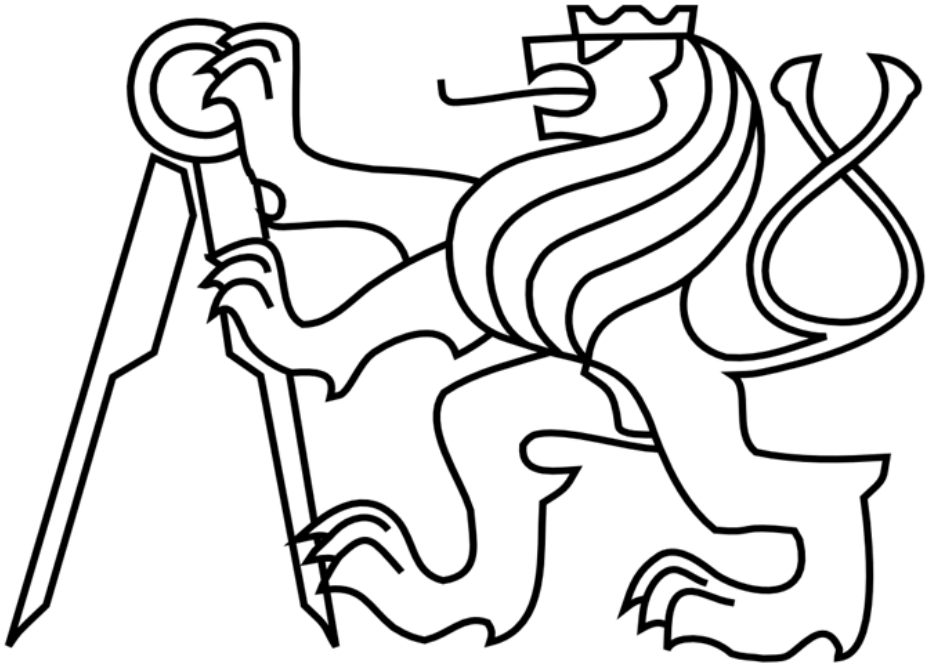
ČESKÉ VYSOKÉ UČENÍ TECHNICKÉ V PRAZE

Fakulta strojní

Ú12101- Ústav techniké matematiky

Matematické modelování v technice



Použití metody konečných prvků pro modální analýzu elastického tělesa

Usage of the finite element method for modal analysis of the elastic body

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ZADÁNÍ DIPLOMOVÉ PRÁCE

pro: Bc. Lukáše Horáčka

v oboru: Matematické modelování v technice

Název: **Použití metody konečných prvků pro modální analýzu elastického tělesa**

Zásady pro vypracování:

1. Popis matematického aparátu užívaného v metodě konečných prvků a při modální analýze.

2. Popis matematického modelu problému lineární elasticity. Klasická formulace okrajové úlohy a její slabá formulace. Popis aproximace pomocí metody konečných prvků.

3. Rozšíření modelu na případ dynamické lineární elasticity, formulace okrajové úlohy, její slabá formulace. Prostorová diskretizace pomocí metody konečných prvků.

4. Modální analýza elastického tělesa užitím metody konečných prvků.

5. Grafické zpracování a popis výsledků.

Rozsah diplomové práce: do 40–80 stran včetně obrázků a grafů.

Seznam odborné literatury:

1. Brdička, M., Samek, L., Sopko, B., Mechanika kontinua, Academia, Prague, 2000.

2. Zienkiewicz, O.C., Taylor, R.L., The Finite Element Method, Butterworth-Heinemann, Oxford, 2000.

3. Sváček P., Feistauer M., Metoda konečných prvků, Skripta FS ČVUT v Praze, Vydavatelství ČVUT v Praze.

Vedoucí diplomové práce: Doc. RNDr. Petr Sváček, Ph.D.

Datum zadání diplomové práce: 31. 3. 2015

Datum odevzdání diplomové práce: 19. 6. 2015

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V Praze dne 31. 3. 2015

Neodevzdá-li student diplomovou práci včas, je povinen tuto skutečnost předem písemně zdůvodnit, pokud bude omluva (předaná prostřednictvím studijního oddělení děkanovi) děkanem uznána, určí děkan studentovi náhradní termín konání státní závěrečné zkoušky (zůstávají dva termíny SZZ). Pokud tuto skutečnost student řádně neomluví nebo omluva nebude děkanem uznána, určí děkan studentovi termín pro opakování státní závěrečné zkoušky. SZZ je možné opakovat pouze jednou (SZŘ čl. 22, odst. 3, 4).

Student bere na vědomí, že je povinen vypracovat diplomovou práci samostatně, bez cizí pomoci, s výjimkou poskytnutých konzultací. Seznam použité literatury, jiných pramenů a jmen konzultantů je třeba uvést v diplomové práci.

Zadání diplomové práce převzal dne:

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Anotační list

Jméno autora:	Lukáš Horáček	
Název diplomové práce:	Použití metody konečných prvků pro modální analýzu elastického tělesa	
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Anotace:

Práce obsahuje teoretické odvození vztahů pro výpočet vlastních tvarů a modálních participačních faktorů s použitím metody konečných prvků. Dále pak popis jejich využití pro výpočet odpovědi elastického tělesa na externí buzení. Přibližná odpověď tohoto systému je počítána pomocí tak zvaného reduced order model postupu. Druhá část práce pak popisuje realizaci praktického výpočtu na komerčně vyráběné motorové pile.

Anotation:

Thesis contains the theoretical derivation of the relations for the calculation of the unit normal eigen modes and the modal participation factors with the aid of finite element method. Further a description of their application for calculation of an elastic body response to an external excitation is presented. The response of the system is calculated with the aid of the reduced order model approach. The second part of the thesis then describes the practical application of this method on a model case of a commercially produced chainsaw.

Prohlášení o samostatném vypracovaní

Diplomovou práci jsem vypracoval samostatně výhradně s použitím zdrojů uvedených v seznamu na konci práce.

V Roudnici nad Labem .....

19.6.2015 Lukáš Horáček

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I would like to thank my supervisor doc. RNDr. Petr Sváček Ph.D. for his professional consultations regarding the theoretical part of this diploma thesis as well for his patient lead during my whole Masters studies.

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1. Introduction

This diploma thesis was created with cooperation with Andreas STIHL AG & Co. KG, company that develops professional handheld power equipment. Professional machines have to fulfill a great quality demands. One of those demands is that the usage of the machine should be easy and comfortable. From this point of view the important criterions are the weight of the machine and also the level of vibration on the handles of the machine. Machines marked as "professional tools" i.e. machines predetermined to be used during 8-hours work shift have maximal level of vibration on handles established by ISO norm [1]. Problem with high vibration, which does not fulfill this ISO norm, occurs during the design process very often. In such a case it is necessary to find modifications of the machine structure which have no influence on its function and minimal influence to the stiffness of the machine, but the vibration level is much more damped. Prototype creating and vibration measuring proved to be expensive and time consuming way for finding these changes. Numerical simulation of the vibration is on the other hand much quicker and cheaper way and therefore is used very often.

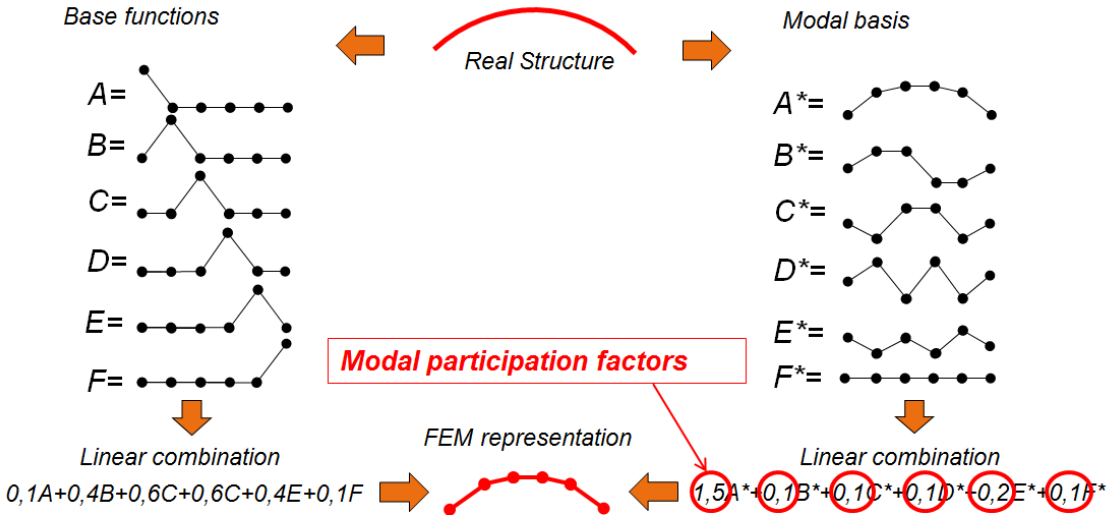
Calculation of the vibration is based on the theory of linear elasticity, which was described by many different authors, for example [2], [3] or [4]. Theory of the linear elasticity mathematically models the dynamic behavior of the elastic structure with the aid of partial differential equations. For solution of these equations, numerical techniques like the finite element method can be used. The numerical techniques are described for example in [5], [6], [7], [8] or [9]. Solving the dynamic equations obtained by discretizing linear elasticity problem using the finite element method is a very complex and complicated process. Therefore many authors try to find a ways how to simplify it. One of the method, that is described for example in [10], [11] is modal decomposition and reduce order modelling. The main part of the thesis focuses on the description of this method.

1.1. Basic concepts of modal decomposition

Any physical system can vibrate. The frequencies at which vibration naturally occurs and the modal shapes which the vibrating system assumes are properties of the system, and can be determined analytically using modal analysis. For elastic body are these frequencies usually called eigen frequencies and these shapes are called the unit normal eigen modes. The unit eigen modes can be identified experimentally, see [10] or can be calculated using the modal decomposition, see [5]. Theoretically, for the continuous model, there exists an infinite number of the eigen modes. Nevertheless in order to approximately describe the dynamic behavior of such system, it can be sufficient to use only a few of the eigen modes. Such an approach is usually called the reduced order modelling (ROM) [11]. Each

of these modes is associated with a fixed oscillation frequency. The response of the system to an external loading then can be approximately described as linear combination of these eigen modes. The coefficients used in this linear combination are called the modal participation factors [12]. Precision of such computation is obviously dependent on the number of used modes [10]. The other possibility for the dynamical system behavior modeling is the application of the finite element method and solving directly the time dependent problem. In such a case, the solution is obtained as a linear combination of unit basis functions [9]. Difference between the modal decomposition and this direct orthogonal decomposition is shown in Figure 1.1.

Figure 1.1 Modal decomposition



1.2. Goals of the diploma thesis

This diploma thesis is created with the cooperation with Andreas STIHL AG & Co. KG. The main goal is to improve the methodology standardly used by STIHL to calculate vibrations on chainsaw handles that occur as a response to the excitation from the crankshaft revolving. The standardly used methodology is based on the direct calculations with aid of the commerce finite element method solver called PERMAS. Information about this software can be found on developer's website [13] or in the software manual [12]. The newly developed methodology, based on the modal decomposition, uses the PERMAS solver for the computation of the unit normal eigen modes and the modal participation factors (see [11], [12]). Post processing of these results is then done in my own program called Participation Magic, programed within the Matlab GUI. Vibration curve is in Participation Magic recalculated as a linear combination of the unit eigen modes. This procedure allows us to use only the dominant modes of the calculated vibration curve. Such an approach is usually called the reduced order modelling [5], [11]. Moreover this methodology allows to identify the

sources of the maximal vibration, i.e. to find the dominant unit eigen modes which cause highest vibration. Identifying of the dominant unit eigen modes provides an important information, which can help the engineer responsible for the structure design to take measures exactly aimed to the damping of those vibrations.

1.3. Structure of the diploma thesis

The diploma thesis is divided into the theoretical part and the practical. The theoretical part starts with the description of the needed mathematical background, then the basic equations of the linear elasticity model are shown. Next part then contains the full description of the linear elasticity problem (dynamic and static) from the viewpoint of the structural mechanics. Further the description of the discretization using the FEM is presented and the derivation of all equations needed for the computation of the modal basis and system response to the external excitation is shown.

The practical part contains description of the test model used for computations. This model is not part of the thesis as it is owned by Andreas STIHL AG & Co. KG, Further the description of the preprocessing of the model in accordance with the inner STIHL standards is presented. For the preprocessing procedure the commerce software called Medina was used. For more information about this software see developer's website [14] or the software manual [15]. Follows description of all inputs required by PERMAS solver for the computation. Next part describes the post processing of the results performed by the developed program Participation Magic i.e. calculating vibration curve from modal basis and modal participation factors. Last part describes the tools of Participation Magic that can be used for finding dominant eigen frequencies as well as for examining needed size of the modal basis.

2. Mathematical background

This part of diploma thesis presents short overview of well-known definitions and theorems from the linear algebra and the functional analysis, that can be found for example in [16], [9], [17]. First, eigen values and eigen vectors as well as generalized eigen values and eigen vectors are described. These are needed further in the text for solution of the dynamic problem of linear elasticity. Next section is about a numerical method that can be used for finding eigen values. Section about function spaces and their properties follow. In the last part of this chapter several important theorems needed for finite element method are included.

2.1. Eigen values and eigen vectors

Eigen values are involved in solution of many practical problems. In mathematics they are used for solution of ordinary and partial differential equations. In physics eigen values are usually used in problems regarding oscillations and oscillators. In mechanics, their use involves a wide range of applications from simple ones as finding of the principal axes of stress and strain, to more complicated, as finding normal eigen modes of a component or a structure.

2.1.1. Definition of basic matrix properties

In this part, an overview of well-known results from linear algebra is presented, see [16]. Here, we shall work with vectors in $\boldsymbol{R^n}$ and matrices in $\boldsymbol{R^{m\times n}}$. In the practical part of diploma thesis, we mostly consider symmetric positive definite matrices. Definition of these properties follows.

First let us define the transposition of a matrix.

Definition 2.1. *Matrix* $\boldsymbol{B} = (b_{ij}) \in R^{m\times n}$ *is called transposed matrix to a matrix* $\boldsymbol{A} = (a_{ij}) \in R^{m\times n}$ *if* $a_{ij} = b_{ji}$ *holds for all* i,j. The transposed matrix is denoted by $\boldsymbol{A^T}$.

Further, we recall the definition of symmetric matrix. We shall see later that symmetry of the matrix is a very strong property.

Definition 2.2. *Matrix* $\boldsymbol{A} \in R^{n\times n}$ *is called symmetric if* $a_{ij} = a_{ji}$ *holds for all* i,j, *i.e.* $\boldsymbol{A} = \boldsymbol{A^T}$

As already mentioned another very important property of a matrix is being positive definite matrix.

Definition 2.3. *Matrix* $\boldsymbol{A} \in R^{n\times n}$ *is called positive definite, if* $\boldsymbol{x^T A x} = \sum_{\boldsymbol{i,j}} x_i a_{ij} x_j > 0$,

holds for arbitrary vector $\boldsymbol{x} \in \boldsymbol{R^n}; \boldsymbol{x} \neq \boldsymbol{0}$.

In order to decide whether a symmetric matrix is positive definite, the Sylvester's criterion can be used.

Lemma 2.1. ***(*Sylvester's criterion)** *Let matrix* $\boldsymbol{A} \in R^{n\times n}$ *be symmetric. Then it is positive definite if and only if all the main minors* $D_1, D_2 \dots D_n$ *of the matrix* $\boldsymbol{A}$ *are positive. Here, the main minors mean*

$$D_1 = a_{11}\,; D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}; \dots .;\ D_n = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}. \tag{2.1}$$

2.1.2. Eigen values and eigen vectors of matrices

For calculations of the vibration eigen values and eigen vectors of a matrix can be. We start with the definition of the eigen values.

Definition 2.4. *Let* $\boldsymbol{A}$ *be a real matrix* $\boldsymbol{A} \in \boldsymbol{R}^{n\times n}$. *A complex number* $\lambda\,; \lambda \in \mathbb{C}$ *is called an eigen value of the matrix* $\boldsymbol{A}$, *if there exists a nonzero vector* $\boldsymbol{u} \in \mathbb{C}^n$ *such that*

$$\boldsymbol{Au} = \lambda \boldsymbol{u}. \tag{2.2}$$

The vector $\boldsymbol{u}$ *is called the eigen vector, which corresponds to the eigen value* λ.

It is clear that for an eigen value λ the corresponding eigen vector is not unique. Usually, the normalized eigen vectors $\|\boldsymbol{u}\|_2 = 1$ are used. Using equation (2.2), it can be easily shown that the eigen values of the matrix $\boldsymbol{A}$ are the roots of a characteristic polynomial.

Lemma 2.2. *Let matrix* $\boldsymbol{A}$ *be a matrix* $\boldsymbol{A} \in \boldsymbol{R}^{n\times n}$. *Then the eigen values* λ *of the matrix* $\boldsymbol{A}$ *are roots of equation*

$$det(\boldsymbol{A} - \lambda \boldsymbol{E}) = 0, \tag{2.3}$$

where $\boldsymbol{E}$ *denotes the identity* $n \times n$ *matrix.*

For eigen values λ of matrix $\boldsymbol{A}$ it is easy to show that they satisfy following properties, see [16], [18]:

- The matrix $\boldsymbol{A}$ has eigen value 0 if and only if it is singular.
- The eigen values of identity matrix $\boldsymbol{E}$ are ones and every vector is eigen vector.
- Let matrix $\boldsymbol{A}$ has eigen value λ then matrix $\boldsymbol{B} = (\boldsymbol{A} + \boldsymbol{E})$ has eigen value $\lambda + 1$.

- Let λ be an eigen value of matrix $\boldsymbol{A}$ and $\boldsymbol{u}$ the corresponding eigen vector, then the inverse matrix $\boldsymbol{A}^{-1}$ (if it exists) has the eigen value $1/\lambda$ and the corresponding eigen vector is the same vector $\boldsymbol{u}$.
- The eigen vectors corresponding to different eigen values of the same matrix $\boldsymbol{A}$ are always linearly independent.
- Let λ be the eigen value of matrix $\boldsymbol{A}$ and $\boldsymbol{u}$ be the corresponding eigen vector. Then the eigen value of a matrix $\boldsymbol{A}^2$ is λ^2 and the corresponding eigen vector is $\boldsymbol{u}$.
- Let $\boldsymbol{A}$ be a real matrix and λ be its real eigen value. Then there exists an eigen vector whose components are real numbers.

It is well known that the characteristic polynomial of a matrix $\boldsymbol{A} \in \boldsymbol{R}^{n\times n}$has in complex plane n solutions. In practical problems we often rather seek real eigen values. To guarantee that the eigen values of a matrix are real numbers the following theorems can be used, see [9].

Lemma 2.3. *Let $\boldsymbol{A}$ be a symmetric matrix. Then all its eigen values are real.*

Moreover eigen vectors of symmetric matrix have another important property.

Lemma 2.4. *Let $\boldsymbol{A}$ be a symmetric matrix. Then the eigen vectors corresponding to different eigen values are orthogonal. Moreover, there exists an orthonormal base formed of eigen vectors of the matrix $\boldsymbol{A}$.*

Furthermore, if the matrix is also symmetric and positive definite, it has another property.

Lemma 2.5. *Let $\boldsymbol{A}$ be a symmetric matrix. Then it is positive definite if and only if all of its eigen values are real positive numbers.*

For a symmetric positive definite matrix its square root can be defined.

Lemma 2.6. *Let matrix $\boldsymbol{A} \in R^{n\times n}$ be a symmetric and positive definite. Then there exists a matrix $\boldsymbol{B}$, such that $\boldsymbol{BB} = \boldsymbol{A}$. The matrix $\boldsymbol{B}$ is called the square root of the matrix $\boldsymbol{A}$ and it is denoted by $\boldsymbol{A}^{1/2}$.*

Proof of this statement

As matrix $\boldsymbol{A} \in R^{n\times n}$ is symmetric, it can be written as (see Lemma 2.4.)

$$\boldsymbol{U}^T\boldsymbol{A}\boldsymbol{U} = diag(\lambda_1, \lambda_2 \dots \lambda_n) = \boldsymbol{\Lambda},$$

where $\lambda_1, \lambda_2 \dots \lambda_n$ are the positive eigen values of the matrix $\boldsymbol{A}$ and $\boldsymbol{U}$ is a matrix formed from orthonormal base of eigen vectors. As $\boldsymbol{U^T U} = \boldsymbol{E}$, the matrix $\boldsymbol{A}$ reads

$$\boldsymbol{A} = \boldsymbol{U\Lambda U^T}.$$

Now the matrix $\boldsymbol{B}$ defined by

$$\boldsymbol{B} = \boldsymbol{U}\boldsymbol{\Lambda}^{\boldsymbol{1/2}}\boldsymbol{U}^{\boldsymbol{T}}, \text{ where } \boldsymbol{\Lambda}^{\boldsymbol{1/2}} = diag\left(\lambda_1^{1/2}, \lambda_2^{1/2} \dots \lambda_n^{1/2}\right),$$

satisfies

$$\boldsymbol{B}\boldsymbol{B} = \boldsymbol{A}. \qquad \blacksquare$$

2.1.5. Generalized eigen values and eigen vectors

For a solution of a dynamic problem of linear elasticity, generalized eigen values problem is used. Let us start with the definition of generalized eigen values and eigen vectors, see [17].

Definition 2.5. *Let $\boldsymbol{A}, \boldsymbol{B}$ be matrices $\boldsymbol{A}, \boldsymbol{B} \in R^{n \times n}$. Let $\boldsymbol{B}$ be a regular matrix. Complex number λ; $\lambda \in \mathbb{C}$ is called generalized eigen value of matrices $\boldsymbol{A}, \boldsymbol{B}$, if there is a nonzero vector $\boldsymbol{u} \in \mathbb{C}^n$ such that*

$$\boldsymbol{A}\boldsymbol{u} = \lambda \boldsymbol{B}\boldsymbol{u}. \tag{2.4}$$

The vector $\boldsymbol{u}$ is called generalized eigen vector corresponding to generalized eigen value $\lambda \in \mathbb{C}$.

Similarly, as for eigen values problem, the generalized eigen values are roots of the characteristic equation.

Lemma 2.7. *Let $\boldsymbol{A}, \boldsymbol{B}$ be matrices $\boldsymbol{A}, \boldsymbol{B} \in R^{n \times n}$. The generalized eigen values λ of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are the roots of equation*

$$\det(\boldsymbol{A} - \lambda \boldsymbol{B}) = 0. \tag{2.5}$$

For generalized eigen values and eigen vectors of symmetric and positive definite matrices following lemma can be proved.

Lemma 2.8. *Let $\boldsymbol{A}, \boldsymbol{B} \in R^{n \times n}$ be a symmetric positive definite matrices. Then there exist n real positive eigen values λ_i with the corresponding real eigen vectors $\boldsymbol{u_i}$, such that $\boldsymbol{A}\boldsymbol{u_i} = \lambda_i \boldsymbol{B}\boldsymbol{u_i}$ and moreover*

$$\boldsymbol{u_i^T}\boldsymbol{B}\boldsymbol{u_j} = 0 \quad i \neq j, \tag{2.6}$$

$$\boldsymbol{u_i^T}\boldsymbol{A}\boldsymbol{u_j} = 0 \quad i \neq j. \tag{2.7}$$

Proof

Let us consider equation $\boldsymbol{A}\boldsymbol{u} = \lambda \boldsymbol{B}\boldsymbol{u}$.

As $\boldsymbol{B}$ is symmetric positive definite matrix, the square root of the matrix $\boldsymbol{B}$ exists. Now, multiplying (2.4) by $\boldsymbol{B^{-1/2}}$ from the left we get the equation

$$\boldsymbol{B^{-1/2}}\boldsymbol{A}\boldsymbol{B^{-1/2}}\boldsymbol{B^{1/2}}\boldsymbol{u} = \lambda \boldsymbol{B^{1/2}}\boldsymbol{u},$$

Now using the substitution $\boldsymbol{B^{1/2}u = z}$ we get

$$\boldsymbol{B^{-1/2}AB^{-1/2}z = \lambda z.}$$

Finally denoting $\boldsymbol{C = B^{-1/2}AB^{-1/2}}$ we can see that equation (2.4) is equivalent to equation

$$\boldsymbol{Cz} = \lambda \boldsymbol{z},$$

where $\boldsymbol{C}$ is a symmetric positive definite matrix.

Following Lemmas 2.1.3. and 2.1.4. there exists n eigen values $\lambda_i > 0$ and the corresponding real eigen vectors $\boldsymbol{z_i}$ of the matrix $\boldsymbol{C}$. Thus generalized eigen vectors are given by $u_i = \boldsymbol{B^{-1/2}z_i}$ ■

2.2 Numerical techniques for finding eigen values and eigen vectors

Analytical ways of finding eigen values of a matrix are not suitable for finding the eigen values and eigen vectors of large matrices because they requires large amount of mathematical operations. In the practical part we often work with stiffness matrix and mass matrix. Size of the stiffness matrix and mass matrix is proportional to the number of elements of FE model and therefore its size can be in millions. In most cases calculating all eigen values and eigen vectors is not necessary. Calculating only the small number of them (hundreds) is sufficient. Therefore numerical techniques allowing to find easily only several eigen values are being used.

For overview of numerical methods see e.g. [6]. Here, only the idea of the power method is described.

Power method

Let us consider symmetric positive definite matrix $\boldsymbol{A} \in R^{n \times n}$. Such a matrix has n linearly independent real eigen vectors which form a base of n dimensional vector space $\boldsymbol{R^n}$. An arbitrary vector $v_0 \in \boldsymbol{R^n}$ thus can be written as a linear combination of the basis vectors, i.e.

$$\boldsymbol{v_0} = \sum_{i=1}^{n} \alpha_i \boldsymbol{x_i}, \tag{2.8}$$

where $\boldsymbol{x_i}$ are the eigen vectors of matrix $\boldsymbol{A}$. Using the definition of eigen values (2.2), the product $\boldsymbol{Av_0}$ can be written as

$$\boldsymbol{Av_0} = \sum_{i=1}^{n} \alpha_i \lambda_i \boldsymbol{x_i}. \tag{2.9}$$

Using this repeatedly we get the m^{th} iterated vector $\boldsymbol{v_m}$ as

$$\boldsymbol{v_m} := \boldsymbol{A^m v_0} = \sum_{i=1}^{n} \alpha_i {\lambda_i}^m \boldsymbol{x_i}. \tag{2.10}$$

Here we can see, that the influence of the largest eigen value, compared to the others increases for $m \to \infty$. Therefore we can, according to [6], use Von Mises theorem to evaluate highest eigen value.

Theorem 2.1. (Von Mises theorem) *Let us assume that the matrix* $\boldsymbol{A}$ *has n linearly independent eigen vectors and eigen value* λ_1 *that has the highest absolute value i.e.* $|\lambda_1| > |\lambda_i|, i \neq 1$*, then*

$$\lim_{m\to\infty} \frac{1}{{\lambda_1}^m} \boldsymbol{A^m v_0} = \alpha_1 \boldsymbol{x_1}. \quad \alpha_1 \neq 0 \tag{2.11}$$

Let's assume an arbitrary vector $\boldsymbol{y}$ that is not perpendicular to the eigen vector $\boldsymbol{x_1}$.

From the Von Mises theorem we get

$$\lambda_1 = \lim_{n\to\infty} \frac{\boldsymbol{y^T v_{m+1}}}{\boldsymbol{y^T v_m}}, \tag{2.12}$$

where the values $\boldsymbol{y^T v_{m+1}}, \boldsymbol{y^T A v_m}$ are usually called Schwarz's constants, see also [6].

Calculating other eigen values

The above described procedure allows us to calculate only the highest (dominant) eigen value. In order to compute also the other eigen values, the concept of a reduced matrix can be used, see [6].

Theorem 2.2. *Let* $\boldsymbol{A}$ *be a matrix which has an eigen value* λ_1*. Let* $\boldsymbol{x_1}$ *be the eigen vector corresponding to the eigen value* $\boldsymbol{\lambda_1}$*. Let us assume an arbitrary vector* $\boldsymbol{v}$*, such that* $\boldsymbol{v^T x_1} = 1$*. Then the matrix*

$$\boldsymbol{W_1} = \boldsymbol{A} - \lambda_1 \boldsymbol{x_1 v^T} \tag{2.13}$$

(called the reduced matrix) has the same eigen values as the original matrix $\boldsymbol{A}$ *with the only exception of the eigenvalue* λ_1 *that is replaced by 0.*

The above described procedure then is used repeatedly to calculate the other eigen values of the matrix $\boldsymbol{A}$*.*

2.3. Function spaces

In this section the definition of the vector space is given and its basic properties are described. For more information see [9] [19].

Vector spaces

Let us recall here that a set $\boldsymbol{V}$ is a real linear vector space, if the operations addition of two vectors and multiplication of a vector by a real number are defined on $\boldsymbol{V}$ and satisfy the following relations. For arbitrary vectors $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}$ and real number $\alpha \in \boldsymbol{R}$ hold $\boldsymbol{u} + \boldsymbol{v} \in \boldsymbol{V}$, $\alpha \boldsymbol{u} \in \boldsymbol{V}$. Further, for an arbitrary element $\boldsymbol{v} \in \boldsymbol{V}$ there exists an opposite element $-\boldsymbol{v} \in \boldsymbol{V}$ for which holds $\boldsymbol{v} + (-\boldsymbol{v}) = 0$. For more details see [20].

In order to define convergence, the normed linear spaces are used.

Definition 2.6. (Norm definition) *Let $\boldsymbol{V}$ be a vector space. Function $\|.\|_V : \boldsymbol{v} \in \boldsymbol{V} \to \boldsymbol{R}$ is called norm if*

$$\|\boldsymbol{v}\|_V \geq 0 \text{ for all } \boldsymbol{v} \in V,$$

$$\|\alpha \boldsymbol{v}\|_V = |\alpha| \|\boldsymbol{v}\|_V \text{ for all } \boldsymbol{v} \in V;\ \alpha \in \boldsymbol{R},$$

$$\|\boldsymbol{v} + \boldsymbol{u}\|_V \leq \|\boldsymbol{v}\|_V + \|\boldsymbol{u}\|_V \text{ for all } \boldsymbol{v} \in \boldsymbol{V}.$$

Then the space is called normed linear space. If additionally the normed linear vector space $\boldsymbol{V}$ is complete, then it is called Banach's space. The definition of the complete space can be found for example in [19] or in [20].

In next section we will often work with linear and bilinear forms.

Definition 2.6. (Linear form) *Let $\boldsymbol{V}$ be a Banach's space. Then a mapping $L : \boldsymbol{V} \to \boldsymbol{R}$ is called a linear form if for arbitrary vectors $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}$ and a real number $\alpha \in \boldsymbol{R}$ holds*

$$L(\boldsymbol{u} + \boldsymbol{v}) = L(\boldsymbol{u}) + L(\boldsymbol{v}),$$

$$L(\alpha \boldsymbol{u}) = \alpha L(\boldsymbol{u}),$$

The linear form $L: \boldsymbol{V} \to \boldsymbol{R}$ is called bounded if there exists

$$c \in \boldsymbol{R}, c > 0 \text{ such that} |L(\boldsymbol{u})| \leq c \|\boldsymbol{u}\|_V, \text{ for any } \boldsymbol{u} \in \boldsymbol{V}$$

The linear form $L: \boldsymbol{V} \to \boldsymbol{R}$ is called bounded if and only if it is continuous, i.e. holds

$$L(\boldsymbol{u_n}) \to L(\boldsymbol{u}) \text{ for arbitrary convergent sequence } \boldsymbol{u_n} \to \boldsymbol{u}.$$

Definition 2.7. (Bilinear forms) *Let V be a Banach's space. Then a mapping $a: \boldsymbol{V} \times \boldsymbol{V} \to \boldsymbol{R}$ is called a bilinear form if for arbitrary vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$, and for arbitr1ary real number $\alpha \in \boldsymbol{R}$ holds*

$$a(\boldsymbol{u}+\boldsymbol{v}, \boldsymbol{w}) = a(\boldsymbol{u}, \boldsymbol{w}) + a(\boldsymbol{v}, \boldsymbol{w}),$$

$$a(\boldsymbol{w}, \boldsymbol{u}+\boldsymbol{v}) = a(\boldsymbol{w}, \boldsymbol{u}) + a(\boldsymbol{w}, \boldsymbol{v}),$$

$$a(\alpha\boldsymbol{u}, \boldsymbol{v}) = \alpha\boldsymbol{a}(\boldsymbol{u}, \boldsymbol{v}),$$

$$a(\boldsymbol{u}, \alpha\boldsymbol{v}) = \alpha\boldsymbol{a}(\boldsymbol{u}, \boldsymbol{v}),$$

Similarly, the bilinear form $a: \boldsymbol{V} \to \boldsymbol{R}$ is called bounded if there exists

$$c \in \boldsymbol{R}, c > 0 \text{ such that} |a(\boldsymbol{u}, \boldsymbol{v})| \leq c\|\boldsymbol{u}\|_V \|\boldsymbol{v}\|_V, \text{ for any } \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}$$

The bilinear form is bounded if and only if it is continuous.

For the application of FEM the special function spaces called the Lebesgue's spaces are used, see e.g. [20].

Lebesgue's spaces

The space of all functions such that $\int_\Omega |u|^p \, dx < +\infty, 1 \leq p < \infty$ is called the Lebesgue's space $L^n(\Omega)$. Here Ω is a bounded domain with the Lipschitz`s continuous boundary, see* [20] *for the definition of the Lipschitz`s continuous boundary. This space is Banach`s space with the norm defined by $\|u\|_{0,p,\Omega} = \left(\int_\Omega |u|^p \, dx\right)^{1/p}$. Let us mention that the concept of the Lebesgue's space is complicated, e.g. the involved integral is so called Lebesgue's integral, see [9].

We shall also work with Sobolev's spaces, which are another example of Banach`s spaces, for more information see [9].

Sobolev's spaces

The Sobolev's space is defined by the relation

$$W^{k,p}(\Omega) = \{v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega) \; for \; |\alpha| \leq k\},$$

where $1 \leq p < \infty$, $\alpha = (\alpha_1, \dots \alpha_n)$ is the multi-index, $D^\alpha v$ are the partial derivatives $D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1 \dots \partial x_n}$ and Ω is a bounded domain with the Lipschitz`s continuous boundary. The space $W^{k,p}(\Omega)$ is the Banach`s space with norm defined by

$$\|\boldsymbol{v}\|_{k,p,\Omega} = \|\boldsymbol{v}\|_{W^{k,p,(\Omega)}} = \left(\sum_{|\alpha| \leq k} \|D^\alpha \boldsymbol{v}\|^p_{0,p,\Omega}\right)^{1/p}.$$

The semi-norm of the space $W^{k,p}(\Omega)$ is defined by

$$|\boldsymbol{v}|_{k,p,\Omega} = |\boldsymbol{v}|_{W^{k,p,(\Omega)}} = \left(\sum_{|\alpha|=k} \|D^{\alpha}\boldsymbol{v}\|^{p}_{0,p,\Omega}\right)^{1/p}.$$

The Sobolev's space $\boldsymbol{W}^{k,n}(\Omega)$ with the special choice $p = 2$ are at the same time Hilbert's spaces, see [9]. *We use the notation $W^{k,2}(\Omega) = H^{k}(\Omega)$.*

2.4. Important theorems for finite element method

In order to introduce a weak formulation of a boundary value problems, the Green`s Theorem is usually applied.

Theorem 2.3. (Green's theorem)

Let $\Omega \subset \boldsymbol{R}^{d}$ be a bounded domain with the Lipschitz's continuous boundary, then for arbitrary functions $u, v \in H^{1}(\Omega)$ holds

$$\int_{\Omega} \frac{\partial u}{\partial x_i} v dx = \int_{\partial\Omega} u\, v\, n_i\, dS - \int_{\Omega} \frac{\partial v}{\partial x_i} u\, dx \tag{2.14}$$

where $\boldsymbol{n} = (n_1, n_2 \ldots . n_d)$ is the outward unit normal to the boundary of Ω.

Further, for the finite element method the concept of the reference element is used, where the integration needs to be transformed on the reference element using the substitution theorem.

Theorem 2.4. (Substitution theorem)

Let $F: \tilde{\Omega} \to \Omega$ be a continuously differentiable mapping of the domain $\tilde{\Omega}$ to the domain Ω. Let f be a function $f \in L^{1}(\Omega)$, then holds

$$\int_{\Omega} f(x) dx = \int_{\Omega} f\big(F(\tilde{x})\big) |F'(\tilde{x})|\, d\tilde{x}, \tag{2.15}$$

where $|F'(\tilde{x})|$ is the absolute value of the determinant of the Jacobi's matrix of the mapping F.

In order to show some important properties as V-ellipticity, the generalized Poincare's inequality can be used.

Theorem 2.5. (Generalized Poincare's inequality)

Let $\Omega \subset \boldsymbol{R}^{d}$ be a bounded domain with the Lipschitz's continuous boundary $\partial\Omega$. Let $\Gamma \subset \partial\Omega$ be a part of the boundary of Ω that has positive $(d-1)$ dimensional measure. Then there exists a constant $C_p > 0$ such that

$$\|\boldsymbol{u}\|_{1,2,\Omega} \leq C_p \left(|\boldsymbol{u}|_{1,2,\Omega} + \left| \int_{\Gamma} \boldsymbol{u}\, dS \right|^{2} \right) \text{ holds for all } u \in H^{1}(\Omega). \tag{2.16}$$

Very important theorem for showing existence and uniqueness of the solution of the linear elasticity problem is Lax-Milgram theorem.

Theorem 2.6. (Lax-Milgram theorem) *Let V be a Banachs space, L be a linear form on* $\boldsymbol{V}$ *and* $\boldsymbol{a}$ *be a symmetric bilinear form on* $\boldsymbol{V}$*. Let* M, m, C *be such positive constants that for arbitrary* $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}$ *holds*

$$|a(\boldsymbol{u}, \boldsymbol{v})| \leq M\|\boldsymbol{u}\|_V \|\boldsymbol{v}\|_V,$$

$$a(\boldsymbol{u}, \boldsymbol{u}) \geq m\|\boldsymbol{u}^2\|_V,$$

$$|L(\boldsymbol{v})| \leq C\|\boldsymbol{v}\|_V.$$

Then there exists exactly one $\boldsymbol{u}^* \in \boldsymbol{V}$ *such, that for any* $\boldsymbol{v} \in \boldsymbol{V}$ *holds*

$$a(\boldsymbol{u}^*, \boldsymbol{v}) = L(\boldsymbol{v}).$$

Moreover the norm of $\boldsymbol{u}^*$ *is bounded by*

$$\|\boldsymbol{u}^*\|_V \leq \frac{C}{m}.$$

For Lax-Milgram theorem and its proof, see [9].

3. Mathematical description and numerical approximation

This part deals with the deformation of an elastic body. In order to mathematically model its behavior, the continuum assumption is made. Then the mathematical model describing the static and the dynamic problems of elasticity is presented. Continuum assumption means that we neglect the molecular interactions in material and assume a continuous behavior, more details can be found in [2]. Further we shall use the assumptions that the deformation is small and also that the behavior of the material is linear, see also [2].

3.1. Static problem of linear elasticity.

First, we start with a formulation of the static problem of linear elasticity, assuming that only steady state solution is sought.

3.1.1. Hooke's law, tensors of deformation, Lamé-Navier's equation

Elasticity can be considered as the ability of material to reversibly deform itself in response to the action of external forces. If the deformation is completely reversible and time independent, then we speak about ideal elasticity. If the relationship between the force and the deformation is linear, we speak about linear elasticity. This is usually true if the deformation is small. Under these assumptions the basic equations of linear elasticity describing the elastic structure deformation can be written, see [2].

In order to describe a deformation of an elastic body, we start with the definition of a displacement vector.

Displacement vector

Let us consider an arbitrary point $P(\boldsymbol{x})$ of the undeformed body represented by a bounded domain $\Omega \subset \boldsymbol{R}^3$ in the cartesian coordinate system $(x_1, x_2, x_3) = \boldsymbol{x}$. By acting of external forces the point $P(\boldsymbol{x})$ is shifted to a point $P'(\boldsymbol{y})$, where $\boldsymbol{y} = (y_1, y_2, y_3)$ denotes the coordinates of the point $P'(\boldsymbol{y})$. The displacement vector $\boldsymbol{u} = (u_1, u_2, u_3)$ at the point $P(\boldsymbol{x})$ is then given by

$$u_i(\boldsymbol{x}) = y_i - x_i. \tag{3.1}$$

Tensor of finite deformation

Deformation of the body Ω means a change of its shape, due to the action of the external forces. The external forces change the mutual position of the different particles of the body. In case that after removal of the external forces the body returns to its original shape, we speak about reversible

(elastic) deformation, otherwise we speak about irreversible deformation [2]. The deformation of the body, according to [2] can be characterized by the tensor of the finite deformation ϵ given by

$$\epsilon_{lk} = \frac{1}{2} \cdot \left[\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} + \left(\frac{\partial u_j}{\partial x_l} \right) \left(\frac{\partial u_j}{\partial x_k} \right) \right]. \tag{3.2}$$

In the case of small deformation the derivatives of the displacement vector $\boldsymbol{u}$ are small $\left| \frac{\partial u_j}{\partial x_l} \right| \ll 1$. Then the term $\left(\frac{\partial u_j}{\partial x_l} \right) \left(\frac{\partial u_j}{\partial x_k} \right)$ is negligible compared to $\frac{\partial u_j}{\partial x_l}$, In such a case, the deformation can be described using only (simplified) linear tensor of small deformation see [2].

Tensor of small deformation

The tensor of small deformation is given by

$$\varepsilon_{lk} = \frac{1}{2} \left[\frac{\partial u_k}{\partial u_l} + \frac{\partial u_l}{\partial u_k} \right]. \tag{3.3}$$

In this diploma thesis the components of the tensor of the finite deformation are denoted by ϵ_{ij} and the components of the tensor of small deformation is denoted by ε_{ij}.

The inner body forces are described by stress.

The stress tensor

The stress tensor represents the density of the inner forces in the body acting between two parts of this body at the point x_i. The stress at point x_i is fully determined by 9 components written as.

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}, \tag{3.4}$$

see e.g. [21] . These components form a second order tensor $\boldsymbol{\sigma}$, which is called the stress tensor, see [19] or [4].

The three diagonal components of the stress tensor correspond to the stresses in the normal directions, 6 remaining components then represent the shear stresses. It can be shown (see [2]), that for the non-diagonal components of the stress tensor following relationships hold

$$\sigma_{12} = \sigma_{21}\,;\ \sigma_{13} = \sigma_{31}\,;\ \sigma_{32} = \sigma_{23},$$

i.e. the stress tensor is symmetric and has only 6 independent components.

Under the assumption of small deformations, a linear relation between the stress tensor and the deformation tensor given by extended Hooke's law can be used, see [2].

Extended Hooke's Law

Under the assumption of the linear behavior of the material, the stress tensor $\boldsymbol{\sigma}$ modelled by the linear relation

$$\boldsymbol{\sigma} = \boldsymbol{C_m}\boldsymbol{\varepsilon}, \tag{3.5}$$

where $\boldsymbol{\sigma} = [\,\sigma_{11}\ \sigma_{22}\ \sigma_{33}\ \sigma_{12}\ \sigma_{13}\ \sigma_{23}\,]^T$, $\boldsymbol{\varepsilon} = [\,\varepsilon_{11}\ \varepsilon_{22}\ \varepsilon_{33}\ \varepsilon_{12}\ \varepsilon_{13}\ \varepsilon_{23}\,]^T$ and $\boldsymbol{C_m} \in \boldsymbol{R^{6\times 6}}$ is the matrix of a material model, i.e.

$$\boldsymbol{C_m} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1123} & C_{1131} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3331} & C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2331} & C_{2312} \\ C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3131} & C_{3112} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1231} & C_{1212} \end{bmatrix}, \tag{3.6}$$

where C_{ijkl} are components of the 4^{th} order tensor again denoted by $\boldsymbol{C_m}$, which is symmetric, with the respect to the first and the second pair of indices, which means $C_{ijkl} = C_{klij}$ for all i,j,k,l, see [2].

In the most general case there are 21 independent coefficients C_{ijkl} of the extended Hook's Law [2]. However this is only the case of materials, that have different behavior in every direction, for example some composites or material with the trigeminal crystal lattice. In what follows, we shall restrict ourselves to the case of an isotropic material.

The extended Hooke`s Law for the isotropic material we shall write as

$$\boldsymbol{\sigma} = \boldsymbol{C_L}\boldsymbol{\varepsilon}, \tag{3.7}$$

Equation (3.7) can be also written for the components of the tensor $\boldsymbol{\sigma} = (\sigma_{ij})$, i.e.

$$\sigma_{ij}(\boldsymbol{u}) = C_{ijkl}\varepsilon_{kl}. \tag{3.8}$$

3.1.2. Hooke`s law for isotropic material

For an isotropic material only two independent coefficients of the extended Hooke's Law exists. These two coefficients are called Lame's constants and are denoted by λ and μ. The matrix of the isotropic material model $\boldsymbol{C_L}$ can be written as

$$\boldsymbol{C_L} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix},$$

or for the components

$$\sigma_{ij}(\boldsymbol{u}) = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}. \tag{3.9}$$

Behavior of the isotropic materials is often described by another set of constants E_Y,σ_P. Constant E_Y is called tensile modulus or Young's modulus, σ_P is called Poisson's constant. The constant λ is also often called shear modulus and marked as G. The following relations between E_Y, σ_P, λ, μ can be found for example see [2], i.e.

$$E_Y(\lambda,\mu) = \frac{\mu+(3\lambda+2\mu)}{\mu+\lambda}, \tag{3.10}$$

$$\sigma_P(\lambda,\mu) = \frac{\lambda}{2(\mu+\lambda)}. \tag{3.11}$$

$$\lambda(\sigma_P,E_Y) = \frac{E_Y\sigma_P}{(1+\sigma_P)(1-2\sigma_P)} \tag{3.12}$$

$$\mu(\sigma_P,E_Y) = \frac{E_y}{2(1+\sigma_P)} \tag{3.13}$$

Lame-Navier's equation

The elastic body is in steady state solution (or in equilibrium) if each infinitesimal element within it is in equilibrium. The equilibrium of the structure represented by Ω is described by Lame-Navier's equation, i.e. at every point $x \in \Omega$ holds

$$\frac{\partial\sigma_{ij}}{\partial x_j} + b_i = 0, \tag{3.14}$$

where $b_i = b_i(\boldsymbol{x})$ denotes the volume forces acting at the point $\boldsymbol{x}$.

By combining the Lame-Navier's equation (3.14), the Hooke's law (3.5) and the relation for the tensor of small deformation (3.3) we get Lame's equation

$$\frac{\partial}{\partial x_k}\left(\mu\frac{\partial u_i}{\partial x_k}\right)+\frac{\partial}{\partial x_k}\left(\mu\frac{\partial u_i}{\partial x_i}\right)+\frac{\partial}{\partial x_k}\left(\lambda\frac{\partial u_i}{\partial x_i}\right)+b_i=0, \tag{3.15}$$

where λ and μ are Lame`s constants.

This equation can be written in the vector form

$$\mu\Delta\boldsymbol{u}+(\lambda+\mu)\mathrm{grad}(\mathrm{div}\,\boldsymbol{u})+\boldsymbol{b}=\boldsymbol{0}. \tag{3.16}$$

3.1.2. Static linear elasticity - problem description

We assume a solid body represented by bounded domain Ω that is loaded by volume and surface external forces at boundary $\partial\Omega$. In order to describe the steady state of the structure Ω, we assume that the deformation is small and we use the isotropic material model.

For the description of the steady state of the structure Ω we have the following equations: the Lame-Navier's equations (3.14), extended Hooke's law for isotropic material (3.5) and equation for tensor of small deformation (3.3). In order to enclose the model, the boundary conditions on $\partial\Omega$ needs to be specified.

The boundary $\partial\Omega$ is divided into two parts $\partial\Omega=\Gamma_N\cup\Gamma_D$. On Γ_D the boundary condition for the displacement $\boldsymbol{u}$ is used. On Γ_N the external surface forces are applied.

Dirichlet boundary condition

This kind of boundary condition prescribes the value of the displacements $\boldsymbol{u}$ on the Dirichlet part of boundary Γ_D i.e.

$$\boldsymbol{u}(\boldsymbol{x})=\boldsymbol{u_D}(\boldsymbol{x})\,,\ \forall\,\boldsymbol{x}=(x_1,x_2,x_3)\in\Gamma_D, \tag{3.17}$$

where $\boldsymbol{x}=(x_1,x_2,x_3)$. Particularly, the condition $\boldsymbol{u_D}(\boldsymbol{x})=0$ is being used to suppress any deformation on this part of the boundary.

Neumann boundary condition

The Neumann type of boundary conditions is used to describe the action of external surface forces.

$$\boldsymbol{\sigma n}=\boldsymbol{f_s}(\boldsymbol{x})\quad\forall\,\boldsymbol{x}\in\Gamma_N, \tag{3.18}$$

where $\boldsymbol{n}=(n_1,n_2,n_3)$ is the vector of the unit outer normal to the $\partial\Omega$ at the point $\boldsymbol{x}\in\Gamma_N$ and $\boldsymbol{f_s}$ are the surface forces.

Static elasticity problem

We seek a vector function $\boldsymbol{u}: \Omega \to \mathbf{R^3}; \boldsymbol{u} \in \boldsymbol{C^2}(\Omega) \cap \mathbf{C^1}(\bar{\Omega})$ such that satisfies equation (3.14) in Ω and the boundary conditions (3.17), (3.18).

3.1.3. Weak formulation of the static problem of linear elasticity

The partial differential equation (3.14) has one disadvantage, its solution with continuous 2^{nd} order partial derivatives are not guaranteed to exists, even in the case that the formulation of the problem is correct and the "physical" solution exists. However it is possible to define the problem more generally with the aid of the so-called weak formulation, see [9].

We start with the definition of the space of test functions $\boldsymbol{V_f} = \{\boldsymbol{v} \in [\, H^1(\Omega)\,]^3 : \boldsymbol{v} = 0 \; on \; \Gamma_u\}$. Further we multiply the i^{th} equation of (3.14) by the i^{th} component of a test function $\boldsymbol{v} \in \boldsymbol{V_f}$ and integrate over Ω. We get

$$\int_\Omega -\frac{\partial \sigma_{ij}(\boldsymbol{u})}{\partial x_j} v_i \, d\Omega = \int_\Omega b_j v_i \, d\Omega,$$

where the components $\sigma_{ij}(\boldsymbol{u})$ are given by (3.8).

Further we apply Green's theorem (2.14) on the left hand-side of equation and get

$$\int_{\partial\Omega} \left(-\sigma_{ij}(\boldsymbol{u}) n_j\right) v_i \, dS \; + \; \int_\Omega \sigma_{ij}(\boldsymbol{u}) \frac{\partial v_i}{\partial x_j} \, d\Omega = \int_\Omega b_i v_i \, d\Omega.$$

Now, we use that the function $\boldsymbol{v} = 0$ on Γ_u and the boundary condition (3.18) on Γ_σ. We get

$$\int_\Omega \sigma_{ij}(\boldsymbol{u}) \frac{\partial v_i}{\partial x_j} \, d\Omega = \int_\Omega b_i v_i \, d\Omega + \int_{\Gamma_\sigma} f_{s_i} v_i \, dS.$$

Now we rewrite the first term using the symmetry of the tensor $\boldsymbol{\sigma}$ and tensor of small deformations $\boldsymbol{\varepsilon}$

$$\sigma_{ij}(\boldsymbol{u}) \frac{\partial v_i}{\partial x_j} = \frac{1}{2}\left(\sigma_{ij}(\boldsymbol{u}) \frac{\partial v_i}{\partial x_j}\right) + \frac{1}{2}\left(\sigma_{ij}(\boldsymbol{u}) \frac{\partial v_i}{\partial x_j}\right) = \frac{1}{2}\left(\sigma_{ij}(\boldsymbol{u}) \frac{\partial v_i}{\partial x_j}\right) + \frac{1}{2}\left(\sigma_{ji}(\boldsymbol{u}) \frac{\partial v_i}{\partial x_j}\right) =$$
$$\frac{1}{2}\left(\sigma_{ij}(\boldsymbol{u}) \frac{\partial v_i}{\partial x_j}\right) + \frac{1}{2}\left(\sigma_{ij}(\boldsymbol{u}) \frac{\partial v_j}{\partial x_i}\right) = \sigma_{ij}(\boldsymbol{u}) \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right),$$

where we denote $\varepsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}\left(\frac{\partial v_k}{\partial v_l} + \frac{\partial v_l}{\partial v_k}\right)$.

By substituting back into equation we get the final relation

$$\int_{\Omega} \sigma_{ij}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{v})\, d\Omega = \int_{\Omega} b_i v_i \, d\Omega + \int_{\Gamma_N} f_{s_i} v_i \, dS, \tag{3.19}$$

which holds for an arbitrary $\boldsymbol{v} \in \boldsymbol{V_f}$. Equation (3.19) is a weak formulation of the static problem of linear elasticity.

Denoting

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_{\Omega} \sigma_{ij}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{v})\, d\Omega, \tag{3.20}$$

and

$$L(\boldsymbol{v}) = \int_{\Omega} b_i v_i d\Omega + \int_{\Gamma_N} f_{s_i} v_i dS, \tag{3.21}$$

equation (3.19) then can be written as

$$a(\boldsymbol{u},\boldsymbol{v}) = L(\boldsymbol{v}). \tag{3.22}$$

Nonzero Dirichlet boundary condition

Generally the Dirichlet condition can be nonzero $\boldsymbol{u_D} \neq 0$ on $\partial\Omega$. In such case, we assume the function $\boldsymbol{u_D}$ is the trace of a function $\widehat{\boldsymbol{u}_{\boldsymbol{D}}} \in [H^1(\Omega)]^3$. The solution of the weak static problem of linear elasticity is sought in the form $\boldsymbol{u} = \widehat{\boldsymbol{u}_{\boldsymbol{D}}} + \boldsymbol{u_0}$, where $\boldsymbol{u_0} \in \boldsymbol{V_f}$ is an unknown function, which satisfies

$$a(\boldsymbol{u_0} + \boldsymbol{u_D}, \boldsymbol{v}) = L(\boldsymbol{v})$$

for all $\boldsymbol{v} \in \boldsymbol{V_f}$.

Using the linearity of the form a we get the formulation: Find $u_0 \in \boldsymbol{V_f}$, such that

$$a(\boldsymbol{u_0}, \boldsymbol{v}) = L(\boldsymbol{v}) - a(\boldsymbol{u_D}, \boldsymbol{v}) \tag{3.23}$$

holds for all $\boldsymbol{v} \in \boldsymbol{V_f}$.

We can see that the problem with the nonzero Dirichlet boundary condition can be reformulated as a problem with the zero Dirichlet boundary condition and a modified right-hand side. Further in this diploma thesis only the zero Dirichlet boundary condition shall be considered.

Static weak problem formulation

We seek function $\boldsymbol{u} \in [H^1(\Omega)]^3$ such that for arbitrary function $\boldsymbol{v}$; $\boldsymbol{v} \in \boldsymbol{V_f}$: satisfies equation $\int_{\Omega} \sigma_{ij}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{v})\, d\Omega = \int_{\Omega} b_i v_i \, d\Omega + \int_{\Gamma_N} f_{s_i} v_i \, dS$, where $\sigma_{ij}(\boldsymbol{u})$ is given by (3.8) and $\boldsymbol{u} = \boldsymbol{u_D}$ at Γ_D.

3.1.4. Discretized weak problem of static linear elasticity

We would like to approximate continuous problem of the linear elasticity by discrete problem. Let us consider a finite dimensional subspace $\boldsymbol{V_h} \subset \boldsymbol{V_f}$, with $\dim \boldsymbol{V_h} = n < \infty$. Let functions $\boldsymbol{\varphi_1}, \boldsymbol{\varphi_2} \ldots \boldsymbol{\varphi_n} \in \boldsymbol{V}_h$ form basis of the function space $\boldsymbol{V_h}$. An arbitrary function $\boldsymbol{v_h} \in \boldsymbol{V_h}$ then can be written as a linear combination of the base functions of the function space $\boldsymbol{V_h}$. We call function $\boldsymbol{u_h}$ the solution of the discrete problem of linear elasticity if

$$a(\boldsymbol{u_h}, \boldsymbol{v_h}) = L(\boldsymbol{v_h}) \text{ holds for all } v_h \in \boldsymbol{V_h}, \tag{3.24}$$

where $a(\boldsymbol{u_h}, \boldsymbol{v_h})$ and $L(\boldsymbol{v_h})$ are given by

$$a(u_h, v_h) = \int_\Omega \sigma_{ij}\,(\boldsymbol{u_h})\varepsilon_{ij}(\,\boldsymbol{v}_\mathrm{h})\; d\Omega\,, \tag{3.25}$$

$$L(\boldsymbol{v_h}) = \int_\Omega b_i v_i d\Omega + \int_{\Gamma_\sigma} f_{s_i} v_i dS. \tag{3.26}$$

As the function $\boldsymbol{v_h} \in \boldsymbol{V_h}$ is a linear combination of the base functions $\boldsymbol{\varphi_k} \epsilon \boldsymbol{V_h}$, the discrete weak problem of static linear elasticity can be formulated for the basis functions instead of for $\boldsymbol{v_h}$. In such a case the discrete problem reads

$$a(\boldsymbol{u_h}, \boldsymbol{\varphi_k}) = L(\boldsymbol{\varphi_k}). \tag{3.27}$$

Stiffness matrix, right-hand side vector

In order to find a solution of the discrete problem (3.24), we write $\boldsymbol{u_h}$ written as a linear combination of the basis functions i.e.

$$\boldsymbol{u_h} = \sum_{l=1}^{n} \alpha_l \boldsymbol{\varphi_l}. \tag{3.28}$$

By substituting relation (3.28) into equation (3.27) and using the linearity of the form a we get

$$\sum_{l=1}^{n} \alpha_l a(\boldsymbol{\varphi_l}, \boldsymbol{\varphi_k}) = L(\boldsymbol{\varphi_k}).$$

or in the matrix notation

$$\boldsymbol{K\alpha} = \boldsymbol{F}, \tag{3.29}$$

Where $\boldsymbol{K}$ is the stiffness matrix given by

$$\boldsymbol{K} = \begin{pmatrix} a(\boldsymbol{\varphi_1}, \boldsymbol{\varphi_1}) & a(\boldsymbol{\varphi_2}, \boldsymbol{\varphi_1}) & \cdots & a(\boldsymbol{\varphi_n}, \boldsymbol{\varphi_1}) \\ a(\boldsymbol{\varphi_1}, \boldsymbol{\varphi_2}) & a(\boldsymbol{\varphi_2}, \boldsymbol{\varphi_2}) & \cdots & a(\boldsymbol{\varphi_n}, \boldsymbol{\varphi_2}) \\ \vdots & \vdots & \ddots & \vdots \\ a(\boldsymbol{\varphi_1}, \boldsymbol{\varphi_n}) & a(\boldsymbol{\varphi_2}, \boldsymbol{\varphi_n}) & \cdots & a(\boldsymbol{\varphi_n}, \boldsymbol{\varphi_n}) \end{pmatrix}$$

And F is the load vector given by

$$\boldsymbol{F} = [L(\boldsymbol{\varphi_1})\;\; L(\boldsymbol{\varphi_1})\;\; \ldots\;\; L(\boldsymbol{\varphi_n})]^T.$$

The vector of unknowns reads

$$\boldsymbol{\alpha} = [\alpha_1 \alpha_2 \dots \alpha_n]^T.$$

We can see, that the solution of the discrete problem of the static linear elasticity (3.24) is equivalent to the solution of the system of linear equations (3.29).

By solving this linear system we get the coefficients α_i. The solution $\boldsymbol{u_h}$ of the discrete problem (3.24) is given by relation (3.28), see [9], [5].

It can be shown that, under the considered boundary conditions, the matrix $\boldsymbol{K}$ is symmetric and positive definite. See [5].

Equation (3.19) holds for an arbitrary set of base functions. In order to obtain matrix $\boldsymbol{K}$ which is sparse, the base function with small supports are usually selected. The support of the function is a part of the domain of the function, where this function is nonzero. Such a choice leads to $a(\boldsymbol{\varphi_k}, \boldsymbol{\varphi_l}) = 0$ for most of the indices $k, l\, k \neq l$. Obtaining sparse matrix saves computer memory, simplifies matrix multiplication and allows us to use iterative method for solution of the system of equations.

3.2. Dynamic problem of linear elasticity

Similarly to the static problem of linear elasticity we assume a solid body represented by bounded domain Ω that is loaded by volume and surface external forces. Now, the considered forces can be time dependent. Moreover for the dynamic problem initial conditions need to be specified.

In order to describe the dynamic problem of linear elasticity, see [2], we include the inertia forces into equation (3.14) for static equilibrium.

$$-\frac{\partial \sigma_{ij}}{\partial x_j} + \rho \frac{\partial^2 u_i}{\partial t^2} = b_i, \tag{3.30}$$

where $\rho = \rho(\boldsymbol{x})$ represents the density of the material in the point $\boldsymbol{x}$. In this diploma thesis we assume that density is piecewise constant and positive $\rho(\boldsymbol{x}) > 0$. In the dynamic problem stress, the displacement and the external force load are also functions of $\boldsymbol{x}$ and t, i.e.$\sigma_{ij} = \sigma_{ij}(\boldsymbol{x}, t)$, $u_i = u_i(\boldsymbol{x}, t)$,$b_i = b_i(\boldsymbol{x}, t)$.

By combining equation (3.30), the Hooke's law (3.9) and the tensor of small deformation (3.3) we can obtain Lame's equation for dynamic problem that contains only u_i as a variable

$$\frac{\partial}{\partial x_k}\left(\mu \frac{\partial u_i}{\partial x_k}\right) + \frac{\partial}{\partial x_k}\left(\mu \frac{\partial u_i}{\partial x_i}\right) + \frac{\partial}{\partial x_k}\left(\lambda \frac{\partial u_i}{\partial x_i}\right) + b_i = \rho \frac{\partial^2 u_i}{\partial t^2}, \tag{3.31}$$

where $\lambda = \lambda(\boldsymbol{x}), \mu = \mu(\boldsymbol{x})$ are the Lamé`s coefficients of the material at the point $\boldsymbol{x}$. Again λ and μ are assumed to be piecewise constants constant.

Boundary conditions

Similarly to the static case we consider only the Dirichlet or Neumann boundary conditions, which now can be time dependent.

Dirichlet boundary condition

This kind of boundary condition prescribes the value of the displacements u on the Dirichlet part of boundary Γ_u i.e.

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u_D}(\boldsymbol{x}, t)\ ,\ \forall\, \boldsymbol{x} \in\ \Gamma_u. \tag{3.32}$$

Similarly to the static case $\boldsymbol{u_d}$ is the trace of a function $\widehat{\boldsymbol{u}}_{\boldsymbol{D}} \in [H^1(\Omega)]^3 \times \bar{I};\ I = (0, T)$.

Neumann boundary condition

The Neumann type of boundary conditions is used to describe the action of external surface forces and is given by

$$\boldsymbol{\sigma}(\boldsymbol{x}, t) \cdot \boldsymbol{n} = \boldsymbol{f_s}(\boldsymbol{x}, t) \forall\, x \in\ \Gamma_\sigma, \tag{3.33}$$

where $\boldsymbol{n}$ is the vector of the unit outer normal to the $\partial\Omega$ at point $x \in\ \Gamma_\sigma$ and $\boldsymbol{f_s}$ are the surface forces.

Initial conditions

In order to enclose the dynamic model of the linear elasticity it is necessary to add a state of the structure at time $t = 0$ on the whole domain Ω i.e. initial conditions

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u^I}(\boldsymbol{x}), \qquad \forall\, x \in\ \Omega, \tag{3.34}$$

$$\frac{\partial \boldsymbol{u}(\boldsymbol{x}, 0)}{\partial t} = \overline{\boldsymbol{u^I}}(\boldsymbol{x}), \qquad \forall\, x \in\ \Omega. \tag{3.35}$$

Dynamic elasticity problem formulation

We seek a function $\boldsymbol{u} : Q_T \to \mathbf{R^3}$; $Q_T = \bar{\Omega} \times \bar{I};\ I = (0, T)$ such that satisfies equation (3.30) for any $[\boldsymbol{x}, t] \in Q_T$, initial conditions (3.34), (3.35) and boundary conditions (3.32), (3.33).

3.2.1. Weak problem formulation

Similarly to the static problem we start with the weak formulation of the dynamic problem. Again we denote the space of test functions $\boldsymbol{V_f} = \{\boldsymbol{v} \in [H^1(\Omega)]^3 : \boldsymbol{v} = 0 \text{ on } \Gamma_u\}$ and multiply the i^{th} equation (3.11.) at arbitrary time $t \in \bar{I}$ by the i^{th} component of a test function $\boldsymbol{v} \in \boldsymbol{V_f}$ and integrate over Ω. We get

$$-\int_\Omega \frac{\partial \sigma_{ij}}{\partial x_j} v_i d\Omega + \int_\Omega \rho \frac{\partial^2 u_i}{\partial t^2} v_i d\Omega = \int_\Omega b_i v_i d\Omega. \tag{3.36}$$

Further we apply Green's theorem (2.4.4.), boundary conditions, use the symmetry of the tensor $\boldsymbol{\sigma}$ and we get

$$\int_\Omega \sigma_{ij}(\boldsymbol{u}) \varepsilon_{ij}(\boldsymbol{v}) d\Omega + \int_\Omega \rho \frac{\partial^2 u_i}{\partial t^2} . v_i = \int_\Omega b_i \, v_i d\Omega + \int_{\Gamma_\sigma} f_{s_i} v_i dS, \tag{3.37}$$

where $b_i = b_i(\boldsymbol{x}, t)$ and $f_{s_i} = f_{s_i}(\boldsymbol{x}, t)$.

Let us denote

$$a(u, v) = \int_\Omega \sigma_{ij}(\boldsymbol{u}) \varepsilon_{ij}(\boldsymbol{v}) \, d\Omega, \tag{3.38}$$

$$L(\boldsymbol{v}; t) = \int_\Omega b_i(t) v_i d\Omega + \int_{\Gamma_\sigma} f_{s_i}(t) v_i dS \tag{3.39}$$

and

$$m\left(\frac{\partial^2 u_i}{\partial t^2}, \boldsymbol{v_i}\right) = \int_\Omega \rho \frac{\partial^2 u_i}{\partial t^2} . v_i. \tag{3.40}$$

Then the problem (3.36) can be written as

$$m\left(\frac{\partial^2 u_i}{\partial t^2}, v_i\right) + a(\boldsymbol{u}, \boldsymbol{v}) = L(\boldsymbol{v}; t). \tag{3.41}$$

Weak dynamic problem formulation

We seek a function $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$, which for any t satisfy $\boldsymbol{u}(\cdot, t) \in \boldsymbol{V_f} = [H^1(\Omega)]^3$, $\frac{\partial^2 \boldsymbol{u_i}}{\partial t^2} \in [L^2(\Omega)]^3$ such that for any $t \in \bar{I}$ equation (3.41) holds for any test function $\boldsymbol{v} \in \boldsymbol{V_f}$. Further, $\boldsymbol{u}$ is assumed to satisfy the boundary condition $\boldsymbol{u} = \boldsymbol{u_D}$ at Γ_D and the initial conditions (3.34), (3.35).

3.2.2. Semi-discrete weak dynamic problem

Similarly to the static problem, we consider finite dimensional function subspace $\boldsymbol{V_h} \subset \boldsymbol{V_f}$, $\dim \boldsymbol{V_h} = n < \infty$. Basis function of $\boldsymbol{V}_h$ are denoted by $\boldsymbol{\varphi_1}, \boldsymbol{\varphi_2} \ldots \boldsymbol{\varphi_n}$ and arbitrary function $\boldsymbol{v_h} \in \boldsymbol{V_h}$ can be written as a linear combination of base functions. We seek at every $t \in I$ for a solution $\boldsymbol{u_h} = \boldsymbol{u_h}(\cdot, t) \in \boldsymbol{V_h}$ in the form $\boldsymbol{u_h}(\boldsymbol{x}, t) = \sum_{i=1}^{n} \alpha_i(t)\varphi_i(\boldsymbol{x})$. By substituting this form into equation (3.41) we get the following relation

$$\sum_{l=1}^{n} \ddot{\alpha}_l m(\varphi_k, \varphi_l) + \sum_{l=1}^{n} \alpha_k a(\varphi_k, \varphi_l) = L(\varphi_k; t) \tag{3.42}$$

where $\ddot{\alpha}_k$ denotes the second (time) derivative of the coefficient α_k, i.e. $\ddot{\alpha}_k = \frac{d^2\alpha}{dt^2}$.

The right-hand side form L and the coefficients α_k are functions of time t. Problem is now semi-discretized i.e. discrete in space and continuous in time.

Equation (3.42) can be written in the matrix notation as

$$\boldsymbol{M}\ddot{\boldsymbol{\alpha}} + \boldsymbol{K}\boldsymbol{\alpha} = \boldsymbol{F}, \tag{3.43}$$

where the mass matrix $\boldsymbol{M}$ is given by

$$\boldsymbol{M} = \begin{pmatrix} m(\boldsymbol{\varphi_1}, \boldsymbol{\varphi_1}) & m(\boldsymbol{\varphi_2}, \boldsymbol{\varphi_1}) & \cdots & m(\boldsymbol{\varphi_n}, \boldsymbol{\varphi_1}) \\ m(\boldsymbol{\varphi_1}, \boldsymbol{\varphi_2}) & m(\boldsymbol{\varphi_2}, \boldsymbol{\varphi_2}) & \cdots & m(\boldsymbol{\varphi_n}, \boldsymbol{\varphi_2}) \\ \vdots & \vdots & \ddots & \vdots \\ m(\boldsymbol{\varphi_1}, \boldsymbol{\varphi_n}) & m(\boldsymbol{\varphi_2}, \boldsymbol{\varphi_n}) & \cdots & m(\boldsymbol{\varphi_n}, \boldsymbol{\varphi_n}) \end{pmatrix},$$

And $\boldsymbol{K}$ is the stiffness matrix already defined by

$$\boldsymbol{K} = \begin{pmatrix} a(\boldsymbol{\varphi_1}, \boldsymbol{\varphi_1}) & a(\boldsymbol{\varphi_2}, \boldsymbol{\varphi_1}) & \cdots & a(\boldsymbol{\varphi_n}, \boldsymbol{\varphi_1}) \\ a(\boldsymbol{\varphi_1}, \boldsymbol{\varphi_2}) & a(\boldsymbol{\varphi_2}, \boldsymbol{\varphi_2}) & \cdots & a(\boldsymbol{\varphi_n}, \boldsymbol{\varphi_2}) \\ \vdots & \vdots & \ddots & \vdots \\ a(\boldsymbol{\varphi_1}, \boldsymbol{\varphi_n}) & a(\boldsymbol{\varphi_2}, \boldsymbol{\varphi_n}) & \cdots & a(\boldsymbol{\varphi_n}, \boldsymbol{\varphi_n}) \end{pmatrix}.$$

The load vector $\boldsymbol{F}$ reads

$$\boldsymbol{F} = \boldsymbol{F}(t) = [L(\boldsymbol{\varphi_1}; t)\ L(\boldsymbol{\varphi_1}; t)\ \ldots\ L(\boldsymbol{\varphi_n}; t)]^T$$

and the vector of $\boldsymbol{\alpha}$ unknowns is given by

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}(t) = [\alpha_1 \alpha_2 \ldots \alpha_n]^T.$$

3.2.3. Damping of the system

Damping is an influence within or upon an oscillating system that has the effect of reducing oscillations over time. In the physical processes it usually represents the dissipation of energy stored in the considered system. For the FEM simulations usually a linear model of damping is used. This model is represented by including the velocity dependent term $\boldsymbol{C\dot{\alpha}}$ into equation (3.43). Here, the damping matrix $\boldsymbol{C}$ is considered as a linear combination of stiffness and mass matrix [5], i.e.

$$\boldsymbol{C} = \varepsilon_1 \boldsymbol{M} + \varepsilon_2 \boldsymbol{K}, \tag{3.44}$$

where $\varepsilon_1, \varepsilon_2 \in \boldsymbol{R}, \varepsilon_1, \varepsilon_2 > 0$. Such a damping is called proportion damping, see [5] [7]**.** The coefficients $\varepsilon_1, \varepsilon_2$ are usually chosen to be small.

Including the damping matrix into equation (3.43) we get equation for the damped dynamic system excited by external forces, i.e.

$$\boldsymbol{M\ddot{\alpha}} + \boldsymbol{C\dot{\alpha}} + \boldsymbol{K\alpha} = \boldsymbol{F}. \tag{3.45}$$

3.2.4. Free vibration analysis

In order to find the solution of equation (3.25.) let us start with a simplified problem without damping and external forces. Such problem is usually called free vibration analysis. In this case the so called eigen frequencies and the so called normal eigen modes of the structure can be determined. The simplified equation reads

$$\boldsymbol{M\ddot{\alpha}} + \boldsymbol{K\alpha} = \boldsymbol{0}. \tag{3.46}$$

Solution of equation (3.46) is expected to be periodic. The system of equations (3.46) is linear, general solution is linear combination of the functions of the fundamental system [22]. The fundamental system is formed from functions

$$\boldsymbol{\alpha} = \boldsymbol{V}\,[\cos(\omega t) + i\sin(\omega t)] = \boldsymbol{V}e^{i\omega t}, \tag{3.47}$$

where $\omega \in R$ and $\boldsymbol{V} \in \mathbb{C}^n$.

Now, by taking α given by (3.47) and substituting it into equation (3.46), we get

$$-\boldsymbol{MV}\omega^2 e^{i\omega t} + \boldsymbol{KV}e^{i\omega t} = 0,$$

which is equivalent to equation

$$(\boldsymbol{K} - \boldsymbol{M}\omega^2)\boldsymbol{V} = 0.$$

This equation represents a generalized eigen value problem described in section 2.1.5. It can be shown that there exist n eigen values representing the eigen frequencies $\omega_j \quad j = 1,2 \ldots n$. For each eigen frequency ω_j there exists a corresponding eigen vector $\boldsymbol{V_j}$ representing a normal mode of the system. Usually the normalized vectors $\widehat{\boldsymbol{V}}$ are used, according to [5], i.e.

$$\widehat{\boldsymbol{V}}_{\boldsymbol{j}}^{\boldsymbol{T}} \boldsymbol{M} \widehat{\boldsymbol{V}}_{\boldsymbol{j}} = 1 \quad \text{for} \quad j = 1,2 \ldots n \tag{3.48}$$

We call $\widehat{\boldsymbol{V}_{\boldsymbol{j}}}$ the unit normal eigen mode corresponding to the eigen frequency ω_j. In the following text we shall denote the unit normal eigen modes by $\boldsymbol{V_j}$ (without ˆ).

3.2.5. Damped free vibration analysis

Furthermore a damped system given by equation (3.45) is considered, without external forces (i.e. zero right-hand side). The system reads

$$\boldsymbol{M}\ddot{\boldsymbol{\alpha}} + \boldsymbol{C}\dot{\boldsymbol{\alpha}} + \boldsymbol{K}\boldsymbol{\alpha} = \boldsymbol{0}. \tag{3.49}$$

In order to solve the problem (3.49) the transformation of this second order system to the first order system can be used, see [5].

Let us denote $\beta = \dot{\alpha}$, we get

$$\boldsymbol{M}\dot{\beta} + \boldsymbol{C}\beta + \boldsymbol{K}\boldsymbol{\alpha} = \boldsymbol{0}, \tag{3.50}$$

which is equivalent to equation

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{M} \end{bmatrix} \cdot \begin{bmatrix} \dot{\beta} \\ \dot{\alpha} \end{bmatrix} + \begin{bmatrix} \boldsymbol{C} & \boldsymbol{K} \\ \boldsymbol{M} & \boldsymbol{0} \end{bmatrix} \cdot \begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}. \tag{3.51}$$

Similarly to the non-damped case we consider a solution in form

$$\boldsymbol{\alpha} = \boldsymbol{V}\,[\cos(\omega t) + i\sin(\omega t)] = \boldsymbol{V}e^{i\omega t}, \tag{3.52}$$

$$\boldsymbol{\beta} = \widetilde{\mathbf{V}}\,[\cos(\omega t) + i\sin(\omega t)] = \widetilde{\mathbf{V}}e^{i\omega t},$$

Now substituting relations (3.52) into equation (3.37) we get

$$\left(\omega \begin{bmatrix} \boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{M} \end{bmatrix} + \begin{bmatrix} \boldsymbol{C} & \boldsymbol{K} \\ \boldsymbol{M} & \boldsymbol{0} \end{bmatrix}\right) \cdot \begin{bmatrix} \widetilde{\mathbf{V}} \\ \mathbf{V} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}. \tag{3.53}$$

Equation (3.53) is again the general eigen value problem, but now the matrices $\begin{bmatrix} \boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{M} \end{bmatrix}$ and $\begin{bmatrix} \boldsymbol{C} & \boldsymbol{K} \\ \boldsymbol{M} & \boldsymbol{0} \end{bmatrix}$ are unsymmetric and not positive definite, Under the assumption that the damping is the described with the aid of the damping matrix given by equation (3.44) the situation is much simpler.

3.2.6. Eigen frequencies of damped system

Let us consider equation (3.49) with the damping matrix $\boldsymbol{C}$ given by (3.44), i.e.

$$\boldsymbol{M\ddot{\alpha}} + \boldsymbol{C\dot{\alpha}} + \boldsymbol{K\alpha} = \boldsymbol{0} \tag{3.54}$$

According to [5], a solution of equation (3.49) is expected to be in the form

$$\boldsymbol{\alpha} = \boldsymbol{V}e^{(\Psi+i\Omega)t} = \boldsymbol{V}e^{zt}. \tag{3.55}$$

Now substitute relation (3.55) into equation (3.49), which gives relation for $z \in \mathbb{C}$ and vector $\boldsymbol{V}$.

$$(z^2\boldsymbol{MV} + z\boldsymbol{CV} + \boldsymbol{KV})\boldsymbol{e}^{zt} = 0. \tag{3.56}$$

Furthermore using matrix $\boldsymbol{C}$ given by (3.44) leads to

$$(z^2 + \varepsilon_1 z)\boldsymbol{MV} + (1 + \varepsilon_2 z)\boldsymbol{KV} = 0,$$

or equivalently after rearrangement

$$\frac{z^2 + \varepsilon_1 z}{1 + \varepsilon_2 z}\boldsymbol{MV} + \boldsymbol{KV} = 0. \tag{3.57}$$

Denoting

$$\frac{z^2 + \varepsilon_1 z}{1 + \varepsilon_2 z} = -\omega^2 \tag{3.58}$$

in equation (3.57) we get again the generalized eigen values problem described in section 2.1.5.

$$-\omega^2\boldsymbol{MV} + \boldsymbol{KV} = 0 \tag{3.59}$$

Here ω represents the eigen frequency of the non-damped system.

Now let us decide under which assumptions are eigen values z of equation (3.59) real, complex or purely complex.

Rearranging (3.58) we get

$$z^2 + (\varepsilon_1 + \varepsilon_2\omega^2)z + \omega^2 = 0, \tag{3.60}$$

which is a quadratic equation with the discriminant given by

$$D = (\varepsilon_1 + \varepsilon_2\omega^2)^2 - 4\omega^2$$

In case $\varepsilon_1, \varepsilon_2$ are small enough, such that $D < 0$, the roots $z_{1,2}$ of equation (3.60) are complex numbers, the solution is a goniometric function and system vibrates. On the other hand, if $D > 0$, then the roots $z_{1,2}$ of equation (3.60) are real, solutions are exponential functions and no vibrations occur. In the case that $D = 0$, i.e. $(\varepsilon_1 + \varepsilon_2\Omega^2)^2 = 4\Omega^2$, we speak about the critical damping.

Let us focus on the of low damped vibration case, i.e. $D < 0$. In this case the roots of equation (3.60) are

$$z_{1,2} = \frac{-\varepsilon_1 - \varepsilon_2\omega^2 \pm \sqrt{D}}{2} = \frac{-\varepsilon_1 - \varepsilon_2\omega^2}{2} \pm i\sqrt{\omega^2 - \frac{(\varepsilon_1 + \varepsilon_2\omega^2)^2}{4}} = \Psi \pm i\Omega,$$

where $\Psi = \frac{-\varepsilon_1 - \varepsilon_2\omega^2}{2}$ and $\Omega = \sqrt{\omega^2 - \frac{(\varepsilon_1 + \varepsilon_2\omega^2)^2}{4}}$. Here, Ψ is a damping coefficient and Ω is the eigen frequency of the damped system (a modified eigen frequency of the undamped system ω).

Now, the fundamental system of solutions is formed from functions

$$\boldsymbol{V}_k e^{(\Psi_k + i\Omega_k)t} = \boldsymbol{V}_k e^{\Psi t}[\cos(\Omega_k t + i\sin(\Omega_k t)],$$

$$\boldsymbol{V}_k e^{(\Psi_k + i\Omega_k)t} = \boldsymbol{V}_k e^{\Psi t}[\cos(\Omega_k t - i\sin(\Omega_k t)],$$

where $k = 1,2 \ldots n$ and $\boldsymbol{V_k}$ is the real eigen vector corresponding to the eigen frequency ω_k.

The real fundamental system reads

$$\boldsymbol{V}_k e^{\Psi_k t}\cos(\Omega_k t),$$

$$\boldsymbol{V}_k e^{\Psi_k t}\sin(\Omega_k t).$$

Any solution of equation (3.49) can be written as a linear combination of functions from the fundamental system, i.e.

$$\boldsymbol{\alpha} = \sum_{k=1}^{n} \boldsymbol{V}_k e^{\Psi_k t}\ [A_k \cos(\Omega_k t) + \mathrm{B}_k \sin(\Omega_k t)] \tag{3.61}$$

We have shown that the real positive solution of the problem given by equation (3.49) exists in case that the structure of damping matrix is given by equation (3.44). Solving system (3.49) we can get n eigen frequencies Ω and n unit eigen vectors $\boldsymbol{V}$, corresponding to these eigen frequencies. Eigen vectors $\boldsymbol{V}$ represent unit normal eigen modes of damped system.

3.2.7. Frequency response analysis and modal participation factors

Now the full discrete problem of dynamic linear elasticity given by equation (3.45) is considered. The damping matrix $\boldsymbol{C}$ is given by relation (3.44) and a special choice of the right-hand side term (described further) is taken into account. We seek the solution of the problem as a linear combination of the unit eigen modes with time dependent coefficients. This procedure, described for example in [5], is called the decomposition to the modal basis. The complex coefficients used in this linear combination are then called the modal participation factors. Decomposing to the modal basis using the modal participation factors is a widely used way to evaluate any kind of vibrations [5].

Let us consider the solution, according to [5], of equation (3.45) as a linear combination of the unit normal eigen modes, i.e.

$$\boldsymbol{\alpha}(t) = \sum_i^n \mathbf{V_i}\, y_i(t), \tag{3.62}$$

where $y_i(t)$ are unknown functions and where $\boldsymbol{V_i}$ denote the unit normal eigen modes of the system represented by equation (3.45), see section 3.2.5.

Now by differentiating of $\boldsymbol{\alpha}(t)$ given by equation (3.62) and substituting the result into equation (3.45) we get

$$\boldsymbol{M}\left(\sum_{i=1}^n \mathbf{V_i}\, \ddot{y_i}\right) + \boldsymbol{C}\left(\sum_i^n \mathbf{V_i}\, \dot{y}_i(t)\right) + \boldsymbol{K}\left(\sum_i^n \mathbf{V_i}\, y_i(t)\right) + \boldsymbol{F} = 0. \tag{3.63}$$

Further, multiplying equation (3.63) by $\boldsymbol{V_j^T}$ from the left we get

$$\boldsymbol{V_j^T M}\left(\sum_{i=1}^n \boldsymbol{V_i}\, \ddot{y}_i\right) + \boldsymbol{V_j^T C}\left(\sum_i^n \boldsymbol{V_i}\, \dot{y}_i(t)\right) + \boldsymbol{V_j^T K}\left(\sum_j^n \boldsymbol{V_i}\, y_i(t)\right) + \boldsymbol{V^T F} = 0. \tag{3.64}$$

System (3.64) is a second order system of n linear differential equations with n unknown functions $y_i(t)$. System (3.64) is simplified by using relations, see [5]

$$\boldsymbol{V_j^T M V_i} = 0;\ \ \boldsymbol{V_j^T C V_i} = 0;\ \ \boldsymbol{V_j^T K V_i} = 0 \ \text{ for } i \neq j, \tag{3.65}$$

and

$$\boldsymbol{V_i^T M V_i} = m_i = 1;\quad \boldsymbol{V_i^T C V_i} = c_i;\ \ \boldsymbol{V_i^T K V_i} = k_i. \tag{3.66}$$

System (3.64) now leads to n second order linear ordinary differential equations for unknown functions $y_i = y_i(t)$, see [5]

$$m_i \ddot{y}_i + c_i \dot{y}_i + k_i y_i = f_i. \qquad i = 1,2\ldots.n, \tag{3.67}$$

where $\boldsymbol{V^T F} = f_i$.

We can see that each equation of system (3.67) can be now solved separately. Any general solution of such second order ordinary differential equation can be written as a sum of a particular y_{ip} and the general solution of the homogenous equation y_{ih}, i.e.

$$y_i = y_{ip} + y_{ih}$$

The general solution of the homogenous equation reads (see section 3.2.6)

$$y_{ih} = c_{i1} e^{\Psi_i t} \cos(\Omega_i t) + c_{i2} e^{\Psi_i t} \sin(\Omega_i t), \tag{3.68}$$

Where Ψ_i is the damping coefficient and Ω_i is the eigen frequency, see section 3.2.6.

We can see, that solution (3.68) contains the exponential function $e^{\Psi t}$, where $\Psi < 0$, this part of the solution approaches zero with increasing time. As we are interested in long term solution, we shall focus more on the other part of solution.

In order to calculate the particular solution y_p the linearity of the system (3.67) is used. The i^{th} right-hand side term of (3.67) can be decomposed to x, y, z direction. See the derivation of the right-hand side term in section 3.2.1. and 3.2.2.

$$f_i = f_{ix} + f_{iy} + f_{iz} \tag{3.69}$$

In the practical part of this thesis the right-hand side term in equation (3.67) represents the excitation from the revolving of the crankshaft, which is periodical and almost sinusoidal. Therefore we consider that right-hand side term of equation (3.67) can be approximated with the aid of the discrete Fourier transformation. Applying the discrete Fourier transformation to f_i we get

$$f_i = \sum_{j=1}^{m} \left[r_{i,j,x} \sin(n_j \varpi t) + s_{i,j,x} \cos(n_j \varpi t) \right] + \left[r_{i,j,y} \sin(n_j \varpi t) + s_{i,j,y} \cos(n_j \varpi t) \right] + \left[r_{i,j,z} \sin(n_j \varpi t) + s_{i,j,z} \cos(n_j \varpi t) \right], \tag{3.70}$$

where $m \in \boldsymbol{N}$ is the number of orders, $r_{i,j}, s_{i,j} \in \boldsymbol{R}$ are coefficients of the Fourier transform and ϖ is the excitation force frequency, i.e. engine frequency. Using 3 orders for modeling excitation from crankshaft revolving was found to be sufficient for the considered case.

Now using linearity of the system (3.67) we can solve the i^{th} equation of the system (3.67) separately for each direction and each order, which means that we consider the f_i in form

$$f_i = r_i \cos(n\varpi t) + \sin(n\varpi t). \tag{3.71}$$

Using this decomposition originally one equation was decomposed into $3m$ equations. In what follows, solution of equation for one order and one direction is shown, the process is similar for other directions and orders. Particular solution is sought, according to [5], in the shape

$$y_{ip} = \theta_i \sin(n\varpi t) + \vartheta_i \cos(n\varpi t), \tag{3.72}$$

where $\theta_i, \vartheta_i \in \boldsymbol{R}$ are unknown constants and $\varpi \in \boldsymbol{R}$ is frequency, obtained from decomposing excitation force using Fourier transform, see equation (3.70).

Now, let us once and twice derivate equation (3.72). Substituting the result into equation (3.67) we get

$$\begin{aligned} -m_i n^2 \varpi^2 [\theta_i \sin(n\varpi t) + \vartheta_i \cos(n\varpi t)] + \\ c_i n\varpi [\theta_i \cos(n\varpi t) - \vartheta_i \sin(n\varpi t)] + k_i [\theta_i \sin(n\varpi t) + \vartheta_i \cos(n\varpi t)] = \\ r_i \sin(n\varpi t) + s_i \cos(n\varpi t). \end{aligned} \tag{3.73}$$

From this equation we can get conditions for θ_i, ϑ_i.

$$r_i = -\theta_i m_i n^2 \varpi^2 - \vartheta_i c_i n\varpi + \theta_i k, \tag{3.74}$$

$$s_i = -\vartheta_i m_i n^2 \varpi^2 + \theta_i c_i n\varpi + \vartheta_i k. \tag{3.75}$$

Finally, solving this system of equations we get the result

$$\theta_i = \frac{c_i n\varpi s_i + r_i (k_i - m_i n^2 \varpi^2)}{(c_i n\varpi)^2 + {k_i}^2 - 2k_i m_i n^2 \varpi^2 + (m_i n^2 \varpi^2)^2}, \tag{3.76}$$

$$\vartheta_i = \frac{-c_i n\varpi r_i + s_i (k_i - m_i n^2 \varpi^2)}{(c_i n\varpi)^2 + {k_i}^2 - 2k_i m_i n^2 \varpi^2 + (m_i n^2 \varpi^2)^2}. \tag{3.77}$$

The i^{th} solution of equation (3.67) for f_i given by (3.71)(i.e. for one order and one direction) then reads

$$\begin{aligned} y_i = c_{i1} e^{\Psi t} \cos(\Omega t) + c_{i2} e^{\Psi t} \sin(\Omega t) + \theta_i \sin(n\varpi t) + \\ \vartheta_i \cos(n\varpi t). \end{aligned} \tag{3.78}$$

Considering only the solution without the transient component given by relation (3.68) we get

$$y_i = \theta_i \sin(n\varpi t) + \vartheta_i \cos(n\varpi t). \tag{3.79}$$

or

$$y_i = \theta_i \sin(n\varpi t) + \vartheta_i \cos(n\varpi t) = A_i \sin(n\varpi t + \varphi_i), \tag{3.80}$$

where $A_i = \sqrt{{\theta_i}^2 + {\vartheta_i}^2}$ and $\varphi_i = atan\frac{\theta_i}{\vartheta_i}$. The coefficients A_i, φ_i are the modal participation factor for the considered order n, the considered direction and the given excitation frequency ϖ. Substituting relation (3.80) into the considered solution shape (3.62) we get the steady state solution of equation (3.67).

$$\boldsymbol{\alpha}(t) = \sum_i^n \mathbf{V_i}\, A_i \sin(n\varpi t + \varphi_i) \tag{3.81}$$

3.2.8. Modal participation of points of interest

Relation (3.81) represents solution at all points of the discretized structure, particularly is possible to obtain solution for a specific point by multiplying relation (3.81) by a vector representation of the point of interest $\boldsymbol{q_{PoI}}$, i.e.

$$\boldsymbol{\alpha_{PoI}}(t) = \boldsymbol{q_{PoI}}\, \boldsymbol{\alpha}(t) = \sum_i^n \boldsymbol{q_{PoI}}\, \mathbf{V_i}\, y_i(t),$$

where $\boldsymbol{\alpha_{PoI}}(t) = \boldsymbol{q_{PoI}}$. Denoting $q_i = \boldsymbol{q_{PoI}}\, \mathbf{V_i}$ and $\boldsymbol{\alpha_{PoI}}(t) = \boldsymbol{q_{PoI}}\, \boldsymbol{\alpha}(t)$ we get

$$\boldsymbol{\alpha_{PoI}}(t) = \sum_i^n q_i\, y_i(t). \tag{3.82}$$

Now, expecting excitation force $\boldsymbol{F}$ decomposed, we can use the relation (3.80) to get equation

$$\boldsymbol{\alpha_{PoI}}(t) = \sum_i^n q_i\, \theta_i \sin(n\varpi t) + q_i \vartheta_i \cos(n\varpi t) \tag{3.83}$$

or

$$\boldsymbol{\alpha_{PoI}}(t) = \sum_i^n q_i\, A_i \sin(n\varpi t + \varphi_i). \tag{3.84}$$

Substituting $\widehat{A_\iota} = q_i A_i$ we get

$$\boldsymbol{\alpha_{PoI}}(t) = \sum_{\boldsymbol{i}}^{\boldsymbol{n}} \widehat{A_\iota} \sin(n\varpi t + \varphi_i), \tag{3.85}$$

where coefficients $\widehat{A_i}, \varphi_i$ represent modal participation factors of one point of interest of the model. Here, the excitation forces decomposed into the orders and directions is considered (see equation (3.70)), therefore this solution is valid only for one order and one direction. In praxis we are interested in solution for all direction and all orders combined with many different excitation frequencies. How to work with the modal participation factors in this situation is described in practical part.

3.3. Conclusion of theoretical part

In theoretical part we have derived equations needed for the computation of the unit eigen modes, modal participation factors and frequency response of a structure to external excitation. Now let us move on to the practical part, where the theoretical knowledge will be used for solving practical problems.

4. Practical realization

In this part of the diploma thesis, created in cooperation with Andreas STIHL AG & Co. KG is described the practical realization of the modal decomposition and usage of the modal participation factors in the problem of computing vibration. Main goal, assigned by STIHL, was to improve the STIHL standard methodology for the vibration computation on the chainsaw handles. This vibration occurs as a response to the excitation from the crankshaft revolving. For the computations commerce software called PERMAS is standardly used by STIHL. PERMAS is an internationally established FE analysis system developed by INTES GmbH. PERMAS enables us to perform comprehensive analyses and simulations in many fields of applications like stiffness and stress analysis, contact analysis, vibration computations, acoustic simulations or electromagnetic fields simulations. The data management and the built-in solution algorithm in PERMAS are suitable for the calculation of very large models. Standardly project with well over 20 000 000 unknown degrees of freedom can be solved. PERMAS also supports more than 50 element types for static and dynamic calculations including linear and non-linear elements. For this diploma thesis only the modules for linear static and dynamic computations are used. Information about PERMAS can be found on the INTES website [13]or in PERMAS user manual [12].

Description of the standard method used by STIHL

Machines marked as “professional tools” i.e. machines predetermined to be used during 8-hours work shift have a maximal level of vibration on handles established by ISO norm [1]. To check if the demands of this ISO norm are fulfilled, STIHL has a standard methodology for vibration computation. This methodology is based on the modal vibration calculations with the aid of PERMAS, which computes a vibration curve at a selected point, for the whole range of the engine speeds. In PERMAS this computation is called Frequency response analysis (FRA), see [12]. The vibration curve obtained from FRA is then compared with the allowed value to assess if the limits for vibrations are fulfilled or not. The standard STIHL methodology does not give any answers, what measures to take, in case that the limits given by ISO norm are not fulfilled. That is the concern of the newly developed methodology.

Description of the newly developed method

The main idea of the newly developed method is to analyze the influence of the individual eigen modes to the total result and then approximately reconstruct vibration curve, using only the modes with high influence. Such approach is called reduced order modelling (ROM), see [5] or [11]. Using only the high influence modes for vibration curve reconstruction allows to

identify the sources of the maximal vibration, i.e. find the dominant unit eigen modes which cause the highest vibration. Identifying of the dominant unit eigen modes provides an important information, which should help engineers responsible for the structure design to take measures exactly aimed to the reducing those vibrations.

The newly developed method uses PERMAS to calculate the modal basis and the modal participation factors and then it reconstructs the vibration curve as a linear combination of the modal basis vectors. For theoretical derivation of this procedure see section 3.2. For the postprocessing of the results, an existing tool, Participation Magic, was thoroughly revised and reprogramed. Original Participation Magic was designed only for visualization of the modal participation factor amplitudes. New, redesigned Participation Magic allows us to reconstruct the vibration curve from the modal participation factors and analyze influence of the individual eigen modes to the total result in a user friendly graphic interface.

Before we start to describe how the curve reconstruction in Participation Magic works, let us describe the used FEM model and other inputs required by PERMAS for computations.

4.1. Model description and preprocessing

STIHL develops and manufactures many different chainsaws and other tools. The newly developed method was tested on a FEM model of the chainsaw MS661, see [23]. CAD model of this chainsaw is in **Figure 4.1**. All computation results presented in this thesis were performed on this model. In next sections the test model and other inputs to the simulation including the load forces, material data and bonds between the parts are described. PERMAS typically uses two kinds of inputs - binary input file (BIF) that contains the meshed structure model and text files containing information about the material and loading.

4.1.1. Structure model and finite element mesh realization

The used CAD model is owned by Andreas STIHL AG & Co. KG. Model was preprocessed according to the internal standards of STIHL. For the meshing and other preprocessing procedures like rigid body elements (RBE) creating and bonds creating, the commerce software MEDINA was used. For more information about MEDINA see [14] or [15]. Small parts with negligible impact to the total stiffness of the structure were replaced by mass points connected to the structure by rigid body elements (RBE). The smallest parts, with mass less than 1 gram, were neglected. Used type of elements is, according to the inner STIHL standards, mainly quadratic tetrahedron elements (in PERMAS terminology called TET10). For some

parts (like the guiderail) linear hexahedral and prismatic elements were used (in PERMAS terminology called penta or hexa elements). Preprocessed model is shown in the **Figure 4.2**. The number of nodes and elements of the used mesh is shown in **Table 4.1**.

Figure 4.1 CAD model

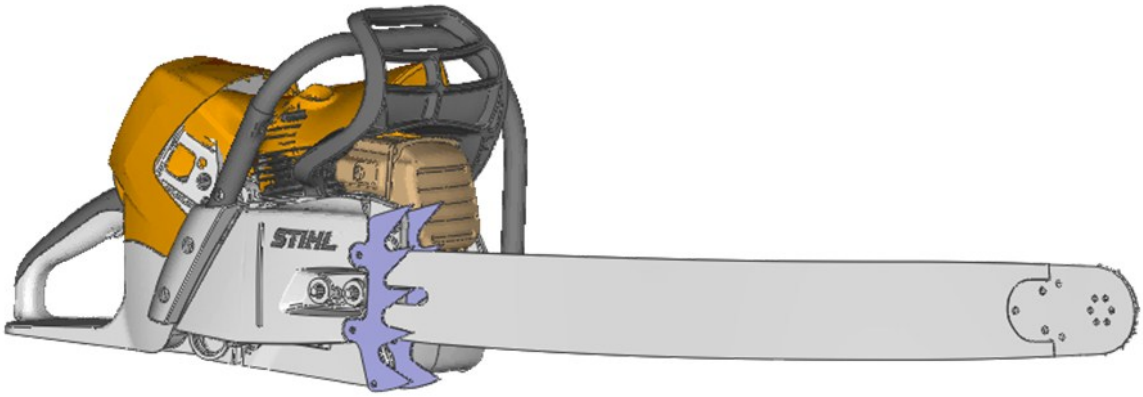


Figure 4.2 Preprocessed model

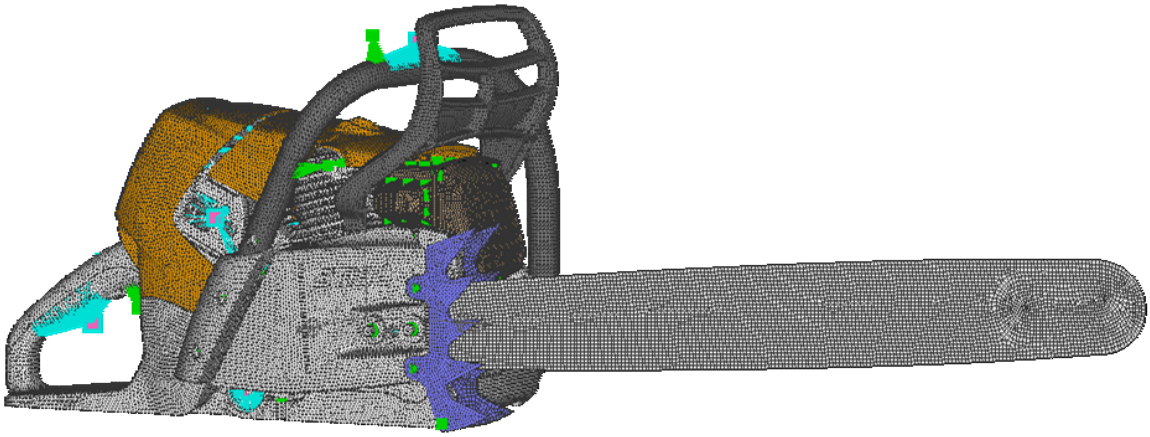


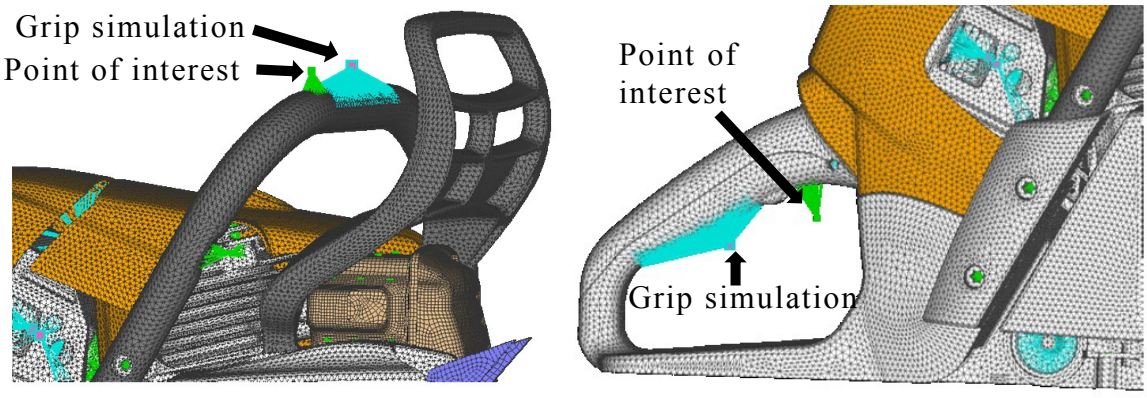
Table 4.1 Mesh quantities

	Number
Nodes	3167518
Tetra 10 elements	1774738
Hexa Elements	11167
Penta Elements	1630
RBE 2	71
RBE 3	42

4.1.2. Points of interest

Our concern is to calculate the vibration on the handles, while the chainsaw is being used, i.e. is held by user and the engine is running. Simulation should be in accordance with the reality, therefore we simulate the holding of the machine by adding an additional RBE bonds. On the tops of these RBE bonds are sets of springs and masses which simulate hand, forearm and arm. The configuration is typically in accordance with DIN45677 [24]. Experience shows, that such simulated holding of the machine is in good accordance with reality. ISO norm [1] specifies points on the handles which have to satisfy vibration limits - we would called them points of interest. These points are on the handles next to the spots used for holding of the machine. Therefore we add another RBE bonds next to the spots used for holding. On their tops are points evaluated in the vibration analysis- points of interest. Positions of the RBE bonds are shown in **Figure 4.3**.

Figure 4.3 Points for vibration evaluating

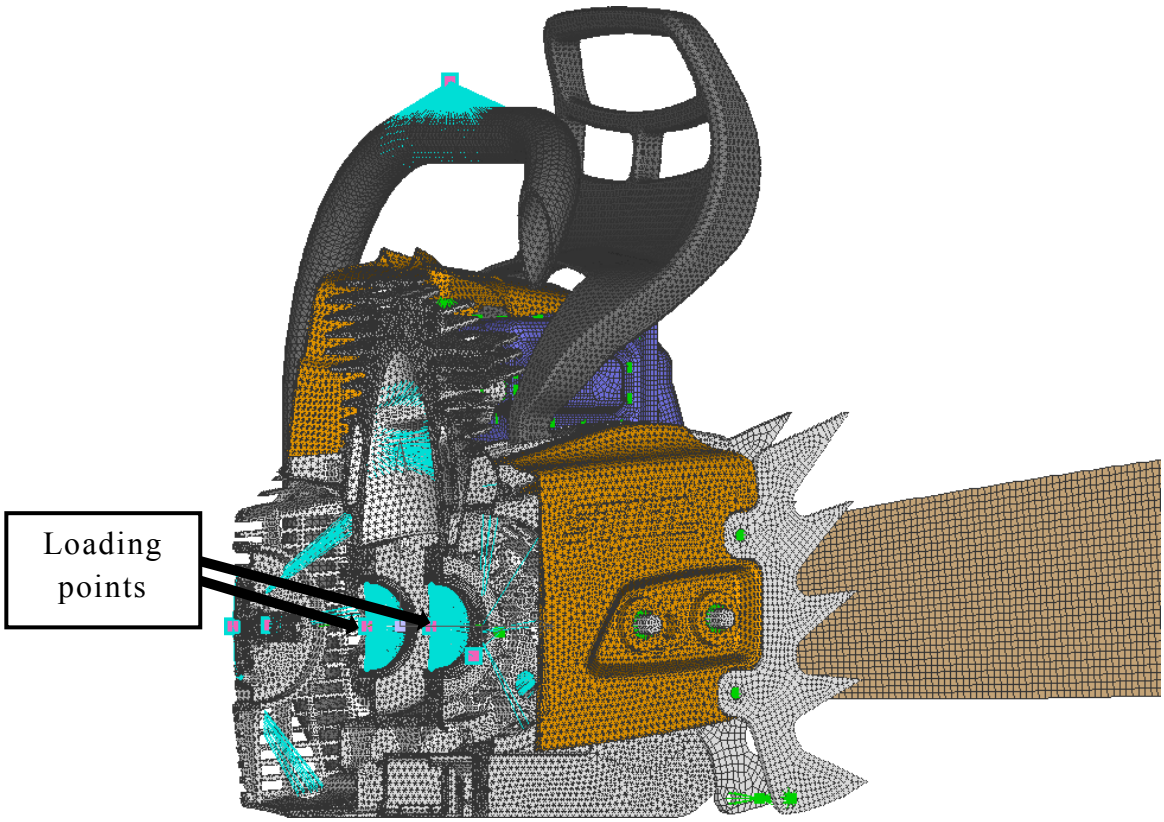


4.1.3. Input text files

In addition to the preprocessed model stored in binary input files, PERMAS requires additional text files. These text files contain material data and information about loading.

Material data text file was created in accordance with PERMAS nomenclature and contains material data for every part of the model. Loading due to the crankshaft revolving is stored in another text file. This loading was calculated externally. STIHL intern software called PANAMA [25] was used for this calculation. This software uses crankshaft geometry and gas pressure in combustion chamber as inputs and analytically calculates loading on the crankshaft bearings. These bearings are simulated by the RBE bonds and their positions are shown in **Figure 4.4**. Loading calculated by PANAMA is decomposed to the x, y, z directions and decomposed to orders with the aid of a discrete Fourier transformation. Such decomposition is required for the modal participation factors calculation, see section 3.2.8. Part of the loading text file calculated by PANAMA and text file with material data are included in the appendix.

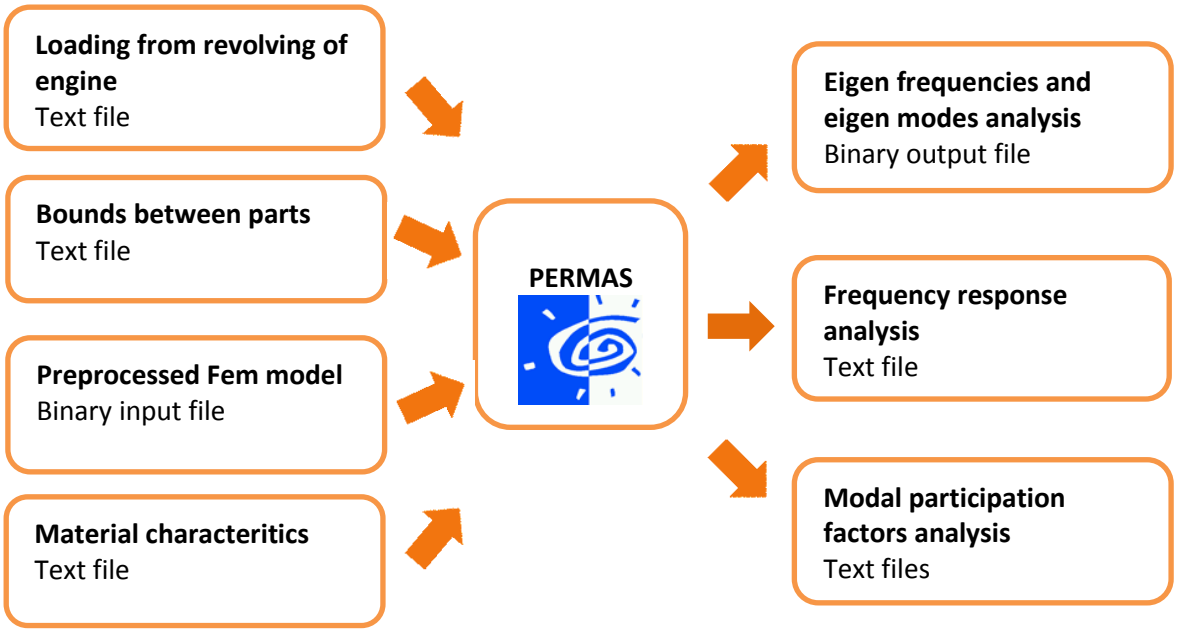
Figure 4.4 Loading points



4.2. Computing in PERMAS

In the previous section, all inputs required by PERMAS for the vibration computation were described. In this section analyses performed by PERMAS and their results are described. Both methods- standard STIHL and newly developed method, uses PERMAS for vibration computations. The newly developed method additionally auses a program called Participation Magic programed in Matlab. Analyses required (by standard or new method) are – Frequency response analysis (FRA), Modal participation factor analysis and Eigen frequencies and eigen modes analysis (EFEMA). More detailed description of the individual analyses follows. Scheme of PERMAS inputs and outputs is shown in **Figure 4.5**

Figure 4.5 Inputs and outputs scheme of computing performed by PERMAS



4.2.1. Eigen frequencies and eigen modes analysis

Eigen frequencies and eigen modes analysis performed in PERMAS solves the free vibrations problem, which was theoretically described in section 3.2.6. Results of the free vibration analysis are eigen frequencies and normal unit eigen modes, that are stored in the binary output file. These results are readable by post processing software, where they can be animated. As was derived in theoretical section (see 3.2.6. and 3.2.7.), the number of eigen frequencies depends on the size of the stiffness matrix and the mass matrix. For an approximation of the machines dynamic behaviour it is not necessary to compute all the eigen frequencies, therefore a frequency limit for the calculation has to be chosen. Usually, an upper limit for eigen frequencies is prescribed. Upper limit means that PERMAS

calculates only the eigen modes with eigen frequency lower then this limit. INTES recommend the use the following relation for calculating frequency limit [13]

$$f_{lim} = f_{elim} \times Nr_{ord} \times 2{,}5,$$

where f_{lim} is frequency limit for eigen frequencies, f_{elim} is engine frequency limit and Nr_{ord} means number of orders used in discrete Fourier transformation to decompose the excitation forces (see section 3.2.4. or section 4.1.3.).

For the test model, the frequency limit has been set to

$$f_{lim} = 270 \times 3 \times 2{,}5 = 2025Hz.$$

For the test model 254 eigen frequencies have been found in this frequency limit. Further 254 normal unit eigen modes belonging to the computed eigen frequencies were calculated. In **Figure 4.6** a few illustration pictures of the computed normal eigen modes computed is shown. The visualisation of these modes was done in postprocessing program MEDINA [14], [15] .

Note: Normal unit eigen modes have no size. They show only the ratio of the deformation in the model, i.e. colours in **Figure 4.6** are only for better visualisation and no scale is included.

4.2.2. Frequency response analysis (FRA)

FRA performed by PERMAS computes maximal acceleration, which occurs as a response to the external excitation in one selected point for the whole range of engine speeds. The result of this analysis is a vibration curve stored in a text file. Calculating frequency response was theoretically derived in section 3.2.7. The **Figure 4.7** shows the vibration curve calculated for the test model.

Figure 4.6 Normal eigen modes examples

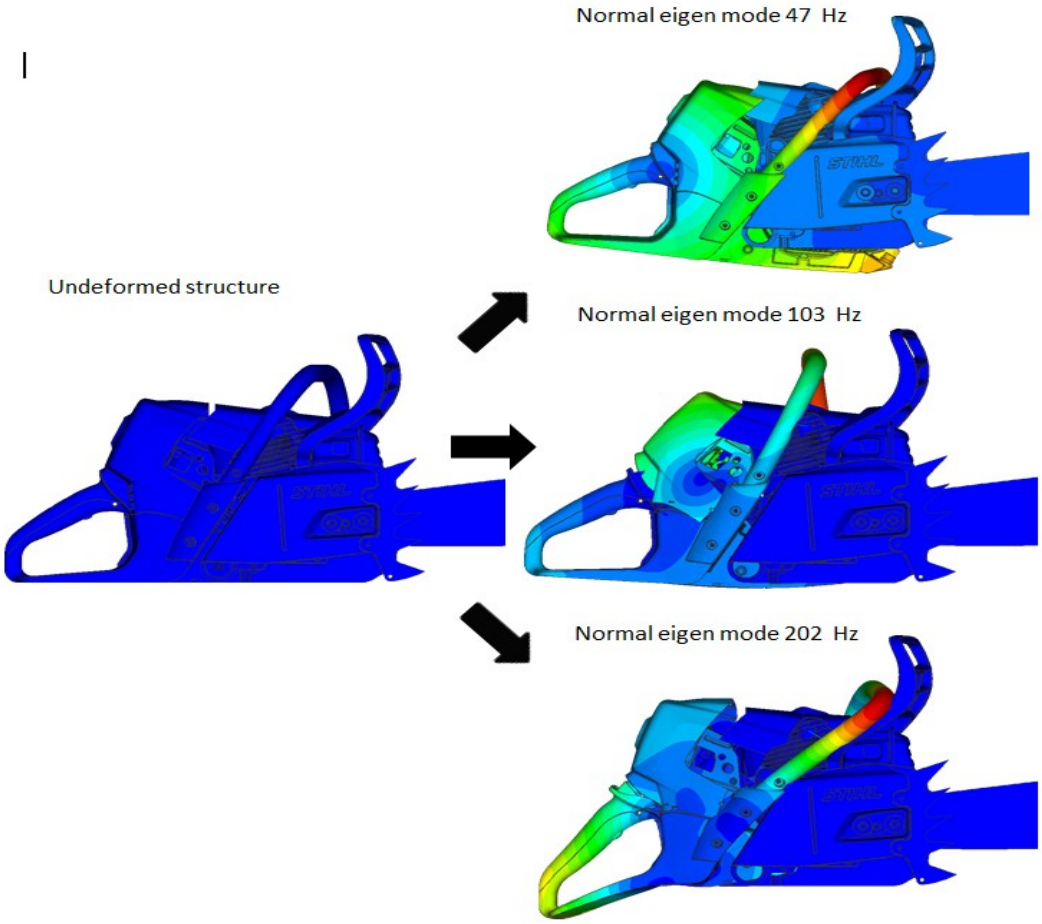
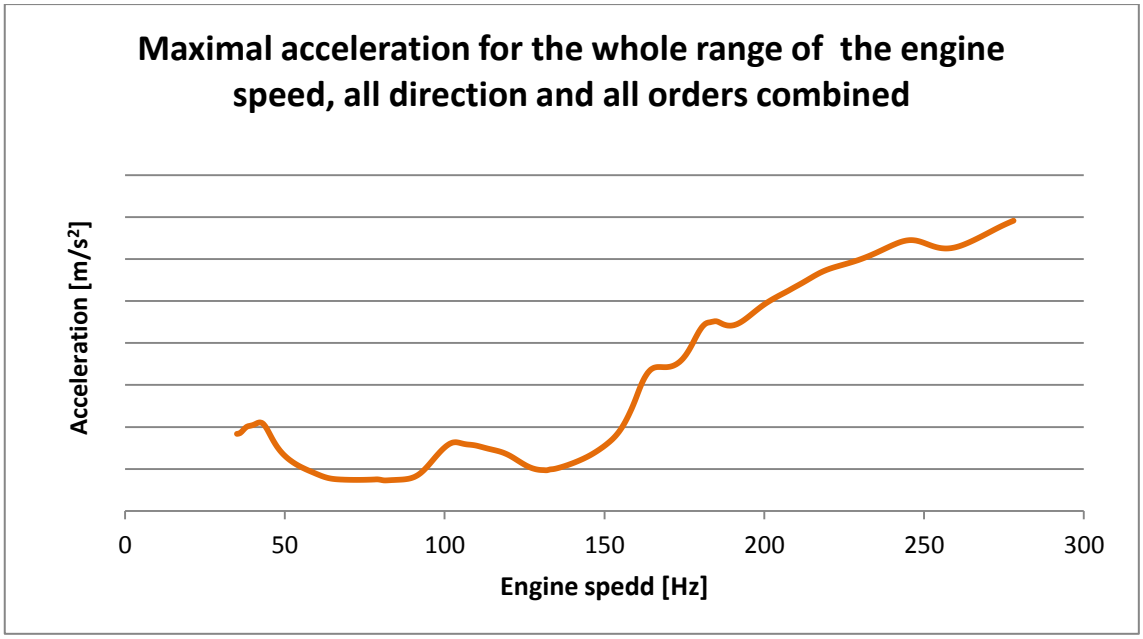


Figure 4.7 Frequency response curve



Note: The scale of vertical axis in **Figure 4.7**. has been deleted due to the protection of data owned by Andreas STIHL AG & Co. KG

Curve in **Figure 4.7** shows the absolute values of the vibration calculated by PERMAS on the test model. It is known that different frequencies of the vibration are perceived by the human body with different intensity. Usually the low frequencies between 5Hz and 30Hz are the most uncomfortable for human body, whereas people are not so sensitive to higher frequencies. For assessing of the vibration intensity, according to ISO norm [1], it is usual to apply a rescale function on the absolute vibration curve. Using the scaling function we obtain so called human sensitivity value curve (HSV).

Vibration curve recalculated to HSV

Norm ISO 5349-1 [1] specifies the conversion from absolute values to HSV values. HSV scale function in logarithmic coordinates is displayed in **Figure 4.8**. The graph shows that scale-value is always less than 1. Result vibration recalculated according the ISO 5349-1 is therefore always lower than absolute values. Figure **Figure 4.9** shows the frequency response curve for the test model recalculated to the HSV scale. For more information about recalculating to HSV see [1].

Figure 4.8 AHV curve

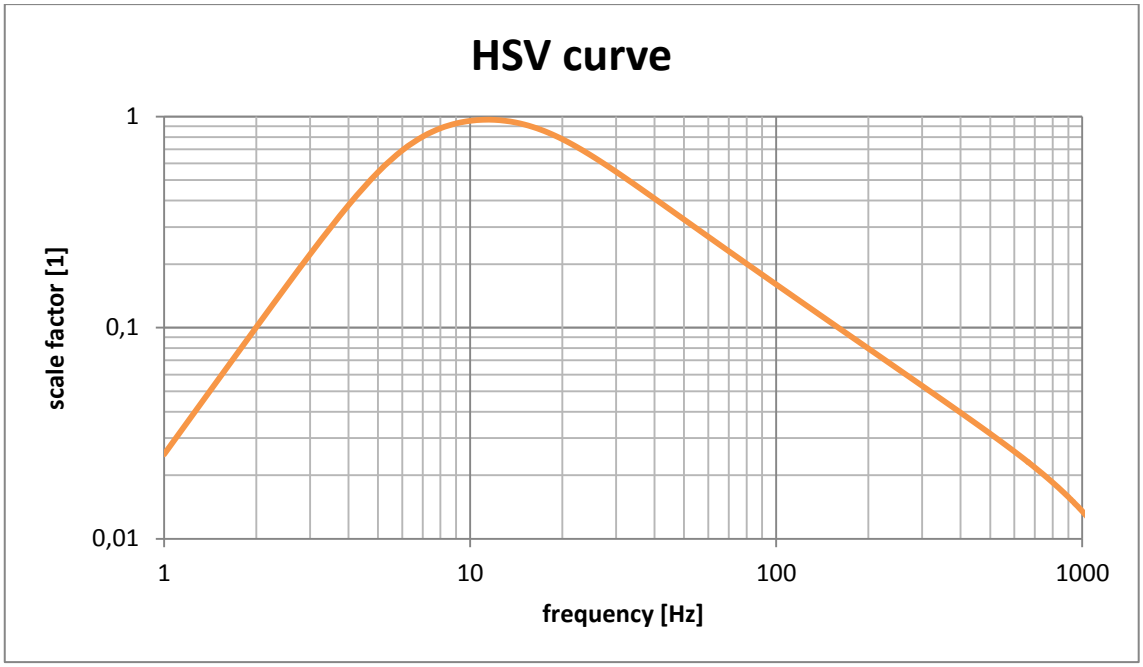
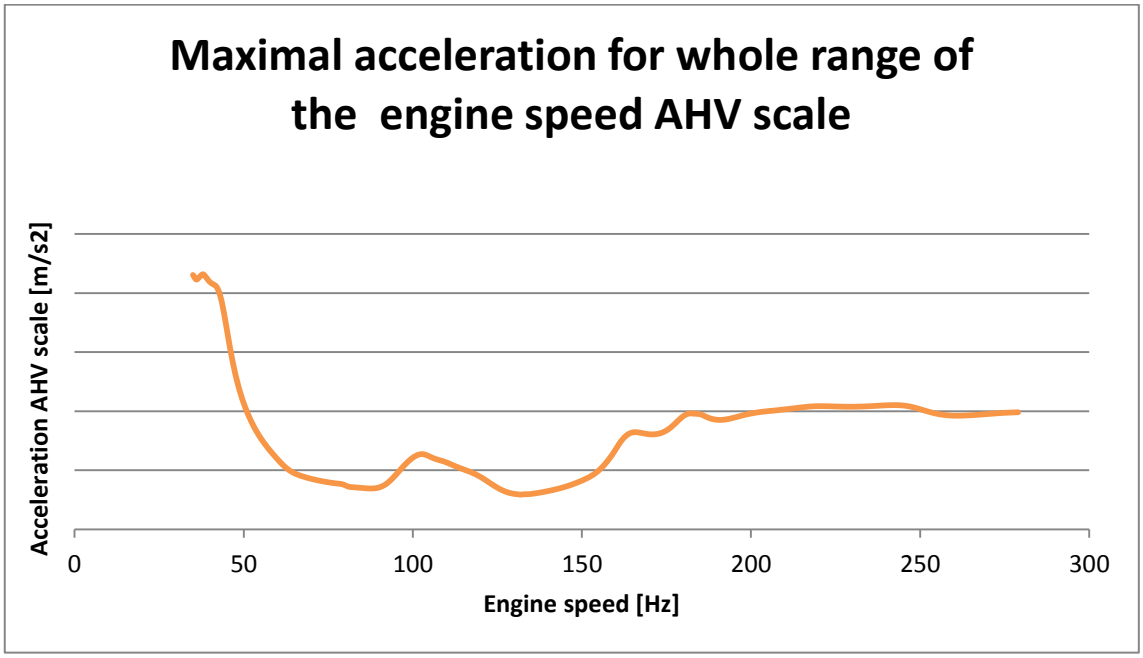


Figure 4.9 Frequency response curve, rescaled into AHV



Note: Due to data protection, owned by Andreas STIHL AG & Co. KG, was deleted scale of vertical axis in **Figure 4.9**.

4.2.3. Modal participation factors analysis (MPFA)

MPFA performed by PERMAS calculates modal participation factors within the frequency limit (see section 4.2.1.) for the whole range of engine speed. MPFA uses modal decomposition, which was theoretically described in section 3.2.7. Output from the MPFA is a set of text files containing modal participation factors for the individual eigen frequencies. Modal participation factors are written in output text file as complex numbers (see section 3.2.7.).

Output file contains one modal participation factor (one complex number, which represents one sine curve) for every combination of eigen frequency, order, direction, point of interest and engine frequency. The total number of modal participation factors is

$$Nr_{MPF} = Nr_{dir} \times Nr_{ord} \times Nr_{poi} \times Nr_{ess} \times Nr_{ef},$$

where Nr_{MPF} means number of modals participation factors, Nr_{dir} means number of, directions, Nr_{ord} means number of orders, Nr_{poi} means number of points of interest, Nr_{ess} means number of engine speed steps and Nr_{ef} means number of eigen frequencies calculated in the chosen frequency limit.

Number of calculated modal participation factors for the test model was

$$Nr_{MPF} = 3 \times 3 \times 2 \times 256 \times 265 = 814\,080.$$

In the newly developed method, all these modal participation factors have to be analysed to approximately reconstruct the vibration curve using reduced order modelling. It is obvious, that such a number may not be analysed manually. It is difficult to even open the output files with a text editor. For this task, new software called Participation Magic was created. Before we start with the reduced order modelling description, let us describe how to reconstruct a vibration curve from all calculated eigen modes. For a better understanding of how the vibration curve reconstruction works, it is good to recalculate modal participation factor from the complex numbers to sine curves.

Recalculation from modal participation factor to sin curve

As already mentioned, every modal participation factor calculated by PERMAS is a complex number, which can be represented by a vector in complex plane or a sine curve. For some applications it is better to have the modal participation factors represented by vectors, for some application it is better to work with sine curves. Description of the recalculation from vector shape to sine curve follows.

Shape of a modal participation factor obtained from PERMAS is

$$z = pi + q, \tag{4.1}$$

where $p, q \in R$.

General sine curve is described by equation

$$y = A \sin\,(ft + \varphi), \tag{4.2}$$

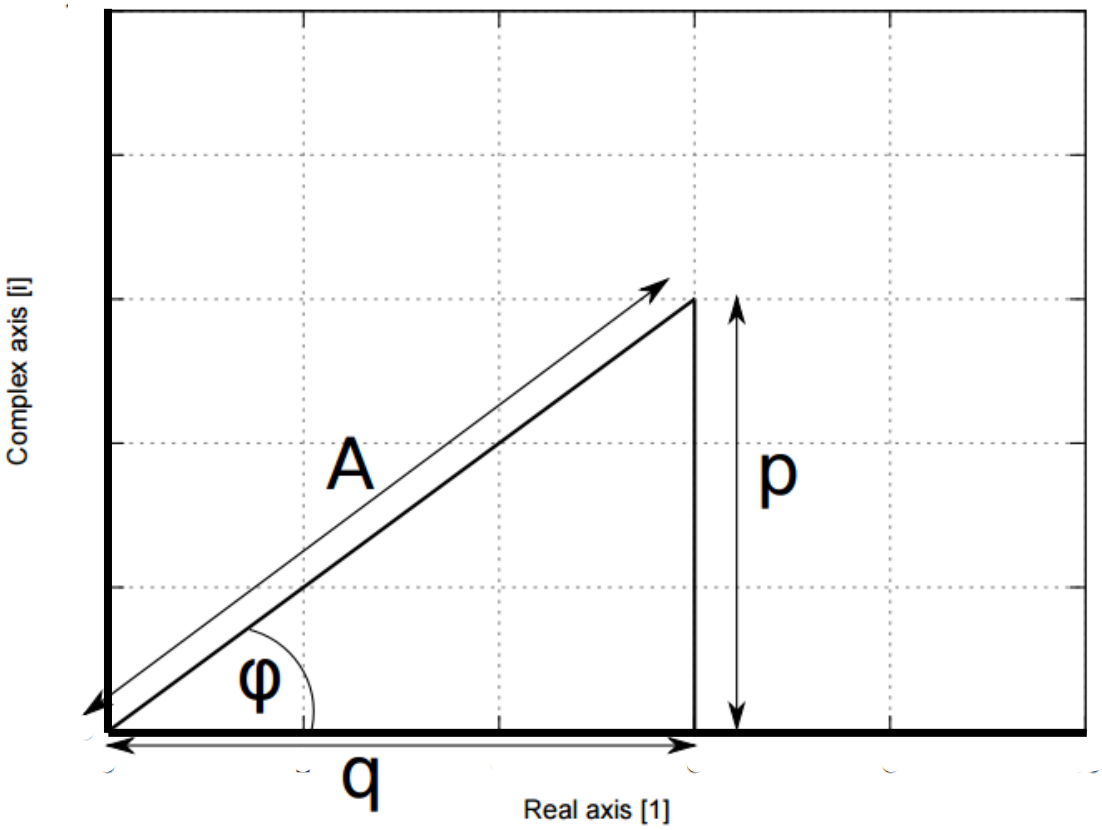
where f represents excitation frequency, i.e. engine frequency in the actual situation and coefficients A, φ can be calluclated from modal participation factors

$$A = \sqrt{p^2 + q^2}, \tag{4.3}$$

$$\varphi = \operatorname{atan}\left(\frac{p}{q}\right). \tag{4.4}$$

Relations (4.3) and (4.4) are obvious from **Figure 4.10.**

Figure 4.10 Recalculation of the MPF from vector in complex plane to sine curve



Vibration curve reconstruction from modal participation factors

In this section it is described how to reconstruct vibration curve from modal participation factors, using all of them. As already mentioned, PERMAS calculated one modal participation factor for every combination of eigen frequency, order, direction, point of interest and engine frequency. Now let us select a fixed engine speed, direction, point and order- we will call this a situation. For every situation we get a number of modal participation factors equal the number of calculated eigen frequencies.

When we add up all the modal participation factors in one situation and recalculate the result into a sine curve, we get one sine curve that represents a waveform of acceleration over one revolution of crankshaft in the respective situation.

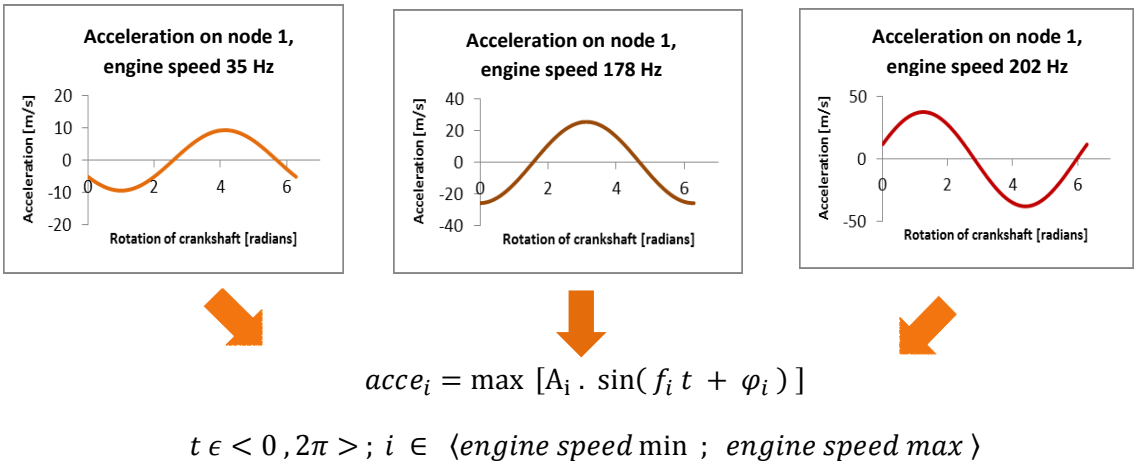
$$z = \sum_j p_j i + q_j \qquad j \in <1;n> \tag{4.5}$$

$$z \rightarrow A\sin(ft + \varphi) \tag{4.6}$$

Our concern now is to obtain a curve of maximal acceleration for the whole range of engine speed for one order and one direction. Therefore, in the next step we take maximal value from each of the previously computed sine

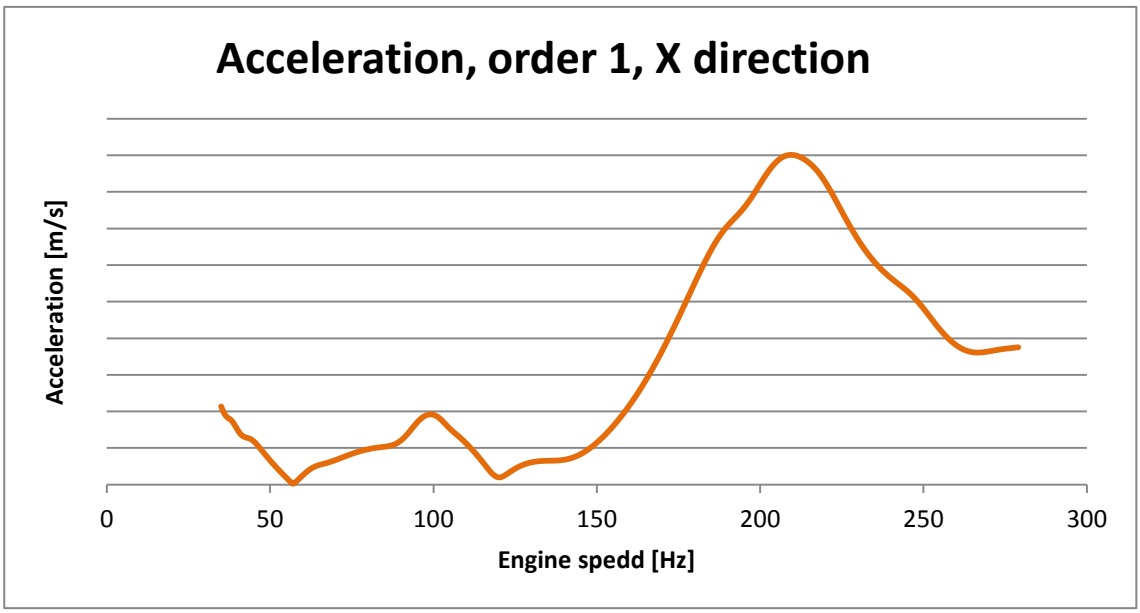
curves and plot it into the engine frequency/ acceleration graph. Result is a frequency response curve for one order and one direction. Described procedure is shown in **Figure 4.11**.

Figure 4.11 Summation of the MPF in one situation



$$acce_i = \max\,[\mathrm{A_i}\,.\,\sin(f_i\,t\,+\,\varphi_i)\,]$$

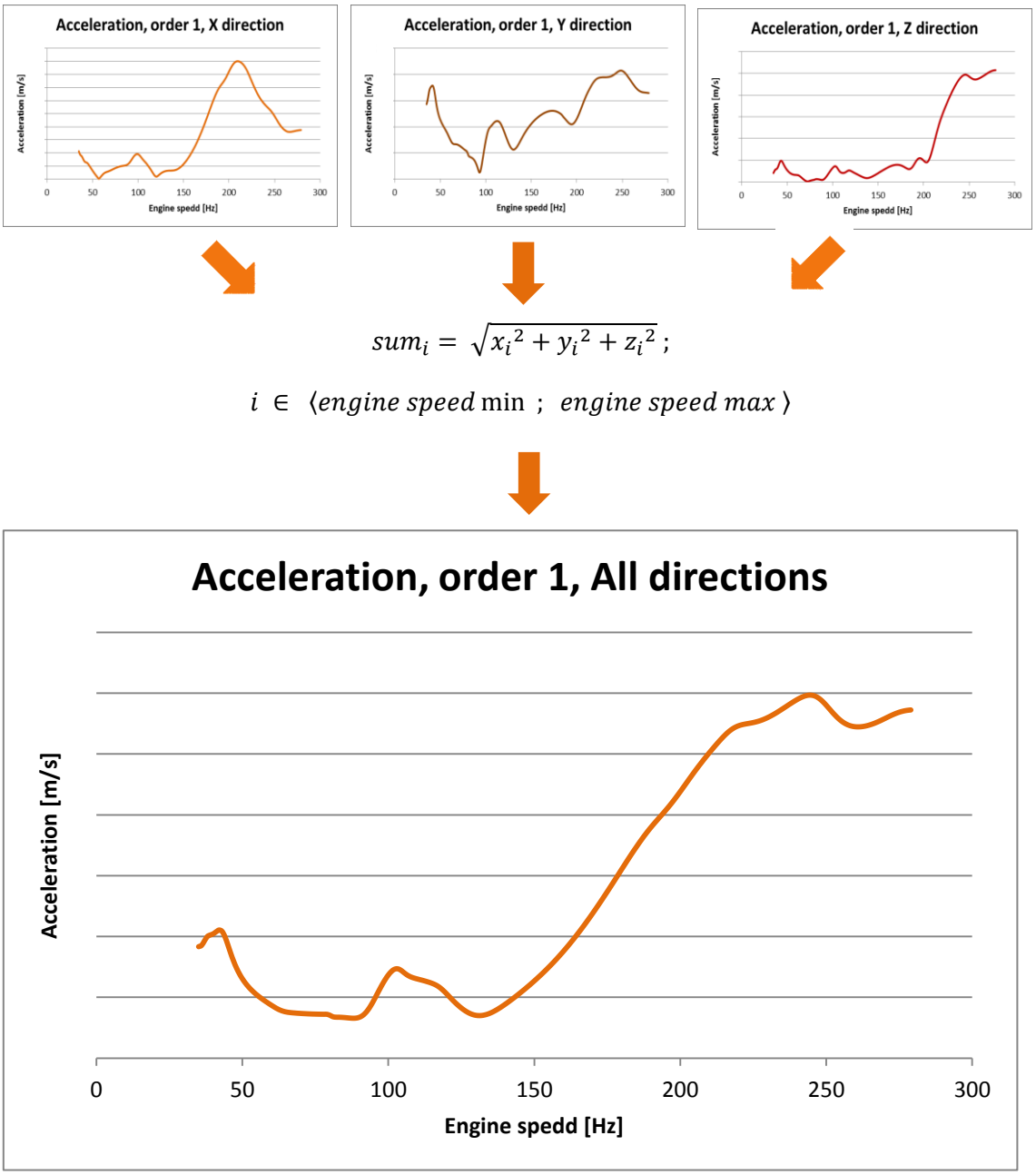
$$t\,\epsilon < 0\,, 2\pi >;\; i\,\in\,\langle engine\ speed\ \min\;;\; engine\ speed\ max\,\rangle$$



Note: Due to data protection, owned by Andreas STIHL Ag & Co. Kg, was deleted scale of vertical axis in **Figure 4.11**

We can compute the frequency response curve, as described above, for every direction. Sum of these curves represents the acceleration in all directions. For the summation the vector summation rule was used, according to [1]. This is also in accordance with the way in which the experiments are evaluated. Again scheme of the summation is shown in **Figure 4.12**.

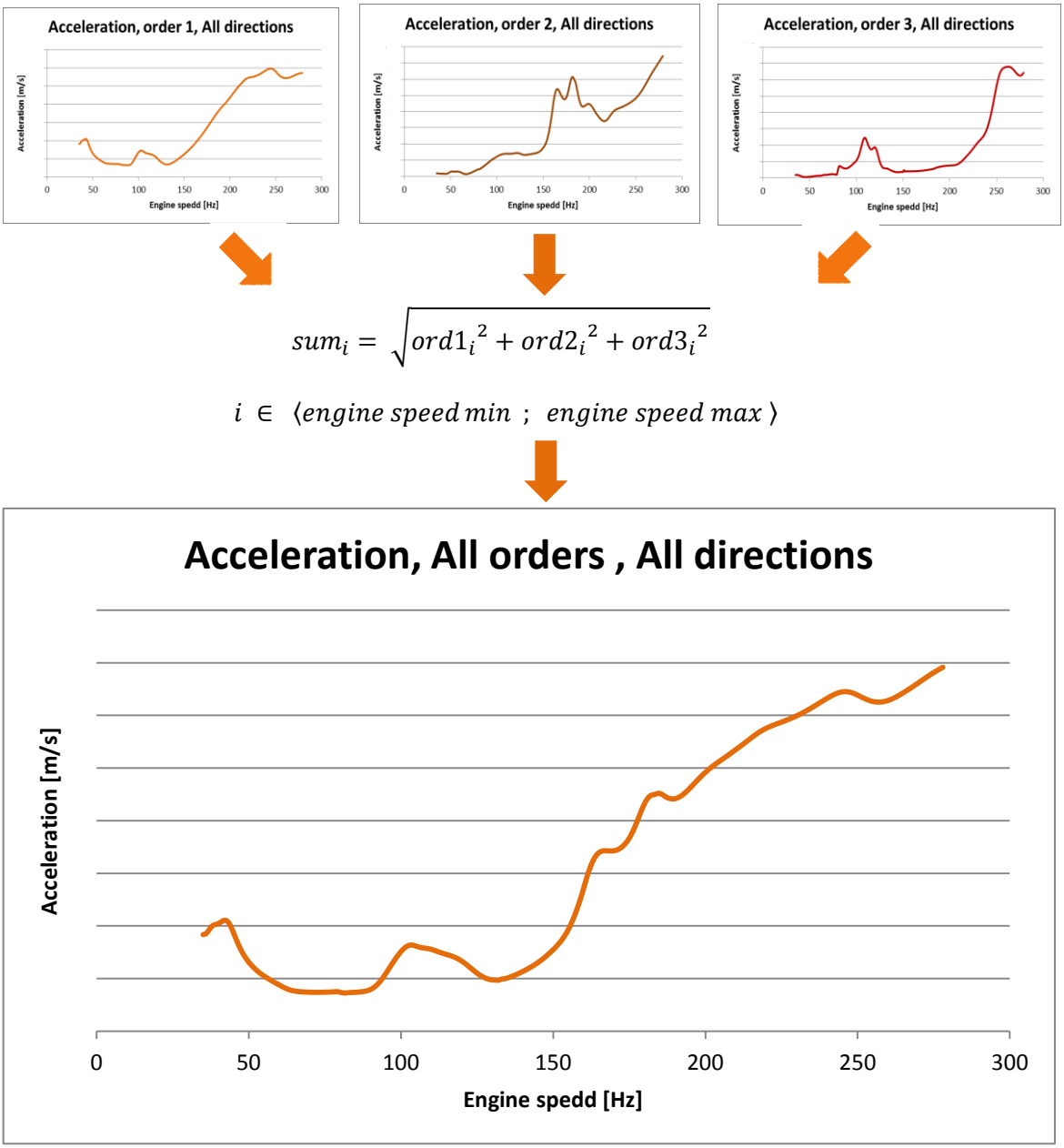
Figure 4.12 Summation frequency response curves over directions



Note: Due to data protection, owned by Andreas STIHL AG & Co. KG, was deleted scale of vertical axis in **Figure 4.12**.

Same summation rule can be used for the orders to get the complete frequency response curve over all orders and directions. For summation the vector summation rule was used again, according to [1] Scheme of the summation is shown in **Figure 4.13**.

Figure 4.13 Summation frequency response curves over orders



Note: Due to data protection, owned by Andreas STIHL AG & Co. KG, was deleted scale of vertical axis in **Figure 4.13**.

4.2.4. Reduce order modelling

The procedure described in the section 4.2.3. allows the calculation of the frequency response curve, which is exactly the same as the frequency response curve obtained from the FRA (see section 4.2.2.). Our concern now is to find the dominant eigen modes and approximately reconstruct the frequency response curve using only these dominant eigen modes. Reconstruction of the frequency response curve using only the modes with very high amplitudes proved to be a wrong way. Frequency response curve created only from the modes with a high amplitude was totally different

than the frequency response curve calculated from all modes. Reason of this high inaccuracy is simple, the modal participation factors contain two pieces of information- the amplitude and the phase shift. This means, that two similar modes with a very high amplitude and a phase shifted by 180 degrees may almost completely cancel each other out. When added together the influence of such modes on the vibration curve is negligible. Experience shows, that these cases are very common and eigen modes with the highest amplitudes are rarely impacting the result the most. Therefore, in the newly developed method, another approach to the reduce order modelling was chosen. In the first step, the frequency response curve is calculated using all eigen modes. Then the frequency response curve is computed again, but some eigen modes are neglected. The difference between these two curves is calculated and converted to a percentage format and displayed in a plot. Using this process repeatedly, we can obtain a plot that contains information about the impact of different eigen modes to the total result.

The total number of combinations for such procedure is unbearably high. For example, the test model has 254 eigen modes within the given frequency limit. This means that the total number of combination is

$$Nr_{com} = \sum_{n=1}^{254} \frac{254!}{(254-n)!} \approx 10^{504}. \tag{4.7}$$

This is of course incredibly high number and therefore some simplifications are needed. The finding of high impact modes is in the newly developed method performed by Participation Magic. The description of used simplification is described in the section 4.3.

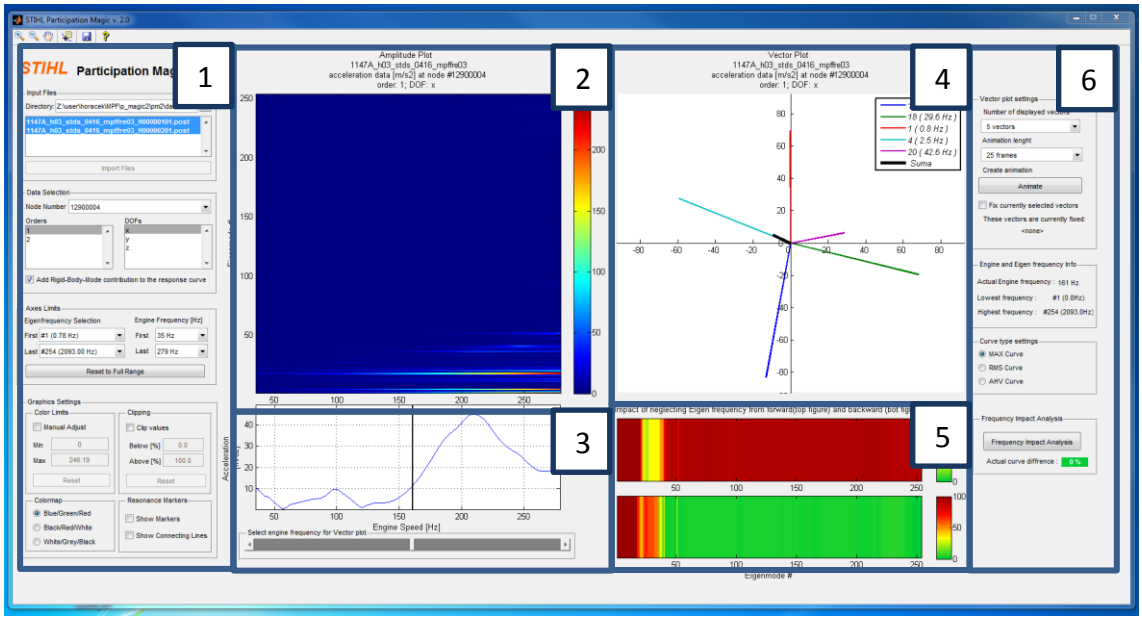
4.3. Participation Magic

Participation Magic is a tool, programed in Matlab GUI interface, developed for the postprocessor analysis of the modal participation factors. It allows us to plot the modal participation factors, reconstruct the frequency response curve from the modal participation factors and most importantly can be used to analyse the influence of the modal participation factors and the corresponding eigen frequencies to the reconstructed vibration curve. Identifying of the dominant unit eigen modes provides an important information, which can help the engineer responsible for the structure design to take measures exactly aimed to the damping of those vibrations.

4.3.1. Participation Magic interface

The Participation Magic interface is designed to be user friendly. It consist of two user controls panels **Figure 4.14-(1)** and **Figure 4.14-(4)** and four plots- Amplitude plot **Figure 4.14-(2)**, Frequency Response plot **Figure 4.14-(3)**, Vector plot **Figure 4.14-(4)** and frequency impact plot **Figure 4.14-(5).** Full Participation Magic documentation is in the appendix, here only the main functions of the program, plot description and basic interface description are shown.

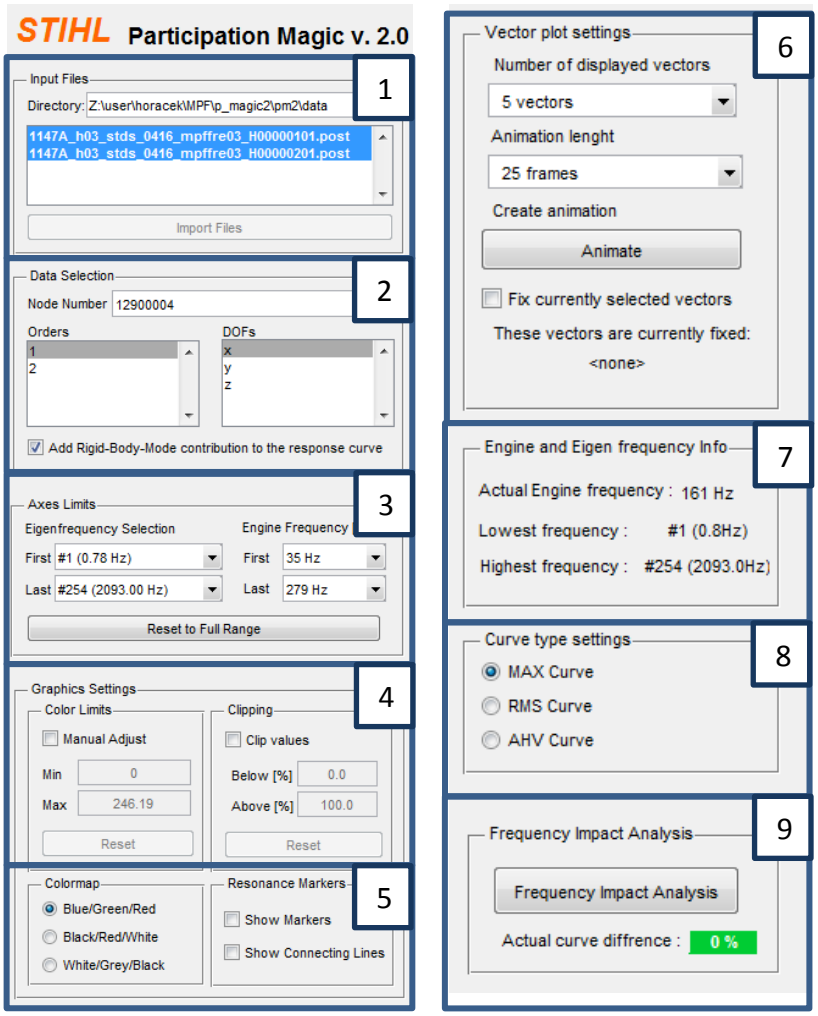
Figure 4.14 Participation magic interface



4.3.6. Participation Magic user control panels

Main function of the Participation Magic is to postprocess and display the modal participation factors obtained from PERMAS. User controls allows us to read PERMAS data files - **Figure 4.15-(1)** and configure plots **Figure 4.15-(2-9).** Participation Magic strictly requests PERMAS data format. Panels with plot settings, **Figure 4.15-(2-9)**, are used for selecting data displayed in plots and for their graphics customization. For closer description of the control panels see Participation Magic documentation in the appendix.

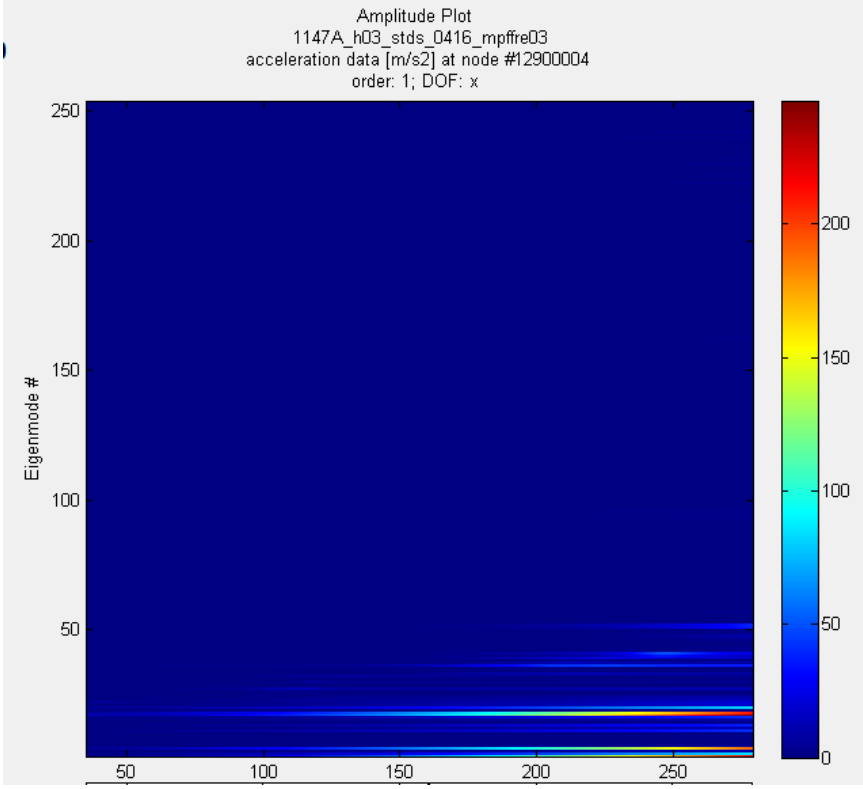
Figure 4.15 Participation Magic- user control Panels



4.3.2. Amplitudes of the modal participation factors plot

Amplitude plot is a tool that gives the user a general overview of the influence of the different eigen modes to the total result (i.e. influence to vibration curve). It displays amplitudes of the modal participation factors, defined by equation (4.3), as a function of the engine frequency and the eigen frequency. The vertical axis represents the number of the given eigen mode, the horizontal axis represents the engine frequency. Amplitude values are represented by colours. Every point in the graph represents one amplitude of one modal participation factor. From this plot we can easily see, which amplitudes are the highest. However, the eigen modes with high amplitudes are actually rarely impacting result the most. Experience shows that there is only a low number of the eigen modes that significantly affect the final result. Other tools in Participation Magic are aimed to finding these dominant eigen modes. Amplitude graph for the test model is in the **Figure 4.16**.

Figure 4.16 Amplitude plot in Participation Magic



4.3.3. Frequency response plot

The newly developed methodology is based on the vibration curve reconstruction from the modal participation factors. Therefore obviously, one of the basic functions of the Participation Magic is to compute and display the vibration curve from the modal participation factors. Example of the vibration curve calculated in Participation Magic is in the **Figure 4.17**. Participation Magic uses the procedure described in the section 4.2.3. to calculate the frequency response curve from the modal participation factors. This allows us to compute frequency response curve for any direction, any order or any combination- for example it is possible to calculate a vibration curve for the first and second order in z direction. Participation Magic also allows us to recalculate the vibration curve from absolute values into HSV values (see section 4.2.2.). Another advantage of this tool is that it allows us to compute the frequency response curve only from a certain section of the eigen modes and compare it with the curve computed from all modal participation factors. This function allows a much deeper insight into the inner behaviour of the structure. Example of a comparison of two curves performed in Participation Magic is in the **Figure 4.15** Switching between directions allows us to find for example dominant direction or order of vibration (if such exists).

Figure 4.17 Frequency response plot in Participation Magic

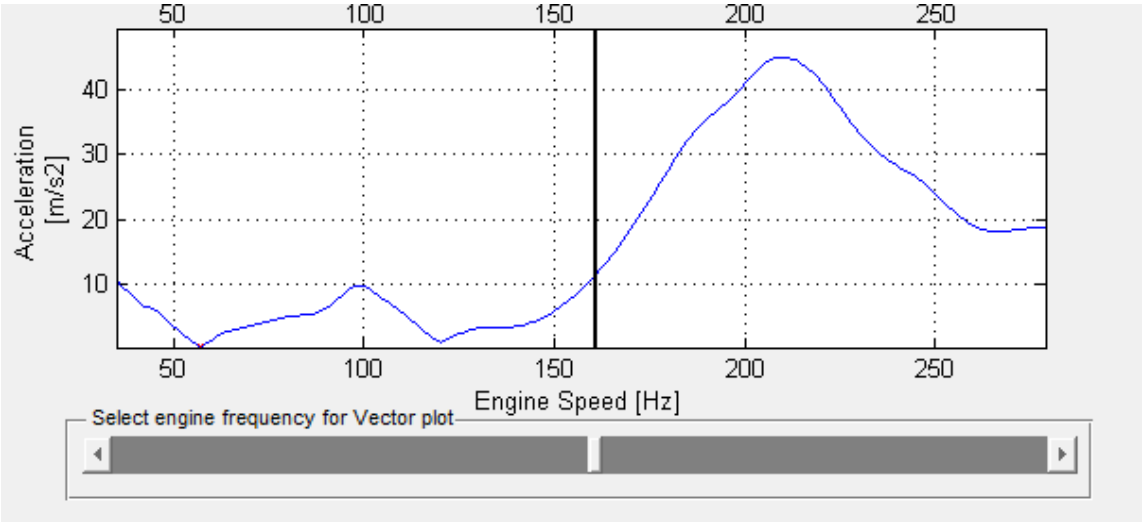
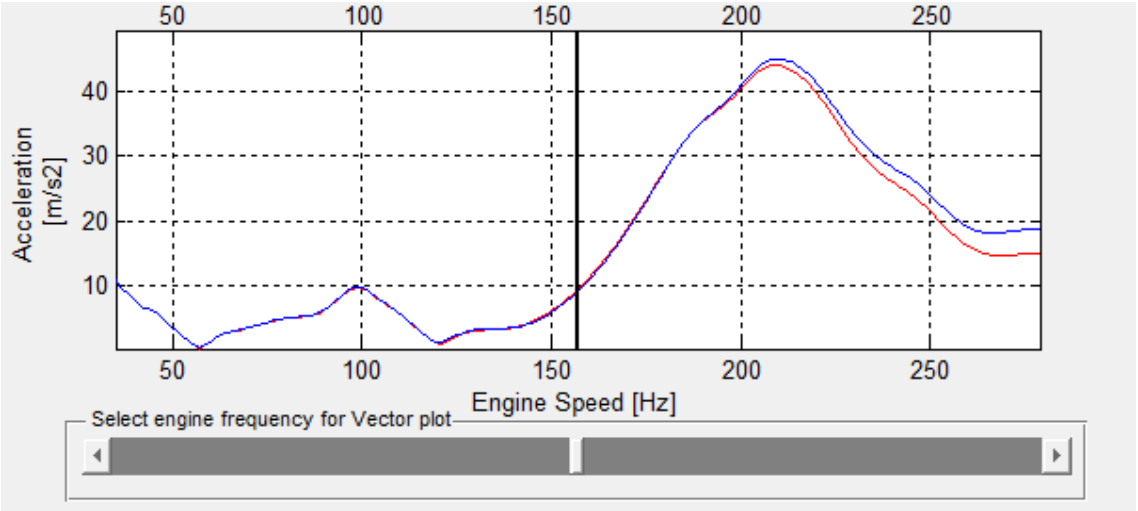


Figure 4.18 Frequency response curves comparison



In the **Figure 4.18** two frequency response curves calculated for the test model are compared. The blue curve is computed using a full spectrum of eigen frequencies and corresponding eigen modes, the red one neglect all eigen frequencies higher than 1500Hz. Is it obvious that impact of the neglected eigen frequencies is negligible, especially in the area of the low frequencies that are most important (see the chapter about AHV 4.2.2.).

4.3.4. Frequency impact analysis plot

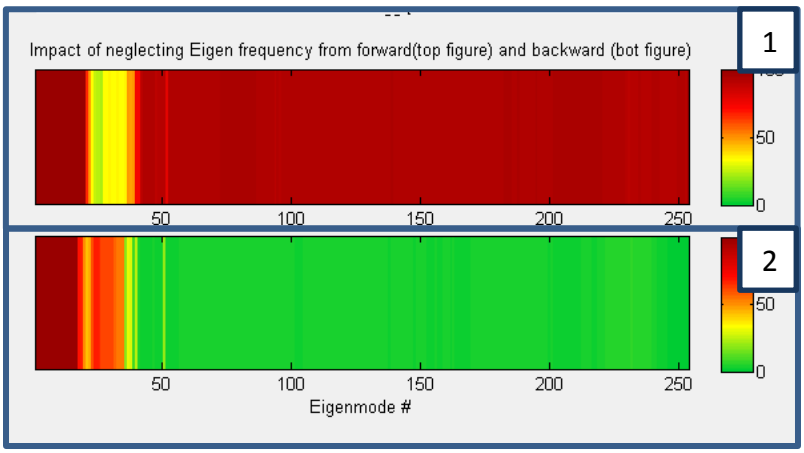
Frequency impact analysis is a tool of Participation Magic, which helps the user to find the dominant eigen modes. The dominant eigen modes are such modes, that create major part of the vibration curve, i.e. our concern is to find a low number of eigen modes and approximately reconstruct the vibration curve using only these modes. As already mentioned in section 4.2.4. a vibration curve reconstruction using only the modes with the highest amplitudes proved to be a wrong way. Therefore, Participation Magic uses another approach to find the dominant eigen modes. First, the vibration curve is calculated using all eigen modes. Then the vibration curve is computed again, but some eigen modes are neglected. The difference between these two curves is calculated and converted to the percentage format and displayed in a plot. Using this process repeatedly we can obtain a plot that contains information about the impact of the different eigen modes to the total result. Due to the incredibly high number of combinations, it is impossible to calculate all of them. Therefore, Participation Magic uses the following simplification.

First, the vibration curve is calculated using all eigen modes. Then is vibration curve calculated again but the first eigen frequency is neglect. In the next step, the difference between these two curves is calculated, converted to percentage format and displayed in **Figure 4.19-(1)** as one vertical line. In the next loop, the first and the second eigen frequencies are neglected and so on. Result of this calculation for the test models is in the **Figure 4.19-(1).**

The same procedure is then applied in the opposite direction, i.e. in the first loop the last eigen frequency is neglected. In second loop the last frequency and the last but one eigen frequencies are neglected and so on. The result is again converted to the percentage format and displayed in plot. Result of this calculation for the test models is in the **Figure 4.19-(2).**

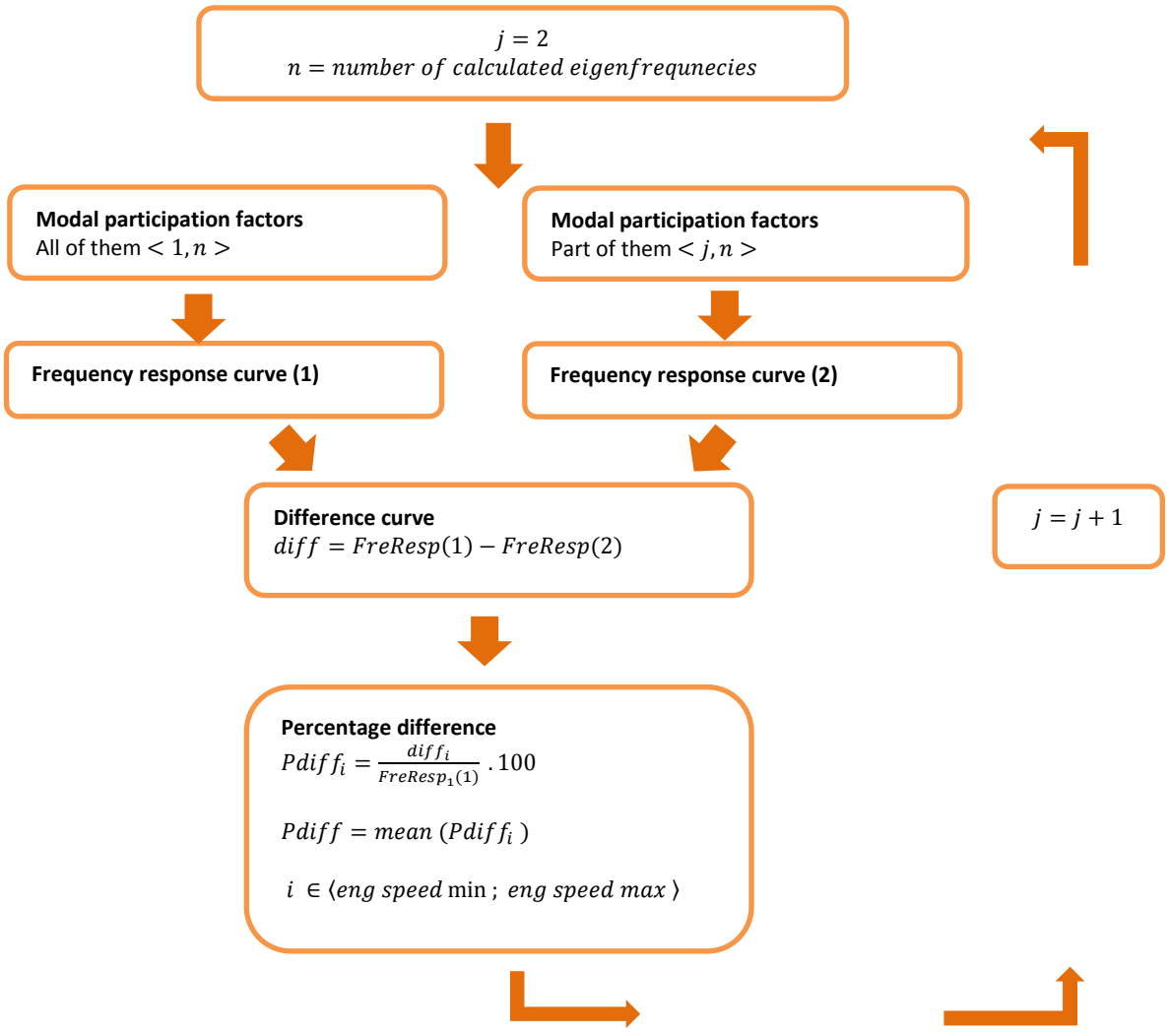
The frequency impact analysis plots help the user to make the correct eigen modes restrictions for the vibration curve reconstruction. For example in the test case displayed in **Figure 4.19-(1-2)** we can see that the frequency limit (see 4.2.1.) used for the calculation of eigen modes is unnecessarily high (large green area in the **Figure 4.19-(2)**). Neglecting the eigen modes above eigen mode #45 has almost zero influence on the total result. We can also see (green strip in the **Figure 4.19-(1)**, that neglecting the modes lower than #25 also has only very small influence on the total result. This means that, in the test case, we can approximately reconstruct the vibration curve using only eigen modes #25-45.

Figure 4.19 Frequency impact analysis plot in Participation Magic



In the **Figure 4.20** the scheme of creating frequency impact analysis plot in Participation Magic is once again shown.

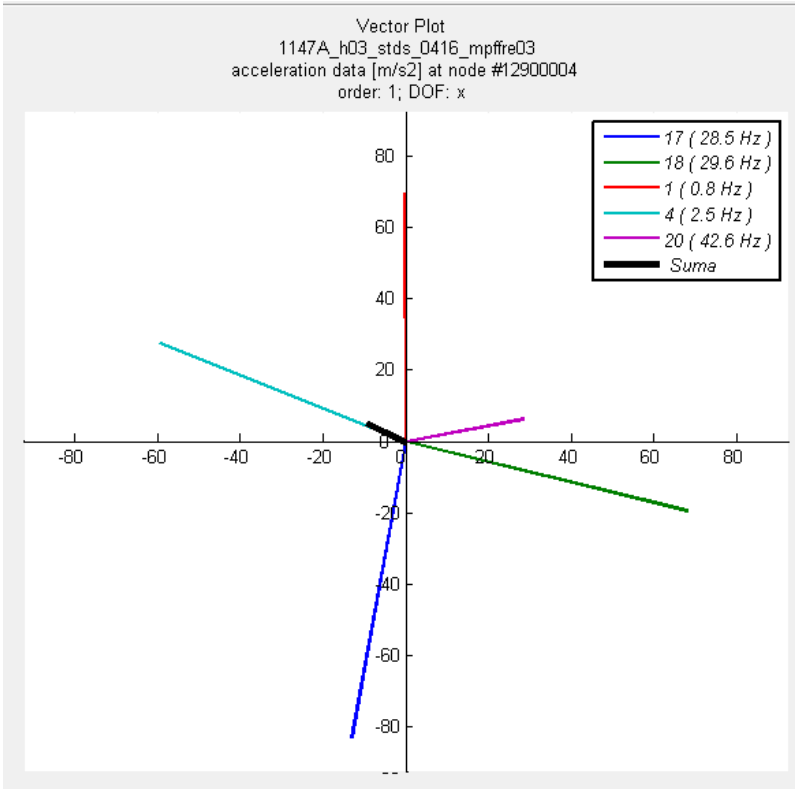
Figure 4.20 Frequency impact analysis computation scheme



4.3.5. Vector plot

Every modal participation factor consists of two pieces of information- amplitude and phase shift. Therefore, it is not possible to display both of these information for the whole range of the engine speed and for all eigen frequencies in one plot (4 dimensional plot would be needed- eigen frequency/engine frequency/ amplitude/ phase shift). However, if a fixed engine frequency is selected, it is possible to display the modal participation factors as vectors in a complex plane, i.e. for the selected engine speed the vector plot allows us to display both parts of modal participation factor- amplitude and phase shift. This is very useful, because it allows us to display relations between the eigen modes and allows to assess their impact on the final result. It is recommended to use this tool after using the Frequency impact analysis (see section 4.3.4.) to decreasing the number of the considered eigen modes. This plot can be displayed for any engine speed. The example of vector plot for the test model is in the **Figure 4.21**.

Figure 4.21 Vector plot in Participation Magic



4.3.5. Participation Magic license

The development of Participation magic was financed by Andreas STIHL AG & Co. KG and is intended for STIHL internal use only. For this reason, it was not possible to include the compiled program in the appendix.

5. Conclusion

The main goal of this diploma thesis, defined by the client - Andreas STIHL AG & Co. KG, was to improve the methodology standardly used by STIHL for calculation of vibration on the chainsaw handles that occurs as a response to the excitation from the crankshaft revolving. The standard methodology is based on the direct calculations with aid of the commerce finite element method solver called PERMAS. The result of this STIHL standard methodology is the vibration curve for the whole range of engine speed. This curve is used, according to ISO norm [1], to assess whether the calculated machine fulfills the vibration limits or not.

Within this thesis a new methodology was developed for vibration calculations. The newly developed methodology uses the reduced order modeling approach to approximately reconstruct the vibration curve from the modal participation factors and the unit normal eigen modes. During the creating of this new methodology, a complete theoretical background was derived. All relations needed for the calculation of the modal participation factors and unit eigen modes were described from the spatially discretized equations of the linear elasticity. For the space discretization, the finite element method was used. In praxis, the unit normal eigen modes and the modal participation factors were calculated by PERMAS (see [12], [13]). Further, the thesis contains the description of vibration curve reconstruction procedure from the modal participation factors and the unit normal eigen modes. The reconstruction of the vibration curve is performed by newly implemented software called Participation Magic. This software uses the unit normal eigen modes and the modal participation factors calculated in PERMAS as an input to reconstruct the vibration curve. Participation Magic also allows us to evaluate the influence of the individual modes and approximately reconstruct the vibration curve with only the dominant modes, using the so called reduced order model. An easy identification of the dominant unit eigen modes is the biggest advantage of the newly developed method because it provides an important information, which helps engineers responsible for the structure design to take measures exactly aimed at reducing those vibrations.

Within this thesis, the newly developed method was tested on a model of the chainsaw MS661, see [23]. Realization of the test computations also required the solving of technical issues during the model preprocessing in accordance with the internal STIHL standards (see section 4.1) or assessing the results in accordance with the ISO norm [1].

The main output of this diploma thesis is the complete methodology, which uses PERMAS and Participation Magic, for computing vibrations on the chainsaw handles. The method is universal and can be used for any hand held power equipment. Thanks to the user friendly graphic interface and

high level of automation in Participation Magic, it can be used without deeper understanding of the concepts of the reduced order modeling and modal decomposition. This should allow a quick and easy use by any customer. Currently (June 2015), the newly developed method and Participation Magic are being tested in simulations performed for STIHL.

6. Used marking overview

A_i- Amplitude of the modal participation factor

$\boldsymbol{C}$- Damping matrix

$\boldsymbol{C_m}$ - Matrix of material model

$\boldsymbol{C_L}$ - Isotropic material model matrix

$\mathbb{C}$ - Set of complex numbers

D- Discriminant

D_i - Main minors

$\boldsymbol{E}$ - Identity matrix

$\boldsymbol{f_s}$- Surface forces

$\boldsymbol{F}$ - Load vector

i - Complex unit

$\boldsymbol{H^k}$ - Hilbert's space

$\boldsymbol{K}$ - Stiffness matrix

$\boldsymbol{M}$ - Mass matrix

$\boldsymbol{n}$ - Outer normal vector

$\boldsymbol{N}$ - Set of natural numbers

$\boldsymbol{R}$ - Set of real numbers

t - Time

$\boldsymbol{u}$ - Displacement vector

$\boldsymbol{u_h}$ **-** Discretized displacement vector

$\boldsymbol{u_D}$ - Dirichlet boundary condition

$\boldsymbol{u_N}$ - Neumann boundary condition

$\boldsymbol{u_0}$ – Zero Dirichlet boundary condition

$\boldsymbol{v}$- Continues test function

$\boldsymbol{v_h}$ - Discrete test function

$\boldsymbol{V_f}$ - Space of continues test functions

$\boldsymbol{V_h}$ - Space of continues test functions

$\boldsymbol{V}$ - Eigen vector

$\boldsymbol{W^{k,p}}$ - Sobolev`s space

$\boldsymbol{x}$ **–** Spatial coordinates

z - Complex number

Γ_D - Domain with Dirichlet boundary condition

Γ_N - Domain with Neumann boundary condition

$\boldsymbol{\varepsilon}$ - Tensor of small deformation

$\varepsilon_1, \varepsilon_2$**-** Coefficient of the damping model

$\boldsymbol{\epsilon}$ - Tensor of finite deformation

μ - Lame constant

λ –Lame constant

$\boldsymbol{\Lambda}$ - Matrix of eigenvalues

ρ - Density

$\boldsymbol{\sigma}$ - stress tensor

φ - Phase shift of the modal participation factor

φ_i - Base functions

ω - Eigen frequency of non-damped system

ϖ - Excitation frequency

Ω - Eigen frequencies of damped system

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8. List of attachements

Enclosed CD contains following files

- Electronic version of the diploma Thesis (pdf)
- Participation Magic documentation (ppt)
- Example of the loading file used for computations in PERMAS (dat1)
- Example of the file with the model bonds (dat1)
- Part of the file containing modal participation factors computed by PERMAS (txt)

Content