Int. J. Theor. Phys. (2011) 50:3750–3760
DOI 10.1007/s10773-011-0738-6

Characterizations of spectral automorphisms and a Stone-type theorem in orthomodular lattices

Dan Caragheorgheopol[1,2], **Josef Tkadlec**[3]

Received: 30 December 2010 / Accepted: 23 February 2011 / Published online: 17 March 2011

Abstract The notion of spectral automorphism of an orthomodular lattice was introduced by Ivanov and Caragheorgheopol [3] to create an analogue of the Hilbert space spectral theory in the abstract framework of orthomodular lattices. We develop the theory of spectral automorphisms finding previously missing characterizations of spectral automorphisms, discussing several examples and the possibility to construct such automorphisms in direct products or horizontal sums of lattices. A factorization of the spectrum of a spectral automorphism is found. The last part of the paper addresses the problem of the unitary time evolution of a system from the point of view of the spectral automorphisms theory. An analogue of the Stone theorem concerning strongly continuous one-parameter unitary groups is given.

Keywords Spectral automorphism · spectral theory · orthomodular lattice · horizontal sum · direct product · Stone theorem

1 Introduction

In the classical Hilbert-space formulation of quantum mechanics, observables are represented by self-adjoint operators, the spectral values of which are interpreted as possible outcomes of the measurements of the observable. By spectral resolution, observables may be replaced by the corresponding projection-valued spectral measures (see, e.g., [7, 2]) with values in the set $\mathcal{P}(H)$ of projection operators on the Hilbert space H associated to the quantum system under investigation. Since projection operators on H can be interpreted as "yes-no"

[1] Department of Mathematics and Informatics, Technical University of Civil Engineering in Bucharest, 124 Lacul Tei blv., RO-020396, Romania; e-mail: dancaraghe@gmail.com
[2] "Ilie Murgulescu" Institute of Physical Chemistry, Romanian Academy, 202 Splaiul Independentei, 060021 Bucharest, Romania
[3] Department of Mathematics, Faculty of Electrical Engineering, Czech Technical University in Prague, 166 27 Prague, Czech Republic; e-mail: tkadlec@fel.cvut.cz

propositions about the quantum system, we consider the orthomodular lattice structure, which is the most natural abstraction of the important properties of the lattice of projection operators, as the appropriate "logic" associated to the quantum system (see, e.g., [6, 5, 1]).

Spectral automorphisms of orthomodular lattices were introduced by Ivanov an Caragheorgheopol [3] with the intention to obtain an analogue of the Hilbert space spectral theory in the more general framework of orthomodular lattices and by this to show to which extent Hilbert space specific tools are needed. The aforementioned paper's focus is on introducing the new notion of spectral automorphism, as well as showing its relevance to the declared purpose, e.g., by proving an analogue of the finite dimensional spectral theorem in the framework of orthomodular lattices.

In this follow-up paper, we further develop the theory of spectral automorphisms in orthomodular lattices. The next section is dedicated to basic definitions and some well-known, but useful facts. In the third section we recall the heuristics behind the spectral automorphism notion and some of the proven facts about spectral automorphisms. We also show various examples of spectral automorphisms. The fourth part of the paper is devoted to the study of spectral automorphisms of a product or horizontal sum of orthomodular lattices. A factorization of the spectrum of a spectral automorphism is also discussed. In the fifth section, we present (previously missing) characterizations of spectral automorphisms, along with a necessary and a sufficient condition for an automorphism to be spectral. In the last section, properties of spectral families of automorphisms are discussed and an analogue in the framework of orthomodular lattices of the Stone theorem concerning strongly continuous uniparametric groups of unitary operators is obtained.

2 Basic notions and properties

Let us recall some basic facts about orthomodular lattices that will be needed in the sequel. For full details and proofs of the following statements, we refer to [4, 6].

Definition 2.1 An *orthomodular lattice* is a structure $(L, \leq, ', \mathbf{0}, \mathbf{1})$ with $\mathbf{0} \neq \mathbf{1}$ such that:

(1) $(L, \leq)$ is a bounded lattice, with the least element $\mathbf{0}$ and the greatest element $\mathbf{1}$;
(2) the unary operation $'$ on L is an orthocomplementation, i.e., (i) $(a')' = a$, (ii) $a \leq b$ implies $a' \geq b'$, (iii) $a \vee a' = \mathbf{1}$, for every $a, b \in L$;
(3) $a \leq b$ implies $b = a \vee (a' \wedge b)$ for every $a, b \in L$ (*orthomodular law*).

For simplicity, we shall use the notation L for an orthomodular lattice.

Elements a, b of an orthomodular lattice are *orthogonal* (denoted by $a \perp b$) if $a \leq b'$ (i.e., $b \leq a'$). An *atom* of an orthomodular lattice L is a minimal non-zero element of L. A lattice L is *atomic* if every element of L dominates an atom of L. Every atomic orthomodular lattice is *atomistic*, i.e., every its element is the least upper bound of the set of atoms it dominates.

The *trivial* orthomodular lattice is the smallest possible orthomodular lattice $\{\mathbf{0}, \mathbf{1}\}$. An orthomodular lattice L is *complete* if the supremum (hence also the infimum) of arbitrary subsets exists in L. A *subortholattice* of L is a subset of L containing $\{\mathbf{0}, \mathbf{1}\}$ and closed under the operations $\wedge$ and $'$ (and therefore $\vee$, too) with the operations inherited from L.

A one-to-one mapping $\varphi\colon L_1 \to L_2$ is an *isomorphism* of orthomodular lattices L_1, L_2 if (i) $a \leq b$ is equivalent to $\varphi(a) \leq \varphi(b)$ for every $a, b \in L_1$, and (ii) $\varphi(a') = \varphi(a)'$ for every $a \in L_1$. An *automorphism* of an orthomodular lattice L is an isomorphism $L \to L$.

If L is an orthomodular lattice and $a \in L \setminus \{\mathbf{0}\}$, then $([\mathbf{0}, a], \leq|_{[\mathbf{0},a]}, ^*, \mathbf{0}, a)$ with $[\mathbf{0}, a] = \{b \in L\colon b \leq a\}$ and $^*\colon b \mapsto b' \wedge a$ for every $b \in [\mathbf{0}, a]$ is an orthomodular lattice.

Definition 2.2 Elements a, b of an orthomodular lattice L *commute* (are *compatible*) if $a = (a \wedge b) \vee (a \wedge b')$. We write $a \leftrightarrow b$ in such a case.

Commutativity is symmetric in orthomodular lattices. In an orthomodular lattice L, for $a \in L$ and $M, N \subseteq L$ we write $a \leftrightarrow M$ if $a \leftrightarrow b$ for every $b \in M$ and $M \leftrightarrow N$ if $a \leftrightarrow b$ for every $a \in M$ and $b \in N$. If L_1 is a subortholattice of an orthomodular lattice L_2 and $a, b \in L_1$ with $a \leftrightarrow b$ in L_1 then $a \leftrightarrow b$ in L_2. An orthomodular lattice in which every pair of elements commutes is a Boolean algebra. The set of elements of an orthomodular lattice L that commute with all elements of L is the *center* of L, denoted by $C(L)$, and it is a Boolean subalgebra of L. For $M \subset L$, the set of all elements commuting with every element of M is the *commutant* of M (in L) and is denoted by $K(M)$. A *block* of L is a maximal set of pairwise commuting elements or, equivalently, a maximal Boolean subalgebra of L. Every orthomodular lattice is the set-theoretical union of its blocks.

The following result gives an equivalent formulation for the commutativity:

Proposition 2.3 *Let L be an orthomodular lattice and $a, b \in L$. Then $a \leftrightarrow b$ if and only if there are pairwise orthogonal $a_1, b_1, c \in L$ such that $a = a_1 \vee c$ and $b = b_1 \vee c$.*

The following properties shall be used in the sequel.

Proposition 2.4 *[4, Theorem 5, Section §3, Ch. 1] Let L be an orthomodular lattice and $a, b, c \in L$ such that $a \leftrightarrow b$ and $a \leftrightarrow c$. Then the sublattice of L generated by $\{a, b, c\}$ is distributive.*

Proposition 2.5 *[6, Proposition 1.3.10] If L is an orthomodular lattice, $M \subseteq L$ such that $\bigvee M$ exists in L and $b \in L$ with $b \leftrightarrow M$ then $b \leftrightarrow \bigvee M$ and $b \wedge \bigvee M = \bigvee_{a \in M}(b \wedge a)$.*

The *product* $\prod_{i \in I} L_i$ of a nonempty collection $(L_i)_{i \in I}$ of orthomodular lattices is obtained by endowing their Cartesian product with the "component-wise" partial order and orthocomplementation (i.e., for $a, b \in \prod_{i \in I} L_i$, we have $a \leq b$ if $a_i \leq b_i$ for all $i \in I$ and $a = b'$ if $a_i = b_i'$ for all $i \in I$). The *horizontal sum* of a nonempty collection $(L_i)_{i \in I}$ of orthomodular lattices is constructed as the disjoint union of all L_i's with identifying their least (greatest) elements to obtain the least (greatest) element of the horizontal sum, the order and orthocomplementation in the horizontal sum are inherited from L_i (if a_i, a_j are not elements of the same summand then, in the horizontal sum, $a_i \wedge a_j = \mathbf{0}$ and $a_i \vee a_j = \mathbf{1}$). A summand that is not a horizontal sum of at least two nontrivial orthomodular lattices is *minimal*. (Every orthomodular lattice is the horizontal sum of itself and of an arbitrary collection of trivial orthomodular lattices). Every summand is a subortholattice of the horizontal sum. Both the product and the horizontal sum of a collection of orthomodular lattices are orthomodular lattices.

Lemma 2.6 *Let $(L_i)_{i \in I}$ be a collection of orthomodular lattices and let L be their product. The elements $a = (a_i)_{i \in I}, b = (b_i)_{i \in I}$ of L commute if and only if $a_i \leftrightarrow b_i$ for all $i \in I$.*

Proof The proof is straightforward and therefore we leave it to the reader (see [6]). □

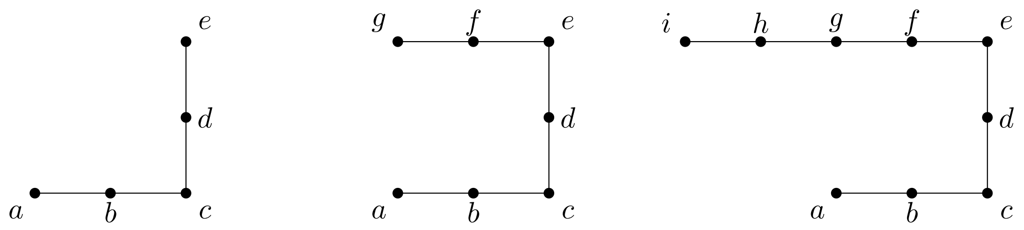

Figure 1: Greechie diagrams of orthomodular lattices used in Examples 3.3, 3.4 and 6.9.

3 Basics about spectral automorphisms

Let us briefly recall considerations that led to the introduction of spectral automorphisms as well as their definition and a few important properties proved in [3].

Let H be a Hilbert space and $\mathcal{P}(H)$ the orthomodular lattice of projection operators on H. According to a version of Wigner's theorem due to Wright [9], automorphisms of $\mathcal{P}(H)$ are of the form $\varphi_U : \mathcal{P}(H) \to \mathcal{P}(H)$, $\varphi_U(P) = UPU^{-1}$, with U being a unitary or an antiunitary operator on H. Let us assume that U is unitary and B_U is the Boolean subalgebra of $\mathcal{P}(H)$ that is the range of the spectral measure associated to U. Then $P \in \mathcal{P}(H)$ is φ_U-invariant if and only if $UP = PU$ if and only if P commutes with B_U (i.e., commutes with every projection operator in B_U) if and only if $P \leftrightarrow B_U$ (for the last two equivalences, see [2]). This inspired naturally the definition of spectral automorphisms in orthomodular lattices.

Definition 3.1 Let L be an orthomodular lattice and φ be an automorphism of L. The automorphism φ is *spectral* if there is a Boolean subalgebra B of L such that

$$\varphi(a) = a \text{ if and only if } a \leftrightarrow B\,. \tag{P1}$$

A Boolean subalgebra of L satisfying condition (P1) is a *spectral algebra* of φ. The greatest such Boolean algebra is *the spectrum* of φ, denoted by σ_φ. The set of φ-invariant elements of L (which is a subortholattice of L) is denoted by L_φ.

Let us remark that the spectrum of a spectral automorphism of an orthomodular lattice exists [3, Proposition 2.7].

Proposition 3.2 *[3, see Proposition 2.9 and Corollary 2.2] An automorphism φ of an orthomodular lattice L is spectral if and only if $K\big(C(L_\varphi)\big) \subseteq L_\varphi$. In this case, $\sigma_\varphi = C(L_\varphi)$.*

Let us remark that obviously $K\big(C(L_\varphi)\big) \supseteq L_\varphi$, hence the condition $K\big(C(L_\varphi)\big) = L_\varphi$ might be used in Proposition 3.2.

Let us discuss some examples of spectral automorphisms. We shall use the technique of Greechie diagrams—see, e.g., [4]. A Greechie diagram of an orthomodular lattice L consists of a set of points and a set of lines such that points are in one-to-one correspondence with atoms of L and lines are in one-to-one correspondence with blocks of L. An automorphism of a finite orthomodular lattice is completely determined by its values on atoms.

Example 3.3 Let L be the orthomodular lattice described by the first Greechie diagram in Fig. 1. It is the union of two blocks, one determined by atoms $\{a, b, c\}$ and the other determined by atoms $\{c, d, e\}$. Let φ be an automorphism of L such that a, b, c are φ-invariant and d, e are permuted by φ. Then the set L_φ of φ-invariant elements of L is the block of

L determined by atoms $\{a,b,c\}$. Since L_φ is already Boolean, it coincides with its center $C(L_\varphi)$ and since it's a block, $x \leftrightarrow L_\varphi$ if and only if $x \in L_\varphi$ for every $x \in L$. Therefore φ is spectral and $\sigma_\varphi = L_\varphi = C(L_\varphi)$.

Example 3.4 Let L be the orthomodular lattice described by the second Greechie diagram in Fig. 1. It is the union of three blocks, the first determined by atoms $\{a,b,c\}$, the second determined by atoms $\{c,d,e\}$ and the last determined by atoms $\{e,f,g\}$. Let φ be an automorphism of L such that a,b,c,d,e are φ-invariant and f,g are permuted by φ. Then the set L_φ of φ-invariant elements of L is not a block but it is precisely the lattice from Example 3.3 which is the union of blocks determined by atoms $\{a,b,c\}$ and $\{c,d,e\}$. Its center is $C(L_\varphi) = \{\mathbf{0},\mathbf{1},c,c'\}$ and for an element $x \in L$, $x \leftrightarrow C(L_\varphi)$ if and only if $x \leftrightarrow c$ if and only if $x \in L_\varphi$. Therefore φ is spectral and $\sigma_\varphi = C(L_\varphi) = \{\mathbf{0},\mathbf{1},c,c'\}$.

Example 3.5 Let H be an n-dimensional complex Hilbert space and $\mathcal{P}(H)$ be the set of its projection operators. Let Q be a 1-dimensional projection on H and Q' be its orthogonal complement. We define $U\colon H \to H$ as the symmetry of H with respect to the hyperplane corresponding to Q'. It is easy to see that U is a unitary operator, therefore $\varphi\colon \mathcal{P}(H) \to \mathcal{P}(H)$ defined by $\varphi(P) = UPU^{-1}$ is an automorphism of $\mathcal{P}(H)$. $\mathcal{B} = \{\mathbf{0}, Q, Q', \mathbf{1}\}$ is a Boolean subalgebra of $\mathcal{P}(H)$ fulfilling condition (P1) in Definition 3.1. Indeed, the set of φ-invariant elements, as well as the set of elements that commute with $\mathcal{B}$, is $\mathcal{P}_0 \cup \mathcal{P}_0'$, where $\mathcal{P}_0 = \{A \in \mathcal{P}(H)\colon A \le Q'\}$ and $\mathcal{P}_0'$ denotes the set of orthocomplements of the elements of $\mathcal{P}_0$.

4 Spectral automorphisms in products and horizontal sums

We shall discuss the construction of spectral automorphisms in products and horizontal sums of orthomodular lattices. This will show new ways to obtain spectral automorphisms, proving the richness of this class of automorphisms. A factorization of the spectra of spectral automorphisms is also discussed.

Theorem 4.1 *Let L be the product of a collection $(L_i)_{i\in I}$ of orthomodular lattices and, for every $i \in I$, φ_i be an automorphism of L_i. Let us define the mapping $\varphi\colon L \to L$ by $\varphi\big((a_i)_{i\in I}\big) = \big(\varphi_i(a_i)\big)_{i\in I}$. Then:*

(1) *φ is an automorphism of L;*
(2) *φ is spectral if and only if φ_i is spectral for every $i \in I$; in this case, $\sigma_\varphi = \prod_{i\in I} \sigma_{\varphi_i}$.*

Proof (1) It is a routine verification.

(2) An element $(a_i)_{i\in I} \in L$ is φ-invariant if and only if $\varphi_i(a_i) = a_i$ for every $i \in I$, i.e., $L_\varphi = \prod_{i\in I} L_{\varphi_i}$. According to Lemma 2.6, $C(L_\varphi) = \prod_{i\in I} C(L_{\varphi_i})$ and $K\big(C(L_\varphi)\big) = \prod_{i\in I} K\big(C(L_{\varphi_i})\big)$. According to Proposition 3.2, φ is spectral if and only if $K\big(C(L_\varphi)\big) = L_\varphi$, i.e., $\prod_{i\in I} K\big(C(L_{\varphi_i})\big) = \prod_{i\in I} L_{\varphi_i}$, i.e., $K\big(C(L_{\varphi_i})\big) = L_{\varphi_i}$ for every $i \in I$, i.e., φ_i is spectral for every $i \in I$. Moreover, in such a case, $\sigma_\varphi = C(L_\varphi) = \prod_{i\in I} C(L_{\varphi_i}) = \prod_{i\in I} \sigma_{\varphi_i}$. □

We shall turn now to spectral automorphisms in horizontal sums of orthomodular lattices. Let us begin with some preparatory remarks.

Remark 4.2 Let L be an orthomodular lattice and φ be a spectral automorphism of L. If $L_\varphi = \{\mathbf{0},\mathbf{1}\}$ then $L = \{\mathbf{0},\mathbf{1}\}$. This follows from the fact that $\sigma_\varphi = C(L_\varphi) = \{\mathbf{0},\mathbf{1}\}$ and

$\{\mathbf{0},\mathbf{1}\} \leftrightarrow L$, hence $L \subseteq L_\varphi = \{\mathbf{0},\mathbf{1}\}$.

Remark 4.3 Let L be the horizontal sum of a collection $(L_i)_{i\in I}$ of orthomodular lattices. If $a_i \in L_i \setminus \{\mathbf{0},\mathbf{1}\}$ and $a_j \in L_j \setminus \{\mathbf{0},\mathbf{1}\}$ for some $i, j \in I$, $i \neq j$, then $a_i \not\leftrightarrow a_j$. Indeed, $a_i \wedge a_j = \mathbf{0}$ and $a_i \wedge a_j' = \mathbf{0}$, hence $(a_i \wedge a_j) \vee (a_i \wedge a_j') = \mathbf{0} \neq a_i$. Hence, every nontrivial block (i.e., every block if L is nontrivial) of L is a subset of exactly one summand.

Lemma 4.4 *Let L be the horizontal sum of a collection $(L_i)_{i\in I}$ of orthomodular lattices such that every summand is minimal, φ be an automorphism of L such that $\varphi(L_i) \cap L_i \neq \{\mathbf{0},\mathbf{1}\}$ for some $i \in I$. Then the restriction φ_i of φ to L_i is an automorphism of L_i.*

Proof The horizontal sum L is nontrivial, hence, according to Remark 4.3, every block of L is a subset of exactly one summand. It is worth noting that for a block B of L, $\varphi(B)$ is also a block of L. Let us denote $L_{i,j} = \{\mathbf{0},\mathbf{1}\} \cup \bigcup\{B \subseteq L_i\colon \text{B is a block in } L,\ \varphi(B) \subseteq L_j\}$. It is easy to see that L_i is the horizontal sum of $(L_{i,j})_{j\in I}$. Since L_i is a minimal summand and, due to the condition $\varphi(L_i) \cap L_i \neq \{\mathbf{0},\mathbf{1}\}$, $L_{i,i}$ is nontrivial, we obtain $L_{i,i} = L_i$ and therefore $\varphi(L_i) \subseteq L_i$. By applying the same reasoning to φ^{-1}, which also satisfies the condition $\varphi^{-1}(L_i) \cap L_i \neq \{\mathbf{0},\mathbf{1}\}$, we find that $\varphi^{-1}(L_i) \subseteq L_i$. Hence φ_i is a bijection on L_i and, since φ is an automorphism of L, φ_i is an automorphism of L_i. □

Theorem 4.5 *Let L be the horizontal sum of a collection $(L_i)_{i\in I}$ of orthomodular lattices such that every summand is minimal and φ be an automorphism of L.*

(1) *If φ is spectral then there is an $i \in I$ such that $L_\varphi \subset L_i$, $\sigma_\varphi \subset L_i$ and the restriction φ_i of φ to L_i is a spectral automorphism of L_i with $L_{\varphi_i} = L_\varphi$ and $\sigma_{\varphi_i} = \sigma_\varphi$.*

(2) *If $L_\varphi \neq \{\mathbf{0},\mathbf{1}\}$ and there is an $i \in I$ such that $L_\varphi \subset L_i$ and the restriction φ_i of φ to L_i is spectral then φ is a spectral automorphism of L and $L_\varphi = L_{\varphi_i}$, $\sigma_\varphi = \sigma_{\varphi_i}$.*

Proof (1) It is obvious if L is trivial. Let us suppose that L is nontrivial. The spectrum σ_φ of φ is a Boolean subalgebra of L. According to Remark 4.3, there is an $i \in I$ such that $\sigma_\varphi \subset L_i$. For every $a \in L_\varphi$, we have $a \leftrightarrow \sigma_\varphi$ and, according to Remark 4.3, $a \in L_i$. Hence, $L_\varphi \subset L_i$. According to Remark 4.2, there is an $a \in L_\varphi \setminus \{\mathbf{0},\mathbf{1}\}$ and, obviously, $\varphi(a) = a \in L_i$. Hence, $\varphi(L_i) \cap L_i \neq \{\mathbf{0},\mathbf{1}\}$ and, according to Lemma 4.4, the restriction φ_i of φ to L_i is an automorphism of L_i. Since $L_\varphi \subset L_i$, $L_{\varphi_i} = L_\varphi$ and $C(L_{\varphi_i}) = C(L_\varphi) = \sigma_\varphi$. Therefore, since φ is spectral, φ_i is spectral, too, and $\sigma_{\varphi_i} = C(L_{\varphi_i}) = \sigma_\varphi$.

(2) According to our assumptions, $\varphi(L_i) \cap L_i \neq \{\mathbf{0},\mathbf{1}\}$. According to Lemma 4.4, the restriction φ_i of φ to L_i is an automorphism of L_i which we assume by hypothesis to be spectral. Since $a \in L_\varphi = L_{\varphi_i}$ if and only if $a \leftrightarrow \sigma_{\varphi_i}$, φ is spectral and $\sigma_\varphi = C(L_\varphi) = C(L_{\varphi_i}) = \sigma_{\varphi_i}$. □

Theorem 4.6 *Let L be an orthomodular lattice, φ be a spectral automorphism of L and $a \in L \setminus \{\mathbf{0},\mathbf{1}\}$ be φ-invariant. Let us denote by $\varphi_a, \varphi_{a'}$ the restrictions of φ to $[\mathbf{0}, a]$ and $[\mathbf{0}, a']$, respectively, and $B_x = x \wedge \sigma_\varphi = \{x \wedge b\colon b \in \sigma_\varphi\}$ for every $x \in L$. Then:*

(1) *φ_a is a spectral automorphism of $[\mathbf{0}, a]$ and B_a is its spectral algebra;*
(2) *if $a \in \sigma_\varphi$, then $\sigma_{\varphi_a} = B_a$;*
(3) *σ_φ is isomorphic to the product $B_a \times B_{a'}$.*

Proof (1) Let us denote by ${}^*\colon b \mapsto b' \wedge a$ the orthocomplementation in $[\mathbf{0}, a]$ and let us verify that φ_a is an automorphism of $[\mathbf{0}, a]$. For every $b \in [\mathbf{0}, a]$, $\varphi_a(b) = \varphi(b) \leq \varphi(a) = a$. Hence, φ_a is a mapping into $[\mathbf{0}, a]$. Since φ is an automorphism of L, for every $b \in [\mathbf{0}, a]$ there is a

$c \in L$ such that $\varphi(c) = b \le a = \varphi(a)$ and therefore $c \le a$. Hence, φ_a is a mapping onto $[\mathbf{0}, a]$. Since φ is an automorphism of L, $\varphi_a, \varphi_a^{-1}$ preserve the ordering and, for every $b \in [\mathbf{0}, a]$, $\varphi_a(b^*) = \varphi(b' \wedge a) = \varphi(b)' \wedge \varphi(a) = \varphi_a(b)^*$.

Clearly, $B_a \supseteq \{\mathbf{0}, a\}$ and is closed under the operation $\wedge$. Moreover, for every $b \in \sigma_\varphi$, $a \leftrightarrow b$ and, using Proposition 2.4, $(a \wedge b)^* = (a \wedge b)' \wedge a = \big((a \wedge b) \vee a'\big)' = \big((a \vee a') \wedge (b \vee a')\big)' = (b \vee a')' = b' \wedge a \in B_a$. Hence B_a is closed under the operation * and therefore B_a is a subortholattice of $[\mathbf{0}, a]$. For every $b_1, b_2 \in \sigma_\varphi$, elements a, b_1, b_2 pairwise commute, hence $a \wedge b_1 \leftrightarrow a \wedge b_2$ and therefore B_a is a Boolean subalgebra of $[\mathbf{0}, a]$.

Let us prove now that B_a is a spectral algebra of φ_a. Every φ_a-invariant $c \in [\mathbf{0}, a]$ is φ-invariant, hence $c \leftrightarrow \sigma_\varphi$. Since $c \le a$, we have $c \leftrightarrow a$ and therefore $c \leftrightarrow a \wedge b$ for every $b \in \sigma_\varphi$, i.e., $c \leftrightarrow B_a$. Conversely, let $c \in [\mathbf{0}, a]$ with $c \leftrightarrow B_a$. Hence $c \leftrightarrow a \wedge b$ for every $b \in \sigma_\varphi$. Since $c \le a \le a \vee b' = (a' \wedge b)'$, $c \leftrightarrow a' \wedge b$. Using Proposition 2.5 and the fact that $a \leftrightarrow b$, we find $c \leftrightarrow (a \wedge b) \vee (a' \wedge b) = b$ for every $b \in \sigma_\varphi$. Hence c is φ-invariant and therefore φ_a-invariant.

(2) Since B_a is a spectral algebra of φ_a, $B_a \subseteq \sigma_{\varphi_a}$. Let $b \in \sigma_{\varphi_a} = C(L_{\varphi_a})$. Since $a \in \sigma_\varphi = C(L_\varphi)$, for every $c \in L_\varphi$ we have $c \leftrightarrow a$, i.e., $c = (c \wedge a) \vee (c \wedge a')$. Since $c \wedge a \in L_{\varphi_a}$ and $b \le a$, we obtain $b \leftrightarrow (c \wedge a), (c \wedge a')$ and therefore, according to Proposition 2.5, $b \leftrightarrow (c \wedge a) \vee (c \wedge a') = c$ for every $c \in L_\varphi$. Hence, $b \in C(L_\varphi) = \sigma_\varphi$ and, since $b \le a$, $b = a \wedge b$, i.e., $b \in B_a$.

(3) We shall prove that $f\colon\ \sigma_\varphi \to B_a \times B_{a'}$ defined by $f(b) = (a \wedge b, a' \wedge b)$ is an isomorphism and $g\colon\ B_a \times B_{a'} \to \sigma_\varphi$ with $g(c_1, c_2) = c_1 \vee c_2$ is its inverse. Since $a \leftrightarrow \sigma_\varphi$, we get $g\big(f(b)\big) = (a \wedge b) \vee (a' \wedge b) = b$ for every $b \in \sigma_\varphi$. For every $c_1 \in B_a$ and $c_2 \in B_{a'}$ there are $b_1, b_2 \in \sigma_\varphi$ such that $c_1 = b_1 \wedge a$ and $c_2 = b_2 \wedge a'$. Then, $f\big(g(c_1, c_2)\big) = f\big((b_1 \wedge a) \vee (b_2 \wedge a')\big) = \big(a \wedge ((b_1 \wedge a) \vee (b_2 \wedge a')), a' \wedge ((b_1 \wedge a) \vee (b_2 \wedge a'))\big)$. Since clearly $\{a, a'\} \leftrightarrow \{b_1 \wedge a, b_2 \wedge a'\}$, we obtain, according to Proposition 2.4, $\big(a \wedge ((b_1 \wedge a) \vee (b_2 \wedge a')), a' \wedge ((b_1 \wedge a) \vee (b_2 \wedge a'))\big) = (a \wedge b_1, a' \wedge b_2) = (c_1, c_2)$. Hence, f and g are bijective mappings and $g = f^{-1}$. Clearly, both f, g preserve the operation $\wedge$ and therefore the ordering. It remains to prove that f preserves the orthocomplementation. Let $^\sharp$ denote the orthocomplementation in $B_a \times B_{a'}$ and * the orthocomplementations in $B_a, B_{a'}$. For every $b \in \sigma_\varphi$, we obtain $f(b)^\sharp = (a \wedge b, a' \wedge b)^\sharp = \big((a \wedge b)^*, (a' \wedge b)^*\big) = (a \wedge b', a' \wedge b') = f(b')$. □

Question 4.7 Is it possible to omit the condition $a \in \sigma_\varphi$ in Theorem 4.6 (2)?

5 Characterizations of spectral automorphisms

Theorem 5.1 *Let L be an orthomodular lattice. An automorphism φ of L is spectral if and only if $a \wedge b \in L_\varphi$ for every $a \in C(L_\varphi)$ and $b \in K\big(C(L_\varphi)\big)$.*

Proof $\Rightarrow$: If φ is spectral then, according to Proposition 3.2, $K\big(C(L_\varphi)\big) \subseteq L_\varphi$. If $a \in C(L_\varphi)$ and $b \in K\big(C(L_\varphi)\big)$ then $a, b \in L_\varphi$ and, since L_φ is a subortholattice of L, we obtain $a \wedge b \in L_\varphi$.

$\Leftarrow$: Let $b \in K\big(C(L_\varphi)\big)$. For every $a \in C(L_\varphi)$, $a' \in C(L_\varphi)$ and, due to the hypothesis, $a \wedge b, a' \wedge b \in L_\varphi$ and, since $b \leftrightarrow a$ and L_φ is a subortholattice of L, $b = (a \wedge b) \vee (a' \wedge b) \in L_\varphi$. Hence, $K\big(C(L_\varphi)\big) \subseteq L_\varphi$ and, according to Proposition 3.2, φ is spectral. □

Definition 5.2 Let L be an orthomodular lattice and φ an automorphism of L. An element $a \in L$ is *totally φ-invariant* if $\varphi(b) = b$ for every $b \in L$ with $b \le a$.

In the sequel, we shall need the following result:

Proposition 5.3 *[3, Proposition 2.4] If B is an atomic Boolean subalgebra of the orthomodular lattice L, a is an atom of B and $b \in L$ with $b \leq a$, then $b \leftrightarrow B$.*

Lemma 5.4 *Let L be a complete orthomodular lattice and φ be an automorphism of L such that $C(L_\varphi)$ is atomic with the set A of atoms. Then $\bigvee A = \mathbf{1}$.*

Proof Since L is complete, $\bigvee A$ exists in L. Since φ is an automorphism of L and $A \subseteq L_\varphi$, $\varphi(\bigvee A) = \bigvee_{a\in A} \varphi(a) = \bigvee_{a\in A} a = \bigvee A$ and therefore $\bigvee A \in L_\varphi$. Since $L_\varphi \leftrightarrow A$, according to Proposition 2.5, $L_\varphi \leftrightarrow \bigvee A$ and therefore $\bigvee A \in C(L_\varphi)$. Since $C(L_\varphi)$ is atomic, $\bigvee A = \mathbf{1}$. □

Theorem 5.5 *Let L be a complete orthomodular lattice and φ be an automorphism of L such that $C(L_\varphi)$ is atomic. Then φ is spectral if and only if all atoms of $C(L_\varphi)$ are totally φ-invariant.*

Proof $\Rightarrow$: Let a be an atom of $C(L_\varphi)$ and $b \in L$ with $b \leq a$. According to Propositions 5.3 and 3.2, $b \leftrightarrow C(L_\varphi) = \sigma_\varphi$ and therefore b is φ-invariant. Hence, a is totally φ-invariant.

$\Leftarrow$: Let $b \in K\big(C(L_\varphi)\big)$ and A be the set of atoms of $C(L_\varphi)$. Then, according to Lemma 5.4, $\bigvee A = \mathbf{1}$ and, according to Proposition 2.5, since $b \leftrightarrow A$, $b = b \wedge \bigvee A = \bigvee_{a\in A}(b \wedge a)$. Since φ is an automorphism and $b \wedge a$ is φ-invariant for every $a \in A$, we obtain $\varphi(b) = \bigvee_{a\in A} \varphi(b \wedge a) = \bigvee_{a\in A}(b \wedge a) = b$ and therefore $b \in L_\varphi$. Hence, $K\big(C(L_\varphi)\big) \subseteq L_\varphi$ and therefore, according to Proposition 3.2, φ is spectral. □

Corollary 5.6 *Let L be a complete orthomodular lattice and φ be an automorphism of L such that $C(L_\varphi)$ is atomic. If all atoms of $C(L_\varphi)$ are atoms of L then φ is spectral.*

Proof It follows easily from Theorem 5.5 because atoms of $C(L_\varphi)$ are φ-invariant and, being atoms of L, they are totally φ-invariant. □

Theorem 5.7 *Let L be a complete orthomodular lattice and φ be an automorphism of L such that L_φ and $C(L_\varphi)$ are atomic. If φ is spectral then all atoms of L_φ are atoms of L.*

Proof First, let us prove that every atom of L_φ is dominated by an atom of $C(L_\varphi)$. Let us suppose that b is an atom of L_φ that is not dominated by any atom of $C(L_\varphi)$ and seek a contradiction. For every atom a of $C(L_\varphi)$ we have $b \leftrightarrow a$ and therefore $b = (b \wedge a) \vee (b \wedge a')$. Since b is an atom in L_φ, either $b \leq a$ or $b \leq a'$. Since we supposed that the first inequality is not satisfied, we obtain $b \leq a'$, i.e., $a \leq b'$, hence $\bigvee\{a \in L\colon a \text{ is an atom in } C(L_\varphi)\} \leq b'$. According to Lemma 5.4, $\bigvee\{a \in L\colon a \text{ is an atom in } C(L_\varphi)\} = \mathbf{1}$ and therefore $b = \mathbf{0}$, which is a contradiction.

According to Theorem 5.5, all atoms of $C(L_\varphi)$ are totally φ-invariant. Since all atoms of L_φ are dominated by atoms of $C(L_\varphi)$, all atoms of L_φ are totally φ-invariant and therefore they are atoms of L. □

6 Spectral families of automorphisms and a Stone-type theorem

Definition 6.1 Let L be an orthomodular lattice and Φ be a family of automorphisms of L. The family Φ is *spectral* if there is a Boolean subalgebra B of L such that:

$$(\varphi(a) = a \text{ for every } \varphi \in \Phi) \text{ if and only if } a \leftrightarrow B\,. \tag{P2}$$

A Boolean algebra B satisfying condition (P2) is a *spectral algebra of* Φ. The set of Φ-invariant elements of L (which is a subortholattice of L) is denoted by L_Φ.

Let us remark that a family of (more than one) spectral automorphisms need not be a spectral family of automorphisms—see Example 6.5.

Proposition 6.2 *For every spectral family Φ of automorphisms of an orthomodular lattice L there is the greatest spectral algebra of the family Φ.*

Proof Let $\{B_i\colon i \in I\}$ be the set of spectral algebras of the family Φ. If $a \in B_i$ then $a \leftrightarrow B_i$, hence $a \in L_\Phi$ and $a \leftrightarrow B_j$, for every $i, j \in I$. Hence $B_i \leftrightarrow B_j$ for every $i, j \in I$ and the subortholattice of L generated by $\{B_i\colon i \in I\}$ is a Boolean algebra (see [8]) and satisfies condition (P2). Obviously, it is the greatest spectral algebra of the family Φ. $\square$

Definition 6.3 Let Φ be a spectral family of automorphisms of an orthomodular lattice. The *spectrum* σ_Φ *of the family* Φ is the greatest spectral algebra of the family Φ.

Proposition 6.4 *Let Φ be a spectral family of automorphisms of an orthomodular lattice L. Then:*

(1) $\sigma_\Phi = C(L_\Phi)$*;*
(2) $\sigma_\Phi = C\big(K(\sigma_\Phi)\big)$ *(i.e., σ_Φ is C-maximal [3]);*
(3) $\sigma_\Phi = K\big(K(\sigma_\Phi)\big)$*.*

Proof (1) According to Definition 6.1, B is a spectral algebra of Φ if and only if $L_\Phi = K(B)$; in such a case $B \leftrightarrow B$ and therefore $B \subseteq L_\Phi$ and $B \subseteq C(L_\Phi)$. In particular, $L_\Phi = K(\sigma_\Phi)$ and $\sigma_\Phi \subseteq C(L_\Phi)$. Obviously, $L_\Phi \subseteq K\big(C(L_\Phi)\big)$ and, since $\sigma_\Phi \subseteq C(L_\Phi)$, $K\big(C(L_\Phi)\big) \subseteq K(\sigma_\Phi) = L_\Phi$, hence $C(L_\Phi)$ is a spectral algebra of Φ. Since $\sigma_\Phi \subseteq C(L_\Phi)$ and σ_Φ is the greatest spectral algebra of Φ, we obtain $\sigma_\Phi = C(L_\Phi)$.

(2) According to part (1) and Definition 6.1, $\sigma_\Phi = C(L_\Phi) = C\big(K(\sigma_\Phi)\big)$.

(3) According to [3, Theorem 3.1], $\sigma_\Phi = C\big(K(\sigma_\Phi)\big)$ if and only if $\sigma_\Phi = K\big(K(\sigma_\Phi)\big)$. $\square$

Example 6.5 Let L be the lattice from Example 3.3, φ, ψ be automorphisms of L such that a, b, c are φ-invariant, d, e are permuted by φ, c, d, e are ψ-invariant and a, b are permuted by ψ. Then $\Phi = \{\varphi, \psi\}$ is a nonspectral family of spectral automorphisms. Indeed, we have shown in Example 3.3 that φ is spectral, and, similarly, ψ is spectral, too. On the other hand, $L_\Phi = L_\varphi \cap L_\psi = \{\mathbf{0}, c, c', \mathbf{1}\} = C(L_\Phi)$, hence $K(C(L_\Phi)) = L \neq L_\Phi$ and therefore, according to Proposition 6.4, Φ is not a spectral family.

Theorem 6.6 *Let L be an orthomodular lattice and Φ be a family of spectral automorphisms of L. Then Φ is a spectral family if and only if $\sigma_\varphi \leftrightarrow \sigma_\psi$ for every $\varphi, \psi \in \Phi$. In this case, the spectrum σ_Φ of the family contains all spectra σ_φ, $\varphi \in \Phi$.*

Proof $\Rightarrow$: Let $\varphi \in \Phi$. For every $a \in K(\sigma_\Phi) = L_\Phi$ we have $\varphi(a) = a$ and therefore $a \leftrightarrow \sigma_\varphi$. Hence, $\sigma_\varphi \leftrightarrow K(\sigma_\Phi)$, and thus, using Proposition 6.4, $\sigma_\varphi \subseteq K\big(K(\sigma_\Phi)\big) = \sigma_\Phi$. Since σ_Φ is a Boolean algebra, we get $\sigma_\varphi \leftrightarrow \sigma_\psi$ for every $\varphi, \psi \in \Phi$.

$\Leftarrow$: Let us denote by B the subortholattice of L generated by $\bigcup_{\varphi\in\Phi} \sigma_\varphi$. A subortholattice of an orthomodular lattice generated by a family of pairwise commuting Boolean algebras is a Boolean algebra (see [8]). The following statements are equivalent: $\varphi(a) = a$ for every $\varphi \in \Phi$, $a \leftrightarrow \sigma_\varphi$ for every $\varphi \in \Phi$, $a \leftrightarrow \bigcup_{\varphi\in\Phi} \sigma_\varphi$, $a \leftrightarrow B$. Hence B is a spectral algebra and Φ

is a spectral family. □

From the very beginning, the purpose of introducing and studying spectral automorphisms has been to construct something similar to the Hilbert space spectral theory without using the specific instruments available in a Hilbert space setting, but using only the abstract orthomodular lattice structure. The next result is intended as an analogue of the Stone theorem concerning strongly continuous uniparametric groups of unitary operators. Before stating it, we should notice the following easily verifiable facts:

Remark 6.7 (1) The identity $\mathrm{id} : L \to L$ is a spectral automorphism and $\sigma_{\mathrm{id}} = C(L)$.

(2) The inverse φ^{-1} of a spectral automorphism φ of L is also spectral and $\sigma_{\varphi^{-1}} = \sigma_\varphi$.

Theorem 6.8 *Let L be an orthomodular lattice and Φ be a family of spectral automorphisms of L. If Φ is an Abelian group and $\varphi(L_\psi) = L_{\varphi\psi}$ for every $\varphi, \psi \in \Phi$ with $\psi \notin \{\mathrm{id}, \varphi^{-1}\}$ then:*

(1) *$L_\varphi = L_\psi$ for every $\varphi, \psi \in \Phi \setminus \{\mathrm{id}\}$;*
(2) *$\sigma_\varphi = \sigma_\psi$ for every $\varphi, \psi \in \Phi \setminus \{\mathrm{id}\}$;*
(3) *Φ is a spectral family.*

Proof (1) Let $\varphi, \psi \in \Phi$. The following statements are equivalent for every $a \in L$: $a \in L_\varphi$, $\varphi(a) = a$, $\psi\big(\varphi(a)\big) = \psi(a)$, $\varphi\big(\psi(a)\big) = \psi(a)$, $\psi(a) \in L_\varphi$. Hence $\psi(L_\varphi) = L_\varphi$.

Let $\varphi, \psi \in L \setminus \{\mathrm{id}\}$ be different. If $\psi = \varphi^{-1}$ then, obviously, $L_\varphi = L_\psi$. Let us suppose that $\psi \neq \varphi^{-1}$ and let us denote $\chi = (\varphi\psi)^{-1} = \psi^{-1}\varphi^{-1}$. Since $\chi \in \Phi \setminus \{\mathrm{id}, \varphi^{-1}, \psi^{-1}\}$, we obtain $L_\varphi = L_{\chi^{-1}\varphi\chi} = \chi^{-1}\big(\varphi(L_\chi)\big) = \chi^{-1}\big(\psi(L_\chi))\big) = L_{\chi^{-1}\psi\chi} = L_\psi$.

(2) According to part (1) and Proposition 6.4 (1), $\sigma_\varphi = C(L_\varphi) = C(L_\psi) = \sigma_\psi$ for every $\varphi, \psi \in \Phi \setminus \{\mathrm{id}\}$.

(3) According to Proposition 3.2, $\sigma_{\mathrm{id}} = C(L)$ and therefore $\sigma_{\mathrm{id}} \leftrightarrow \sigma_\varphi$ for every $\varphi \in \Phi$. Since, according to part (2), the set $\{\sigma_\varphi \colon \varphi \in \Phi\}$ is at most 2-element, $\sigma_\varphi \leftrightarrow \sigma_\psi$ for every $\varphi, \psi \in \Phi$. According to Theorem 6.6, Φ is a spectral family. □

Let us show a non-trivial example of an abelian group of spectral automorphisms satisfying the conditions of the Theorem 6.8.

Example 6.9 Let L be the orthomodular lattice described by the third Greechie diagram in Fig. 1. L is the union of three blocks, the first determined by atoms $\{a, b, c\}$, the second determined by atoms $\{c, d, e\}$ and the last determined by atoms $\{e, f, g, h, i\}$. Let φ be an automorphism of L such that a, b, c, d, e are φ-invariant and φ performs a cyclic permutation on the atoms f, g, h, i (i.e., $\varphi(f) = g$, $\varphi(g) = h$, $\varphi(h) = i$ and $\varphi(i) = f$). Clearly, $\varphi^4 = \mathrm{id}$ and $\Phi = \{\mathrm{id}, \varphi, \varphi^2, \varphi^3\}$ is an abelian group of automorphisms of L. $L_\varphi = L_{\varphi^2} = L_{\varphi^3} = L_\Phi$ is the set-theoretical union of the blocks determined by $\{a, b, c\}$ and $\{c, d, e\}$. $C(L_\varphi) = C(L_{\varphi^2}) = C(L_{\varphi^3}) = C(L_\Phi) = \{\mathbf{0}, c, c', \mathbf{1}\}$ is the spectrum of automorphisms $\varphi, \varphi^2, \varphi^3$, and therefore of the family Φ, because for $x \in L$, $x \leftrightarrow \{\mathbf{0}, c, c', \mathbf{1}\}$ if and only if $x \leftrightarrow c$ if and only if $x \in L_\Phi$.

Let us remark that Theorem 6.8 gives purely algebraic conditions for a family of automorphisms to have a spectrum. The last hypothesis (namely, that $\varphi(L_\psi) = L_{\varphi\psi}$) can be seen as a replacement for the continuity condition in the original Stone theorem.

Acknowledgments

The authors thank to Professor Mirko Navara for pointing their attention to spectral automorphisms in products and horizontal sums. The first author gratefully acknowledges the scholarship of the Czech Technical University in Prague. The work was supported by the research plan of the Ministry of Education of the Czech Republic no. 6840770010.

References

1. Beltrametti, E. G., Casinelli, G.: *The Logic of Quantum Mechanics.* Addison-Wesley, Reading (1981).
2. Halmos, P. R.: *Introduction to Hilbert Space and Theory of Spectral Multiplicity.* Chelsea, New York (1951).
3. Ivanov, A., Caragheorgheopol, D.: *Spectral automorphisms in quantum logics.* Int. J. Theor. Phys. **49**, 3146–3152 (2010). doi: 10.1007/s10773-010-0369-3.
4. Kalmbach, G.: *Orthomodular lattices.* Academic Press, London (1983).
5. Piron, C.: *Foundations of Quantum Physics.* Benjamin, Reading (1976).
6. Pták, P., Pulmannová, S.: *Orthomodular Structures as Quantum Logics.* VEDA/Kluwer Academic, Bratislava/Dordrecht (1991).
7. Reed, M., Simon, B.: *Methods of Modern Mathematical Physics, vol. I.* Academic Press, New York (1975).
8. Varadarajan, V. S.: *Geometry of Quantum Theory, vol. I.* van Nostrand, Princeton (1968).
9. Wright, R.: *The structure of projection-valued states: a generalization of Wigner's theorem*, Int. J. Theor. Phys. **16**, 567–573 (1977).