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Atomic Sequential Effect Algebras

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Abstract Various conditions ensuring that a sequential effect algebra or the set of sharp elements of a sequential effect algebra is a Boolean algebra are presented.

Keywords Effect algebra · Boolean algebra · Sequential product · Atomic · Sharp

The basic algebraic object for studying quantum structures is the effect algebra—see, e.g., Foulis and Bennet [2]. Gudder (see, e.g., [5]) introduced the notion of a sequential product on an effect algebra as an abstract formalization of a sequential measurement. Sequential effect algebras have the property that sharp elements remain sharp whenever we use an embedding to a greater structure, hence the notion of sharpness is not contextual within sequential effect algebras. In this paper we present several results stating that a sequential effect algebra (or the set of sharp elements of a sequential effect algebra) is a Boolean algebra. Some of them are generalizations of analogous results of Gudder and Greechie [5].

1 Basic Notions

Definition 1.1 An *effect algebra* is an algebraic structure $(E, \oplus, 0, 1)$ such that E is a set, 0 and 1 are different elements of E and $\oplus$ is a partial binary operation on E such that for every $a, b, c \in E$ the following conditions hold (the equalities mean also "if one side exists then the other side exists"):

(1) $a \oplus b = b \oplus a$ (*commutativity*),
(2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (*associativity*),
(3) for every $a \in E$ there is a unique $a' \in E$ such that $a \oplus a' = 1$ (*orthosupplement*),

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(4) $a = 0$ whenever $a \oplus 1$ is defined (*zero-unit law*).

For simplicity, we use the notation E for an effect algebra. A partial ordering on an effect algebra E is defined by $a \leq b$ iff there is a $c \in E$ such that $b = a \oplus c$; such an element c is unique (if it exists) and is denoted by $b \ominus a$. Also, 0 (1, resp.) is the least (the greatest, resp.) element of E with respect to this partial ordering. An *orthogonality* relation on E is defined by $a \perp b$ iff $a \oplus b$ exists (i.e., iff $a \leq b'$). It can be shown that $a \oplus 0 = a$ for every $a \in E$ and that the *cancellation law* is valid: for every $a, b, c \in E$ with $a \oplus b \leq a \oplus c$ we have $b \leq c$.

For $a \leq b$ we denote $[a, b] = \{c \in E\colon\ a \leq c \leq b\}$. A *chain* in E is a nonempty linearly (totally) ordered subset of E.

Definition 1.2 Let E be an effect algebra. An element $a \in E$ is called

- *sharp*, if $a \wedge a' = 0$;
- *principal*, if $b \oplus c \leq a$ for every $b, c \in E$ with $b \perp c$ and $b, c \leq a$;
- *central*, if a and a' are principal and for every $b \in E$ there are $b_1, b_2 \in E$ such that $b_1 \leq a$, $b_2 \leq a'$ and $b = b_1 \oplus b_2$. The set of central elements of E is called the *center* of E.

By definition, every central element is principal, and it is well-known and easy to see that every principal element is sharp. The reverse implications need not be true.

Definition 1.3 An *orthoalgebra* is an effect algebra in which every its element is sharp.
An *orthomodular poset* is an effect algebra in which every its element is principal.

Since every principal element in an effect algebra is sharp, every orthomodular poset is an orthoalgebra.

2 Atomic Effect Algebras

Definition 2.1 Let E be an effect algebra.

- An *atom* in E is a minimal element of $E \setminus \{0\}$.
- The effect algebra E is *atomic* if every nonzero element of E dominates an atom.
- The effect algebra E is *atomistic* if every nonzero element of E is a supremum of a set of atoms (hence dominated by this element).
- The effect algebra E is *determined by atoms* if for different elements $a, b \in E$ the sets of atoms in $[0, a]$ and $[0, b]$ are different.

Let us remark that an effect algebra is atomistic iff the atoms are order determining in the sense that, if every atom in $[0, a]$ belongs to $[0, b]$, then $a \leq b$.

Lemma 2.2 *Every atomistic effect algebra is determined by atoms. Every effect algebra determined by atoms is atomic.*

Proof Let E be an atomistic effect algebra. Since every nonzero element of E is the supremum of atoms it dominates, for different elements we obtain different sets of dominated atoms.

Let E be an effect algebra determined by atoms. Then for every nonzero element $a \in E$ the set of dominated atoms is nonempty. □

Greechie [3] presented examples of atomic orthomodular posets not determined by atoms. Let us present an example of a nonatomistic orthomodular poset determined by atoms.

Example 2.3 Let $X_1 = \{x_1\}$, $X_2 = \{x_2\}$, X_3 and X_4 be mutually disjoint sets, and let X_3, X_4 be infinite. Let us put $X = \bigcup_{i=1}^4 X_i$,

$$E' = \{\emptyset, X_1 \cup X_2, X_2 \cup X_3, X_3 \cup X_4, X_4 \cup X_1, X\},$$
$$E = \{(A \setminus F) \cup (F \setminus A) \colon\ A \in E' \text{ and } F \subset X_3 \cup X_4 \text{ is finite}\}.$$

Then $(E, \oplus, \emptyset, X)$ with $A \oplus B = A \cup B$ for disjoint $A, B \in E$ is an orthomodular poset. The orthosupplement is the set theoretic complement in X, the partial ordering is the inclusion. The atoms in E are $\{x_1, x_2\}$ and one-element subsets of $X_3 \cup X_4$.

E is not atomistic because the set of atoms dominated by the element $X_1 \cup X_4$ is the set of one-element subsets of X_4 that has $X_3 \cup X_4$ as an upper bound and $X_3 \cup X_4 \not\supseteq X_1 \cup X_4$.

Let us prove that E is determined by atoms. Let $A, B \in E$ such that the sets of atoms dominated by A and B coincide. Since $\{x\}$ is an atom for every $x \in X_3 \cup X_4$, we obtain $A \cap (X_3 \cup X_4) = B \cap (X_3 \cup X_4)$. Let us suppose that $A \neq B$ and seek a contradiction. Then, e.g., $A \not\subset B$ and there is an $x \in X_1 \cup X_2$ such that $x \in A \setminus B$. Let, e.g., $x = x_1$. Since $\{x_1, x_2\}$ is an atom not dominated by B, it is not dominated by A and therefore $x_2 \notin A$. Hence $B \cap X_4 = A \cap X_4$ is cofinite and $B \cap X_3 = A \cap X_3$ is finite. Therefore $x_1 \in B$—a contradiction.

It is known that every atomic orthomodular lattice is atomistic—see, e.g., Pták and Pulmannová [6]. Let us present an analogous result for effect algebras (lattice orthoalgebras are orthomodular lattices).

Proposition 2.4 *Every lattice effect algebra determined by atoms is atomistic.*

Proof Let E be a lattice effect algebra determined by atoms, $a \in E \setminus \{0\}$ and A be the set of atoms in $[0, a]$. For every $b \in E$ that dominates all elements of A we obtain that A is the set of atoms in $[0, a \wedge b]$ and therefore, since E is determined by atoms, $a = a \wedge b \leq b$. Hence $a = \bigvee A$. □

Let us remark that for example the 3-chain $C_3 = \{0, a, 1\}$ with $a \oplus a = 1$ and $x \oplus 0 = x$ for every $x \in C_3$ is an atomic lattice effect algebra that is not determined by atoms—different elements $a, 1$ dominate the same set $\{a\}$ of atoms.

Proposition 2.5 *Every effect algebra in which every its nonzero element dominates a nonzero sharp element is an orthoalgebra.*

Proof Let us suppose that the effect algebra E is not an orthoalgebra and seek a contradiction. There is an unsharp element $a \in E$. Hence there is a nonzero element $b \in E$ such that $b \leq a, a'$. According to the assumption, there is a nonzero sharp element $c \in E$ such that $c \leq b$. Then $c \leq a, a'$ and therefore $a \leq c'$. Hence $c \leq a \leq c'$ and therefore $c \wedge c' = c \neq 0$—a contradiction. □

Corollary 2.6 *Every atomic effect algebra in which every its atom is sharp is an orthoalgebra.*

3 Sequential Effect Algebras

Definition 3.1 A *sequential product* on an effect algebra E is a binary operation $\circ$ on E such that for every $a, b, c \in E$ the following conditions hold:

(1) $a \circ (b \oplus c) = (a \circ b) \oplus (a \circ c)$ if $b \oplus c$ exists;
(2) $1 \circ a = a$;
(3) if $a \circ b = 0$ then $a \mid b$ (where $a \mid b$ denotes $a \circ b = b \circ a$);
(4) if $a \mid b$ then $a \mid b'$ and $a \circ (b \circ c) = (a \circ b) \circ c$;
(5) if $c \mid a, b$ then $c \mid a \circ b$ and $c \mid a \oplus b$ (if $a \oplus b$ exists).

An effect algebra with a sequential product is called a *sequential effect algebra*.

For examples of sequential effect algebras see Gudder and Greechie [5]—e.g., every Boolean algebra with $a \circ b = a \wedge b$ forms a sequential effect algebra, the set of positive self-adjoint operators on a Hilbert space bounded by the identity with $A \circ B = A^{1/2}BA^{1/2}$ forms a sequential algebra and there is an atomic sequential effect algebra that is not a Boolean algebra (Sect. 7).

Let us present some results concerning sequential effect algebras.

Proposition 3.2 *Let E be a sequential effect algebra. Then for every $a, b \in E$ the following properties hold:*

(1) $a \circ 0 = 0 \circ a = 0$;
(2) $a \circ 1 = 1 \circ a = a$;
(3) $a \circ b \leq a$;
(4) *if a is sharp then $a \leq b$ iff $a \circ b = b \circ a = a$.*

Proof See Gudder and Greechie [5], Lemma 3.1 and Theorem 3.4. □

Proposition 3.3 *Let E be a sequential effect algebra, $a \in E$ be an atom. Then $a \mid b$ for every $b \in E$.*

Proof Let $b \in E$. Since $a \circ b \leq a$ and a is an atom, we obtain that $a \circ b \in \{0, a\}$. If $a \circ b = 0$ then $a \mid b$ from the definition of a sequential product. Let us suppose that $a \circ b = a$. We obtain that $a \circ b = a = a \circ 1 = a \circ (b \oplus b') = (a \circ b) \oplus (a \circ b')$ and therefore $a \circ b' = 0$. Hence $a \mid b'$ and therefore $a \mid b$. □

Lemma 3.4 *Let E be a sequential effect algebra, $a, b, c \in E$ such that a is sharp, $a \mid c$ and $a \leq b \circ c$. Then $a \leq b, c$. If, moreover, $a \mid (c \circ b)$ then $a \leq c \circ b$.*

Proof According to Proposition 3.2, $a \leq b \circ c \leq b$ and, since a is sharp, $a \mid b$. Since $a \leq b, b \circ c$, a is sharp and $a \mid b, c$, we obtain, according to Proposition 3.2 and the definition of a sequential product, that $a = a \circ (b \circ c) = (a \circ b) \circ c = a \circ c = c \circ a \leq c$. Hence, if moreover $a \mid (c \circ b)$, we obtain that $a = a \circ b = (a \circ c) \circ b = a \circ (c \circ b) = (c \circ b) \circ a \leq c \circ b$. □

Proposition 3.5 *Let E be a sequential effect algebra, $a \in E$ be a sharp atom. Then $a \leq b \circ c$ iff $a \leq c \circ b$ for every $b, c \in E$.*

Proof Let $b, c \in E$ such that $a \leq b \circ c$. According to Proposition 3.3, $a \mid c$ and $a \mid (c \circ b)$. According to Lemma 3.4, $a \leq c \circ b$. The reverse implication can be proved analogously. □

Let us summarize some properties of sequential effect algebras that we will use in the sequel.

Proposition 3.6 *Let E be a sequential effect algebra.*

(1) *The set of sharp elements of E is a sub-effect algebra and forms an orthomodular poset.*
(2) *Let $a, b \in E$ be sharp. If $a \wedge b$ (resp., $a \vee b$) exists in E then $a \wedge b$ is sharp (resp., $a \vee b$ is sharp).*
(3) *If E is chain finite then every atom of E is sharp.*
(4) *If $a \in E$ is an atom then $a \leq b$ or $a \leq b'$ for every $b \in E$.*

Proof See Gudder and Greechie [5], Corollary 3.5, Corollary 4.3, proof of Theorem 5.5 and Lemma 5.2. □

4 Weak Distributivity, Maximality Property

We will present two properties and show that an orthomodular poset with these two properties is a Boolean algebra (see [7]).

Definition 4.1 An effect algebra E is *weakly distributive* if, for every $a, b \in E$, $a = 0$ whenever $a \wedge b = a \wedge b' = 0$.

Obviously, every Boolean algebra is weakly distributive. Let us present an example of a weakly distributive orthomodular poset that is not a Boolean algebra.

Example 4.2 Let X_1, X_2, X_3, X_4 be mutually disjoint infinite sets. Let us put $X = \bigcup_{i=1}^{4} X_i$,

$$E' = \{\emptyset, X_1 \cup X_2, X_2 \cup X_3, X_3 \cup X_4, X_4 \cup X_1, X\},$$
$$E = \{(A \setminus F) \cup (F \setminus A) \colon A \in E' \text{ and } F \subset X \text{ is finite}\}.$$

Then $(E, \oplus, \emptyset, X)$ with $A \oplus B = A \cup B$ for disjoint $A, B \in E$ is an orthomodular poset. The orthosupplement is the set theoretic complement in X, the partial ordering is the inclusion.

E is not a lattice because $X_1 \wedge X_2$ does not exist (the set of lower bounds—the set of finite subsets of $X_1 \cup X_2$—does not have a greatest element).

Let us prove that E is weakly distributive. Let $A, B \in E$ such that $A \wedge B = A \wedge B' = \emptyset$. Since $\{x\} \in E$ for every $x \in X$, we obtain that $A \cap B = A \cap B' = \emptyset$ and therefore $A \cap X = A \cap (B \cup B') = (A \cap B) \cup (A \cap B') = \emptyset$. Hence $A = \emptyset$.

In the sequel we will use the following statement (see also [8], Proposition 2.2).

Proposition 4.3 *Let E be an atomic effect algebra such that $a \leq b$ or $a \leq b'$ for every atom $a \in E$ and for every element $b \in E$. Then E is weakly distributive.*

Proof Let us suppose that E is not weakly distributive and seek a contradiction. There are elements $a, b \in E$ such that $a \wedge b = a \wedge b' = 0$ and $a \neq 0$. Since E is atomic, there is an atom $c \in E$ such that $c \leq a$. Since $a \wedge b = a \wedge b' = 0$, we obtain that $c \not\leq b$ and $c \not\leq b'$—a contradiction. □

Definition 4.4 An effect algebra E has the *maximality property* if $[0, a] \cap [0, b]$ has a maximal element for every $a, b \in E$.

Let us show some examples of effect algebras with the maximality property.

Proposition 4.5 *An effect algebra E has the maximality property if at least one of the following conditions hold:*

(1) *E is a lattice.*
(2) *E is chain finite.*

Proof (1) For every $a, b \in E$, the element $a \wedge b$ is a maximal (even the greatest) element of $[0, a] \cap [0, b]$.

(2) Let $a, b \in E$. According to Zorn's lemma, there is a maximal chain in $[0, a] \cap [0, b]$. According to the assumption, this chain has a maximal element and this element is also a maximal element of $[0, a] \cap [0, b]$. □

Theorem 4.6 *Every weakly distributive orthomodular poset with the maximality property is a Boolean algebra.*

Proof See Tkadlec [7], Theorem 4.2. □

It should be noted that the above theorem cannot be generalized to orthoalgebras (see the so-called Fano plane in [1], Sect. 7).

5 Main Results

Let us recall two known results about the center of an effect algebra.

Theorem 5.1 (1) *The center of an effect algebra is a sub-effect algebra and forms a Boolean algebra.*

(2) *The center of a sequential effect algebra is the set of sharp elements that commute (with respect to the sequential product) with all elements.*

Proof (1) See Greechie et al. [4], Theorem 5.4.
(2) See Gudder and Greechie [5], Theorem 4.4. □

Proposition 5.2 *An atom in a sequential effect algebra is central iff it is sharp.*

Proof It follows from the part (2) of Theorem 5.1 and from Proposition 3.3. □

Proposition 5.3 *Every atomic sequential effect algebra is weakly distributive.*

Proof It follows from Proposition 3.6 and Proposition 4.3. □

Theorem 5.4 *Every atomic sequential orthoalgebra with the maximality property is a Boolean algebra.*

Proof Let E be an atomic sequential orthoalgebra with the maximality property. According to Proposition 3.6, E is an orthomodular poset. According to Proposition 5.3, E is weakly distributive. Since E has the maximality property, we obtain, according to Theorem 4.6, that E is a Boolean algebra. □

Corollary 5.5 *Every chain finite sequential effect algebra is a Boolean algebra.*

Proof Let E be a chain finite sequential effect algebra. Then E is atomic. According to Proposition 3.6, every atom is sharp. According to Corollary 2.6, E is an orthoalgebra. According to Proposition 4.5, E has the maximality property. The rest follows from Theorem 5.4. □

Let us remark that the last corollary was stated in Gudder and Greechie [5], Theorem 5.5(ii), with a different proof.

Theorem 5.6 *Every sequential effect algebra determined by atoms such that every atom is sharp is a Boolean algebra.*

Proof Let E be a sequential effect algebra determined by atoms such that every atom is sharp. Let $b, c \in E$. According to Proposition 3.5, the sets of atoms dominated by $b \circ c$ and $c \circ b$ coincide. Since E is determined by atoms, $b \circ c = c \circ b$. According to Theorem 5.1, the center is the set of sharp elements. According to Corollary 2.6, E is an orthoalgebra, hence every element of E is central. The rest follows from Theorem 5.1. □

The last theorem generalizes the result of Gudder and Greechie [5] that was stated for atomistic sequential orthoalgebras (see Lemma 2.2 and Example 2.3). According to this theorem, the orthomodular poset from Example 2.3 cannot be organized into a sequential effect algebra (every element in an orthomodular poset is sharp).

Theorem 5.7 *The set of sharp elements of a weakly distributive lattice sequential effect algebra is a Boolean algebra.*

Proof Let E be a weakly distributive lattice sequential effect algebra. According to Proposition 3.6, the set $S(E)$ of sharp elements of E is an orthomodular poset and a sublattice of E. Hence, according to Proposition 4.5, $S(E)$ has the maximality property. Moreover, since E is weakly distributive, $S(E)$ is weakly distributive, too. Hence, according to Theorem 4.6, $S(E)$ is a Boolean algebra. □

Corollary 5.8 *The set of sharp elements of an atomic lattice sequential effect algebra is a Boolean algebra.*

Proof According to Proposition 5.3, every atomic sequential effect algebra is weakly distributive. The rest follows from Theorem 5.7. □

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