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COMPLEX COVARIANCE

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Abstract. According to some generalized correspondence principle the classical limit of a non-Hermitian quantum theory describing quantum degrees of freedom is expected to be the well known classical mechanics of classical degrees of freedom in the complex phase space, i.e., some phase space spanned by complex-valued space and momentum coordinates. As special relativity was developed by Einstein merely for real-valued space-time and four-momentum, we will try to understand how special relativity and covariance can be extended to complex-valued space-time and four-momentum. Our considerations will lead us not only to some unconventional derivation of Lorentz transformations for complex-valued velocities, but also to the non-Hermitian Klein-Gordon and Dirac equations, which are to lay the foundations of a non-Hermitian quantum theory.

Keywords: non-Hermitean, quantum theory, relativity, complex plane.

1. Introduction

As has been pointed out on various places (see e.g. [1–10] and references therein), a simultaneous causal, local, analytic and covariant formulation of physical laws requires a non-Hermitian extension of quantum theory (QT), i.e. quantum mechanics and quantum field theory. Since causality expressed in QT by the time-ordering operation infers some small negative imaginary part in self-energies appearing in causal propagators which may be represented by some (in most cases infinitesimal) negative imaginary part in particle masses, even field operators representing electrically neutral causal particles have to be considered non-Hermitian. This leads to the fact that their creation and annihilation operators are not Hermitian conjugate to each other. During the attempt to find some spatial representation of non-Hermitian creation and annihilation operators which preserves analyticity and covariance, it has turned out that the aforementioned non-Hermitian causal field operators are functions of the complex spatial variable z instead of merely the real spatial variable x, while anticausal field operators are functions of the complex conjugate spatial variable z^*. Consequently a causal, local, analytic and covariant formulation of the laws of nature separates into a holomorphic causal sector and an antiholomorphic anticausal sector, which must not interact on the level of causal and anticausal field-operators in the spatial respresentation.

At least since Gregor Wentzel [11] (1926) there exists a formalism (see also [12–18]) nowadays called e.g. the "Quantum-Hamilton-Jacobi Theory" (QHJT) or the "Modified de Broglie-Bohm Approach", which relates a field or wave function by some correspondence principle (see e.g. Eqs. (4.10)) to the trajectories of some "quantum particle" in the whole complex phase space. Wentzel's approach has recently even been fortified by A. Voros [19] (2012) by providing an "exact WKB method" allowing to solve the Schrödinger equation for arbitrary polynomial potentials simultaneously in the whole complex z-plane. Moreover, there exists a rapidly increasing interest [23] of many theoretical and experimental researchers to study solutions of the Schrödinger equation even for non-Hermitian Hamiltonians in the whole complex plane due to a meanwhile confirmed conjecture of D. Bessis (and J. Zinn-Justin) in 1992 on the reality and positivity of spectra for manifestly non-Hermitian Hamiltonians, which was related by C.M. Bender and S. Boettcher in 1997 [20] to the PT-symmetry [21] of these Hamiltonians.

Despite this enormous amount of activities to "make sense of non-Hermitian Hamiltonians" [21, 22] and the fact that we had managed [6] to construct even a Lorentz-boost for complex-mass fields required to formulate non-Hermitian spinors and Dirac-equations, there has remained — as to our understanding — one crucial point neglected and unclear which is in the spirit of the QHJT and which will be the focus of the presented results: *How can the general concept of "covariance" having been formulated by Albert Einstein (1905)* [24] *and colleagues (see also* [25, 26]*) merely for a phase space of real-valued spatial and momentum coordinates be extended to a complex (i.e. complex-valued) phase space?* An answer will be given to some extent in the follwoing text. Certainly one might argue that there are already approaches like e.g. [27–31], which refer to seemingly covariant equations within a non-Hermitian framework. Nonetheless, it turns out that in all of the approaches there remain open questions to the reader to what extent these equations are consistent with some of the aspects of causality, locality or analyticity, and to what extent the fourvectors formed by space-time and momentum-energy coordinates contained in these equations will transform consistently under Lorentz transformations, as it seems at the first sight (e.g. [32, 33]) completely puzzling how to extend the framework of an inertial frame to the complex plane.

2. Space-time covariance for complex-valued velocities

The purpose of this section is to derive generalized Lorentz-Larmor-FitzGerald-Voigt (LLFV) [34, 35] transformations[1] relating the space-time coordinates $\vec{z}, t$ and $\vec{z}', t'$ of two inertial frames $\mathcal{S}$ and $\mathcal{S}'$, respectively, which move with some constant 3-dimensional complex-valued relative velocity $\vec{v} \in \mathbb{C}\times\mathbb{C}\times\mathbb{C}$. While the 3-dimensional space vectors $\vec{z}$ and $\vec{z}'$ are assumed to be complex-valued, i.e. $\vec{z}, \vec{z}' \in \mathbb{C}\times\mathbb{C}\times\mathbb{C}$, our derivation will display how the preferably real-valued time coordinates t and t' will be complexified.

Without loss of generality we will constrain ourselves throughout the derivation to one complex-valued spatial dimension while the generalization to three complex-valued spatial dimensions appears straightforward. Hence we consider in what follows for simplicity generalized LLFV transformations relating the space-time coordinates z, t and z', t' (with $z, z' \in \mathbb{C}$) of two inertial frames $\mathcal{S}$ and $\mathcal{S}'$, respectively, moving with some constant one-dimensional complex-valued relative velocity $v \in \mathbb{C}$.

According to the standard definition, an inertial frame in the absence of gravitation is a system in which the first law of Newton holds. For our purposes we will rephrase this definition in a way which may be used even in some complexified space-time:

An inertial frame in the absence of gravitation is a system whose trajectory in (even complexified) space-time is a straight line.

Our definition of inertial frames implies directly that generalized LLFV transformations relating inertial frames must be *linear*. Making use of this observation we can write down — as a first step in our derivation — the following linear ansatz for the generalized LLFV transformations between the inertial frames $\mathcal{S}$ and $\mathcal{S}'$:

$$z' = \gamma \cdot z + \delta \cdot t + \varepsilon, \tag{2.1}$$

$$t' = \kappa \cdot z + \mu \cdot t + \nu. \tag{2.2}$$

Here γ, δ, ε, κ, μ, ν are yet unspecified eventually complex-valued constants.

In a second step we will perform — without loss of generality — a synchronization of the inertial frames $\mathcal{S}$ and $\mathcal{S}'$ by imposing the following condition:

$$(z, t) = (0, 0) \iff (z', t') = (0, 0). \tag{2.3}$$

[1] Major steps in the LLFV-history: stepwise derivation of the transformations by Voigt (1887), FitzGerald (1889), Lorentz (1895, 1899, 1904), Larmor (1897, 1900); formalistic progress afterwards: Poincaré (1900, 1905) (discovery of Lorentz-group properties and some invariants), Einstein (1905) (derivation of LLFV transformations from first principles), Minkowski (1907-1908) (geometric interpretation of LLFV transformations).

Obviously the synchronisation yields $\varepsilon = \nu = 0$. Inserting this result in Eqs. (2.1) and (2.2) leads to the following equations:

$$z' = \gamma \cdot z + \delta \cdot t = \gamma\Big(z + \frac{\delta}{\gamma} \cdot t\Big), \tag{2.4}$$

$$t' = \kappa \cdot z + \mu \cdot t. \tag{2.5}$$

In the 3rd step we use the relative complex-valued velocity between inertial frames $\mathcal{S}$ and $\mathcal{S}'$: the spatial orgin $z' = \gamma \cdot z + \delta \cdot t = 0$ of $\mathcal{S}'$ moves in $\mathcal{S}$ with constant complex-valued velocity $v \equiv z/t = -\delta/\gamma$ yielding for Eq. (2.4):

$$z' = \gamma(z - v \cdot t) \quad \text{with} \quad \gamma = \gamma(v). \tag{2.6}$$

In writing $\gamma = \gamma(v)$ we point out that the constant γ could be a function of the complex-valued velocity v.

The fourth step is the application of the principle of relativity which states that — in the absence of gravity — *there does not exist any preferred inertial frame of reference implying in particular that the laws of physics take the same mathematical form in all inertial frames.* This provides an essentially unique prescription of how to construct inverse generalized LLFV transformations: interchange the space-time coordinates z, t and z', t', respectively, and replace the complex-valued velocity v by $-v$. As a consequence we obtain for the corresponding "inverse" of Eq. (2.6):

$$z = \overline{\gamma}(z' + v \cdot t') \quad \text{with} \quad \overline{\gamma} \equiv \gamma(-v). \tag{2.7}$$

In order to determine the yet unknown eventually complex-valued constants κ and μ in Eq. (2.4), we solve Eq. (2.7) for t' and apply to the result the identity Eq. (2.6), i.e.:

$$t' = \frac{1}{v}\Big(\frac{z}{\overline{\gamma}} - z'\Big) = \frac{1}{v}\Big(\frac{z}{\overline{\gamma}} - \gamma(z - v \cdot t)\Big) \tag{2.8}$$

or — after some rearrangement —

$$t' = \gamma\bigg(t - \frac{1}{v}\Big(1 - \frac{1}{\gamma\overline{\gamma}}\Big)z\bigg). \tag{2.9}$$

Comparison of Eq. (2.9) with Eq. (2.5) yields $\kappa = -\frac{\gamma}{v}\Big(1 - \frac{1}{\gamma\overline{\gamma}}\Big)$ and $\mu = \gamma$. By the same procedure leading from Eq. (2.6) to Eq. (2.7) the principle of relativity can be used to obtain the corresponding "inverse" of Eq. (2.9), i.e.:

$$t = \overline{\gamma}\bigg(t' + \frac{1}{v}\Big(1 - \frac{1}{\overline{\gamma}\gamma}\Big)z'\bigg). \tag{2.10}$$

Hence, the previous considerations result in the following four identities (with $\gamma \equiv \gamma(v)$, $\overline{\gamma} \equiv \gamma(-v)$):

$$z' = \gamma(z - v \cdot t), \tag{2.11}$$

$$t' = \gamma\bigg(t - \frac{1}{v}\Big(1 - \frac{1}{\gamma\overline{\gamma}}\Big)z\bigg), \tag{2.12}$$

$$z = \overline{\gamma}\Big(z' + v \cdot t'\Big), \tag{2.13}$$

$$t = \overline{\gamma}\bigg(t' + \frac{1}{v}\Big(1 - \frac{1}{\overline{\gamma}\gamma}\Big)z'\bigg). \tag{2.14}$$

In dividing Eq. (2.11) by Eq. (2.12) and Eq. (2.13) by Eq. (2.14) we obtain a generalized velocity addition law Eq. (2.15) and its inverse Eq. (2.16), respectively, i.e.:

$$\frac{z'}{t'} = \frac{\frac{z}{t} - v}{1 - \frac{1}{v}\left(1 - \frac{1}{\gamma\overline{\gamma}}\right)\frac{z}{t}}, \tag{2.15}$$

$$\frac{z}{t} = \frac{\frac{z'}{t'} + v}{1 + \frac{1}{v}\left(1 - \frac{1}{\overline{\gamma}\gamma}\right)\frac{z'}{t'}}. \tag{2.16}$$

A final fifth step makes use of the principle of constancy inferred by Albert Einstein stating that *light in vacuum is propagating in all inertial frames with the same speed independent of the movement of the light source and the propagation direction.* For our purposes we will generalize and simplify this principle of constancy by simply claiming that *the velocity addition law and its inverse possess an eventually complex-valued fixed point c whose modulus $|c|$ coincides with the vacuum speed of light.* Or, in other words: *there exists some eventually complex-valued velocity c which is not modified by the application of the velocity addition law and its inverse while the modulous $|c|$ coincides with the vacuum speed of light.* Application of this generalized principle of constancy to the addition laws Eq. (2.15) and Eq. (2.16) yields the following identity Eq. (2.17):

$$c = \frac{c \mp v}{1 \mp \frac{1}{v}\left(1 - \frac{1}{\gamma\overline{\gamma}}\right)c} \implies \gamma\overline{\gamma} = \frac{1}{1 - \frac{v^2}{c^2}}. \tag{2.17}$$

As $\gamma\overline{\gamma}$ depends on c^2, i.e., the square of c, we can conclude that — besides the fixed point $+c$ of the velocity addition law Eq. (2.15) and its inverse — there simultaneously exists a second fixed point $-c$. Eq. (2.17) can be used to transform Eqs. (2.11)–(2.16) to their final form. As LLFV transformations should reduce to the identity in the limit $v \to 0$, we take the "positive" complex square root of Eq. (2.17), i.e., $(\gamma\overline{\gamma})^{-1/2} = +\sqrt{1 - \frac{v^2}{c^2}}$, and invoke it together with $\gamma = \sqrt{\gamma\overline{\gamma}} \cdot \sqrt{\frac{\gamma}{\overline{\gamma}}}$ and $\overline{\gamma} = \sqrt{\gamma\overline{\gamma}} \cdot \sqrt{\frac{\overline{\gamma}}{\gamma}}$ to Eqs. (2.11)–(2.14) and obtain the following generalized LLFV transformations (with $\gamma \equiv \gamma(v)$, $\overline{\gamma} \equiv \gamma(-v)$):

$$z' = \sqrt{\frac{\gamma}{\overline{\gamma}}} \cdot \frac{z - v \cdot t}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{2.18}$$

$$t' = \sqrt{\frac{\gamma}{\overline{\gamma}}} \cdot \frac{t - \frac{v}{c^2} \cdot z}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{2.19}$$

$$z = \sqrt{\frac{\overline{\gamma}}{\gamma}} \cdot \frac{z' + v \cdot t'}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{2.20}$$

$$t = \sqrt{\frac{\overline{\gamma}}{\gamma}} \cdot \frac{t' + \frac{v}{c^2} \cdot z'}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{2.21}$$

and to Eqs. (2.15), (2.16) to arrive at the generalized velocity addition law and its inverse:

$$\frac{z'}{t'} = \frac{\frac{z}{t} - v}{1 - \frac{v}{c^2} \cdot \frac{z}{t}}, \tag{2.22}$$

$$\frac{z}{t} = \frac{\frac{z'}{t'} + v}{1 + \frac{v}{c^2} \cdot \frac{z'}{t'}}. \tag{2.23}$$

Two comments are in order here: even though the previous Eqs. (2.18)–(2.23) look very similar to the well known text-book equations appearing in the context of the standard formalism of special relativity, they are completely non-trivial as they hold not only in a real-valued space-time and for real-valued velocities, yet also *in a complex-valued space-time and for complex-valued velocities.* Moreover, the extension of the aforementioned equations to three spatial dimensions is achieved by replacing the complex-valued quantities z, z' and v by complex-valued 3-dimensional vectors $\vec{z}$, $\vec{z}'$ and $\vec{v}$, respectively.

3. On the choice of the invariant velocities $\pm c$ and the complexification of time

As we allow complex-valued velocities we face more freedom than Albert Einstein in defining the invariant eventually complex-valued invariant velocities $\pm c$. We will discuss here two specific options for defining c of which the first is our preferred choice due to the arguments given below:

- **Option 1:** *Choose $c = \pm|c|$ real-valued with $|c| =$* 299792458 m/s [38] *being the vacuum speed of light and set $\gamma = \overline{\gamma}$* (See the discussion of Eq. (4.14)!). Performing this choice Eqs. (2.18)–(2.23) read:

$$z' = \frac{z - v \cdot t}{\sqrt{1 - \frac{v^2}{|c|^2}}}, \qquad z = \frac{z' + v \cdot t'}{\sqrt{1 - \frac{v^2}{|c|^2}}}, \tag{3.1}$$

$$t' = \frac{t - \frac{v}{|c|^2} \cdot z}{\sqrt{1 - \frac{v^2}{|c|^2}}}, \qquad t = \frac{t' + \frac{v}{|c|^2} \cdot z'}{\sqrt{1 - \frac{v^2}{|c|^2}}}, \tag{3.2}$$

$$\frac{z'}{t'} = \frac{\frac{z}{t} - v}{1 - \frac{v}{|c|^2} \cdot \frac{z}{t}}, \qquad \frac{z}{t} = \frac{\frac{z'}{t'} + v}{1 + \frac{v}{|c|^2} \cdot \frac{z'}{t'}}. \tag{3.3}$$

With the exception of the square-roots all these equations are manifestly analytic. On the worldline $z = v \cdot t$ of $\mathcal{S}'$ in $\mathcal{S}$ we have with $t \in \mathbb{R}$:

$$t' = \frac{t - \frac{v}{|c|^2} \cdot v \cdot t}{\sqrt{1 - \frac{v^2}{|c|^2}}} = t \cdot \sqrt{1 - \frac{v^2}{|c|^2}} \in \mathbb{C} \quad \text{for} \quad v \in \mathbb{C}. \tag{3.4}$$

Hence, the attractive feature of analyticity would be obtained at the price of multiplying time in the boosted frame by some complex-valued constant.

- **Option 2:** *Choose c and v to be (anti)parallel in the complex plane, i.e., $c = \pm|c| \cdot \frac{v}{|v|} = \pm\left|\frac{c}{v}\right| \cdot v$ with $|c| = 299792458\,\mathrm{m/s}$ [38] being the vacuum speed of light, and set $\gamma = \overline{\gamma}$* (See the discussion of Eq. (4.14)!).

  Performing this choice Eqs. (2.18)–(2.23) read:

$$z' = \frac{z - v \cdot t}{\sqrt{1 - \left|\frac{v}{c}\right|^2}}, \qquad z = \frac{z' + v \cdot t'}{\sqrt{1 - \left|\frac{v}{c}\right|^2}}, \tag{3.5}$$

$$t' = \frac{t - \frac{v^*}{|c|^2} \cdot z}{\sqrt{1 - \left|\frac{v}{c}\right|^2}}, \qquad t = \frac{t' + \frac{v^*}{|c|^2} \cdot z'}{\sqrt{1 - \left|\frac{v}{c}\right|^2}}, \tag{3.6}$$

$$\frac{z'}{t'} = \frac{\frac{z}{t} - v}{1 - \frac{v^*}{|c|^2} \cdot \frac{z}{t}}, \qquad \frac{z}{t} = \frac{\frac{z'}{t'} + v}{1 + \frac{v^*}{|c|^2} \cdot \frac{z'}{t'}}. \tag{3.7}$$

All these equations are manifestly non-analytic. On the world-line $z = v \cdot t$ of $\mathcal{S}'$ in $\mathcal{S}$ we have with $t \in \mathbb{R}$:

$$t' = \frac{t - \frac{v^*}{|c|^2} \cdot v \cdot t}{\sqrt{1 - \left|\frac{v}{c}\right|^2}} = t \cdot \sqrt{1 - \left|\frac{v}{c}\right|^2} \in \mathbb{R} \quad \text{for} \quad \left|\frac{v}{c}\right| \leq 1. \tag{3.8}$$

Hence, the attractive feature of a real-valued time in $\mathcal{S}$ and $\mathcal{S}'$ would be obtained at the price of interferring manifest non-analyticity to the theory.

4. Momentum-energy covariance for complex-valued velocities

It seems to be one of the greatest mysteries in theoretical physics that — as to our understanding — the most straightforward derivation of Einstein's [36] famous seemingly classical identity $E = mc^2$ is based on the correspondence between classical and quantum physics, finding its manifestation in the concept of Louis de Broglie's [37] (1923) particle-wave duality (see also [25, 26]). In our words:[2] *In the process of quantisation the point particle of classical mechanics propagating in complex-valued space-time is replaced by energy quanta (quantum particles) being represented by some wave function ψ evolving also in complex-valued space-time.* Quantum particles, i.e. energy quanta, can be — depending on the spatial spread of the wave function and circumstances — localized or delocalized. Moreover, — according to Liouville's complementarity and Heisenberg's uncertainty principle — they display some simultaneous spread in complex-valued momentum space.

[2] It should be stressed that the concept of "electromagnetic mass" [25, 26] involving names like J.J. Thomson (1881), FitzGerald, Heaviside (1888), Searle (1896, 1897), Lorentz (1899), Wien (1900), Poincaré (1900), Kaufmann (1902-1904), Abraham (1902-1905), Hasenöhrl (1905) had revealed already before Einstein (1905) a proportionality between energy and some (eventually velocity dependent) mass, i.e. $E \propto mc^2$. Einstein himself considered a moving body in the presence of e.m. radiation to derive $E = mc^2$.

In the interaction-free case, the wave function of a quantum particle with sharply defined momentum is a plane wave with angular frequency ω and wave number k (or wave vector $\vec{k}$ in more than one dimension). For a real-valued space coordinate x the functional behaviour of a plane wave is known to be $\psi(x,t) \propto \exp\big(i(kx - \omega t)\big)$, yielding obviously:

$$\omega = +i\frac{\partial \ln \psi}{\partial t}, \qquad k = -i\frac{\partial \ln \psi}{\partial x}. \tag{4.1}$$

As we extend our formalism to the complex plane we replace the real-valued coordinate x by the complex-valued coordinate z (or the complex-conjugate z^*). Instead of performing partial derivatives with respect to x, we will now perform partial derivatives with respect to z (or z^*), which are known as Wirtinger derivatives in one complex dimension and Dolbeault operators in several complex dimensions. They are used in the context of (anti)holomorphic functions and have the following fundamental properties, being some special case of the famous Cauchy-Riemann differential equations:

$$\frac{\partial z^*}{\partial z} = \frac{\partial z}{\partial z^*} = 0, \quad \frac{\partial z}{\partial z} = \frac{\partial z^*}{\partial z^*} = 1. \tag{4.2}$$

On this formalistic ground we denote now generalized relations within a holomorphic framework to determine the eventually complex-valued angular frequency ω and wave number k for a plane wave propagating in some complexified phase space:

$$\omega = +i\frac{\partial \ln \psi}{\partial t}, \qquad k = -i\frac{\partial \ln \psi}{\partial z}. \tag{4.3}$$

Integration of these equations results in the following wave function for a plane wave in some some holomorphic phase space:

$$\psi(z,t) \propto \exp\big(i(kz - \omega t)\big). \tag{4.4}$$

As a key postulate (let us call it e.g. plane-wave-phase-covariance postulate (PWPCP)) in our derivation we claim at this point that *the eventually complex-valued phase of a plane wave should be a Lorentz scalar.* Or, in other words: *the eventually complex-valued phase of a plane wave should not change when boosted from one inertial frame to another.*[3] For the previously considered inertial frames $\mathcal{S}$ and $\mathcal{S}'$ this implies in particular:

$$kz - \omega t = k'z' - \omega' t'. \tag{4.5}$$

We may now insert into the left-hand side of this equation our generalized LLFV transformations Eqs. (2.20)

[3] One could use these considerations even to define inertial frames on the basis of quantum particles: *An inertial frame in the absence of gravitation is some reference frame in complex-valued space-time in which the wave function describing a non-interacting quantum particle with sharply defined momentum has the mathematical form of a plane wave.*

and (2.21), i.e.:

$$k \cdot \sqrt{\frac{\overline{\gamma}}{\gamma}} \cdot \frac{z' + v \cdot t'}{\sqrt{1 - \frac{v^2}{c^2}}} - \omega \cdot \sqrt{\frac{\overline{\gamma}}{\gamma}} \cdot \frac{t' + \frac{v}{c^2} \cdot z'}{\sqrt{1 - \frac{v^2}{c^2}}} = k'z' - \omega' t'. \quad (4.6)$$

Comparison of the left- and right-hand side of this equation yields the following two equations:

$$k' = \sqrt{\frac{\overline{\gamma}}{\gamma}} \cdot \frac{k - \frac{v}{c^2} \cdot \omega}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad \omega' = \sqrt{\frac{\overline{\gamma}}{\gamma}} \cdot \frac{\omega - v \cdot k}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad (4.7)$$

and by application of the principle of relativity interchanging k, ω and k', ω', respectively, and replacing v by $-v$ the two inverse equations:

$$k = \sqrt{\frac{\gamma}{\overline{\gamma}}} \cdot \frac{k' + \frac{v}{c^2} \cdot \omega'}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad \omega = \sqrt{\frac{\gamma}{\overline{\gamma}}} \cdot \frac{\omega' + v \cdot k'}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (4.8)$$

These four equations state that an eventually complex-valued frequency ω and some — 3-dimensionally generalized — wave vector $\vec{k}$ are transforming like a fourvector under *inverse* generalized LLFV transformations.

While the wave representation of particles has brought us to the quantum formalism without even involving Planck's quantum of action $\hbar = h/(2\pi) = 1.054571726(47) \cdot 10^{-34}\,\mathrm{J\,s}$ [38] it is the following two highly non-trivial and fundamental identities conjectured by Louis de Broglie [37] (1923) to be applicable even to *massive* particles which will bring us back to the seemingly classical quantities momentum p (here for simplicity in one dimension) and energy E, i.e. (with $k = 2\pi/\lambda$):[4]

$$p = \hbar k, \qquad E = \hbar\omega \quad (4.9)$$

and — when combined with our Eqs. (4.3) — to the following two fundamental identities representing even for wave functions of interacting quantum particles (being not of plane wave form) the correspondence principle of QHJT, i.e.:

$$E = +i\hbar \frac{\partial \ln \psi}{\partial t}, \qquad p = -i\hbar \frac{\partial \ln \psi}{\partial z}. \quad (4.10)$$

In multiplying Eqs. (4.7) and (4.8) by $\hbar$ it is now straightforward to obtain via Eqs. (4.9) the seemingly classical generalized Lorentz-Planck (LP) transformations (see also Planck [39] (1906)) relating here some even eventually complex-valued momentum and energy in inertial frames $\mathcal{S}$ and $\mathcal{S}'$:[5]

$$p' = \sqrt{\frac{\overline{\gamma}}{\gamma}} \cdot \frac{p - \frac{v}{c^2} \cdot E}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad E' = \sqrt{\frac{\overline{\gamma}}{\gamma}} \cdot \frac{E - v \cdot p}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad (4.11)$$

$$p = \sqrt{\frac{\gamma}{\overline{\gamma}}} \cdot \frac{p' + \frac{v}{c^2} \cdot E'}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad E = \sqrt{\frac{\gamma}{\overline{\gamma}}} \cdot \frac{E' + v \cdot p'}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (4.12)$$

Hence, a particle with zero momentum ($p' = 0$) in $\mathcal{S}'$ will have in the frame $\mathcal{S}$ of a resting observer the eventually complex-valued velocity v and appear with eventually complex-valued momentum p and energy E given by:[6]

$$p = \sqrt{\frac{\gamma}{\overline{\gamma}}} \cdot \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad E = \sqrt{\frac{\gamma}{\overline{\gamma}}} \cdot \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad (4.13)$$

with $m_0 \equiv E'/c^2$ being — for the limit $\gamma = \overline{\gamma}$ being seemingly favoured by experiment — some eventually complex-valued rest mass and Lorentz invariant, as there obviously holds the generalized dispersion relation:

$$E^2 - (pc)^2 = \frac{\gamma}{\overline{\gamma}} (m_0 c^2)^2. \quad (4.14)$$

5. Non-Hermitian Klein-Gordon-Fock equation

At this point we would like to recall Eqs. (4.10) expressing the correspondence principle in QHJT and being even valid for *interacting* quantum particles:

$$E = +i\hbar \frac{\partial \psi}{\partial t} \cdot \frac{1}{\psi}, \qquad p = -i\hbar \frac{\partial \psi}{\partial z} \cdot \frac{1}{\psi}, \quad (5.1)$$

yielding obviously

$$E^2 = (+i\hbar)^2 \left(\frac{\partial \psi}{\partial t}\right)^2 \cdot \frac{1}{\psi^2}, \quad p^2 = (-i\hbar)^2 \left(\frac{\partial \psi}{\partial z}\right)^2 \cdot \frac{1}{\psi^2}, \quad (5.2)$$

Simultaneously there are the following two identities holding for *non-interacting* quantum particles being described by a plane wave $\psi \propto \exp\bigl(\frac{i}{\hbar}(pz - Et)\bigr)$ (obtained by combining Eq. (4.4) with Eqs. (4.9)):

$$E^2 = (+i\hbar)^2 \frac{\partial^2 \psi}{\partial t^2} \cdot \frac{1}{\psi}, \quad p^2 = (-i\hbar)^2 \frac{\partial^2 \psi}{\partial z^2} \cdot \frac{1}{\psi}, \quad (5.3)$$

Like Klein [40], Gordon [41] and Fock [42] in 1926 we can insert Eqs. (5.3) in the dispersion relation

[4] It is of course known that the former identity $E = hf$ (with $f = \omega/(2\pi)$) had been derived earlier — using an energy-discretisation trick of Boltzmann — by Planck (1900) in the context of the e.m. radiation of a black body and by Einstein (1905) to determine the energy of his *massless* photon, while the latter identity had been used in the form $p = hf/c$ for *massless* photons for the first time by Stark (1909) and later by Einstein (1916, 1918), while Compton (1923) and Debye (1923) had finally confirmed the proportionality of the suggested three-momentum and wave vector of a *massless* photon by famous experiments.

[5] Without loss of generality, we display the equations here only for one complex-valued momentum dimension.

[6] In the limit $\gamma = \overline{\gamma}$ we recover the famous relativistic identities $\vec{p} = m\vec{v}$ (Planck [39] (1906)) and $E = mc^2$ (Einstein [36] (1905)) with $m = \frac{m_0}{\sqrt{1-(v/c)^2}}$.

Eq. (4.14) to obtain the generalized Klein-Gordon-Fock (KGF) equation describing a *non-interacting* relativistic quantum particle:

$$\overbrace{(+i\hbar)^2\frac{\partial^2\psi}{\partial t^2}\cdot\frac{1}{\psi}}^{E^2}-\overbrace{(-i\hbar c)^2\frac{\partial^2\psi}{\partial z^2}\cdot\frac{1}{\psi}}^{(pc)^2}=\frac{\gamma}{\overline{\gamma}}(m_0c^2)^2 \tag{5.4}$$

$$\Longrightarrow\ (+i\hbar)^2\frac{\partial^2\psi}{\partial t^2}-(-i\hbar c)^2\frac{\partial^2\psi}{\partial z^2}=\frac{\gamma}{\overline{\gamma}}(m_0c^2)^2\psi \tag{5.5}$$

$$\Longrightarrow\ (+i\hbar)^2\frac{\partial^2\psi}{\partial t^2}=(-i\hbar c)^2\frac{\partial^2\psi}{\partial z^2}+\frac{\gamma}{\overline{\gamma}}(m_0c^2)^2\psi. \tag{5.6}$$

As usual, the solution $\psi=\psi^{(+)}+\psi^{(-)}$ of the KGF Eq. (5.6) can be decomposed into a sum of a retarded solution $\psi^{(+)}$ and an advanced solution $\psi^{(-)}$ solving not only the KGF Eq. (5.6), but also respectively the following relativistic retarded or advanced interaction free Schrödinger [43] (1926) equations:

$$\begin{aligned}\pm i\hbar\frac{\partial\psi^{(\pm)}}{\partial t}&=\sqrt{(-i\hbar c)^2\frac{\partial^2}{\partial z^2}+\frac{\gamma}{\overline{\gamma}}(m_0c^2)^2}\,\psi^{(\pm)}\\&\approx\left(\sqrt{\frac{\gamma}{\overline{\gamma}}}m_0c^2-\sqrt{\frac{\overline{\gamma}}{\gamma}}\frac{\hbar^2}{2m_0}\frac{\partial^2}{\partial z^2}+\ldots\right)\psi^{(\pm)}\end{aligned} \tag{5.7}$$

In the last line we performed the non-relativistic limit well known for $\gamma=\overline{\gamma}$.

6. Non-Hermitian Dirac-equation

Each of the four components of the Dirac spinor ψ of a *non-interacting* Dirac-quantum particle should individually respect the KGF Eq. (5.4). Returning to three eventually complex-valued space and momentum dimensions, this condition is formally denoted by the following equivalent identities:

$$0=\left(E^2-(\vec{p}c)^2-\frac{\gamma}{\overline{\gamma}}(m_0c^2)^2\right)\psi \tag{6.1}$$

$$0=\left((+i\hbar)^2\frac{\partial^2}{\partial t^2}-(-i\hbar c)^2\frac{\partial}{\partial\vec{z}}\cdot\frac{\partial}{\partial\vec{z}}-\frac{\gamma}{\overline{\gamma}}(m_0c^2)^2\right)\psi \tag{6.2}$$

$$0=\left(\left[\beta\left(+i\hbar\frac{\partial}{\partial t}-(-i\hbar c)\vec{\alpha}\cdot\frac{\partial}{\partial\vec{z}}\right)\right]^2-\frac{\gamma}{\overline{\gamma}}(m_0c^2)^2\right)\psi \tag{6.3}$$

$$\begin{aligned}0=&\left(\beta\left(+i\hbar\frac{\partial}{\partial t}-(-i\hbar c)\vec{\alpha}\cdot\frac{\partial}{\partial\vec{z}}\right)-\sqrt{\frac{\gamma}{\overline{\gamma}}}m_0c^2\right)\\&\cdot\left(\beta\left(+i\hbar\frac{\partial}{\partial t}-(-i\hbar c)\vec{\alpha}\cdot\frac{\partial}{\partial\vec{z}}\right)+\sqrt{\frac{\gamma}{\overline{\gamma}}}m_0c^2\right)\psi.\end{aligned} \tag{6.4}$$

Throughout the factorization of Eq. (6.2) we made use of the four well known 4×4 Dirac matrices $\vec{\alpha}$ and β, which are defined as follows with the help of the Pauli matrices $\vec{\sigma}$, the 2×2 unit matrix 1_2 and the 2×2 zero matrix 0_2:

$$\vec{\alpha}\equiv\begin{pmatrix}0_2 & \vec{\sigma}\\ \vec{\sigma} & 0_2\end{pmatrix},\qquad \beta\equiv\begin{pmatrix}1_2 & 0_2\\ 0_2 & -1_2\end{pmatrix}. \tag{6.5}$$

By simple inspection of Eq. (6.4) and use of the identity $\beta^2=1_4$ it is now straightforward to denote the retarded and advanced Dirac [44] (1928) equations for the retarded component $\psi^{(+)}$ and advanced component $\psi^{(-)}$ of solution $\psi=\psi^{(+)}+\psi^{(-)}$ of the *interaction free* KGF Eq. (6.1), i.e.:

$$0=\left(\beta\left(+i\hbar\frac{\partial}{\partial t}-(-i\hbar c)\vec{\alpha}\cdot\frac{\partial}{\partial\vec{z}}\right)\mp\sqrt{\frac{\gamma}{\overline{\gamma}}}m_0c^2\right)\psi^{(\pm)}, \tag{6.6}$$

$$+i\hbar\frac{\partial\psi^{(\pm)}}{\partial t}=\left(-i\hbar c\vec{\alpha}\cdot\frac{\partial}{\partial\vec{z}}\pm\sqrt{\frac{\gamma}{\overline{\gamma}}}\beta m_0c^2\right)\psi^{(\pm)}, \tag{6.7}$$

$$\pm i\hbar\frac{\partial\psi^{(\pm)}}{\partial t}=\left(\pm(-i\hbar c)\vec{\alpha}\cdot\frac{\partial}{\partial\vec{z}}+\sqrt{\frac{\gamma}{\overline{\gamma}}}\beta m_0c^2\right)\psi^{(\pm)}. \tag{6.8}$$

Once more we stress that these generalized Dirac equations, the generalized Schrödinger Eq. (5.7) and the generalized KGF Eqs. (5.6), (6.2) do hold even in complex-valued space-time and for complex-valued rest mass m_0.

7. Final remarks

The purpose of the considerations presented here has been to extend the concept of covariance to complex-valued space-time. It is remarkable that this can be achieved in some analytical way on the basis of and in accordance with the correspondence principle of QHJT. After extending the concept of inertial frames to the complex plane we have constructed on one hand generalized LLFV and LP transformations relating the fourvectors of complex-valued space-time and momentum-energy beween two inertial frames with an eventually complex-valued relative velocity, and on the other hand a complex generalization of Einstein's energy-mass equivalence $E=mc^2$. It turned out that the complexification of time is — in the limit $\gamma=\overline{\gamma}$ — not a severe problem, as a boost will multiply the time at most by a complex constant. Moreover, it has been possible to derive on the basis of a generalized concept of covariance generalized KGF, Schrödinger and Dirac differential equations, which can be used to formulate a non-Hermitian QT describing the apparently complex laws of physics. As had already been pointed out earlier (e.g. [2]) it is the advanced Schrödinger (or Dirac) equation which plays the role of Benders hardly constructable CPT-transformed Schrödinger (or Dirac) equation required to construct some positive semidefinite CPT-inner product [45] for some PT-symmetric QT. The possibility to obtain — via covariance — directly the underlying advanced Schrödinger (or Dirac) equation, as described in the present paper, will make the tedious search and construction of a unique CPT-inner product in non-Hermitian QT needless.

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References

[1] F. Kleefeld, Czech. J. Phys. **56** (2006) 999, arXiv:quant-ph/0606070.

[2] F. Kleefeld, arXiv:hep-th/0408028, arXiv:hep-th/0408097.

[3] F. Kleefeld, arXiv:hep-th/0312027.

[4] F. Kleefeld, eConf C **0306234** (2003) 1367 [Proc. Inst. Math. NAS Ukraine **50** (2004) 1367], arXiv:hep-th/0310204.

[5] F. Kleefeld, Few Body Syst. Suppl. **15** (2003) 201, arXiv:nucl-th/0212008.

[6] F. Kleefeld, AIP Conf. Proc. **660** (2003) 325, arXiv:hep-ph/0211460.

[7] F. Kleefeld, E. van Beveren and G. Rupp, Nucl. Phys. A **694** (2001) 470, arXiv:hep-ph/0101247.

[8] F. Kleefeld: PhD Thesis (Univ. of Erlangen-Nürnberg, Germany, 1999).

[9] F. Kleefeld, Proc. XIV ISHEPP 98 (17-22 August, 1998, Dubna), Eds. A.M. Baldin et al., JINR, Dubna, 2000, Part 1, 69-77, arXiv:nucl-th/9811032.

[10] F. Kleefeld, Acta Phys. Polon. B **30** (1999) 981, arXiv:nucl-th/9806060.

[11] G. Wentzel, Z. Phys. **38** (1926) 518.

[12] C. D. Yang, Annals Phys. **319** (2005) 399, 444.

[13] M. V. John, Found. Phys. Lett. **15** (2002) 329, arXiv:quant-ph/0109093; quant-ph/0102087.

[14] A. E. Faraggi and M. Matone, Int. J. Mod. Phys. A **15** (2000) 1869 arXiv:hep-th/9809127.

[15] R. A. Leacock and M. J. Padgett, Phys. Rev. Lett. **50** (1983) 3.

[16] C. M. Bender, K. Olaussen, P. S. Wang, Phys. Rev. D **16** (1977) 1740.

[17] Chapter III in A. S. Dawydow, *Quantenmechanik* (7. Auflage), VEB Deutscher Verlag d. Wissenschaften 1987, ISBN 3-326-00095-2 (Revision of the german translation of the russian original edition, Moskau 1973).

[18] J. L. Dunham, Phys. Rev. **41** (1932) 713.

[19] A. Voros, arXiv:1202.3100.

[20] C. M. Bender, S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998) arXiv:physics/9712001.

[21] C. M. Bender, Rept. Prog. Phys. **70** (2007) 947 arXiv:hep-th/0703096.

[22] C. M. Bender, D. W. Hook, P. N. Meisinger and Q. H. Wang, Annals Phys. **325** (2010) 2332 arXiv:0912.4659.

[23] See e.g. the PT Symmeter page http://ptsymmetry.net/ and for PT-symmetry-meetings http://gemma.ujf.cas.cz/~znojil/conf/index.html.

[24] A. Einstein, Annalen Phys. **17** (1905) 891 [Ann. Phys. **14** (2005) 194].

[25] A. Pais, *Raffiniert ist der Herrgott, Albert Einstein. Eine wissenschaftliche Biographie*, Spektrum, Heidelberg 2000, ISBN 3-8274-0529-7 (German translation of: A. Pais, *Subtle is the Lord: The Science and the Life of Albert Einstein*, Oxford Univ. Press, New York 1982).

[26] Wikipedia, (*History of special relativity*, http://en.wikipedia.org/wiki/History_of_relativity [August 30, 2012].

[27] N. Nakanishi, Phys. Rev. D **5** (1972) 1968.

[28] N. Nakanishi, Prog. Theor. Phys. Suppl. **51** (1972) 1.

[29] T. D. Lee and G. C. Wick, Phys. Rev. D **2** (1970) 1033.

[30] A. Mostafazadeh, Int.J.Mod.Phys. A **21** (2006) 2553, arXiv:quant-ph/0307059.

[31] C. M. Bender and P. D. Mannheim, Phys. Rev. D **84** (2011) 105038 arXiv:1107.0501.

[32] N. Nakanishi, Phys. Rev. D **3** (1971) 811.

[33] A. M. Gleeson, R. J. Moore, H. Rechenberg and E. C. G. Sudarshan, Phys. Rev. D **4**, 2242 (1971).

[34] H.A. Lorentz, Proc. Roy. Netherlands Acad. of Arts a.Sci., **1** (1899) 427.

[35] H. Poincaré, C.R. Hebd. Séances Acad. Sci., **140** (1905b) 1504.

[36] A. Einstein, Annalen Phys. **18** (1905) 639.

[37] L. V. P. R. de Broglie, Thesis, Paris, 1924, Annals Phys. **3** (1925) 22.

[38] P.J. Mohr, B.N. Taylor, D.B. Newell, arXiv:1203.5425.

[39] M. Planck, Verh. Deutsch. Phys. Ges. **4** (1906) 136; see also: Sitzungsber. Preuß. Akad. Wiss. (1907) 542; Annalen Phys. **26** (1908) 1.

[40] O. Klein, Z. Phys. **37** (1926) 895 [Surveys High Energ.Phys. **5** (1986) 241].

[41] W. Gordon, Z. Phys. **40** (1926) 117.

[42] V. Fock, Z. Phys. **38** (1926) 242; Z. Phys. **39** (1926) 226 [Surveys High Energ. Phys. **5** (1986) 245].

[43] E. Schrödinger, Phys. Rev. **28** (1926) 1049 (see also H. Feshbach and F. Villars, Rev. Mod. Phys. **30** (1958) 24).

[44] P. A. M. Dirac, Proc. Roy. Soc. Lond. A **117** (1928) 610, Proc. Roy. Soc. Lond. A **118** (1928) 351.

[45] C. M. Bender, D. C. Brody, H. F. Jones, eConf C **0306234** (2003) 617 [Phys. Rev. Lett. **89** (2002) 270401] [Erratum-ibid. **92** (2004) 119902], arXiv:quant-ph/0208076.