,\alpha_{\mathscr{A}}}"] & \mathbf{t}'_{\mathscr{A}}\mathbf{s}_{\mathscr{A}} \arrow[d, "\mathbf{t}'_{\mathscr{A}}\alpha_{\mathscr{A}}"] \\
\mathbf{t}_{\mathscr{A}}\mathbf{s}'_{\mathscr{A}} \arrow[r, "\beta_{\mathscr{A}}\mathbf{s}'_{\mathscr{A}}"'] & \mathbf{t}'_{\mathscr{A}}\mathbf{s}'_{\mathscr{A}}
\end{tikzcd}$$

   indexed by the objects of $\mathbf{K}$.
3. The pair $(\mathbf{t}, \rho)$ with $\rho : \alpha \Rightarrow \alpha'$ gets mapped to the collection $\mathbf{t}_{\mathscr{A}}\rho_{\mathscr{A}}$ of 2-cells indexed by the objects of $\mathbf{K}$.
4. The pair $(\sigma, \mathbf{s})$ with $\sigma : \beta \Rightarrow \beta'$ gets mapped to the collection $\sigma_{\mathscr{A}}\mathbf{s}_{\mathscr{A}}$ of 2-cells indexed by the objects of $\mathbf{K}$.

We refer the interested reader to the book [38] for further details on the well-definedness of the **Gray**-category $[\mathbf{K}, \mathbf{L}]$.

Remark 7.1.8. All **Gray**-categories, together with all **Gray**-functors and all **Gray**-natural transformations form a 2-category that we will denote by **Gray**-CAT.

7.2 Equational presentations of algebras

To be able to define *presentations* of **Gray**-categories, we will need to cover a large amount of technical notions. In this section we introduce the basics of presentations in the ordinary setting before moving on to presentations of 2-categories and **Gray**-categories.

We will recall the notion of a presentation of an algebra. Building on the ideas of [51] we introduce the category of algebraic presentations and their morphisms. We will also describe the adjunction between a given category of algebras for a monad and the category of presentations for the monad.

Given an adjunction

$$F \dashv U : \mathscr{A} \to \mathscr{X}$$

of ordinary categories with unit η and counit ε, let us denote by (T, η, μ) its monad. We say that the adjunction $F \dashv U$ is of *descent type* if the canonical comparison functor $K : \mathscr{A} \to \mathsf{Alg}(T)$ is fully faithful.

Equivalently, $F \dashv U$ is of descent type if for every A in $\mathscr{A}$ the diagram

$$\begin{tikzcd} FUFUA \arrow[r, shift left, "FU\varepsilon_A"] \arrow[r, shift right, "\varepsilon_{FUA}"'] & FUA \arrow[r, "\varepsilon_A"] & A \end{tikzcd}$$

is a coequaliser in $\mathscr{A}$.

Assumption 7.2.1. In the rest of this section we shall work with an adjunction $F \dashv U : \mathscr{A} \to \mathscr{X}$ of descent type.

Definition 7.2.2. A *presentation* is a diagram

$$\begin{tikzcd} E \arrow[r, shift left, "l"] \arrow[r, shift right, "r"'] & UFX \end{tikzcd}$$

in $\mathscr{X}$.

Example 7.2.3. Let $\mathscr{A}$ be a finitary variety of one-sorted algebras. Then the adjunction $F \dashv U : \mathscr{A} \to \mathsf{Set}$ given by the free algebra functor and underlying set functor is monadic and the comparison functor $K : \mathscr{A} \to \mathsf{Alg}(T)$ is an equivalence. Therefore $F \dashv U$ is of descent type. A presentation $\begin{tikzcd} E \arrow[r, shift left, "l"] \arrow[r, shift right, "r"'] & UFX \end{tikzcd}$ then amounts to specifying an E-tuple of pairs $(l(e), r(e))$ (where $e \in E$), that is, an E-tuple of *equations in variables X*.

Example 7.2.4. The pair

$$\begin{tikzcd} FUFUA \arrow[r, shift left, "FU\varepsilon_A"] \arrow[r, shift right, "\varepsilon_{FUA}"'] & FUA \end{tikzcd}$$

can be transposed under $F \dashv U$ to the pair

$$\begin{tikzcd} UFUA \arrow[r, "\eta_{UFUA}"] \arrow[rr, bend right, "1_{UFUA}"'] & UFUFUA \arrow[r, "UFU\varepsilon_A"] & UFUA. \end{tikzcd}$$

We call this pair the *canonical presentation of A in $\mathscr{A}$*.

Definition 7.2.5. Given two presentations

$$\begin{tikzcd} E \arrow[r, shift left, "l"] \arrow[r, shift right, "r"'] & UFX \end{tikzcd} \qquad \begin{tikzcd} E' \arrow[r, shift left, "l'"] \arrow[r, shift right, "r'"'] & UFX' \end{tikzcd}$$

we say that $f : X \to UFX'$ is a *morphism of presentations* (l, r) *and* (l', r') if the following property holds: whenever $h : FX' \to A$ coequalises the transposed pair

$$\begin{tikzcd} FE' \arrow[r, shift left, "l'^{\sharp}"] \arrow[r, shift right, "r'^{\sharp}"'] & FX' \end{tikzcd}$$

then $h \cdot f^{\sharp} : FX \to FX' \to A$ coequalises the transposed pair

$$\begin{tikzcd} FE \arrow[r, shift left, "l^{\sharp}"] \arrow[r, shift right, "r^{\sharp}"'] & FX \end{tikzcd}$$

Observe now that we can define composition of morphisms of presentations: given morphisms $f : X \to UFX'$ and $g : X' \to UFX''$, their composite is the morphism

$$X \xrightarrow{\quad f \quad} UFX' \xrightarrow{\quad UFg \quad} UFUFX'' \xrightarrow{\ U\varepsilon FX'' \ } UFX''.$$

Such a composition is associative with identities of the form $\eta_X : X \to FX$. Presentations and their morphisms thus form a category Pres.

Proposition 7.2.6. *Given a category $\mathscr{A}$ with coequalisers and an adjunction $F \dashv U : \mathscr{A} \to \mathscr{X}$ of descent type, there is an adjunction*

$$B \dashv C : \mathscr{A} \to \mathsf{Pres}$$

between $\mathscr{A}$ and the category of presentations.

Proof. The functor C is defined on objects as follows:

$$\begin{tikzcd} CA = & UFUA \arrow[r, "\eta_{UFUA}"] \arrow[rr, bend right, "1_{UFUA}"'] & UFUFUA \arrow[r, "UFU\varepsilon_A"] & UFUA. \end{tikzcd}$$

That is, C maps A to its canonical presentation.

Given a morphism $f : A \to A'$, we claim that

$$Cf = \ UA \xrightarrow{\quad \eta_{UA} \quad} UFUA \xrightarrow{\quad UFUf \quad} UFUA'$$

is a morphism of presentations CA and CB. Since the transposes of the presentations CA and CB are

$$\begin{tikzcd} FUFUA \arrow[r, shift left, "FU\varepsilon_A"] \arrow[r, shift right, "\varepsilon_{FUA}"'] & FUA & FUFUA' \arrow[r, shift left, "FU\varepsilon_{A'}"] \arrow[r, shift right, "\varepsilon_{FUA'}"'] & FUA' \end{tikzcd}$$

respectively, and since the transpose of Cf is $FUf : FUA \to FUA'$, we may use that the diagram

$$\begin{tikzcd} FUFUA \arrow[r, shift left, "FU\varepsilon_A"] \arrow[r, shift right, "\varepsilon_{FUA}"'] \arrow[d, "FUFUf"'] & FUA \arrow[d, "FUf"] \\ FUFUA' \arrow[r, shift left, "FU\varepsilon_{A'}"] \arrow[r, shift right, "\varepsilon_{FUA'}"'] & FUA' \end{tikzcd}$$

commutes by naturality of ε: therefore, for any $h : FUA \to A'$ coequalising the pair

$$\begin{tikzcd} FUFUA' \arrow[r, shift left, "FU\varepsilon_{A'}"] \arrow[r, shift right, "\varepsilon_{FUA'}"'] & FUA' \end{tikzcd}$$

the morphism $h \cdot FUf : FUA \to FUA' \to A'$ coequalises the pair

$$\begin{tikzcd} FUFUA \arrow[r, shift left, "FU\varepsilon_A"] \arrow[r, shift right, "\varepsilon_{FUA}"'] & FUA. \end{tikzcd}$$

That C is a functor follows immediately.

The functor B is defined on objects as follows: the presentation

$$\begin{tikzcd} E \arrow[r, shift left, "l"] \arrow[r, shift right, "r"'] & UFX \end{tikzcd}$$

is sent to the coequaliser A as in the following diagram:

$$\begin{tikzcd} FE \arrow[r, shift left, "l^\sharp"] \arrow[r, shift right, "r^\sharp"'] & FX \arrow[r, "c"] & A \end{tikzcd}$$

Given another presentation

$$\begin{tikzcd} E' \arrow[r, shift left, "l'"] \arrow[r, shift right, "r'"'] & UFX' \end{tikzcd}$$

and a morphism of presentations $f : X \to UFX'$, the morphism $Bf : A \to A'$ is given by the universal property of coequalisers as seen in the diagram

$$\begin{tikzcd} FE \arrow[r, shift left, "l^\sharp"] \arrow[r, shift right, "r^\sharp"'] & FX \arrow[r, "c"] \arrow[d, "f^\sharp"] & A \arrow[d, dotted, "Bf"] \\ FE' \arrow[r, shift left, "l'^\sharp"] \arrow[r, shift right, "r'^\sharp"'] & FX' \arrow[r, "c'"'] & A' \end{tikzcd}$$

Since the morphisms are defined by a universal property, it follows that B is a functor.

We will now check that $B \dashv C$ holds. Observe that to give a morphism

$$h : B\left(\begin{tikzcd} E \arrow[r, shift left, "l"] \arrow[r, shift right, "r"'] & UFX \end{tikzcd} \right) \to A'$$

is to give a morphism $h : A \to A'$ if we denote by $c : FX \to A$ the coequaliser of the pair $(l^\sharp, r^\sharp)$, and by the coequaliser property, this amounts to giving a morphism $h{\cdot}c : FX \to A'$ that coequalises $\begin{tikzcd} FE \arrow[r, shift left, "l^\sharp"] \arrow[r, shift right, "r^\sharp"'] & FX \end{tikzcd}$. Then the transpose $(h{\cdot}c)^\flat : X \to UA'$ is a morphism from $\begin{tikzcd} E \arrow[r, shift left, "l"] \arrow[r, shift right, "r"'] & UFX \end{tikzcd}$ to the canonical presentation of A', as the following diagram

$$\begin{tikzcd} FE \arrow[r, shift left, "l^\sharp"] \arrow[r, shift right, "r^\sharp"'] & FX \arrow[r, "c"] \arrow[d, "F(h\cdot c)^\flat"] & A \arrow[d, "h"] \\ FUFUA' \arrow[r, shift left, "FU\varepsilon_{A'}"] \arrow[r, shift right, "\varepsilon_{FUA'}"'] & FUA' \arrow[r, "\varepsilon_A"'] & A' \end{tikzcd}$$

shows. This concludes the proof that $B \dashv C$ holds. ■

Example 7.2.7. There is an adjunction $F \dashv U$: Cat → Graph consisting of the functor U assigning to a category its underlying graph, and the functor F assigning to a graph the free category on that graph.

Giving a presentation

$$\begin{tikzcd} E \arrow[r, shift left, "l"] \arrow[r, shift right, "r"'] & UFX \end{tikzcd}$$

amounts to giving a free category UFX on a graph X (the graph of "generating arrows"), and to giving the pair (l, r) specifying the diagrams that are to commute in the resulting category C of the coequaliser

$$\begin{tikzcd} FE \arrow[r, shift left, "l^\sharp"] \arrow[r, shift right, "r^\sharp"'] & FX \arrow[r, "c"] & C. \end{tikzcd}$$

Example 7.2.8. Consider the graph X consisting of one node x and one arrow $a : x \to x$. The category FX consists of a node x and an arrow $a^n : x \to x$ for all natural numbers n such that $a^0 = 1_x$ and $a^i \cdot a^j = a^{i+j}$. If we take a graph E consisting of a node x and an arrow $e : x \to x$ and a presentation

$$\begin{tikzcd} E \arrow[r, shift left, "l"] \arrow[r, shift right, "r"'] & UFX \end{tikzcd}$$

given by identity on nodes and by the assignments

$$l(e) = a^1, \qquad r(e) = a^2,$$

the resulting category C given by the coequaliser

$$\begin{tikzcd} FE \arrow[r, shift left, "l^\sharp"] \arrow[r, shift right, "r^\sharp"'] & FX \arrow[r, "c"] & C \end{tikzcd}$$

is the "idempotent arrow" category consisting of a node x, the identity morphism on x, and one morphism $a : x \to x$ subject to the equation

$$a \cdot a = a.$$

7.3 Presentations of 2-categories and beyond

In Example 7.2.7 we have seen that ordinary categories can be presented quite easily using the monad arising from the category-graph adjunction $F \dashv U : \mathsf{Cat} \to \mathsf{Graph}$. Indeed, the only "equational data" that are needed to present a category are those that describe the equalities between morphisms, i.e., commutative diagrams. In this sense, the equational data are *1-dimensional*.

When presenting 2-categories, our aim is to mimic the approach for ordinary categories. Of course, every locally discrete 2-category $\mathscr{D}$ can be presented by a graph and the (generating) collection of diagrams commuting in $\mathscr{D}$. However, if we want to specify a 2-category $\mathscr{C}$ that has non-trivial 2-cells, we need to

1. specify a collection of "generating" 2-cells of $\mathscr{C}$,

2. specify a collection of equations between diagrams of 2-cells, i.e., tuples of "pasting diagrams" that are to be equal in $\mathscr{C}$; the pasting diagrams are generated by the generating 2-cells specified earlier.

In [76] Street introduces the notion of a *computad* which allows precisely such a presentation of 2-categories. That is, he defines computads to be graphs with a certain additional structure, such that there is an ordinary underlying functor

$$U : (\mathsf{2\text{-}Cat})_0 \to \mathsf{2\text{-}Comp}$$

to the category $\mathsf{2\text{-}Comp}$ of computads and their morphisms, and U is monadic. Denoting the left adjoint of U by F, this allows us to *present* every 2-category $\mathscr{C}$ as the coequaliser of (the transpose of) some pair

$$\begin{tikzcd} \mathcal{E} \arrow[r, shift left, "l"] \arrow[r, shift right, "r"'] & UF\mathcal{G}, \end{tikzcd}$$

i.e., as a presentation given by computads. We shall describe computads in a much more general context in Section 7.5: here we only sketch the structure of a computad that is necessary for presenting 2-categories, and show a simple example of using computads to present the "free adjunction" 2-category.

Example 7.3.1. To describe a computad, we are to give a graph

$$\begin{tikzcd} G = & G_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G_0 \end{tikzcd},$$

form a free category $\mathscr{G}$ on G and take its underlying graph (denote it by H):

$$\begin{tikzcd} H = & H_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & H_0 \end{tikzcd},$$

Now the elements of H_1 are the morphisms of $\mathscr{G}$, i.e., sequences of formally composable arrows in G. We need H to be able to specify 2-cells: let us say we want to specify a set K of generating 2-cells. Each 2-cell has to have a source and target 1-cell (morphism), so we are forced to specify a tuple

$$\begin{tikzcd} K \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & H_1 \end{tikzcd}.$$

Moreover, the source and target 1-cells have to have a common source and target themselves. That is, in the diagram

$$\begin{tikzcd} K \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & H_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & H_0 \end{tikzcd}$$

we need that the equations $s \cdot s = s \cdot t$ and $t \cdot s = t \cdot t$ hold.

Then the 2-category $\mathscr{C}$ generated by our computad has $\mathscr{G}$ as its underlying ordinary category, and the 2-cells in $\mathscr{C}$ are those generated by the ones specified in K via horizontal and vertical composition. In this sense both the 1-dimensional and 2-dimensional structure of $\mathscr{C}$ is freely generated by the specified data.

We shall postpone explaining the specification of 1-dimensional and 2-dimensional equalities to Section 7.5. We only remark here that the presentation technique using computads formalises and makes sound the following description of the "free adjunction" 2-category:

Example 7.3.2 (The free adjunction). There is a 2-category **adj** such that, given a 2-category $\mathscr{K}$, adjunctions in $\mathscr{K}$ correspond precisely to 2-functors $\mathbf{adj} \to \mathscr{K}$.

The 2-category **adj** is presented by specifying objects A and X, morphisms $F : X \to A$, $U : A \to X$, 2-cells $\eta : 1 \to UF$ and $\varepsilon : FU \to 1$ and by specifying that the composite 2-cells

$$\begin{tikzcd}
X \arrow[r, "F"] \arrow[dr, "1"'] \arrow[r, phantom, shift right=5, "\eta\,\Nearrow"] & A \arrow[d, "U"] \arrow[dr, "1"] & \\
 & X \arrow[r, "F"'] \arrow[r, phantom, shift left=5, "\varepsilon\,\Nearrow"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[d, "U"'] \arrow[dr, "1"] & \\
X \arrow[r, "F"] \arrow[dr, "1"'] \arrow[r, phantom, shift left=5, "\varepsilon\,\Nearrow"] \arrow[r, phantom, shift right=5, "\eta\,\Nearrow"] & X \arrow[d, "U"] \\
 & A
\end{tikzcd}$$

are equal to the identity 2-cells of F and U, respectively.

The above "definitorial" approach is very useful in cases we do not want (or do not need) to study explicitly the category that is being presented. Compare the above specification of **adj** to the concrete description:

Example 7.3.3 (Description of free adjunction [73]). The 2-category **badj** is defined as follows: the objects are finite ordinals $p = \{1, \dots, p-1\}$. For a pair of objects p, q, the morphisms $m : p \to q$ correspond to finite ordinals $p \leqslant m \leqslant q$, and a 2-cell $\theta : m \Rightarrow m'$ in $\mathbf{badj}(p, q)$ corresponds to an order-preserving function $\theta : m \to m'$ such that

$$\theta(i) = \begin{cases} i, \text{ for } 0 \leqslant i < p, \\ m' - m + i, \text{ for } m - q \leqslant i < m. \end{cases}$$

The composition functor $\mathbf{badj}(q, r) \times \mathbf{badj}(p, q) \to \mathbf{badj}(p, r)$ composes $m : p \to q$ and $n : q \to r$ to $m - q + n : p \to r$, and the 2-cells $\varphi : n \Rightarrow n'$ and $\theta : m \Rightarrow m'$ are composed to the 2-cell $\psi : m - q + n \Rightarrow m' - q + n'$ defined as

$$\psi(i) = \begin{cases} \theta(i), \text{ for } i < m, \\ \varphi(i - m + q) + m' - q, \text{ for } i \geqslant m - q. \end{cases}$$

The full sub-2-category of **badj** spanned by the objects 0 and 1 is denoted by **adj**. This 2-category is the free adjunction 2-category.

7.4 Globular operads and clubs

When presenting **Gray**-categories, we will need to postulate equations not only between 1-cells of those categories, but between higher-dimensional cells as well. We shall need to manipulate with higher-dimensional operations, that is, operations that have *diagrams*, instead of tuples, as their input and output. For this we will introduce the notion of an *operad*, or more concretely, a *globular operad* [63], via Kelly's notion of *abstract clubs* from [42].

Remark 7.4.1. Recall from Example 1.1.5 the definition of a graph as a diagram

$$\begin{tikzcd} G_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G_0 \end{tikzcd}$$

in Set. We will generalise this definition to give a notion of a higher-dimensional graph, called a *globular set*.

Definition 7.4.2. A *globular set* G is a diagram

$$\begin{tikzcd} \dots \arrow[r, shift left] \arrow[r, shift right] & G_4 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G_3 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G_2 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G_0 \end{tikzcd}$$

in Set, satisfying the equations $ss = st$ and $ts = tt$ for all parallel pairs (ss, st) and (ts, tt) in the above diagram.

Given a globular set G, we can think of G_n as of the set of n-cells (n-dimensional arrows) α having the $(n-1)$-cells $s(\alpha)$ and $t(\alpha)$ as a source and target, respectively.

Remark 7.4.3. Consider the category $\mathscr{Z}$ generated by

$$\begin{tikzcd} \dots \arrow[r, shift left] \arrow[r, shift right] & 4 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & 3 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & 2 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & 1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & 0 \end{tikzcd}$$

subject to the equations $ss = st$ and $ts = tt$ as above. Then the functor category $[\mathscr{Z}, \mathsf{Set}]$ has globular sets as objects, and homomorphisms of globular sets as arrows. We will denote this category by Glob.

We note that by Appendix F of [63] there is a monadic adjunction

$$\begin{tikzcd} \mathsf{Str}\text{-}\omega\text{-}\mathsf{Cat} \arrow[rr, bend right, "U_\omega"'] \arrow[rr, phantom, "\perp"] & & \mathsf{Glob} \arrow[ll, bend right, "F_\omega"'] \end{tikzcd}$$

between the category $\mathsf{Str}\text{-}\omega\text{-}\mathsf{Cat}$ of *strict ω-categories*[3] and *strict ω-functors* and the category of globular sets with the obvious forgetful functor. Denoting the monad of this adjunction by (S, η, μ), the globular set $S\mathbf{1}$ on the terminal globular set $\mathbf{1}$ will play a crucial role as a (globular) set of the *input shapes* of higher-dimensional operations.

In [42], Kelly studies the adjunction

$$\begin{tikzcd} \mathscr{K} \arrow[rr, bend right, "\Delta"'] \arrow[rr, phantom, "\perp"] & & {[\mathscr{K}, \mathscr{K}]} \arrow[ll, bend right, "ev_{\mathbf{1}}"'] \end{tikzcd}$$

where $\mathscr{K}$ is a category with finite limits (e.g., the category of globular sets), ev_1 is the evaluation functor that evaluates at the terminal object, and Δ is the functor sending X

---

[3] Very informally, strict ω-categories are the natural extension of the notions of a 2-category and a 3-category; an ω-category has objects (0-cells), 1-cells, 2-cells, $\dots$, and n-cells for every natural number n. These cells can compose in a strict way: composition is strictly associative, there are strict identities at every level of cells, and the "exchange laws" are strict at all levels of composition. See, e.g., [79].

to the functor ΔX constant at X. An easy inspection of the above adjunction shows that Δ is fully faithful and that $ev_{\mathbf{1}}$ preserves finite limits. Hence, in the terminology of [24], the category $\mathscr{K}$ is a *localisation* of $[\mathscr{K},\mathscr{K}]$ (see Definition 7.4.4). Thus there exists a *local* factorisation system $(\mathcal{E},\mathcal{M})$ on $[\mathscr{K},\mathscr{K}]$ (explained in Definition 7.4.5).

If, for a fixed object H of $[\mathscr{K},\mathscr{K}]$, we denote by $\mathcal{M}/H$ the full subcategory of $[\mathscr{K},\mathscr{K}]/H$ spanned by $X \to H$ in $\mathcal{M}$, there is a further adjunction

$$\begin{tikzcd} \mathcal{M}/H \arrow[r, bend right] \arrow[r, phantom, "\perp"] & {[\mathscr{K},\mathscr{K}]/H.} \arrow[l, bend right] \end{tikzcd}$$

In what follows, we shall prove that there is an equivalence

$$\mathcal{M}/H \simeq \mathscr{K}/H\mathbf{1}$$

of categories; this allows us to introduce the notion of a *club* (Definition 7.4.8): a club on $\mathscr{K}$ is a monad (S,η,μ) on $\mathscr{K}$ such that the monoidal structure $\otimes$, I on $[\mathscr{K},\mathscr{K}]/S$ defined by

$$\begin{tikzcd} & & & HK \arrow[d, "\alpha*\beta"] & & & \\ H \arrow[d, "\alpha"'] & \otimes & K \arrow[d, "\beta"] & = \quad SS \arrow[d, "\mu"] & , & I = & Id \arrow[d, "\eta"] \\ S & & S & S & & & S \end{tikzcd}$$

restricts to $\mathcal{M}/S$, or, equivalently, to $\mathscr{K}/S1$.

We shall recall some results of [42] to see that the free strict ω-category monad (S,η,μ) on Glob (from Remark 7.4.3) is a club. It will then follow that there exists a monoidal structure on $\mathsf{Glob}/S\mathbf{1}$ (where $S\mathbf{1}$ is the free strict ω-category on $\mathbf{1}$). Monoids in this monoidal structure will be precisely the *globular operads*. From the general theory of [42], it follows that there is a *monoidal* functor

$$\mathsf{Glob}/S\mathbf{1} \simeq \mathcal{M}/S \to [\mathsf{Glob},\mathsf{Glob}]/S \xrightarrow{dom} [\mathsf{Glob},\mathsf{Glob}]$$

or, equivalently, a *monoidal action*

$$@ : \mathsf{Glob}/S\mathbf{1} \times \mathsf{Glob} \to \mathsf{Glob} \tag{7.3}$$

Thus, for a monoid $(P \xrightarrow{p} S\mathbf{1}, i, m)$ in $\mathsf{Glob}/S\mathbf{1}$, the functor $X \mapsto p@X$ bears canonically the structure of a monad on Glob: the *associated monad of a globular operad*, admitting a *pullback description*. We shall make the above more precise in the following subsections.

Localisations and local factorisation systems

In this subsection we will recall the notion of a factorisation system arising from a localisation.

Definition 7.4.4 ([24])**.** A *localisation* of a finitely complete category $\mathscr{B}$ is an adjunction

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend right, "I"'] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, bend right, "R"']
\end{tikzcd}$$

where I is fully faithful and R preserves finite limits.

If we denote by $\rho : Id \to IR$ the unit of $R \dashv I$, then one can define a factorisation system $(\mathcal{E}, \mathcal{M})$ on $\mathscr{B}$ in the following way:

1. $e \in \mathcal{E}$ iff Re is an isomorphism,
2. $m \in \mathcal{M}$ iff

$$\begin{tikzcd}
X \arrow[r, "m"] \arrow[d, "\rho_X"'] & Y \arrow[d, "\rho_Y"] \\
IRX \arrow[r, "IRm"'] & IRY
\end{tikzcd}$$

is a pullback.

Indeed, to factorise $f : X \to Y$ in $\mathscr{B}$, consider the factorisation of the naturality square through a pullback

$$\begin{tikzcd}
X \arrow[rrd, bend left, "f"] \arrow[rd, dotted, "e"] \arrow[rdd, bend right, "\rho_X"'] & & \\
& P \arrow[r, "m"] \arrow[d, "p"'] & Y \arrow[d, "\rho_Y"] \\
& IRX \arrow[r, "IRf"'] & IRY
\end{tikzcd}$$

Since R preserves finite limits (hence pullbacks) it follows that Re is an isomorphism. Hence e is in $\mathcal{E}$. Moreover, the diagram

$$\begin{tikzcd}
P \arrow[r, "m"] \arrow[d, "\rho_P"'] & Y \arrow[d, "\rho_Y"] \\
IRP \arrow[r, "IRm"'] & IRY
\end{tikzcd}$$

is a pullback, since the diagram

$$\begin{tikzcd}
P \arrow[rr, "m"] \arrow[d, "p"'] & & Y \arrow[d, "\rho_Y"] \\
IRX \arrow[r, "IRe"'] & IRP \arrow[r, "IRm"'] & IRY
\end{tikzcd}$$

is a pullback and IRe is an isomorphism.

Definition 7.4.5. The factorisation system $(\mathcal{E}, \mathcal{M})$ on $\mathscr{B}$ is *local*, i.e., it satisfies the following two properties:

1. Whenever $g \cdot f \in \mathcal{E}$ and $g \in \mathcal{E}$, then $f \in \mathcal{E}$,
2. $\mathcal{E}$ is stable under pullbacks.

The results of [24] show that a local factorisation system on a finitely complete category $\mathscr{B}$ yelds a localisation on $\mathscr{B}$. In fact, the correspondence is bijective.

Notation 7.4.6. For a fixed C in $\mathscr{B}$, let us write $\mathcal{M}/C$ for the full subcategory of $\mathscr{B}/C$ spanned by morphisms $m : B \to C$ in $\mathcal{M}$.

Proposition 7.4.7 (see Paragraph 3.1 of [42]). *Suppose $R \dashv I : \mathscr{A} \to \mathscr{B}$ is a localisation of a finitely complete category $\mathscr{B}$ and denote by $(\mathcal{E}, \mathcal{M})$ the corresponding local factorisation system on $\mathscr{B}$. Then the following hold:*

1. *For any object C in $\mathscr{B}$, the category $\mathcal{M}/C$ is a full reflective subcategory of $\mathscr{B}/C$.*
2. *For any object C in $\mathscr{B}$, there is an equivalence*

$$\mathcal{M}/C \simeq \mathscr{A}/RC.$$

Proof. 1. Denote by

$$\begin{tikzcd} B \arrow[rr, "\eta_f"] \arrow[dr, "f"'] & & B^* \arrow[dl, "f^*"] \\ & C & \end{tikzcd}$$

the $(\mathcal{E}, \mathcal{M})$-factorisation of f. Then the $(\mathcal{E}, \mathcal{M})$-diagonalisation property shows that

$$\eta_f : f \to f^*$$

has the desired universal property of a reflection.

2. Define a functor $\mathcal{M}/C \to \mathscr{A}/RC$ by the assignment

$$\begin{tikzcd} B \arrow[d, "f"'] & \mapsto & RB \arrow[d, "Rf"] \\ C & & RC \end{tikzcd}$$

and a functor $\mathscr{A}/RC \to \mathcal{M}/C$ by the assignment

$$\begin{tikzcd} A \arrow[d, "g"'] & \mapsto & B \arrow[d, "m"] \\ RC & & C \end{tikzcd}$$

where the morphism m is defined as the morphism in the pullback

$$\begin{tikzcd}
B \arrow[r, "m"] \arrow[d] & C \arrow[d, "\rho_C"] \\
IA \arrow[r, "Ig"'] & IRC
\end{tikzcd}$$

Then, using that $R \dashv I$ is a localisation, one can see that the above pair of functors establishes an equivalence $\mathcal{M}/C \simeq \mathscr{A}/RC$.

■

Very often it is much easier to work with the category $\mathscr{A}/RC$ in place of the more involved category $\mathcal{M}/C$, as we will see in the following subsection.

Clubs on a finitely complete category

The results and notions of the previous subsection will now be applied to a localisation

$$\begin{tikzcd}
\mathscr{K} \arrow[r, bend right, "\Delta"'] \arrow[r, phantom, "\perp"] & {[\mathscr{K}, \mathscr{K}]} \arrow[l, bend right, "ev_{\mathbf{1}}"']
\end{tikzcd}$$

where $\mathscr{K}$ is finitely complete, Δ sends X to ΔX – the functor constant at X, and ev_1 evaluates a functor $H : \mathscr{K} \to \mathscr{K}$ at the terminal object $\mathbf{1}$ of $\mathscr{K}$.

Since Δ is fully faithful and since $ev_{\mathbf{1}}$ preserves finite limits, the adjunction $ev_{\mathbf{1}} \dashv \Delta$ is a localisation.

By the previous subsection there exists a local factorisation system $(\mathcal{E}, \mathcal{M})$ on $[\mathscr{K}, \mathscr{K}]$ that is described as follows:

1. The natural transformation $\alpha : H \to K$ is in $\mathcal{E}$ if and only if $\alpha_{\mathbf{1}} : H\mathbf{1} \to K\mathbf{1}$ is invertible.

2. The natural transformation $\alpha : H \to K$ is in $\mathcal{M}$ if and only if the diagram

$$\begin{tikzcd}
HX \arrow[r, "\alpha_X"] \arrow[d, "H!"'] & KX \arrow[d, "K!"] \\
H\mathbf{1} \arrow[r, "\alpha_{\mathbf{1}}"'] & K\mathbf{1}
\end{tikzcd}$$

is a pullback for every X, or, equivalently, if and only if

$$\begin{tikzcd}
HX \arrow[r, "\alpha_X"] \arrow[d, "Hf"'] & KX \arrow[d, "Kf"] \\
HY \arrow[r, "\alpha_Y"'] & KY
\end{tikzcd}$$

is a pullback for every $f : X \to Y$. Thus $\alpha : H \to K$ is in $\mathcal{M}$ if and only if α is a *cartesian* natural transformation.

The unit of $ev_{\mathbf{1}} \dashv \Delta$, having the component $\rho_H : H \to \Delta H\mathbf{1}$ for an endofunctor $H : \mathscr{K} \to \mathscr{K}$, is $H! : HX \to H\mathbf{1}$ for every X in $\mathscr{K}$.

The $(\mathcal{E}, \mathcal{M})$-factorisation of $\alpha : H \to K$ is given by the pullback

$$\begin{tikzcd}
H \arrow[rrd, bend left, "\alpha"] \arrow[rd, dotted, "e"] \arrow[rdd, bend right, "\rho_H"'] & & \\
& P \arrow[r, "m"] \arrow[d] & K \arrow[d, "\rho_K"] \\
& \Delta H\mathbf{1} \arrow[r, "\Delta\alpha_{\mathbf{1}}"'] & \Delta K\mathbf{1}
\end{tikzcd} \tag{7.4}$$

which means that we have for every X in $\mathscr{K}$ a diagram

$$\begin{tikzcd}
HX \arrow[rrd, bend left, "\alpha_X"] \arrow[rd, dotted, "e_X"] \arrow[rdd, bend right, "H!"'] & & \\
& PX \arrow[r, "m_X"] \arrow[d] & KX \arrow[d, "K!"] \\
& H\mathbf{1} \arrow[r, "\alpha_{\mathbf{1}}"'] & K\mathbf{1}.
\end{tikzcd}$$

Hence, by Proposition 7.4.7 we have for every $K : \mathscr{K} \to \mathscr{K}$ an adjunction

$$\begin{tikzcd}
\mathcal{M}/K \arrow[r, bend right] \arrow[r, phantom, "\perp"] & {[\mathscr{K}, \mathscr{K}]/K.} \arrow[l, bend right]
\end{tikzcd}$$

where the reflection of $\alpha : H \to K$ is given by the $(\mathcal{E}, \mathcal{M})$-factorisation

$$\begin{tikzcd}
H \arrow[rr, "\eta_\alpha"] \arrow[rd, "\alpha"'] & & P \arrow[ld, "\alpha^*"] \\
& K &
\end{tikzcd}$$

as in Diagram (7.4).

Definition 7.4.8 (see Paragraph 3.3 of [42]). Suppose $\mathscr{K}$ has finite limits and let (S, η^S, μ^S) be a monad on $\mathscr{K}$. Consider the monoidal structure $\otimes$, I on $[\mathscr{K}, \mathscr{K}]/S$ given by

$$\begin{tikzcd}
 & & & & HK \arrow[dd, "\alpha*\beta"] & & & \\
H \arrow[dd, "\alpha"'] & & K \arrow[dd, "\beta"] & & & & & Id \arrow[dd, "\eta^S"] \\
 & \otimes & & = & SS \arrow[dd, "\mu^S"] & , & I = & \\
S & & S & & & & & S \\
 & & & & S & & &
\end{tikzcd}$$

We say that (S, η^S, μ^S) is a *club* **S** if the above monoidal structure on $[\mathscr{K}, \mathscr{K}]/S$ restricts to $\mathcal{M}/S$.

We shall now show that *cartesian* monads are clubs; this is useful since the example of our interest is given by a cartesian monad.

Definition 7.4.9. Let $\mathscr{K}$ be a category with pullbacks. A monad (S, η, μ) is *cartesian* if S preserves pullbacks and both η and μ are cartesian, i.e., for each $f : A \to B$ in $\mathscr{K}$ the naturality squares

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] & SSA \arrow[r, "SSf"] \arrow[d, "\mu_A"'] & SSB \arrow[d, "\mu_B"] \\
SA \arrow[r, "Sf"'] & SB & SA \arrow[r, "Sf"'] & SB
\end{tikzcd}$$

are pullbacks.

Proposition 7.4.10 (see Proposition 3.1 of [42]). *For a monad (S, η^S, μ^S) on $\mathscr{K}$, the following are equivalent:*

1. *The monad (S, η^S, μ^S) is a club.*
2. *The natural transformations η^S and μ^S are cartesian natural transformations, and S preserves morphisms in $\mathcal{M}$. Equivalently, S preserves all pullbacks of the form*

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & SX \arrow[d, "S!"] \\
Y \arrow[r, "f"'] & S\mathbf{1}
\end{tikzcd}$$

In particular, every cartesian *monad (S, η^S, μ^S) on $\mathscr{K}$ is a club.*

Proof. 1 implies 2: Since $I = \eta^S : 1 \to S$, the natural transformation η^S has to be cartesian by the description of $(\mathcal{E}, \mathcal{M})$ on $[\mathscr{K}, \mathscr{K}]$. Since $1_S : S \to S$ is in $\mathcal{M}$, we need

$$\begin{tikzcd}
S \arrow[d, "1_S"'] & \otimes & S \arrow[d, "1_S"] & = & SS \arrow[d, "\mu^S"] \\
S & & S & & S
\end{tikzcd}$$

to be in $\mathcal{M}$. Hence μ^S has to be cartesian. Since

$$\begin{tikzcd}
 & & & & & & HK \arrow[dd, "\alpha K"] \\
 & & & & HK \arrow[dd, "\alpha * \beta"] & & \\
H \arrow[dd, "\alpha"'] & & K \arrow[dd, "\beta"] & & & & SK \arrow[dd, "S\beta"] \\
 & \otimes & & = & SS \arrow[dd, "\mu^S"] & = & \\
S & & S & & & & SS \arrow[dd, "\mu^S"] \\
 & & & & S & & \\
 & & & & & & S,
\end{tikzcd}$$

then it follows that if both α and β are in $\mathcal{M}$, also $\mu^S \cdot S\beta \cdot \alpha K$ is in $\mathcal{M}$. But we know that both μ^S and αK are in $\mathcal{M}$. It would therefore suffice that $S\beta$ be in $\mathcal{M}$. By taking $\alpha = 1_S$, this condition is also necessary. Therefore S has to preserve all pullbacks of the form

$$\begin{tikzcd}
KX \arrow[r, "\beta_X"] \arrow[d, "K!"'] & SX \arrow[d, "S!"] \\
K\mathbf{1} \arrow[r, "\beta_{\mathbf{1}}"'] & S\mathbf{1}.
\end{tikzcd}$$

Since, by Proposition 7.4.7, there is an equivalence $\mathcal{M}/S \simeq \mathscr{K}/S\mathbf{1}$, the above pullback-preservation condition is equivalent to the condition that S preserves all pullbacks of the form

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & SX \arrow[d, "S!"] \\
Y \arrow[r, "f"'] & S\mathbf{1}.
\end{tikzcd}$$

2 implies 1: This is trivial.

Finally, to assert that every cartesian monad is a club is trivial. ■

Example 7.4.11. The free monoid monad (T, η^T, μ^T) on Set is a cartesian monad, and consequently a club.

The category of collections for a club

In this subsection we fix a finitely complete category $\mathscr{K}$ and a club $\mathbf{S} = (S, \eta^S, \mu^S)$ on $\mathscr{K}$.

Notation 7.4.12. Let us define

$$\mathsf{Coll}(\mathbf{S}) = \mathscr{K}/S\mathbf{1}.$$

and call $\mathsf{Coll}(\mathbf{S})$ the *category of collections* for $\mathbf{S}$.

Roughly speaking, a collection $X \to S\mathbf{1}$ is an "abstract signature"; the composition of collections that we are about to introduce, is a "substitution of signatures". See Example 7.4.13.

Since $\mathbf{S}$ is a club, there is a monoidal functor

$$\mathscr{K}/S\mathbf{1} \simeq \mathcal{M}/S \to [\mathscr{K},\mathscr{K}]/S \xrightarrow{dom} [\mathscr{K}\mathscr{K}] \tag{7.5}$$

with respect to a monoidal structure $\circ$, i on $\mathscr{K}/S\mathbf{1}$, transported via the equivalence $\mathscr{K}/S\mathbf{1} \simeq \mathcal{M}/S$ to the (restriction of the) monoidal structure $\otimes$, I on $[\mathscr{K},\mathscr{K}]/S$.

We give an explicit description of $\circ$, i in this section and we also describe the *monoidal action*

$$@ : \mathsf{Coll}(\mathbf{S}) \times \mathscr{K} \to \mathscr{K}$$

resulting by uncurrying (7.5).

Since the equivalence of $\mathcal{M}/S$ and $\mathscr{K}/S\mathbf{1}$ is, by Proposition 7.4.7, given by the functor $\mathcal{M}/S \to \mathscr{K}/S\mathbf{1}$ acting by

$$\begin{tikzcd} H \arrow[d, "\alpha"'] & \mapsto & H\mathbf{1} \arrow[d, "\alpha_{\mathbf{1}}"] \\ S & & S\mathbf{1}, \end{tikzcd}$$

we define i to be the morphism $\eta^S_{\mathbf{1}} : \mathbf{1} \to S\mathbf{1}$. To define the composition

$$\begin{tikzcd} X \arrow[d, "f"'] & \circ & Y \arrow[d, "g"] \\ S\mathbf{1} & & S\mathbf{1} \end{tikzcd}$$

in $\mathscr{K}/S\mathbf{1}$, consider first the natural transformations $\alpha : H \to S$ and $\beta : K \to S$ that correspond to f and g via the equivalence $\mathscr{K}/S\mathbf{1} \to \mathcal{M}/S$. That is, we have the pullbacks

$$\begin{tikzcd} H \arrow[r, "\alpha"] \arrow[d] & S \arrow[d, "\rho_S"] & K \arrow[r, "\beta"] \arrow[d] & S \arrow[d, "\rho_S"] \\ \Delta X \arrow[r, "\Delta f"'] & \Delta S\mathbf{1} & \Delta Y \arrow[r, "\Delta g"'] & \Delta S\mathbf{1} \end{tikzcd}$$

and we evaluate the composite

$$HK \xrightarrow{\alpha * \beta} SS \xrightarrow{\mu^S} S$$

at the terminal object.

The above process can be understood better if we introduce the following notation: For every collection $f : X \to S\mathbf{1}$ and every Y in $\mathscr{K}$ let us define $(X,f)@Y$ to be the vertex of the pullback

$$\begin{tikzcd}
(X,f)@Y \arrow[r, "\partial_1^f Y"] \arrow[d, "\partial_0^f Y"'] & SY \arrow[d, "S!"] \\
X \arrow[r, "f"'] & S\mathbf{1}.
\end{tikzcd} \tag{7.6}$$

Then the pullback

$$\begin{tikzcd}
H \arrow[r, "\alpha"] \arrow[d] & S \arrow[d, "\rho_S"] \\
\Delta X \arrow[r, "\Delta f"'] & \Delta S\mathbf{1}
\end{tikzcd}$$

is given by the pullback in Diagram (7.6) for every Y. Hence $\alpha_Y : HY \to SY$ can be *defined* by putting $HY = (X,f)@Y$ and by putting $\alpha_Y = \partial_1^f Y$.

Clearly, the assignment $Y \mapsto HY$ extends to a functor by the universal property of pullbacks; for $g : Y \to Z$ we have

$$\begin{tikzcd}
 & (X,f)@Y \arrow[rr, "\partial_1^f Y"] \arrow[dl, "(X,f)@g"'] \arrow[dd, "\partial_0^f Y"] & & SY \arrow[dl, "Sg"'] \arrow[dd, "S!"] \\
(X,f)@Z \arrow[rr, "\partial_1^f Z"] \arrow[dd, "\partial_0^f Z"'] & & SZ \arrow[dd, "S!"] & \\
 & X \arrow[rr, "f"] \arrow[dl, equal] & & S\mathbf{1} \arrow[dl, equal] \\
X \arrow[rr, "f"'] & & S\mathbf{1} &
\end{tikzcd}$$

Therefore, for a fixed collection (X,f), we have defined a functor $(X,f)@- : \mathscr{K} \to \mathscr{K}$ that is the object assignment of the desired monoidal action

$$@ : \mathsf{Coll}(\mathbf{S}) \times \mathscr{K} \to \mathscr{K}.$$

Observe that for any collection (X,f) we have a morphism $f : (X,f) \to (S\mathbf{1}, 1_{S\mathbf{1}})$ in $\mathsf{Coll}(\mathbf{S})$, and that the equalities

$$(S\mathbf{1}, 1_{S\mathbf{1}})@Y = SY, \qquad (S\mathbf{1}, 1_{S\mathbf{1}})@g = Sg$$

hold for any Y and any $g : Y \to Z$. This allows us to define

$$f@- : (X,f)@- \to (S\mathbf{1}, 1_{S\mathbf{1}})@-$$

by rewriting the pullback (7.6) as

$$\begin{tikzcd}
(X,f)@Y \arrow[r, "f@Y"] \arrow[d, "(X{,}f)@!"'] & (S\mathbf{1},1_{S\mathbf{1}})@Y \arrow[d, "(S\mathbf{1}{,}1_{S\mathbf{1}})@!"] \\
(X,f)@\mathbf{1} \arrow[r, "f@\mathbf{1}"] \arrow[d, "\cong"'] & (S\mathbf{1},1_{S\mathbf{1}})@\mathbf{1} \arrow[d, "\cong"] \\
X \arrow[r, "f"'] & S\mathbf{1}
\end{tikzcd}$$

since $(S\mathbf{1},1_{S\mathbf{1}})@\mathbf{1} = S\mathbf{1}$ and $(X,f)@\mathbf{1} = X$.

Since $(S\mathbf{1},1_{S\mathbf{1}})$ is the terminal collection, the above diagram defines

$$h@- : (X,f)@- \to (Y,g)@-$$

for any morphism $h : (X,f) \to (Y,g)$ of collections by pasting pullbacks together:

$$\begin{tikzcd}
(X,f)@Z \arrow[r, dotted, "h@Z"] \arrow[d, "(X{,}f)@!"'] \arrow[rr, bend left, "f@Z"] & (Y,g)@Z \arrow[r, dotted, "g@Z"] \arrow[d, "(Y{,}g)@!"] & (S\mathbf{1},1_{S\mathbf{1}})@Z \arrow[d, "(S\mathbf{1}{,}1_{S\mathbf{1}})@!"] \\
(X,f)@\mathbf{1} \arrow[d, "\cong"'] & (Y,g)@\mathbf{1} \arrow[d, "\cong"] & (S\mathbf{1},1_{S\mathbf{1}})@\mathbf{1} \arrow[d, "\cong"] \\
X \arrow[r, "h"] \arrow[rr, bend right, "f"'] & Y \arrow[r, "g"] & S\mathbf{1}
\end{tikzcd}$$

Thus, we have (see Proposition 3.5 of [42])

$$\begin{tikzcd}
 & & & & (X,f)@Y \arrow[d, "f@g"] \\
X \arrow[d, "f"'] & \circ & Y \arrow[d, "g"] & = & (S\mathbf{1},1_{S\mathbf{1}})@S\mathbf{1} \arrow[d, "\mu^S_{\mathbf{1}}"] \\
S1 & & S1 & & S\mathbf{1}.
\end{tikzcd}$$

Example 7.4.13. Consider the club $\mathbf{T} = (T, \eta^T, \mu^T)$ on Set given by the free monoid monad (T, η^T, μ^T). The set $T1$ is (isomorphic to) the set of natural numbers $\mathbb{N}$ The category $\mathsf{Coll}(\mathbf{T})$ is the slice category $\mathsf{Set}/T1 = \mathsf{Set}/\mathbb{N}$ having morphisms $f : X \to \mathbb{N}$ as objects, i.e., families $(X_i)_{i\in\mathbb{N}}$ of sets. Given $f : X \to \mathbb{N}$ and $g : X \to \mathbb{N}$, their composition $f \circ g : (X, f)@Y \to T1$ is given by the family $(Z_i)_{i\in\mathbb{N}}$, where

$$Z_n = \coprod_{n_1+\cdots+n_i=n} X_i \times Y_{n_1} \times \cdots \times Y_{n_i}.$$

Example 7.4.14 (Globular operads). We can use the above theory in the case when $\mathscr{K} = \mathsf{Glob}$ is the category of globular sets (see Remark 7.4.3) and $\mathbf{S} = (S, \eta^S, \mu^S)$ is the strict-ω-category monad on Glob. Since $\mathbf{S}$ is cartesian (see Appendix F of [63]), it is a club on Glob by Proposition 7.4.10.

The category $\mathsf{Coll}(\mathbf{S})$ is $\mathsf{Glob}/S\mathbf{1}$ and monoids in $(\mathsf{Coll}(\mathbf{S}), \circ, i)$ are called *globular operads*. We will show examples of globular operads (and *computads*) in the following section.

7.5 Gray-computads via globular computads

In Section 2 of [76] Street introduces a *monadic* adjunction

$$F \dashv U : (\text{2-}\mathsf{Cat})_0 \to \text{2-}\mathsf{Comp}$$

between the *category* of 2-categories and 2-functors and the *category* of 2-computads.

As we hinted at in Section 7.3, one obtains an equational presentation of any 2-category $\mathscr{X}$, i.e., a coequaliser

$$\begin{tikzcd} F(\mathcal{E}) \arrow[r, shift left, "l"] \arrow[r, shift right, "r"'] & F(\mathcal{G}) \arrow[r, "c"] & \mathscr{X} \end{tikzcd}$$

for suitable 2-computads $\mathcal{G}$ (of "generators", or "basic operations") and $\mathcal{E}$ (of "equations"). We shall give a definition of 2-computads that is only formally different from the original Street's definition.

Assumption 7.5.1. Since we will work with various algebraic categories $\mathsf{Alg}(T)$ for monads on differing base categories $\mathscr{X}$, we will denote the algebraic categories $\mathsf{Alg}(T)$ as $\mathscr{X}^T$ in this section to stress the base category.

First we introduce truncated globular sets:

Definition 7.5.2. Let $\mathscr{Z}_n$ be the generated by

$$\begin{tikzcd} n \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \dots \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & 2 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & 1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & 0 \end{tikzcd}$$

subject to the equations $ss = st$ and $ts = tt$ where applicable. Then we denote by $\mathsf{Glob}_n = [\mathscr{Z}_n, \mathsf{Set}]$ the category of all *globular sets truncated at stage n.*

Let us denote by

$$F_1 \dashv U_1 : \mathsf{Cat} \to \mathsf{Graph}$$

the monadic adjunction between the category of all categories and all functors and the category of all graphs and all graph homomorphisms.

Observe now that $\mathsf{Graph} = \mathsf{Glob}_1$: graphs are globular sets truncated at stage 1. Similarly, sets are just globular sets truncated at stage 0 (i.e., $\mathsf{Glob}_0 = \mathsf{Set}$). Then we have an another adjunction

$$\begin{tikzcd}
\mathsf{Glob}_0 \arrow[r, bend left, "L_1"] \arrow[r, phantom, "\perp"] & \mathsf{Glob}_1 \arrow[l, bend left, "R_1"]
\end{tikzcd}$$

with

$$L_1 : X \mapsto (\ \varnothing \overset{s}{\underset{t}{\rightrightarrows}} X\ )$$

$$R_1 : (\ G_1 \overset{s}{\underset{t}{\rightrightarrows}} G_0\ ) \mapsto G_0$$

(thus L_1 being the "free discrete graph" functor and R_1 being the "underlying set of vertices" functor).

Let (M_1, η_1, μ_1) be the monad of $F_1 \dashv U_1$ on Glob_1 and consider the monad (M_0, η_0, μ_0) on Glob_0 that arises from (M_1, η_1, μ_1) by *transport* along R_1. That is, M_0 is defined as the dotted composite in

$$\begin{tikzcd}
\mathsf{Glob}_0 \arrow[r, dotted, "M_0"] \arrow[d, "L_1"'] & \mathsf{Glob}_0 \\
\mathsf{Glob}_1 \arrow[r, "M_1"'] & \mathsf{Glob}_1 \arrow[u, "R_1"']
\end{tikzcd}$$

the unit $\eta_0 : Id \Rightarrow M_0$ is defined as the composite

$$\begin{tikzcd}
\mathsf{Glob}_0 \arrow[r, "L_1"] & \mathsf{Glob}_1 \arrow[r, bend left, "M_1"] \arrow[r, bend right, "Id"'] \arrow[r, phantom, "\Uparrow \eta_1"] & \mathsf{Glob}_1 \arrow[r, "R_1"] & \mathsf{Glob}_0
\end{tikzcd}$$

and the multiplication $\mu_0 : M_0 \cdot M_0 \Rightarrow M_0$ is defined as the composite

$$\begin{tikzcd}
\mathsf{Glob}_0 \arrow[r, "L_1"] & \mathsf{Glob}_1 \arrow[r, bend left, "M_1"] \arrow[r, bend right, "M_1 \cdot M_1"'] \arrow[r, phantom, "\Uparrow \mu_1"] & \mathsf{Glob}_1 \arrow[r, "R_1"] & \mathsf{Glob}_0
\end{tikzcd}$$

In this case (M_0, η_0, μ_0) is the identity monad on Glob_0. There is a square

$$\begin{tikzcd}
\mathsf{Glob}_0^{M_0} \arrow[d, "U_0"'] & \mathsf{Glob}_1^{M_1} \arrow[l, "R_1^\sharp"'] \arrow[d, "U_1"] \\
\mathsf{Glob}_0 & \mathsf{Glob}_1 \arrow[l, "R_1"]
\end{tikzcd}$$

where R_1 sends the M_1-algebra $M_1 X \xrightarrow{a} X$ to the composite

$$R_1 M_1 L_1 R_1 X \xrightarrow{R_1 M_1 \varepsilon_1 X} R_1 M_1 X \xrightarrow{R_1 a} R_1 X$$

(where ε_1 is the counit of $L_1 \dashv R_1$).

Observe now that the morphism

$$R_1 M_1 \varepsilon_1 : R_1 M_1 L_1 R_1 \Rightarrow R_1 M_1$$

is an isomorphism, and denote it by $\beta_1 : M_0 R_1 \Rightarrow R_1 M_1$ (compare to Definition 3.1 in [11]).

We can now iterate the above construction "one dimension higher", i.e., we start with an adjunction

$$F_2 \dashv U_2 : (\mathsf{2\text{-}Cat})_0 \to \mathsf{Glob}_2$$

where Glob_2 is the category of globular sets truncated at stage 2: the generic object of Glob_2 being a diagram

$$\begin{tikzcd} G_2 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G_0 \end{tikzcd}$$

in Set satisfying the globularity conditions.

For a 2-category $\mathscr{X}$, the object $U_2\mathscr{X}$ is defined in an obvious way:

1. $(U_2\mathscr{X})_0$ is the set of objects of $\mathscr{X}$,
2. $(U_2\mathscr{X})_1$ is the set of 1-cells of $\mathscr{X}$,
3. $(U_2\mathscr{X})_2$ is the set of 2-cells of $\mathscr{X}$.

The maps s and t are the domain and codomain maps; the globularity equations are satisfied.

The free 2-category $F_2 G$ on a 2-globular set is constructed in the same way as a 2-category is constructed out of a category-enriched graph.

This adjunction gives rise to a monad (M_2, η_2, μ_2) on Glob_2. Since there is an adjunction

$$L_2 \dashv R_2 : \mathsf{Glob}_2 \to \mathsf{Glob}_1$$

we can transport M_2 along R_2: the resulting transported monad is M_1, since

$$R_2 M_2 L_2 \Big(\begin{tikzcd} G_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G_0 \end{tikzcd} \Big)$$

is the graph of the free category on $\begin{tikzcd} G_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G_0 \end{tikzcd}$.

As in the case of $\beta_1 : M_0 R_1 \Rightarrow R_1 M_1$, observe that $\beta_2 : M_1 R_2 \Rightarrow R_2 M_2$, defined as the morphism $R_2 M_2 \varepsilon_2 : R_2 M_2 L_2 R_2 \Rightarrow R_2 M_2$, is again an isomorphism.

We therefore have a chain (M_2, M_1, M_0) of finitary monads on Glob_2, Glob_1 and Glob_0, respectively. Following the development in [11], we can now define categories

$$\mathsf{Comp}(M_0), \qquad \mathsf{Comp}(M_1), \qquad \mathsf{Comp}(M_2)$$

of computads together with *monadic* adjunctions

$$\begin{tikzcd} \mathsf{Glob}_0^{M_0} \arrow[d, bend right, "\mathcal{U}_0"'] \arrow[d, phantom, "\vdash"] & \mathsf{Glob}_1^{M_1} \arrow[d, bend right, "\mathcal{U}_1"'] \arrow[d, phantom, "\vdash"] & \mathsf{Glob}_2^{M_2} \arrow[d, bend right, "\mathcal{U}_2"'] \arrow[d, phantom, "\vdash"] \\ \mathsf{Comp}(M_0) \arrow[u, bend right, "\mathcal{F}_0"'] & \mathsf{Comp}(M_1) \arrow[u, bend right, "\mathcal{F}_1"'] & \mathsf{Comp}(M_2) \arrow[u, bend right, "\mathcal{F}_2"'] \end{tikzcd}$$

The monadicity of the above adjunctions follows from Theorem 5.1 in [11], since M_2 is a cartesian monad on Glob_2 and since both β_2 and β_1 are isomorphisms (i.e., M_2 is *truncable*, see Definition 3.1 of [11]).

We shall now describe the above mentioned categories and adjunctions and show how they give rise to **Gray**-computads.

We start at stage 0 and proceed up to 2.

1. The category $\mathsf{Comp}(M_0)$ is equivalent to Glob_0, i.e., it is just the category Set. The underlying functor $\mathcal{U}_0 : \mathsf{Glob}_0^{M_0} \to \mathsf{Glob}_0$ is the underlying functor $U_0 : \mathsf{Glob}_0^{M_0} \to \mathsf{Glob}_0$ (i.e., the identity). Therefore $\mathcal{F}_0 = F_0$ is the identity as well.

2. An M_1-computad is a tuple (G, φ, X) consisting of an object G in Glob_1 (i.e., G being a graph $\begin{tikzcd} G_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G_0 \end{tikzcd}$), the object X in $\mathsf{Comp}(M_0)$ (thus being a set), and
$$\varphi : U_0\mathcal{F}_0X \to R_1G$$
being an isomorphism in Glob_0: φ is a bijection between X and G_0, stating that the set of vertices of G is X.

   A morphism from (G, φ, X) to (G', φ', X') is a pair (f, Φ) where $f : G \to G'$ is a morphism in Glob_1 (a graph homomorphism), and $\Phi : X \to X'$ is a morphism of M_0-computads (a mapping) such that the square
$$\begin{tikzcd} U_0\mathcal{F}_0X \arrow[r, "U_0\mathcal{F}_0\Phi"] \arrow[d, "\varphi"'] & U_0\mathcal{F}_0X' \arrow[d, "\varphi'"] \\ R_1G \arrow[r, "R_1f"'] & R_1G' \end{tikzcd}$$
commutes.

3. An M_2-computad is a tuple (G, φ, X) where G is an object in Glob_2, X is an M_1-computad and φ is an isomorphism
$$\varphi : U_1\mathcal{F}_1X \to R_2G$$
in Glob_1. The morphisms in $\mathsf{Comp}(M_2)$ are defined analogously to the definition of morphisms in $\mathsf{Comp}(M_1)$.

   In order to define $\mathsf{Comp}(M_2)$ we need to have a description of the free functor $\mathcal{F}_1 : \mathsf{Comp}(M_1) \to \mathsf{Glob}_1^{M_1}$. By definition, $\mathsf{Glob}_1 \cong \mathsf{Comp}(M_1)$, and the category $\mathsf{Glob}_1^{M_1}$ is the category Cat. Thus $\mathcal{F}_1$ is the functor that constructs from a graph a free category on that graph.

   Therefore, an M_2-computad is a 2-globular set
$$\begin{tikzcd} G_2 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G_0 \end{tikzcd}$$
such that
$$\begin{tikzcd} G_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G_0 \end{tikzcd}$$

is the underlying graph of a free category on a graph

$$\begin{tikzcd} X_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & X_0. \end{tikzcd}$$

This means precisely that elements of G_1 are words of elements of X_1 and elements of G_2 can therefore be drawn as "2-cells"

$$\begin{tikzcd}
 & * \arrow[r] & * & \cdots & * \arrow[r] & * \arrow[dr] & \\
* \arrow[ur] \arrow[dr] & & & \Downarrow & & & * \\
 & * \arrow[r] & * & \cdots & * \arrow[r] & * \arrow[ur] &
\end{tikzcd}$$

Of course, G_0 is (isomorphic to) X_0. Thus the category $\mathsf{Comp}(M_2)$ is isomorphic to the category $2\text{-}\mathsf{Comp}$ of [76].

We need to establish the adjunction $\mathcal{F}_2 \dashv \mathcal{U}_2 : (2\text{-}\mathsf{Cat})_0 \to \mathsf{Comp}(M_2)$. This can be done using adjunctions $\mathcal{F}_0 \dashv \mathcal{U}_0$, $\mathcal{F}_1 \dashv \mathcal{U}_1$, and using restriction functors

$$\begin{tikzcd}
\mathsf{Glob}_0^{M_0} & \mathsf{Glob}_1^{M_1} \arrow[l, "R_1^\sharp"'] & \mathsf{Glob}_2^{M_2} \arrow[l, "R_2^\sharp"'] \\
\mathsf{Glob}_0 & \mathsf{Glob}_1 \arrow[l, "R_1"] & \mathsf{Glob}_2. \arrow[l, "R_2"]
\end{tikzcd}$$

Our construction says that $R^\sharp \mathscr{X}$ is the underlying 1-category of the 2-category $\mathscr{X}$. We define $\mathcal{U}_2 X = (G, \varphi, X)$ as follows: First observe that the counit of $\mathcal{F}_1 \dashv \mathcal{U}_1$ yields

$$r_1 : \mathcal{F}_1 \mathcal{U}_1 R_2^\sharp \mathscr{X} \to R_2^\sharp \mathscr{X}$$

and we define the M_1-computad X by $X = \mathcal{U}_1 R_2^\sharp \mathscr{X}$. That is, X is the graph

$$\begin{tikzcd} X_1 \arrow[r, shift left, "dom"] \arrow[r, shift right, "cod"'] & X_0 \end{tikzcd}$$

with X_1 being the set of 1-cells of $\mathscr{X}$ and X_0 being the set of objects of $\mathscr{X}$. We now want to define G in Glob_2 such that

$$\begin{tikzcd} G_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G_0 \end{tikzcd} = U_1 \mathcal{F}_1 \mathcal{U}_1 R_2^\sharp \mathscr{X}$$

and we need to define

$$\begin{tikzcd} G_2 \arrow[r, shift left, "dom"] \arrow[r, shift right, "cod"'] & G_1. \end{tikzcd}$$

Since G_1 consists of words of 1-cells of $\mathscr{X}$, we define G_2 to consist of 2-cells

$$\begin{tikzcd}
 & * \arrow[r] & * & \cdots & * \arrow[r] & * \arrow[dr] & \\
* \arrow[ur] \arrow[dr] & & & \Downarrow & & & * \\
 & * \arrow[r] & * & \cdots & * \arrow[r] & * \arrow[ur] &
\end{tikzcd}$$

with the obvious source and target maps. The above can be expressed as follows: we put

$$\begin{tikzcd}
G_1 \arrow[r, shift left, "dom"] \arrow[r, shift right, "cod"'] & G_0 \; = U_1\mathcal{F}_1\mathcal{U}_1R_2^{\sharp}\mathscr{X}
\end{tikzcd}$$

and we consider $\mathscr{X}$ as an algebra for M_2 on a 2-globular set Z:

$$\mathscr{X} = \begin{tikzcd}
M_2Z \arrow[d, "a"] \\
Z.
\end{tikzcd}$$

In particular, there is a set mapping $a_1 : (M_2Z)_1 \to Z_1$ and we form the pullback

$$\begin{tikzcd}
G_2 \arrow[r, "\langle s{,}t\rangle"] \arrow[d] & (M_2Z)_1 \times (M_2Z)_1 \arrow[d, "a_1\times a_1"] \\
Z_2 \arrow[r, "\langle dom{,}cod\rangle"'] & Z_1 \times Z_1
\end{tikzcd}$$

This yields the correct result since $U_1R_2^{\sharp}\mathscr{X} = R_2U_2\mathscr{X} = \; Z_1 \overset{s}{\underset{t}{\rightrightarrows}} Z_0$, and hence $(M_2Z)_1 = G_1$.

We put

$$\mathcal{U}_2\mathscr{X} = (\; G_2 \overset{s}{\underset{t}{\rightrightarrows}} G_1 \overset{s}{\underset{t}{\rightrightarrows}} G_0 \; , id, \mathcal{U}_1R_2^{\sharp}\mathscr{X});$$

and this is precisely how the forgetful functor $(\mathsf{2\text{-}Cat})_0 \to \mathsf{2\text{-}Comp}$ is defined in [76].

To define $\mathcal{F}_2 : \mathsf{Comp}(M_2) \to \mathsf{Glob}_2^{M_2}$, we first define

$$\begin{tikzcd}
X \arrow[d, mapsto] & \mathsf{Comp}(M_0) \arrow[d, "V_0"] & (G{,}\varphi{,}X) \arrow[d, mapsto] & \mathsf{Comp}(M_1) \arrow[d, "V_1"] & (G{,}\varphi{,}X) \arrow[d, mapsto] & \mathsf{Comp}(M_2) \arrow[d, "V_2"] \\
X & \mathsf{Glob}_0 & G & \mathsf{Glob}_1 & G & \mathsf{Glob}_2
\end{tikzcd}$$

and truncation functors

$$\begin{tikzcd}
\mathsf{Comp}(M_0) & \mathsf{Comp}(M_1) \arrow[l, hook', "tr_1"'] & \mathsf{Comp}(M_2) \arrow[l, hook', "tr_2"'] \\
 & X & (G,\varphi,X) \arrow[l, mapsto] \\
X & (G,\varphi,X) \arrow[l, mapsto] &
\end{tikzcd}$$

Further, we define natural transformations

$$\begin{tikzcd}
\mathsf{Glob}_0^{M_0} \arrow[r, "\mathcal{U}_0"] \arrow[dr, "U_0"'] & \mathsf{Comp}(M_0) \arrow[d, "V_0"] \arrow[dl, phantom, "\Swarrow \Theta_0"] \\
 & \mathsf{Glob}_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Glob}_1^{M_1} \arrow[r, "\mathcal{U}_1"] \arrow[dr, "U_1"'] & \mathsf{Comp}(M_1) \arrow[d, "V_1"] \arrow[dl, phantom, "\Swarrow \Theta_1"] \\
 & \mathsf{Glob}_1
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{Glob}_2^{M_2} \arrow[r, "\mathcal{U}_2"] \arrow[dr, "U_2"'] & \mathsf{Comp}(M_2) \arrow[d, "V_2"] \arrow[dl, phantom, "\Swarrow \Theta_2"] \\
 & \mathsf{Glob}_2
\end{tikzcd}$$

and we use them to construct coequalisers

$$\begin{tikzcd}
\dots \arrow[r, shift left] \arrow[r, shift right] & F_iV_i \arrow[r] & \mathcal{F}_i
\end{tikzcd} \qquad i = 0,1,2$$

The only interesting case is $i = 2$, since both V_1 and V_2 are identities.

To define the natural transformation $\Theta_2(\mathscr{X}) : V_2\mathcal{U}_2\mathscr{X} \to U_2\mathscr{X}$ at a 2-category $\mathscr{X}$, we express $\mathscr{X}$ as an M_2-algebra $a : M_2Z \to Z$. Then we get that

$$V_2\mathcal{U}_2\mathscr{X} = V_2(\; G_2 \rightrightarrows G_1 \rightrightarrows G_0 \;, id, \mathcal{U}_1R_2^\sharp) = \; G_2 \rightrightarrows G_1 \rightrightarrows G_0$$

and

$$U_2\mathscr{X} = U_2(M_2Z \xrightarrow{a} Z) = \; Z_2 \rightrightarrows Z_1 \rightrightarrows Z_0 \;,$$

and we define $\Theta_2(\mathscr{X})$ to be the triple of horizontal morphisms in

$$\begin{tikzcd}
G_2 \arrow[r, dotted] \arrow[d] & Z_2 \arrow[d, shift left] \arrow[d, shift right] \\
G_1 \arrow[r, "\rho_1"] \arrow[d, shift left] \arrow[d, shift right] & Z_1 \arrow[d, shift left] \arrow[d, shift right] \\
G_0 \arrow[r, "\rho_0"] & Z_0
\end{tikzcd}$$

where $(\rho_1, \rho_0) = U_1r_1$ and the dotted morphism is the pullback projection.

Define the morphism Ξ_1 by means of the following bijections

$$\begin{array}{c} V_1\mathcal{U}_1 \xrightarrow{\Theta_1} U_1 \\ \hline V_1 \to U_1\mathcal{F}_1 \\ \hline F_1V_1 \xrightarrow{\Xi_1} \mathcal{F}_1 \end{array}$$

and define Ψ as the dotted arrow in the following commutative diagram:

$$\begin{tikzcd}
M_1M_1V_1tr_1 \arrow[r, dotted, "\Psi"] \arrow[d, "M_1U_1\Xi_1tr_1"'] & R_2M_2V_2 \\
M_1U_1\mathcal{F}_1tr_1 \arrow[r, "M_1\varphi"'] & R_2M_2L_2R_2 \arrow[u, "R_2M_2"']
\end{tikzcd}$$

Then we can take the (non-commutative) pair of composites

$$\begin{tikzcd}
M_1V_1tr_1 \arrow[r, shift left, "\eta_1M_1V_1tr_1"] \arrow[r, shift right, "M_1\eta_1V_1tr_1"'] & M_1M_1V_1tr_1 \arrow[r, "\Psi"] & R_2M_2V_2
\end{tikzcd}$$

that mates under $L_2 \dashv R_2$ with a pair

$$\begin{tikzcd}
L_2M_1V_1tr_1 \arrow[r, shift left] \arrow[r, shift right] & M_2V_2
\end{tikzcd}$$

and that, in turn, mates under $F_2 \dashv U_2$ with a pair

$$\begin{tikzcd}
F_2L_2M_1V_1tr_1 \arrow[r, shift left] \arrow[r, shift right] & F_2V_2
\end{tikzcd}$$

whose coequaliser is $\mathcal{F}_2$.

We have thus recovered Street's notion of a computad from [76] in a more general setting.

7.5.1 Globular computads

We can generalise the above two-step procedure by induction to an n-step procedure:

1. Consider the obvious chain of adjunctions

$$\begin{tikzcd}
\mathsf{Glob}_0 \arrow[r, bend left, "L_1"] \arrow[r, phantom, "\perp"] & \mathsf{Glob}_1 \arrow[l, bend left, "R_1"] \arrow[r, bend left, "L_2"] \arrow[r, phantom, "\perp"] & \cdots \arrow[l, bend left, "R_2"] & \cdots \arrow[r, bend left, "L_n"] \arrow[r, phantom, "\perp"] & \mathsf{Glob}_n \arrow[l, bend left, "R_n"]
\end{tikzcd}$$

   and a finitary monad (M_n, η_n, μ_n) on Glob_n for a fixed n.

2. Transport M_n along the above adjunctions to obtain finitary monads (M_k, η_k, μ_k) on Glob_k for all $0 \leqslant k \leqslant n$. Furthermore, consider adjunctions $F_k \dashv U_k : \mathsf{Glob}_k^{M_k} \to \mathsf{Glob}_k$ and lift the restriction functors

$$\begin{tikzcd}
\mathsf{Glob}_{k-1}^{M_{k-1}} \arrow[d, "U_{k-1}"'] & \mathsf{Glob}_k^{M_k} \arrow[l, dotted, "R_k^\sharp"'] \arrow[d, "U_k"] \\
\mathsf{Glob}_{k-1} & \mathsf{Glob}_k \arrow[l, "R_k"]
\end{tikzcd}$$

for all $1 \leqslant k \leqslant n$.

3. Define M_k-computads for $0 \leqslant k \leqslant n$ inductively:
   (a) For $k = 0$:
$$\mathcal{U}_0 = U_0 : \mathsf{Glob}_0^{M_0} \to \mathsf{Comp}(M_0) = \mathsf{Glob}_0$$
$$\mathcal{F}_1 = F_1 : \mathsf{Comp}(M_0) \to \mathsf{Glob}_0^{M_0}$$
   (b) For $k > 0$: An $(k+1)$-computad is a tuple (G, φ, X) where G is in Glob_{k+1} and X is an M_k-computad, and φ is an isomorphism
$$\varphi : U_k \mathcal{F}_k X \to R_{k+1} G$$
   in Glob_k. Morphisms from (G, φ, X) to (G', φ', X') are pairs (f, Φ), where $f : G \to G'$ is in Glob_{k+1}, $\Phi : X \to X'$ is in $\mathsf{Comp}(M_k)$, and the following square
$$\begin{tikzcd} U_k\mathcal{F}_kX \arrow[r, "U_k\mathcal{F}_k\Phi"] \arrow[d, "\varphi"'] & U_k\mathcal{F}_kX' \arrow[d, "\varphi'"] \\ R_{k+1}G \arrow[r, "R_{k+1}f"'] & R_{k+1}G' \end{tikzcd}$$
   commutes in Glob_k.

   The adjunctions $\mathcal{F}_k \dashv \mathcal{U}_k$ are defined by induction in the same way as in the previous section.

In [11], Batanin calls M_n *truncable at* $k-1$ if the canonical morphism $\beta_k : M_{k-1}R_k \to R_kM_k$ is an isomorphism. And M_n is called *truncable* if it is truncable at every $k < n$.

The main result of [11] then reads as follows.

Theorem 7.5.3 (Theorem 5.1 of [11]). *If M_n is a truncable cartesian monad on Glob_n, then the adjunction*
$$\mathcal{F}_m \dashv \mathcal{U}_n : \mathsf{Glob}_n^{M_n} \to \mathsf{Glob}_n$$
is monadic.

Now we can almost describe the **Gray**-computads. To capture the 3-dimensional structure of **Gray**-categories, we first introduce the operads for *sesquicategories* and for **Gray**-categories.

7.5.2 The operad for sesquicategories

Recall from Example 9.3.4 of [63] that a category $\mathscr{X}$ is called a *sesquicategory*, provided it comes equipped with a functor HOM : $\mathscr{X}^{op} \times \mathscr{X} \to \mathsf{Cat}$ such that the triangle
$$\begin{tikzcd} \mathscr{X}^{op} \times \mathscr{X} \arrow[r, "\mathrm{HOM}"] \arrow[dr, "\mathscr{X}(-,=)"'] & \mathsf{Cat} \arrow[d, "\mathrm{ob}"] \\ & \mathsf{Set} \end{tikzcd}$$

commutes. The objects and arrows of every $\mathscr{X}(X, X')$ are called *1-cells* and *2-cells*. More prosaically sesquicategories are “2-categories without the middle-four interchange law”.

Let us now take $\mathbf{S} : \mathsf{Glob}_2 \to \mathsf{Glob}_2$ to be the monad of strict 2-categories. Since $\mathbf{S} = (S, \eta^S, \mu^S)$ is cartesian, one can form the category

$$\mathsf{Coll}(\mathbf{S}) = \mathsf{Glob}_2/S\mathbf{1}$$

of collections for $\mathbf{S}$. Since $\mathbf{S}$ is a club, there is a monoidal structure on $\mathsf{Coll}(\mathbf{S})$ and we will pick a particular monoid $((P, p), j, m)$.

1. The collection $p : P \to S1$ is defined as follows:
   (a) $P_0 = \{a\}$,
   (b) $p_0 = id$ (since $(S1)_0 = \{\bullet\}$),
   (c) $P_1 = \{f\}$,
   (d) $p_1 = id$ (since $(S1)_1 = \{\bullet\}$),
   (e) a typical element of P_2 can be drawn as

$$\begin{tikzcd}
 & {} \arrow[rr] & & {} & & & & {} \arrow[rr] & & {} & \\
 & & \Downarrow a_{11} & & & & & & \Downarrow a_{n1} & & \\
 & {} \arrow[rr] & & {} & & & & {} \arrow[rr] & & {} & \\
\bullet & & \vdots & & \bullet & \cdots & \bullet & & \vdots & & \bullet \\
 & {} \arrow[rr] & & {} & & & & {} \arrow[rr] & & {} & \\
 & & \Downarrow a_{1k_1} & & & & & & \Downarrow a_{nk_n} & & \\
 & {} \arrow[rr] & & {} & & & & {} \arrow[rr] & & {} &
\end{tikzcd}$$

together with a total order on the disjoint union $k_1 + \cdots + k_n$ that restricts to the usual order on each k_i. The total order represents the bracketing. For example, the pasting diagram

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow a_{11}"] & \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow a_{21}"] & \bullet
\end{tikzcd}$$

together with the order $11 \leqslant 21$ represents the bracketing

$$\begin{tikzcd}
\text{first} & \bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow a_{11}"] & \bullet \arrow[r, "g"] & \bullet, & \text{then} & \bullet \arrow[r, "f'"] & \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow a_{21}"] & \bullet
\end{tikzcd}$$

Analogously, the ordering $21 \leqslant 11$ represents the bracketing

$$\begin{tikzcd}
\text{first} & \bullet \arrow[r, "f"] & \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow a_{21}"] & \bullet, & \text{then} & \bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow a_{11}"] & \bullet \arrow[r, "g'"] & \bullet
\end{tikzcd}$$

2. The unit $j : (\mathbf{1}, \eta^S_{\mathbf{1}}) \to (P, p)$ is given by

$$\begin{tikzcd}
\mathbf{1} \arrow[rr, "j"] \arrow[dr, "\eta^S_{\mathbf{1}}"'] & & P \arrow[dl, "p"] \\
& S\mathbf{1} &
\end{tikzcd}$$

where the only non-trivial component is $j_2 : \mathbf{1}_2 \to P_2$ with the action

$$\begin{tikzcd}
\bullet \arrow[r, phantom, "\mapsto"] & \bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow 1"] & \bullet
\end{tikzcd}$$

3. The multiplication $m : (P, p) \cdot (P, p) \to (P, p)$ is given by the diagram

$$\begin{tikzcd}
& (P,p)@P \arrow[rrr, "m"] \arrow[dl, "\partial_1^{(P,p)}P"'] \arrow[dr, "p@p"] & & & P \arrow[dd, "p"] \\
SP \arrow[r, "Sp"'] & SS\mathbf{1} \arrow[r, equal] & (S\mathbf{1}, 1_{S\mathbf{1}})@S\mathbf{1} \arrow[drr, "\mu^S_{\mathbf{1}}"'] & & \\
& & & & S\mathbf{1}
\end{tikzcd}$$

in Glob_2, where

$$\begin{tikzcd}
(P,p)@P \arrow[r, "\partial_1^{(P,p)}P"] \arrow[d, "\partial_0^{(P,p)}P"'] & SP \arrow[d, "S!"] \\
P \arrow[r, "p"'] & S\mathbf{1}
\end{tikzcd}$$

is a pullback in Glob_2. Since both P_0 and P_1 have only one element, we need to describe $m_2 : ((P, p)@P)_2 \to P_2$. Since

$$\begin{tikzcd}
((P,p)@P)_2 \arrow[r, "(\partial_1^{(P,p)}P)_2"] \arrow[d, "(\partial_0^{(P,p)}P)_2"'] & (SP)_2 \arrow[d, "(S!)_2"] \\
P_2 \arrow[r, "p_2"'] & (S\mathbf{1})_2
\end{tikzcd}$$

is a pullback in Set, an element of $((P,p)@P)_2$ is a pair

$$\left(\bullet \begin{array}{c} \longrightarrow \\ \Downarrow a_{11} \\ \longrightarrow \\ \vdots \\ \longrightarrow \\ \Downarrow a_{1k_1} \\ \longrightarrow \end{array} \bullet \quad \cdots \quad \bullet \begin{array}{c} \longrightarrow \\ \Downarrow a_{n1} \\ \longrightarrow \\ \vdots \\ \longrightarrow \\ \Downarrow a_{nk_n} \\ \longrightarrow \end{array} \bullet \text{ with bracketing, } \begin{tikzcd} a \arrow[r, bend left, "f^i"] \arrow[r, bend right, "f^j"'] \arrow[r, phantom, "\Downarrow \alpha"] & a \end{tikzcd} \text{ in } (SP)_2 \right)$$

where $P_0 = \{a\}$ and $P_1 = \{f\}$, and the pullback condition states that $\alpha : f^i \to f^j$ has "the same pasting shape" as the left-hand-side diagram. The mapping $m_2 : ((P,p)@P)_2 \to P_2$ then substitutes α into the left-hand-side.

7.5.3 The operad for Gray-categories

We already know that the restriction functor $R_3 : \mathsf{Glob}_3 \to \mathsf{Glob}_2$ has a left adjoint L_3, as well as a right adjoint I_3. In fact, R_3 is given by composition with the inclusion

$$E_3 : G_2 = \begin{tikzcd} 2 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ 1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ 0 \end{tikzcd} \quad \to \quad \begin{tikzcd} 3 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ 2 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ 1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ 0 \end{tikzcd} = G_3$$

The functor L_3 is thus given as a *left* Kan extension along E_3. Therefore, given an object

$$X = \ X_2 \rightrightarrows X_1 \rightrightarrows X_0$$

in Glob_2, we get that $(L_3(X))(3) = \int^{i \in G_2} G_3(i,3) \bullet X(i) = \varnothing$, while $L_3(X)(j) = X_j$ for $0 \leqslant j \leqslant 2$.

The functor I_3 is given as a *right* Kan extension along E_3. Therefore $I_3(X)(j) = X_j$ for $0 \leqslant j \leqslant 2$, and

$$\begin{aligned}(I_3(X))(3) &= \int_{i \in G_2} G_3(3,i) \pitchfork X(i) \\ &= \{(a,b) \in X_2 \times X_2 \mid s(a) = s(b) \text{ and } t(a) = t(b)\}.\end{aligned}$$

We can think of $(I_3(X))(3)$ as of the set $\mathrm{par}(X_2)$ of parallel pairs in X_2. The source and target maps are given by the projection maps:

$$\mathrm{par}(X_2) \underset{t=\pi_2}{\overset{s=\pi_1}{\rightrightarrows}} X_2.$$

Let us denote by $\mathbf{Sesq} = (Sesq, \eta, \mu)$ the cartesian monad on Glob_2 given by the operad for sesquicategories from subsection 7.5.3.

We can now transport $\mathbf{Sesq}$ along $I_3 : \mathsf{Glob}_2 \to \mathsf{Glob}_3$ since I_3 is a right adjoint. More in detail, the transported monad assigns to the 3-globular set

$$X = \; X_3 \rightrightarrows X_2 \rightrightarrows X_1 \rightrightarrows X_0$$

the 3-globular set

$$Z = \; \mathrm{par}(Z_2) \rightrightarrows Z_2 \rightrightarrows Z_1 \rightrightarrows Z_0,$$

where $Z_2 \rightrightarrows Z_1 \rightrightarrows Z_0$ is the 2-globular set underlying the free sesquicategory on X. Let us denote the resulting monad on Glob_3 by $\mathbf{M_3} = (M_3, \eta_3, \mu_3)$. Observe that $\mathbf{M_3}$ is *cartesian* since $\mathbf{Sesq}$ is (as $\mathbf{Sesq}$ comes from a 2-globular operad).

By construction, $\mathbf{M_3}$ is *truncable*. Thus Batanin's theory from [11] applies and we can describe $\mathbf{M_3}$-computads. This is desirable, since $\mathsf{Glob}_3^{\mathbf{M_3}} = \mathbf{Gray}$. To see this, consider an $\mathbf{M_3}$-algebra $a : M_3(X) \to X$, i.e., a morphism

$$\begin{tikzcd}
\mathrm{par}(Z_2) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "a_3"] & Z_2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "a_2"] & Z_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "a_1"] & Z_0 \arrow[d, "a_0"] \\
X_3 \arrow[r, shift left] \arrow[r, shift right] & X_2 \arrow[r, shift left] \arrow[r, shift right] & X_1 \arrow[r, shift left] \arrow[r, shift right] & X_0
\end{tikzcd}$$

of 3-globular sets subject to algebra axioms. The unit axiom states that trivial pasting diagrams are computed trivially. The associativity axiom states the following:

0. Dimension 0: the condition is void.
1. Dimension 1: associativity axiom for the composition of 1-cells.
2. Dimension 2: associativity axiom for the composition of 2-cells.
3. Dimension 3: Given a 2-pasting diagram

   $$\begin{tikzcd}
   \bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow a_{11}"] & \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow a_{21}"] & \bullet
   \end{tikzcd}$$

   with $11 \leqslant 21$ we get the pasting

   $$\begin{tikzcd}
   \bullet \arrow[r, bend left, "gf"] \arrow[r, bend right, "g'f'"'] \arrow[r, phantom, "\Downarrow a"] & \bullet
   \end{tikzcd}$$

   given by composing

   $$\begin{tikzcd}
   \bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow a_{11}"] & \bullet \arrow[r, "g"] & \bullet
   \end{tikzcd}$$

with

$$\begin{tikzcd}\bullet \arrow[r, "f'"] & \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow a_{21}"] & \bullet.\end{tikzcd}$$

The same pasting diagram with ordering $21 \leqslant 11$ yields

$$\begin{tikzcd}\bullet \arrow[r, bend left, "gf"] \arrow[r, bend right, "g'f'"'] \arrow[r, phantom, "\Downarrow b"] & \bullet\end{tikzcd}$$

given by composing

$$\begin{tikzcd}\bullet \arrow[r, "f"] & \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow a_{21}"] & \bullet\end{tikzcd}$$

with

$$\begin{tikzcd}\bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow a_{11}"] & \bullet \arrow[r, "g'"] & \bullet.\end{tikzcd}$$

Thus we get pairs (a, b) and (b, a) in $\mathrm{par}(Z_2)$ that yield mutually inverse 3-cells $a \to b$ and $b \to a$ in X_3. The category $\mathsf{Glob}_3^{\mathbf{M_3}}$ is therefore the *ordinary* category of **Gray**-categories and **Gray**-functors.

7.5.4 Gray-computads

By transport along restrictions we can form the chain $(\mathbf{M_3}, \mathbf{M_2}, \mathbf{M_1}, \mathbf{M_0})$ with $\mathsf{Glob}_0^{\mathbf{M_0}} = \mathsf{Glob}_0 = \mathsf{Set}$, $\mathsf{Glob}_1^{\mathbf{M_1}} = \mathsf{Cat}$ and $\mathsf{Glob}_2^{\mathbf{M_2}} = \mathsf{Sscat}$ (the category of sesquicategories), and $\mathsf{Glob}_3^{\mathbf{M_3}} = \mathbf{Gray}$.

By definition, an $\mathbf{M_3}$-computad C is a triple (G, φ, X) consisting of a 3-globular set

$$G_3 \rightrightarrows G_2 \rightrightarrows G_1 \rightrightarrows G_0 \ ,$$

an $\mathbf{M_2}$-computad X and an isomorphism

$$\varphi : U_2\mathcal{F}_2X \to (\ G_2 \rightrightarrows G_1 \rightrightarrows G_0\ ).$$

The $\mathbf{M_2}$-computad X is a triple (H, ψ, Y) where $H = (\ H_2 \rightrightarrows H_1 \rightrightarrows H_0\ )$ is a 2-globular set, Y is an $\mathbf{M_1}$-computad and ψ is an isomorphism

$$\psi : U_1\mathcal{F}_1Y \to (\ H_1 \rightrightarrows H_0\ ).$$

The $\mathbf{M_1}$-computad Y is a triple (K, ξ, Z) where $K = \ K_1 \rightrightarrows K_0\ $ is a 1-globular set (graph), Z is an $\mathbf{M_0}$-computad and ξ is an isomorphism $\xi : U_0\mathcal{F}_0Z \to K_0$. Thus Y is a 1-globular set $K_1 \rightrightarrows K_0$ with $Z = K_0$. The $\mathbf{M_2}$-computad X is the 2-globular set $H_2 \rightrightarrows H_1 \rightrightarrows H_0$ where the underlying graph of H is the graph of the free category on K. Finally, the $\mathbf{M_3}$-computad C can be described by the following data:

$$\begin{array}{rcccccc}
G_3 & \rightrightarrows & G_2 & \rightrightarrows & G_1 & \rightrightarrows & G_0 \\
 & & H_2 & \rightrightarrows & H_1 & \rightrightarrows & H_0 \\
 & & & & K_1 & \rightrightarrows & K_0 \\
 & & & & & & Z,
\end{array}$$

Z being K_0, the underlying 1-globular set of H being the free category on K, and the underlying 2-globular set on G being the free sesquicategory on H.

Having **Gray**-computads at hand, we can give presentations of several important **Gray**-categories in the following chapter.

Chapter 8

Pseudoadjunctions, pseudomonads and their presentations

We have shown in Chapter 7 that **Gray**-categories can be presented in a manner similar to presentations of ordinary categories, or 2-categories. In this chapter we give concrete presentations of certain **Gray**-categories; these will be, e.g., **Gray**-categories **psa** and **psm** that will play the role of "free pseudoadjunction" and "free pseudomonad" objects. That is, a pseudoadjunction in a **Gray**-category **K** will correspond precisely to a **Gray**-functor

$$\mathbf{psa} \to \mathbf{K}.$$

The technique of presenting **Gray**-categories is very useful in situations as is the one above. In this case we are more interested in the *existence* of a **Gray**-category **psa** with the property that it detects pseudoadjunctions than with the peculiarities of the inner structure of the presented **Gray**-category **psa**. Some basic results on the relation between pseudoadjunctions and pseudomonads can be deduced just with the presentations in hand. Namely, we can show that every pseudoadjunction gives rise to a pseudomonad by showing that there is a **Gray**-functor

$$\mathbf{psm} \to \mathbf{psa}$$

given by a morphism of presentations of **Gray**-categories **psm** and **psa**. This is the subject of the present chapter.

Structure of the chapter.

- In Section 8.1 we define pseudoadjunctions, pseudomonads, and their "KZ"-variants: KZ-pseudoadjunctions and KZ-pseudomonads.
- In Section 8.2 we give examples of presentations of **Gray**-categories **psa** and **psm** that detect pseudoadjunctions and pseudomonads in **Gray**-categories, and presentations of **Gray**-categories **kza** and **kzm** that detect KZ-pseudoadjunctions and KZ-pseudomonads.

The definitions contained in this chapter are standard and the results are known. However, we have not found any explicitly computed example of a presentation of a **Gray**-category, and we thus remedy this omission in detail.

8.1 Pseudoadjunctions and pseudomonads

We will first state the notion of a pseudoadjunction and a pseudomonad *in* a general **Gray**-category.

Definition 8.1.1 (Pseudoadjunctions in Gray-categories). Let **K** be a **Gray**-category. We say that 1-cells $U : \mathscr{A} \to \mathscr{X}$, $F : \mathscr{X} \to \mathscr{A}$ together with the data

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "F"] \arrow[dr, "1"'] & \mathscr{A} \arrow[d, "U"] \arrow[dl, phantom, "\nearrow \eta"] & \mathscr{A} \arrow[d, "U"'] \arrow[dr, "1"] & & F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "1_F"'] & FUF \arrow[d, Rightarrow, "\varepsilon F"] \arrow[dl, phantom, "\Rrightarrow s"] & U \arrow[r, Rightarrow, "\eta U"] \arrow[dr, Rightarrow, "1_U"'] & UFU \arrow[d, Rightarrow, "U\varepsilon"] \arrow[dl, phantom, "\Lleftarrow t"] \\
& \mathscr{X} & \mathscr{X} \arrow[r, "F"'] \arrow[ur, phantom, "\nearrow \varepsilon"] & \mathscr{A} & & F & & U
\end{tikzcd}$$

(with s and t being isomorphisms) constitute a *pseudoadjunction* in **K** with *unit* η and *counit* ε if these data satisfy two coherence identities: the 3-cell

$$\begin{tikzcd}
& UF \arrow[dr, Rightarrow, "UF\eta"] \arrow[rrd, Rightarrow, bend left, "1_{UF}"] & \Downarrow Us & \\
1_X \arrow[ur, Rightarrow, "\eta"] \arrow[dr, Rightarrow, "\eta"'] & \Downarrow \eta_\eta & UFUF \arrow[r, Rightarrow, "U\varepsilon F"] & UF \\
& UF \arrow[ur, Rightarrow, "\eta UF"] \arrow[rru, Rightarrow, bend right, "1_{UF}"'] & \Downarrow tF &
\end{tikzcd}$$

has to be equal to the identity 3-cell on η, and the 3-cell

$$\begin{tikzcd}
& \Downarrow sU & FU \arrow[dr, Rightarrow, "\varepsilon"] & \\
FU \arrow[r, Rightarrow, "F\eta U"] \arrow[rru, Rightarrow, bend left, "1_{FU}"] \arrow[rrd, Rightarrow, bend right, "1_{FU}"'] & FUFU \arrow[ur, Rightarrow, "\varepsilon FU"] \arrow[dr, Rightarrow, "FU\varepsilon"'] & \Downarrow \varepsilon_\varepsilon & 1_A \\
& \Downarrow Ft & FU \arrow[ur, Rightarrow, "\varepsilon"'] &
\end{tikzcd}$$

has to be equal to the identity 3-cell on ε. We write $F \dashv U : \mathscr{A} \to \mathscr{X}$ for this pseudoadjunction.

Definition 8.1.2 (Pseudomonads in Gray-categories). A *pseudomonad* in a **Gray**-category **K** *on* an object $\mathscr{X}$ of **K** is a 1-cell $T : \mathscr{X} \to \mathscr{X}$ together with the data

$$\begin{tikzcd}
\mathscr{X} \arrow[dr, bend left, "T"] \arrow[dr, bend right, "1"'] \arrow[dr, phantom, "\nearrow \eta"] & & \mathscr{X} \arrow[d, "T"'] \arrow[dr, "T"] & & T \arrow[r, Rightarrow, "T\eta"] \arrow[dr, Rightarrow, "1_T"'] & TT \arrow[d, Rightarrow, "\mu"] \arrow[dl, phantom, "\Rrightarrow s"] & T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, Rightarrow, "1_T"'] & TT \arrow[d, Rightarrow, "\mu"] \arrow[dl, phantom, "\Lleftarrow t"] \\
& \mathscr{X} & \mathscr{X} \arrow[r, "T"'] \arrow[ur, phantom, "\nearrow \mu"] & \mathscr{X} & & T & & T
\end{tikzcd}$$

and

$$\begin{tikzcd}
TTT \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"'] \arrow[dr, phantom, "\Swarrow m"] & TT \arrow[d, Rightarrow, "\mu"] \\
TT \arrow[r, Rightarrow, "\mu"'] & T
\end{tikzcd}$$

where η is the *unit* of the pseudomonad, μ is the *multiplication* of the pseudomonad, and l, r and t are isomorphisms subject to the following axioms:

1. The 3-cell

$$\begin{tikzcd}
 & TTT \arrow[r, Rightarrow, "T\mu"] \arrow[d, phantom, "\Downarrow Tm"] & TT \arrow[dr, Rightarrow, "\mu"] \arrow[dd, phantom, "\Downarrow m"] & \\
TTTT \arrow[ur, Rightarrow, "TT\mu"] \arrow[r, Rightarrow, "T\mu T"] \arrow[dr, Rightarrow, "\mu TT"'] & TTT \arrow[ur, Rightarrow, "T\mu"] \arrow[dr, Rightarrow, "\mu T"'] \arrow[d, phantom, "\Downarrow mT"] & & T \\
 & TTT \arrow[r, Rightarrow, "\mu T"'] & TT \arrow[ur, Rightarrow, "\mu"'] &
\end{tikzcd}$$

is equal to the 3-cell

$$\begin{tikzcd}
 & TTT \arrow[r, Rightarrow, "T\mu"] \arrow[dr, Rightarrow, "\mu T"'] \arrow[dd, phantom, "\Downarrow {\mu_{T\mu}}^{-1}"] & TT \arrow[dr, Rightarrow, "\mu"] \arrow[d, phantom, "\Downarrow m"] & \\
TTTT \arrow[ur, Rightarrow, "TT\mu"] \arrow[dr, Rightarrow, "\mu TT"'] & & TT \arrow[r, Rightarrow, "\mu"] \arrow[d, phantom, "\Downarrow m"] & T \\
 & TTT \arrow[ur, Rightarrow, "T\mu"] \arrow[r, Rightarrow, "\mu T"'] & TT \arrow[ur, Rightarrow, "\mu"'] &
\end{tikzcd}$$

2. The 3-cell

$$\begin{tikzcd}
 & & TT \arrow[dr, Rightarrow, "\mu"] \arrow[dd, phantom, "\Downarrow m"] & \\
TT \arrow[r, Rightarrow, "T\eta T"] & TTT \arrow[ur, Rightarrow, "T\mu"] \arrow[dr, Rightarrow, "\mu T"'] & & T \\
 & & TT \arrow[ur, Rightarrow, "\mu"'] &
\end{tikzcd}$$

is equal to the 3-cell

$$\begin{tikzcd}
 & TTT \arrow[dr, Rightarrow, "T\mu"] \arrow[d, phantom, "\Downarrow Tt"] & & \\
TT \arrow[ur, Rightarrow, "T\eta T"] \arrow[rr, Rightarrow, "1"] \arrow[dr, Rightarrow, "T\eta T"'] & {} & TT \arrow[r, Rightarrow, "\mu"] & T \\
 & TTT \arrow[ur, Rightarrow, "\mu T"'] \arrow[u, phantom, "\Downarrow sT"] & &
\end{tikzcd}$$

Such a pseudomonad will be denoted by (T, η, μ).

We shall now introduce the notion of a *KZ-pseudoadjunction* and *KZ-pseudomonad*. These notions will be studied in greater detail in Chapter 10.

Definition 8.1.3 (KZ-pseudoadjunction [23]). A pseudoadjunction $F \dashv U : \mathscr{A} \to \mathscr{X}$ is a *KZ-pseudoadjunction* if

1. the 3-cell

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "1_F"'] \arrow[dr, phantom, shift left=3, "s \Rrightarrow"] & FUF \arrow[d, Rightarrow, "\varepsilon F"] \\
 & F
\end{tikzcd}$$

   is the unit of the (ordinary) adjunction $F\eta \dashv \varepsilon F$ and if

2. the 3-cell

$$\begin{tikzcd}
U \arrow[r, Rightarrow, "\eta U"] \arrow[dr, Rightarrow, "1_U"'] \arrow[dr, phantom, shift left=3, "t \Lleftarrow"] & UFU \arrow[d, Rightarrow, "U\varepsilon"] \\
 & U
\end{tikzcd}$$

   is the counit of the adjunction $\eta U \dashv U\varepsilon$.

Definition 8.1.4 (KZ-pseudomonad [50, 82]). A pseudomonad (T, η, μ) with the triangle isomorphisms

$$\begin{tikzcd}
T \arrow[r, Rightarrow, "T\eta"] \arrow[dr, Rightarrow, "1_T"'] \arrow[dr, phantom, shift left=3, "\Rrightarrow s"] & TT \arrow[d, Rightarrow, "\mu"] & T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, Rightarrow, "1_T"'] \arrow[dr, phantom, shift left=3, "\Lleftarrow t"] & TT \arrow[d, Rightarrow, "\mu"] \\
 & T & & T
\end{tikzcd}$$

is a *KZ-pseudomonad* if there is an adjunction $T\eta \dashv \mu$ with unit s and an adjunction $\mu \dashv \eta T$ with counit t (both s and t being invertible by virtue of (T, η, μ) being a pseudomonad).

8.2 Presentations of important Gray-categories

We know from Chapter 7 that every **Gray**-category $\mathbf{K}$ admits a coequaliser presentation

$$\begin{tikzcd}
\mathcal{F}_3(\mathcal{E}) \arrow[r, shift left, "l^\sharp"] \arrow[r, shift right, "r^\sharp"'] & \mathcal{F}_3(\mathcal{G}) \arrow[r, "\mathbf{c}"] & \mathbf{K}
\end{tikzcd}$$

by means of a **Gray**-computad $\mathcal{G}$ specifying the data of "generators" of $\mathbf{K}$, a **Gray**-computad $\mathcal{E}$ of "equation data", and equations

$$\begin{tikzcd}
\mathcal{E} \arrow[r, shift left, "l"] \arrow[r, shift right, "r"'] & \mathcal{U}_3\mathcal{F}_3(\mathcal{G})
\end{tikzcd}$$

specifying the equalities that are to "hold freely" in $\mathbf{K}$.

We are going to use these presentations to describe **Gray**-categories **psa**, **psm**, **kzm** and **kza** for pseudoadjunctions, pseudomonads, KZ-pseudomonads and KZ-pseudoadjunctions such that, e.g., **Gray**-functors **psm** → $\mathbf{K}$ into a **Gray**-category $\mathbf{K}$ are in bijection with pseudomonads in $\mathbf{K}$.

We use the notation of Subsection 7.5.4 when describing the **Gray**-computads in each example.

Example 8.2.1 (**The Gray-category psa for pseudoadjunctions).** The data for $\mathcal{G}$, the computad of generators for **psa**, consist of:

1. The set $Z = \{X, A\}$ of designated 0-cells.
2. The set $K_1 = \{X \xrightarrow{F} A,\ A \xrightarrow{U} X\}$ of designated 1-cells.
3. The set $H_2 = \{\eta, \varepsilon\}$ of designated 2-cells, where η and ε are the 2-cells

$$\begin{tikzcd}
X \arrow[r, "F"] \arrow[dr, "1"'] \arrow[dr, phantom, shift left=3, "\Nearrow \eta"] & A \arrow[d, "U"] & A \arrow[r, "U"] \arrow[dr, "1"'] \arrow[dr, phantom, shift left=3, "\Swarrow \varepsilon"] & X \arrow[d, "F"] \\
 & X & & A
\end{tikzcd}$$

4. The set $G_3 = \{\tau, \omega, \tau', \omega'\}$ of designated 3-cells

$$\begin{tikzcd}
U \arrow[r, Rightarrow, "\eta U"] \arrow[dr, Rightarrow, "1"'] \arrow[dr, phantom, shift left=3, "\Nearrow \tau"] & UFU \arrow[d, Rightarrow, "U\varepsilon"] & F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "1"'] \arrow[dr, phantom, shift left=3, "\Swarrow \omega"] & FUF \arrow[d, Rightarrow, "\varepsilon F"] \\
 & U & & F
\end{tikzcd}$$

$$\begin{tikzcd}
U \arrow[r, Rightarrow, "\eta U"] \arrow[dr, Rightarrow, "1"'] \arrow[dr, phantom, shift left=3, "\Swarrow \tau'"] & UFU \arrow[d, Rightarrow, "U\varepsilon"] & F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "1"'] \arrow[dr, phantom, shift left=3, "\Nearrow \omega'"] & FUF \arrow[d, Rightarrow, "\varepsilon F"] \\
 & U & & F
\end{tikzcd}$$

The data for $\mathcal{E}$ consist of Z, K_1, H_2 as in the case of $\mathcal{G}$, and the set G_3 consists of six 3-cells; four "invertibility" 3-cells

$$\begin{tikzcd}
U \arrow[r, Rightarrow, "\eta U"] \arrow[d, Rightarrow, "\eta U"'] \arrow[dr, phantom, "\Nearrow \alpha"] & UFU \arrow[d, Rightarrow, "U\varepsilon"] \\
UFU \arrow[r, Rightarrow, "U\varepsilon"'] & U
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "1"'] \arrow[dr, phantom, "\Nearrow \beta"] & U \arrow[d, Rightarrow, "1"] \\
U \arrow[r, Rightarrow, "1"'] & U
\end{tikzcd}$$

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[d, Rightarrow, "F\eta"'] \arrow[dr, phantom, "\Nearrow \gamma"] & FUF \arrow[d, Rightarrow, "\varepsilon F"] \\
FUF \arrow[r, Rightarrow, "\varepsilon F"'] & F
\end{tikzcd}
\qquad
\begin{tikzcd}
F \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "1"'] \arrow[dr, phantom, "\Nearrow \delta"] & F \arrow[d, Rightarrow, "1"] \\
F \arrow[r, Rightarrow, "1"'] & F
\end{tikzcd}$$

and two 3-cells

$$\begin{tikzcd}
FU \arrow[r, Rightarrow, bend left, "\varepsilon"] \arrow[r, Rightarrow, bend right, "\varepsilon"'] \arrow[r, phantom, "\Downarrow\varphi"] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, Rightarrow, bend left, "\eta"] \arrow[r, Rightarrow, bend right, "\eta"'] \arrow[r, phantom, "\Downarrow\psi"] & UF
\end{tikzcd}$$

The presentation

$$\begin{tikzcd}
\mathcal{E} \arrow[r, shift left, "l"] \arrow[r, shift right, "r"'] & \mathcal{U}_3\mathcal{F}_3(\mathcal{G})
\end{tikzcd}$$

specifies the equalities to hold in **psa**; the 3-cells α and β postulate that τ is invertible with the inverse τ'.

$$l: \begin{tikzcd}
U \arrow[r, Rightarrow, "\eta U"] \arrow[d, Rightarrow, "\eta U"'] \arrow[dr, phantom, "\Nearrow \alpha"] & UFU \arrow[d, Rightarrow, "U\varepsilon"] \\
UFU \arrow[r, Rightarrow, "U\varepsilon"'] & U
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
U \arrow[r, Rightarrow, "\eta U"] \arrow[d, Rightarrow, "\eta U"] \arrow[dr, Rightarrow, "1"'] & UFU \arrow[d, Rightarrow, "U\varepsilon"] \arrow[dl, phantom, "\Nearrow \tau"] \\
UFU] \arrow[r, Rightarrow, "U\varepsilon"'] \arrow[ur, phantom, "\Nearrow \tau'"] & U
\end{tikzcd}$$

$$r: \begin{tikzcd}
U \arrow[r, Rightarrow, "\eta U"] \arrow[d, Rightarrow, "\eta U"'] \arrow[dr, phantom, "\Nearrow \alpha"] & UFU \arrow[d, Rightarrow, "U\varepsilon"] \\
UFU \arrow[r, Rightarrow, "U\varepsilon"'] & U
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
U \arrow[r, Rightarrow, "\eta U"] \arrow[d, Rightarrow, "\eta U"] \arrow[dr, phantom, "\Nearrow 1"] & UFU \arrow[d, Rightarrow, "U\varepsilon"] \\
UFU \arrow[r, Rightarrow, "U\varepsilon"'] & U
\end{tikzcd}$$

For β we define

$$l: \begin{tikzcd}
U \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "1"'] \arrow[dr, phantom, "\Nearrow \beta"] & U \arrow[d, Rightarrow, "1"] \\
U \arrow[r, Rightarrow, "1"'] & U
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
U \arrow[r, Rightarrow, "\eta U"] \arrow[rr, Rightarrow, bend left=60, "1"] \arrow[rr, Rightarrow, bend right=60, "1"'] & UFU \arrow[r, Rightarrow, "U\varepsilon"] \arrow[u, phantom, shift left=3, "\Uparrow\tau'"] \arrow[d, phantom, shift right=3, "\Uparrow\tau"] & U
\end{tikzcd}$$

$$\begin{tikzcd}
r : & U \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "1"'] \arrow[dr, phantom, "\Rrightarrow \beta"] & U \arrow[d, Rightarrow, "1"] & \mapsto & U \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "1"'] \arrow[dr, phantom, "\Rrightarrow 1"] & U \arrow[d, Rightarrow, "1"] \\
 & U \arrow[r, Rightarrow, "1"'] & U & & U \arrow[r, Rightarrow, "1"'] & U
\end{tikzcd}$$

Similarly, we postulate that ω is invertible:

$$\begin{tikzcd}
l : & F \arrow[r, Rightarrow, "F\eta"] \arrow[d, Rightarrow, "F\eta"'] \arrow[dr, phantom, "\Rrightarrow \gamma"] & FUF \arrow[d, Rightarrow, "\varepsilon F"] & \mapsto & F \arrow[r, Rightarrow, "F\eta"] \arrow[d, Rightarrow, "F\eta"'] \arrow[dr, Rightarrow, "1"'] & FUF \arrow[d, Rightarrow, "\varepsilon F"] \arrow[dl, phantom, "\Rrightarrow \omega'"] \\
 & FUF \arrow[r, Rightarrow, "\varepsilon F"'] & F & & FUF \arrow[r, Rightarrow, "\varepsilon F"'] \arrow[ur, phantom, "\Rrightarrow \omega"] & F
\end{tikzcd}$$

$$\begin{tikzcd}
r : & F \arrow[r, Rightarrow, "F\eta"] \arrow[d, Rightarrow, "F\eta"'] \arrow[dr, phantom, "\Rrightarrow \gamma"] & FUF \arrow[d, Rightarrow, "\varepsilon F"] & \mapsto & F \arrow[r, Rightarrow, "F\eta"] \arrow[d, Rightarrow, "F\eta"'] \arrow[dr, phantom, "\Rrightarrow 1"] & FUF \arrow[d, Rightarrow, "\varepsilon F"] \\
 & FUF \arrow[r, Rightarrow, "\varepsilon F"'] & F & & FUF \arrow[r, Rightarrow, "\varepsilon F"'] & F
\end{tikzcd}$$

$$\begin{tikzcd}
l : & F \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "1"'] \arrow[dr, phantom, "\Rrightarrow \delta"] & F \arrow[d, Rightarrow, "1"] & \mapsto & F \arrow[r, Rightarrow, "F\eta"] \arrow[rr, Rightarrow, bend left=50, "1"] \arrow[rr, Rightarrow, bend right=50, "1"'] & FUF \arrow[r, Rightarrow, "\varepsilon F"] \arrow[u, phantom, "\Uparrow \omega"] \arrow[d, phantom, "\Uparrow \omega'"] & F \\
 & F \arrow[r, Rightarrow, "1"'] & F & & & &
\end{tikzcd}$$

$$\begin{tikzcd}
r : & F \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "1"'] \arrow[dr, phantom, "\Rrightarrow \delta"] & F \arrow[d, Rightarrow, "1"] & \mapsto & F \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "1"'] \arrow[dr, phantom, "\Rrightarrow 1"] & F \arrow[d, Rightarrow, "1"] \\
 & F \arrow[r, Rightarrow, "1"'] & F & & F \arrow[r, Rightarrow, "1"'] & F
\end{tikzcd}$$

The 3-cells φ and ψ specify the pseudoadjunction identities

$$\begin{tikzcd}
 & \Downarrow \omega' U & FU \arrow[dr, Rightarrow, "\varepsilon"] & & & & \\
FU \arrow[urr, Rightarrow, bend left, "1"] \arrow[r, Rightarrow, "F\eta U"] \arrow[drr, Rightarrow, bend right, "1"'] & FUFU \arrow[ur, Rightarrow, "\varepsilon FU"] \arrow[dr, Rightarrow, "FU\varepsilon"'] & \Downarrow \varepsilon_{\varepsilon} & 1 & = & FU \arrow[r, Rightarrow, bend left, "\varepsilon"] \arrow[r, Rightarrow, bend right, "\varepsilon"'] \arrow[r, phantom, "\Downarrow 1"] & 1 \\
 & \Downarrow F\tau' & FU \arrow[ur, Rightarrow, "\varepsilon"'] & & & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & & & UF \arrow[dr, Rightarrow, "UF\eta"] \arrow[drr, equal, bend left, "1"] \arrow[drr, phantom, "\Downarrow U\omega'"] & & \\
1 \arrow[r, Rightarrow, bend left, "\eta"] \arrow[r, Rightarrow, bend right, "\eta"'] \arrow[r, phantom, "\Downarrow 1"] & UF & = & 1 \arrow[ur, Rightarrow, "\eta"] \arrow[dr, Rightarrow, "\eta"'] \arrow[rr, phantom, "\Downarrow \eta_\eta"] & & UFUF \arrow[r, Rightarrow, "U\varepsilon F"] & UF \\
 & & & & UF \arrow[ur, Rightarrow, "\eta UF"'] \arrow[urr, equal, bend right, "1"'] \arrow[urr, phantom, "\Downarrow \tau' F"] & &
\end{tikzcd}$$

The coequaliser

$$\begin{tikzcd}
\mathcal{F}_3(\mathcal{E}) \arrow[r, shift left, "l^\sharp"] \arrow[r, shift right, "r^\sharp"'] & \mathcal{F}_3(\mathcal{G}) \arrow[r, "\mathbf{c}"] & \mathbf{psa}
\end{tikzcd}$$

then yields the category **psa** for which **Gray**-functors $\mathbf{T} : \mathbf{psa} \to \mathbf{K}$ correspond precisely to pseudoadjunctions $F \dashv U$ in $\mathbf{K}$.

Example 8.2.2 (The Gray-category psm for pseudomonads). The **Gray**-category **psm** for pseudomonads is presented by the diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r, shift left, "l"] \arrow[r, shift right, "r"'] & \mathcal{U}_3\mathcal{F}_3(\mathcal{G})
\end{tikzcd}$$

where the data for $\mathcal{G}$ consist of:

1. The set $Z = \{K\}$ of designated 0-cells.
2. The set $K_1 = \{K \xrightarrow{T} K\}$ of designated 1-cells.
3. The set $H_2 = \{\eta, \mu\}$ of designated 2-cells, where η and μ are the 2-cells

$$\begin{tikzcd}
K \arrow[r, bend left, "T"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Uparrow \eta"] & K ,
\end{tikzcd}
\qquad
\begin{tikzcd}
K \arrow[r, "T"] \arrow[dr, "T"'] & K \arrow[d, "T"] \arrow[dl, phantom, "\swarrow \mu"] \\
 & K
\end{tikzcd}$$

4. The set $G_3 = \{\tau, \omega, m, \tau', \omega', m'\}$ of designated 3-cells

$$\begin{tikzcd}
T \arrow[r, Rightarrow, "T\eta"] \arrow[dr, Rightarrow, "1"'] & TT \arrow[d, Rightarrow, "\mu"] \arrow[dl, phantom, "\nearrow \omega'"] \\
 & T
\end{tikzcd} ,
\quad
\begin{tikzcd}
T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, Rightarrow, "1"'] & TT \arrow[d, Rightarrow, "\mu"] \arrow[dl, phantom, "\swarrow \tau'"] \\
 & T
\end{tikzcd} ,
\quad
\begin{tikzcd}
TTT \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"'] \arrow[dr, phantom, "\swarrow m"] & TT \arrow[d, Rightarrow, "\mu"] \\
TT \arrow[r, Rightarrow, "\mu"'] & T
\end{tikzcd}$$

and

$$\begin{tikzcd} T \arrow[r, Rightarrow, "T\eta"] \arrow[dr, Rightarrow, "1"'] & TT \arrow[d, Rightarrow, "\mu"] \arrow[dl, phantom, "\Lleftarrow\omega" near start] \\ & T \end{tikzcd} , \quad \begin{tikzcd} T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, Rightarrow, "1"'] & TT \arrow[d, Rightarrow, "\mu"] \arrow[dl, phantom, "\Rrightarrow\tau" near start] \\ & T \end{tikzcd} , \quad \begin{tikzcd} TTT \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"'] \arrow[dr, phantom, "\Rrightarrow m'"] & TT \arrow[d, Rightarrow, "\mu"] \\ TT \arrow[r, Rightarrow, "\mu"'] & T \end{tikzcd}$$

The data for $\mathcal{E}$ consist of Z, K_1, H_2 as in the case of $\mathcal{G}$ again, and the set G_3 of postulated 3-cells has 8 elements:

$$G_3 = \{a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2\}$$

The 3-cells a_i, b_i and c_i specify invertibility of ω, τ and m respectively, and the 3-cells d_1 and d_2 specify the coherence conditions for pseudomonads. Thus the 3-cells must be of types

$$\begin{tikzcd} T \arrow[r, Rightarrow, "T\eta"] \arrow[d, Rightarrow, "T\eta"'] \arrow[dr, phantom, "\Rrightarrow a_1"] & TT \arrow[d, Rightarrow, "\mu"] \\ TT \arrow[r, Rightarrow, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "1"'] \arrow[dr, phantom, "\Rrightarrow a_2"] & T \arrow[d, Rightarrow, "1"] \\ T \arrow[r, Rightarrow, "1"'] & T \end{tikzcd}$$

$$\begin{tikzcd} T \arrow[r, Rightarrow, "\eta T"] \arrow[d, Rightarrow, "\eta T"'] \arrow[dr, phantom, "\Rrightarrow b_1"] & TT \arrow[d, Rightarrow, "\mu"] \\ TT \arrow[r, Rightarrow, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "1"'] \arrow[dr, phantom, "\Rrightarrow b_2"] & T \arrow[d, Rightarrow, "1"] \\ T \arrow[r, Rightarrow, "1"'] & T \end{tikzcd}$$

$$\begin{tikzcd} TTT \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "T\mu"'] \arrow[dr, phantom, "\Rrightarrow c_1"] & TT \arrow[d, Rightarrow, "\mu"] \\ TT \arrow[r, Rightarrow, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} TTT \arrow[r, Rightarrow, "\mu T"] \arrow[d, Rightarrow, "\mu T"'] \arrow[dr, phantom, "\Rrightarrow c_2"] & TT \arrow[d, Rightarrow, "\mu"] \\ TT \arrow[r, Rightarrow, "\mu"'] & T \end{tikzcd}$$

$$\begin{tikzcd} & TTT \arrow[r, Rightarrow, "T\mu"] & TT \arrow[dr, Rightarrow, "\mu"] & \\ TTTT \arrow[ur, Rightarrow, "TT\mu"] \arrow[dr, Rightarrow, "\mu TT"'] & \arrow[r, phantom, "\Downarrow d_1"] & \; & T \\ & TTT \arrow[r, Rightarrow, "\mu T"'] & TT \arrow[ur, Rightarrow, "\mu"'] & \end{tikzcd}$$

$$\begin{tikzcd}
& TTT \arrow[r, Rightarrow, "T\mu"] & TT \arrow[dr, Rightarrow, "\mu"] & \\
TT \arrow[ur, Rightarrow, "T\eta T"] \arrow[dr, Rightarrow, "T\eta T"'] & \Downarrow d_2 & & T \\
& TTT \arrow[r, Rightarrow, "\mu T"'] & TT \arrow[ur, Rightarrow, "\mu"'] &
\end{tikzcd}$$

The 3-cells a_1 and a_2 are sent to the obvious composites in $\mathcal{U}_3\mathcal{F}_3(\mathcal{G})$.

$$l : \begin{tikzcd}
T \arrow[r, Rightarrow, "T\eta"] \arrow[d, Rightarrow, "T\eta"'] \arrow[dr, phantom, "\Nearrow a_1"] & TT \arrow[d, Rightarrow, "\mu"] \\
TT \arrow[r, Rightarrow, "\mu"'] & T
\end{tikzcd}
\mapsto
\begin{tikzcd}
T \arrow[r, Rightarrow, "T\eta"] \arrow[d, Rightarrow, "T\eta"'] \arrow[dr, Rightarrow, "1"'] \arrow[dr, phantom, "\Nearrow \omega'", shift left=3ex] \arrow[dr, phantom, "\Nearrow \omega"', shift right=3ex] & TT \arrow[d, Rightarrow, "\mu"] \\
TT \arrow[r, Rightarrow, "\mu"'] & T
\end{tikzcd}$$

$$r : \begin{tikzcd}
T \arrow[r, Rightarrow, "T\eta"] \arrow[d, Rightarrow, "T\eta"'] \arrow[dr, phantom, "\Nearrow a_1"] & TT \arrow[d, Rightarrow, "\mu"] \\
TT \arrow[r, Rightarrow, "\mu"'] & T
\end{tikzcd}
\mapsto
\begin{tikzcd}
T \arrow[r, Rightarrow, "T\eta"] \arrow[d, Rightarrow, "T\eta"'] \arrow[dr, phantom, "\Nearrow 1"] & TT \arrow[d, Rightarrow, "\mu"] \\
TT \arrow[r, Rightarrow, "\mu"'] & T
\end{tikzcd}$$

$$l : \begin{tikzcd}
T \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "1"'] \arrow[dr, phantom, "\Nearrow a_2"] & T \arrow[d, Rightarrow, "1"] \\
T \arrow[r, Rightarrow, "1"'] & T
\end{tikzcd}
\mapsto
\begin{tikzcd}
T \arrow[r, Rightarrow, "T\eta"] \arrow[rr, Rightarrow, bend left=50, "1"] \arrow[rr, Rightarrow, bend right=50, "1"'] & TT \arrow[r, Rightarrow, "\mu"] \arrow[u, phantom, "\Uparrow \omega"] \arrow[d, phantom, "\Uparrow \omega'"] & T
\end{tikzcd}$$

$$r : \begin{tikzcd}
T \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "1"'] \arrow[dr, phantom, "\Nearrow a_2"] & T \arrow[d, Rightarrow, "1"] \\
T \arrow[r, Rightarrow, "1"'] & T
\end{tikzcd}
\mapsto
\begin{tikzcd}
T \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "1"'] \arrow[dr, phantom, "\Nearrow 1"] & T \arrow[d, Rightarrow, "1"] \\
T \arrow[r, Rightarrow, "1"'] & T
\end{tikzcd}$$

The 3-cells b_1, b_2, c_1, c_2 are similarly sent by l and r to the following composites in

$\mathcal{U}_3\mathcal{F}_3(\mathcal{G})$:

$$\begin{aligned}
l : b_1 &\mapsto \tau \cdot \tau' \\
r : b_1 &\mapsto 1 \\
l : b_2 &\mapsto \tau' \cdot \tau \\
r : b_2 &\mapsto 1 \\
l : b_1 &\mapsto m \cdot m' \\
r : b_1 &\mapsto 1 \\
l : b_2 &\mapsto m' \cdot m \\
r : b_2 &\mapsto 1
\end{aligned}$$

The 3-cell d_1 gets mapped by l and r as follows:

$$\begin{tikzcd}
 & & & TTT \arrow[r, Rightarrow, "T\mu"] \arrow[d, phantom, "\Downarrow Tm"] & TT \arrow[dr, Rightarrow, "\mu"] & \\
l : d_1 & \mapsto & TTTT \arrow[ur, Rightarrow, "TT\mu"] \arrow[r, Rightarrow, "T\mu T"] \arrow[dr, Rightarrow, "\mu TT"'] & TTT \arrow[ur, Rightarrow, "T\mu"] \arrow[dr, Rightarrow, "\mu T"] & \Downarrow m & T \\
 & & & TTT \arrow[r, Rightarrow, "\mu T"'] \arrow[u, phantom, "\Downarrow mT"] & TT \arrow[ur, Rightarrow, "\mu"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & TTT \arrow[r, Rightarrow, "T\mu"] \arrow[dr, Rightarrow, "\mu T"] & TT \arrow[dr, Rightarrow, "\mu"] \arrow[d, phantom, "\Downarrow m"] & \\
r : d_1 & \mapsto & TTTT \arrow[ur, Rightarrow, "TT\mu"] \arrow[dr, Rightarrow, "\mu TT"'] & \Downarrow {\mu_{T\mu}}^{-1} & TT \arrow[r, Rightarrow, "\mu"] & T \\
 & & & TTT \arrow[r, Rightarrow, "\mu T"'] \arrow[ur, Rightarrow, "T\mu"] & TT \arrow[ur, Rightarrow, "\mu"'] \arrow[u, phantom, "\Downarrow m"] &
\end{tikzcd}$$

Similarly, the 3-cell d_2 establishes the condition that

$$\begin{tikzcd}
 & & TT \arrow[dr, Rightarrow, "\mu"] & \\
TT \arrow[r, Rightarrow, "T\eta T"] & TTT \arrow[ur, Rightarrow, "T\mu"] \arrow[dr, Rightarrow, "\mu T"'] & \Downarrow m & T \\
 & & TT \arrow[ur, Rightarrow, "\mu"'] &
\end{tikzcd}$$

equals

$$\begin{tikzcd}
 & TTT \arrow[dr, Rightarrow, "T\mu"] \arrow[d, phantom, "\Downarrow T\tau'"] & & \\
TT \arrow[ur, Rightarrow, "T\eta T"] \arrow[rr, Rightarrow, "1"] \arrow[dr, Rightarrow, "T\eta T"'] & {} & TT \arrow[r, Rightarrow, "\mu"] & T. \\
 & TTT \arrow[ur, Rightarrow, "\mu T"'] \arrow[u, phantom, "\Downarrow \omega' T"] & &
\end{tikzcd}$$

The resulting category **psm** then has the property that **Gray**-functors $\mathbf{T} : \mathbf{psm} \to \mathbf{K}$ amount precisely to specifying pseudomonads T in $\mathbf{K}$.

Remark 8.2.3. Compare the above example specifying the category **psm** for pseudomonads with the construction of **psm** in [54]. While proving the existence of **psm** in [54] amounts to a considerable effort, with the **Gray**-computad presentation approach this becomes easy.

Example 8.2.4 (The Gray-category kzm for KZ-pseudomonads). The **Gray**-category **kzm** for KZ-pseudomonads (see Definition 8.1.4) is presented by the diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r, Rightarrow, shift left, "l"] \arrow[r, Rightarrow, shift right, "r"'] & \mathcal{U}_3\mathcal{F}_3(\mathcal{G})
\end{tikzcd}$$

where the data for $\mathcal{G}$ consist of:

1. The set $Z = \{K\}$ of designated 0-cells.
2. The set $K_1 = \{K \xrightarrow{T} K\}$ of designated 1-cells.
3. The set $H_2 = \{\eta, \mu\}$ of designated 2-cells, where η and μ are the 2-cells

$$\begin{tikzcd}
K \arrow[r, bend left, "T"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Uparrow \eta"] & K ,
\end{tikzcd}
\qquad
\begin{tikzcd}
K \arrow[r, "T"] \arrow[dr, "T"'] & K \arrow[d, "T"] \arrow[dl, phantom, "\Swarrow \mu"] \\
 & K
\end{tikzcd}$$

4. The set $G_3 = \{\omega', \tau', \alpha, \varepsilon, \omega, \tau\}$ $G_3 = \{\eta, \beta, \alpha, \varepsilon, \eta', \beta'\}$ of designated 3-cells

$$\begin{tikzcd}
T \arrow[r, Rightarrow, "T\eta"] \arrow[dr, Rightarrow, "1"'] & TT \arrow[d, Rightarrow, "\mu"] \arrow[dl, phantom, "\omega' \Rrightarrow"] \\
 & T
\end{tikzcd} ,
\quad
\begin{tikzcd}
T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, Rightarrow, "1"'] & TT \arrow[d, Rightarrow, "\mu"] \arrow[dl, phantom, "\tau' \Lleftarrow"] \\
 & T
\end{tikzcd} ,
\quad
\begin{tikzcd}
TT \arrow[r, Rightarrow, "\mu"] \arrow[dr, Rightarrow, "1"'] & T \arrow[d, Rightarrow, "T\eta"] \arrow[dl, phantom, "\varepsilon \Lleftarrow"] \\
 & TT
\end{tikzcd}$$

and

$$\begin{tikzcd}
T \arrow[r, Rightarrow, "T\eta"] \arrow[dr, Rightarrow, "1"'] & TT \arrow[d, Rightarrow, "\mu"] \arrow[dl, phantom, "\omega \Lleftarrow"] \\
 & T
\end{tikzcd} ,
\quad
\begin{tikzcd}
T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, Rightarrow, "1"'] & TT \arrow[d, Rightarrow, "\mu"] \arrow[dl, phantom, "\tau \Rrightarrow"] \\
 & T
\end{tikzcd} ,
\quad
\begin{tikzcd}
TT \arrow[r, Rightarrow, "\mu"] \arrow[dr, Rightarrow, "1"'] & T \arrow[d, Rightarrow, "\eta T"] \arrow[dl, phantom, "\alpha \Rrightarrow"] \\
 & TT
\end{tikzcd}$$

The data for $\mathcal{E}$ consist of Z, K_1, H_2 as in the case of $\mathcal{G}$ again, and the set G_3 of postulated 3-cells has 7 elements:

$$G_3 = \{a_1, a_2, d_1, d_2, e_1, e_2, k\}$$

The 3-cells a_1, a_2 specify invertibility of ω, and therefore they must be of types

$$\begin{tikzcd}
T \arrow[r, Rightarrow, "T\eta"] \arrow[d, Rightarrow, "T\eta"'] \arrow[dr, phantom, "a_1 \Rrightarrow"] & TT \arrow[d, Rightarrow, "\mu"] & T \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "1"'] \arrow[dr, phantom, "a_2 \Rrightarrow"] & T \arrow[d, Rightarrow, "1"] \\
TT \arrow[r, Rightarrow, "\mu"'] & T & T \arrow[r, Rightarrow, "1"'] & T
\end{tikzcd}$$

These 3-cells are sent to the following composites in $\mathcal{U}_3\mathcal{F}_3(\mathcal{G})$:

$$l : \begin{tikzcd}
T \arrow[r, Rightarrow, "T\eta"] \arrow[d, Rightarrow, "T\eta"'] \arrow[dr, phantom, "a_1 \Rrightarrow"] & TT \arrow[d, Rightarrow, "\mu"] \\
TT \arrow[r, Rightarrow, "\mu"'] & T
\end{tikzcd} \mapsto \begin{tikzcd}
T \arrow[r, Rightarrow, "T\eta"] \arrow[d, Rightarrow, "T\eta"'] \arrow[dr, Rightarrow, "1"'] \arrow[dr, phantom, shift left=4ex, "\omega' \Rrightarrow"] \arrow[dr, phantom, shift right=4ex, "\omega \Rrightarrow"] & TT \arrow[d, Rightarrow, "\mu"] \\
TT \arrow[r, Rightarrow, "\mu"'] & T
\end{tikzcd}$$

$$r : \begin{tikzcd}
T \arrow[r, Rightarrow, "T\eta"] \arrow[d, Rightarrow, "T\eta"'] \arrow[dr, phantom, "a_1 \Rrightarrow"] & TT \arrow[d, Rightarrow, "\mu"] \\
TT \arrow[r, Rightarrow, "\mu"'] & T
\end{tikzcd} \mapsto \begin{tikzcd}
T \arrow[r, Rightarrow, "T\eta"] \arrow[d, Rightarrow, "T\eta"'] \arrow[dr, phantom, "1 \Rrightarrow"] & TT \arrow[d, Rightarrow, "\mu"] \\
TT \arrow[r, Rightarrow, "\mu"'] & T
\end{tikzcd}$$

$$l : \begin{tikzcd}
T \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "1"'] \arrow[dr, phantom, "a_2 \Rrightarrow"] & T \arrow[d, Rightarrow, "1"] \\
T \arrow[r, Rightarrow, "1"'] & T
\end{tikzcd} \mapsto \begin{tikzcd}
T \arrow[r, Rightarrow, "T\eta"] \arrow[rr, Rightarrow, bend left=60, "1", "\Uparrow\omega"'] \arrow[rr, Rightarrow, bend right=60, "1"', "\Uparrow\omega'"] & TT \arrow[r, Rightarrow, "\mu"] & T
\end{tikzcd}$$

$$r : \begin{tikzcd}
T \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "1"'] \arrow[dr, phantom, "a_2 \Rrightarrow"] & T \arrow[d, Rightarrow, "1"] \\
T \arrow[r, Rightarrow, "1"'] & T
\end{tikzcd} \mapsto \begin{tikzcd}
T \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "1"'] \arrow[dr, phantom, "1 \Rrightarrow"] & T \arrow[d, Rightarrow, "1"] \\
T \arrow[r, Rightarrow, "1"'] & T
\end{tikzcd}$$

The 3-cells d_1, d_2 specify that $T\eta$ and μ form an adjunction with unit ω' and counit ε, and are of the form

$$\begin{tikzcd}
T \arrow[r, Rightarrow, "T\eta"] \arrow[d, Rightarrow, "1"'] \arrow[dr, phantom, "d_1 \Rrightarrow"] & TT \arrow[d, Rightarrow, "1"] & TT \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "\mu"'] \arrow[dr, phantom, "d_2 \Rrightarrow"] & TT \arrow[d, Rightarrow, "\mu"] \\
T \arrow[r, Rightarrow, "T\eta"'] & TT & T \arrow[r, Rightarrow, "1"'] & T
\end{tikzcd}$$

They are sent by l and r as follows:

$$\begin{tikzcd}
& T \arrow[r, Rightarrow, "T\eta"] \arrow[d, Rightarrow, "1"'] \arrow[dr, phantom, "d_1 \Rrightarrow"] & TT \arrow[d, Rightarrow, "1"] & & T \arrow[r, Rightarrow, "T\eta"] \arrow[dr, Rightarrow, "1"'] \arrow[r, phantom, shift right=4, "\omega' \Rrightarrow"'] & TT \arrow[d, Rightarrow, "\mu"'] \arrow[dr, Rightarrow, "1"] \arrow[dr, phantom, shift right=3, "\varepsilon \Rrightarrow"'] & \\
l : & T \arrow[r, Rightarrow, "T\eta"'] & TT & \mapsto & & T \arrow[r, Rightarrow, "T\eta"'] & TT
\end{tikzcd}$$

$$\begin{tikzcd}
& T \arrow[r, Rightarrow, "T\eta"] \arrow[d, Rightarrow, "1"'] \arrow[dr, phantom, "d_1 \Rrightarrow"] & TT \arrow[d, Rightarrow, "1"] & & T \arrow[r, Rightarrow, "T\eta"] \arrow[d, Rightarrow, "1"'] \arrow[dr, phantom, "1 \Rrightarrow"] & TT \arrow[d, Rightarrow, "1"] \\
r : & T \arrow[r, Rightarrow, "T\eta"'] & TT & \mapsto & T \arrow[r, Rightarrow, "T\eta"'] & TT
\end{tikzcd}$$

$$\begin{tikzcd}
& TT \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "\mu"'] \arrow[dr, phantom, "d_2 \Rrightarrow"] & TT \arrow[d, Rightarrow, "\mu"] & & TT \arrow[d, Rightarrow, "\mu"'] \arrow[dr, Rightarrow, "1"] \arrow[dr, phantom, shift right=3, "\varepsilon \Rrightarrow"'] & \\
l : & T \arrow[r, Rightarrow, "1"'] & T & \mapsto & T \arrow[r, Rightarrow, "T\eta"] \arrow[dr, Rightarrow, "1"'] \arrow[dr, phantom, shift left=3, "\omega' \Rrightarrow"] & TT \arrow[d, Rightarrow, "\mu"] \\
& & & & & T
\end{tikzcd}$$

$$\begin{tikzcd}
& TT \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "\mu"'] \arrow[dr, phantom, "d_2 \Rrightarrow"] & TT \arrow[d, Rightarrow, "\mu"] & & TT \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "\mu"'] \arrow[dr, phantom, "1 \Rrightarrow"] & TT \arrow[d, Rightarrow, "\mu"] \\
r : & T \arrow[r, Rightarrow, "1"'] & T & \mapsto & T \arrow[r, Rightarrow, "1"'] & T
\end{tikzcd}$$

Similarly, the 3-cells

$$\begin{tikzcd}
T \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "\eta T"'] \arrow[dr, phantom, "e_1 \Rrightarrow"] & T \arrow[d, Rightarrow, "\eta T"] & TT \arrow[r, Rightarrow, "\mu"] \arrow[d, Rightarrow, "1"'] \arrow[dr, phantom, "e_2 \Rrightarrow"] & T \arrow[d, Rightarrow, "1"] \\
TT \arrow[r, Rightarrow, "1"'] & TT & TT \arrow[r, Rightarrow, "\mu"'] & T
\end{tikzcd}$$

specify that μ and ηT form an adjunction with unit α and counit τ'.

The 3-cell k is of the form

$$\begin{tikzcd}
& T \arrow[r, Rightarrow, "\eta T"] & TT \arrow[dr, Rightarrow, "\mu"] & \\
1 \arrow[ur, Rightarrow, "\eta"] \arrow[dr, Rightarrow, "\eta"'] & \arrow[r, phantom, "\Downarrow k"] & & T \\
& T \arrow[r, Rightarrow, "T\eta"'] & TT \arrow[ur, Rightarrow, "\mu"'] &
\end{tikzcd}$$

and specifies the KZ-axiom

$$\begin{tikzcd}
& T \arrow[dr, Rightarrow, "\eta T"] & & & & & TT \arrow[dr, Rightarrow, "\mu"] & \\
1 \arrow[ur, Rightarrow, "\eta"] \arrow[dr, Rightarrow, "\eta"'] & (\eta_\eta)^{-1} \Downarrow & TT \arrow[r, Rightarrow, "\mu"] & T \arrow[r, phantom, "="] & 1 \arrow[r, Rightarrow, "\eta"] & T \arrow[ur, Rightarrow, "\eta T"] \arrow[rr, Rightarrow, "1"] \arrow[dr, Rightarrow, "T\eta"'] & & T. \\
& T \arrow[ur, Rightarrow, "T\eta"'] & & & & & TT \arrow[ur, Rightarrow, "\mu"'] &
\end{tikzcd}$$

with $\tau' \Downarrow$ in the upper triangle and $\omega' \Downarrow$ in the lower triangle of the right-hand diagram.

Example 8.2.5 (The Gray-category kza for KZ-pseudoadjunctions). When we want to give a presentation for the **Gray**-category **kza** for KZ-pseudoadjunctions, the data for $\mathcal{G}$ consist of:

1. The set $Z = \{X, A\}$ of designated 0-cells.
2. The set $K_1 = \{X \xrightarrow{F} A,\ A \xrightarrow{U} X\}$ of designated 1-cells.
3. The set $H_2 = \{\eta, \epsilon\}$ of designated 2-cells, where η and ε are the 2-cells

$$\begin{tikzcd}
X \arrow[r, "F"] \arrow[dr, "1"'] & A \arrow[d, "U"] \arrow[dl, phantom, "\eta \Nearrow"] \\
& X
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "U"] \arrow[dr, "1"'] & X \arrow[d, "F"] \arrow[dl, phantom, "\Swarrow \varepsilon"] \\
& A
\end{tikzcd}$$

4. The set $G_3 = \{\tau, \omega, \tau', \omega', \sigma', \chi'\}$ of designated 3-cells

$$\begin{tikzcd}
U \arrow[r, Rightarrow, "\eta U"] \arrow[dr, Rightarrow, "1"'] & UFU \arrow[d, Rightarrow, "U\varepsilon"] \arrow[dl, phantom, "\tau \Nearrow"] \\
& U
\end{tikzcd}
\qquad
\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "1"'] & FUF \arrow[d, Rightarrow, "\varepsilon F"] \arrow[dl, phantom, "\omega \Swarrow"] \\
& F
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[r, Rightarrow, "\eta U"] \arrow[dr, Rightarrow, "1"'] & UFU \arrow[d, Rightarrow, "U\varepsilon"] \arrow[dl, phantom, "\tau' \Swarrow"] \\
& U
\end{tikzcd}$$

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "1"'] \arrow[dr, phantom, shift left=3, "\omega' \Rrightarrow"] & FUF \arrow[d, Rightarrow, "\varepsilon F"] \\
 & F
\end{tikzcd}
\quad
\begin{tikzcd}
U \arrow[r, Rightarrow, "U\varepsilon"] \arrow[dr, Rightarrow, "1"'] \arrow[dr, phantom, shift left=3, "\sigma' \Rrightarrow"] & UFU \arrow[d, Rightarrow, "\eta U"] \\
 & U
\end{tikzcd}
\quad
\begin{tikzcd}
FUF \arrow[r, Rightarrow, "\varepsilon F"] \arrow[dr, Rightarrow, "1"'] \arrow[dr, phantom, shift left=3, "\chi' \Lleftarrow"] & F \arrow[d, Rightarrow, "F\eta"] \\
 & FUF
\end{tikzcd}$$

The data for $\mathcal{E}$ consist of Z, K_1, H_2 as in $\mathcal{G}$; the set G_3 consists of 3-cells $\alpha, \beta, \gamma, \delta, \varphi, \psi$ as in Example 8.2.1 together with 3-cells d_1, d_2, e_1, e_2 of the types

$$\begin{tikzcd}
UFU \arrow[r, Rightarrow, "U\varepsilon"] \arrow[d, Rightarrow, "1"'] \arrow[dr, phantom, "d_2 \Rrightarrow"] & U \arrow[d, Rightarrow, "1"] \\
UFU \arrow[r, Rightarrow, "U\varepsilon"'] & U
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "\eta U"'] \arrow[dr, phantom, "d_1 \Rrightarrow"] & U \arrow[d, Rightarrow, "\eta U"] \\
UFU \arrow[r, Rightarrow, "1"'] & UFU
\end{tikzcd}$$

$$\begin{tikzcd}
FUF \arrow[r, Rightarrow, "1"] \arrow[d, Rightarrow, "\varepsilon F"'] \arrow[dr, phantom, "e_1 \Rrightarrow"] & FUF \arrow[d, Rightarrow, "\varepsilon F"] \\
F \arrow[r, Rightarrow, "1"'] & F
\end{tikzcd}
\qquad
\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[d, Rightarrow, "1"'] \arrow[dr, phantom, "e_2 \Rrightarrow"] & FUF \arrow[d, Rightarrow, "1"] \\
F \arrow[r, Rightarrow, "F\eta"'] & FUF
\end{tikzcd}$$

that specify that τ' is the counit of $U\varepsilon \dashv \eta U$ (with d_1 and d_2) and that ω' is the unit of $F\eta \dashv \varepsilon F$ (with e_1 and e_2). The specification is analogous to that of specifying adjunctions in Example 8.2.4.

Example 8.2.6 (Every pseudoadjunction gives rise to a pseudomonad). We show that there is a **Gray**-functor

$$\mathbf{M} : \mathbf{psm} \to \mathbf{psa}$$

that gives for a pseudoadjunction $\mathbf{A} : \mathbf{psa} \to \mathbf{K}$ in $\mathbf{K}$ a pseudomonad $\mathbf{A} \cdot \mathbf{M} : \mathbf{psm} \to \mathbf{psa} \to \mathbf{K}$. Moreover, the approach via presentations shows quickly that such a **Gray**-functor $\mathbf{M} : \mathbf{psm} \to \mathbf{psa}$ indeed exists: consider the presentation

$$\begin{tikzcd}
\mathcal{E}_1 \arrow[r, shift left, "l_1"] \arrow[r, shift right, "r_1"'] & \mathcal{U}_3\mathcal{F}_3(\mathcal{G}_1)
\end{tikzcd}$$

of the **Gray**-category **psa** from Example 8.2.1, and the presentation

$$\begin{tikzcd}
\mathcal{E}_2 \arrow[r, shift left, "l_2"] \arrow[r, shift right, "r_2"'] & \mathcal{U}_3\mathcal{F}_3(\mathcal{G}_2)
\end{tikzcd}$$

of the **Gray**-category **psm** from Example 8.2.2. We can now define

$$m : \mathcal{G}_1 \to \mathcal{U}_3\mathcal{F}_3(\mathcal{G}_2)$$

by the following assignments:

$$\begin{aligned} K &\mapsto X \\ T &\mapsto UF \\ \eta &\mapsto \eta \\ \mu &\mapsto U\varepsilon F \\ \omega &\mapsto U\omega \\ \omega' &\mapsto U\omega' \\ \tau &\mapsto \tau F \\ \tau' &\mapsto \tau' F \\ m &\mapsto U\varepsilon_{\varepsilon F}. \end{aligned}$$

It is immediately seen that $m : \mathcal{G}_1 \to \mathcal{U}_3\mathcal{F}_3(\mathcal{G}_2)$ is a morphism of presentations. The transpose $m^\sharp : \mathcal{F}_3(\mathcal{G}_1) \to \mathcal{F}_3(\mathcal{G}_2)$ of m thus gives rise to the dotted arrow $\mathbf{M}$ in the following diagram by the couniversal property of coequalisers:

$$\begin{tikzcd}
\mathcal{F}_3(\mathcal{E}_1) \arrow[r, shift left, "{l_1}^\sharp"] \arrow[r, shift right, "{r_1}^\sharp"'] & \mathcal{F}_3(\mathcal{G}_1) \arrow[r, "\mathbf{c_1}"] \arrow[d, "m^\sharp"] & \mathbf{psm} \arrow[d, dotted, "\mathbf{M}"] \\
\mathcal{F}_3(\mathcal{E}_2) \arrow[r, shift left, "{l_2}^\sharp"] \arrow[r, shift right, "{r_2}^\sharp"'] & \mathcal{F}_3(\mathcal{G}_2) \arrow[r, "\mathbf{c_2}"] & \mathbf{psa}.
\end{tikzcd}$$

Remark 8.2.7. The fact that $m : \mathcal{G}_1 \to \mathcal{U}_3\mathcal{F}_3(\mathcal{G}_2)$ was a morphism of presentations in Example 8.2.6 corresponds to the observation that every pseudoadjunction gives rise to a pseudomonad. The assignments by which m is defined correspond to the pseudoadjunction data used to define a pseudomonad.

In the same spirit we could use the known fact that every KZ-pseudoadjunction gives rise to a KZ-pseudomonad to define a **Gray**-functor

$$\mathbf{kzm} \to \mathbf{kza}$$

which expresses that fact abstractly. Even further, there is a commutative square

$$\begin{tikzcd}
\mathbf{psm} \arrow[r] \arrow[d] & \mathbf{psa} \arrow[d] \\
\mathbf{kzm} \arrow[r] & \mathbf{kza}
\end{tikzcd}$$

of **Gray**-functors expressing that every KZ-pseudoadjunction gives rise a KZ-pseudomonad, that every KZ-pseudoadjunction is a pseudoadjunction, that every pseudoadjunction gives rise to a pseudomonad, and that every KZ-pseudomonad is a pseudomonad.

We will study KZ-pseudoadjunctions and KZ-pseudomonads in greater detail in Chapter 10.

Chapter 9

Formal adjoint functor theorem in Gray-categories

Bénabou showed in [13] that an adjunction

$$\begin{tikzcd} \mathscr{A} \arrow[d, bend left, "U"] \\ \mathscr{X} \arrow[u, bend left, "F"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

(with unit η) between categories $\mathscr{A}$ and $\mathscr{X}$ can be characterised as an *absolute left Kan extension*

$$\begin{tikzcd} \mathscr{X} \arrow[r, "F"] \arrow[dr, "1"'] & \mathscr{A} \arrow[d, "U"] \arrow[dl, phantom, "\Nearrow \eta"] \\ & \mathscr{X} \end{tikzcd}$$

of 1 *along* F. In this chapter we are interested in proving a correspondence very similar to the above: a *pseudoadjunction* $F \dashv U : \mathscr{A} \to \mathscr{X}$ (below left)

$$\begin{tikzcd} \mathscr{A} \arrow[d, bend left, "U"] & & \mathscr{X} \arrow[r, "F"] \arrow[dr, "1"'] & \mathscr{A} \arrow[d, "U"] \arrow[dl, phantom, "\Nearrow \eta"] \\ \mathscr{X} \arrow[u, bend left, "F"] \arrow[u, phantom, "\dashv"] & & & \mathscr{X} \end{tikzcd}$$

in a **Gray**-*category* **K** is precisely an *absolute left pseudoexension* (above right) *of* 1 *along* F in **K**. The notion of a pseudoextension already appears in [71].

Thus our aim is to reproduce Bénabou's result for the *weaker* notion of a pseudoadjunction. Pseudoadjunctions are interesting: they abound, e.g., in the study of pseudomonads. Recall from Example 8.2.6 that every pseudoadjunction gives rise to a pseudomonad. Important examples of pseudomonads arise in the theory of free cocompletions of categories, see Remark 2.1.20 (and [41]).

Instead of working with 2-categories, pseudofunctors and pseudonatural transformations and studying pseudoadjunctions in this setting, we work in the framework of **Gray**-categories, that is, categories enriched in the category $\mathscr{V} = \mathbf{Gray}$ of 2-categories and 2-functors, equipped with **Gray**-tensor product [39] as introduced in Chapter 7.

Structure of the chapter.

- The necessary background and the definitions of pseudoextensions and pseudoliftings are covered in Section 9.1.
- The proof of the formal adjoint functor theorem appears in Section 9.2.

The results of this chapter appear in the preprint [29]. The wording of the chapter is a slight modification of the text of the preprint.

9.1 Pseudoextensions and duality of Gray-categories

We first introduce the notion of a pseudoextension. It is the appropriate weakening of the usual notion of a (left) Kan extension.

Definition 9.1.1 (Left pseudoextension [35, 71]). In a **Gray**-category $\mathbf{K}$, we say that

$$\begin{tikzcd} \mathscr{X} \arrow[r, "J"] \arrow[dr, "H"'] \arrow[dr, phantom, "\Nearrow \eta" above right] & \mathscr{A} \arrow[d, "L"] \\ & \mathscr{B} \end{tikzcd}$$

exhibits L as a *left pseudoextension* of H along J if for each

$$\begin{tikzcd} \mathscr{X} \arrow[r, "J"] \arrow[dr, "H"'] \arrow[dr, phantom, "\Nearrow f" above right] & \mathscr{A} \arrow[d, "K"] \\ & \mathscr{B} \end{tikzcd}$$

(i.e., $f : H \Rightarrow KJ$) there is a 2-cell $f^\sharp : L \Rightarrow K$ and an isomorphism 3-cell

$$\begin{tikzcd} H \arrow[r, Rightarrow, "\eta"] \arrow[dr, Rightarrow, "f"'] \arrow[dr, phantom, "\Lleftarrow \mu(f)" above right] & LJ \arrow[d, Rightarrow, "f^\sharp J"] \\ & KJ \end{tikzcd}$$

such that for each $k : L \Rightarrow K$ and a 3-cell

$$\begin{tikzcd} H \arrow[r, Rightarrow, "\eta"] \arrow[dr, Rightarrow, "f"'] \arrow[dr, phantom, "\Lleftarrow \omega" above right] & LJ \arrow[d, Rightarrow, "kJ"] \\ & KJ \end{tikzcd}$$

there is a unique 3-cell $\widehat{\omega} : k \Rrightarrow f^\sharp$ such that

$$\begin{tikzcd}
H \arrow[r, Rightarrow, "\eta"] \arrow[dr, Rightarrow, "f"'] & LJ \arrow[d, Rightarrow, "f^\sharp J"'] \arrow[d, Rightarrow, bend left=90, "kJ"] \arrow[dl, phantom, "\Lleftarrow \mu(f)"] \arrow[d, phantom, shift left=25, "\overset{\Lleftarrow}{\widehat{\omega} J}"] \\
& KJ
\end{tikzcd}
=
\begin{tikzcd}
H \arrow[r, Rightarrow, "\eta"] \arrow[dr, Rightarrow, "f"'] & LJ \arrow[d, Rightarrow, "kJ"] \arrow[dl, phantom, "\Lleftarrow \omega"] \\
& KJ
\end{tikzcd}$$

We say that the pseudoextension $\eta : H \Rightarrow LJ$ is *preserved by* $G : \mathscr{B} \to \mathscr{C}$ if the 2-cell

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "J"] \arrow[dr, "GH"'] & \mathscr{A} \arrow[d, "GL"] \arrow[dl, phantom, "G\eta \Nearrow"] \\
& \mathscr{C}
\end{tikzcd}$$

exhibits GL as a left pseudoextension of GH along J.

The pseudoextension $\eta : H \Rightarrow LJ$ is said to be *absolute* if it is preserved by *any* 1-cell $G : \mathscr{B} \to \mathscr{C}$.

Remark 9.1.2. In the ordinary case a left Kan extension is a value of a certain left adjoint (i.e., a “free object”). In the case of a left pseudoextension it is a value of a left pseudoadjoint (i.e., a “pseudofree object”). See [35] for a detailed explanation of pseudoadjunctions given by pseudofree objects.

Duality of Gray-categories **Gray**-categories admit dual constructions on a **Gray**-category $\mathbf{K}$, which we introduce now.

- The *horizontal dual* $\mathbf{K}^{op}$ of $\mathbf{K}$ is defined by reversing the 1-cells of $\mathbf{K}$. That is,

  $$\mathbf{K}^{op}(\mathscr{A}, \mathscr{B}) = \mathbf{K}(\mathscr{B}, \mathscr{A}).$$

  Composition in $\mathbf{K}^{op}$ is defined by the symmetry of the **Gray**-tensor product, as is usual in the context of enriched categories.

- The *vertical dual* $\mathbf{K}^{co}$ of $\mathbf{K}$ is defined by reversing the 2-cells of $\mathbf{K}$. That is, we put

  $$\mathbf{K}^{co}(\mathscr{A}, \mathscr{B}) = (\mathbf{K}(\mathscr{A}, \mathscr{B}))^{op};$$

  observe that the 1-cells of $\mathbf{K}$ are not reversed. In this definition we use that $\mathbf{K}(\mathscr{A}, \mathscr{B})$ is a *2-category* and that we can therefore form *its* opposite.

Duality operations with the **Gray**-category $\mathbf{K}$ transform pseudoadjunctions into pseudoadjunctions. The roles of the defining data have to be swapped accordingly.

Remark 9.1.3. Suppose we are given the category $\mathbf{K}$ as in Definition 8.1.1 and the data for the pseudoadjunction $F \dashv U : \mathscr{A} \to \mathscr{X}$.

1. In $\mathbf{K}^{op}$, the same data transform into a pseudoadjunction $U \dashv F : \mathscr{X} \to \mathscr{A}$ due to the reversal of 1-cells. The unit η and counit ε stay the same, as well as the coherence 3-cells s and t.

2. In $\mathbf{K}^{co}$, the same data transform into a pseudoadjunction $U \dashv F : \mathscr{X} \to \mathscr{A}$, but with unit ε and counit η; the coherence 3-cells s and t stay the same, although their role as witnesses for the triangle isomorphisms is swapped.

The various notions of duality for **Gray**-categories also allow us to express compactly the definition of (left/right) pseudoextensions and pseudoliftings via the definition of a left pseudoextension.

Definition 9.1.4. Given a left pseudoextension

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "J"] \arrow[dr, "H"'] \arrow[dr, phantom, shift left=3, "\eta \Nearrow"] & \mathscr{A} \arrow[d, "L"] \\
 & \mathscr{B}
\end{tikzcd}$$

in a **Gray**-category $\mathbf{K}$, we call it

1. a *right* pseudoextension of H along J in $\mathbf{K}^{co}$.
2. a *left* pseudolifting of H through J in $\mathbf{K}^{op}$.
3. a *right* pseudolifting of H through J in $\mathbf{K}^{coop}$.

9.2 The formal adjoint functor theorem

Let us first recall the ordinary formal adjoint functor theorem [13].

Theorem 9.2.1. *For functors $U : \mathscr{A} \to \mathscr{X}$ and $F : \mathscr{X} \to \mathscr{A}$ the following are equivalent:*

1. *$F \dashv U$ holds with unit η.*
2. *η exhibits U as an absolute left extension of $1_{\mathscr{X}}$ along F.*
3. *η exhibits U as a left extension of $1_{\mathscr{X}}$ along F, and this extension is preserved by F.*
4. *η exhibits F as an absolute left lifting of $1_{\mathscr{X}}$ through U.*
5. *η exhibits F as a left lifting of $1_{\mathscr{X}}$ through U, and this lifting is preserved by U.*

The pseudo-version of the formal adjoint functor theorem can be stated in the same way, changing the notions of adjunction and extension/lifting to the notions of pseudoadjunction and pseudoextension/pseudolifting.

Theorem 9.2.2 (The formal adjoint functor theorem in Gray-categories). *Given two 1-cells $U : \mathscr{A} \to \mathscr{X}$ and $F : \mathscr{X} \to \mathscr{A}$ in a* **Gray***-category* $\mathbf{K}$*, the following are equivalent:*

1. *$F \dashv U$ is a pseudoadjunction with unit η.*
2. *η exhibits U as an absolute left pseudoextension of $1_{\mathscr{X}}$ along F.*

3. *η exhibits U as a left pseudoextension of $1_{\mathscr{X}}$ along F, and this extension is preserved by F.*

4. *η exhibits F as an absolute left pseudolifting of $1_{\mathscr{X}}$ through U.*

5. *η exhibits F as a left pseudolifting of $1_{\mathscr{X}}$ through U, and this lifting is preserved by U.*

The proof strategy in the ordinary case and in the pseudo-case is the same: it is enough to prove the implications 1 $\implies$ 2 and 3 $\implies$ 1. This is because 2 $\implies$ 3 is trivial, and because the equivalence of 1, 4 and 5 follows by duality. Moreover, the ordinary proofs can serve as a guidance for the proofs of the pseudo-case.

Lemma 9.2.3 (The implication 3 $\implies$ 1)**.** *Suppose that*

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "F"] \arrow[dr, "1"'] \arrow[dr, phantom, shift left=3ex, "\eta \Nearrow"] & \mathscr{A} \arrow[d, "U"] \\
 & \mathscr{X}
\end{tikzcd}$$

is a left (Kan) pseudoextension preserved by F. Then η can be made a unit of a pseudo-adjunction $F \dashv U$.

Proof. Recall that

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "F"] \arrow[dr, "1"'] \arrow[dr, phantom, shift left=3ex, "\eta \Nearrow"] & \mathscr{A} \arrow[d, "U"] \\
 & \mathscr{X}
\end{tikzcd}$$

is a left pseudoextension if for each

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "F"] \arrow[dr, "1"'] \arrow[dr, phantom, shift left=3ex, "f \Nearrow"] & \mathscr{A} \arrow[d, "K"] \\
 & \mathscr{X}
\end{tikzcd}$$

(i.e., $f : 1_X \Rightarrow KF$) there is a 2-cell $f^\sharp : U \Rightarrow K$ and an isomorphism 3-cell

$$\begin{tikzcd}
1_X \arrow[r, Rightarrow, "\eta"] \arrow[dr, Rightarrow, "f"'] \arrow[dr, phantom, shift left=3ex, "\Swarrow \mu(f)"] & UF \arrow[d, Rightarrow, "f^\sharp F"] \\
 & KF
\end{tikzcd}$$

such that for each $k : U \Rightarrow K$ and a 3-cell

$$\begin{tikzcd}
1_X \arrow[r, Rightarrow, "\eta"] \arrow[dr, Rightarrow, "f"'] \arrow[dr, phantom, "\Lleftarrow \omega", near start, shift left=3] & UF \arrow[d, Rightarrow, "kF"] \\
& KF
\end{tikzcd}$$

there is a unique 3-cell $\widehat{\omega} : k \Rrightarrow f^\sharp$ such that

$$\begin{tikzcd}
1_X \arrow[r, Rightarrow, "\eta"] \arrow[dr, Rightarrow, "f"'] \arrow[dr, phantom, "\Lleftarrow \mu(f)", near start, shift left=3] & UF \arrow[d, Rightarrow, "f^\sharp F"] \arrow[d, Rightarrow, bend left=90, looseness=2, "kF"] \arrow[d, phantom, "\Lleftarrow \widehat{\omega}F", shift left=6] \\
& KF
\end{tikzcd}
=
\begin{tikzcd}
1_X \arrow[r, Rightarrow, "\eta"] \arrow[dr, Rightarrow, "f"'] \arrow[dr, phantom, "\Lleftarrow \omega", near start, shift left=3] & UF \arrow[d, Rightarrow, "kF"] \\
& KF
\end{tikzcd}$$

For the purpose of establishing notation, we describe the data concerning the left pseudoextension

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "F"] \arrow[dr, "F"'] \arrow[dr, phantom, "F\eta \Nearrow", near start, shift left=3] & \mathscr{A} \arrow[d, "FU"] \\
& \mathscr{A}
\end{tikzcd}$$

Given, e.g., the identity

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "F"] \arrow[dr, "F"'] \arrow[dr, phantom, "1_F \Nearrow", near start, shift left=3] & \mathscr{A} \arrow[d, "1_A"] \\
& \mathscr{A}
\end{tikzcd}$$

(the 2-cell $1_F : F \Rightarrow F$), we have a 2-cell $(1_A)^\sharp : FU \Rightarrow 1_A$ that we will denote by ε and which will be the counit of the pseudoadjunction we construct. With this counit comes an isomorphism

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "1_F"'] \arrow[dr, phantom, "\Lleftarrow s^{-1}", near start, shift left=3] & FUF \arrow[d, Rightarrow, "\varepsilon F"] \\
& F
\end{tikzcd}$$

such that for each $h : FU \Rightarrow 1_A$ and a 3-cell

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "1_F"'] \arrow[dr, phantom, "\Lleftarrow \nu", near start, shift left=3] & FUF \arrow[d, Rightarrow, "hF"] \\
& F
\end{tikzcd}$$

there is a unique $\widehat{\nu} : h \Rrightarrow \varepsilon$ satisfying

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "1_F"'] \arrow[dr, phantom, shift left=4ex, "\Lleftarrow s^{-1}"] & FUF \arrow[d, Rightarrow, "\varepsilon F"'] \arrow[d, Rightarrow, bend left=90, looseness=2, "kF"] \arrow[d, phantom, shift left=5ex, "\overset{\Leftarrow}{\widehat{\nu}F}"] \\
& F
\end{tikzcd}
=
\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "1_F"'] \arrow[dr, phantom, shift left=4ex, "\Lleftarrow \nu"] & UF \arrow[d, Rightarrow, "hF"] \\
& F
\end{tikzcd}$$

Let us first observe that s^{-1} (or, equivalently, its inverse s) witnesses the first triangle axiom of a pseudoadjunction, see Definition 8.1.1. To obtain the second triangle isomorphism, consider that $\eta : 1_X \Rightarrow UF$ lifts to the identity $1_U = \eta^\sharp : U \Rightarrow U$ with the identity 3-cell ($\mu(\eta) = 1_\eta$)

$$\begin{tikzcd}
1_X \arrow[r, Rightarrow, "\eta"] \arrow[dr, Rightarrow, "\eta"'] \arrow[dr, phantom, shift left=4ex, "\Lleftarrow \mu(\eta)"] & UF \arrow[d, Rightarrow, "1_{UF}"] \\
& UF
\end{tikzcd}$$

By the universal property of the left pseudoextension given by η we get that for the 2-cell

$$\begin{tikzcd}
U \arrow[r, Rightarrow, "\eta U"] & UFU \arrow[r, Rightarrow, "U\varepsilon"] & U
\end{tikzcd}$$

and the 3-cell

$$\begin{tikzcd}
1_X \arrow[r, Rightarrow, "\eta"] \arrow[d, Rightarrow, "\eta"'] \arrow[dr, phantom, "\Lleftarrow {\eta_\eta}^{-1}"] & UF \arrow[d, Rightarrow, "\eta UF"] \\
UF \arrow[r, Rightarrow, "UF\eta"] \arrow[dr, Rightarrow, "1_{UF}"'] \arrow[dr, phantom, shift left=4ex, "\Lleftarrow Us^{-1}"] & UFUF \arrow[d, Rightarrow, "U\varepsilon F"] \\
& UF
\end{tikzcd}$$

there is a unique 3-cell

$$\begin{tikzcd}
U \arrow[r, Rightarrow, "\eta U"] \arrow[dr, Rightarrow, "1_U"'] \arrow[dr, phantom, shift left=4ex, "t \Lleftarrow"] & UFU \arrow[d, Rightarrow, "U\varepsilon"] \\
& U
\end{tikzcd}$$

such that the 3-cell

$$\begin{tikzcd}
1_X \arrow[r, Rightarrow, "\eta"] & UF \arrow[r, Rightarrow, "\eta UF"] \arrow[dr, Rightarrow, "1_{UF}"'] \arrow[dr, phantom, shift left=4ex, "tF \Lleftarrow"] & UFUF \arrow[d, Rightarrow, "U\varepsilon F"] \\
& & UF
\end{tikzcd}$$

equals

$$\begin{tikzcd}
1_X \arrow[r, Rightarrow, "\eta"] \arrow[d, Rightarrow, "\eta"'] \arrow[dr, phantom, "\Lleftarrow {\eta_\eta}^{-1}"] & UF \arrow[d, Rightarrow, "\eta UF"] \\
UF \arrow[r, Rightarrow, "UF\eta"] \arrow[dr, Rightarrow, "1_{UF}"'] \arrow[dr, phantom, shift left=3ex, "Us^{-1} \Lleftarrow"] & UFUF \arrow[d, Rightarrow, "U\varepsilon F"] \\
 & UF
\end{tikzcd}$$

or, written differently, that the 3-cell

$$\begin{tikzcd}
 & UF \arrow[dr, Rightarrow, "UF\eta"] \arrow[drr, Rightarrow, bend left, "1_{UF}"] \arrow[dr, phantom, shift left=4ex, "\Downarrow Us"] & & \\
1_X \arrow[ur, Rightarrow, "\eta"] \arrow[dr, Rightarrow, "\eta"'] \arrow[r, phantom, "\Downarrow \eta_\eta"] & {} & UFUF \arrow[r, Rightarrow, "U\varepsilon F"] & UF \\
 & UF \arrow[ur, Rightarrow, "\eta UF"] \arrow[urr, Rightarrow, bend right, "1_{UF}"'] \arrow[ur, phantom, shift right=4ex, "\Downarrow tF"] & &
\end{tikzcd} \tag{9.1}$$

is equal to identity. This is precisely the first coherence axiom for pseudoadjunctions.

For the other coherence axiom, we need the 3-cell

$$\begin{tikzcd}
 & & FU \arrow[dr, Rightarrow, "\varepsilon"] & \\
FU \arrow[urr, Rightarrow, bend left, "1_{FU}"] \arrow[r, Rightarrow, "F\eta U"] \arrow[drr, Rightarrow, bend right, "1_{FU}"'] & FUFU \arrow[ur, Rightarrow, "\varepsilon FU"] \arrow[dr, Rightarrow, "FU\varepsilon"'] \arrow[u, phantom, "\Downarrow sU"] \arrow[d, phantom, "\Downarrow Ft"] \arrow[rr, phantom, "\Downarrow \varepsilon_\varepsilon"] & & 1_A \\
 & & FU \arrow[ur, Rightarrow, "\varepsilon"'] &
\end{tikzcd}$$

to be equal to identity as well.

We shall use that for each $h : FU \Rightarrow 1_A$ and a 3-cell

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "1_F"'] \arrow[dr, phantom, shift left=3ex, "\nu \Lleftarrow"] & FUF \arrow[d, Rightarrow, "hF"] \\
 & F
\end{tikzcd}$$

there is a *unique* $\widehat{\nu} : h \Rrightarrow \varepsilon$ satisfying

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "1_F"'] \arrow[dr, phantom, shift left=3ex, "s^{-1} \Lleftarrow"] & FUF \arrow[d, Rightarrow, "\varepsilon F"'] \arrow[d, Rightarrow, bend left=90, looseness=2, "kF"] \arrow[d, phantom, shift left=4ex, "\overset{\Lleftarrow}{\widehat{\nu}F}"] \\
& F
\end{tikzcd}
=
\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "1_F"'] \arrow[dr, phantom, shift left=3ex, "\nu \Lleftarrow"] & UF \arrow[d, Rightarrow, "hF"] \\
& F
\end{tikzcd}$$

Thus if we find *two* 3-cells $\alpha, \beta : h \Rrightarrow \varepsilon$ with

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "1_F"'] \arrow[dr, phantom, shift left=3ex, "s^{-1} \Lleftarrow"] & FUF \arrow[d, Rightarrow, "\varepsilon F"'] \arrow[d, Rightarrow, bend left=90, looseness=2, "kF"] \arrow[d, phantom, shift left=4ex, "\overset{\Lleftarrow}{\alpha F}"] \\
& F
\end{tikzcd}
=
\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "1_F"'] \arrow[dr, phantom, shift left=3ex, "s^{-1} \Lleftarrow"] & FUF \arrow[d, Rightarrow, "\varepsilon F"'] \arrow[d, Rightarrow, bend left=90, looseness=2, "kF"] \arrow[d, phantom, shift left=4ex, "\overset{\Lleftarrow}{\beta F}"] \\
& F
\end{tikzcd}$$

it means that $\alpha = \beta$. Take now the 3-cell

$$\begin{tikzcd}
FU \arrow[r, Rightarrow, "F\eta U"] \arrow[dr, Rightarrow, "1_{FU}"'] \arrow[dr, phantom, shift left=3ex, "Ft \Lleftarrow"] & FUFU \arrow[d, Rightarrow, "FU\varepsilon"] \\
& FU \arrow[r, Rightarrow, "\varepsilon"] & 1_A
\end{tikzcd}$$

for α and the 3-cell

$$\begin{tikzcd}
FU \arrow[r, Rightarrow, "F\eta U"] \arrow[dr, Rightarrow, "1_{FU}"'] \arrow[dr, phantom, shift left=3ex, "s^{-1}U \Lleftarrow"] & FUFU \arrow[d, Rightarrow, "\varepsilon FU"] \arrow[r, Rightarrow, "FU\varepsilon"] \arrow[dr, phantom, "\Lleftarrow \varepsilon_{\varepsilon}{}^{-1}"] & FU \arrow[d, Rightarrow, "\varepsilon"] \\
& FU \arrow[r, Rightarrow, "\varepsilon"'] & 1_A
\end{tikzcd}$$

for β. We ask whether the 3-cells

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[ddrr, Rightarrow, "1_F"'] & FUF \arrow[r, Rightarrow, "F\eta UF"] \arrow[dr, Rightarrow, "1_{FUF}"'] \arrow[dr, phantom, shift left=3ex, "FtF \Lleftarrow"] & FUFUF \arrow[d, Rightarrow, "FU\varepsilon F"] \\
& \Lleftarrow s^{-1} & FUF \arrow[d, Rightarrow, "\varepsilon F"] \\
& & F
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & FUF \arrow[r, Rightarrow, "F\eta UF"] \arrow[dr, Rightarrow, "1_{FUF}"'] \arrow[r, phantom, shift right=6, "s^{-1}UF \Swarrow"] & FUFUF \arrow[r, Rightarrow, "FU\varepsilon F"] \arrow[d, Rightarrow, "\varepsilon FUF"'] \arrow[dr, phantom, "\Swarrow \varepsilon_{\varepsilon F}{}^{-1}"] & FUF \arrow[d, Rightarrow, "\varepsilon F"] \\
F \arrow[ur, Rightarrow, "F\eta"] \arrow[rrr, Rightarrow, bend right, "1_F"'] \arrow[r, phantom, "\Swarrow s^{-1}"] & & FUF \arrow[r, Rightarrow, "\varepsilon F"'] & F
\end{tikzcd}$$

are equal. Pasting s and $F\eta_\eta$, we can equivalently ask whether the 3-cells

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[d, Rightarrow, "F\eta"'] \arrow[dr, phantom, "\Swarrow F\eta_\eta"] & FUF \arrow[d, Rightarrow, "FUF\eta"] & \\
FUF \arrow[r, Rightarrow, "F\eta UF"] \arrow[dr, Rightarrow, "1_{FUF}"'] \arrow[r, phantom, shift right=6, "FtF \Swarrow"] & FUFUF \arrow[d, Rightarrow, "FU\varepsilon F"] & \\
 & FUF \arrow[r, Rightarrow, "\varepsilon F"] & F
\end{tikzcd}$$

(9.2)

and

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[d, Rightarrow, "F\eta"'] \arrow[dr, phantom, "\Swarrow F\eta_\eta"] & FUF \arrow[d, Rightarrow, "FUF\eta"] & \\
FUF \arrow[r, Rightarrow, "F\eta UF"] \arrow[dr, Rightarrow, "1_{FUF}"'] \arrow[r, phantom, shift right=6, "s^{-1}UF \Swarrow"] & FUFUF \arrow[r, Rightarrow, "FU\varepsilon F"] \arrow[d, Rightarrow, "\varepsilon FUF"] \arrow[dr, phantom, "\Swarrow \varepsilon_{\varepsilon F}{}^{-1}"] & FUF \arrow[d, Rightarrow, "\varepsilon F"] \\
 & FUF \arrow[r, Rightarrow, "\varepsilon F"'] & F
\end{tikzcd}$$

(9.3)

are equal. Using the first coherence axiom (9.1), the diagram (9.2) is equal to

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] & FUF \arrow[r, Rightarrow, "FUF\eta"] \arrow[dr, Rightarrow, "1_{FUF}"'] \arrow[r, phantom, shift right=6, "\Swarrow FUs^{-1}"] & FUFUF \arrow[d, Rightarrow, "FU\varepsilon F"] & \\
 & & FUF \arrow[r, Rightarrow, "\varepsilon F"] & F
\end{tikzcd}$$

(9.4)

Let us take diagrams (9.3) and (9.4) and paste $\varepsilon_{\varepsilon F}$ and $\varepsilon_{F\eta}$. The resulting diagrams

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[d, Rightarrow, "F\eta"'] \arrow[dr, phantom, "\Swarrow F\eta_\eta"] & FUF \arrow[d, Rightarrow, "FUF\eta"'] \arrow[dr, Rightarrow, "\varepsilon F"] & \\
FUF \arrow[r, Rightarrow, "F\eta UF"] \arrow[dr, Rightarrow, "1_{FUF}"'] \arrow[dr, phantom, bend left, "\Swarrow s^{-1}UF"] & FUFUF \arrow[d, Rightarrow, "\varepsilon FUF"] \arrow[r, phantom, "\Swarrow \varepsilon_{F\eta}"] & F \arrow[dl, Rightarrow, "F\eta"] \\
& FUF \arrow[r, Rightarrow, "\varepsilon F"'] & F
\end{tikzcd} \tag{9.5}$$

and

$$\begin{tikzcd}
& & F \arrow[r, Rightarrow, "F\eta"] \arrow[d, phantom, "\Downarrow \varepsilon_{F\eta}"] & FUF \arrow[dr, Rightarrow, "\varepsilon F"] \arrow[d, phantom, "\Downarrow \varepsilon_{\varepsilon F}"] & \\
F \arrow[r, Rightarrow, "F\eta"] & FUF \arrow[ur, Rightarrow, "\varepsilon F"] \arrow[r, Rightarrow, "F\eta UF"] \arrow[rr, Rightarrow, bend right=60, "1_{FUF}"'] \arrow[rr, phantom, bend right=30, "\Downarrow FUs^{-1}"] & FUFUF \arrow[ur, Rightarrow, "\varepsilon FUF"] \arrow[r, Rightarrow, "FU\varepsilon F"] & FUF \arrow[r, Rightarrow, "\varepsilon F"] & F
\end{tikzcd} \tag{9.6}$$

are equal by using the identities

$$\begin{tikzcd}
FUF \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[dd, Rightarrow, bend right=90] \arrow[dr, phantom, "\overset{\varepsilon_{F\eta}}{\Swarrow}"] & F \arrow[d, Rightarrow] \\
FUFUF \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[l, phantom, "\overset{FUs^{-1}}{\Swarrow}"] \arrow[dr, phantom, "\overset{\varepsilon_{\varepsilon F}}{\Swarrow}"] & FUF \arrow[d, Rightarrow] \\
FUF \arrow[r, Rightarrow] & F
\end{tikzcd}
=
\begin{tikzcd}
FUF \arrow[d, Rightarrow, "\varepsilon F"] & \\
F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "1_F"'] \arrow[dr, phantom, bend left, "s^{-1} \Swarrow"] & FUF \arrow[d, Rightarrow, "\varepsilon F"] \\
& F
\end{tikzcd}$$

and

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[d, Rightarrow, "F\eta"'] \arrow[dr, phantom, "\overset{F\eta_\eta}{\Swarrow}"] & FUF \arrow[r, Rightarrow, "\varepsilon F"] \arrow[d, Rightarrow, "FUF\eta"] \arrow[dr, phantom, "\overset{\varepsilon_{F\eta}}{\Swarrow}"] & F \arrow[d, Rightarrow, "F\eta"] \\
FUF \arrow[r, Rightarrow, "F\eta UF"'] \arrow[rr, Rightarrow, bend right=60, "1_{FUF}"'] \arrow[rr, phantom, bend right=30, "\Downarrow s^{-1}UF"] & FUFUF \arrow[r, Rightarrow, "\varepsilon FUF"'] & FUF
\end{tikzcd}
=
\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "1_F"'] \arrow[dr, phantom, bend left, "s^{-1} \Swarrow"] & FUF \arrow[d, Rightarrow, "\varepsilon F"] \\
& F \arrow[d, Rightarrow, "F\eta"] \\
& FUF
\end{tikzcd}$$

The proof is therefore finished. ■

Lemma 9.2.4 (The implication 1 $\implies$ 2)**.** *Suppose that $F \dashv U : \mathscr{A} \to \mathscr{X}$ is a pseudoadjunction in a* **Gray***-category* **K** *with*

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "F"] \arrow[dr, "1"'] & \mathscr{A} \arrow[d, "U"] \arrow[dl, phantom, "\Nearrow \eta"] & \mathscr{A} \arrow[d, "U"'] \arrow[dr, "1_A"] \arrow[dr, phantom, "\Nearrow \varepsilon"'] & & F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "1_F"'] & FUF \arrow[d, Rightarrow, "\varepsilon F"] \arrow[dl, phantom, "s \Rrightarrow"] & U \arrow[r, Rightarrow, "\eta U"] \arrow[dr, Rightarrow, "1_U"'] & UFU \arrow[d, Rightarrow, "U\varepsilon"] \arrow[dl, phantom, "t \Lleftarrow"] \\
& \mathscr{X} & \mathscr{X} \arrow[r, "F"'] & \mathscr{A} & & F & & U
\end{tikzcd}$$

Then

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "F"] \arrow[dr, "1"'] & \mathscr{A} \arrow[d, "U"] \arrow[dl, phantom, "\Nearrow \eta"] \\
& \mathscr{X}
\end{tikzcd}$$

is an absolute left pseudolifting.

Proof. We need to show that for each $G : \mathscr{X} \to \mathscr{Y}$ and $g : G \Rightarrow KF$ there is a 2-cell $g^\sharp : GU \Rightarrow K$ and an isomorphism

$$\begin{tikzcd}
G \arrow[r, Rightarrow, "G\eta"] \arrow[dr, Rightarrow, "G"'] & GUF \arrow[d, Rightarrow, "g^\sharp F"] \arrow[dl, phantom, "\Lleftarrow \mu(g)"] \\
& KF
\end{tikzcd}$$

satisfying that for each $k : GU \Rightarrow K$ and

$$\begin{tikzcd}
G \arrow[r, Rightarrow, "G\eta"] \arrow[dr, Rightarrow, "g"'] & GUF \arrow[d, Rightarrow, "kF"] \arrow[dl, phantom, "\omega \Lleftarrow"] \\
& KF
\end{tikzcd}$$

there is a unique 3-cell $\widehat{\omega} : k \Rrightarrow g^\sharp$ such that

$$\begin{tikzcd}
G \arrow[r, Rightarrow, "G\eta"] \arrow[dr, Rightarrow, "g"'] & GUF \arrow[d, Rightarrow, "g^\sharp F"] \arrow[dl, phantom, "\Lleftarrow \mu(g)"] \arrow[d, Rightarrow, bend left=90, looseness=2, "kF"] \arrow[d, phantom, bend left=45, "\overset{\Lleftarrow}{\widehat{\omega} F}"] & = & G \arrow[r, Rightarrow, "G\eta"] \arrow[dr, Rightarrow, "g"'] & GUF \arrow[d, Rightarrow, "kF"] \arrow[dl, phantom, "\omega \Lleftarrow"] \\
& KF & & & KF
\end{tikzcd}$$

We shall define $g^\sharp : GU \Rightarrow K$ as the 2-cell

$$GU \xRightarrow{gU} KFU \xRightarrow{K\varepsilon} K$$

Then, for the 2-cell $g : G \Rightarrow KF$ we define $\mu(g)$ to be the 3-cell

$$\begin{tikzcd}
G \arrow[r, Rightarrow, "G\eta"] \arrow[d, Rightarrow, "g"'] \arrow[dr, phantom, "{g_\eta}^{-1} \Lleftarrow"] & GUF \arrow[d, Rightarrow, "gUF"] \\
KF \arrow[r, Rightarrow, "KF\eta"] \arrow[dr, Rightarrow, "1_{KF}"'] & KFUF \arrow[d, Rightarrow, "K\varepsilon F"] \arrow[dl, phantom, "\Lleftarrow Ks^{-1}"] \\
 & KF
\end{tikzcd}$$

Now given a 3-cell

$$\begin{tikzcd}
G \arrow[r, Rightarrow, "G\eta"] \arrow[dr, Rightarrow, "g"'] & GUF \arrow[d, Rightarrow, "kF"] \arrow[dl, phantom, "\omega \Lleftarrow"] \\
 & KF
\end{tikzcd}$$

we will show that the “lifted” 3-cell $\widehat{\omega} : k \Rrightarrow g^\sharp$ is the 3-cell

$$\begin{tikzcd}
GU \arrow[r, Rightarrow, "1_{GU}"] \arrow[d, Rightarrow, "gU"'] \arrow[dr, phantom, "\Lleftarrow \omega U"] & GU \arrow[d, Rightarrow, "G\eta U"] \arrow[dd, Rightarrow, bend left=80, "1_{GU}"] & {} \\
KFU \arrow[d, Rightarrow, "K\varepsilon"'] \arrow[dr, phantom, "\Lleftarrow {k_\varepsilon}^{-1}"] & GUFU \arrow[l, Rightarrow, "kFU"] \arrow[d, Rightarrow, "GU\varepsilon"] \arrow[r, phantom, "Gt^{-1} \Leftarrow"] & {} \\
K & GU \arrow[l, Rightarrow, "k"] & {}
\end{tikzcd}$$

Indeed, observe that the 3-cell

$$\begin{tikzcd}
G \arrow[r, Rightarrow, "G\eta"] \arrow[dr, Rightarrow, "g"'] & GUF \arrow[d, Rightarrow, "g^\sharp F"] \arrow[d, Rightarrow, bend left=80, "kF"] \arrow[dl, phantom, "\Lleftarrow \mu(g)"] \\
 & KF
\end{tikzcd}
\qquad \Leftarrow \widehat{\omega}F$$

is the composite

$$\begin{tikzcd}
G \arrow[r, Rightarrow, "G\eta"] \arrow[d, Rightarrow, "g"'] \arrow[dr, phantom, "{g_\eta}^{-1} \Lleftarrow"] & GUF \arrow[r, Rightarrow, "1_{GUF}"] \arrow[d, Rightarrow, "gUF"] \arrow[dr, phantom, "\omega UF \Leftarrow"] & GUF \arrow[d, Rightarrow, "G\eta UF"] \arrow[dd, Rightarrow, bend left=80, "1_{GUF}"] & {} \\
KF \arrow[r, Rightarrow, "KF\eta"] \arrow[dr, Rightarrow, "1_{KF}"'] & KFUF \arrow[d, Rightarrow, "K\varepsilon F"] \arrow[dl, phantom, "\Lleftarrow Ks^{-1}"] \arrow[dr, phantom, "\Lleftarrow {k_{\varepsilon F}}^{-1}"] & GUFUF \arrow[l, Rightarrow, "kFUF"] \arrow[d, Rightarrow, "GU\varepsilon F"] \arrow[r, phantom, "GtF^{-1} \Leftarrow"] & {} \\
 & KF & GUF \arrow[l, Rightarrow, "kF"] & {}
\end{tikzcd}$$

which is equal to the 3-cell

$$\begin{tikzcd}
 & & G \arrow[dll, Rightarrow, "g"'] \arrow[dl, Rightarrow, "G\eta"] \arrow[dd, Rightarrow, "G\eta"] \arrow[dll, phantom, bend left=10, "\overset{\omega}{\Lleftarrow}"] \\
KF \arrow[d, Rightarrow, "KF\eta"'] \arrow[dd, Rightarrow, bend right=80, "1_{KF}"'] \arrow[dd, phantom, bend right=40, "\overset{Ks^{-1}}{\Lleftarrow}"] \arrow[dr, phantom, "\Lleftarrow k_{F\eta}{}^{-1}"] & GUF \arrow[l, Rightarrow, "kF"] \arrow[d, Rightarrow, "GUF\eta"] \arrow[dr, phantom, "\Lleftarrow G\eta_\eta{}^{-1}"] & \\
KFUF \arrow[d, Rightarrow, "K\varepsilon F"'] \arrow[dr, phantom, "\Lleftarrow k_{\varepsilon F}{}^{-1}"] & GUFUF \arrow[l, Rightarrow, "kFUF"] \arrow[d, Rightarrow, "GU\varepsilon F"] \arrow[r, phantom, shift right=4, "\Lleftarrow GtF^{-1}"] & GUF \arrow[l, Rightarrow, "G\eta UF"'] \arrow[dl, Rightarrow, "1_{GUF}"] \\
KF & GUF \arrow[l, Rightarrow, "kF"] &
\end{tikzcd}$$

and, transforming Ks^{-1} to GUs^{-1}, the latter is equal to

$$\begin{tikzcd}
G \arrow[rrr, Rightarrow, "G\eta"] \arrow[dr, Rightarrow, "G\eta"] \arrow[dddrrr, Rightarrow, bend right=40, "g"'] & & & GUF \arrow[dl, Rightarrow, "G\eta UF"] \arrow[ddd, Rightarrow, "1_{GUF}"] \arrow[dll, phantom, "\overset{G\eta_\eta{}^{-1}}{\Downarrow}"] \\
 & GUF \arrow[r, Rightarrow, "GUF\eta"] \arrow[dr, Rightarrow, "1_{GUF}"'] \arrow[r, phantom, shift right=4, "\overset{GUs^{-1}}{\Downarrow}"] & GUFUF \arrow[d, Rightarrow, "GU\varepsilon F"] \arrow[dr, phantom, "\overset{GtF^{-1}}{\Downarrow}"] & \\
 & \overset{\omega}{\Downarrow} & GUF \arrow[dr, Rightarrow, "kF"] & \\
 & & & KF
\end{tikzcd}$$

The above diagram simplifies to ω, showing that our choice of $\widehat{\omega}$ was correct. Indeed, the choice of $\widehat{\omega}$ is even the only possible one: each diagram in the following series is equal to $\widehat{\omega}$.

$$\begin{tikzcd}
GU \arrow[r, Rightarrow, bend left, "k"] \arrow[r, Rightarrow, bend right, "g^\sharp"'] \arrow[r, phantom, "\widehat{\omega}\Downarrow"] & K
\end{tikzcd}$$ (9.7)

$$\begin{tikzcd}
 & \Downarrow Gt^{-1} & & \\
 & GUFU \arrow[rd, Rightarrow, "GU\varepsilon"] \arrow[d, phantom, "\Downarrow Gt"] & & \\
GU \arrow[ru, Rightarrow, "G\eta U"] \arrow[rr, Rightarrow, "1_{GU}"'] \arrow[rr, Rightarrow, bend left=90, "1_{GU}"] \arrow[d, Rightarrow, "gU"'] \arrow[rrd, phantom, "\Swarrow 1_{gU}"] & {} & GU \arrow[r, Rightarrow, "k"] \arrow[d, Rightarrow, "gU"'] \arrow[rd, phantom, "\overset{\widehat{\omega}}{\Searrow}"] & K \\
KFU \arrow[rr, Rightarrow, "1_{KFU}"'] & & KFU \arrow[ru, Rightarrow, "K\varepsilon"'] & {}
\end{tikzcd} \tag{9.8}$$

$$\begin{tikzcd}
 & \Downarrow Gt^{-1} & & \\
 & GUFU \arrow[rd, Rightarrow, "GU\varepsilon"] \arrow[d, Rightarrow, "gUFU"] & & \\
GU \arrow[ru, Rightarrow, "G\eta U"] \arrow[rr, Rightarrow, bend left=90, "1_{GU}"] \arrow[d, Rightarrow, "gU"'] \arrow[r, phantom, "\Swarrow {g_{\eta U}}^{-1}"] & KFUFU \arrow[rd, Rightarrow, "KFU\varepsilon"] \arrow[d, phantom, "\Downarrow KFt"] \arrow[r, phantom, "\Swarrow {g_{U\varepsilon}}^{-1}"] & GU \arrow[r, Rightarrow, "k"] \arrow[d, Rightarrow, "gU"'] \arrow[rd, phantom, "\overset{\widehat{\omega}}{\Searrow}"] & K \\
KFU \arrow[ru, Rightarrow, "KF\eta U"] \arrow[rr, Rightarrow, "1_{KFU}"'] & {} & KFU \arrow[ru, Rightarrow, "K\varepsilon"'] & {}
\end{tikzcd} \tag{9.9}$$

$$\begin{tikzcd}
 & \Downarrow Gt^{-1} & & \\
 & GUFU \arrow[r, Rightarrow, "GU\varepsilon"] \arrow[rd, Rightarrow, "kFU"] \arrow[d, Rightarrow, "gUFU"'] & GU \arrow[rd, Rightarrow, "k"] \arrow[d, phantom, "\Downarrow {k_{\varepsilon}}^{-1}"] & \\
GU \arrow[ru, Rightarrow, "G\eta U"] \arrow[rru, Rightarrow, bend left=60, "1_{GU}"] \arrow[d, Rightarrow, "gU"'] \arrow[r, phantom, "\Swarrow {g_{\eta U}}^{-1}"] & KFUFU \arrow[r, Rightarrow, "K\varepsilon FU"'] \arrow[rd, Rightarrow, "KFU\varepsilon"] \arrow[d, phantom, "\Downarrow KFt"] \arrow[ru, phantom, "\overset{\widehat{\omega}FU}{\Swarrow}"] & KFU \arrow[r, Rightarrow, "K\varepsilon"'] \arrow[d, phantom, "\Downarrow K\varepsilon_{\varepsilon}"] & K \\
KFU \arrow[ru, Rightarrow, "KF\eta U"] \arrow[rr, Rightarrow, "1_{KFU}"'] & {} & KFU \arrow[ru, Rightarrow, "K\varepsilon"'] & {}
\end{tikzcd} \tag{9.10}$$

$$\begin{tikzcd}
 & GUFU \arrow[r, Rightarrow, "GU\varepsilon"] \arrow[d, Rightarrow, "gUFU"'] \arrow[dr, Rightarrow, "kFU"] \arrow[dr, phantom, shift right=4, "\widehat{\omega}FU \Swarrow"] & GU \arrow[dr, Rightarrow, "k"] \arrow[d, phantom, "\Downarrow {k_\varepsilon}^{-1}"] & \\
GU \arrow[ur, Rightarrow, "G\eta U"] \arrow[d, Rightarrow, "gU"'] \arrow[urr, Rightarrow, bend left=60, "1_{GU}"] \arrow[urr, phantom, bend left=30, "\Downarrow Gt^{-1}"] \arrow[r, phantom, "\Swarrow {g_{\eta U}}^{-1}"] & KFUFU \arrow[r, Rightarrow, "K\varepsilon FU"'] & KFU \arrow[r, Rightarrow, "K\varepsilon"'] & K \\
KFU \arrow[ur, Rightarrow, "KF\eta U"] \arrow[urrr, Rightarrow, bend right, "K\varepsilon"'] \arrow[urr, phantom, "\Downarrow KsU^{-1}"] & & &
\end{tikzcd} \tag{9.11}$$

$$\begin{tikzcd}
GU \arrow[r, Rightarrow, "1_{GU}"] \arrow[d, Rightarrow, "gU"'] \arrow[dr, phantom, "\overset{\omega U}{\Leftarrow}"] & GU \arrow[d, Rightarrow, "G\eta U"] \arrow[dd, Rightarrow, bend left=90, "1_{GU}"] & \\
KFU \arrow[d, Rightarrow, "K\varepsilon"'] & GUFU \arrow[l, Rightarrow, "kFU"] \arrow[d, Rightarrow, "GU\varepsilon"] \arrow[dl, phantom, "\overset{{k_\varepsilon}^{-1}}{\Nwarrow}"] & \overset{Gt^{-1}}{\Leftarrow} \\
K & GU \arrow[l, Rightarrow, "k"] &
\end{tikzcd} \tag{9.12}$$

The proof is therefore complete. ■

Remark 9.2.5. Having proved the implications $1 \implies 2$ and $3 \implies 1$, the proof of Theorem 9.2.2 is complete.

Remark 9.2.6. An interesting area for further work would be to generalise the formal adjoint functor theorem to a *relative* formal adjoint functor theorem in the sense of [80]. More in detail: a functor $F : \mathscr{Y} \to \mathscr{A}$ is a left adjoint of U, relative to J ($F \dashv_J U$), if there is an absolute left lifting

$$\begin{tikzcd}
\mathscr{Y} \arrow[r, dotted, "F"] \arrow[dr, "J"'] \arrow[dr, phantom, shift left=3, "\Nearrow"] & \mathscr{A} \arrow[d, "U"] \\
 & \mathscr{X} .
\end{tikzcd}$$

The statement of the formal relative adjoint functor theorem in **Gray**-categories requires, however, the concept of *pointwise* pseudoextensions. We defer this to future work.

Chapter 10

KZ-pseudoadjunctions and KZ-pseudomonads

In this chapter we study the properties of a special class of pseudoadjunctions and pseudomonads: the *Kock-Zöberlein pseudoadjunctions and pseudomonads* [50, 82]. KZ-pseudomonads capture formally the essence of colimit cocompletions of categories. Recall from Remark 2.1.20 the non-strict behaviour of the cocompletion process: the forgetful 2-functor

$$U_\Phi : \Phi\text{-}\mathsf{COCTS} \to \mathscr{V}\text{-}\mathsf{CAT}$$

from the 2-category of Φ-cocomplete categories, Φ-cocontinuous functors and all natural transformations admits only a left *pseudoadjoint*, giving rise to a *pseudomonad*. In fact, we get a KZ-pseudoadjunction and a KZ-pseudomonad as defined in Definition 8.1.3 and Definition 8.1.4. By abstraction, KZ-pseudomonads and KZ-pseudoadjunctions thus give us a way to study "colimit-like cocompletions" in a more general setting.

What is the interplay between KZ-pseudoadjunctions and KZ-pseudomonads? When a pseudoadjunction gives rise to a KZ-pseudomonad, is it already a KZ-pseudoadjunction? We shall study the interplay and see that the latter question has a positive answer.

Is the definition of a KZ-pseudoadjunction from literature "minimal" or does it contain any redundancy? KZ-pseudoadjunctions are defined as those pseudoadjunctions whose coherence data form (ordinary) adjunctions. We will show that only half of the usual requirements are sufficient.

What special properties do pseudoalgebras for a KZ-pseudomonad have? In the (stricter) 2-categorical setting Kelly and Lack have shown in [46] that KZ-pseudomonads are *property-like* in a technical sense; we will show that the same characterisation is possible in our weaker setting.

Structure of the chapter.

- We introduce the notion of a pseudoalgebra in Section 10.1 and give a short account of the construction of the **Gray**-category of pseudoalgebras for a pseudomonad, and of the Eilenberg-Moore object in a **Gray**-category.

- In Section 10.2 we show that the definition of a KZ-pseudoadjunction can be weakened and that a pseudoadjunction giving rise to a KZ-pseudomonad is a KZ-pseudoadjunction itself. In the proof we use some of the characterisations of KZ-pseudomonads from Marmolejo [67].

- In Section 10.3 we give a characterisation of KZ-pseudomonads as property-like pseudomonads in the spirit of [46].

The results of this chapter are the "obvious weakenings" of the relevant classical results, but they have not, to the best knowledge of the author, appeared in the literature. The statement and proof of Theorem 10.2.3 were inspired by the nLab entry [74] dealing with lax-idempotent 2-monads. (Some authors study KZ-pseudomonads under the name of lax-idempotent pseudomonads.) Section 10.1 is a review section that draws from [68, 54].

10.1 Pseudoalgebras

Given a pseudomonad T on an object $\mathscr{X}$ of $\mathbf{K}$, we want to define the Eilenberg-Moore object $\mathscr{X}^T$ of algebras for T. There are essentially two problems:

1. The algebraic structure $a : TX \to X$ cannot be a "1-cell in $\mathscr{X}$" since there need not be any 1-cells in $\mathscr{X}$. (Compare to the fact that an object of an abstract 2-category might not have a "categorical" internal structure.)

2. The object $\mathscr{X}^T$ of $\mathbf{K}$ need not exist: we will see that the existence of $\mathscr{X}^T$ is a mild completeness side condition that needs to be imposed on $\mathbf{K}$.

The above problems can be remedied in the usual way:

1. We introduce algebras as 2-cells $a : TX \Rightarrow X$, where $X : \mathscr{W} \to \mathscr{X}$ is a generalised element of "shape $\mathscr{W}$". In fact, this process yields a pseudomonad $\mathbf{K}(\mathscr{W}, T)$ on the 2-category $\mathbf{K}(\mathscr{W}, \mathscr{X})$ and we define algebras (carried by elements of $\mathscr{X}$ of shape $\mathscr{W}$) as the 2-category $\mathbf{K}(\mathscr{W}, \mathscr{X})^{\mathbf{K}(\mathscr{W},T)}$ of Eilenberg-Moore algebras for the monad $\mathbf{K}(\mathscr{W}, T)$ and their pseudohomomorphisms.

2. The above process is functorial in $\mathscr{W}$, i.e., we obtain a **Gray**-functor

   $$\mathbf{K}(-, \mathscr{X})^{\mathbf{K}(-,T)} : \mathbf{K}^{op} \to \mathbf{Gray}$$

   and the Eilenberg-Moore object $\mathscr{X}^T$ is an object that represents the above functor. That is, we have $\mathbf{K}(\mathscr{W}, \mathscr{X}^T) \cong \mathbf{K}(\mathscr{W}, \mathscr{X})^{\mathbf{K}(\mathscr{W},T)}$ naturally in $\mathscr{W}$.

   Since representability of a functor is a limit condition, the existence of $\mathscr{X}^T$ in $\mathbf{K}$ will amount to the existence of a certain limit in $\mathbf{K}$, as we show later.

Fix now a pseudomonad T on an object $\mathscr{X}$ of $\mathbf{K}$, and an object $\mathscr{W}$ of $\mathbf{K}$. We will first define a 2-category $\mathbf{K}(\mathscr{W}, \mathscr{X})^{\mathbf{K}(\mathscr{X},T)}$ of pseudoalgebras with "carrier 1-cells" $\mathscr{W} \to \mathscr{X}$, all pseudohomomorphisms and all homomorphism 3-cells.

Definition 10.1.1. A *pseudoalgebra* for a pseudomonad (T, η, μ) is a tuple (X, a, a_0, a_1) with carrier $X : \mathscr{W} \to \mathscr{X}$, structure 2-cell

$$TX \xRightarrow{a} X$$

and two invertible 3-cells

$$\begin{tikzcd}
X \arrow[r, Rightarrow, "\eta X"] \arrow[dr, Rightarrow, "1"'] & TX \arrow[d, Rightarrow, "a"] \arrow[dl, phantom, "\Lleftarrow a_0"] & , & TTX \arrow[r, Rightarrow, "Ta"] \arrow[d, Rightarrow, "\mu X"'] \arrow[dr, phantom, "\Lleftarrow a_1"] & TX \arrow[d, Rightarrow, "a"] \\
 & X & & TX \arrow[r, Rightarrow, "a"'] & X
\end{tikzcd}$$

that are subject to the following axioms:

$$\begin{tikzcd}
 & & TX \arrow[dr, Rightarrow, "a"] & \\
TX \arrow[r, Rightarrow, "T\eta X"] & TTX \arrow[ur, Rightarrow, "Ta"] \arrow[dr, Rightarrow, "\mu X"'] & \Downarrow a_1 & X \\
 & & TX \arrow[ur, Rightarrow, "a"'] &
\end{tikzcd}$$

is equal to

$$\begin{tikzcd}
 & TTX \arrow[dr, Rightarrow, "Ta"] \arrow[d, phantom, "\Downarrow Ta_0"] & & \\
TX \arrow[ur, Rightarrow, "T\eta X"] \arrow[rr, Rightarrow, "1"] \arrow[dr, Rightarrow, "T\eta X"'] & {} & TX \arrow[r, Rightarrow, "a"] & X \\
 & TTX \arrow[ur, Rightarrow, "\mu X"'] \arrow[u, phantom, "\Downarrow sX"] & &
\end{tikzcd}$$

and the 3-cell

$$\begin{tikzcd}
 & TTX \arrow[r, Rightarrow, "Ta"] \arrow[d, phantom, "\Downarrow Ta_1"] & TX \arrow[dr, Rightarrow, "a"] & \\
TTTX \arrow[ur, Rightarrow, "TTa"] \arrow[r, Rightarrow, "T\mu X"] \arrow[dr, Rightarrow, "\mu TX"'] & TTX \arrow[ur, Rightarrow, "Ta"] \arrow[dr, Rightarrow, "\mu X"'] & \Downarrow a_1 & X \\
 & TTX \arrow[r, Rightarrow, "\mu X"'] \arrow[u, phantom, "\Downarrow aX"] & TX \arrow[ur, Rightarrow, "a"'] &
\end{tikzcd}$$

is equal to

$$\begin{tikzcd}
 & TTX \arrow[r, Rightarrow, "Ta"] \arrow[dr, Rightarrow, "\mu X"] & TX \arrow[dr, Rightarrow, "a"] \arrow[d, phantom, "\Downarrow a_1"] & \\
TTTX \arrow[ur, Rightarrow, "TTa"] \arrow[dr, Rightarrow, "\mu TX"'] \arrow[r, phantom, "\Downarrow {\mu_a}^{-1}"] & & TX \arrow[r, Rightarrow, "a"] \arrow[d, phantom, "\Downarrow a_1"] & X \\
 & TTX \arrow[ur, Rightarrow, "Ta"'] \arrow[r, Rightarrow, "\mu X"'] & TX \arrow[ur, Rightarrow, "a"'] &
\end{tikzcd}$$

We will now give the definitions of a pseudohomomorphism (and lax homomorphism) between two pseudoalgebras, and of a homomorphism 3-cell (between two (lax) homomorphisms).

Definition 10.1.2. Given two pseudoalgebras (X, a, a_0, a_1) and (Y, b, b_0, b_1) with carriers $X, Y : \mathscr{W} \to \mathscr{X}$, a pair (h, ρ) consisting of a 2-cell $h : X \Rightarrow Y$ and a 3-cell

$$\begin{tikzcd}
TX \arrow[r, Rightarrow, "Th"] \arrow[d, Rightarrow, "a"'] \arrow[dr, phantom, "\Swarrow \rho"] & TY \arrow[d, Rightarrow, "b"] \\
X \arrow[r, Rightarrow, "h"'] & Y
\end{tikzcd}$$

is a *lax homomorphism* if the 3-cell

$$\begin{tikzcd}
 & & TY \arrow[dr, Rightarrow, "b"] & \\
X \arrow[r, Rightarrow, "\eta X"] \arrow[drr, Rightarrow, bend right, "1"'] & TX \arrow[ur, Rightarrow, "Th"] \arrow[dr, Rightarrow, "a"] \arrow[d, phantom, "\Downarrow a_0"] & \Downarrow \rho & Y \\
 & {} & X \arrow[ur, Rightarrow, "h"'] &
\end{tikzcd}$$

is equal to the 3-cell

$$\begin{tikzcd}
 & TX \arrow[dr, Rightarrow, "Th"] & & \\
X \arrow[ur, Rightarrow, "\eta X"] \arrow[dr, Rightarrow, "h"'] \arrow[rr, phantom, "\Downarrow \eta_h"] & & TY \arrow[r, Rightarrow, "b"] & Y \\
 & Y \arrow[ur, Rightarrow, "\eta Y"'] \arrow[urr, Rightarrow, bend right, "1"'] & \Downarrow b_0 &
\end{tikzcd}$$

and the 3-cell

$$\begin{tikzcd}
 & TTX \arrow[r, Rightarrow, "TTh"] \arrow[dl, Rightarrow, "\mu X"'] \arrow[d, phantom, "\Swarrow {\mu_h}^{-1}"] & TTY \arrow[dl, Rightarrow, "\mu Y"] \arrow[dr, Rightarrow, "Tb"] & \\
TX \arrow[r, Rightarrow, "Th"] \arrow[dr, Rightarrow, "a"'] & TY \arrow[dr, Rightarrow, "b"] \arrow[d, phantom, "\Downarrow \rho"] & \Swarrow b_1 & TY \arrow[dl, Rightarrow, "b"] \\
 & X \arrow[r, Rightarrow, "h"'] & Y &
\end{tikzcd}$$

is equal to the 3-cell

$$\begin{tikzcd}
 & TTX \arrow[r, Rightarrow, "TTh"] \arrow[dl, Rightarrow, "\mu X"'] \arrow[dr, Rightarrow, "Ta"'] & TTY \arrow[dr, Rightarrow, "Tb"] \arrow[d, phantom, "\Downarrow T\rho"] & \\
TX \arrow[dr, Rightarrow, "a"'] & \Swarrow a_1 & TX \arrow[r, Rightarrow, "Th"] \arrow[dl, Rightarrow, "a"] \arrow[d, phantom, "\Swarrow \rho"] & TY \arrow[dl, Rightarrow, "b"] \\
 & X \arrow[r, Rightarrow, "h"'] & Y &
\end{tikzcd}$$

We say that (h, ρ) is a *pseudohomomorphism* if ρ is invertible.

Given lax homomorphisms

$$\begin{tikzcd}
TX \arrow[r, Rightarrow, "Th"] \arrow[d, Rightarrow, "a"'] \arrow[dr, phantom, "\Swarrow \rho"] & TY \arrow[d, Rightarrow, "b"] & \text{and} & TX \arrow[r, Rightarrow, "Tk"] \arrow[d, Rightarrow, "a"'] \arrow[dr, phantom, "\Swarrow \sigma"] & TY \arrow[d, Rightarrow, "b"] \\
X \arrow[r, Rightarrow, "h"'] & Y & & X \arrow[r, Rightarrow, "k"'] & Y
\end{tikzcd}$$

a 3-cell

$$\begin{tikzcd}
X \arrow[r, Rightarrow, bend left, "h"] \arrow[r, Rightarrow, bend right, "k"'] \arrow[r, phantom, "\Downarrow \alpha"] & Y
\end{tikzcd}$$

from (h, ρ) to (k, σ) is a *homomorphism 3-cell* if

$$\begin{tikzcd}
TX \arrow[r, Rightarrow, "Th"] \arrow[d, Rightarrow, "a"'] \arrow[dr, phantom, "\Swarrow \rho"] & TY \arrow[d, Rightarrow, "b"] & = & TX \arrow[r, Rightarrow, bend left, "Th"] \arrow[r, Rightarrow, bend right, "Tk"'] \arrow[r, phantom, "\Downarrow \alpha"] \arrow[d, Rightarrow, "a"'] \arrow[dr, phantom, "\Swarrow \sigma"] & TY \arrow[d, Rightarrow, "b"] \\
X \arrow[r, Rightarrow, bend left, "h"] \arrow[r, Rightarrow, bend right, "k"'] \arrow[r, phantom, "\Downarrow \alpha"] & Y & & X \arrow[r, Rightarrow, "k"'] & Y
\end{tikzcd}$$

holds.

Remark 10.1.3. The above definitions form the data for a 2-category

$$\mathbf{K}(\mathscr{W},\mathscr{X})^{\mathbf{K}(\mathscr{W},T)}$$

that has pseudoalgebras with carriers of the form $\mathscr{W} \to \mathscr{X}$ as objects, pseudohomomorphisms between pseudoalgebras as 1-cells, and homomorphism 3-cells as 2-cells. It is straightforward to show that these data indeed form a 2-category with the obvious definitions of composition. For any object $\mathscr{W}$ of $\mathbf{K}$ there is a forgetful 2-functor

$$U^{\mathbf{K}(\mathscr{W},T)} : \mathbf{K}(\mathscr{W},\mathscr{X})^{\mathbf{K}(\mathscr{W},T)} \to \mathbf{K}(\mathscr{W},\mathscr{X})$$

mapping a pseudoalgebra to its carrier.

Consider another object $\mathscr{Z}$ of $\mathbf{K}$ and a 1-cell $Z : \mathscr{Z} \to \mathscr{W}$. This 1-cell induces a 2-functor

$$\mathbf{K}(Z,\mathscr{X})^{\mathbf{K}(Z,T)} : \mathbf{K}(\mathscr{W},\mathscr{X})^{\mathbf{K}(\mathscr{W},T)} \to \mathbf{K}(\mathscr{Z},\mathscr{X})^{\mathbf{K}(\mathscr{Z},T)}$$

that acts as a *change of base* by precomposition with Z. Moreover, these assignments can be extended functorially to yield a **Gray**-functor

$$\mathbf{K}(-,\mathscr{X})^{\mathbf{K}(-,T)} : \mathbf{K}^{op} \to \mathbf{Gray}$$

The forgetful 2-functors $U^{\mathbf{K}(\mathscr{W},T)} : \mathbf{K}(\mathscr{W},\mathscr{X})^{\mathbf{K}(\mathscr{W},T)} \to \mathbf{K}(\mathscr{W},\mathscr{X})$ are the components of a **Gray**-natural transformation $U^{\mathbf{K}(-,T)} : \mathbf{K}(-,\mathscr{X})^{\mathbf{K}(-,T)} \to \mathbf{K}(-,\mathscr{X})$. See Section 3 of [68] for more details concerning the constructions in this remark. (Marmolejo uses the notation T-Alg for $\mathbf{K}(-,\mathscr{X})^{\mathbf{K}(-,T)}$ in [68].)

Having defined pseudoadjunctions and pseudomonads in a **Gray**-category $\mathbf{K}$ in Definitions 8.1.1 and 8.1.2, we recall from Chapter 8 that there exist **Gray**-categories **psa** and **psm**, the "walking pseudoadjunction" and "walking pseudomonad" **Gray**-categories. These categories have the property that **Gray**-functors of the form

$$\mathbf{P} : \mathbf{psa} \to \mathbf{K}, \qquad \mathbf{T} : \mathbf{psm} \to \mathbf{K}$$

correspond precisely to pseudoadjunctions and pseudomonads in $\mathbf{K}$. We now give an explicit description of these **Gray**-categories.

Definition 10.1.4 ([33, 54])**.** The **Gray**-category **psa** consists of two objects a and x; the hom-2-categories $\mathbf{psa}(a,a)$, $\mathbf{psa}(a,x)$, $\mathbf{psa}(x,x)$ and $\mathbf{psa}(x,a)$ are as follows:

1. The 2-category $\mathbf{psa}(a,a)$ is locally preordered and it is isomorphic to the *augmented pseudosimplicial scheme*. That is, $\mathbf{psa}(a,a)$ is the 2-category on the graph

$$\begin{tikzcd}[column sep=large]
\cdots & {[2]} \arrow[r, shift left=8, "d^2_0" description] \arrow[r, leftarrow, shift left=4, "s^2_1" description] \arrow[r, "d^2_1" description] \arrow[r, leftarrow, shift right=4, "s^2_0" description] \arrow[r, shift right=8, "d^2_2" description] & {[1]} \arrow[r, shift left=4, "d^1_0" description] \arrow[r, leftarrow, "s^1_0" description] \arrow[r, shift right=4, "d^1_1" description] & {[0]} \arrow[r, "d^0_0" description] & {[-1]}
\end{tikzcd}$$

that is equipped by the *coherence isomorphism 2-cells*

$$\begin{array}{rlr}
d_i^n \cdot d_j^{n+1} \cong d_{j-1}^n \cdot d_i^{n+1} & & \text{for } 0 \leqslant i < j \leqslant n+1 \\
s_i^{n+1} \cdot s_j^n \cong s_{j+1}^{n+1} \cdot s_i^n & & \text{for } 0 \leqslant i < j \leqslant n-1 \\
d_i^{n+1} \cdot s_j^{n+1} \cong s_{j-1}^n \cdot d_i^n & & \text{for } 0 \leqslant i < j \leqslant n \\
d_i^{n+1} \cdot s_j^{n+1} \cong 1_{[n]} & & \text{for } i = j \text{ and } i = j+1 \\
d_i^{n+1} \cdot s_j^{n+1} \cong s_j^n \cdot d_{i-1}^n & & \text{for } 0 \leqslant j+1 < i \leqslant n+1 \\
d_0^0 \cdot d_0^1 \cong d_0^0 \cdot d_1^1 & &
\end{array}$$

that are encoding the pseudo-version of the *simplicial identities.* We shall use the “adjoint notation”

$$\begin{tikzcd}[column sep=huge]
\cdots & fufufu \arrow[r, shift left=12, "fufu\varepsilon"] \arrow[r, shift left=6, leftarrow, "f\eta ufu"] \arrow[r, "fu\varepsilon fu"] \arrow[r, shift right=6, leftarrow, "fuf\eta u"] \arrow[r, shift right=12, "\varepsilon fufu"] & fufu \arrow[r, shift left=6, "fu\varepsilon"] \arrow[r, leftarrow, "f\eta u"] \arrow[r, shift right=6, "\varepsilon fu"] & fu \arrow[r, "\varepsilon"] & 1_a
\end{tikzcd}$$

for the objects and 1-cells of **psa**(a, a). With this notation, there is a unique isomorphism 2-cell between two 1-cells precisely when an equality between these 1-cells can be derived from the triangle equations for η and ε.

2. The 2-category **psa**(a, x) is isomorphic to the *augmented split pseudosimplicial scheme.* That is, **psa**(a, a) is the 2-category on the graph

$$\begin{tikzcd}[column sep=huge]
\cdots & {[2]} \arrow[r, shift left=12, "d_0^2"] \arrow[r, shift left=6, leftarrow, "s_1^2"] \arrow[r, "d_1^2"] \arrow[r, shift right=6, leftarrow, "s_0^2"] \arrow[r, shift right=12, "d_2^2"] \arrow[r, shift right=18, leftarrow, "s_2^2"] & {[1]} \arrow[r, shift left=6, "d_0^1"] \arrow[r, leftarrow, "s_0^1"] \arrow[r, shift right=6, "d_1^1"] \arrow[r, shift right=18, leftarrow, "s_1^1"] & {[0]} \arrow[r, "d_0^0"] \arrow[r, shift right=18, leftarrow, "s_0^0"] & {[-1]}
\end{tikzcd}$$

that is equipped by the *coherence isomorphism 2-cells*

$$\begin{array}{rlr}
d_i^n \cdot d_j^{n+1} \cong d_{j-1}^n \cdot d_i^{n+1} & & \text{for } 0 \leqslant i < j \leqslant n+1 \\
s_i^{n+1} \cdot s_j^n \cong s_{j+1}^{n+1} \cdot s_i^n & & \text{for } 0 \leqslant i < j \leqslant n-1 \\
d_i^{n+1} \cdot s_j^{n+1} \cong s_{j-1}^n \cdot d_i^n & & \text{for } 0 \leqslant i < j \leqslant n \\
d_i^{n+1} \cdot s_j^{n+1} \cong 1_{[n]} & & \text{for } i = j \text{ and } i = j+1 \\
d_i^{n+1} \cdot s_j^{n+1} \cong s_j^n \cdot d_{i-1}^n & & \text{for } 0 \leqslant j+1 < i \leqslant n+1 \\
d_0^0 \cdot d_0^1 \cong d_0^0 \cdot d_1^1 & & \\
d_i^{n+1} \cdot s_{n+1}^{n+1} \cong s_n^n \cdot d_i^n & & \text{for } 0 \leqslant i < n \\
d_{n+1}^{n+1} \cdot s_{n+1}^{n+1} \cong 1_{[n]} & & \\
s_i^{n+1} \cdot s_n^n \cong s_{n+1}^{n+1} \cdot s_i^n & & \text{for } 0 \leqslant i < n \\
s_{n+1}^{n+1} \cdot s_n^n \cong s_n^{n+1} \cdot s_n^n & &
\end{array}$$

encoding the simplicial and splitting "pseudoidentities". We shall again use the "adjoint notation"

$$\begin{tikzcd}[column sep=huge]
\cdots & fufufu \arrow[r, shift left=10, "fufu\varepsilon" description] \arrow[r, <-, shift left=6, "f\eta ufu" description] \arrow[r, shift left=2, "fu\varepsilon fu" description] \arrow[r, <-, shift right=2, "fuf\eta u" description] \arrow[r, shift right=6, "\varepsilon fufu" description] \arrow[r, <-, shift right=10, "\eta ufufu" description] & fufu \arrow[r, shift left=4, "fu\varepsilon" description] \arrow[r, <-, "f\eta u" description] \arrow[r, shift right=4, "\varepsilon fu" description] \arrow[r, <-, shift right=10, "\eta ufu" description] & fu \arrow[r, "\varepsilon" description] \arrow[r, <-, shift right=10, "\eta u" description] & 1_a
\end{tikzcd}$$

for the objects and 1-cells of **psa**(a, x). The unique isomorphism 2-cell between two 1-cells then again witnesses that these 1-cells can be derived to be equal by the triangle equations for η and ε.

3. The 2-category **psa**(x, x) is the opposite of **psa**(a, a):

$$\mathbf{psa}(x, x) = \mathbf{psa}(a, a)^{op}.$$

4. The 2-category **psa**(x, a) is the opposite of **psa**(a, x):

$$\mathbf{psa}(x, a) = \mathbf{psa}(a, x)^{op}.$$

Remark 10.1.5. We have shown in Chapter 8 (Example 8.2.1) that **Gray**-functors

$$\mathbf{P} : \mathbf{psa} \to \mathbf{K}$$

correspond precisely to pseudoadjunctions in **K**. This fact was first proved in [54].

We will use the notational convention that such a **Gray**-functor has as its image a pseudoadjunction

$$\begin{tikzcd}
\mathscr{A} \arrow[d, bend left=40, "U"] \arrow[d, phantom, "\dashv"] \\
\mathscr{X} \arrow[u, bend left=40, "F"]
\end{tikzcd}$$

with unit η, counit ε, and the coherence isomorphisms s and t.

Remark 10.1.6. The **Gray**-category **psm** can be regarded as the full subcategory of **psa** spanned by the object x. We have defined **psm** equivalently in Chapter 8 and denoted the inclusion **Gray**-functor by

$$\mathbf{M} : \mathbf{psm} \to \mathbf{psa}$$

in Example 8.2.6. Recall that the **Gray**-category **psm** has the property that **Gray**-functors

$$\mathbf{T} : \mathbf{psm} \to \mathbf{K}$$

correspond precisely to pseudomonads in **K** (see also [54].

We can now introduce the notion of the Eilenberg-Moore pseudoadjunction (and, dually, the Kleisli pseudoadjunction) in a **Gray**-category.

Definition 10.1.7. Given a pseudomonad $\mathbf{T} : \mathbf{psm} \to \mathbf{K}$ in a **Gray**-category $\mathbf{K}$, we say that

1. $\mathbf{T}$ admits the *Kleisli construction* if $\mathrm{Lan}_{\mathbf{M}}\mathbf{T} : \mathbf{psa} \to \mathbf{K}$ exists.
2. $\mathbf{T}$ admits the *Eilenberg-Moore construction* if $\mathrm{Ran}_{\mathbf{M}}\mathbf{T} : \mathbf{psa} \to \mathbf{K}$ exists.

We can weaken the completeness and cocompleteness conditions in the above definitions:

Lemma 10.1.8. *Given a* **Gray**-*category* $\mathbf{K}$ *and a pseudomonad* $\mathbf{T} : \mathbf{psa} \to \mathbf{K}$*,*

1. $\mathbf{T}$ *admits the Kleisli construction if* $\mathbf{psa}(\mathbf{M}-, a) * \mathbf{T}$ *exists.*
2. $\mathbf{T}$ *admits the Eilenberg-Moore construction if* $\{\mathbf{psa}(a, \mathbf{M}-), \mathbf{T}\}$ *exists.*

Proof. We know that $\mathbf{T}$ admits the Kleisli construction *if and only if* both $\mathbf{psa}(\mathbf{M}-, a) * \mathbf{T}$ and $\mathbf{psa}(\mathbf{M}-, x) * \mathbf{T}$ exist. We will show that $\mathbf{psa}(\mathbf{M}-, x) * \mathbf{T}$ exists always. Since $\mathbf{M}$ is fully faithful and x lies in $\mathbf{psm}$, we have the following series of isomorphisms:

$$\mathbf{psa}(\mathbf{M}-, x) * \mathbf{T} \cong \mathbf{psm}(-, x) * \mathbf{T} \cong \mathbf{T}(x).$$

Dually, the limit $\{\mathbf{psa}(x, \mathbf{M}-), \mathbf{T}\}$ exists always as well, hence $\mathbf{T}$ admits the Eilenberg-Moore construction whenever $\{\mathbf{psa}(a, \mathbf{M}-), \mathbf{T}\}$ exists. ■

Remark 10.1.9. The above weighted colimits (and Kan extensions) are to be considered as colimits in the 2-category **Gray**-CAT, see Remark 7.1.8

Remark 10.1.10. Suppose $\mathbf{T} : \mathbf{psa} \to \mathbf{K}$ is a pseudomonad on $\mathscr{X}$, i.e., $\mathbf{T}(x) = \mathscr{X}$. We introduce the notation $\mathsf{Kl}(\mathbf{T})$ for the *Kleisli object* $\mathbf{psa}(\mathbf{M}-, a) * \mathbf{T}$ and the notation $\mathscr{X}^T$ for the *Eilenberg-Moore object* $\{\mathbf{psa}(a, \mathbf{M}-), \mathbf{T}\}$. The pseudoadjunctions given by $\mathrm{Lan}_{\mathbf{M}}\mathbf{T}$ and $\mathrm{Ran}_{\mathbf{M}}\mathbf{T}$ will be denoted by

$$\begin{tikzcd}
\mathsf{Kl}(T) \arrow[d, bend left=50, "U_T"] \arrow[d, phantom, "\dashv"] & & \mathscr{X}^T \arrow[d, bend left=50, "U^T"] \arrow[d, phantom, "\dashv"] \\
\mathscr{X} \arrow[u, bend left=50, "F_T"] & , & \mathscr{X} \arrow[u, bend left=50, "F^T"]
\end{tikzcd}$$

with units and counits η_T, ε_T and η^T, ε^T respectively, and similarly s_T, t_T and s^T, t^T for the coherence 3-cells.

In Remark 10.1.3 we have already introduced the notation $\mathbf{K}(-, \mathscr{X})^{\mathbf{K}(-,T)}$ for a **Gray**-functor from $\mathbf{K}^{op}$ to **Gray**. In cases where $\mathbf{T} : \mathbf{psm} \to \mathbf{K}$ does not admit the Eilenberg-Moore construction, we may consider the Yoneda embedding $Y : \mathbf{K} \to [\mathbf{K}^{op}, \mathbf{Gray}]$ and form the Eilenberg-Moore object in the **Gray**-category $[\mathbf{K}^{op}, \mathbf{Gray}]$ of presheaves on $\mathbf{K}$ as the limit $\{\mathbf{psa}(a, \mathbf{M}-), Y \cdot \mathbf{T}\}$. This limit is, in elementary terms, precisely the **Gray**-functor $\mathbf{K}(-, \mathscr{X})^{\mathbf{K}(-,T)} : \mathbf{K}^{op} \to \mathbf{Gray}$. In cases where $\mathbf{T} : \mathbf{psm} \to \mathbf{K}$ *does* admit the Eilenberg-Moore construction, $\{\mathbf{psa}(a, \mathbf{M}-), Y \cdot \mathbf{T}\}$ is representable and is precisely $\mathbf{K}(-, \mathscr{X}^T)$. See also Section 4 of [54].

10.2 KZ-pseudoadjunctions

In this section we give a characterisation of KZ-pseudoadjunctions. Let us quickly recall the notion of a KZ-pseudoadjunction.

Remark 10.2.1. A pseudoadjunction $F \dashv U : \mathscr{A} \to \mathscr{X}$ is a *KZ-pseudoadjunction* (see Definition 8.1.3) if

1. the 3-cell

$$\begin{tikzcd} F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "1_F"'] & FUF \arrow[d, Rightarrow, "\varepsilon F"] \arrow[dl, phantom, "s \Rrightarrow"] \\ & F \end{tikzcd}$$

   is the unit of the (ordinary) adjunction $F\eta \dashv \varepsilon F$ and if

2. the 3-cell

$$\begin{tikzcd} U \arrow[r, Rightarrow, "\eta U"] \arrow[dr, Rightarrow, "1_U"'] & UFU \arrow[d, Rightarrow, "U\varepsilon"] \arrow[dl, phantom, "t \Lleftarrow"] \\ & U \end{tikzcd}$$

   is the counit of the adjunction $\eta U \dashv U\varepsilon$.

Example 10.2.2. Given a class Φ of weights, the pseudoadjunction

$$\begin{tikzcd} \Phi\text{-COCTS} \arrow[d, bend left, "U_\Phi"] \\ \mathscr{V}\text{-CAT} \arrow[u, bend left, "F_\Phi"] \end{tikzcd}$$

(see Remark 2.1.20) giving rise to the free Φ-cocompletion pseudomonad is a KZ-pseudoadjunction.

Observe that Definition 8.1.3 of a KZ-pseudoadjunction is tailored to prove that its induced pseudomonad is a KZ-pseudomonad. Analogously to the case of KZ-pseudomonads (see [67], Theorem 11.1), the definition of a KZ-pseudoadjunction also contains redundancy. In fact, for a pseudoadjunction to be KZ it is enough to satisfy only one of the requirements of Remark 10.2.1. In the following lemma we give various characterisations of KZ-pseudoadjunctions that prove this fact.

Theorem 10.2.3 (A characterisation of KZ-pseudoadjunctions). *For a pseudoadjunction $F \dashv U : \mathscr{A} \to \mathscr{X}$ the following are equivalent:*

1. *$F \dashv U : \mathscr{A} \to \mathscr{X}$ is a KZ-pseudoadjunction.*

2. *The 3-cell*

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "1_F"'] \arrow[dr, phantom, shift left=4ex, "s \Rrightarrow"] & FUF \arrow[d, Rightarrow, "\varepsilon F"] \\
& F
\end{tikzcd}$$

is the unit of the (ordinary) adjunction $F\eta \dashv \varepsilon F$.

3. *The 3-cell*

$$\begin{tikzcd}
UF \arrow[r, Rightarrow, "UF\eta"] \arrow[dr, Rightarrow, "1_{UF}"'] \arrow[dr, phantom, shift left=4ex, "Us \Rrightarrow"] & UFUF \arrow[d, Rightarrow, "U\varepsilon F"] \\
& UF
\end{tikzcd}$$

is the unit of the (ordinary) adjunction $UF\eta \dashv U\varepsilon F$.

4. *There is a 3-cell*

$$\begin{tikzcd}
UF \arrow[r, Rightarrow, bend left, "UF\eta"] \arrow[r, Rightarrow, bend right, "\eta UF"'] \arrow[r, phantom, "\Downarrow d"] & UFUF
\end{tikzcd}$$

satisfying the equalities

$$\begin{tikzcd}
1 \arrow[r, Rightarrow, "\eta"] & UF \arrow[r, Rightarrow, bend left, "UF\eta"] \arrow[r, Rightarrow, bend right, "\eta UF"'] \arrow[r, phantom, "\Downarrow d"] & UFUF
\end{tikzcd}
=
\begin{tikzcd}
1 \arrow[r, Rightarrow, "\eta"] \arrow[d, Rightarrow, "\eta"'] \arrow[dr, phantom, "\eta_\eta \Lleftarrow"] & UF \arrow[d, Rightarrow, "UF\eta"] \\
UF \arrow[r, Rightarrow, "\eta UF"'] & UFUF
\end{tikzcd}$$

and

$$\begin{tikzcd}
UF \arrow[r, Rightarrow, bend left, "UF\eta"] \arrow[r, Rightarrow, bend right, "\eta UF"'] \arrow[r, phantom, "\Downarrow d"] & UFUF \arrow[r, Rightarrow, "U\varepsilon F"] & UF
\end{tikzcd}
=
\begin{tikzcd}
UF \arrow[r, Rightarrow, "UF\eta"] \arrow[d, Rightarrow, "\eta UF"'] \arrow[dr, Rightarrow, "1_{UF}"] \arrow[dr, phantom, shift left=4ex, "Us^{-1} \Lleftarrow"] \arrow[dr, phantom, shift right=4ex, "t^{-1}F \Lleftarrow"] & UFUF \arrow[d, Rightarrow, "U\varepsilon F"] \\
UFUF \arrow[r, Rightarrow, "U\varepsilon F"'] & UF.
\end{tikzcd}$$

5. *There is a 3-cell*

$$\begin{tikzcd}
UFU \arrow[r, Rightarrow, bend left, "UF\eta U"] \arrow[r, Rightarrow, bend right, "\eta UFU"'] \arrow[r, phantom, "\Downarrow d'"] & UFUFU
\end{tikzcd}$$

satisfying the equalities

$$\begin{tikzcd}
U \arrow[r, Rightarrow, "\eta U"] & UFU \arrow[r, Rightarrow, bend left, "UF\eta U"] \arrow[r, Rightarrow, bend right, "\eta UFU"'] \arrow[r, phantom, "\Downarrow d'"] & UFUFU
\end{tikzcd}
=
\begin{tikzcd}
U \arrow[r, Rightarrow, "\eta U"] \arrow[d, Rightarrow, "\eta U"'] \arrow[dr, phantom, "\eta_\eta U \swarrow"] & UFU \arrow[d, Rightarrow, "UF\eta U"] \\
UFU \arrow[r, Rightarrow, "\eta UFU"'] & UFUFU
\end{tikzcd}$$

and

$$\begin{tikzcd}
UFU \arrow[r, Rightarrow, bend left, "UF\eta U"] \arrow[r, Rightarrow, bend right, "\eta UFU"'] \arrow[r, phantom, "\Downarrow d'"] & UFUFU \arrow[r, Rightarrow, "U\varepsilon FU"] & UFU
\end{tikzcd}
=
\begin{tikzcd}
UFU \arrow[r, Rightarrow, "UF\eta U"] \arrow[d, Rightarrow, "\eta UFU"'] \arrow[dr, Rightarrow, "1_{UFU}"] & UFUFU \arrow[d, Rightarrow, "U\varepsilon FU"] \arrow[dl, phantom, "Us^{-1}U \swarrow"] \\
UFUFU \arrow[r, Rightarrow, "U\varepsilon FU"'] \arrow[ur, phantom, "t^{-1}FU \swarrow"] & UFU.
\end{tikzcd}$$

6. *The 3-cell*

$$\begin{tikzcd}
U \arrow[r, Rightarrow, "\eta U"] \arrow[dr, Rightarrow, "1_U"'] \arrow[dr, phantom, bend left, "t \swarrow"] & UFU \arrow[d, Rightarrow, "U\varepsilon"] \\
& U
\end{tikzcd}$$

is the counit of the adjunction $\eta U \dashv U\varepsilon$.

Proof. We will prove the implication chain $2 \implies 3 \implies 4 \implies 5 \implies 6$. The chain $6 \implies 5 \implies 4 \implies 3 \implies 2$ follows from duality. Condition 1 is equivalent to the conjunction of 2 and 6 by definition.

To have 2 is to have an adjunction $F\eta \dashv \varepsilon F$ with unit s and counit r,

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "1_F"'] \arrow[dr, phantom, bend left, "s \nearrow"] & FUF \arrow[d, Rightarrow, "\varepsilon F"] \\
& F
\end{tikzcd}
\qquad
\begin{tikzcd}
FUF \arrow[r, Rightarrow, "\varepsilon F"] \arrow[dr, Rightarrow, "1_{FUF}"'] \arrow[dr, phantom, bend left, "r \swarrow"] & F \arrow[d, Rightarrow, "F\eta"] \\
& FUF
\end{tikzcd}$$

respectively. Applying U, we get the adjunction $UF\eta \dashv U\varepsilon F$ with unit Us and counit Ur,

$$\begin{tikzcd}
UF \arrow[r, Rightarrow, "UF\eta"] \arrow[dr, Rightarrow, "1_{UF}"'] \arrow[dr, phantom, bend left, "Us \nearrow"] & UFUF \arrow[d, Rightarrow, "U\varepsilon F"] \\
& F
\end{tikzcd}
\qquad
\begin{tikzcd}
UFUF \arrow[r, Rightarrow, "U\varepsilon F"] \arrow[dr, Rightarrow, "1_{UFUF}"'] \arrow[dr, phantom, bend left, "Ur \swarrow"] & UF \arrow[d, Rightarrow, "UF\eta"] \\
& UFUF
\end{tikzcd}$$

respectively. Thus 3 holds.

From the data in 3 we can construct a 3-cell

$$\begin{tikzcd}
UF \arrow[r, Rightarrow, bend left, "UF\eta"] \arrow[r, Rightarrow, bend right, "\eta UF"'] \arrow[r, phantom, "\Downarrow d"] & UFUF
\end{tikzcd}$$

as in the diagram (2) of Marmolejo's paper [67]: d is the composite

$$\begin{tikzcd}
UF \arrow[rr, Rightarrow, "1_{UF}"] \arrow[ddr, Rightarrow, "\eta UF"'] & & UF \arrow[ddr, Rightarrow, "UF\eta"] & \\
& t^{-1}F \Downarrow & Ur \Downarrow & \\
& UFUF \arrow[uur, Rightarrow, "U\varepsilon F"'] \arrow[rr, Rightarrow, "1_{UFUF}"'] & & UFUF
\end{tikzcd}$$

By Proposition 3.1 and Lemma 3.2 of [67] we know that the equalities

$$\begin{tikzcd}
1 \arrow[r, Rightarrow, "\eta"] & UF \arrow[r, Rightarrow, bend left, "UF\eta"] \arrow[r, Rightarrow, bend right, "\eta UF"'] \arrow[r, phantom, "\Downarrow d"] & UFUF
\end{tikzcd}
=
\begin{tikzcd}
1 \arrow[r, Rightarrow, "\eta"] \arrow[d, Rightarrow, "\eta"'] \arrow[dr, phantom, "\eta_\eta \Swarrow"] & UF \arrow[d, Rightarrow, "UF\eta"] \\
UF \arrow[r, Rightarrow, "\eta UF"'] & UFUF
\end{tikzcd}$$

and

$$\begin{tikzcd}
UF \arrow[r, Rightarrow, bend left, "UF\eta"] \arrow[r, Rightarrow, bend right, "\eta UF"'] \arrow[r, phantom, "\Downarrow d"] & UFUF \arrow[r, Rightarrow, "U\varepsilon F"] & UF
\end{tikzcd}
=
\begin{tikzcd}
UF \arrow[r, Rightarrow, "UF\eta"] \arrow[d, Rightarrow, "\eta UF"'] \arrow[dr, Rightarrow, "1_{UF}"] & UFUF \arrow[d, Rightarrow, "U\varepsilon F"] \arrow[dl, phantom, "Us^{-1} \Swarrow", pos=0.2] \\
UFUF \arrow[r, Rightarrow, "U\varepsilon F"'] \arrow[ur, phantom, "t^{-1}F \Swarrow", pos=0.2] & UF
\end{tikzcd}$$

hold. This proves the implication 3 $\implies$ 4.

Condition 4 implies 5 by whiskering, where $d' = dU$.

To prove 5 $\implies$ 6, we define the unit of $U\varepsilon \dashv \eta U$ to be the composite

$$\begin{tikzcd}
& UFt^{-1}\Downarrow & \\
UFU \arrow[rr, Rightarrow, bend left=70, "1_{UFU}"] \arrow[r, Rightarrow, bend left, "UF\eta U"] \arrow[r, Rightarrow, bend right, "\eta UFU"'] \arrow[r, phantom, "\Downarrow dU"] \arrow[ddr, Rightarrow, bend right, "U\varepsilon"'] & UFUF \arrow[r, Rightarrow, "UFU\varepsilon"] & UFU \\
& \eta U_{FU\varepsilon} \Downarrow & \\
& U \arrow[uur, Rightarrow, bend right, "\eta U"'] &
\end{tikzcd}$$

Indeed, the triangle identities hold: the composite

$$\begin{tikzcd}
& & UFt^{-1}\Downarrow & \\
U \arrow[r, Rightarrow, "\eta U"] \arrow[ddrr, Rightarrow, bend right, "1_U"'] & UFU \arrow[rr, Rightarrow, bend left=70, "1_{UFU}"] \arrow[r, Rightarrow, bend left, "UF\eta U"] \arrow[r, Rightarrow, bend right, "\eta UFU"'] \arrow[r, phantom, "\Downarrow dU"] \arrow[ddr, Rightarrow, bend right, "U\varepsilon"'] & UFUF \arrow[r, Rightarrow, "UFU\varepsilon"] & UFU \\
& t \Downarrow & \eta U_{FU\varepsilon} \Downarrow & \\
& & U \arrow[uur, Rightarrow, bend right, "\eta U"'] &
\end{tikzcd}$$

is equal to

$$\begin{tikzcd}
 & UFU \arrow[dr, Rightarrow, "UF\eta U"] \arrow[drr, Rightarrow, bend left, "1_{UFU}"] & \arrow[d, phantom, "UFt^{-1}\Downarrow"] & \\
U \arrow[ur, Rightarrow, "\eta U"] \arrow[r, Rightarrow, "\eta U"] \arrow[dr, Rightarrow, bend right, "1_U"'] & UFU \arrow[ul, phantom, "\eta_{\eta U}\Downarrow"] \arrow[r, Rightarrow, "\eta UFU"'] \arrow[d, Rightarrow, bend left, "U\varepsilon"] \arrow[dl, phantom, "t\Downarrow"] & UFUF \arrow[r, Rightarrow, "UFU\varepsilon"] \arrow[d, phantom, "\eta U_{FU\varepsilon}\Downarrow"] & UFU \\
 & & U \arrow[ur, Rightarrow, bend right, "\eta U"'] &
\end{tikzcd}$$

which in turn is equal to

$$\begin{tikzcd}
U \arrow[r, Rightarrow, "\eta U"] \arrow[rr, Rightarrow, bend left=50, "1_U"] \arrow[rr, Rightarrow, bend right=50, "1_U"'] \arrow[rr, phantom, bend left=30, "t^{-1}\Downarrow"] \arrow[rr, phantom, bend right=30, "t\Downarrow"] & UFU \arrow[r, Rightarrow, "U\varepsilon"] & U
\end{tikzcd}$$

and therefore equal to the identity 3-cell on U. For the second triangle equality see that the 3-cell

$$\begin{tikzcd}
UFU \arrow[rr, Rightarrow, bend left=60, "1_{UFU}"] \arrow[r, Rightarrow, bend left, "UF\eta U"] \arrow[r, Rightarrow, bend right, "\eta UFU"'] \arrow[r, phantom, "\Downarrow dU"] \arrow[dr, Rightarrow, bend right, "U\varepsilon"'] & UFUF \arrow[u, phantom, "UFt^{-1}\Downarrow"] \arrow[r, Rightarrow, "UFU\varepsilon"] \arrow[d, phantom, "\eta U_{FU\varepsilon}\Downarrow"] & UFU \arrow[d, Rightarrow, "U\varepsilon"] \\
 & U \arrow[ur, Rightarrow, "\eta U"] \arrow[r, Rightarrow, "1_U"'] \arrow[ur, phantom, shift right=4, "t\Downarrow"] & U
\end{tikzcd}$$

is equal to

$$\begin{tikzcd}
 & UFt^{-1}\Downarrow & & \\
 & UFUFU \arrow[r, Rightarrow, "UFU\varepsilon"] \arrow[dr, Rightarrow, "U\varepsilon FU"'] \arrow[d, phantom, "Us^{-1}U \Downarrow"] & UFU \arrow[dr, Rightarrow, "U\varepsilon"] \arrow[d, phantom, "U\varepsilon_{\varepsilon}^{-1}\Downarrow"] & \\
UFU \arrow[ur, Rightarrow, "UF\eta U"] \arrow[rr, Rightarrow, "1_{UFU}"] \arrow[dr, Rightarrow, "\eta UFU"'] \arrow[ddr, Rightarrow, bend right, "U\varepsilon"'] \arrow[urr, Rightarrow, bend left=60, "1_{UFU}"] & & UFU \arrow[r, Rightarrow, "U\varepsilon"] & U \\
 & UFUFU \arrow[ur, Rightarrow, "U\varepsilon FU"] \arrow[r, Rightarrow, "UFU\varepsilon"'] \arrow[u, phantom, "t^{-1}FU \Downarrow"] & UFU \arrow[ur, Rightarrow, "U\varepsilon"'] \arrow[u, phantom, "U\varepsilon_{\varepsilon} \Downarrow"] \arrow[d, phantom, "t \Downarrow"] & \\
 & U \arrow[ur, Rightarrow, "\eta U"'] \arrow[uurr, Rightarrow, bend right, "1_U"'] \arrow[u, phantom, "\eta U_{FU\varepsilon} \Downarrow"] & &
\end{tikzcd}$$

and by the properties of pseudoadjunctions this 3-cell is equal to

$$\begin{tikzcd}
UFU \arrow[r, Rightarrow, "\eta_{UFU}"] \arrow[d, Rightarrow, "U\varepsilon"'] \arrow[rr, Rightarrow, bend left=50, "1_{UFU}"] \arrow[dr, phantom, "\eta U_{FU\varepsilon} \swarrow"] & UFUFU \arrow[r, Rightarrow, "U\varepsilon FU"] \arrow[d, Rightarrow, "UFU\varepsilon"] \arrow[dr, phantom, "U\varepsilon_{\varepsilon} \swarrow"] \arrow[u, phantom, "t^{-1}FU \Downarrow"] & UFU \arrow[d, Rightarrow, "U\varepsilon"] \\
U \arrow[r, Rightarrow, "\eta U"'] \arrow[rr, Rightarrow, bend right=50, "1_U"'] & UFU \arrow[r, Rightarrow, "U\varepsilon"'] \arrow[d, phantom, "t \Downarrow"] & U \\
 & &
\end{tikzcd}$$

which in turn is equal to the identity on $U\varepsilon$. We have thus proved the implication 5 $\implies$ 6. ■

Remark 10.2.4. Suppose we have a pseudoadjunction $F \dashv U$ such that its corresponding pseudomonad T is KZ. A natural question arises: is $F \dashv U$ necessarily KZ? Theorem 10.2.3 gives a positive answer and the reasoning is as follows. Construct the 3-cell

$$\begin{tikzcd}
UF \arrow[r, Rightarrow, bend left, "UF\eta"] \arrow[r, Rightarrow, bend right, "\eta UF"'] \arrow[r, phantom, "\Downarrow d"] & UFUF
\end{tikzcd}$$

as in [67]. By construction, the 3-cell d satisfies the equalities demanded in point 4 of Theorem 10.2.3, and the implication 4 $\implies$ 6 thus proves that $F \dashv U$ is KZ.

Example 10.2.5. It is perhaps more customary to say that the free Φ-cocompletion pseudomonad is KZ than to say that the pseudoadjunction

$$\begin{tikzcd}
\Phi\text{-}\mathsf{COCTS} \arrow[d, bend left, "U_\Phi"] \\
\mathscr{V}\text{-}\mathsf{CAT} \arrow[u, bend left, "F_\Phi"]
\end{tikzcd}$$

is KZ. Theorem 10.2.3 states that these observations are equivalent.

10.3 KZ-pseudomonads and property-like pseudomonads

In this section we will give a characterisation of KZ-pseudomonads by means of properties of their algebras and homomorphisms (or more precisely, of the "pseudo"-counterparts of these notions). We will characterise KZ-pseudomonads as those pseudomonads that are *property-like*: this means that to give a T-pseudoalgebra $TX \Rightarrow X$ for a KZ-pseudomonad is to give an object with a certain property specified by T; the basic example being the free Φ-cocompletion pseudomonad $\Phi : \mathscr{V}\text{-}\mathsf{CAT} \to \mathscr{V}\text{-}\mathsf{CAT}$ for a class Φ of weights. We shall need to introduce a technical condition called *(AEL)* that was introduced in the context of 2-monads in [46]

Definition 10.3.1. A pseudomonad (T, η, μ) satisfies the *(AEL)* condition if for *all* pseudoalgebras (X, a, a_0, a_1), (Y, b, b_0, b_1), and all $h : X \to Y$ there *exists* a unique 2-cell

$$\begin{tikzcd}
TX \arrow[r, Rightarrow, "Th"] \arrow[d, Rightarrow, "a"'] \arrow[dr, phantom, "\swarrow \rho"] & TY \arrow[d, Rightarrow, "b"] \\
X \arrow[r, Rightarrow, "h"'] & Y
\end{tikzcd}$$

making (h, ρ) a *lax* homomorphism.

Remark 10.3.2. The emphasized notions in Definition 10.3.1 explain the origin of the name (AEL) for this condition.

Observe that the cocompletion pseudomonads have the (AEL) property: every functor between Φ-cocomplete categories can be thought of as a lax homomorphism. The 3-cell ρ is then the pointwise canonical comparison $\varphi * hD \to h(\varphi * D)$.

Pseudomonads that satisfy the (AEL) condition should remind the reader of idempotent monads: for idempotent (ordinary) monads $T : \mathscr{X} \to \mathscr{X}$ we know that the forgetful functor $U^T : \mathsf{Alg}(T) \to \mathscr{X}$ is fully faithful, i.e., that every morphism $f : X \to Y$ between the carriers of the algebras $a : TX \to X$ and $b : TY \to Y$ is a homomorphism of these algebras. The (AEL) condition is a weakening of this property.

We now turn to the characterisation theorem. Some parts of this characterisation are already known from [67].

Theorem 10.3.3 (A characterisation of KZ-pseudomonads.). *Let (T, η, μ) be a pseudomonad in* $\mathbf{K}$. *Then the following are equivalent:*

1. *(T, η, μ) is a KZ-pseudomonad.*
2. *(T, η, μ) satisfies the (AEL) condition.*
3. *There is a 3-cell $d : T\eta \Rrightarrow \eta T$ with*

$$\begin{tikzcd} 1 \arrow[r, Rightarrow, "\eta"] & T \arrow[r, Rightarrow, bend left, "T\eta"] \arrow[r, Rightarrow, bend right, "\eta T"'] \arrow[r, phantom, "\Downarrow d"] & T \end{tikzcd} = \begin{tikzcd} 1 \arrow[r, Rightarrow, "\eta"] \arrow[d, Rightarrow, "\eta"'] \arrow[dr, phantom, "\eta_\eta \Lleftarrow"] & T \arrow[d, Rightarrow, "T\eta"] \\ T \arrow[r, Rightarrow, "\eta T"'] & TT \end{tikzcd}$$

and

$$\begin{tikzcd} T \arrow[r, Rightarrow, bend left, "T\eta"] \arrow[r, Rightarrow, bend right, "\eta T"'] \arrow[r, phantom, "\Downarrow d"] & TT \arrow[r, Rightarrow, "\mu"] & T \end{tikzcd} = \begin{tikzcd} T \arrow[r, Rightarrow, "T\eta"] \arrow[d, Rightarrow, "\eta T"'] \arrow[dr, Rightarrow, "1_T"] & TT \arrow[d, Rightarrow, "\mu"] \arrow[dl, phantom, near start, "s^{-1} \Lleftarrow"] \\ TT \arrow[r, Rightarrow, "\mu"'] \arrow[ur, phantom, near start, "t^{-1} \Lleftarrow"] & T \end{tikzcd}$$

4. *Every pseudoalgebra (X, a, a_0, a_1) satisfies $a \dashv \eta X$ with*

$$\begin{tikzcd} X \arrow[r, Rightarrow, "\eta X"] \arrow[dr, Rightarrow, "1"'] & TX \arrow[d, Rightarrow, "a"] \arrow[dl, phantom, "\Lleftarrow a_0"] \\ & X \end{tikzcd}$$

as a counit.

Proof. The equivalence of the conditions 1, 3 and 4 follows from [67]. We begin by proving that 2 $\implies$ 1.

Observe first that we can deduce from the axioms of a pseudomonad that

$$\begin{tikzcd} T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, Rightarrow, "1_T"'] & TT \arrow[d, Rightarrow, "\mu"] \arrow[dl, phantom, "t \Lleftarrow"] \\ & T \end{tikzcd} \qquad \begin{tikzcd} TTT \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"'] \arrow[dr, phantom, "\Lleftarrow m"] & TT \arrow[d, Rightarrow, "\mu"] \\ TT \arrow[r, Rightarrow, "\mu"'] & T \end{tikzcd}$$

make (T, μ, t, m) into a pseudoalgebra. Similarly, $(TT, \mu T, tT, mT)$ is a pseudoalgebra via the 3-cells

$$\begin{tikzcd} TT \arrow[r, Rightarrow, "\eta TT"] \arrow[dr, Rightarrow, "1_{TT}"'] & TT \arrow[d, Rightarrow, "\mu T"] \arrow[dl, phantom, "tT \Lleftarrow"] \\ & TT \end{tikzcd} \qquad \begin{tikzcd} TTTT \arrow[r, Rightarrow, "T\mu T"] \arrow[d, Rightarrow, "\mu TT"'] \arrow[dr, phantom, "\Lleftarrow mT"] & TTT \arrow[d, Rightarrow, "\mu T"] \\ TTT \arrow[r, Rightarrow, "\mu T"'] & TT. \end{tikzcd}$$

Therefore, for the 2-cell $\eta T : T \Rightarrow TT$ there exists a unique lax homomorphism

$$\begin{tikzcd}
TT \arrow[r, Rightarrow, "T\eta T"] \arrow[d, Rightarrow, "\mu"'] & TTT \arrow[d, Rightarrow, "\mu T"] \arrow[dl, phantom, "\Swarrow \overline{\eta T}"] \\
T \arrow[r, Rightarrow, "\eta T"'] & TT
\end{tikzcd}$$

from (T, μ, t, m) to $(TT, \mu T, tT, mT)$ by 2. By the properties of lax homomorphisms we get that the 3-cell

$$\begin{tikzcd}
 & & TTT \arrow[dr, Rightarrow, "\mu T"] & \\
T \arrow[r, Rightarrow, "\eta T"] \arrow[drr, Rightarrow, bend right, "1"'] & TT \arrow[ur, Rightarrow, "T\eta T"] \arrow[dr, Rightarrow, "\mu"] & \Rrightarrow \overline{\eta T} & TT \\
 & \Rrightarrow t & T \arrow[ur, Rightarrow, "\eta T"'] &
\end{tikzcd}$$

is equal to the 3-cell

$$\begin{tikzcd}
 & TT \arrow[dr, Rightarrow, "T\eta T"] & & \\
T \arrow[ur, Rightarrow, "\eta T"] \arrow[dr, Rightarrow, "\eta T"'] & \Rrightarrow \eta_{\eta T} & TTT \arrow[r, Rightarrow, "\mu T"] & TT \\
 & TT \arrow[ur, Rightarrow, "\eta TT"'] \arrow[urr, Rightarrow, bend right, "1"'] & \Rrightarrow tT &
\end{tikzcd}$$

and the 3-cell

$$\begin{tikzcd}
 & TTT \arrow[r, Rightarrow, "TT\eta T"] \arrow[dl, Rightarrow, "\mu T"'] & TTTT \arrow[dl, Rightarrow, "\mu TT"] \arrow[dr, Rightarrow, "T\mu T"] & \\
TT \arrow[r, Rightarrow, "T\eta T"] \arrow[dr, Rightarrow, "\mu"'] & TTT \arrow[dr, Rightarrow, "\mu T"] \arrow[u, phantom, "\Swarrow \mu_{\eta T}{}^{-1}"] & \Swarrow mT & TTT \arrow[dl, Rightarrow, "\mu T"] \\
 & T \arrow[r, Rightarrow, "\eta T"'] \arrow[u, phantom, "\Rrightarrow \overline{\eta T}"] & TT &
\end{tikzcd}$$

is equal to the 3-cell

$$\begin{tikzcd}
& TTT \arrow[r, Rightarrow, "TT\eta T"] \arrow[dl, Rightarrow, "\mu T"'] \arrow[dr, Rightarrow, "T\mu"'] & TTTT \arrow[dr, Rightarrow, "T\mu T"] \arrow[d, phantom, "\Downarrow T\overline{\eta T}"] & \\
TT \arrow[dr, Rightarrow, "\mu"'] \arrow[r, phantom, "\Lleftarrow m"] & {} & TT \arrow[r, Rightarrow, "T\eta T"] \arrow[dl, Rightarrow, "\mu"] & TTT \arrow[dl, Rightarrow, "\mu T"] \\
& T \arrow[r, Rightarrow, "\eta T"'] & TT \arrow[ur, phantom, "\Lleftarrow \overline{\eta T}"] &
\end{tikzcd}$$

We claim that

$$\begin{tikzcd}
T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, Rightarrow, "1_T"'] & TT \arrow[d, Rightarrow, "\mu"] \arrow[dl, phantom, "t \Lleftarrow"] \\
& T
\end{tikzcd}$$

is the *invertible* counit of $\mu \dashv \eta T$ with the unit

$$\begin{tikzcd}
TT \arrow[d, Rightarrow, "\mu"'] \arrow[dr, Rightarrow, "1"] & \\
T \arrow[r, Rightarrow, "\eta T"'] \arrow[ur, phantom, "u \Lleftarrow"] & TT
\end{tikzcd}$$

given by the pasting

$$\begin{tikzcd}
TT \arrow[r, Rightarrow, "T\eta T"] \arrow[rr, Rightarrow, bend left=60, "1", "sT\Downarrow"'] \arrow[dr, Rightarrow, "\mu"'] & TTT \arrow[r, Rightarrow, "\mu T"] \arrow[d, phantom, "\Downarrow\overline{\eta T}"] & TT. \\
& T \arrow[ur, Rightarrow, "\eta T"'] &
\end{tikzcd}$$

Observe that

$$\begin{tikzcd}
T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, Rightarrow, "1_T"'] & TT \arrow[d, Rightarrow, "\mu"] \arrow[dr, Rightarrow, "1"] \arrow[dl, phantom, "t \Lleftarrow"] & \\
& T \arrow[r, Rightarrow, "\eta T"'] \arrow[ur, phantom, "u \Lleftarrow"] & TT
\end{tikzcd}$$

is, by definition of u, the 3-cell

$$\begin{tikzcd}
 & & sT\Ddownarrow & \\
 & TT \arrow[r, Rightarrow, "T\eta T"] \arrow[rr, Rightarrow, bend left=70, "1"] \arrow[ddr, Rightarrow, "\mu"'] & TTT \arrow[r, Rightarrow, "\mu T"] & TT \\
 & t\Ddownarrow & \Ddownarrow\overline{\eta T} & \\
T \arrow[uur, Rightarrow, "\eta T"] \arrow[rr, Rightarrow, "1"'] & & T \arrow[uur, Rightarrow, "\eta T"'] &
\end{tikzcd}$$

and by the unit law for $\overline{\eta T}$ this equals

$$\begin{tikzcd}
 & T \arrow[dr, Rightarrow, "T\eta"] \arrow[drr, Rightarrow, bend left, "1_T"] \arrow[drr, phantom, "\Ddownarrow s"] & & \\
1 \arrow[ur, Rightarrow, "\eta"] \arrow[dr, Rightarrow, "\eta"'] & \Ddownarrow\eta_\eta & TT \arrow[r, Rightarrow, "\mu"] & T \\
 & T \arrow[ur, Rightarrow, "\eta T"] \arrow[urr, Rightarrow, bend right, "1_T"'] \arrow[urr, phantom, "\Ddownarrow t"] & &
\end{tikzcd}$$

which is (by virtue of T being a pseudomonad) equal to the identity 3-cell

$$\begin{tikzcd}
T \arrow[r, Rightarrow, bend left, "\eta T"] \arrow[r, Rightarrow, bend right, "\eta T"'] \arrow[r, phantom, "\Ddownarrow 1"] & TT.
\end{tikzcd}$$

For the other triangle equality, observe that

$$\begin{tikzcd}
TT \arrow[d, Rightarrow, "\mu"'] \arrow[dr, Rightarrow, "1"] \arrow[dr, phantom, "u\Swarrow"' near end] & \\
T \arrow[r, Rightarrow, "\eta T"'] \arrow[dr, Rightarrow, "1"'] \arrow[dr, phantom, "t\Swarrow" near start] & TT \arrow[d, Rightarrow, "\mu"] \\
 & T
\end{tikzcd}$$

is equal to

$$\begin{tikzcd}
 & sT\Ddownarrow & & \\
TT \arrow[r, Rightarrow, "T\eta T"] \arrow[rr, Rightarrow, bend left=70, "1"] \arrow[ddr, Rightarrow, "\mu"'] & TTT \arrow[r, Rightarrow, "\mu T"] & TT \arrow[ddr, Rightarrow, "\mu"] & \\
 & \Ddownarrow\overline{\eta T} & t\Ddownarrow & \\
 & T \arrow[uur, Rightarrow, "\eta T"'] \arrow[rr, Rightarrow, "1"'] & & T.
\end{tikzcd} \tag{10.1}$$

Now since

$$\begin{tikzcd}
TT \arrow[r, Rightarrow, "T\eta T"] \arrow[d, Rightarrow, "\mu"'] \arrow[dr, phantom, "\Swarrow \overline{\eta T}"] & TTT \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"] \arrow[dr, phantom, "\Swarrow m"] & TT \arrow[d, Rightarrow, "\mu"] \\
T \arrow[r, Rightarrow, "\eta T"'] \arrow[rr, Rightarrow, bend right=60, "1"'] \arrow[rr, phantom, bend right=30, "t\Downarrow"] & TT \arrow[r, Rightarrow, "\mu"'] & T
\end{tikzcd}$$

equals

$$\begin{tikzcd}
 & TTT \arrow[dr, Rightarrow, "T\mu"] \arrow[d, phantom, "Tt\Downarrow"] & & \\
TT \arrow[ur, Rightarrow, "T\eta T"] \arrow[rr, Rightarrow, "1"] \arrow[dr, Rightarrow, "T\eta T"'] \arrow[ddr, Rightarrow, bend right, "\mu"'] & {} & TT \arrow[r, Rightarrow, "\mu"] \arrow[r, phantom, shift right=3, "\Downarrow t"'] & T, \\
 & TTT \arrow[ur, Rightarrow, "\mu T"'] \arrow[u, phantom, "sT\Downarrow"] \arrow[d, phantom, "\Downarrow\overline{\eta T}"] & & \\
 & T \arrow[uur, Rightarrow, bend right, "\eta T"'] \arrow[uurr, Rightarrow, bend right, "1"'] & &
\end{tikzcd}$$

the diagram (10.1) equals

$$\begin{tikzcd}
TT \arrow[rr, Rightarrow, bend left=60, "1"] \arrow[rr, phantom, bend left=30, "Tt^{-1}\Downarrow"] \arrow[r, Rightarrow, "T\eta T"] \arrow[d, Rightarrow, "\mu"'] \arrow[dr, phantom, "\Swarrow \overline{\eta T}"] & TTT \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"] \arrow[dr, phantom, "\Swarrow m"] & TT \arrow[d, Rightarrow, "\mu"] \\
T \arrow[r, Rightarrow, "\eta T"'] \arrow[rr, Rightarrow, bend right=60, "1"'] \arrow[rr, phantom, bend right=30, "t\Downarrow"] & TT \arrow[r, Rightarrow, "\mu"'] & T
\end{tikzcd}$$

and by 2 the above diagram in turn equals

$$\begin{tikzcd}
TT \arrow[rr, Rightarrow, "1"] \arrow[d, Rightarrow, "\mu"'] \arrow[drr, phantom, "\Swarrow 1"] & & TT \arrow[d, Rightarrow, "\mu"] \\
T \arrow[rr, Rightarrow, "1"'] \arrow[dr, Rightarrow, "\eta T"'] \arrow[rr, Rightarrow, bend right=90, looseness=2.5, "1"'] & & T \\
& TT \arrow[ur, Rightarrow, "\mu"'] \arrow[u, phantom, "t^{-1}\Downarrow"] \arrow[d, phantom, "t\Downarrow"] & \\
& {} &
\end{tikzcd}$$

and thus is the identity 3-cell on $\mu : TT \Rightarrow T$. Hence the implication 2 $\implies$ 1 holds.

To complete the proof of the theorem, we prove the implication 4 $\implies$ 2. Consider two pseudoalgebras (X, a, a_0, a_1) and (Y, b, b_0, b_1) We assume that $a \dashv \eta X$ and $b \dashv \eta Y$ have counits a_0 and b_0, and units u and v, respectively. Now consider a 2-cell $h : X \Rightarrow Y$. To give

$$\begin{tikzcd}
TX \arrow[r, Rightarrow, "Th"] \arrow[d, Rightarrow, "a"'] \arrow[dr, phantom, "\Swarrow \rho"] & TY \arrow[d, Rightarrow, "b"] \\
X \arrow[r, Rightarrow, "h"'] & Y
\end{tikzcd}$$

is equivalently to give its mate

$$\begin{tikzcd}
TX \arrow[r, Rightarrow, "Th"] \arrow[dr, phantom, "\Searrow \widehat{\rho}"] & TY & & X \arrow[r, Rightarrow, "\eta X"] \arrow[dr, Rightarrow, "1"'] & TX \arrow[r, Rightarrow, "Th"] \arrow[d, Rightarrow, "a"] \arrow[l, phantom, shift right=3, "\Swarrow a_0"] \arrow[dr, phantom, "\Swarrow \rho"] & TY \arrow[d, Rightarrow, "b"] \arrow[dr, Rightarrow, "1"] & \\
X \arrow[r, Rightarrow, "h"'] \arrow[u, Rightarrow, "\eta X"] & Y \arrow[u, Rightarrow, "\eta Y"'] & = & & X \arrow[r, Rightarrow, "h"'] & Y \arrow[r, Rightarrow, "\eta Y"'] \arrow[ur, phantom, "\Swarrow v"] & TY.
\end{tikzcd}$$

Since any lax homomorphism (h, ρ) has to satisfy that

$$\begin{tikzcd}
& & TY \arrow[dr, Rightarrow, "b"] & \\
X \arrow[r, Rightarrow, "\eta X"] \arrow[drr, Rightarrow, bend right, "1"'] & TX \arrow[ur, Rightarrow, "Th"] \arrow[dr, Rightarrow, "a"] \arrow[d, phantom, "\Downarrow a_0"] \arrow[rr, phantom, "\Downarrow \rho"] & & Y \\
& {} & X \arrow[ur, Rightarrow, "h"'] &
\end{tikzcd}$$

is equal to

$$\begin{tikzcd}
 & TX \arrow[dr, Rightarrow, "Th"] & & \\
X \arrow[ur, Rightarrow, "\eta X"] \arrow[dr, Rightarrow, "h"'] & \Downarrow \eta_h & TY \arrow[r, Rightarrow, "b"] & Y \\
 & Y \arrow[ur, Rightarrow, "\eta Y"'] \arrow[urr, Rightarrow, bend right, "1"'] & \Downarrow b_0 &
\end{tikzcd}$$

we see that the only way to define ρ is to take it to be the mate of η_h; that is,

$$\begin{tikzcd}
TX \arrow[r, Rightarrow, "Th"] \arrow[d, Rightarrow, "a"'] \arrow[dr, phantom, "\swarrow \rho"] & TY \arrow[d, Rightarrow, "b"] & & & TX \arrow[r, Rightarrow, "Th"] \arrow[dr, phantom, "\eta_h \searrow"] & TY \arrow[r, Rightarrow, "b"] & Y \\
X \arrow[r, Rightarrow, "h"'] & Y & = & TX \arrow[ur, Rightarrow, "1"] \arrow[r, Rightarrow, "a"'] \arrow[ur, phantom, "\searrow u"', shift right=3] & X \arrow[u, Rightarrow, "\eta X"'] \arrow[r, Rightarrow, "h"'] & Y \arrow[u, Rightarrow, "\eta Y"'] \arrow[ur, Rightarrow, "1"'] \arrow[ur, phantom, "b_0 \searrow", shift left=3] &
\end{tikzcd}$$

We now have to show that such a definition of (h, ρ) indeed gives a lax homomorphism. One part follows easily: the 3-cell

$$\begin{tikzcd}
 & TX \arrow[r, Rightarrow, "Th"] \arrow[dr, phantom, "\eta_h \searrow"] & TY \arrow[r, Rightarrow, "b"] & Y \\
TX \arrow[ur, Rightarrow, "1"] \arrow[r, Rightarrow, "a"'] \arrow[ur, phantom, "\searrow u"', shift right=3] & X \arrow[u, Rightarrow, "\eta X"'] \arrow[r, Rightarrow, "h"'] & Y \arrow[u, Rightarrow, "\eta Y"'] \arrow[ur, Rightarrow, "1"'] \arrow[ur, phantom, "b_0 \searrow", shift left=3] & \\
X \arrow[u, Rightarrow, "\eta X"] \arrow[ur, Rightarrow, "1"'] \arrow[ur, phantom, "a_0 \searrow", shift left=3] & & &
\end{tikzcd}$$

is equal to

$$\begin{tikzcd}
TX \arrow[r, Rightarrow, "Th"] \arrow[dr, phantom, "\eta_h \searrow"] & TY \arrow[r, Rightarrow, "b"] & Y \\
X \arrow[u, Rightarrow, "\eta X"] \arrow[r, Rightarrow, "h"'] & Y \arrow[u, Rightarrow, "\eta Y"'] \arrow[ur, Rightarrow, "1"'] \arrow[ur, phantom, "b_0 \searrow", shift left=3] &
\end{tikzcd}$$

since $a \dashv \eta X$. For the second identity of lax homomorphisms we need to check that the

3-cell

$$\begin{tikzcd}
TTX \arrow[rr, Rightarrow, "TTh"] \arrow[dd, Rightarrow, "\mu X"'] \arrow[drr, phantom, "\Downarrow {\mu_h}^{-1}"] & & TTY \arrow[r, Rightarrow, "Tb"] \arrow[d, Rightarrow, "\mu Y"'] \arrow[dr, phantom, "\Swarrow b_1"] & TY \arrow[d, Rightarrow, "b"] \\
& TX \arrow[r, Rightarrow, "Th"] & TY \arrow[r, Rightarrow, "b"] \arrow[dr, phantom, "b_0 \Searrow"] & Y \\
TX \arrow[r, Rightarrow, "a"'] \arrow[ur, Rightarrow, "1"] \arrow[r, phantom, shift left=4, "\Searrow u"] & X \arrow[r, Rightarrow, "h"'] \arrow[u, Rightarrow, "\eta X"] \arrow[ur, phantom, "\eta_h \Searrow"] & Y \arrow[u, Rightarrow, "\eta Y"] \arrow[ur, Rightarrow, "1"']
\end{tikzcd} \tag{10.2}$$

equals

$$\begin{tikzcd}
& TTX \arrow[r, Rightarrow, "TTh"] \arrow[dl, Rightarrow, "\mu X"'] \arrow[dr, Rightarrow, "Ta"] \arrow[dd, phantom, "\Swarrow a_1"] & TTY \arrow[dr, Rightarrow, "Tb"] \arrow[d, phantom, "\Swarrow T\rho"] & \\
TX \arrow[dr, Rightarrow, "a"'] & & TX \arrow[r, Rightarrow, "Th"] \arrow[dl, Rightarrow, "a"] \arrow[d, phantom, "\Downarrow \rho"] & TY \arrow[dl, Rightarrow, "b"] \\
& X \arrow[r, Rightarrow, "h"'] & Y &
\end{tikzcd} \tag{10.3}$$

To prove this, we show that the mate of (10.2) is equal to the mate of (10.3), i.e., that

$$\begin{tikzcd}
TX \arrow[r, Rightarrow, "T\eta X"] \arrow[ddr, Rightarrow, bend right, "1"'] \arrow[ddr, phantom, "\Swarrow s^{-1}X"] & TTX \arrow[rr, Rightarrow, "TTh"] \arrow[dd, Rightarrow, "\mu X"] \arrow[drr, phantom, "\Downarrow {\mu_h}^{-1}"] & & TTY \arrow[r, Rightarrow, "Tb"] \arrow[d, Rightarrow, "\mu Y"'] \arrow[dr, phantom, "\Swarrow b_1"] & TY \arrow[d, Rightarrow, "b"] \arrow[dr, Rightarrow, "1"] \arrow[dr, phantom, shift right=3, "\Swarrow v"] & \\
& & TX \arrow[r, Rightarrow, "Th"] & TY \arrow[r, Rightarrow, "b"] \arrow[dr, phantom, "b_0 \Searrow"] & Y \arrow[r, Rightarrow, "\eta Y"'] & TY \\
& TX \arrow[r, Rightarrow, "a"'] \arrow[ur, Rightarrow, "1"] \arrow[r, phantom, shift left=4, "\Searrow u"] & X \arrow[r, Rightarrow, "h"'] \arrow[u, Rightarrow, "\eta X"] \arrow[ur, phantom, "\eta_h \Searrow"] & Y \arrow[u, Rightarrow, "\eta Y"] \arrow[ur, Rightarrow, "1"'] & &
\end{tikzcd}$$

equals

$$\begin{tikzcd}
TX \arrow[r, Rightarrow, "T\eta X"] \arrow[d, Rightarrow, "1"'] \arrow[dr, phantom, "\Swarrow s^{-1}X"] & TTX \arrow[r, Rightarrow, "TTh"] \arrow[dl, Rightarrow, "\mu X"] \arrow[dr, Rightarrow, "Ta"] \arrow[dd, phantom, "\Swarrow a_1"] & TTY \arrow[dr, Rightarrow, "Tb"] \arrow[d, phantom, "\Downarrow T\rho"] & \\
TX \arrow[dr, Rightarrow, "a"'] & & TX \arrow[r, Rightarrow, "Th"] \arrow[dl, Rightarrow, "a"] \arrow[d, phantom, "\Swarrow \rho"] & TY \arrow[dl, Rightarrow, "b"'] \arrow[d, Rightarrow, "1"] \arrow[dl, phantom, shift left=4, "\Swarrow v"] \\
& X \arrow[r, Rightarrow, "h"'] & Y \arrow[r, Rightarrow, "\eta Y"'] & TY.
\end{tikzcd}$$

The latter diagram is equal to

$$\begin{tikzcd}
 & TTX \arrow[r, Rightarrow, "TTh"] \arrow[dr, Rightarrow, "Ta"'] \arrow[d, phantom, "\Downarrow Ta_0"] & TTY \arrow[dr, Rightarrow, "Tb"] \arrow[d, phantom, "\Downarrow T\rho"] & \\
TX \arrow[ur, Rightarrow, "T\eta X"] \arrow[rr, Rightarrow, "1"'] & & TX \arrow[r, Rightarrow, "Th"] \arrow[dl, Rightarrow, "a"] \arrow[d, phantom, "\Swarrow \rho"] & TY \arrow[dl, Rightarrow, "b"'] \arrow[d, Rightarrow, "1"] \arrow[dl, phantom, "\Swarrow v" near end] \\
 & X \arrow[r, Rightarrow, "h"'] & Y \arrow[r, Rightarrow, "\eta Y"'] & TY
\end{tikzcd}$$

by the unit axiom of the pseudoalgebra a. Expanding this diagram by substituting the definition of ρ, we get the diagram

$$\begin{tikzcd}
TX \arrow[r, Rightarrow, "T\eta X"] \arrow[dr, Rightarrow, "1"'] & TTX \arrow[d, Rightarrow, "Ta"] \arrow[dr, Rightarrow, "1"] \arrow[dl, phantom, "\Swarrow Ta_0"] \arrow[dr, phantom, "\Swarrow Tu"'] & & \\
 & TX \arrow[r, Rightarrow, "T\eta X"] \arrow[d, Rightarrow, "a"'] \arrow[dr, Rightarrow, "1"] \arrow[dr, phantom, "\Swarrow u"'] & TTX \arrow[ddr, Rightarrow, bend left, "TTh"] \arrow[d, phantom, "\Swarrow T\eta_h"] & \\
 & X \arrow[r, Rightarrow, "\eta X"] \arrow[d, Rightarrow, "h"'] \arrow[dr, phantom, "\Swarrow \eta_h"] & TX \arrow[d, Rightarrow, "Th"] & \\
 & Y \arrow[r, Rightarrow, "\eta Y"] \arrow[dr, Rightarrow, "1"'] & TY \arrow[r, Rightarrow, "T\eta Y"] \arrow[d, Rightarrow, "b"] \arrow[dr, Rightarrow, "1"] \arrow[l, phantom, "\Swarrow b_0" near start] \arrow[dr, phantom, "\Swarrow v"'] & TTY \arrow[d, Rightarrow, "Tb"] \arrow[dl, phantom, "\Swarrow Tb_0"] \\
 & & Y \arrow[r, Rightarrow, "\eta Y"'] & TY,
\end{tikzcd}$$

and since

$$\begin{tikzcd}
TX \arrow[r, Rightarrow, "T\eta X"] \arrow[dr, Rightarrow, "1"'] & TTX \arrow[d, Rightarrow, "Ta"] \arrow[dr, Rightarrow, "1"] \arrow[dl, phantom, "\Swarrow Ta_0"] \arrow[dr, phantom, "\Swarrow Tu"'] & \\
 & TX \arrow[r, Rightarrow, "T\eta X"] & TTX
\end{tikzcd} = TX \xRightarrow{T\eta X} TTX,$$

we get

$$\begin{tikzcd}
TX \arrow[r, Rightarrow, "T\eta X"] \arrow[d, Rightarrow, "a"'] \arrow[dr, Rightarrow, "1"] \arrow[dr, phantom, "\Swarrow u"', shift right=4] & TTX \arrow[ddr, Rightarrow, bend left, "TTh"] \arrow[dr, phantom, "\Swarrow T\eta_h"] & \\
X \arrow[r, Rightarrow, "\eta X"] \arrow[d, Rightarrow, "h"'] \arrow[dr, phantom, "\Swarrow \eta_h"] & TX \arrow[d, Rightarrow, "Th"] & \\
Y \arrow[r, Rightarrow, "\eta Y"] \arrow[dr, Rightarrow, "1"'] \arrow[dr, phantom, "\Swarrow b_0", shift left=4] & TY \arrow[r, Rightarrow, "T\eta Y"] \arrow[d, Rightarrow, "b"] \arrow[dr, Rightarrow, "1"] \arrow[dr, phantom, "\Swarrow Tb_0", shift left=4] \arrow[dr, phantom, "\Swarrow v"', shift right=4] & TTY \arrow[d, Rightarrow, "Tb"] \\
& Y \arrow[r, Rightarrow, "\eta Y"'] & TY.
\end{tikzcd}$$

Removing the 3-cell

$$\begin{tikzcd}
& TX \arrow[r, Rightarrow, "Th"] & TY \arrow[r, Rightarrow, "b"] & Y \\
TX \arrow[ur, Rightarrow, "1"] \arrow[r, Rightarrow, "a"'] \arrow[ur, phantom, "\Searrow u"', shift right=4] & X \arrow[u, Rightarrow, "\eta X"] \arrow[r, Rightarrow, "h"'] \arrow[ur, phantom, "\eta_h \Searrow"] & Y \arrow[u, Rightarrow, "\eta Y"] \arrow[ur, Rightarrow, "1"'] \arrow[ur, phantom, "b_0 \Searrow", shift left=4] &
\end{tikzcd}$$

from both diagrams, we ask whether

$$\begin{tikzcd}
TX \arrow[r, Rightarrow, "T\eta X"] \arrow[dr, Rightarrow, "1"'] \arrow[dr, phantom, "\Swarrow s^{-1}X", shift left=4] & TTX \arrow[r, Rightarrow, "TTh"] \arrow[d, Rightarrow, "\mu X"'] \arrow[dr, phantom, "\Swarrow {\mu_h}^{-1}"] & TTY \arrow[r, Rightarrow, "Tb"] \arrow[d, Rightarrow, "\mu Y"] \arrow[dr, phantom, "\Swarrow b_1"] & TY \arrow[d, Rightarrow, "b"] \arrow[dr, Rightarrow, "1"] \arrow[dr, phantom, "\Swarrow v"', shift right=4] & \\
& TX \arrow[r, Rightarrow, "Th"'] & TY \arrow[r, Rightarrow, "b"'] & Y \arrow[r, Rightarrow, "\eta Y"'] & TY
\end{tikzcd}$$

equals

$$\begin{tikzcd}
TX \arrow[r, Rightarrow, "T\eta X"] \arrow[d, Rightarrow, "Th"] \arrow[dr, phantom, "\Swarrow T\eta_h"] & TTX \arrow[d, Rightarrow, "TTh"] \\
TY \arrow[r, Rightarrow, "T\eta Y"] \arrow[d, Rightarrow, "b"] \arrow[dr, Rightarrow, "1"] \arrow[dr, phantom, "\Swarrow Tb_0", shift left=4] \arrow[dr, phantom, "\Swarrow v"', shift right=4] & TTY \arrow[d, Rightarrow, "Tb"] \\
Y \arrow[r, Rightarrow, "\eta Y"'] & TY.
\end{tikzcd}$$

Using that

$$\begin{tikzcd}
TX \arrow[d, Rightarrow, "Th"'] & & \\
TY \arrow[rr, Rightarrow, "1"] \arrow[dr, Rightarrow, "T\eta Y"'] & {\Downarrow sY} & TY \\
& TTY \arrow[ur, Rightarrow, "\mu Y"'] &
\end{tikzcd}$$

equals

$$\begin{tikzcd}
TX \arrow[rr, Rightarrow, bend left=40, "1"] \arrow[r, Rightarrow, "T\eta X"] \arrow[d, Rightarrow, "Th"'] \arrow[dr, phantom, "\Swarrow T\eta_h"] & TTX \arrow[r, Rightarrow, "\mu X"] \arrow[d, Rightarrow, "TTh"'] \arrow[dr, phantom, "\Swarrow \mu_h"] \arrow[r, phantom, shift left=5ex, "\Downarrow sX"] & TX \arrow[d, Rightarrow, "Th"] \\
TY \arrow[r, Rightarrow, "T\eta Y"'] & TTY \arrow[r, Rightarrow, "\mu Y"'] & TY,
\end{tikzcd}$$

we may equivalently ask whether

$$\begin{tikzcd}
TX \arrow[r, Rightarrow, "Th"] & TY \arrow[r, Rightarrow, "T\eta Y"] \arrow[dr, Rightarrow, "1"'] & TTY \arrow[r, Rightarrow, "Tb"] \arrow[d, Rightarrow, "\mu Y"] \arrow[dl, phantom, "\Swarrow s^{-1}Y"] \arrow[dr, phantom, "\Swarrow b_1"] & TY \arrow[d, Rightarrow, "b"] \arrow[dr, Rightarrow, "1"] & \\
& & TY \arrow[r, Rightarrow, "b"'] & Y \arrow[r, Rightarrow, "\eta Y"'] \arrow[u, phantom, shift right=4ex, "\Swarrow v"] & TY
\end{tikzcd}$$

equals

$$\begin{tikzcd}
TX \arrow[r, Rightarrow, "Th"] & TY \arrow[r, Rightarrow, "T\eta Y"] \arrow[d, Rightarrow, "b"] \arrow[dr, Rightarrow, "1"] & TTY \arrow[d, Rightarrow, "Tb"] \arrow[dl, phantom, "\Swarrow Tb_0"] \\
& Y \arrow[r, Rightarrow, "\eta Y"'] \arrow[ur, phantom, "\Swarrow v"] & TY.
\end{tikzcd}$$

This follows from the unit pseudoalgebra axiom; therefore the proof is complete. ■

We can now state an extended characterisation of KZ-pseudoadjunctions and KZ-pseudomonads.

Theorem 10.3.4 (A characterisation of KZ-pseudoadjunctions and KZ-pseudomonads). *Given a pseudoadjunction $F \dashv U : \mathscr{A} \to \mathscr{X}$, the following conditions are equivalent:*

1. *$F \dashv U : \mathscr{A} \to \mathscr{X}$ is a KZ-pseudoadjunction.*

2. *The 3-cell*

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "1_F"'] \arrow[dr, phantom, "s \Rrightarrow" {above right, pos=0.4}] & FUF \arrow[d, Rightarrow, "\varepsilon F"] \\
& F
\end{tikzcd}$$

is the unit of the (ordinary) adjunction $F\eta \dashv \varepsilon F$*.*

3. *The 3-cell*

$$\begin{tikzcd}
UF \arrow[r, Rightarrow, "UF\eta"] \arrow[dr, Rightarrow, "1_{UF}"'] \arrow[dr, phantom, "Us \Rrightarrow" {above right, pos=0.4}] & UFUF \arrow[d, Rightarrow, "U\varepsilon F"] \\
& UF
\end{tikzcd}$$

is the unit of the (ordinary) adjunction $UF\eta \dashv U\varepsilon F$*.*

4. *There is a 3-cell*

$$\begin{tikzcd}
UF \arrow[r, Rightarrow, bend left, "UF\eta"] \arrow[r, Rightarrow, bend right, "\eta UF"'] \arrow[r, phantom, "\Downarrow d"] & UFUF
\end{tikzcd}$$

satisfying the equalities

$$\begin{tikzcd}
& & & 1 \arrow[r, Rightarrow, "\eta"] \arrow[d, Rightarrow, "\eta"'] \arrow[dr, phantom, "\eta_\eta \Lleftarrow"] & UF \arrow[d, Rightarrow, "UF\eta"] \\
1 \arrow[r, Rightarrow, "\eta"] & UF \arrow[r, Rightarrow, bend left, "UF\eta"] \arrow[r, Rightarrow, bend right, "\eta UF"'] \arrow[r, phantom, "\Downarrow d"] & UFUF & = & \\
& & & UF \arrow[r, Rightarrow, "\eta UF"'] & UFUF
\end{tikzcd}$$

and

$$\begin{tikzcd}
& & & & UF \arrow[r, Rightarrow, "UF\eta"] \arrow[d, Rightarrow, "\eta UF"'] \arrow[dr, Rightarrow, "1_{UF}"] & UFUF \arrow[d, Rightarrow, "U\varepsilon F"] \\
UF \arrow[r, Rightarrow, bend left, "UF\eta"] \arrow[r, Rightarrow, bend right, "\eta UF"'] \arrow[r, phantom, "\Downarrow d"] & UFUF \arrow[r, Rightarrow, "U\varepsilon F"] & UF & = & & \\
& & & & UFUF \arrow[r, Rightarrow, "U\varepsilon F"'] & UF.
\end{tikzcd}$$

$$Us^{-1} \Lleftarrow \qquad t^{-1}F \Lleftarrow$$

5. *There is a 3-cell*

$$\begin{tikzcd}
UFU \arrow[r, Rightarrow, bend left, "UF\eta U"] \arrow[r, Rightarrow, bend right, "\eta UFU"'] \arrow[r, phantom, "\Downarrow d'"] & UFUFU
\end{tikzcd}$$

satisfying the equalities

$$\begin{tikzcd} U \arrow[r, Rightarrow, "\eta U"] & UFU \arrow[r, Rightarrow, bend left, "UF\eta U"] \arrow[r, Rightarrow, bend right, "\eta UFU"'] \arrow[r, phantom, "\Downarrow d'"] & UFUFU \end{tikzcd} = \begin{tikzcd} U \arrow[r, Rightarrow, "\eta U"] \arrow[d, Rightarrow, "\eta U"'] \arrow[dr, phantom, "\eta_\eta U \Lleftarrow"] & UFU \arrow[d, Rightarrow, "UF\eta U"] \\ UFU \arrow[r, Rightarrow, "\eta UFU"'] & UFUFU \end{tikzcd}$$

and

$$\begin{tikzcd} UFU \arrow[r, Rightarrow, bend left, "UF\eta U"] \arrow[r, Rightarrow, bend right, "\eta UFU"'] \arrow[r, phantom, "\Downarrow d'"] & UFUFU \arrow[r, Rightarrow, "U\varepsilon FU"] & UFU \end{tikzcd} = \begin{tikzcd} UFU \arrow[r, Rightarrow, "UF\eta U"] \arrow[d, Rightarrow, "\eta UFU"'] \arrow[dr, Rightarrow, "1_{UFU}"'] \arrow[dr, phantom, shift left=4, "Us^{-1}U \Lleftarrow"] \arrow[dr, phantom, shift right=4, "t^{-1}FU \Lleftarrow"] & UFUFU \arrow[d, Rightarrow, "U\varepsilon FU"] \\ UFUFU \arrow[r, Rightarrow, "U\varepsilon FU"'] & UFU. \end{tikzcd}$$

6. *The 3-cell*

$$\begin{tikzcd} U \arrow[r, Rightarrow, "\eta U"] \arrow[dr, Rightarrow, "1_U"'] \arrow[dr, phantom, shift left=3, "t \Lleftarrow"] & UFU \arrow[d, Rightarrow, "U\varepsilon"] \\ & U \end{tikzcd}$$

is the counit of the adjunction $\eta U \dashv U\varepsilon$.

7. *The pseudomonad $(UF, \eta, U\varepsilon F)$ induced by $U \dashv F$ is a KZ-pseudomonad.*

8. *The pseudomonad $(UF, \eta, U\varepsilon F)$ induced by $U \dashv F$ satisfies the (AEL) condition.*

9. *The 3-cell*

$$\begin{tikzcd} FU \arrow[r, Rightarrow, "F\eta U"] \arrow[dr, Rightarrow, "1_{FU}"'] \arrow[dr, phantom, shift left=3, "Ft \Lleftarrow"] & FUFU \arrow[d, Rightarrow, "FU\varepsilon"] \\ & FU \end{tikzcd}$$

is the counit of the adjunction $F\eta U \dashv FU\varepsilon$.

Remark 10.3.5 (KZ-pseudoadjunctions in Pos). When considering KZ-pseudoadjunctions and KZ-pseudomonads in Pos-enriched categories, their definitions can be significantly simplified. A KZ-pseudoadjunction in a Pos-category $\mathscr{K}$ is an adjunction $F \dashv U : A \to X$ with unit η and counit ε such that there is any of the following two inequalities

$$\begin{tikzcd} FUF \arrow[d, "\varepsilon F"'] \arrow[dr, "1"] & \\ F \arrow[r, "F\eta"'] \arrow[ur, phantom, "\Swarrow"] & FUF, \end{tikzcd} \qquad \begin{tikzcd} UFU \arrow[d, "U\varepsilon"'] \arrow[dr, "1"] & \\ U \arrow[r, "\eta U"'] \arrow[ur, phantom, "\Nearrow"] & UFU \end{tikzcd}$$

(and thus, by Theorem 10.3.4, both of them). Every such KZ-pseudoadjunction gives rise to a KZ-pseudomonad, which is, in the case of Pos-enriched categories, a monad with unit η such that $T\eta \leqslant \eta T$ holds (compare to condition 4 of Theorem 10.3.4). These conditions are equivalent to the (AEL) condition, which is in the case of Pos-enriched categories equivalent to the fact that that the obvious forgetful functor from the category of algebras and lax homomorphisms for T is locally fully faithful.

Applications of KZ-pseudomonads (and the dual notion of co-KZ-pseudomonads, for which $\eta T \leqslant T\eta$ holds) in the context of Pos-enrichment arise in theoretical computer science, e.g. in domain theory. See for example [34] and its study of semantic domains via KZ-pseudomonads.

Chapter 11

Wreaths for pseudomonads

Consider the ordinary situation of a category $\mathscr{X}$ together with a monad (T, η^T, μ^T) on $\mathscr{X}$. Suppose moreover that we are given an endofunctor $H : \mathscr{X} \to \mathscr{X}$. When does H admit a *lifting* to a functor $\widetilde{H} : \mathsf{Alg}(T) \to \mathsf{Alg}(T)$ such that the diagram

$$\begin{tikzcd}
\mathsf{Alg}(T) \arrow[r, "\widetilde{H}"] \arrow[d, "U^T"'] & \mathsf{Alg}(T) \arrow[d, "U^T"] \\
\mathscr{X} \arrow[r, "H"'] & \mathscr{T}
\end{tikzcd}$$

commutes? In this ordinary situation the answer is easy: H admits such a lifting precisely when there is a natural transformation

$$\chi : TH \Rightarrow HT$$

that satisfies the unit axiom

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "H"] \arrow[d, "T"'] \arrow[dr, phantom, "\swarrow\chi"] & \mathscr{T} \arrow[d, bend right=50, "T"'] \arrow[d, bend left=50, "1"] \arrow[d, phantom, "\overset{\eta^T}{\Leftarrow}"] & & \mathscr{X} \arrow[d, bend right=50, "T"'] \arrow[d, bend left=50, "1"] \arrow[d, phantom, "\overset{\eta^T}{\Leftarrow}"] & \\
\mathscr{X} \arrow[r, "H"'] & \mathscr{T} & = & \mathscr{X} \arrow[r, "H"'] & \mathscr{T}
\end{tikzcd}$$

and the multiplication axiom

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "H"] \arrow[dd, "T"'] & \mathscr{X} \arrow[dd, "T"'] \arrow[dr, "T"] & & & \mathscr{X} \arrow[r, "H"] \arrow[d, "T"'] \arrow[dd, bend right=60, "T"'] \arrow[dr, phantom, "\swarrow\chi"] & \mathscr{X} \arrow[d, "T"] \\
\arrow[r, phantom, "\swarrow\chi"] & {} \arrow[r, phantom, "\swarrow\mu^T"] & \mathscr{X} \arrow[dl, "T"] & = & \mathscr{X} \arrow[r, "H"'] \arrow[d, "T"'] \arrow[l, phantom, "\swarrow\mu^T"] \arrow[dr, phantom, "\swarrow\chi"] & \mathscr{X} \arrow[d, "T"] \\
\mathscr{X} \arrow[r, "H"'] & \mathscr{X} & & & \mathscr{X} \arrow[r, "H"'] & \mathscr{X}
\end{tikzcd}$$

that make H together with $\chi : TH \Rightarrow HT$ into a *monad functor* in the sense of Street's [75].

Suppose H is moreover equipped with the structure of a monad. When does the monad structure lift to the category of algebras, i.e., when is $\widetilde{H}$ equipped with a monad structure derived from the monad structure of H? The answer is well known: this happens precisely when $\chi : TH \Rightarrow HT$ is a distributive law [12].

Lack and Street studied an even more general situation in [58]: given $H : \mathscr{X} \to \mathscr{X}$ (not necessarily carrying a monad structure itself), characterise the situations when the lifted functor $\widetilde{H}$ carries a monad structure. The resulting notion of a *wreath* generalises the notion of a distributive law.

In this chapter we will continue the study started by Marmolejo in [68] where distributive laws for pseudomonads were studied, and we will give a description of wreaths for pseudomonads. We will first recall the ordinary theory of wreaths in Section 11.1 and then use the approach taken in the ordinary setting in the setting of pseudomonads: we will construct a **Gray**-category of liftings in a **Gray**-category **K** to define wreaths, and then use a *tricategory triequivalent* to the **Gray**-category of liftings to obtain an "elementary" description of wreaths.

Structure of the chapter.

1. We review ordinary wreaths in Section 11.1.
2. Section 11.2 introduces the **Gray**-category of liftings in a **Gray**-category **K**.
3. Section 11.3 introduces the tricategory of transitions in a **Gray**-category **K**.
4. We describe the triequivalence between the tricategories of liftings and transitions in Section 11.4.
5. We use the triequivalence from Section 11.4 to obtain an elementary description of wreaths in Section 11.5.

The description of wreaths for pseudomonads is novel and has not appeared elsewhere yet.

11.1 Wreaths in ordinary categories

In this section we will give an overview of wreaths in the ordinary setting. No results are original here; we only expand [58]. However, we proceed in such a way that will allow us to generalise to wreaths in **Gray**-categories rather straightforwardly. We will consider monads in a general 2-category **K** (satisfying a slight completeness property – having Eilenberg-Moore objects). Then we will construct two *isomorphic* 2-categories out of **K**: the 2-category $\mathsf{LIFT}(\mathbf{K})$ of *liftings* in **K**, and the 2-category $\mathsf{TRANS}(\mathbf{K})$ of *transitions* in **K**.

The 2-category $\mathsf{LIFT}(\mathbf{K})$ will allow us to describe easily a situation when, given a monad T, a 1-cell $H : \mathscr{X} \to \mathscr{X}$ in **K** admits a lifting $\widetilde{H} : \mathscr{X}^T \to \mathscr{X}^T$ of H and a monad $(\widetilde{H}, \widetilde{\sigma}, \widetilde{\nu})$ on the Eilenberg-Moore object $\mathscr{X}^T$ for the monad T. That is, it will allow us to describe when a *wreath* of H around T exists.

The 2-category $\mathsf{TRANS}(\mathbf{K})$ (isomorphic to $\mathsf{LIFT}(\mathbf{K})$) will then be used to obtain an *elementary* description of wreaths.

The 2-category $\mathsf{LIFT}(\mathbf{K})$ of liftings in K Given a 2-category $\mathbf{K}$ that has all Eilenberg-Moore objects [75], $\mathsf{LIFT}(\mathbf{K})$ is defined in elementary terms as follows:

1. The objects of $\mathsf{LIFT}(\mathbf{K})$ are the 1-cells

$$\begin{tikzcd} \mathscr{X}^T \arrow[d, "U^T"'] \\ \mathscr{X}, \end{tikzcd}$$

that is, the "forgetful" 1-cells from the Eilenberg-Moore object $\mathscr{X}^T$ to the underlying object $\mathscr{X}$, where T is a monad on $\mathscr{X}$.

2. The 1-cells of $\mathsf{LIFT}(\mathbf{K})$ from U^T to U^S, *liftings*, are pairs $(H, \widetilde{H})$ with H and $\widetilde{H}$ being

$$H : \mathscr{X} \to \mathscr{Y}, \qquad \widetilde{H} : \mathscr{X}^T \to \mathscr{Y}^S$$

that make the following diagram

$$\begin{tikzcd} \mathscr{X}^T \arrow[r, "\widetilde{H}"] \arrow[d, "U^T"'] & \mathscr{Y}^S \arrow[d, "U^S"] \\ \mathscr{X} \arrow[r, "H"'] & \mathscr{Y} \end{tikzcd} \tag{11.1}$$

commute in $\mathbf{K}$.

3. The 2-cells $\widetilde{\rho} : (H, \widetilde{H}) \to (K, \widetilde{K})$ in $\mathsf{LIFT}(\mathbf{K})$, *lifting morphisms*, are the 2-cells

$$\begin{tikzcd} \mathscr{X}^T \arrow[r, bend left, "\widetilde{H}", ""{name=U, below}] \arrow[r, bend right, "\widetilde{K}"', ""{name=D, above}] & \mathscr{Y}^S \arrow[Rightarrow, from=U, to=D, "\widetilde{\rho}"] \end{tikzcd}$$

in $\mathbf{K}$.

The composition in $\mathsf{LIFT}(\mathbf{K})$ is retained from $\mathbf{K}$.

Remark 11.1.1. We see that the 1-cells capture precisely the situations where a 1-cell H can be lifted to a 1-cell "on algebras". However, the 2-cells are defined in a perhaps surprising way: they are *not* pairs $(\rho, \widetilde{\rho})$ satisfying the "obvious" equality

$$\begin{tikzcd} \mathscr{X}^T \arrow[r, bend left, "\widetilde{H}", ""{name=U, below}] \arrow[r, bend right, "\widetilde{K}"', ""{name=D, above}] & \mathscr{Y}^S \arrow[d, "U^S"] \arrow[Rightarrow, from=U, to=D, "\widetilde{\rho}"] \\ & \mathscr{Y} \end{tikzcd} = \begin{tikzcd} \mathscr{X}^T \arrow[d, "U^T"'] & \\ \mathscr{X} \arrow[r, bend left, "H", ""{name=A, below}] \arrow[r, bend right, "K"', ""{name=B, above}] & \mathscr{Y} \arrow[Rightarrow, from=A, to=B, "\rho"] \end{tikzcd}$$

meaning that $\widetilde{\rho}$ is a *lifted* 2-cell ρ. This definition is deliberate. We are interested in situations where H admits a monad $(\widetilde{H}, \widetilde{\sigma}, \widetilde{\nu})$ "on algebras" even when there is *not* a monad (H, σ, ν) of which $(\widetilde{H}, \widetilde{\sigma}, \widetilde{\nu})$ would be a lifting.

The 2-category $\mathsf{TRANS}(\mathbf{K})$ of transitions in $\mathbf{K}$ The 2-category $\mathsf{TRANS}(\mathbf{K})$ is defined in elementary terms as follows:

1. The objects are pairs $(\mathscr{X}, T)$, where $\mathscr{X}$ is an object of $\mathbf{K}$ and T is a monad on $\mathscr{X}$.

2. The 1-cells $(H, \chi) : (\mathscr{X}, T) \to (\mathscr{Y}, S)$ are *transitions* that consist of the data

$$H : \mathscr{X} \to \mathscr{Y}, \qquad \chi : SH \Rightarrow HT$$

subject to the commutativity of diagrams

$$\begin{tikzcd} H \arrow[r, Rightarrow, "\eta^S H"] \arrow[dr, Rightarrow, "H\eta^T"'] & SH \arrow[d, Rightarrow, "\chi"] \\ & HT \end{tikzcd}$$

and

$$\begin{tikzcd} SSH \arrow[r, Rightarrow, "S\chi"] \arrow[d, Rightarrow, "\mu^S H"'] & SHT \arrow[r, Rightarrow, "\chi T"] & HTT \arrow[d, Rightarrow, "H\mu^T"] \\ SH \arrow[rr, Rightarrow, "\chi"'] & & HT. \end{tikzcd}$$

3. The 2-cells $\rho : (H, \chi) \Rightarrow (K, \kappa)$ are *transition morphisms*, i.e., 2-cells

$$\rho : H \Rightarrow KT$$

such that the diagram

$$\begin{tikzcd} SH \arrow[rr, Rightarrow, "S\rho"] \arrow[dd, Rightarrow, "\chi"'] & & SKT \arrow[d, Rightarrow, "\kappa T"] \\ & & KTT \arrow[d, Rightarrow, "K\mu^T"] \\ HT \arrow[r, Rightarrow, "\rho T"'] & KTT \arrow[r, Rightarrow, "K\mu^T"'] & KT \end{tikzcd}$$

commutes.

Given transitions

$$(H,\chi) : (\mathscr{X},T) \to (\mathscr{Y},S), \qquad\qquad (K,\kappa) : (\mathscr{Y},S) \to (\mathscr{Z},U),$$

the composite transition is given by

$$(\mathscr{X},T) \xrightarrow{(KH,K\chi\cdot\kappa H)} (\mathscr{Z},U).$$

It is easy to check that the required diagrams indeed commute.

Given a situation

$$\begin{tikzcd}
(\mathscr{X},T) \arrow[r, bend left=50, "{(H,\chi)}"] \arrow[r, "{(K,\kappa)}"] \arrow[r, bend right=50, "{(L,\lambda)}"'] \arrow[r, phantom, shift left=3, "\Downarrow \rho"] \arrow[r, phantom, shift right=3, "\Downarrow \sigma"] & (\mathscr{Y},S),
\end{tikzcd}$$

the composite transition morphism $\sigma \cdot \rho$ is defined as

$$H \xRightarrow{\rho} KT \xRightarrow{\sigma T} LTT \xRightarrow{L\mu^T} LT.$$

It is again easy to check that this morphism satisfies the requirements of a transition morphism.

Given

$$\begin{tikzcd}
(\mathscr{X},T) \arrow[r, "{(G,\gamma)}"] & (\mathscr{Y},S) \arrow[r, bend left=40, "{(H,\chi)}"] \arrow[r, bend right=40, "{(H',\chi')}"'] \arrow[r, phantom, "\Downarrow \rho"] & (\mathscr{Z},U)
\end{tikzcd}$$

i.e., having

$$\rho : H \Rightarrow H'S$$

satisfying

$$\begin{tikzcd}
UH \arrow[rr, Rightarrow, "U\rho"] \arrow[dd, Rightarrow, "\chi"'] & & UH'S \arrow[d, Rightarrow, "\chi' S"] \\
& & HSS \arrow[d, Rightarrow, "H'\mu^S"] \\
HS \arrow[r, Rightarrow, "\rho S"'] & H'SS \arrow[r, Rightarrow, "H'\mu^S"'] & H'S,
\end{tikzcd}$$

We describe the whiskering

$$\begin{tikzcd}
(\mathscr{X},T) \arrow[r, bend left=40] \arrow[r, bend right=40] \arrow[r, phantom, "\Downarrow (\rho,\rho_1)(G,\gamma)"] & (\mathscr{Z},U).
\end{tikzcd}$$

The domain morphism is $(HG, H\gamma\cdot\chi G)$ and the codomain is $(H'G, H'\gamma\cdot\chi' G)$. The cell $(\rho,\rho_1)(G,\gamma)$ is defined as

$$\begin{tikzcd} HG \arrow[r, Rightarrow, "\rho G"] & H'SG \arrow[r, Rightarrow, "H'\gamma"] & H'GT, \end{tikzcd}$$

and it satisfies the transition morphism equation

$$\begin{tikzcd}
UHG \arrow[rr, Rightarrow, "U\rho G"] \arrow[dd, Rightarrow, "\chi G"'] & & UH'SG \arrow[r, Rightarrow, "UH'\gamma"] \arrow[d, Rightarrow, "\chi' SG"] & UH'GT \arrow[d, Rightarrow, "\chi' GT"] \\
 & & H'SSG \arrow[r, Rightarrow, "H'S\gamma"] \arrow[d, Rightarrow, "H'\mu^S"] & H'SGT \arrow[d, Rightarrow, "H'\gamma T"] \\
HSG \arrow[r, Rightarrow, "\rho SG"] \arrow[d, Rightarrow, "H\gamma"'] & H'SSG \arrow[r, Rightarrow, "H'\mu^S G"] \arrow[d, Rightarrow, "H'S\gamma"] & H'SG \arrow[dr, Rightarrow, "H'\gamma"] & H'GTT \arrow[d, Rightarrow, "H'G\mu^T"] \\
HGT \arrow[r, Rightarrow, "\rho GT"'] & H'SGT \arrow[r, Rightarrow, "H'\gamma T"'] & H'GTT \arrow[r, Rightarrow, "H'G\mu^T"'] & H'GT.
\end{tikzcd}$$

Given a situation

$$\begin{tikzcd}
(\mathscr{Y}, S) \arrow[r, bend left, "{(H,\chi)}"] \arrow[r, bend right, "{(H',\chi')}"'] \arrow[r, phantom, "\Downarrow \rho"] & (\mathscr{Z}, U) \arrow[r, "{(L,\lambda)}"] & (\mathscr{A}, V),
\end{tikzcd}$$

the whiskering

$$\begin{tikzcd}
(\mathscr{Y}, S) \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow (L,\lambda)\rho"] & (\mathscr{A}, V)
\end{tikzcd}$$

has the domain $(LH, L\chi\cdot\lambda H)$ and codomain $(LH', L\chi'\cdot\lambda H')$ and is defined as

$$\begin{tikzcd} LH \arrow[r, Rightarrow, "L\rho"] & LH'S \end{tikzcd}$$

satisfying the transition morphism equation:

$$\begin{tikzcd}
VLH \arrow[rr, Rightarrow, "VL\rho"] \arrow[d, Rightarrow, "\lambda H"'] & & VLH'S \arrow[d, Rightarrow, "\lambda H'S"] \\
LUH \arrow[rr, Rightarrow, "LU\rho"] \arrow[dd, Rightarrow, "L\chi"'] & & LUH'S \arrow[d, Rightarrow, "L\chi' S"] \\
& & LH'SS \arrow[d, Rightarrow, "LH'\mu^S"] \\
LHS \arrow[r, Rightarrow, "L\rho S"'] & LH'SS \arrow[r, Rightarrow, "LH'\mu^S"'] & LH'S.
\end{tikzcd}$$

These whiskerings give rise to horizontal composition of 2-cells in $\mathsf{TRANS}(\mathbf{K})$ that, together with the vertical composition of 2-cells, satisfies the middle-four interchange law.

The isomorphism between $\mathsf{LIFT}(\mathbf{K})$ and $\mathsf{TRANS}(\mathbf{K})$ The 2-categories $\mathsf{LIFT}(\mathbf{K})$ and $\mathsf{TRANS}(\mathbf{K})$ are isomorphic. We will sketch a proof of this fact for $\mathbf{K} = \mathsf{CAT}$, the 2-category of all categories, functors, and natural transformations, noting that the ideas transfer to the general proof.

We will show that the objects, 1-cells and 2-cells of $\mathsf{LIFT}(\mathsf{CAT})$ and $\mathsf{TRANS}(\mathsf{CAT})$ are in bijective correspondences. Showing the rest, i.e., that composition in both 2-categories corresponds precisely, is easy and we omit the proofs.

Lemma 11.1.2. *There is a bijection*

$$\Phi : \mathrm{ob}(\mathsf{LIFT}(\mathsf{CAT})) \to \mathrm{ob}(\mathsf{TRANS}(\mathsf{CAT})).$$

Proof. Immediate: forgetful functors $U^T : \mathscr{X}^T \to \mathscr{X}$ are in one-to-one correspondence to monads in CAT, i.e., pairs $(\mathscr{X}, T)$, where $\mathscr{X}$ is a category and T a monad on $\mathscr{X}$. ■

Lemma 11.1.3. *Liftings (1-cells in* $\mathsf{LIFT}(\mathsf{CAT})$*) correspond precisely to transitions (i.e., to 1-cells in* $\mathsf{TRANS}(\mathsf{CAT})$*).*

Remark 11.1.4. Unveiling the above lemma, we want to prove that to give a lifting $\widetilde{H} : \mathscr{X}^T \to \mathscr{Y}^S$ of $H : \mathscr{X} \to \mathscr{Y}$, i.e., to give a pair $(H, \widetilde{H})$ such that

$$\begin{tikzcd}
\mathscr{X}^T \arrow[r, "\widetilde{H}"] \arrow[d, "U^T"'] & \mathscr{Y}^S \arrow[d, "U^S"] \\
\mathscr{X} \arrow[r, "H"'] & \mathscr{Y}
\end{tikzcd}$$

commutes is to give a 2-cell (natural transformation)

$$\chi : SH \Rightarrow HT$$

such that

$$\begin{tikzcd}
H \arrow[r, Rightarrow, "\eta^S H"] \arrow[dr, Rightarrow, "H\eta^T"'] & SH \arrow[d, Rightarrow, "\chi"] \\
 & HT
\end{tikzcd}$$

and

$$\begin{tikzcd}
SSH \arrow[r, Rightarrow, "S\chi"] \arrow[d, Rightarrow, "\mu^S H"'] & SHT \arrow[r, Rightarrow, "\chi T"] & HTT \arrow[d, Rightarrow, "H\mu^T"] \\
SH \arrow[rr, Rightarrow, "\chi"'] & & HT
\end{tikzcd}$$

commute.

Proof. Consider the free algebra $\mu^T X : TTX \to TX$ and denote $\widetilde{H}(TX, \mu^T X)$ by $aX : SHTX \to HTX$. This way we obtain from the natural transformation $\mu^T : TT \to T$ a natural transformation $a : SHT \to HT$.

Define χ to be the composite

$$SH \xRightarrow{SH\eta^T} SHT \xRightarrow{a} HT.$$

Since

$$\begin{tikzcd}
H \arrow[r, Rightarrow, "\eta^S H"] \arrow[d, Rightarrow, "H\eta^T"'] & SH \arrow[d, Rightarrow, "SH\eta^T"] \\
HT \arrow[r, Rightarrow, "\eta^S HT"] \arrow[dr, Rightarrow, "1"'] & SHT \arrow[d, Rightarrow, "a"] \\
 & HT
\end{tikzcd}$$

commutes, we have shown that

$$\begin{tikzcd}
H \arrow[r, Rightarrow, "\eta^S H"] \arrow[dr, Rightarrow, "H\eta^T"'] & SH \arrow[d, Rightarrow, "\chi"] \\
 & HT
\end{tikzcd}$$

commutes. Since by naturality $\widetilde{H}(TTX, \mu^T TX) = aTX : SHTTX \to HTTX$, and since the homomorphism $\mu^T X : (TTX, \mu^T TX) \to (TX, \mu^T X)$ is mapped to $S\mu^T X$:

$(HTTX, aTX) \to (HTX, aX)$, the diagram

$$\begin{tikzcd}
SHTTX \arrow[r, Rightarrow, "SH\mu^T X"] \arrow[d, Rightarrow, "aTX"'] & SHTX \arrow[d, Rightarrow, "aX"] \\
HTTX \arrow[r, Rightarrow, "H\mu^T X"'] & HTX
\end{tikzcd}$$

commutes, and therefore

$$\begin{tikzcd}
SHTT \arrow[r, Rightarrow, "SH\mu^T"] \arrow[d, Rightarrow, "aT"'] & SHT \arrow[d, Rightarrow, "a"] \\
HTT \arrow[r, Rightarrow, "H\mu^T"'] & HT
\end{tikzcd}$$

commutes as well. Consider now the diagram

$$\begin{tikzcd}
SSH \arrow[r, Rightarrow, "SSH\eta^T"] \arrow[d, Rightarrow, "\mu^S H"'] & SSHT \arrow[r, Rightarrow, "Sa"] \arrow[dr, Rightarrow, "\mu^S HT"'] & SHT \arrow[r, Rightarrow, "SH\eta^T T"] \arrow[d, Rightarrow, "1"'] & SHTT \arrow[r, Rightarrow, "aT"] \arrow[dl, Rightarrow, "SH\mu^T"] & HTT \arrow[d, Rightarrow, "H\mu^T"] \\
SH \arrow[rr, Rightarrow, "SH\eta^T"'] & & SHT \arrow[rr, Rightarrow, "a"'] & & HT
\end{tikzcd}$$

that proves the commutativity of

$$\begin{tikzcd}
SSH \arrow[r, Rightarrow, "S\chi"] \arrow[d, Rightarrow, "\mu^S H"'] & SHT \arrow[r, Rightarrow, "\chi T"] & HTT \arrow[d, Rightarrow, "H\mu^T"] \\
SH \arrow[rr, Rightarrow, "\chi"'] & & HT
\end{tikzcd}$$

Conversely, define $\widetilde{H}$ on objects as follows: an algebra $a : TX \to X$ is mapped to the composite

$$SHX \xrightarrow{\chi X} HTX \xrightarrow{Ha} HX \ .$$

This is an algebra since the diagrams

$$\begin{tikzcd}
HX \arrow[r, "\eta^S HX"] \arrow[dr, "H\eta^T X"] \arrow[ddr, "1"'] & SHX \arrow[d, "\chi X"] \\
& HTX \arrow[d, "Ha"] \\
& HX
\end{tikzcd}$$

and

$$\begin{tikzcd}
SSHX \arrow[r, "S\chi X"] \arrow[dd, "\mu^S HX"'] & SHTX \arrow[r, "SHa"] \arrow[d, "\chi TX"'] & SHX \arrow[d, "\chi X"] \\
 & HTTX \arrow[r, "HTa"] \arrow[d, "H\mu^T X"'] & HTX \arrow[d, "Ha"] \\
SHX \arrow[r, "\chi X"'] & HTX \arrow[r, "Ha"] & HX
\end{tikzcd}$$

commute. (We have used both axioms for a transition.) Given a homomorphism

$$\begin{tikzcd}
TX \arrow[r, "Th"] \arrow[d, "a"'] & TY \arrow[d, "b"] \\
X \arrow[r, "h"'] & Y
\end{tikzcd},$$

we define $\widetilde{H}(h)$ to be $Hh : HX \to HY$: this is a homomorphism since

$$\begin{tikzcd}
SHX \arrow[r, "SHh"] \arrow[d, "\chi X"'] & SHY \arrow[d, "\chi Y"] \\
HTX \arrow[r, "HTh"] \arrow[d, "Ha"'] & HTY \arrow[d, "Hb"] \\
HX \arrow[r, "h"'] & HY
\end{tikzcd}$$

commutes.

The above two processes are clearly inverse to each other. ■

Lemma 11.1.5. *Lifting morphisms (2-cells in* $\mathsf{LIFT}(\mathsf{CAT})$*) correspond precisely to transition morphisms (2-cells in* $\mathsf{TRANS}(\mathsf{CAT})$*).*

Remark 11.1.6. Unveiling again, we need to prove that to give a 2-cell (natural transformation)

$$\widetilde{\rho} : \widetilde{H} \Rightarrow \widetilde{K}$$

in $\mathsf{LIFT}(\mathsf{CAT})$ is to give a 2-cell (natural transformation)

$$\sigma : H \Rightarrow KT$$

in $\mathsf{TRANS}(\mathsf{CAT})$ such that the diagram

$$\begin{tikzcd}
SH \arrow[rr, Rightarrow, "SS\rho"] \arrow[dd, Rightarrow, "\chi"'] & & SKT \arrow[d, Rightarrow, "\kappa T"] \\
& & KTT \arrow[d, Rightarrow, "K\mu^T"] \\
HT \arrow[r, Rightarrow, "\rho T"'] & KTT \arrow[r, Rightarrow, "K\mu^T"'] & KT
\end{tikzcd}$$

commutes.

Proof. As in the proof of Lemma 11.1.3, denote the image of the free algebra $(TX, \mu^T X) : TTX \to TX$ under $\widetilde{H}$ by $a^H X : SHTX \to HTX$ and under $\widetilde{K}$ by $a^K X : SKTX \to KTX$. These assignments give rise to natural transformations $a^H : SHT \Rightarrow HT$ and $a^K : SKT \Rightarrow KT$.

Given $\widetilde{\rho} : \widetilde{H} \Rightarrow \widetilde{K}$, define ρ to be the composite

$$\begin{tikzcd}
H \arrow[r, Rightarrow, "H\eta^T"'] \arrow[rr, Rightarrow, bend left, "\rho"] & HT \arrow[r, Rightarrow, "\widetilde{\rho}"'] & KT.
\end{tikzcd}$$

By naturality of $\widetilde{\rho}$ the diagram

$$\begin{tikzcd}
HTT \arrow[r, Rightarrow, "\widetilde{\rho}T"] \arrow[d, Rightarrow, "H\mu^T"'] & KTT \arrow[d, Rightarrow, "K\mu^T"] \\
HT \arrow[r, Rightarrow, "\widetilde{\rho}"'] & KT
\end{tikzcd}$$

commutes. Since $\widetilde{\rho}$ is a 2-cell of $\mathsf{LIFT}(\mathsf{CAT})$,

$$\begin{tikzcd}
SHT \arrow[r, Rightarrow, "S\widetilde{\rho}"] \arrow[d, Rightarrow, "a^H"'] & SKT \arrow[d, Rightarrow, "a^K"] \\
HT \arrow[r, Rightarrow, "\widetilde{\rho}"'] & KT
\end{tikzcd}$$

commutes. Therefore the diagram

$$\begin{tikzcd}
SH \arrow[r, Rightarrow, "SH\eta^T"] \arrow[d, Rightarrow, "SH\eta^T"'] & SHT \arrow[dl, Rightarrow, "1"] \arrow[rr, Rightarrow, "S\widetilde{\rho}"] & & SKT \arrow[dd, Rightarrow, "1"'] \arrow[dr, Rightarrow, "SK\eta^T T"] & \\
SHT \arrow[dd, Rightarrow, "a^H"'] & & & & SKTT \arrow[dl, Rightarrow, "SK\mu^T"] \arrow[d, Rightarrow, "a^K T"] \\
 & & & SKT \arrow[d, Rightarrow, "a^K"'] & KTT \arrow[dl, Rightarrow, "K\mu^T"] \\
HT \arrow[rr, Rightarrow, "1"] \arrow[dr, Rightarrow, "H\eta^T T"'] & & HT \arrow[r, Rightarrow, "\widetilde{\rho}"] & KT & \\
 & HTT \arrow[ur, Rightarrow, "H\mu^T"'] \arrow[r, Rightarrow, "\widetilde{\rho}T"'] & KTT \arrow[ur, Rightarrow, "K\mu^T"'] & &
\end{tikzcd}$$

also commutes; and the definition of ρ makes sense.

Conversely, given $\rho : H \Rightarrow KT$, define $\widetilde{\rho}$ to be

$$HT \xRightarrow{\rho T} KTT \xRightarrow{K\mu^T} KT.$$

Naturality of $\widetilde{\rho}$ is easy, and to show that for any algebra $a : TX \to X$ the diagram

$$\begin{tikzcd}
SHX \arrow[r, "S\rho X"] \arrow[d, "\chi X"'] & SKTX \arrow[r, "SKa"] & SKX \arrow[d, "\kappa X"] \\
HTX \arrow[d, "Ha"'] & & KTX \arrow[d, "Ka"] \\
HX \arrow[r, "\rho X"'] & KTX \arrow[r, "Ka"'] & KX
\end{tikzcd}$$

commutes, observe that the diagram

$$\begin{tikzcd}
SHX \arrow[rr, "S\rho X"] \arrow[dd, "\chi X"'] & & SKTX \arrow[r, "SKa"] \arrow[d, "\kappa TX"] & SKX \arrow[d, "\kappa X"] \\
& & KTTX \arrow[r, "KTa"] \arrow[d, "K\mu^T T"] & KTX \arrow[dd, "Ka"] \\
HTX \arrow[r, "\rho TX"] \arrow[d, "Ha"'] & KTTX \arrow[r, "K\mu^T X"] \arrow[d, "KTa"] & KTX \arrow[dr, "a"] & \\
HX \arrow[r, "\rho X"'] & KTX \arrow[rr, "Ka"'] & & KX
\end{tikzcd}$$

commutes (as the diagram in the top right corner commutes by assumption).

The above two processes are clearly inverse to each other. ■

Taking together these results, we may state the main theorem of this section.

Theorem 11.1.7. *There is an isomorphism of 2-categories*

$$\Phi : \mathsf{LIFT}(\mathsf{CAT}) \to \mathsf{TRANS}(\mathsf{CAT}).$$

We may now define wreaths in a 2-category $\mathbf{K}$ using the 2-category $\mathsf{LIFT}(\mathbf{K})$.

Definition 11.1.8. An (ordinary) *wreath* is a monad in $\mathsf{LIFT}(\mathbf{K})$.

Remark 11.1.9. More concretely, a wreath of $H : \mathscr{X} \to \mathscr{X}$ around a monad T on $\mathscr{X}$ is a 1-cell $\widetilde{H} : \mathscr{X}^T \to \mathscr{X}^T$ together with 2-cells

$$\begin{tikzcd}
\mathscr{X}^T \arrow[r, bend left, "1"] \arrow[r, bend right, "\widetilde{H}"'] \arrow[r, phantom, "\Downarrow \widetilde{\sigma}"] & \mathscr{X}^T,
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{X}^T \arrow[r, bend left, "\widetilde{H}\cdot\widetilde{H}"] \arrow[r, bend right, "\widetilde{H}"'] \arrow[r, phantom, "\Downarrow \widetilde{\nu}"] & \mathscr{X}^T
\end{tikzcd}$$

satisfying the usual monad axioms.

Since $\mathsf{LIFT}(\mathbf{K})$ and $\mathsf{TRANS}(\mathbf{K})$ are isomorphic, a wreath is equivalently a monad in $\mathsf{TRANS}(\mathbf{K})$. This allows us to state an elementary definition of a wreath.

Remark 11.1.10. In elementary terms, we say that a wreath of $H : \mathscr{X} \to \mathscr{X}$ around a monad T on $\mathscr{X}$ in a 2-category $\mathbf{K}$ is thus a transition (H, χ), i.e., a 2-cell $\chi : TH \Rightarrow HT$ satisfying equalities (see [58])

$$\begin{tikzcd}
H \arrow[r, Rightarrow, "\eta^T H"] \arrow[dr, Rightarrow, "H\eta^T"'] & TH \arrow[d, Rightarrow, "\chi"] \\
& HT
\end{tikzcd}
\qquad
\begin{tikzcd}
TTH \arrow[r, Rightarrow, "T\chi"] \arrow[d, Rightarrow, "\mu^T H"'] & THT \arrow[r, Rightarrow, "\chi T"] & HTT \arrow[d, Rightarrow, "H\mu^T"] \\
TH \arrow[rr, Rightarrow, "\chi"'] & & HT,
\end{tikzcd}$$

transition morphisms $\sigma : (1, 1) \to (H, \chi)$ and $\nu : (H, \chi) \cdot (H, \chi) \to (H, \chi)$, i.e., 2-cells $\sigma : 1 \Rightarrow HT$ and $\nu : HH \Rightarrow HT$ satisfying equalities

$$\begin{tikzcd}
T \arrow[rr, Rightarrow, "T\sigma"] \arrow[dd, Rightarrow, "1"'] & & THT \arrow[d, Rightarrow, "\chi T"] \\
 & & HTT \arrow[d, Rightarrow, "H\mu^T"] \\
T \arrow[r, Rightarrow, "\sigma T"'] & HTT \arrow[r, Rightarrow, "H\mu^T"'] & HT
\end{tikzcd}
\qquad
\begin{tikzcd}
THH \arrow[rr, Rightarrow, "T\nu"] \arrow[d, Rightarrow, "\chi H"'] & & THT \arrow[d, Rightarrow, "\chi T"] \\
HTH \arrow[d, Rightarrow, "H\chi"'] & & HTT \arrow[d, Rightarrow, "H\mu^T"] \\
HHT \arrow[r, Rightarrow, "\nu T"'] & HTT \arrow[r, Rightarrow, "H\mu^T"'] & HT,
\end{tikzcd}$$

subject to monad axioms

$$\begin{tikzcd}
H \arrow[r, Rightarrow, "\sigma H"] \arrow[ddrr, Rightarrow, "H\eta^T"'] & HTH \arrow[r, Rightarrow, "H\chi"] & HHT \arrow[d, Rightarrow, "\nu T"] \\
 & & HTT \arrow[d, Rightarrow, "H\mu^T"] \\
 & & HT,
\end{tikzcd}
\qquad
\begin{tikzcd}
HHT \arrow[d, Rightarrow, "\nu T"'] & H \arrow[l, Rightarrow, "H\sigma"'] \arrow[ddl, Rightarrow, "H\eta^T"] \\
HTT \arrow[d, Rightarrow, "H\mu^T"'] & \\
HT, &
\end{tikzcd}$$

and

$$\begin{tikzcd}
HHH \arrow[rrr, Rightarrow, "H\nu"] \arrow[dd, Rightarrow, "\nu H"'] & & & HHT \arrow[d, Rightarrow, "\nu T"] \\
 & & & HTT \arrow[d, Rightarrow, "H\mu^T"] \\
HTH \arrow[r, Rightarrow, "H\chi"'] & HHT \arrow[r, Rightarrow, "\nu T"'] & HTT \arrow[r, Rightarrow, "H\mu^T"] & HT.
\end{tikzcd}$$

Remark 11.1.11. All the proofs in the current section used the fact that CAT admits an Eilenberg-Moore object $\mathscr{X}^T$ for a monad T. Working in a general 2-category $\mathbf{K}$, one need not (as we did) assume that Eilenberg-Moore objects exist. The trick is to pass to "lifting diagrams" of the form

$$\begin{tikzcd}
\mathbf{K}(-, \mathscr{X})^{\mathbf{K}(-,T)} \arrow[r, "\tilde{H}"] \arrow[d, "U^{\mathbf{K}(-,T)}"'] & \mathbf{K}(-, \mathscr{Y})^{\mathbf{K}(-,S)} \arrow[d, "U^{\mathbf{K}(-,S)}"] \\
\mathbf{K}(-, \mathscr{X}) \arrow[r, "\mathbf{K}(-,H)"'] & \mathbf{K}(-, \mathscr{Y})
\end{tikzcd}$$

and work within $[\mathbf{K}^{op}, \mathsf{CAT}]$. Of course, the above diagram reduces to

$$\begin{tikzcd}
\mathbf{K}(-,\mathscr{X}^T) \arrow[r, "\mathbf{K}(-,\tilde{H})"] \arrow[d, "\mathbf{K}(-,U^T)"'] & \mathbf{K}(-,\mathscr{Y}^S) \arrow[d, "\mathbf{K}(-,U^S)"] \\
\mathbf{K}(-,\mathscr{X}) \arrow[r, "\mathbf{K}(-,H)"'] & \mathbf{K}(-,\mathscr{Y}),
\end{tikzcd}$$

i.e., to the image of diagram (11.1) under the Yoneda embedding. This approach allows us to work with generalised elements.

The case of ordinary wreaths helps with gaining intuition for the case of wreaths of 2-functors around pseudomonads. There the axioms for ordinary wreaths become two-dimensional data; and additional coherence conditions concerning these data need to be specified.

We will now turn to the case of **Gray**-categories and to the notion of a wreath in this setting.

11.2 The Gray-category $\mathsf{LIFT}(\mathbf{K})$ for a Gray-category K

The purpose of this section is to define, in the spirit of Section 11.1, a **Gray**-category $\mathsf{LIFT}(\mathbf{K})$ of *liftings* in a **Gray**-category **K**, generalising straightforwadly the ideas from the ordinary case. The reason for introducing $\mathsf{LIFT}(\mathbf{K})$ remains the same – to be able to define wreaths in **K**.

In [54] Lack defines a **Gray**-category $\mathsf{PSM}(\mathbf{K})$ of pseudomonads in a **Gray**-category **K** using a nice and alterable construction. Having the **Gray**-category $\mathsf{PSM}(\mathbf{K})$ at hand, it is then easy to *define* a distributive law between pseudomonads as a pseudomonad *in* $\mathsf{PSM}(\mathbf{K})$. We shall define, by tweaking Lack's approach, the **Gray**-category $\mathsf{LIFT}(\mathbf{K})$ of liftings in **K**, and consequently define wreaths as pseudomonads in $\mathsf{LIFT}(\mathbf{K})$. This rather abstract construction of $\mathsf{LIFT}(\mathbf{K})$ yields, when unfolded, the expected generalisation of the 2-category $\mathsf{LIFT}(\mathbf{K})$ for a 2-category **K**.

Of course, our definition of wreaths around pseudomonads will need to be unfolded, and we will give the elementary description of wreaths in the subsequent sections of this chapter.

Let **K** be a **Gray**-category. Consider the Yoneda embedding

$$\mathbf{Y} : \mathbf{K} \to [\mathbf{K}^{op}, \mathbf{Gray}]$$

and its factorisation

$$\begin{tikzcd}
\mathbf{K} \arrow[r, "\mathbf{Z}"] \arrow[dr, "\mathbf{Y}"'] & \overline{\mathbf{K}} \arrow[d, "\mathbf{W}"] \\
 & {[\mathbf{K}^{op}, \mathbf{Gray}]}
\end{tikzcd}$$

through the closure $\overline{\mathbf{K}}$ of $\mathbf{K}$ in $[\mathbf{K}^{op}, \mathbf{Gray}]$ under $\mathbf{psa}(a, M-)$-limits (i.e., the free completion of $\mathbf{K}$ under Eilenberg-Moore objects, recall, e.g., Remark 10.1.10).

Lack [54] defines $\mathsf{PSM}(\mathbf{K})$ as the object in the factorisation

$$\begin{tikzcd}
\mathbf{Gray}\text{-}\mathsf{CAT}_0(\mathbf{psm}, \mathbf{K}) \arrow[r, "\text{b.o.}"] \arrow[dr, dotted] & \mathsf{PSM}(\mathbf{K}) \arrow[d, "\text{f.f.}"] \\
 & {[\mathbf{2}, \overline{\mathbf{K}}]}
\end{tikzcd}$$

where the dotted arrow denotes the $\mathbf{Gray}$-functor that assigns to a pseudomonad T on $\mathscr{X}$ the forgetful arrow $U^T : \mathscr{X}^T \to \mathscr{X}$. We shall define $\mathsf{LIFT}(\mathbf{K})$ via a similar factorisation

$$\begin{tikzcd}
\mathbf{Gray}\text{-}\mathsf{CAT}_0(\mathbf{psm}, \mathbf{K}) \arrow[r, "\text{b.o.}"] \arrow[dr, dotted] & \mathsf{LIFT}(\mathbf{K}) \arrow[d, "\text{f.f.}"] \\
 & \mathsf{two}(\overline{\mathbf{K}})
\end{tikzcd}$$

replacing $[\mathbf{2}, \overline{\mathbf{K}}]$ by the $\mathbf{Gray}$-category $\mathsf{two}(\overline{\mathbf{K}})$ which we define now.

Definition 11.2.1. For any $\mathbf{Gray}$-category $\mathbf{K}$, define $\mathsf{two}(\mathbf{K})$ as follows. Take the $\mathbf{Gray}$-functor $\ulcorner 0 \urcorner : \mathbf{1} \to \mathbf{2}$ and consider $[\ulcorner 0 \urcorner, \mathbf{K}] : [\mathbf{2}, \mathbf{K}] \to [\mathbf{1}, \mathbf{K}]$.

1. 0-cells in $\mathsf{two}(\mathbf{K})$ are 0-cells in $[\mathbf{2}, \mathbf{K}]$.
2. 1-cells in $\mathsf{two}(\mathbf{K})$ are 1-cells in $[\mathbf{2}, \mathbf{K}]$.
3. To define 2-cells and 3-cells in $\mathsf{two}(\mathbf{K})$, perform a factorisation

$$\begin{tikzcd}
{[\mathbf{2}, \mathbf{K}](U, V)} \arrow[r, "\text{b.o.}"] \arrow[dr, "{[\ulcorner 0 \urcorner, \mathbf{K}]_{U,V}}"'] & \mathsf{two}(\mathbf{K})(U, V) \arrow[d, "\text{f.f.}"] \\
 & {[\mathbf{1}, \mathbf{K}](\mathrm{dom}(U), \mathrm{dom}(V))}
\end{tikzcd}$$

in $\mathsf{2}\text{-}\mathsf{Cat}$, for every pair U, V of objects of $[\mathbf{2}, \mathbf{K}]$.

To define composition in $\mathsf{two}(\mathbf{K})$, consider the factorisation

$$\begin{tikzcd}
{[\mathbf{2}, \mathbf{K}](V, W) \otimes [\mathbf{2}, \mathbf{K}](U, V)} \arrow[r, "\text{comp}"] \arrow[d, "\text{b.o.}"'] \arrow[dd, bend right=60, "\mathrm{dom} \otimes \mathrm{dom}"'] & {[\mathbf{2}, \mathbf{K}](U, W)} \arrow[d] \arrow[dd, bend left=60, "\mathrm{dom}"] \\
\mathsf{two}(\mathbf{K})(V, W) \otimes \mathsf{two}(\mathbf{K})(U, V) \arrow[r, dotted, "\text{comp}"] \arrow[d] & \mathsf{two}(\mathbf{K})(U, W) \arrow[d, "\text{f.f.}"] \\
{[\mathbf{1}, \mathbf{K}](\mathrm{dom}V, \mathrm{dom}W) \otimes [\mathbf{1}, \mathbf{K}](\mathrm{dom}U, \mathrm{dom}V)} \arrow[r, "\text{comp}"'] & {[\mathbf{1}, \mathbf{K}](\mathrm{dom}U, \mathrm{dom}W)}
\end{tikzcd}$$

Thus we define composition in $\mathsf{two}(\mathbf{K})$ by the above diagonal. An easy verification shows that we indeed obtain a **Gray**-category. Moreover, the construction shows that there is an obvious **Gray**-functor $[\mathbf{2}, \mathbf{K}] \to \mathsf{two}(\mathbf{K})$ that is an identity on 0-cells and 1-cells.

Since both $\mathsf{PSM}(\mathbf{K})$ and $\mathsf{LIFT}(\mathbf{K})$ are defined by factorisation, we may now see that there is a diagonal (dotted arrow $\mathbf{D}$) in the following diagram:

$$\begin{tikzcd}
\mathbf{Gray}\text{-}\mathsf{CAT}_0(\mathbf{psm}, \mathbf{K}) \arrow[r, "\text{b.o.}"] \arrow[dr, "\text{b.o.}"'] & \mathsf{PSM}(\mathbf{K}) \arrow[r, "\text{f.f.}"] \arrow[d, dotted, "\mathbf{D}"] & {[\mathbf{2}, \overline{\mathbf{K}}]} \arrow[d] \\
 & \mathsf{LIFT}(\mathbf{K}) \arrow[r, "\text{f.f.}"'] & \mathsf{two}(\overline{\mathbf{K}})
\end{tikzcd}$$

This construction allows a slick definition of a wreath:

Definition 11.2.2 (Wreaths in Gray-categories). A *wreath* is a pseudomonad in $\mathsf{LIFT}(\mathbf{K})$.

Remark 11.2.3. This definition is the obvious generalisation of ordinary wreaths. Recall from Definition 11.1.8 that a wreath in a 2-category $\mathbf{K}$ is a monad in $\mathsf{LIFT}(\mathbf{K})$.

We may use the **Gray**-functor $\mathbf{D} : \mathsf{PSM}(\mathbf{K}) \to \mathsf{LIFT}(\mathbf{K})$ to show quickly that every distributive law is a wreath: given a distributive law $\mathbf{H} : \mathbf{psm} \to \mathsf{PSM}(\mathbf{K})$, its induced wreath is given by

$$\mathbf{psm} \xrightarrow{\mathbf{H}} \mathsf{PSM}(\mathbf{K}) \xrightarrow{\mathbf{D}} \mathsf{LIFT}(\mathbf{K}).$$

Remark 11.2.4 (Elementary description of $\mathsf{LIFT}(\mathbf{K})$). We will show now in elementary terms the structure of $\mathsf{LIFT}(\mathbf{K})$ for a **Gray**-category $\mathbf{K}$ that has Eilenberg-Moore objects. (In general, one would have to work with a free completion of $\mathbf{K}$ under Eilenberg-Moore objects, working thus in a subcategory of $[\mathbf{K}^{op}, \mathbf{Gray}]$. We will not do that; the transition is easy and not illuminating.)

1. The objects of $\mathsf{LIFT}(\mathbf{K})$ are the 1-cells

$$\begin{tikzcd}
\mathscr{X}^T \arrow[d, "U^T"'] \\
\mathscr{X},
\end{tikzcd}$$

the "forgetful" 1-cells from the Eilenberg-Moore object $\mathscr{X}^T$ to the underlying object $\mathscr{X}$, where T is a pseudomonad on $\mathscr{X}$.

2. The 1-cells of $\mathsf{LIFT}(\mathbf{K})$ are *liftings*, i.e., pairs $(H, \widetilde{H})$ that make the following diagram

$$\begin{tikzcd}
\mathscr{X}^T \arrow[r, "\widetilde{H}"] \arrow[d, "U^T"'] & \mathscr{Y}^S \arrow[d, "U^S"] \\
\mathscr{X} \arrow[r, "H"'] & \mathscr{Y}
\end{tikzcd}$$

commute.

3. The 2-cells of $\mathsf{LIFT}(\mathbf{K})$ are *lifting morphisms*, i.e., 2-cells $\widetilde{\rho} : (H, \widetilde{H}) \to (K, \widetilde{K})$ of the form

$$\begin{tikzcd}
\mathscr{X}^T \arrow[r, bend left, "\widetilde{H}"] \arrow[r, bend right, "\widetilde{K}"'] \arrow[r, phantom, "\Downarrow \widetilde{\rho}"] & \mathscr{Y}^S
\end{tikzcd}$$

in $\mathbf{K}$.

4. The 3-cells of $\mathsf{LIFT}(\mathbf{K})$ are 3-cells $\widetilde{\tau} : \widetilde{\rho} \to \widetilde{\rho}^*$

$$\begin{tikzcd}
\mathscr{X}^T \arrow[r, bend left=50, "\widetilde{H}"] \arrow[r, bend right=50, "\widetilde{K}"'] \arrow[r, phantom, "\widetilde{\rho} \Downarrow \overset{\widetilde{\tau}}{\Rrightarrow} \Downarrow \widetilde{\rho}^*"] & \mathscr{Y}^S
\end{tikzcd}$$

in $\mathbf{K}$.

The composition is inherited from $\mathbf{K}$.

Remark 11.2.5. Recall that since we are working in the setting of an abstract **Gray**-category $\mathbf{K}$, the Eilenberg-Moore object need not have an "inner categorical structure" consisting of algebras as objects, homomorphisms as arrows, etc. However, we may work with generalised elements of $\mathscr{X}^T$ as with algebras: recall that this was the case in the ordinary setting from Remark 11.1.11, and recall the definition of a pseudoalgebra and pseudohomomorphism from Section 10.1 of Chapter 10.

Remark 11.2.6. As in the ordinary case, we now see that a wreath in $\mathbf{K}$, or more concretely, a wreath of $H : \mathscr{X} \to \mathscr{X}$ around a pseudomonad T on $\mathscr{X}$ is a pseudomonad $\widetilde{H}$ on $\mathscr{X}^T$. We shall not expand the definition of a wreath further at this point. In the following section we will introduce the *tricategory* $\mathsf{TRANS}(\mathbf{K})$ of *transitions* in $\mathbf{K}$ that will allow us, as in the ordinary setting, to give an elementary description of wreaths.

11.3 The tricategory $\mathsf{TRANS}(\mathbf{K})$ for a Gray-category $\mathbf{K}$

Recall from Section 11.1 that in the case of ordinary wreaths in a 2-category $\mathbf{K}$ we used the 2-category $\mathsf{LIFT}(\mathbf{K})$ of liftings in $\mathbf{K}$ to *define* wreaths, and then we defined an another 2-category $\mathsf{TRANS}(\mathbf{K})$ of transitions in $\mathbf{K}$ that was *isomorphic* to $\mathsf{LIFT}(\mathbf{K})$ in order to obtain an *elementary* description of wreaths.

We already started in Section 11.2 to use the same approach to define and describe wreaths in the context of **Gray**-categories. We defined wreaths with the help of the **Gray**-category $\mathsf{LIFT}(\mathbf{K})$ of liftings in a **Gray**-category $\mathbf{K}$. A logical next step would be to define a **Gray**-category $\mathsf{TRANS}(\mathbf{K})$ of transitions in $\mathbf{K}$ that would be **Gray**-isomorphic to $\mathsf{LIFT}(\mathbf{K})$. However, while there is a way to define $\mathsf{TRANS}(\mathbf{K})$ in the spirit of the definition of $\mathsf{TRANS}(\mathbf{K})$ for a *2-category* $\mathbf{K}$, the resulting structure is *no longer a* **Gray**-*category*, but an honest *tricategory*.

Working with tricategories poses substantial technical difficulties (compared to working with **Gray**-categories), but it is still possible to use the tricategory $\mathsf{TRANS}(\mathbf{K})$ to obtain an elementary description of wreaths as $\mathsf{TRANS}(\mathbf{K})$ is *triequivalent* to $\mathsf{LIFT}(\mathbf{K})$.

As the definition of a tricategory (and of triequivalence) is quite involved, we *will not* state it in full here. Instead, we will show the important data that constitute the tricategory $\mathsf{TRANS}(\mathbf{K})$ in this section, and show how the triequivalence $\mathsf{LIFT}(\mathbf{K}) \simeq \mathsf{TRANS}(\mathbf{K})$ works in the next section.

In short, $\mathsf{TRANS}(\mathbf{K})$ consists of objects, 1-cells, 2-cells and 3-cells, where

1. the objects are pseudomonads in $\mathbf{K}$ (i.e., pairs $(\mathscr{X}, T)$ with $\mathscr{X}$ an object of $\mathbf{K}$ and T a pseudomonad on $\mathscr{X}$),
2. the 1-cells are transitions, the 2-cells are transition morphisms (similarly as in the 2-category of transitions in a 2-category $\mathbf{K}$),
3. and higher transition cells between transition morphisms.

The *homs* of $\mathsf{TRANS}(\mathbf{K})$ will not be 2-categories as in the case of $\mathsf{LIFT}(\mathbf{K})$, but *bicategories.* We will first describe the hom-bicategories of $\mathsf{TRANS}(\mathbf{K})$.

The hom-bicategory of transitions

Consider a **Gray**-category $\mathbf{K}$, a pseudomonad T on an object $\mathscr{X}$ in $\mathbf{K}$ and a pseudomonad S on $\mathscr{Y}$ in $\mathbf{K}$. Then there is a bicategory $\mathbf{B} = \mathsf{TRANS}(\mathbf{K})((\mathscr{X}, T), (\mathscr{Y}, S))$ of *transitions, transition morphisms and transition 2-cells* between $(\mathscr{X}, T)$ and $(\mathscr{Y}, S)$. We shall examine the structure of this bicategory.

The underlying structure of $\mathbf{B}$ consists of

1. objects $(H, \chi, \chi_0, \chi_1)$ being *transitions*, consisting of $H : \mathscr{X} \to \mathscr{Y}$, $\chi : SH \Rightarrow HT$ and isomorphisms

$$\begin{tikzcd}
H \arrow[r, Rightarrow, "\eta^S H"] \arrow[dr, Rightarrow, "H\eta^T"'] & SH \arrow[d, Rightarrow, "\chi"] \arrow[dl, phantom, "\chi_0 \Lleftarrow"] \\
 & HT
\end{tikzcd} ,
\qquad
\begin{tikzcd}
SSH \arrow[r, Rightarrow, "S\chi"] \arrow[dd, Rightarrow, "\mu^S H"'] & SHT \arrow[d, Rightarrow, "\chi T"] \\
\chi_1 \Lleftarrow & HTT \arrow[d, Rightarrow, "H\mu^T"] \\
SH \arrow[r, Rightarrow, "\chi"'] & HT
\end{tikzcd}$$

satisfying coherence conditions

$$\begin{tikzcd}
SH \arrow[r, Rightarrow, "S\eta^S H"] \arrow[dr, Rightarrow, bend right, "1"'] \arrow[dr, phantom, "\Swarrow s^S H^{-1}" near start] & SSH \arrow[r, Rightarrow, "S\chi"] \arrow[d, Rightarrow, "\mu^S H"] \arrow[dr, phantom, "\Swarrow \chi_1"] & SHT \arrow[dd, Rightarrow, "\chi T"] \\
 & SH \arrow[d, Rightarrow, "\chi"] & \\
 & HT & HTT \arrow[l, Rightarrow, "H\mu^T"]
\end{tikzcd}
=
\begin{tikzcd}
SH \arrow[r, Rightarrow, "S\eta^S H"] \arrow[dr, Rightarrow, "SH\eta^T"'] \arrow[dd, Rightarrow, "\chi"'] & SSH \arrow[d, Rightarrow, "S\chi"] \arrow[dl, phantom, "\Swarrow S\chi_0" near start] \\
 & SHT \arrow[d, Rightarrow, "\chi T"] \arrow[dl, phantom, "\Swarrow \chi_{\eta^T}"] \\
HT \arrow[r, Rightarrow, "HT\eta^T"] \arrow[dr, Rightarrow, bend right, "1"'] & HTT \arrow[d, Rightarrow, "H\mu^T"] \arrow[dl, phantom, "\Swarrow Hs^{T-1}" near start] \\
 & HT
\end{tikzcd}
\tag{11.2}$$

and

$$\begin{tikzcd}
 & SSHT \arrow[r, Rightarrow, "S\chi T"] & SHTT \arrow[r, Rightarrow, "SH\mu^T"] & SHT \arrow[d, Rightarrow, "\chi T"] \\
 & \Downarrow S\chi_1 & SSH \arrow[ur, Rightarrow, "S\chi"] \arrow[dd, Rightarrow, "\mu^S H"'] & HTT \arrow[d, Rightarrow, "H\mu^T"] \\
SSSH \arrow[uur, Rightarrow, "SS\chi"] \arrow[urr, Rightarrow, "S\mu^S H"] \arrow[dr, Rightarrow, "\mu^S SH"'] & \Downarrow m^S H & \Downarrow \chi_1 & HT \\
 & SSH \arrow[r, Rightarrow, "\mu^S H"'] & SH \arrow[ur, Rightarrow, "\chi"'] &
\end{tikzcd}$$

$$=\tag{11.3}$$

$$\begin{tikzcd}
 & SSHT \arrow[r, Rightarrow, "S\chi T"] \arrow[dd, Rightarrow, "\mu^S HT"] & SHTT \arrow[r, Rightarrow, "SH\mu^T"] \arrow[d, Rightarrow, "\chi TT"'] \arrow[dr, phantom, "\Swarrow \chi\mu^T"] & SHT \arrow[d, Rightarrow, "\chi T"] \\
 & \Swarrow \chi_1 T & HTTT \arrow[r, Rightarrow, "HT\mu^T"] \arrow[d, Rightarrow, "H\mu^T T"'] \arrow[dr, phantom, "\Swarrow Hm^T"] & HTT \arrow[d, Rightarrow, "H\mu^T"] \\
SSSH \arrow[uur, Rightarrow, "SS\chi"] \arrow[dr, Rightarrow, "\mu^S SH"'] \arrow[r, phantom, "\Downarrow \mu^S_\chi"] & SHT \arrow[r, Rightarrow, "\chi T"'] & HTT \arrow[r, Rightarrow, "H\mu^T"'] & HT. \\
 & SSH \arrow[u, Rightarrow, "S\chi"] \arrow[r, Rightarrow, "\mu^S H"'] \arrow[ur, phantom, "\Downarrow \chi_1"] & SH \arrow[ur, Rightarrow, "\chi"'] &
\end{tikzcd}$$

2. For each pair $(H, \chi, \chi_0, \chi_1)$, $(K, \kappa, \kappa_0, \kappa_1)$ of transitions, a *category* of *transition morphisms* and *transition 2-cells*, where:

   (a) The objects are pairs

$$(\rho, \rho_1) : (H, \chi, \chi_0, \chi_1) \to (K, \kappa, \kappa_0, \kappa_1)$$

   where $\rho : H \Rightarrow KT$ and ρ_1 is an isomorphism

$$\begin{tikzcd}
SH \arrow[rr, Rightarrow, "S\rho"] \arrow[dd, Rightarrow, "\chi"'] & & SKT \arrow[d, Rightarrow, "\kappa T"] \\
& \rho_1 \Lleftarrow & KTT \arrow[d, Rightarrow, "K\mu^T"] \\
HT \arrow[r, Rightarrow, "\rho T"'] & KTT \arrow[r, Rightarrow, "K\mu^T"'] & KT
\end{tikzcd}$$

   satisfying coherence conditions

$$\begin{tikzcd}
H \arrow[r, Rightarrow, "\eta^S H"] \arrow[ddr, Rightarrow, "H\eta^T"'] \arrow[dr, phantom, "\chi_0 \Lleftarrow"] & SH \arrow[rr, Rightarrow, "S\rho"] \arrow[dd, Rightarrow, "\chi"'] & & SKT \arrow[d, Rightarrow, "\kappa T"] \\
& & \rho_1 \Lleftarrow & KTT \arrow[d, Rightarrow, "K\mu^T"] \\
& HT \arrow[r, Rightarrow, "\rho T"'] & KTT \arrow[r, Rightarrow, "K\mu^T"'] & KT
\end{tikzcd}$$

$$=$$

$$\begin{tikzcd}
& SH \arrow[dr, Rightarrow, "S\rho"] & \\
H \arrow[ur, Rightarrow, "\eta^S H"] \arrow[d, Rightarrow, "H\eta^T"'] \arrow[dr, Rightarrow, "\rho"] & \Downarrow \eta^S_\rho & SKT \arrow[d, Rightarrow, "\kappa T"] \\
HT \arrow[dr, Rightarrow, "\rho T"'] \arrow[r, phantom, "\Lleftarrow \rho_{\eta^T}"] & KT \arrow[ur, Rightarrow, "\eta^S KT"] \arrow[r, Rightarrow, "K\eta^T T"] \arrow[d, Rightarrow, "KT\eta^T"'] \arrow[dr, Rightarrow, "1"] & KTT \arrow[d, Rightarrow, "K\mu^T"] \arrow[u, phantom, "\Downarrow \kappa_0 T"'] \\
& KTT \arrow[r, Rightarrow, "K\mu^T"'] \arrow[ur, phantom, "\Lleftarrow Ks^T", "\Lleftarrow Kt^T"'] & KT
\end{tikzcd} \tag{11.4}$$

and

$$\begin{tikzcd}
SSH \arrow[rr, Rightarrow, "SS\rho"] \arrow[dddd, Rightarrow, "\mu^S H"'] \arrow[ddr, Rightarrow, "S\chi"] & & SSKT \arrow[dr, Rightarrow, "S\kappa T"] & & \\
& & \Swarrow S\rho_1 & SKTT \arrow[dr, Rightarrow, "SK\mu^T"] & \\
& SHT \arrow[rr, Rightarrow, "S\rho T"] \arrow[dd, Rightarrow, "\chi T"'] & & SKTT \arrow[r, Rightarrow, "SK\mu T"] \arrow[d, Rightarrow, "\kappa TT"] \arrow[dr, phantom, "\Swarrow \kappa_{\mu^T}{}^{-1}"] & SKT \arrow[d, Rightarrow, "\kappa T"] \\
\Swarrow \chi_1 & & \Swarrow \rho_1 T & KTTT \arrow[r, Rightarrow, "KT\mu^T"] \arrow[d, Rightarrow, "K\mu^T T"] & KTT \arrow[dd, Rightarrow, "K\mu^T"] \\
SH \arrow[dr, Rightarrow, "\chi"'] & HTT \arrow[r, Rightarrow, "\rho TT"] \arrow[d, Rightarrow, "H\mu^T"'] \arrow[dr, phantom, "\Swarrow \rho_{\mu^T}"] & KTTT \arrow[r, Rightarrow, "K\mu^T T"] \arrow[d, Rightarrow, "KT\mu^T"] \arrow[dr, phantom, "\Swarrow Km^{T-1}"] & KTT \arrow[dr, Rightarrow, "K\mu^T"] \arrow[r, phantom, "\Swarrow Km^T"] & \\
& HT \arrow[r, Rightarrow, "\rho T"'] & KTT \arrow[rr, Rightarrow, "K\mu^T"'] & & KT
\end{tikzcd} \tag{11.5}$$

$$=$$

$$\begin{tikzcd}
SSH \arrow[r, Rightarrow, "SS\rho"] \arrow[dd, Rightarrow, "\mu^S H"'] \arrow[ddr, phantom, "\Swarrow \mu_\rho^{S-1}"] & SSKT \arrow[dr, Rightarrow, "S\kappa T"] \arrow[dd, Rightarrow, "\mu^S KT"'] & & \\
& & SKTT \arrow[d, Rightarrow, "\kappa TT"] \arrow[dr, Rightarrow, "SK\mu^T"] & \\
SH \arrow[r, Rightarrow, "S\rho"] \arrow[ddr, Rightarrow, "\chi"'] & SKT \arrow[dr, Rightarrow, "\kappa T"] \arrow[ur, phantom, "\Swarrow \kappa_1 T"] & KTTT \arrow[d, Rightarrow, "K\mu^T T"] \arrow[dr, Rightarrow, "KT\mu^T"] \arrow[r, phantom, "\Swarrow \kappa_{\mu^T}{}^{-1}"] & SKT \arrow[d, Rightarrow, "\kappa T"] \\
& \Swarrow \rho_1 & KTT \arrow[dr, Rightarrow, "K\mu^T"] \arrow[r, phantom, "\Swarrow Km^T"] & KTT \arrow[d, Rightarrow, "K\mu^T"] \\
& HT \arrow[r, Rightarrow, "\rho T"] & KTT \arrow[r, Rightarrow, "K\mu^T"] & KT.
\end{tikzcd}$$

(b) The morphisms are cells

$$\begin{tikzcd}
(H,\chi,\chi_0,\chi_1) \arrow[rr, bend left, "{(\rho,\rho_1)}"] \arrow[rr, bend right, "{(\rho^*,\rho_1^*)}"'] \arrow[rr, phantom, "\Downarrow \tau"] & & (K,\kappa,\kappa_0,\kappa_1),
\end{tikzcd}$$

where

$$\begin{tikzcd}
H \arrow[rr, Rightarrow, bend left, "\rho"] \arrow[rr, Rightarrow, bend right, "\rho^*"'] \arrow[rr, phantom, "\Downarrow \tau"] & & KT
\end{tikzcd}$$

is subject to the equality

$$\begin{tikzcd}
SH \arrow[rr, Rightarrow, bend left, "S\rho"] \arrow[rr, Rightarrow, bend right, "S\rho^*"'] \arrow[rr, phantom, "\Downarrow S\tau"] \arrow[dd, Rightarrow, "\chi"'] & & SKT \arrow[d, Rightarrow, "\kappa T"] \\
& \swarrow \rho_1^* & KTT \arrow[d, Rightarrow, "K\mu^T"] \\
KT \arrow[r, Rightarrow, "\rho^* T"'] & KTT \arrow[r, Rightarrow, "K\mu^T"'] & KT
\end{tikzcd}$$

$$= \tag{11.6}$$

$$\begin{tikzcd}
SH \arrow[rr, Rightarrow, "S\rho"] \arrow[dd, Rightarrow, "\chi"'] & & SKT \arrow[d, Rightarrow, "\kappa T"] \\
& \swarrow \rho_1 & KTT \arrow[d, Rightarrow, "K\mu^T"] \\
KT \arrow[r, Rightarrow, bend left, "\rho T"] \arrow[r, Rightarrow, bend right, "\rho^* T"'] \arrow[r, phantom, "\Downarrow \tau T"] & KTT \arrow[r, Rightarrow, "K\mu^T"'] & KT.
\end{tikzcd}$$

That these data form a category (together with the obvious definition of composition) is immediate. We denote the category by $\mathbf{B}((H,\chi,\chi_0,\chi_1),(K,\kappa,\kappa_0,\kappa_1))$.

3. For each triple (H,χ,χ_0,χ_1), $(K,\kappa,\kappa_0,\kappa_1)$, $(L,\lambda,\lambda_0,\lambda_1)$ of transitions, a composi-

tion functor[1]

$$* : \mathbf{B}((K,\kappa),(L,\lambda)) \times \mathbf{B}((H,\chi),(K,\kappa)) \to \mathbf{B}((H,\chi),(L,\lambda))$$

This functor is defined on objects by the assignment

$$((\sigma,\sigma_1),(\rho,\rho_1)) \mapsto (\sigma,\sigma_1) * (\rho,\rho_1) = (\tau,\tau_1),$$

where

$$\tau = \ H \xRightarrow{\rho} KT \xRightarrow{\sigma T} LTT \xRightarrow{L\mu^T} LT$$

and where τ_1 is the 2-cell

$$\begin{tikzcd}
SH \arrow[r, Rightarrow, "S\rho"] \arrow[dddd, Rightarrow, "\chi"'] & SKT \arrow[rr, Rightarrow, "S\sigma T"] \arrow[dd, Rightarrow, "\kappa T"'] \arrow[ddr, phantom, "\Lleftarrow \sigma_1 T"] & & SLTT \arrow[r, Rightarrow, "SL\mu T"] \arrow[d, Rightarrow, "\lambda TT"] \arrow[dr, phantom, "\Lleftarrow {\lambda_{\mu^T}}^{-1}"] & SLT \arrow[d, Rightarrow, "\lambda T"] \\
& & & LTTT \arrow[r, Rightarrow, "LT\mu^T"] \arrow[d, Rightarrow, "L\mu^T T"] & LTT \arrow[dd, Rightarrow, "L\mu^T"] \\
{} \arrow[r, phantom, "\Lleftarrow \rho_1"] & KTT \arrow[r, Rightarrow, "\sigma TT"] \arrow[d, Rightarrow, "K\mu^T"'] \arrow[dr, phantom, "\Lleftarrow \sigma_{\mu^T}"] & LTTT \arrow[r, Rightarrow, "L\mu^T T"] \arrow[d, Rightarrow, "LT\mu^T"] \arrow[drr, phantom, "\Lleftarrow {Lm^T}^{-1}"] & LTT \arrow[dr, Rightarrow, "L\mu^T"] \arrow[r, phantom, "\Lleftarrow Lm^T"] & {} \\
& KT \arrow[r, Rightarrow, "\sigma T"'] \arrow[dr, phantom, "\Rrightarrow {\sigma_{\mu^T}}^{-1}"] & LTT \arrow[rr, Rightarrow, "L\mu^T"'] \arrow[dr, phantom, "\Rrightarrow Lm^T"] & & LT. \\
HT \arrow[r, Rightarrow, "\rho T"'] & KTT \arrow[u, Rightarrow, "K\mu^T"] \arrow[r, Rightarrow, "\sigma TT"'] & LTTT \arrow[u, Rightarrow, "LT\mu^T"'] \arrow[r, Rightarrow, "L\mu^T T"'] & LTT \arrow[ur, Rightarrow, "L\mu^T T"'] &
\end{tikzcd}$$

On arrows, $*$ sends a pair

$$\begin{tikzcd}
(\ H \arrow[r, bend left, Rightarrow, "\rho"] \arrow[r, bend right, Rightarrow, "\rho*"'] \arrow[r, phantom, "\Downarrow\tau"] & KT\ ,\ K \arrow[r, bend left, Rightarrow, "\sigma"] \arrow[r, bend right, Rightarrow, "\sigma*"'] \arrow[r, phantom, "\Downarrow\nu"] & LT\ )
\end{tikzcd}$$

to

$$\nu * \tau = \begin{tikzcd}
H \arrow[r, bend left, Rightarrow, "\rho"] \arrow[r, bend right, Rightarrow, "\rho*"'] \arrow[r, phantom, "\Downarrow\tau"] & KT \arrow[r, bend left, Rightarrow, "\sigma T"] \arrow[r, bend right, Rightarrow, "\sigma * T"'] \arrow[r, phantom, "\Downarrow\nu T"] & LTT \arrow[r, Rightarrow, "L\mu^T"] & LT.
\end{tikzcd}$$

4. For each object (transition) (H,χ,χ_0,χ_1) an identity functor

$$id_{(H,\chi,\chi_0,\chi_1)} : \mathbf{1} \to \mathbf{B}((H,\chi,\chi_0,\chi_1),(H,\chi,\chi_0,\chi_1))$$

---

[1] To make the rest of the chapter easily readable, we lighten up the notation: transitions, i.e., tuples (H,χ,χ_0,χ_1) are denoted only by the pair (H,χ) (or even by H) whenever it is not important to stress the additional structure.

sending the unique element of $\mathbf{1}$ to the transition morphism

$$H\eta^T : H \Rightarrow HT,$$

$$\begin{tikzcd}
SH \arrow[r, Rightarrow, "SH\eta^T"] \arrow[d, Rightarrow, "\chi"'] \arrow[dr, phantom, "\Swarrow \chi_{\eta^T}{}^{-1}"] & SHT \arrow[d, Rightarrow, "\chi T"] \\
HT \arrow[r, Rightarrow, "HT\eta^T"] \arrow[d, Rightarrow, "H\eta^T T"'] \arrow[dr, Rightarrow, "1"] & HTT \arrow[d, Rightarrow, "H\mu^T"] \\
HTT \arrow[r, Rightarrow] & HT
\end{tikzcd}$$

$$\Swarrow Hs^{T-1} \qquad \Swarrow Ht^{T-1}_{H\mu^T}$$

The underlying structure of $\mathbf{B}$ introduced above is equipped with the following natural isomorphisms:

5. Natural isomorphisms

$$\begin{tikzcd}
\mathbf{1} \times \mathbf{B}(((H,\chi),(K,\kappa)) \arrow[d, "id_{(K,\kappa)}\times 1"'] \arrow[dr, phantom, "\Searrow l_{(((H,\chi),(K,\kappa))}"] & \mathbf{B}(((H,\chi),(K,\kappa)) \arrow[l, "\cong"'] \arrow[d, "1"] \\
\mathbf{B}(((K,\kappa),(K,\kappa)) \times \mathbf{B}(((H,\chi),(K,\kappa)) \arrow[r, "*"'] & \mathbf{B}(((H,\chi),(K,\kappa))
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{B}(((H,\chi),(K,\kappa)) \arrow[r, "\cong"] \arrow[d, "1"] \arrow[dr, phantom, "\Swarrow r_{(((H,\chi),(K,\kappa))}"] & \mathbf{B}(((H,\chi),(K,\kappa)) \times \mathbf{1} \arrow[d, "1\times id_H"] \\
\mathbf{B}(((H,\chi),(K,\kappa)) & \mathbf{B}(((H,\chi),(K,\kappa)) \times \mathbf{B}(((H,\chi),(H,\chi)) \arrow[l, "*"]
\end{tikzcd}$$

i.e., for each transition morphism (ρ,ρ_1) a pair

$$l_{(\rho,\rho_1)} : id_K * \rho \Rightarrow \rho, \qquad r_{(\rho,\rho_1)} : \rho * id_H \Rightarrow \rho;$$

The transition 2-cell $r_{(\rho,\rho_1)}$ is defined as

$$\begin{tikzcd}
H \arrow[r, Rightarrow, "\rho"] \arrow[d, Rightarrow, "H\eta^T"'] \arrow[dr, phantom, "\Nearrow \rho_{\eta^T}{}^{-1}"] & KT \arrow[d, Rightarrow, "KT\eta^T"'] \arrow[dr, Rightarrow, bend left, "1"] \arrow[dr, phantom, "\Nearrow Ks^{T-1}"] & \\
HT \arrow[r, Rightarrow, "\rho T"'] & KTT \arrow[r, Rightarrow, "K\mu^T"'] & KT
\end{tikzcd}$$

and the transition 2-cell $l_{(\rho,\rho_1)}$ is defined as

$$\begin{tikzcd}
H \arrow[r, Rightarrow, "\rho"] & KT \arrow[d, Rightarrow, "K\eta^T T"'] \arrow[dr, Rightarrow, bend left, "1"] \arrow[dr, phantom, "\Nearrow Kt^T"] & \\
& KTT \arrow[r, Rightarrow, "K\mu^T"'] & KT.
\end{tikzcd}$$

6. Natural isomorphism

$$\begin{tikzcd}
\mathbf{B}(K,L)\times\mathbf{B}(H,K)\times\mathbf{B}(G,H) \arrow[r, "1\times *"] \arrow[d, "*\times 1"'] \arrow[dr, phantom, "\nearrow a_{G,H,K,L}"] & \mathbf{B}(K,L)\times\mathbf{B}(G,K) \arrow[d, "*"] \\
\mathbf{B}(H,L)\times\mathbf{B}(G,H) \arrow[r, "*"'] & \mathbf{B}(G,L),
\end{tikzcd}$$

i.e., a transition 2-cell

$$a_{\sigma,\rho,\varphi} : ((\sigma,\sigma_1) * (\rho,\rho_1)) * (\varphi,\varphi_1) \Rightarrow (\sigma,\sigma_1) * ((\rho,\rho_1) * (\varphi,\varphi_1))$$

defined for each triple $((\sigma,\sigma_1),(\rho,\rho_1),(\varphi,\varphi_1))$ of transition 2-cells as

$$\begin{tikzcd}
G \arrow[r, Rightarrow, "\varphi"] & HT \arrow[r, Rightarrow, "\rho T"] & KTT \arrow[r, Rightarrow, "K\mu^T"] \arrow[d, Rightarrow, "\sigma TT"'] \arrow[dr, phantom, "\Nearrow \sigma_{\mu^T}"] & KT \arrow[d, Rightarrow, "\sigma T"] \\
& & LTTT \arrow[r, Rightarrow, "LT\mu^T"'] \arrow[d, Rightarrow, "L\mu^T T"'] \arrow[dr, phantom, "\Nearrow {Lm^T}^{-1}"] & LTT \arrow[d, Rightarrow, "L\mu^T"] \\
& & LTT \arrow[r, Rightarrow, "L\mu^T"'] & LT.
\end{tikzcd}$$

7. These data are subject to the following equalities:

   - The triangle

$$\begin{tikzcd}
((\sigma,\sigma_1) * id_K) * (\rho,\rho_1) \arrow[rr, Rightarrow, "a"] \arrow[dr, Rightarrow, "r*1"'] & & (\sigma,\sigma_1) * (id_K * (\rho,\rho_1)) \arrow[dl, Rightarrow, "1*l"] \\
& (\sigma,\sigma_1) * (\rho,\rho_1) &
\end{tikzcd}$$

   commutes. This triangle commutes since T is a pseudomonad: the diagram

$$\begin{tikzcd}
H \arrow[r, Rightarrow, "\rho"] & KT \arrow[r, Rightarrow, "\rho T"] \arrow[d, Rightarrow, "K\eta^T T"'] \arrow[dr, phantom, "\Nearrow {\sigma_{\eta^T T}}^{-1}"] & LTT \arrow[d, Rightarrow, "LT\eta^T T"'] \arrow[dr, Rightarrow, bend left, "1"] \arrow[dr, phantom, "\Nearrow {Ls^T T}^{-1}"'] & & \\
& KTT \arrow[r, Rightarrow, "\sigma TT"'] & LTTT \arrow[r, Rightarrow, "L\mu^T T"'] & LTT \arrow[r, Rightarrow, "L\mu^T"] & LT
\end{tikzcd}$$

equals

$$\begin{tikzcd}
H \arrow[r, Rightarrow, "\rho"] & KT \arrow[d, Rightarrow, "K\eta^T T"'] \arrow[dr, Rightarrow, "1"] \arrow[dr, phantom, "\Nearrow Kt^T"', near end] & \\
& KTT \arrow[r, Rightarrow, "K\mu^T"'] \arrow[d, Rightarrow, "\sigma TT"'] \arrow[dr, phantom, "\Nearrow \sigma_{\mu^T}"] & KT \arrow[d, Rightarrow, "\sigma T"] \\
& LTTT \arrow[r, Rightarrow, "LT\mu^T"] \arrow[d, Rightarrow, "L\mu^T T"'] \arrow[dr, phantom, "\Nearrow Lm^{T^{-1}}"] & LT \arrow[d, Rightarrow, "L\mu^T"] \\
& LTT \arrow[r, Rightarrow, "L\mu^T"'] & LT.
\end{tikzcd}$$

- The pentagon

$$\begin{tikzcd}
((\delta * \gamma) * \beta) * \alpha \arrow[r, Rightarrow, "a*1"] \arrow[d, Rightarrow, "a"'] & (\delta * (\gamma * \beta)) * \alpha \arrow[r, Rightarrow, "a"] & \delta * ((\gamma * \beta) * \alpha) \arrow[d, Rightarrow, "1*a"] \\
(\delta * \gamma) * (\beta * \alpha) \arrow[rr, Rightarrow, "a"'] & & \delta * (\gamma * (\beta * \alpha))
\end{tikzcd}$$

commutes. This is again true since T is a pseudomonad.

The tricategorical structure of $\mathsf{TRANS}(\mathbf{K})$

We have shown in the previous subsection that for any two fixed pseudomonads T on $\mathscr{X}$ and S on $\mathscr{Y}$ in $\mathbf{K}$ the collection of all transitions from $(\mathscr{X}, T)$ to $(\mathscr{Y}, S)$, all transition morphisms and all transition 2-cells forms a bicategory. However, we can obtain additional structure by observing that transitions themselves can be composed. This additional structure turns out to make $\mathsf{TRANS}(\mathbf{K})$ an honest tricategory. That is,

1. the objects of $\mathsf{TRANS}(\mathbf{K})$ are pairs $(\mathscr{X}, T)$, where $\mathscr{X}$ is an object of $\mathbf{K}$ and T is a pseudomonad $(T, \eta^T, \mu^T, s^T, t^T, m^T)$ on $\mathscr{X}$, and
2. given two objects $(\mathscr{X}, T)$ and $(\mathscr{Y}, S)$, the hom-bicategory $\mathsf{TRANS}(\mathbf{K})((\mathscr{X}, T), (\mathscr{Y}, S))$ is precisely the bicategory $\mathbf{B}$ described in the previous section, consisting of transitions, transition morphisms and transition 2-cells.

The 1-cells $(\mathscr{X}, T) \to (\mathscr{Y}, S)$ in $\mathsf{TRANS}(\mathbf{K})$ are precisely the objects of the bicategory $\mathsf{TRANS}(\mathbf{K})((\mathscr{X}, T), (\mathscr{Y}, S))$, the 2-cells are 1-cells of $\mathsf{TRANS}(\mathbf{K})((\mathscr{X}, T), (\mathscr{Y}, S))$, and similarly the 3-cells are the 2-cells of the hom-bicategory.

We will now describe the most important structure that is added by considering transitions to be composable.

Composition of 1-cells Given transitions $(H,\chi) : (\mathscr{X},T) \to (\mathscr{Y},S)$ and $(K,\kappa) : (\mathscr{Y},S) \to (\mathscr{Z},U)$, we define the composite transition $(L,\lambda,\lambda_0,\lambda_1)$ as follows:

1. The 1-cell L is the composite $KH : \mathscr{X} \to \mathscr{Z}$.
2. The 2-cell $\lambda : UKH \Rightarrow KHT$ is the composite
$$UKH \xRightarrow{\kappa H} KSH \xRightarrow{K\chi} KHT.$$
3. The isomorphism λ_0 is the pasting
$$\begin{tikzcd}
UKH \arrow[r, Rightarrow, "\kappa H"] \arrow[rr, Rightarrow, bend left, "\lambda"] & KSH \arrow[r, Rightarrow, "K\chi"] & KHT. \\
& KH \arrow[ul, Rightarrow, "\eta^U KH"] \arrow[u, Rightarrow, "K\eta^S H"'] \arrow[ur, Rightarrow, "KH\eta^T"'] \arrow[ul, phantom, "\rotatebox{-45}{$\Rrightarrow$}\,\kappa_0 H"'] \arrow[ur, phantom, "\rotatebox{-45}{$\Rrightarrow$}\,K\chi_0"] &
\end{tikzcd}$$
4. The isomorphism λ_1 is the pasting
$$\begin{tikzcd}
& & UKHT \arrow[dr, Rightarrow, "\kappa HT"] & & \\
UUKH \arrow[r, Rightarrow, "U\kappa H"] \arrow[dd, Rightarrow, "\mu^U KH"'] & UKSH \arrow[ur, Rightarrow, "UK\chi"] \arrow[dr, Rightarrow, "\kappa SH"'] & \rotatebox{-90}{$\Rrightarrow$}{\kappa_\chi}^{-1} & KSHT \arrow[r, Rightarrow, "K\chi T"] & KHTT \arrow[dd, Rightarrow, "KH\mu^T"] \\
& \rotatebox{45}{$\Lleftarrow$}\,\kappa_1 H & KSSH \arrow[ur, Rightarrow, "KS\chi"'] \arrow[d, Rightarrow, "K\mu^S H"'] & \rotatebox{45}{$\Lleftarrow$}\,K\chi_1 & \\
UKH \arrow[rr, Rightarrow, "\kappa H"'] & & KSH \arrow[rr, Rightarrow, "K\chi"'] & & KHT
\end{tikzcd}$$

It amounts to a laborious computation to check that such a composite indeed yields a transition again.

The composition of transitions is strictly associative and unital. This is immediate from the definition of composition and from observing that the identity of the composition is the identity transition $(1,1,1,1)$.

Whiskering We shall need to compose 2-cells and 3-cells in $\mathsf{TRANS}(\mathbf{K})$ that are whiskered from left or right with 1-cells.

Consider the situation
$$\begin{tikzcd}
(\mathscr{X},T) \arrow[r, "{(G,\gamma,\gamma_0,\gamma_1)}"] & (\mathscr{Y},S) \arrow[r, bend left, "{(H,\chi,\chi_0,\chi_1)}"] \arrow[r, bend right, "{(H',\chi',\chi_0',\chi_1')}"'] \arrow[r, phantom, "{\Downarrow (\rho,\rho_1)}"] & (\mathscr{Z},U)
\end{tikzcd}$$

i.e., having

$$\rho : H \Rightarrow H'S$$

and

$$\begin{tikzcd}
UH \arrow[rr, Rightarrow, "U\rho"] \arrow[dd, Rightarrow, "\chi"'] & & UH'S \arrow[d, Rightarrow, "\chi' S"] \\
& \rho_1 \Lleftarrow & HSS \arrow[d, Rightarrow, "H'\mu^S"] \\
HS \arrow[r, Rightarrow, "\rho S"'] & H'SS \arrow[r, Rightarrow, "H'\mu^S"'] & H'S.
\end{tikzcd}$$

We wish to describe the whiskering

$$\begin{tikzcd}
(\mathscr{X}, T) \arrow[rr, bend left] \arrow[rr, bend right] \arrow[rr, phantom, "\Downarrow (\rho, \rho_1)(G, \gamma)"] & & (\mathscr{Z}, U).
\end{tikzcd}$$

The domain morphism is $(HG, H\gamma \cdot \chi G)$ and the codomain is $(H'G, H'\gamma \cdot \chi' G)$. The cell $(\rho, \rho_1)(G, \gamma)$ is defined as

$$\begin{tikzcd}
HG \arrow[r, Rightarrow, "\rho G"] & H'SG \arrow[r, Rightarrow, "H'\gamma"] & H'GT,
\end{tikzcd}$$

while $((\rho, \rho_1)(G, \gamma))_1$ is defined as

$$\begin{tikzcd}
UHG \arrow[rr, Rightarrow, "U\rho G"] \arrow[dd, Rightarrow, "\chi G"'] & & UH'SG \arrow[r, Rightarrow, "UH'\gamma"] \arrow[d, Rightarrow, "\chi' SG"] \arrow[dr, phantom, "\Lleftarrow {\chi'_\gamma}^{-1}"] & UH'GT \arrow[d, Rightarrow, "\chi' GT"] \\
& \Lleftarrow \rho_1 G & H'SSG \arrow[r, Rightarrow, "H'S\gamma"] \arrow[d, Rightarrow, "H'\mu^S"] & H'SGT \arrow[d, Rightarrow, "H'\gamma T"] \\
HSG \arrow[r, Rightarrow, "\rho SG"] \arrow[d, Rightarrow, "H\gamma"'] \arrow[dr, phantom, "\Lleftarrow \rho_\gamma"] & H'SSG \arrow[r, Rightarrow, "H'\mu^S G"] \arrow[d, Rightarrow, "H'S\gamma"] \arrow[dr, phantom, "\Lleftarrow {H'\gamma_1}^{-1}"] & H'SG \arrow[r, phantom, "\Lleftarrow H'\gamma_1"] \arrow[dr, Rightarrow, "H'\gamma"] & H'GTT \arrow[d, Rightarrow, "H'G\mu^T"] \\
HGT \arrow[r, Rightarrow, "\rho GT"'] & H'SGT \arrow[r, Rightarrow, "H'\gamma T"'] & H'GTT \arrow[r, Rightarrow, "H'G\mu^T"'] & H'GT.
\end{tikzcd}$$

Given a 3-cell $\tau : (\rho, \rho_1) \Rrightarrow (\rho^*, \rho_1^*)$, i.e.,

$$\begin{tikzcd}
H \arrow[r, Rightarrow, bend left, "\rho"] \arrow[r, Rightarrow, bend right, "\rho^*"'] \arrow[r, phantom, "\Downarrow \tau"] & H'S,
\end{tikzcd}$$

the whiskered cell $\tau(G, \gamma)$ is defined as

$$\begin{tikzcd}
HG \arrow[r, Rightarrow, bend left, "\rho G"] \arrow[r, Rightarrow, bend right, "\rho^* G"'] \arrow[r, phantom, "\Downarrow\!\!\!\Downarrow \tau G"] & H'SG \arrow[r, Rightarrow, "H'\gamma"] & H'GT.
\end{tikzcd}$$

On the other hand, given a situation

$$\begin{tikzcd}
(\mathscr{Y}, S) \arrow[r, bend left, "{(H,\chi,\chi_0,\chi_1)}"] \arrow[r, bend right, "{(H',\chi',\chi_0',\chi_1')}"'] \arrow[r, phantom, "{\Downarrow (\rho, \rho_1)}"] & (\mathscr{Z}, U) \arrow[r, "{(L,\lambda)}"] & (\mathscr{A}, V),
\end{tikzcd}$$

we may form the whiskered cell

$$\begin{tikzcd}
(\mathscr{Y}, S) \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "{\Downarrow (L, \lambda)(\rho, \rho_1)}"] & (\mathscr{A}, V)
\end{tikzcd}$$

with the domain $(LH, L\chi \cdot \lambda H)$ and codomain $(LH', L\chi' \cdot \lambda H')$ which is defined as

$$\begin{tikzcd}
LH \arrow[r, Rightarrow, "L\rho"] & LH'S
\end{tikzcd}$$

with $((L, \lambda)(\rho, \rho_1))_1$ being

$$\begin{tikzcd}
VLH \arrow[rr, Rightarrow, "VL\rho"] \arrow[d, Rightarrow, "\lambda H"'] \arrow[drr, phantom, "\Swarrow \lambda_\rho{}^{-1}"] & & VLH'S \arrow[d, Rightarrow, "\lambda H'S"] \\
LUH \arrow[rr, Rightarrow, "LU\rho"] \arrow[ddd, Rightarrow, "L\chi"'] \arrow[dddrr, phantom, "\Swarrow L\rho_1"] & & LUH'S \arrow[d, Rightarrow, "L\chi' S"] \\
 & & LH'SS \arrow[dd, Rightarrow, "LH'\mu^S"] \\
 & & \\
LHS \arrow[r, Rightarrow, "L\rho S"'] & LH'SS \arrow[r, Rightarrow, "LH'\mu^S"'] & LH'S.
\end{tikzcd}$$

Given a 3-cell $\tau : (\rho, \rho_1) \Rrightarrow (\rho^*, \rho_1^*)$, i.e.,

$$\begin{tikzcd}
H \arrow[r, Rightarrow, bend left, "\rho"] \arrow[r, Rightarrow, bend right, "\rho^*"'] \arrow[r, phantom, "\Downarrow\!\!\!\Downarrow \tau"] & H'S,
\end{tikzcd}$$

the whiskering $(L, \lambda)\tau$ is defined as

$$\begin{tikzcd}
LH \arrow[r, Rightarrow, bend left, "L\rho"] \arrow[r, Rightarrow, bend right, "L\rho^*"'] \arrow[r, phantom, "\Downarrow\!\!\!\Downarrow L\tau"] & LH'S.
\end{tikzcd}$$

The "Gray" 3-cells Consider now the situation of two horizontal 2-cells as in the diagram

$$\begin{tikzcd}
(\mathscr{X},T) \arrow[r, bend left, "{(G,\gamma)}"] \arrow[r, bend right, "{(G',\gamma')}"'] \arrow[r, phantom, "\Downarrow (\pi,\pi_1)"] & (\mathscr{Y},S) \arrow[r, bend left, "{(H,\chi)}"] \arrow[r, bend right, "{(H',\chi')}"'] \arrow[r, phantom, "\Downarrow (\rho,\rho_1)"] & (\mathscr{Z},U).
\end{tikzcd}$$

There are two ways in which (π,π_1) and (ρ,ρ_1) can be horizontally composed. We may compose $(\rho,\rho_1)(G,\gamma)$ with $(H',\chi')(\pi,\pi_1)$, i.e.,

$$HG \xRightarrow{\rho G} H'SG \xRightarrow{H'\gamma} H'GT$$

with

$$H'G \xRightarrow{H'\pi} H'G'T$$

to obtain

$$HG \xRightarrow{\rho G} H'SG \xRightarrow{H'\gamma} H'GT \xRightarrow{H'\pi T} H'G'TT \xRightarrow{H'G'\mu^T} H'G'T,$$

or we may compose $(H,\chi)(\pi,\pi_1)$ with $(\rho,\rho_1)(G',\gamma')$, i.e.,

$$HG \xRightarrow{H\pi} HG'T$$

with

$$HG' \xRightarrow{\rho G'} H'SG' \xRightarrow{H'\gamma'} H'G'T$$

to obtain

$$HG \xRightarrow{H\pi} HG'T \xRightarrow{\rho G'T} H'SG'T \xRightarrow{H'\gamma'T} H'G'TT \xRightarrow{H'G'\mu^T} H'G'T.$$

As in **Gray**-categories, these two ways of composing the 2-cells are related by a 3-cell isomorphism. We denote it by $(\rho,\rho_1)_{(\pi,\pi_1)}$ and define it to be the cell

$$\begin{tikzcd}
HG \arrow[r, Rightarrow, "\rho G"] \arrow[d, Rightarrow, "H\pi"'] \arrow[dr, phantom, "\Swarrow \rho_\pi"] & H'SF \arrow[r, Rightarrow, "H'\gamma"] \arrow[d, Rightarrow, "H'S\pi"'] & H'GT \arrow[r, Rightarrow, "H'\pi T"] \arrow[d, phantom, "\Swarrow H'{\pi_1}^{-1}"] & H'G'TT \arrow[d, Rightarrow, "H'G'\mu^T"] \\
HG'T \arrow[r, Rightarrow, "\rho G'T"'] & H'SG'T \arrow[r, Rightarrow, "H'\gamma'T"'] & H'G'TT \arrow[r, Rightarrow, "H'G'\mu^T"'] & H'G'T.
\end{tikzcd}$$

11.4 Triequivalence of $\mathsf{LIFT}(\mathbf{K})$ and $\mathsf{TRANS}(\mathbf{K})$

In this section we will show how the **Gray**-category $\mathsf{LIFT}(\mathbf{K})$ and the tricategory $\mathsf{TRANS}(\mathbf{K})$ are related. There is a *homomorphism*

$$\Phi : \mathsf{LIFT}(\mathbf{K}) \to \mathsf{TRANS}(\mathbf{K})$$

of tricategories that is a *triequivalence.* The notion of a homomorphism is a suitably weakened analogue of the notion of a functor between categories, similar to homomorphisms of bicategories. The notion of triequivalence is a categorification of the notion of equivalence of categories. We refer the reader to [38] for the technical details: here we only comment that similarly to ordinary equivalence (where one needs to check that a functor is essentially surjective and an isomorphism on hom-sets), to have a triequivalence is to have a homomorphism that is a *biequivalence* on hom-bicategories, and that is *triessentially surjective*. Since the objects of $\mathsf{LIFT}(\mathbf{K})$ and $\mathsf{TRANS}(\mathbf{K})$ correspond bijectively, the most important part of checking the triequivalence corresponds to checking the biequivalence on hom-bicategories.

Define Φ to be essentially an identity on objects, i.e.,

$$\Phi\left(\begin{tikzcd}\mathscr{X}^T \arrow[d, "U^T"'] \\ \mathscr{X}\end{tikzcd}\right) = (\mathscr{X}, T).$$

The biequivalence

$$\Phi_{U^T,U^S} : \mathsf{LIFT}(\mathbf{K})(U^T, U^S) \to \mathsf{TRANS}(\mathbf{K})((\mathscr{X}, T), (\mathscr{Y}, S))$$

sends

$$\begin{tikzcd}
\mathscr{X}^T \arrow[r, "\widetilde{H}"] \arrow[d, "U^T"'] & \mathscr{Y}^S \arrow[d, "U^S"] \\
\mathscr{X} \arrow[r, "H"'] & \mathscr{Y}
\end{tikzcd}$$

to

$$(H, \chi, \chi_0, \chi_1) : (\mathscr{X}, T) \to (\mathscr{Y}, S),$$

where χ is defined in the following steps:

$$\widetilde{H}(\mathscr{X}) : \begin{tikzcd}TT \arrow[d, Rightarrow, "\mu^T"'] \\ T\end{tikzcd} \mapsto \left(\begin{tikzcd}SHT \arrow[d, Rightarrow, "a"] \\ HT\end{tikzcd}, a_0, a_1\right)$$

where a_0 and a_1 are cells

$$\begin{tikzcd}
HT \arrow[r, Rightarrow, "\eta^S HT"] \arrow[dr, Rightarrow, "1"'] & SHT \arrow[d, Rightarrow, "a"] \arrow[dl, phantom, "\Lleftarrow a_o"] \\
& HT
\end{tikzcd}
\qquad
\begin{tikzcd}
SSHT \arrow[r, Rightarrow, "Sa"] \arrow[d, Rightarrow, "\mu^S HT"'] \arrow[dr, phantom, "\Lleftarrow a_1"] & SHT \arrow[d, Rightarrow, "a"] \\
SHT \arrow[r, Rightarrow, "a"'] & HT
\end{tikzcd}$$

and we put

$$\chi = \begin{tikzcd} SH \arrow[r, Rightarrow, "SH\eta^T"] & SHT \arrow[r, Rightarrow, "a"] & HT\end{tikzcd}$$

together with

$$\begin{tikzcd}
H \arrow[r, Rightarrow, "\eta^S H"] \arrow[d, Rightarrow, "H\eta^T"'] \arrow[dr, phantom, "\Swarrow \eta^S_{H\eta^T}"] & SH \arrow[d, Rightarrow, "SH\eta^T"] \\
HT \arrow[r, Rightarrow, "\eta^S HT"] \arrow[dr, Rightarrow, "1"'] & SHT \arrow[d, Rightarrow, "a"] \arrow[ld, phantom, "\Swarrow a_0" near start] \\
& HT
\end{tikzcd}$$

and

$$\begin{tikzcd}
SSH \arrow[r, Rightarrow, "SSH\eta^T"] \arrow[dd, Rightarrow, "\mu^S H"'] \arrow[ddr, phantom, "\Swarrow {\mu^S_{H\eta^T}}^{-1}"] & SSHTX \arrow[r, Rightarrow, "Sa"] \arrow[dd, Rightarrow, "\mu^S HT"] \arrow[ddrr, phantom, "\Swarrow a_1"] & SHT \arrow[rr, Rightarrow, "SH\eta^T T"] \arrow[dr, Rightarrow, "1"'] \arrow[rr, phantom, "\Swarrow SHt^T"', shift right=4] & & SHTT \arrow[r, Rightarrow, "aT"] \arrow[dl, Rightarrow, "SH\mu^T"] \arrow[ddr, phantom, "\Swarrow {\widetilde{H}m^T}^{-1}"] & HTT \arrow[dd, Rightarrow, "H\mu^T"] \\
& & & SHT \arrow[d, Rightarrow, "a"] & & \\
SH \arrow[r, Rightarrow, "SH\eta^T"'] & SHT \arrow[rr, Rightarrow, "a"'] & & HT \arrow[rr, Rightarrow, "1"'] & & HT.
\end{tikzcd}$$

That $(H, \chi, \chi_0, \chi_1)$ is a 1-cell from $(\mathscr{X}, T)$ to $(\mathscr{Y}, S)$ is proved in [69] (in particular, see Proposition 3.4 of [69]).

Given

$$\begin{tikzcd}
\mathscr{X}^T \arrow[r, bend left, "\widetilde{H}", ""{name=U, below}] \arrow[r, bend right, "\widetilde{K}"', ""{name=D, above}] & \mathscr{Y}^S, \arrow[from=U, to=D, Rightarrow, "\widetilde{\rho}"]
\end{tikzcd}$$

we define $\Phi(\widetilde{\rho})$ to be the pair (ρ, ρ_1), where we first consider

$$\widetilde{H}(T, \mu^T) \xRightarrow{\widetilde{\rho}(T,\mu^T)} \widetilde{K}(T, \mu^T)$$

which equals

$$(\widetilde{\rho}, \widetilde{\rho}_1) : \left(\begin{tikzcd} SHT \arrow[d, Rightarrow, "a^H"'] \\ HT \end{tikzcd} , a_0^H, a_1^H \right) \to \left(\begin{tikzcd} SKT \arrow[d, Rightarrow, "a^K"'] \\ KT \end{tikzcd} , a_0^K, a_1^K \right),$$

and

$$\widetilde{H}(TT, \mu^T T) \xRightarrow{\widetilde{\rho}(TT,\mu^T T)} \widetilde{K}(TT, \mu^T T)$$

which equals

$$(\widetilde{\widetilde{\rho}}, \widetilde{\widetilde{\rho}}_1) : \left(\begin{tikzcd} SHTT \arrow[d, Rightarrow, "\overline{a}^H"'] \\ HTT \end{tikzcd} , \overline{a}^H_0, \overline{a}^H_1 \right) \to \left(\begin{tikzcd} SKTT \arrow[d, Rightarrow, "\overline{a}^K"'] \\ KTT \end{tikzcd} , \overline{a}^K_0, \overline{a}^K_1 \right)$$

and the pseudonaturality square

$$\begin{tikzcd} \widetilde{H}(TT, \mu^T T) \arrow[r, Rightarrow, "\widetilde{\rho}(TT,\mu^T T)"] \arrow[d, Rightarrow, "\widetilde{H}(\mu^T, m^T)"'] \arrow[dr, phantom, "\swarrow \widetilde{\rho}_{(\mu^T, m^T)}"] & \widetilde{K}(TT, \mu^T T) \arrow[d, Rightarrow, "\widetilde{K}(\mu^T, m^T)"] \\ \widetilde{H}(T, \mu^T) \arrow[r, Rightarrow, "\widetilde{\rho}(T,\mu^T)"'] & \widetilde{K}(T, \mu^T). \end{tikzcd}$$

Then the above pseudonaturality square is

$$\begin{tikzcd} (a^H T, a^H_0 T, a^H_1 T) \arrow[r, Rightarrow, "(\widetilde{\rho}T, \widetilde{\rho}_1 T)"] \arrow[d, Rightarrow, "(H\mu^T, \widetilde{H}m^T)"'] \arrow[dr, phantom, "\swarrow \widetilde{\rho}_{(\mu^T, m^T)}"] & (a^K T, a^K_0 T, a^K_1 T) \arrow[d, Rightarrow, "(K\mu^T, \widetilde{K}m^T)"] \\ (a^H, a^H_0, a^H_1) \arrow[r, Rightarrow, "(\widetilde{\rho}, \widetilde{\rho}_1)"'] & (a^K, a^K_0, a^K_1) \end{tikzcd}$$

In particular, we have an isomorphism

$$\begin{tikzcd} HTT \arrow[r, Rightarrow, "\widetilde{\rho}T"] \arrow[d, Rightarrow, "H\mu^T"'] \arrow[dr, phantom, "\swarrow \widetilde{\rho}_{(\mu^T, m^T)}"] & KTT \arrow[d, Rightarrow, "K\mu^T"] \\ HT \arrow[r, Rightarrow, "\widetilde{\rho}"'] & KT \end{tikzcd}$$

and we define ρ to be the composite

$$\rho = \begin{tikzcd} H \arrow[r, Rightarrow, "H\eta^T"] & HT \arrow[r, Rightarrow, "\widetilde{\rho}"] & KT \end{tikzcd}$$

and ρ_1 to be the pasting

$$\begin{tikzcd}
SH \arrow[r, Rightarrow, "SH\eta^T"] \arrow[d, Rightarrow, "SH\eta^T"'] & SHT \arrow[rr, Rightarrow, "S\widetilde{\rho}"] \arrow[dl, Rightarrow, "1"] & & SKT \arrow[dr, Rightarrow, "SK\eta^T T"] \arrow[dd, Rightarrow, "1"'] \arrow[ddr, phantom, "\swarrow SKt^T"] & \\
SHT \arrow[ddd, Rightarrow, "a^H"'] \arrow[rrr, phantom, "\swarrow \widetilde{\rho}_1"] & & & & SKTT \arrow[dl, Rightarrow, "SK\mu^T"] \arrow[d, Rightarrow, "a^K T"] \\
& & & SKT \arrow[dd, Rightarrow, "a^K"'] \arrow[r, phantom, "\swarrow \widetilde{K}m^{T^{-1}}"] & KTT \arrow[ddl, Rightarrow, "K\mu^T"] \\
& & & & \\
HT \arrow[rr, Rightarrow, "1"] \arrow[dr, Rightarrow, "H\eta^T T"'] & & HT \arrow[r, Rightarrow, "\widetilde{\rho}"] & KT. & \\
& HTT \arrow[r, Rightarrow, "\widetilde{\rho}T"'] \arrow[ur, Rightarrow, "H\mu^T"] \arrow[u, phantom, "\Downarrow Ht^{T-1}"] & KTT \arrow[ur, Rightarrow, "K\mu^T"'] \arrow[u, phantom, "\Downarrow \widetilde{\rho}^{-1}_{(\mu^T,m^T)}"] & &
\end{tikzcd}$$

It is then easy to check that (ρ,ρ_1) is indeed a morphism from (H,χ,χ_0,χ_1) to $(K,\kappa,\kappa_0,\kappa_1)$.

Given

$$\begin{tikzcd}
\mathscr{X}^T \arrow[rr, bend left=50, "\widetilde{H}"] \arrow[rr, bend right=50, "\widetilde{K}"'] \arrow[rr, phantom, "\widetilde{\rho}\Downarrow \overset{\widetilde{\tau}}{\Rrightarrow} \Downarrow\widetilde{\sigma}"] & & \mathscr{Y}^S,
\end{tikzcd}$$

we have the equality

$$\begin{tikzcd}
\widetilde{H}(TT,\mu^T T) \arrow[rr, Rightarrow, "\widetilde{\rho}(TT,\mu^T T)"] \arrow[d, Rightarrow, "\widetilde{H}(\mu^T,m^T)"'] \arrow[drr, phantom, "\swarrow \widetilde{\rho}_{(\mu^T,m^T)}"] & & \widetilde{K}(TT,\mu^T T) \arrow[d, Rightarrow, "\widetilde{K}(\mu^T,m^T)"] \arrow[from=1-3, to=2-5, phantom, "="] & & \widetilde{H}(TT,\mu^T T) \arrow[rr, Rightarrow, bend left, "\widetilde{\rho}(TT,\mu^T T)"] \arrow[rr, Rightarrow, bend right, "\widetilde{\sigma}(TT,\mu^T T)"'] \arrow[rr, phantom, "\Downarrow\widetilde{\tau}(TT,\mu^T T)"] \arrow[d, Rightarrow, "\widetilde{H}(\mu^T,m^T)"'] & & \widetilde{K}(TT,\mu^T T) \arrow[d, Rightarrow, "\widetilde{K}(\mu^T,m^T)"] \\
\widetilde{H}(T,\mu^T) \arrow[rr, Rightarrow, bend left, "\widetilde{\rho}(T,\mu^T)"] \arrow[rr, Rightarrow, bend right, "\widetilde{\sigma}(T,\mu^T)"'] \arrow[rr, phantom, "\Downarrow\widetilde{\tau}(T,\mu^T)"] & & \widetilde{K}(T,\mu^T) & & \widetilde{H}(T,\mu^T) \arrow[rr, Rightarrow, "\swarrow \widetilde{\sigma}_{(\mu^T,m^T)}", "\widetilde{\sigma}(T,\mu^T)"'] & & \widetilde{K}(T,\mu^T).
\end{tikzcd}$$

In particular, we have the equality

$$\begin{tikzcd}
HTT \arrow[r, Rightarrow, "\widetilde{\rho}T"] \arrow[d, Rightarrow, "H\mu^T"'] \arrow[dr, phantom, "\Swarrow \widetilde{\rho}_{(\mu^T,m^T)}"] & KTT \arrow[d, Rightarrow, "K\mu^T"] & & HTT \arrow[r, Rightarrow, bend left, "\widetilde{\rho}T"] \arrow[r, Rightarrow, bend right, "\widetilde{\sigma}T"'] \arrow[r, phantom, "\Downarrow\widetilde{\tau}T"] \arrow[d, Rightarrow, "H\mu^T"'] \arrow[dr, phantom, "\Swarrow \widetilde{\sigma}_{(\mu^T,m^T)}"] & KTT \arrow[d, Rightarrow, "K\mu^T"] \\
HT \arrow[r, Rightarrow, bend left, "\widetilde{\rho}"] \arrow[r, Rightarrow, bend right, "\widetilde{\sigma}"'] \arrow[r, phantom, "\Downarrow\widetilde{\tau}"] & KT \arrow[r, phantom, "="] & & HT \arrow[r, Rightarrow, "\widetilde{\sigma}"'] & KT
\end{tikzcd}$$

where we put $\widetilde{\tau} = \widetilde{\tau}(T, \mu^T)$. Thus, if we define τ to be

$$\tau = \begin{tikzcd}
H \arrow[r, Rightarrow, "H\eta^T"] & HT \arrow[r, Rightarrow, bend left, "\widetilde{\rho}"] \arrow[r, Rightarrow, bend right, "\widetilde{\sigma}"'] \arrow[r, phantom, "\Downarrow\widetilde{\tau}"] & KT,
\end{tikzcd}$$

we have defined

$$\begin{tikzcd}[column sep=huge]
(\mathscr{X}, T) \arrow[r, bend left=50, "{(H,\chi,\chi_0,\chi_1)}"] \arrow[r, bend right=50, "{(K,\kappa,\kappa_0,\kappa_1)}"'] \arrow[r, phantom, "{(\rho,\rho_1)\Downarrow \overset{\tau}{\Rrightarrow} \Downarrow(\sigma,\sigma_1)}"] & (\mathscr{Y}, S).
\end{tikzcd}$$

That $\Phi_{(\mathscr{X},T),(\mathscr{Y},S)}$ is a biequivalence follows from [69], see in particular Theorem 3.5.

Remark 11.4.1. With the above assignments, the homomorphism

$$\Phi : \mathsf{LIFT}(\mathbf{K}) \to \mathsf{TRANS}(\mathbf{K})$$

is a triequivalence. This will allow us to work with $\mathsf{TRANS}(\mathbf{K})$ in place of $\mathsf{LIFT}(\mathbf{K})$: given a pseudomonad $(\widetilde{H}, \widetilde{\sigma}, \widetilde{\nu}, \widetilde{s}, \widetilde{t}, \widetilde{m})$ in $\mathsf{TRANS}(\mathbf{K})$, the image of this pseudomonad under Φ is again a pseudomonad. This new pseudomonad consists of data and axioms in $\mathbf{K}$ that are necessary and sufficient to reconstruct $(\widetilde{H}, \widetilde{\sigma}, \widetilde{\nu}, \widetilde{s}, \widetilde{t}, \widetilde{m})$, i.e., these are the data that describe in an elementary manner what a wreath is.

11.5 Elementary description of wreaths

We have already introduced wreaths in **Gray**-categories in Section 11.2. However, the definition of a wreath from Definition 11.2.2 is very abstract and does not immediately show the data that are needed in $\mathbf{K}$ in order for a wreath of a 1-cell $H : \mathscr{X} \to \mathscr{X}$ around a pseudomonad T on $\mathscr{X}$ to exist. In this section we remedy the situation and give an elementary description of wreaths.

The approach is straightforward. Since $\mathsf{LIFT}(\mathbf{K}) \simeq \mathsf{TRANS}(\mathbf{K})$ and a wreath in $\mathbf{K}$ is a pseudomonad in $\mathsf{LIFT}(\mathbf{K})$, a wreath is equivalently a pseudomonad in $\mathsf{TRANS}(\mathbf{K})$. We will unveil the definition of a pseudomonad in $\mathsf{TRANS}(\mathbf{K})$; this will be what we call an elementary description of a wreath in $\mathbf{K}$.

To give a wreath in $\mathbf{K}$ is to give the following data from $\mathsf{TRANS}(\mathbf{K})$:

1. An object $(\mathscr{X}, T)$. This amounts to specifying the monad around which the wreath acts.

2. A 1-cell (a transition)

$$(\mathscr{X}, T) \xrightarrow{(H,\chi,\chi_0,\chi_1)} (\mathscr{X}, T),$$

i.e., a 2-cell

$$\chi : TH \Rightarrow HT$$

in **K**, and 3-cells

$$\begin{tikzcd}
H \arrow[r, Rightarrow, "\eta^T H"] \arrow[dr, Rightarrow, "H\eta^T"'] \arrow[r, phantom, shift right=6, "\Lleftarrow \chi_0"'] & TH \arrow[d, Rightarrow, "\chi"] \\
 & HT
\end{tikzcd}$$

and

$$\begin{tikzcd}
TTH \arrow[r, Rightarrow, "T\chi"] \arrow[d, Rightarrow, "\mu^T H"'] & THT \arrow[r, Rightarrow, "\chi T"] \arrow[d, phantom, "\Lleftarrow \chi_1"] & HTT \arrow[d, Rightarrow, "H\mu^T"] \\
TH \arrow[rr, Rightarrow, "\chi"'] & & HT
\end{tikzcd}$$

satisfying the transition coherence conditions. The choice of a transition specifies, among other data, the 1-cell $H : \mathscr{X} \to \mathscr{X}$. Thus in our case we have a wreath of H around T. Observe that the data χ_0 and χ_1 play the role of the transition axioms from the ordinary case.

3. Transition morphisms

$$\begin{tikzcd}
(\mathscr{X}, T) \arrow[r, bend left, "{(1,1,1,1)}"] \arrow[r, bend right, "{(H,\chi,\chi_0,\chi_1)}"'] \arrow[r, phantom, "{\Downarrow (\sigma, \sigma_1)}"] & (\mathscr{X}, T)
\end{tikzcd}$$

and

$$\begin{tikzcd}
(\mathscr{X}, T) \arrow[r, bend left, "{(H,\chi,\chi_0,\chi_1)\cdot(H,\chi,\chi_0,\chi_1)}"] \arrow[r, bend right, "{(H,\chi,\chi_0,\chi_1)}"'] \arrow[r, phantom, "{\Downarrow (\nu, \nu_1)}"] & (\mathscr{X}, T),
\end{tikzcd}$$

i.e., 2-cells

$$\sigma : 1 \Rightarrow HT, \qquad \nu : HH \Rightarrow HT$$

in **K** together with 3-cells

$$\begin{tikzcd}
T \arrow[rr, Rightarrow, "T\sigma"] \arrow[dd, Rightarrow, "1"'] & & THT \arrow[d, Rightarrow, "\chi T"] \\
& \Lleftarrow \sigma_1 & HTT \arrow[d, Rightarrow, "H\mu^T"] \\
T \arrow[r, Rightarrow, "\sigma T"'] & HTT \arrow[r, Rightarrow, "H\mu^T"'] & HT
\end{tikzcd}$$

and

$$\begin{tikzcd}
THH \arrow[rr, Rightarrow, "T\nu"] \arrow[d, Rightarrow, "\chi H"'] & & THT \arrow[d, Rightarrow, "\chi T"] \\
HTH \arrow[d, Rightarrow, "H\chi"'] & \Lleftarrow \nu_1 & HTT \arrow[d, Rightarrow, "H\mu^T"] \\
HHT \arrow[r, Rightarrow, "\nu T"'] & HTT \arrow[r, Rightarrow, "H\mu^T"'] & HT
\end{tikzcd}$$

satisfying the transition morphism coherence conditions. Observe that σ and ν are precisely the 2-cells σ and ν from the ordinary setting, where σ and ν were subject to *equations* σ_1 and ν_1.

4. Isomorphism 3-cells

$$t : \left(\begin{tikzcd}
(\mathscr{X},T) \arrow[r, "{(H,\chi,\chi_0,\chi_1)}"] \arrow[dr, "{(H,\chi,\chi_0,\chi_1)}"'] & (\mathscr{X},T) \arrow[d, "{(H,\chi,\chi_0,\chi_1)}"] \arrow[d, bend left=90, looseness=2, "{(1,1,1,1)}"] \arrow[d, phantom, shift right=5ex, "\Swarrow(\nu,\nu_1)"] \arrow[d, phantom, shift left=5ex, "\Leftarrow(\sigma,\sigma_1)"] \\
& (\mathscr{X},T)
\end{tikzcd}\right) \Rrightarrow \left(\begin{tikzcd}
(\mathscr{X},T) \arrow[r, bend left, "{(H,\chi,\chi_0,\chi_1)}"] \arrow[r, bend right, "{(H,\chi,\chi_0,\chi_1)}"'] \arrow[r, phantom, "\Downarrow id"] & (\mathscr{X},T)
\end{tikzcd}\right),$$

$$s : \left(\begin{tikzcd}
(\mathscr{X},T) \arrow[r, bend left, "{(H,\chi,\chi_0,\chi_1)}"] \arrow[r, bend right, "{(H,\chi,\chi_0,\chi_1)}"'] \arrow[r, phantom, "\Downarrow id"] & (\mathscr{X},T)
\end{tikzcd}\right) \Rrightarrow \left(\begin{tikzcd}
(\mathscr{X},T) \arrow[r, "{(H,\chi,\chi_0,\chi_1)}"] \arrow[r, bend left=90, looseness=2, "{(1,1,1,1)}"] \arrow[r, phantom, shift left=5ex, "\Downarrow(\sigma,\sigma_1)"] \arrow[dr, "{(H,\chi,\chi_0,\chi_1)}"'] & (\mathscr{X},T) \arrow[d, "{(H,\chi,\chi_0,\chi_1)}"] \arrow[d, phantom, shift right=5ex, "\Swarrow(\nu,\nu_1)"] \\
& (\mathscr{X},T)
\end{tikzcd}\right),$$

and

$$m : \left(\begin{tikzcd}
(\mathscr{X},T) \arrow[r, "{(H,\chi)}"] \arrow[drr, bend right, "{(H,\chi)}"'] & (\mathscr{X},T) \arrow[r, "{(H,\chi)}"] \arrow[dr, "{(H,\chi)}"'] \arrow[d, phantom, "\Swarrow(\nu,\nu_1)"] & (\mathscr{X},T) \arrow[d, "{(H,\chi)}"] \arrow[dl, phantom, "\Swarrow(\nu,\nu_1)"] \\
 & & (\mathscr{X},T)
\end{tikzcd}\right) \Rightarrow \left(\begin{tikzcd}
(\mathscr{X},T) \arrow[r, "{(H,\chi)}"] \arrow[dr, "{(H,\chi)}"'] \arrow[ddr, bend right, "{(H,\chi)}"'] & (\mathscr{X},T) \arrow[d, "{(H,\chi)}"] \arrow[dl, phantom, "\Swarrow(\nu,\nu_1)"] \\
\arrow[r, phantom, "\Swarrow(\nu,\nu_1)"] & (\mathscr{X},T) \arrow[d, "{(H,\chi)}"] \\
 & (\mathscr{X},T)
\end{tikzcd}\right)$$

i.e., transition 2-cells

$$\begin{tikzcd}
H \arrow[r, Rightarrow, "\sigma H"] \arrow[ddrr, Rightarrow, "H\eta^T"'] & HTH \arrow[r, Rightarrow, "H\chi"] \arrow[d, phantom, "\Swarrow t"] & HHT \arrow[d, Rightarrow, "\nu T"] \\
 & & HTT \arrow[d, Rightarrow, "H\mu^T"] \\
 & & HT,
\end{tikzcd}
\qquad
\begin{tikzcd}
HHT \arrow[d, Rightarrow, "\nu T"'] & H \arrow[l, Rightarrow, "H\sigma"'] \arrow[ddl, Rightarrow, "H\eta^T"] \arrow[d, phantom, "\Nwarrow s"] \\
HTT \arrow[d, Rightarrow, "H\mu^T"'] & \\
HT, &
\end{tikzcd}$$

and

$$\begin{tikzcd}
HHH \arrow[rrr, Rightarrow, "H\nu"] \arrow[dd, Rightarrow, "\nu H"'] & & & HHT \arrow[d, Rightarrow, "\nu T"] \\
 & \arrow[r, phantom, "\Swarrow m"] & & HTT \arrow[d, Rightarrow, "H\mu^T"] \\
HTH \arrow[r, Rightarrow, "H\chi"'] & HHT \arrow[r, Rightarrow, "\nu T"'] & HTT \arrow[r, Rightarrow, "H\mu^T"] & HT,
\end{tikzcd}$$

subject to the transition 2-cell coherence conditions. These cells correspond to the wreath monad equations from the ordinary setting.

These data are subject to the pseudomonad equations (recall Definition 8.1.2):

1. The unit coherence condition: the diagram

$$\begin{tikzcd}
HH \arrow[r, Rightarrow, "H\sigma H"] \arrow[d, Rightarrow, "H\sigma H"'] \arrow[ddr, Rightarrow, "H\eta^T H"] \arrow[ddrr, Rightarrow, "HH\eta^T"] & HHTH \arrow[r, Rightarrow, "HH\chi"] \arrow[d, phantom, "\Swarrow Ht"] & HHHT \arrow[d, Rightarrow, "H\nu T"] & & \\
HHTH \arrow[d, Rightarrow, "\nu TH"'] \arrow[dr, phantom, "\Swarrow sH"] & \Swarrow H{\chi_0}^{-1} & HHTT \arrow[d, Rightarrow, "HH\mu^T"] & & \\
HTTH \arrow[r, Rightarrow, "H\mu^T H"'] & HTH \arrow[r, Rightarrow, "H\chi"'] & HHT \arrow[r, Rightarrow, "\nu T"'] & HTT \arrow[r, Rightarrow, "H\mu^T"'] & HT
\end{tikzcd}$$

is equal to the diagram

$$\begin{tikzcd}
HH \arrow[d, Rightarrow, "H\sigma H"] & & & & & \\
HHTH \arrow[r, Rightarrow, "HH\chi"] \arrow[dd, Rightarrow, "\nu TH"'] \arrow[ddr, phantom, "\Swarrow {\nu_\chi}^{-1}"] & HHHT \arrow[rrr, Rightarrow, "H\nu T"] \arrow[dd, Rightarrow, "\nu HT"] \arrow[ddrr, phantom, "\Swarrow mT"] & & & HHTT \arrow[r, Rightarrow, "HH\mu^T"] \arrow[d, Rightarrow, "\nu TT"'] \arrow[dr, phantom, "\Swarrow {\nu_{\mu^T}}^{-1}"] & HHT \arrow[d, Rightarrow, "\nu T"] \\
& & & & HTTT \arrow[r, Rightarrow, "HT\mu^T"] \arrow[d, Rightarrow, "H\mu^T T"'] & HTT \arrow[dd, Rightarrow, "H\mu^T"] \\
HTTH \arrow[r, Rightarrow, "HT\chi"] \arrow[d, Rightarrow, "H\mu^T H"'] \arrow[dr, phantom, "\Swarrow H\chi_1"] & HTHT \arrow[r, Rightarrow, "H\chi T"] & HHTT \arrow[r, Rightarrow, "\nu TT"] \arrow[d, Rightarrow, "HH\mu^T"'] \arrow[dr, phantom, "\Swarrow \nu_{\mu^T}"] & HTTT \arrow[r, Rightarrow, "H\mu^T T"] \arrow[d, Rightarrow, "HT\mu^T"] \arrow[dr, phantom, "\Swarrow Hm^{T^{-1}}"] & HTT \arrow[dr, Rightarrow, "H\mu^T"] \arrow[r, phantom, "\Swarrow Hm^T"] & \\
HTH \arrow[rr, Rightarrow, "H\chi"'] & & HHT \arrow[r, Rightarrow, "\nu T"'] & HTT \arrow[rr, Rightarrow, "H\mu^T"'] & & HT.
\end{tikzcd}$$

2. The associativity coherence condition: the diagram

$$\begin{tikzcd}
HHHH \arrow[rrrr, Rightarrow, "HH\nu"] \arrow[dd, Rightarrow, swap, "\nu HH"] & & & & HHHT \arrow[rr, Rightarrow, "H\nu T"] \arrow[dd, Rightarrow, "\nu HT"] & & HHTT \arrow[rr, Rightarrow, "\nu TT"] & & HTTT \arrow[dddd, Rightarrow, "H\mu^T T"] \\
& & \Swarrow {\nu_\nu}^{-1} & & & & & & \\
HTHH \arrow[rrrr, Rightarrow, "HT\nu"] \arrow[dd, Rightarrow, swap, "H\chi H"] & & & & HTHT \arrow[dd, Rightarrow, "H\chi T"] & & \Swarrow mT & & \\
& & & & & & & & \\
HHTH \arrow[dd, Rightarrow, swap, "HH\chi"] & & \Swarrow H\nu_1 & & HHTT \arrow[rr, Rightarrow, "\nu TT"] \arrow[dd, Rightarrow, swap, "HH\mu^T"] & & HTTT \arrow[rr, Rightarrow, "H\mu^T T"] \arrow[dd, Rightarrow, "HT\mu^T"] & & HTT \arrow[dd, Rightarrow, "H\mu^T"] \\
& & & & & \Swarrow \nu_{\mu^T} & & \Swarrow Hm^{T-1} & \\
HHHT \arrow[rr, Rightarrow, "H\nu T"] \arrow[dd, Rightarrow, swap, "\nu HT"] & & HHTT \arrow[rr, Rightarrow, "HH\mu^T"] \arrow[ddrr, Rightarrow, swap, "\nu TT"] & & HHT \arrow[rr, Rightarrow, "\nu T"] & & HTT \arrow[rr, Rightarrow, "H\mu^T"] & & HT \\
& & & & \Downarrow {\nu_{\mu^T}}^{-1} & & \Downarrow Hm^T & & \\
HTHT \arrow[dd, Rightarrow, swap, "H\chi T"] & & \Swarrow mT & & HTTT \arrow[rr, Rightarrow, swap, "H\mu^T T"] \arrow[uurr, Rightarrow, swap, "HT\mu^T"] & & HTT \arrow[uurr, Rightarrow, swap, "H\mu^T"] & & \\
& & & & & & & & \\
HHTT \arrow[rrrr, Rightarrow, swap, "\nu TT"] & & & & HTTT \arrow[uurr, Rightarrow, swap, "H\mu^T T"] & & & &
\end{tikzcd}$$

is equal to the diagram

$$\begin{tikzcd}[arrows=Rightarrow]
HHHH \arrow[r, "HH\nu"] \arrow[d, "\nu HH"'] \arrow[dr, "H\nu H"] & HHHT \arrow[rr, "H\nu T"] & & HHTT \arrow[r, "\nu TT"] \arrow[dr, "HH\mu^T"'] & HTTT \arrow[rr, "H\mu^T T"] \arrow[dr, "HT\mu^T"] \arrow[d, phantom, "\Downarrow\nu_{\mu^T}"] & & HTT \arrow[dddd, "H\mu^T"] \\
HTHH \arrow[d, "H\chi H"'] & HHTH \arrow[d, "\nu TH"] \arrow[dr, "HH\chi"] & \swarrow Hm & & HHT \arrow[r, "\nu T"] \arrow[d, phantom, "\Downarrow{\nu_{\mu^T}}^{-1}"] & HTT \arrow[dddr, "H\mu^T"] & \\
HHTH \arrow[dddd, "HH\chi"'] \arrow[ddr, "\nu TH"] & HTTH \arrow[r, phantom, "\swarrow{\nu_\chi}^{-1}"] \arrow[dr, "HT\chi"] \arrow[ddr, "H\mu^T H"'] & HHHT \arrow[r, "H\nu T"] \arrow[d, "\nu HT"] & HHTT \arrow[r, "\nu TT"] \arrow[ur, "HH\mu^T"] & HTTT \arrow[ur, "HT\mu^T"'] \arrow[ddr, "H\mu^T T"] & \arrow[d, phantom, "\swarrow Hm^T"] & \\
\arrow[from=3-1, to=4-2, phantom, "\swarrow mH"] & & HTHT \arrow[dr, "H\chi T"] & \swarrow mT & & & \\
& HTTH \arrow[r, "H\mu^T H"] \arrow[ddr, "HT\chi"] & HTH \arrow[r, phantom, "\swarrow H\chi_1"] \arrow[dr, "H\chi"] & HHTT \arrow[r, "\nu TT"] \arrow[d, "HH\mu^T"] \arrow[dr, phantom, "\swarrow\nu_{\mu^T}"] & HTTT \arrow[r, "H\mu^T T"] \arrow[d, "HT\mu^T"] \arrow[dr, phantom, "\Downarrow Hm^{T^{-1}}"] & HTT \arrow[r, "H\mu^T"] & HT. \\
\arrow[r, phantom, "\swarrow\nu_\chi"] & & \swarrow{H\chi_1}^{-1} & HHT \arrow[r, "\nu T"] \arrow[dr, phantom, "\searrow{\nu_{\mu^T}}^{-1}"] & HTT \arrow[urr, "H\mu^T"] & \searrow Hm^T & HTT \arrow[u, "H\mu^T"] \\
HHHT \arrow[rr, "\nu TH"'] & & HTHT \arrow[r, "H\chi T"'] & HHTT \arrow[r, "\nu TT"'] \arrow[u, "HH\mu^T"] & HTTT \arrow[u, "HT\mu^T"] \arrow[urr, "H\mu^T T"'] & &
\end{tikzcd}$$

Remark 11.5.1. Let us comment on the nature of the above coherence conditions. For example, the first diagram of the associativity coherence conditions for a pseudomonad (T, η, μ, s, t, m) is the diagram

$$\begin{tikzcd}[arrows=Rightarrow]
& TTT \arrow[r, "T\mu"] \arrow[dr, "\mu T"'] & TT \arrow[dr, "\mu"] \arrow[d, phantom, "\Downarrow m"] & \\
TTTT \arrow[ur, "TT\mu"] \arrow[dr, "\mu TT"'] & \Downarrow{\mu_{T\mu}}^{-1} & TT \arrow[r, "\mu"] \arrow[d, phantom, "\Downarrow m"] & T. \\
& TTT \arrow[ur, "T\mu"] \arrow[r, "\mu T"'] & TT \arrow[ur, "\mu"'] &
\end{tikzcd}$$

Due to the lack of honest associativity of composition in $\mathsf{TRANS}(\mathbf{K})$, we need to decompose the above diagram into 3 parts.

1. The first ("upper") part is

$$\begin{tikzcd}
 & TTT \arrow[r, Rightarrow, "T\mu"] \arrow[dr, Rightarrow, "\mu T"'] & TT \arrow[dr, Rightarrow, "\mu"] \arrow[d, phantom, "\Downarrow m"] & \\
TTTT \arrow[ur, Rightarrow, "TT\mu"] & & TT \arrow[r, Rightarrow, "\mu"] & T.
\end{tikzcd}$$

2. The second ("middle") part is

$$\begin{tikzcd}
 & TTT \arrow[dr, Rightarrow, "\mu T"] & & \\
TTTT \arrow[ur, Rightarrow, "TT\mu"] \arrow[dr, Rightarrow, "\mu TT"'] \arrow[rr, phantom, "\Downarrow {\mu_{T\mu}}^{-1}"] & & TT \arrow[r, Rightarrow, "\mu"] & T. \\
 & TTT \arrow[ur, Rightarrow, "T\mu"'] & &
\end{tikzcd}$$

3. The third ("lower") part is

$$\begin{tikzcd}
TTTT \arrow[dr, Rightarrow, "\mu TT"'] & & TT \arrow[r, Rightarrow, "\mu"] \arrow[d, phantom, "\Downarrow m"] & T. \\
 & TTT \arrow[ur, Rightarrow, "T\mu"] \arrow[r, Rightarrow, "\mu T"'] & TT \arrow[ur, Rightarrow, "\mu"'] &
\end{tikzcd}$$

These three parts correspond to the following diagrams, respectively:

1. The upper part:

$$\begin{tikzcd}
HHHH \arrow[r, Rightarrow, "HH\nu"] & HHHT \arrow[r, Rightarrow, "H\nu T"] \arrow[d, Rightarrow, "\nu HT"] & HHTT \arrow[r, Rightarrow, "\nu TT"] & HTTT \arrow[dd, Rightarrow, "H\mu^T T"] \\
 & HTHT \arrow[d, Rightarrow, "H\chi T"] \arrow[r, phantom, "\Swarrow mT"] & & \\
 & HHTT \arrow[r, Rightarrow, "\nu TT"] & HTTT \arrow[r, Rightarrow, "H\mu^T T"] & HTT \arrow[d, Rightarrow, "H\mu^T"] \\
 & & & HT.
\end{tikzcd}$$

2. The middle part:

$$\begin{tikzcd}
HHHH \arrow[rr, Rightarrow, "HH\nu"] \arrow[d, Rightarrow, "\nu HH"'] \arrow[drr, phantom, "\Swarrow \nu_{\nu}{}^{-1}"] & & HHHT \arrow[d, Rightarrow, "\nu HT"] \\
HTHH \arrow[rr, Rightarrow, "HT\nu"] \arrow[d, Rightarrow, "H\chi H"'] & & HTHT \arrow[d, Rightarrow, "H\chi T"] \\
HHTH \arrow[d, Rightarrow, "HH\chi"'] & \Swarrow H\nu_1 & HHTT \arrow[d, Rightarrow, "HH\mu^T"'] \\
HHHT \arrow[r, Rightarrow, "H\nu T"] & HHTT \arrow[r, Rightarrow, "HH\mu^T"] & HHT \arrow[r, Rightarrow, "\nu T"] & HTT \arrow[r, Rightarrow, "H\mu^T"] & HT.
\end{tikzcd}$$

3. The lower part:

$$\begin{tikzcd}
HHHH \arrow[r, Rightarrow, "\nu HH"] & HTHH \arrow[r, Rightarrow, "H\chi H"] & HHTH \arrow[r, Rightarrow, "HH\chi"] & HHHT \arrow[r, Rightarrow, "H\nu T"] \arrow[d, Rightarrow, "\nu HT"'] & HHTT \arrow[r, Rightarrow, "\nu TT"] & HTTT \arrow[dd, Rightarrow, "H\mu^T T"] \\
 & & & HTHT \arrow[d, Rightarrow, "H\chi T"'] & \Swarrow mT & \\
 & & & HHTT \arrow[r, Rightarrow, "\nu TT"'] & HTTT \arrow[r, Rightarrow, "H\mu^T T"] & HTT \arrow[r, Rightarrow, "H\mu^T"] & HT.
\end{tikzcd}$$

Then one uses the associators

$$\begin{tikzcd}
HHTT \arrow[r, Rightarrow, "\nu TT"] \arrow[d, Rightarrow, "HH\mu^T"'] \arrow[dr, phantom, "\Swarrow \nu_{\mu^T}"] & HTTT \arrow[r, Rightarrow, "H\mu^T T"] \arrow[d, Rightarrow, "HT\mu^T"] \arrow[dr, phantom, "\Swarrow Hm^{T^{-1}}"] & HTT \arrow[d, Rightarrow, "H\mu^T"] \\
HHT \arrow[r, Rightarrow, "\nu T"] & HTT \arrow[r, Rightarrow, "H\mu^T"] & HT
\end{tikzcd}$$

for the above three diagrams (parts) to be composed. Whence comes the resulting diagram

$$\begin{tikzcd}
HHHH \arrow[rr, Rightarrow, "HH\nu"] \arrow[d, Rightarrow, swap, "\nu HH"] \arrow[drr, phantom, "\Lleftarrow {\nu_\nu}^{-1}"] & & HHHT \arrow[r, Rightarrow, "H\nu T"] \arrow[d, Rightarrow, "\nu HT"] & HHTT \arrow[r, Rightarrow, "\nu TT"] & HTTT \arrow[dd, Rightarrow, "H\mu^T T"] \\
HTHH \arrow[rr, Rightarrow, "HT\nu"] \arrow[d, Rightarrow, swap, "H\chi H"] & & HTHT \arrow[d, Rightarrow, "H\chi T"] & \Lleftarrow mT & \\
HHTH \arrow[d, Rightarrow, swap, "HH\chi"] & \Lleftarrow H\nu_1 & HHTT \arrow[r, Rightarrow, "\nu TT"] \arrow[d, Rightarrow, swap, "HH\mu^T"] \arrow[dr, phantom, "\Lleftarrow \nu_{\mu^T}"] & HTTT \arrow[r, Rightarrow, "H\mu^T T"] \arrow[d, Rightarrow, "HT\mu^T"] \arrow[dr, phantom, "\Lleftarrow {Hm^T}^{-1}"] & HTT \arrow[d, Rightarrow, "H\mu^T"] \\
HHHT \arrow[r, Rightarrow, "H\nu T"] \arrow[d, Rightarrow, swap, "\nu HT"] & HHTT \arrow[r, Rightarrow, "HH\mu^T"] \arrow[dr, Rightarrow, swap, "\nu TT"] & HHT \arrow[r, Rightarrow, "\nu T"] \arrow[d, phantom, "\Downarrow {\nu_{\mu^T}}^{-1}"] & HTT \arrow[r, Rightarrow, "H\mu^T"] \arrow[d, phantom, "\Downarrow Hm^T"] & HT. \\
HTHT \arrow[d, Rightarrow, swap, "H\chi T"] & \Lleftarrow mT & HTTT \arrow[ur, Rightarrow, "HT\mu^T"] \arrow[r, Rightarrow, swap, "H\mu^T T"] & HTT \arrow[ur, Rightarrow, swap, "H\mu^T"] & \\
HHTT \arrow[rr, Rightarrow, swap, "\nu TT"] & & HTTT \arrow[ur, Rightarrow, swap, "H\mu^T T"] & &
\end{tikzcd}$$

The rest of the diagrams is generated by a similar process.

Remark 11.5.2. The approach we used in this chapter to obtain an elementary description of wreaths can be used to obtain descriptions of other interesting structures. For example, one could define a KZ-wreath to be a KZ-pseudomonad in $\mathsf{TRANS}(\mathbf{K})$; or we could consider subtricategories of $\mathsf{LIFT}(\mathbf{K})$ and $\mathsf{TRANS}(\mathbf{K})$ spanned by KZ-pseudomonads, and try to obtain simplified descriptions of wreaths around KZ-pseudomonads, or KZ-wreaths around KZ-pseudomonads. We leave these investigations for future work.

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