On the Functional Relation Between Quality Factor and Fractional Bandwidth

Miloslav Capek, Lukas Jelinek, and Pavel Hazdra

***Abstract*—The functional relation between the fractional bandwidth and the quality factor of a radiating system is investigated in this communication. Several widely used definitions of the quality factor are compared with two examples of RLC circuits that serve as a simplified model of a single-resonant antenna tuned to its resonance. It is demonstrated that for a first-order system, only the quality factor based on differentiation of the input impedance has unique proportionality to the fractional bandwidth, whereas, e.g., the classical definition of the quality factor, i.e., the ratio of the stored energy to the lost energy per one cycle, is not uniquely proportional to the fractional bandwidth. In addition, it is shown that for higher order systems, the quality factor based on differentiation of the input impedance ceases to be uniquely related to the fractional bandwidth.**

***Index Terms*—Antenna theory, electromagnetic theory, Q factor.**

I. Introduction

The fractional bandwidth (FBW) is a parameter of primary importance in any oscillating system [1], since it is a relative frequency band in which the system can be effectively driven by an external source. In the case of an antenna, a fractional bandwidth is a frequency band in which the power incident upon the input port can be effectively radiated [2].

Based on an analytical evaluation of the basic RLC circuits in the time-harmonic domain [3], FBW is believed to be inversely proportional to the quality factor, which is commonly defined as 2π times the ratio of the cycle mean stored energy and the lost energy, see e.g., IEEE Std. 145-1993, [4]. This relation is known to be very precise for high values of the quality factor (Q factor), and has been shown to be exact for the Q factor tending to infinity, i.e., a lossless oscillating system cannot be driven by an external source, since its FBW is equal to zero. However, this inverse proportionality is known to fail at low values of the Q factor and, in fact, it is not clear whether there exists any functional relation of FBW and the Q factor, which would be valid in all ranges of the Q factor. It is, however, important to stress that if such a relation were to exist, it would be of crucial importance, since there exists a fundamental lower bound of the Q factor of a lossless electromagnetic radiator [5], [6], which would then imply a fundamental upper bound of its FBW, an essential theoretical limitation for electrically small radiators.

This communication serves two purposes. First, a proof is given of the nonexistence of a general functional relation between traditionally defined FBW and the Q factor. The proof is based on an analytical evaluation of the functional relation for two distinct RLC circuits. It is given by contradiction, and it also covers some other commonly used prescriptions of the Q factor. Second, it is pointed out that the so-called Q_Z quality factor defined in [7] and further generalized in [8] is inversely proportional to FBW for first-order systems, but ceases to have this behavior for higher order (multiresonance) systems with closely spaced resonances [9].

Manuscript received December 04, 2014; revised February 04, 2015; accepted March 14, 2015. Date of publication March 20, 2015; date of current version May 29, 2015. This work was supported in part by the Czech Science Foundation under project 13-09086S and in part by the COST IC1102 (VISTA) action.

The authors are with the Department of Electromagnetic Field, Faculty of Electrical Engineering, Czech Technical University in Prague, 16627 Prague, Czech Republic (e-mail: miloslav.capek@fel.cvut.cz).

Color versions of one or more of the figures in this communication are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAP.2015.2414472

II. Definition of the Q Factor

This section defines several widely used prescriptions of the Q factor that will be used later:

1) classical quality factor Q_{cl} [4];
2) modified quality factor Q_{rev}, based on the concept of recoverable energy [10]–[12];
3) Q_X quality factor, based on differentiation of the input reactance [13];
4) Q_Z quality factor, based on differentiation of the input impedance [7], [8].

A. *Definition of the Classical Quality Factor* Q_{cl}

The classical Q factor is conventionally defined as [4]

$$Q_{\mathrm{cl}} = \frac{\omega_0 W_{\mathrm{sto}}}{P_{\mathrm{lost}}} \tag{1}$$

in which ω_0 is the resonant frequency, W_{sto} is the cycle mean stored energy, and P_{lost} is the cycle mean power loss. This prescription of the Q factor is traditionally encumbered with difficulties in identifying the stored energy of a general electromagnetic radiator [14]. This problem is, however, left aside in this communication, as W_{sto} is used only for nonradiating circuits, for which the concept of stored energy is well established [3]. Namely, the cycle mean stored energy of a nonradiating circuit can generally be written as

$$W_{\mathrm{sto}} = \frac{1}{4}\sum_n \left(L_n \left|I_{\mathrm{L}_n}\right|^2 + C_n \left|U_{\mathrm{C}_n}\right|^2\right) \tag{2}$$

and the cycle mean lost power can be written as

$$P_{\mathrm{lost}} = \frac{1}{2}\sum_n R_n \left|I_{\mathrm{R}_n}\right|^2 . \tag{3}$$

In (2) and (3), L, C, and R are the inductance, capacitance, and resistance of the circuit, and I_{L}, U_{C}, and I_{R} are the corresponding currents and voltages. The convention $\mathcal{F}(t) = \mathrm{Re}\{F(\omega)\exp(\mathrm{j}\omega t)\}$ for time-harmonic quantities has been utilized.

B. *Quality Factor* Q_{rev} *Based on Recoverable Energy*

The original definition of the Q factor (1) can be slightly modified to

$$Q_{\mathrm{rev}} = \frac{\omega_0 W_{\mathrm{rev}}}{P_{\mathrm{lost}}} \tag{4}$$

in which W_{rev} (the so-called recoverable energy) is that part of the stored energy W_{sto}, which can be recovered back from the input port by a matched load. This recoverable energy can in essence be evaluated by bringing the system into a time-harmonic steady state at frequency ω_0 by a voltage source with matched internal impedance, and afterward switching OFF the source and capturing all the energy returned to the internal impedance [10]–[12].

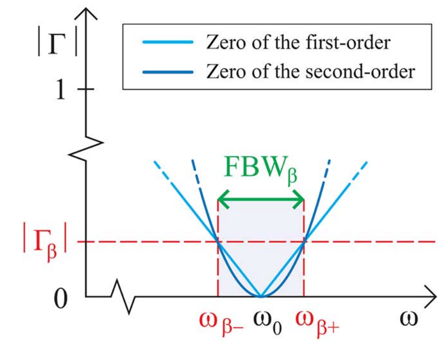

Fig. 1. Course of the reflectance from a device under test in the vicinity of the resonance frequency ω_0. The device under test is assumed to be matched ($Z_{\text{in}}(\omega_0) = R_0$) to the measurement device at the resonance frequency ω_0, so that $|\Gamma(\omega_0)| = 0$.

C. Reactance Quality Factor Q_X

A different approach in defining the Q factor is based on the assumption that Foster's reactance theorem [15] also holds for lossy systems [16], [17]. In that case, the Q factor can be defined by the frequency derivative of the input reactance as

$$Q_X = \frac{\omega_0}{2\,\text{Re}\{Z_{\text{in}}\}} \left| \frac{\partial\, \text{Im}\{Z_{\text{in}}\}}{\partial \omega} \right|_{\omega=\omega_0} \tag{5}$$

where Z_{in} is the input impedance of the circuits. This definition was proposed by Harrington [18], and was refined by Rhodes [13], and it is commonly used even nowadays.

D. Impedance Quality Factor Q_Z

A prescription that is widely used in antenna practice gives the Q factor in terms of the input impedance [7], [8]. The relation reads

$$Q_Z = \frac{\omega_0}{2\,\text{Re}\{Z_{\text{in}}\}} \left| \frac{\partial Z_{\text{in}}}{\partial \omega} \right|_{\omega=\omega_0} \tag{6}$$

and it is known to correspond well to FBW [8].

III. Functional Relation of Q Factor and FBW

The major purpose of this communication is to investigate the functional relation

$$\text{FBW}_\beta = f(Q) \tag{7}$$

where f is an as yet unknown function

$$\text{FBW}_\beta = \frac{\omega_{\beta+} - \omega_{\beta-}}{\omega_0} \tag{8}$$

where $\omega_{\beta+}$ and $\omega_{\beta-}$ delimit the range of frequencies for which the reflectance at the input port of the device under test

$$|\Gamma| = \left| \frac{Z_{\text{in}} - R_0}{Z_{\text{in}} + R_0} \right| \tag{9}$$

is smaller than a given threshold $|\Gamma_\beta|$ (see Fig. 1). The resistance R_0 in (9) belongs to the input port of the measurement device.

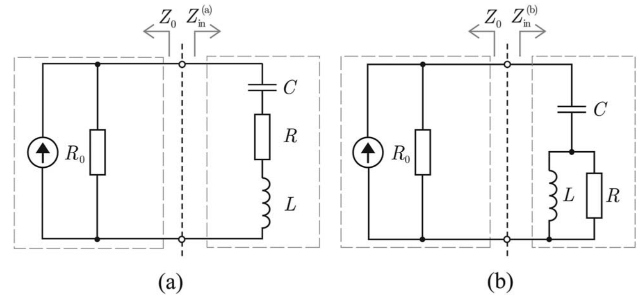

Fig. 2. Studied RLC circuits connected to a voltage source with internal resistance R_0. (a) R, C, and L in series. (b) C in series with parallel L and R.

A. First-Order Systems

Instead of directly analyzing relationship (7) for a complex system such as an antenna, we start with two single-resonant RLC circuits, depicted in Fig. 2. If it is proved that a given definition of the Q factor is not uniquely proportional to FBW for the simple circuits in Fig. 2, it can be concluded that this Q factor is not proportional to FBW at all.

Assuming a circuit tuned to resonance and matched to the measuring device ($Z_{\text{in}}(\omega_0) = R_0$), see Fig. 1, we utilize a simple consideration, in which the reflection coefficient Γ is expanded to its Taylor series around resonance frequency ω_0 and only the first nonzero term is kept. Under such conditions and using (9), the reflectance can be written as

$$\begin{aligned} |\Gamma| &= |\omega - \omega_0| \left| \frac{\partial \Gamma}{\partial \omega} \right|_{\omega=\omega_0} + \mathcal{O}(\omega^2) \\ &= \frac{|\omega - \omega_0|}{\omega_0} \frac{\omega_0}{2\,\text{Re}\{Z_{\text{in}}\}} \left| \frac{\partial Z_{\text{in}}}{\partial \omega} \right|_{\omega=\omega_0} + \mathcal{O}(\omega^2). \end{aligned} \tag{10}$$

Comparing (10) with (6) and (8) gives the required functional relation (7), which reads

$$\text{FBW}_\beta = 2\frac{|\Gamma_\beta|}{Q_Z} \tag{11}$$

and which is valid at least for $|\Gamma_\beta| \to 0$. This means that the quality factor Q_Z (6) is uniquely proportional to the FBW (at least in this differential sense). Furthermore, if the other Q factors are to follow relation (7), they must necessarily be functionally dependent on quality factor Q_Z. This property is investigated as follows.

The quality factor Q_Z of the two circuits under consideration can easily be calculated from the input impedances, which yields

$$Q_Z^{(\text{a})} = \frac{\omega_0^{(\text{a})} L}{R}, \qquad \omega_0^{(\text{a})} = \frac{1}{\sqrt{LC}} \tag{12a}$$

$$Q_Z^{(\text{b})} = \frac{R}{\omega_0^{(\text{b})} L} \cdot \frac{1}{\omega_0^{(\text{b})}\sqrt{LC}}, \qquad \omega_0^{(\text{b})} = \frac{R}{L\sqrt{\frac{CR^2}{L} - 1}} \tag{12b}$$

where superscripts (a) and (b) refer to the two circuits in Fig. 2.

The other Q factors can be evaluated in a straightforward manner as

$$Q_{\text{cl}}^{(\text{a})} = Q_Z^{(\text{a})} \tag{13a}$$

$$Q_X^{(\text{a})} = Q_Z^{(\text{a})} \tag{13b}$$

$$Q_{\text{rev}}^{(\text{a})} = \frac{Q_Z^{(\text{a})}}{2} \tag{13c}$$

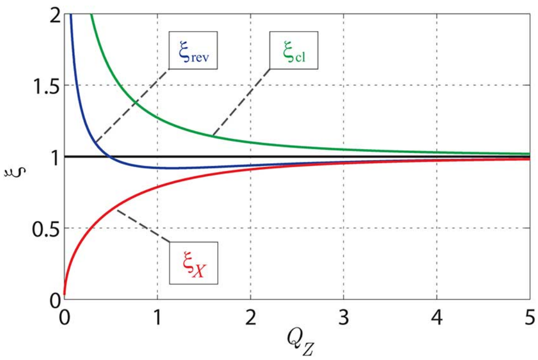

Fig. 3. ξ factors of (15a)–(15c) as a function of quality factor Q_Z.

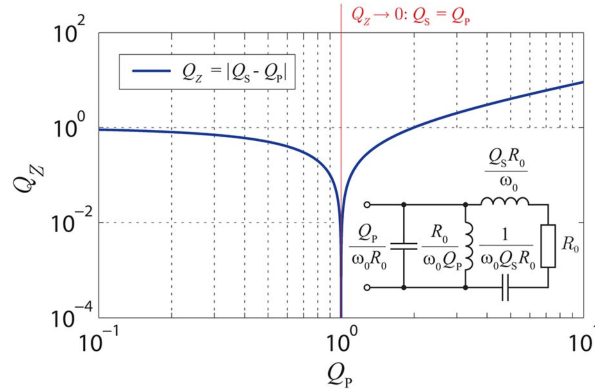

Fig. 4. Quality factor Q_Z of a second-order RLC circuit proposed in [20]. The Q_Z quality factor is evaluated at the resonance frequency ω_0, where the input impedance of the circuit is equal to R_0 irrespective of constants Q_{P} and Q_{S}, which represent the quality factors of the serial and parallel branch of the circuit, respectively. The equality $Q_{\mathrm{S}} = Q_{\mathrm{P}}$ provides a zero of second order in the reflectance (9) and results in a vanishingly small quality factor Q_Z. The depicted curve assumes $Q_{\mathrm{S}} = 1$.

and

$$Q_{\mathrm{cl}}^{(\mathrm{b})} = \xi_{\mathrm{cl}}\, Q_Z^{(\mathrm{b})} \tag{14a}$$
$$Q_X^{(\mathrm{b})} = \xi_X\, Q_Z^{(\mathrm{b})} \tag{14b}$$
$$Q_{\mathrm{rev}}^{(\mathrm{b})} = \xi_{\mathrm{rev}}\, \frac{Q_Z^{(\mathrm{b})}}{2} \tag{14c}$$

where

$$\xi_{\mathrm{cl}} = \sqrt{\chi}, \tag{15a}$$
$$\xi_X = \sqrt{\chi}\, \frac{\chi}{\chi + \left(Q_Z^{(\mathrm{b})}\right)^{-2}} \tag{15b}$$
$$\xi_{\mathrm{rev}} = \sqrt{\chi}\, \frac{\chi + \left(Q_Z^{(\mathrm{b})}\right)^{-2}}{\chi + 2\left(Q_Z^{(\mathrm{b})}\right)^{-2}} \tag{15c}$$
$$\chi = \frac{1 + \sqrt{1 + 4\left(Q_Z^{(\mathrm{b})}\right)^{-2}}}{2}. \tag{15d}$$

The above results offer a simple interpretation. Since the quality factor Q_Z factor has been shown to have a unique functional relation to FBW [see (11)], the other quality factors could have such a unique functional relation only if the functional relations corresponding to circuit (a) and circuit (b) [see (13) and (14)] are the same, i.e., if the corresponding ξ coefficients in (15a)–(15c) are equal to unity. That this is not the case is clear from their analytical prescription, and also from their graphical representation in Fig. 3. By means of contradiction, it must then be stated that there is no general functional relation between FBW and quality factors $Q_{\mathrm{cl}}, Q_X, Q_{\mathrm{rev}}$, the only exception being $Q \to \infty$.

B. Higher Order (Multiresonance) Systems

Section III-A has shown that, in the case of first-order systems, only quality factor Q_Z is a potential candidate for having a general functional relation to FBW. The purpose of this section is to test this property on higher order systems.

Higher order systems offer more degrees of freedom. This in general makes approximation (10) invalid. In fact, it can be shown [19] that a circuit of order n can always be tuned so that the first $n-1$ terms of the Taylor expansion (10) vanish (binomial transformer).

An example of such a second-order system [20] is depicted in Fig. 4, which for $Q_{\mathrm{P}} = Q_{\mathrm{S}}$ results in $Q_Z = 0$. Another example [21], [22] is a thin-strip dipole of length L and width $w = L/100$ (see Fig. 5), which is tuned to resonance by a lumped reactance connected in series

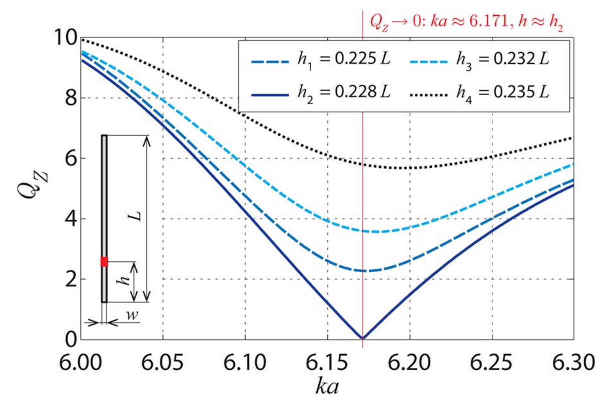

Fig. 5. Quality factor Q_Z of a thin-strip dipole as a function of electrical size and as a function of feeding position. The dipole is kept in resonance by a lumped reactance connected in series with the feed.

with a voltage gap feed. If the voltage gap is placed at $h \approx 0.228L$, it can be seen that $Q_Z = 0$ at $ka \approx 6.171$, in which $k = \omega/c_0$ is the wavenumber, c_0 is the speed of light, and a is the radius of the smallest circumscribing sphere.

The occurrence of $Q_Z \to 0$ mentioned above is an extreme case of the behavior of quality factor Q_Z in a general multiresonance system in which the resonances coincide. The quality factor Q_Z is, however, a smooth function of the frequency distance of the resonances, and there exist a whole range of distances for which it does not represent the fractional bandwidth well [9].

These results, and especially the awkward property of a possibly zero value of quality factor Q_Z in the case of circuits with clearly finite FBW, unfortunately exclude this Q factor from prescriptions with a possibly unique relation to FBW.

IV. Conclusion

It has been shown that, contrary to common belief, the classical quality factor defined by the stored and lost energy is not related to the fractional bandwidth by a general and unambiguous functional relation. This is also true for Q factors resulting from recoverable energy and input reactance. Considering the first-order system, only the Q factor based on differentiation of the input impedance has been shown to be a possible candidate for such a general functional relation. It has, however, been demonstrated that for higher order systems, including elementary radiators such as dipoles, no quality factor has, in general, exact proportionality to the fractional bandwidth.

ACKNOWLEDGMENT

The authors would like to thank to M. Gustafsson from Lund University, Sweden for a fruitful discussion about the topic, and for pointing out the possibility of minimizing quality factor Q_Z by off-center feeding of a dipole. They would also like to thank the two anonymous reviewers whose remarks improved the clarity of this communication.

REFERENCES

[1] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*. New York, NY, USA: McGraw-Hill, 1953.
[2] C. A. Balanis, *Antenna Theory Analysis and Design*, 3rd ed. Hoboken, NJ, USA: Wiley, 2005.
[3] E. Hallen, *Electromagnetic Theory*. London, U.K.: Chapman & Hall, 1962.
[4] *Standard Defintions of Terms for Antennas*, IEEE Antennas and Propagation Society Standard 145, 1993.
[5] L. J. Chu, "Physical limitations of omni-directional antennas," *J. Appl. Phys.*, vol. 19, pp. 1163–1175, 1948.
[6] R. C. Hansen and R. E. Collin, "A new Chu formula for Q," *IEEE Antennas Propag. Mag.*, vol. 51, no. 5, pp. 38–41, Oct. 2009,
[7] D. Kajfez and W. P. Wheless, "Invariant definitions of the unloaded Q factor," *IEEE Antennas Propag. Mag.*, vol. 34, no. 7, pp. 840–841, Jul. 1986.
[8] A. D. Yaghjian and S. R. Best, "Impedance, bandwidth and Q of antennas," *IEEE Trans. Antennas Propag.*, vol. 53, no. 4, pp. 1298–1324, Apr. 2005.
[9] H. R. Stuart, S. R. Best, and A. D. Yaghjian, "Limitations in relating quality factor to bandwidth in a double resonance small antenna," *IEEE Antennas Wireless Propag. Lett.*, vol. 6, pp. 460–463, 2007.
[10] V. G. Polevoi, "Maximum energy extractable from an electromagnetic field," *Radiophys. Quantum Electron.*, vol. 33, no. 7, pp. 603–609, 1990.
[11] C. A. Grimes, G. Liu, F. Tefiku, and D. M. Grimes, "Time-domain measurement of antenna Q," *Microwave Opt. Technol. Lett.*, vol. 25, no. 2, pp. 95–100, Apr. 2000.
[12] R. H. Direen, "Fundamental limitations on the terminal behavior of antennas and nonuniform transmission lines," Ph.D. dissertation, Dept. of Electrical, Computer, and Energy Engineering, Univ. Colorado, Boulder, CO, 2010.
[13] D. R. Rhodes, "Observable stored energies of electromagnetic systems," *J. Franklin Inst.*, vol. 302, no. 3, pp. 225–237, 1976.
[14] G. A. E. Vandenbosch, "Reply to "Comments on 'Reactive energies, impedance, and Q factor of radiating structures," *IEEE Trans. Antennas Propag.*, vol. 61, no. 12, p. 6268, Dec. 2013.
[15] R. M. Foster, "A reactance theorem," *Bell System Tech. J.*, vol. 3, pp. 259–267, 1924.
[16] D. R. Rhodes, "A reactance theorem," *Proc. R. Soc. Lond. A.*, vol. 353, pp. 1–10, Feb. 1977.
[17] W. Geyi, P. Jarmuszewski, and Y. Qi, "The Foster reactance theorem for antennas and radiation Q," *IEEE Trans. Antennas Propag.*, vol. 48, no. 3, pp. 401–408, Mar. 2000.
[18] R. F. Harrington and J. R. Mautz, "Control of radar scattering by reactive loading," *IEEE Trans. Antennas Propag.*, vol. 20, no. 4, pp. 446–454, Jul. 1972.
[19] R. E. Collin, *Foundations for Microwave Engineering*, 2nd ed. Hoboken, NJ, USA: Wiley/IEEE Press, 1992.
[20] M. Gustafsson and S. Nordebo, "Bandwidth, Q factor and resonance models of antennas," *Progr. Electromagn. Res.*, vol. 62, pp. 1–20, 2006.
[21] M. Gustafsson, D. Tayli, and M. Cismasu. (2014). *Q Factors for Antennas in Dispersive Media*. eprint arXiv: 1408.6834v2 [Online]. Available: http://arxiv.org/abs/1408.6834
[22] M. Gustafsson, M. Cismasu, D. Tayli, S. Nordebo, and L. Jonsson. (2013). *An Overview of Current Optimization and Physical Bounds on Antennas* [Online]. Available: www.eit.lth.se/index.php?uhpuid=scd.mgu&hpuid=139&L=1

Design of a Wideband Millimeter Wave Micromachined Rotman Lens

Nathan Jastram and Dejan S. Filipovic

***Abstract*—Design, fabrication, and performance of a micromachined millimeter wave Rotman lens are presented. To achieve wide instantaneous bandwidth with $\pm 30^\circ$ scan range, five different transmission lines with carefully designed transitions are monolithically integrated within the same process. Theoretical bandwidth of over 70 GHz is verified with a narrower bandwidth W-band measurements of VSWR (< 2.75:1), and 65–115 GHz measurements of realized gain (5–11 dBi), 3 dB beamwidth (20°–40°), and beam-peak locations in E-plane. Demonstrated results indicate that wideband Rotman lenses can now be engineered for emerging millimeter and submillimeter wave applications.**

***Index Terms*—Array feeds, micromachining, Rotman lens.**

I. INTRODUCTION

An increased interest in components and subsystems for millimeter and submillimeter wave applications, such as biomedical imaging, automotive anticollision radars, electronic warfare, and communications [1]–[3], has been recently observed. For many of these applications, beam steering is highly desired. The most popular traditional beam-steering approach is phased array which uses a network of active and passive components to set required amplitude and phase distributions for pointing beam in a desired direction. However, at millimeter wave frequencies, active phase shifters can have high loss [4], [5] while wideband passive components are more challenging to realize [6]. An array configuration with lower loss and wider bandwidth is the Rotman lens [7]; a planar electromagnetic lens with inherently broadband performance and frequency invariant beams.

Several Rotman lenses operating above 60 GHz using microstrip [8]–[10] or waveguide [11], [12] based topologies have been reported. In [8], a V-band $3\ (\text{beam port}) \times 5\ (\text{array port})$ device is built using low-temperature cofired ceramic (LTCC). While there is no reported data on the loss of the stand-alone lens, the insertion loss in (only)

Manuscript received July 31, 2014; revised February 04, 2015; accepted March 14, 2015. Date of publication March 19, 2015; date of current version May 29, 2015. This work was supported by the Office of Naval Research under Grant N00014-11-1-0010. PolyStrata is trademark of Nuvotronics, LLC.

The authors are with are with the Department of Electrical, Computer, and Energy Engineering, University of Colorado Boulder, Boulder, CO 80309, USA (e-mail: Nathan.Jastram@colorado.edu; Dejan@colorado.edu).

Color versions of one or more of the figures in this communication are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAP.2015.2414475