Active Adaptive Control

by

Ing. Jan Rathouský

supervised by prof. Ing. Vladimír Havlena, CSc.

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Jan Rathouský

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Abstract

This thesis is concerned with stochastically optimal adaptive control strategies and their so-called active adaptive modifications, which represent computationally feasible approximations of dual control. A control strategy is called stochastically optimal, if it optimally solves a given control problem defined for a stochastic system, i.e. a system, the behavior of which is described by the means of probability theory. The thesis is particularly concerned with analysis of the cautious control strategy. The term active adaptive then means, that the control strategy adapts to new information about the system and at the same time actively examines the system and aims to induce such response from the system that brings as much information as possible, while not violating the control performance more than allowable.

The first part of this thesis contains derivation and analysis of the cautious controller of a general ARMAX model with known MA part. A complete analysis of convergence of the associated cautious Riccati-like equation is presented, which is important when extending the control horizon to infinity to find a steady state controller. It is also shown that a finite steady state control law exists even in the case of divergence of the cautious Riccati-like equation. Because the results are formulated for an ARMAX model, they are applicable to a wide range of linear dynamical systems.

The second part of the thesis proposes novel active adaptive control algorithms. It starts with a single-step algorithm for an ARX system based on cautious control. Extension of this algorithm to multiple step is possible, but has not been studied because of the inconvenient properties of cautious control derived in the first part of the thesis. Multiple step adaptive active algorithms based on information matrix properties are presented next, including the so-called ellipsoid algorithm that is studied in more detail. These algorithms are based on a two-phase bicriterial approach, which means that an initial control is first found using a classical control design method (MPC is usually used throughout the thesis) and this control is afterwards altered to achieve active excitation. The thesis also presents a conservative convexification of the ellipsoid algorithm that makes it solvable for higher dimensional systems, where the original nonconvex algorithm becomes infeasible.

Abstrakt

Tato dizertační práce se zabývá stochasticky optimálními adaptivními strategiemi řízení a jejich takzvanými aktivními adaptivními modifikacemi, jež představují spočitatelné aproximace duálního řízení. Strategie řízení se nazývá stochasticky optimální, pokud optimálně řeší daný problém řízení stochastického systému, tj. systému, jehož chování je popsáno pomocí nástrojů teorie pravděpodobnosti. Práce se zejména zabývá analýzou opatrné strategie řízení. Pojem aktivní adaptivní potom znamená, že se daná strategie řízení přizpůsobuje nově získaným informacím o systému a zároveň systém aktivně zkoumá s cílem vyvolat v systému takovou odezvu, která poskytne co nejvíce informací, aniž by bylo porušeno splnění požadavků na řízení více, než je přípustné.

První část práce obsahuje odvození a analýzu opatrného regulátoru pro obecný ARMAX model se známou MA částí. Uvedena je kompletní analýza konvergence přidružené opatrné Riccatiho rovnice, což je důležité pro prodloužení horizontu řízení do nekonečna a nalezení ustáleného regulátoru. Dále je ukázáno, že konečný ustálený zákon řízení existuje i v případě divergence opatrné Riccatiho rovnice. Jelikož jsou výsledky formulovány pro ARMAX model, jsou aplikovatelné pro širokou třídu lineárních dynamických systémů.

Ve druhé části práce jsou navrženy nové aktivní adaptivní algoritmy řízení. Nejprve je uveden jednokrokový algoritmus pro ARX systém založený na opatrném řízení. Možné rozšíření tohoto algoritmu na vícekrokový je popsáno, ale nebylo studováno kvůli nevhodným vlastnostem opatrného řízení odvozeným v první části práce. Dále jsou odvozeny vícekrokové aktivní adaptivní algoritmy založené na vlastnostech informační matice, včetně takzvaného elipsoidového algoritmu, který je studován detailněji. Tyto algoritmy jsou založené na dvoufázovém postupu, což znamená, že je nejprve klasickou metodou nalezeno prvotní řízení (v celé práci se používá pro tento účel MPC), a toto řízení je následně upraveno tak, aby bylo dosaženo aktivního vybuzení. Práce také navrhuje konzervativní konvexní modifikaci elipsoidového algoritmu, která umožňuje jeho řešení i v případě systémů vyšší dimenze, kde původní algoritmus selhává kvůli výpočetní náročnosti.

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Chapter 1

Introduction

Various techniques and methods exist for designing control algorithms, from rather simple methods based on basic characteristics of the controlled system such as oscillation frequency or bandwidth, to methods exploiting advanced optimization techniques that use sophisticated system models. If the controller design relies on a model of the controlled system, the model quality and accuracy is an important factor influencing the performance of the resulting controller. The model can rarely describe the behavior of the system exactly. Many classical design methods such as pole placing or the classical linear quadratic (LQ) controller assume at the time of design that the model is exact and rely on inherent robustness of the design methods, i.e. on the ability of the controller to cope to a certain extent with different behavior of the controlled system.

Robustness of a controller can be analyzed by determining the nature and amount of uncertainty in the model (e.g. the gain and phase margin) that still does not significantly jeopardize the control objectives such as stability or overshoot. Methods also exist to include the assumed uncertainty of the model into the design process, thus developing a controller that is apriori robust to the modelled uncertainty. These methods include frequency domain based design using additive, multiplicative or even structured uncertainty models and finding the optimal controller via $\mathcal{H}^\infty$ or similar optimization techniques [54, 63].

The uncertainty in the model is not always caused only by inaccurate approximation of the system. Even if the model is quite accurate at the time of design, the system behavior may change over time, which may lead to deteriorated performance. Methods of adaptive control aim to solving these problems by observing the system behavior, detecting its changes, improving the knowledge about the system and adapting the control algorithm accordingly. The use of adaptive methods is obviously not limited to control of time-variant systems, they may be as well convenient for designing self-tuning regulators that improve their performance with the use of the knowledge gained from observation.

Adaptive methods may be divided into two groups – methods that use identification to improve the model and then adapt the control algorithm based on the new model (indirect methods), and methods that directly adapt the algorithm without identification

(direct methods). The former adaptive methods must therefore also include identification algorithms that make the adaptation possible.

This thesis is concerned with stochastically optimal control strategies and their so-called active adaptive modifications. A control strategy is called stochastically optimal, if it optimally solves a given control problem defined for a stochastic system, i.e. a system the behavior of which is described by the means of probability theory. These strategies naturally use discrete-time-domain models described by some parameters that are considered uncertain (or unknown) and the goal of the adaptation process is to identify these parameters with a sufficient accuracy. The term *active* then means, that the control strategy actively examines the system and aims to induce such response from the system that brings as much information about parameters as possible, while not violating the control performance more than allowable.

The goal of the thesis is to examine existing stochastically optimal control strategies and to propose new active adaptive strategies in time domain as computationally feasible approximations of dual control. These strategies should be designed for linear discrete-time system models with uncertain parameters, preferably the ARMAX model. The next goal of the thesis is to analyze properties of cautious control. Although cautious control plays an important role among stochastic control strategies, the goal is to show, that it uses an unrealistic uncertainty model and that the interpretation of its results is problematic, especially when trying to extend the problem to an infinite control horizon. Attention is therefore particularly given to analyzing the limit behavior of the cautious linear quadratic controller, including its convergence to a limit solution and the closed loop stability of this solution, and consequently also the use of cautious control as a basis for developing the active adaptive strategies.

Some of the problems addressed in the thesis, such as cautious or dual control, were defined in the 60' and 70' of the 20th century. The concept of dual control and cautiousness comes from [17, 18] and was further developed in [7, 8, 42] and [43]. The term active adaptive control appeared in [56] and [57]. The problems of controlling an uncertain system, modeling the uncertainty and improving the knowledge about the system are, however, still intensively studied, as for example in the books [10, 19] or more recent publications [26, 33] or [15] and [16].

The next sections of this chapter are concerned with definitions of terminology used in this thesis. We will not use formal mathematical definitions in this section, as the goal is not to define these commonly used objects properly, but rather to put them in the right context and explain their usage. We will particularly focus on time-domain uncertainty modelling and on stochastically optimal control strategies based on these uncertainty models.

1.1 Time-domain system models

The use of probabilistic methods in uncertainty description in time-domain models is usual, however, there may be various sources of uncertainty in the system description and the use of probabilistic methods should be considered carefully. Therefore we will first present

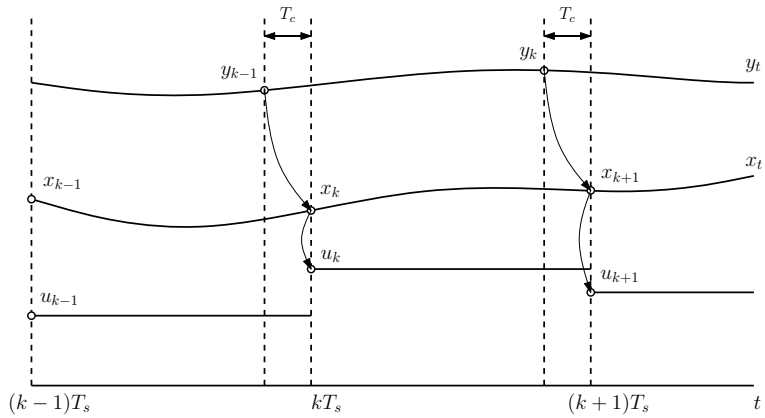


Figure 1.1: Asynchronous sampling of a continuous system with sampling interval T_s. The control law computation indicated by the arrows takes place within the time interval T_c. First, the state estimate is calculated using the output measurement. The control input is then generated according to the control law $u_k^* = \mu_k^*(x_k)$.

a general analysis of uncertainty modelling. It is important to emphasize, that we are considering discrete-time systems according to the Figure 1.1 throughout the thesis. The figure shows, how a deterministic system is created by asynchronous sampling of a continuous system, i.e. the input and output of such system are sampled at different time instants. One consequence is, that we always assume a direct influence of input u_k on output y_k. Second consequence of this assumption is that when estimating x_{k+1} from information including y_k, the y_k is sampled short before x_{k+1} and therefore contains more information than if it were sampled at the same instant as u_k. This information is expressed by correlation between the output noise and the process noise in the stochastic system description later in the chapter.

1.1.1 Deterministic system

A *deterministic discrete-time system* is described by the equations

$$\begin{aligned} x_{k+1} &= f(\theta_k, x_k, u_k), \\ y_k &= g(\theta_k, x_k, u_k), \end{aligned} \tag{1.1}$$

where, as usual, u_k, y_k and x_k denote the system input, output and state, respectively. The variable θ_k represents dependence of the function f on some parameters, which may be generally time varying. An example of such system is a deterministic discrete-time linear system

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k, \\ y_k &= C_k x_k + D_k u_k, \end{aligned} \tag{1.2}$$

where A_k, B_k, C_k and D_k are (generally time-varying) matrices of appropriate dimensions, which are parametrized by θ_k.

1.1.2 Stochastic system

A *stochastic* discrete-time system is a system, where the state transition cannot be described by a deterministic function, but rather by a probability distribution. We will use a conditional probability density function (c.p.d.f.) to describe the joint distribution of x_{k+1} and y_k and their dependence on x_k, u_k and θ_k, i.e.

$$p(y_k, x_{k+1}|\theta_k, x_k, u_k). \tag{1.3}$$

The reason why we use the joint c.p.d.f. is that due to the sampling scheme depicted in Figure 1.1, the output y_k and state x_{k+1} are not conditionally independent. In some situation, it is useful to work with the marginals of the joint distribution (1.3), i.e.

$$p(x_{k+1}|\theta_k, x_k, u_k)$$

for the state and similarly for the system output

$$p(y_k|\theta_k, x_k, u_k).$$

However note, that for the evaluation of the control law $u^*_{k+1} = \mu^*_{k+1}(x_{k+1})$, the full information about the state x_{k+1} is represented by the c.p.d.f $p(x_{k+1}|\theta_k, x_k, u_k, y_k)$, as indicated by Figure 1.1.

Stochastic systems usually model random influences on systems that are not directly explained by the system. These influences may include unmeasurable input noises of various sources like temperature, air pressure or surface unevenness as well as sensor measurement noise and other influences. The use of probability distributions to describe these effects is justified by their usually random and unpredicable nature. An example of a stochastic system is a linear stochastic system

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k + v_k, \\ y_k &= C_k x_k + D_k u_k + e_k, \end{aligned} \tag{1.4}$$

where v_k is the *process noise* which models disturbances affecting the state dynamics, and e_k is the *measurement noise* that models the disturbances affecting the measurement process. These random variables are usually considered to be white gaussian sequences, i.e.

$$\begin{bmatrix} e_k \\ v_k \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}; \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \right).$$

The matrix S is generally nonzero due to the assumed correlation between the process and measurement noise.

We will now introduce some properties of stochastic systems that we assume further. Let us first define the *data set* $\mathcal{D}^k$ as

$$\mathcal{D}^k = \{u_0, \ldots, u_k, y_0, \ldots, y_k\}. \tag{1.5}$$

The state of a stochastic system was defined in [45] as such quantity, that satisfies the *state property*

$$p(y_k, x_{k+1}|\theta, x_k, u_k, \mathcal{D}^{k-1}) = p(y_k, x_{k+1}|\theta, x_k, u_k),$$

i.e. the data set $\mathcal{D}^{k-1}$ cannot improve the information about y_k and x_{k+1} if the state x_k is known. The state thus contains all information about y_k and x_{k+1} that is present in the data set $\mathcal{D}^{k-1}$. The *natural condition of control* introduced by [45] states that

$$p(x_k|\theta, u_k, \mathcal{D}^{k-1}) = p(x_k|\theta, \mathcal{D}^{k-1}),$$

which says that the information about the state x_k cannot be improved by adding the control u_k to the information in $\mathcal{D}^{k-1}$. This holds if the control u_k depends only on $\mathcal{D}^{k-1}$.

1.1.3 Perfect and imperfect state information

The c.p.d.f. (1.3) depends on x_k and θ_k. If both x_k and θ_k are known at time k, then the c.p.d.f. (1.3) can be directly used for modelling the system behavior. In such situation we say that we have a *perfect state information* and the system has no *uncertain parameters*. Let us now assume that the state x_k is unknown at time k and we only have the information about inputs and outputs, i.e. at time k we know the data set $\mathcal{D}^k$ and the input u_k. Let us also assume that the parameters are known and constant, i.e. $\theta_k = \theta$. With this knowledge, we can use the c.p.d.f. (1.3) to express

$$p(y_k, x_{k+1}|\theta, u_k, \mathcal{D}^{k-1}) = \int p(y_k, x_{k+1}|\theta, x_k, u_k) p(x_k|\theta, \mathcal{D}^{k-1}) \, \mathrm{d}\, x_k, \tag{1.6}$$

where we used the state property and the natural conditions of control introduced above.

The expressions on the right-hand side of (1.6) are the model (1.3) and the c.p.d.f. $p(x_k|\theta, \mathcal{D}^{k-1})$ that is called the state estimate. For a linear system (1.4), such c.p.d.f. is calculated by a Kalman filter and it is a p.d.f. of the normal distribution $\mathcal{N}(\hat{x}_k, P_{x,k})$.

We can use (1.6) for two purposes. One is to predict y_k and x_{k+1} in an open loop, based on the data $\mathcal{D}^{k-1}$ and input u_k, which can be done either jointly, using directly (1.6), or marginally, for example as

$$p(x_{k+1}|\theta, u_k, \mathcal{D}^{k-1}) = \int p(y_k, x_{k+1}|\theta, u_k, \mathcal{D}^{k-1}) \, \mathrm{d}\, y_k,$$

which is used when x_{k+1} must be predicted prior to measuring y_k. The second purpose is to express the update of the Kalman filter, i.e. the transition

$$p(x_k|\theta, \mathcal{D}^{k-1}) \to p(x_{k+1}|\theta, \mathcal{D}^k),$$

which can be done formally as

$$p(x_{k+1}|\theta, \mathcal{D}^k) = \frac{p(y_k, x_{k+1}|\theta, u_k, \mathcal{D}^{k-1})}{\int p(y_k, x_{k+1}|\theta, u_k, \mathcal{D}^{k-1}) \, \mathrm{d}\, x_{k+1}}. \tag{1.7}$$

The Kalman filter prediction is a closed loop prediction, as it also uses the output y_k.

Note 1.1. The assumption $\theta_k = \theta$ was made for simplicity of notation. However, we could easily work with time-varying parameters θ_k similarly as with the data, by defining a *parameter history* set

$$\Theta^k = \{\theta_0, \dots, \theta_k\}$$

and conditioning by this set instead of by θ. The derivation would then be analogous to the presented one, adding the assumption

$$p(x_k|\Theta^k, \mathcal{D}^{k-1}) = p(x_k|\Theta^{k-1}, \mathcal{D}^{k-1}),$$

which says that the current parameter cannot influence the estimation of the current state.

Note 1.2. If the parameters are known, they are usually omitted from the condition of c.p.d.f.'s and their influence is assumed to be implicitly given by the function p, e.g.

$$p(x_{k+1}|\theta_k, x_k, u_k) = p_{\theta_k}(x_{k+1}|x_k, u_k) = p_k(x_{k+1}|x_k, u_k).$$

However, we keep the parameters in the condition, because it allows us to naturally proceed to the uncertain parameter case.

.

1.1.4 Uncertain parameters

If the parameters of the model are unknown, we speak about a model with *uncertain parameters*. The concept of uncertain parameters is used to describe those systems, models of which have a given structure, parametrized by a parameter vector θ. For example, the structure of the given system can be a stable linear first order system with gain 1 with an unknown time constant τ as a parameter. The parameters can be constant or time-varying, in which case we also need some model of parameter development. Similarly to the noise (or disturbance) in stochastic systems, uncertain parameters are a way to include a specific kind of uncertainty in the model. Unlike the inherently stochastic nature of noise (disturbance), the uncertain parameters do not model unpredicable events or random dynamics, but they express subjective knowledge about the system at the time of controller design. This lack of knowledge is often described by bayesian probabilistic methods, because the probability theory is a useful tool for uncertainty description. However, we should keep in mind that the parameters are not really random, but only the knowledge about them is modelled in such way. Therefore we should also be careful when interpreting the results of some control strategies, where the stochastic modelling of parameters plays a central role, like for example the cautious control.

Depending on the situation we need to model, we can have both deterministic and stochastic systems with uncertain parameters, both with perfect and imperfect state information. However, it is mostly assumed, that systems with uncertain parameters are also stochastic in the sense of Definition (1.3) and that we do not have perfect state information. We will also assume that the parameters are constant, i.e. $\theta_k = \theta$, or slowly time-varying.

Note 1.3. Similarly to Note 1.1, we could also introduce time-varying parameters θ_k. This time, however, we would have to define the joint c.p.d.f.

$$p(y_k, x_{k+1}, \theta_{k+1}|\theta_k, x_k, u_k)$$

and then proceed analogously, treating the parameter vector similarly as the state. Involving time-varying parameter model leads to increasing the parameter uncertainty by a certain level. For example, assuming a random walk model of parameter development, $\theta_{k+1} = \theta_k + \nu_k$, leads to a constant matrix $V_k = \mathsf{var}\,\nu_k$ being added to the parameter variance matrix in each step of the estimation algorithm, i.e. $P_{k+1} = P_k + V_k$. However, the time-varying parameter model is usually not available and thus cannot be used directly in estimation algorithms. The lack of this knowledge is then solved by introducing some heuristic methods for increasing uncertainty, called forgetting, that keep the uncertainty above a certain level. Forgetting is important, if the estimation algorithm should react on parameter changes – one consequence of the constant parameter assumption is that the uncertainty is only decreasing and after some time the uncertainty is already low enough and new data has little or no impact on the parameter estimate. Forgetting forces the uncertainty to increase and thus also the algorithm to take the new data into account. We will however use constant parameters for simplicity and assume that modification by forgetting may be added later.

For uncertain parameters we have to generalize the equation (1.6) in the following way

$$p(y_k, x_{k+1}|u_k, \mathcal{D}^{k-1}) = \int p(y_k, x_{k+1}|\theta, x_k, u_k) p(\theta, x_k|u_k, \mathcal{D}^{k-1}) \,\mathrm{d}(\theta, x_k), \tag{1.8}$$

where the state property was used. Note that the natural conditions of control cannot be easily used here, as the quality of parameter estimation may depend on the input u_k. Equation (1.8) is important, because it describes the way in which the imperfect state information and parameter uncertainty influence the state prediction. Various stochastically optimal control strategies differ in the way they model this influence, i.e. what assumptions about the c.p.d.f. $p(\theta, x_k|u_k, \mathcal{D}^{k-1})$ are made.

Note 1.4. It might seem that there is formally no difference between the state and parameters, as both of them play a similar role of some internal, hidden variables in the equation (1.8). Indeed, there are situations, where the role of states and parameters can be switched over to obtain interesting results, as will be for example showed in Section 2.1, where the simultaneous state estimator and parameter tracker for ARMAX model is derived. However, an important difference is the state property, i.e.

$$p(y_k, x_{k+1}|x_k, u_k, \mathcal{D}^{k-1}) = p(y_k, x_{k+1}|x_k, u_k).$$

The parameters do not have this property. This is crucial for control design, because the control must use all available information. If the state did not contain this information, any control depending only on the state would be suboptimal. Therefore, the state must contain all important information from the past data $\mathcal{D}^{k-1}$. On the other hand, moving all uncertainty to the state vector is also not

possible. Doing so could lead to needlessly complicated models (e.g. an originally linear model might become nonlinear) or to losing important properties like controllability or observability. Parameters are also often easier to estimate, because they are mostly considered constant.

1.2 Stochastically optimal control strategies

A control problem in time domain is usually specified as finding such manipulated input sequence $u_0, \ldots, u_{N-1}$ that minimizes the *cost function* or *control criterion* in the form of

$$g_N(x_N) + \sum_{i=0}^{N-1} g_i(u_i, x_i), \tag{1.9}$$

where g_i assigns a cost to each combination of u_i and x_i, and N is referred to as *control horizon*. To find the optimal control u_i^*, one should use all information available at time i, therefore the optimal control u_i^* is usually expressed as a function μ_i^* of the state x_i, i.e.

$$u_i^* = \mu_i^*(x_i). \tag{1.10}$$

We can then define the optimal value of the *cost-to-go* function, i.e. the criterion (1.9) calculated from time k to N as

$$J_k^* = g_N(x_N) + \sum_{i=k}^{N-1} g_i(\mu_i^*(x_i), x_i). \tag{1.11}$$

It can be shown that if the criterion is *additive*, e.g. in the form (1.9), the *dynamic programming* approach can be used, making use of the *Bellman equation*, that says

$$J_k^*(x_k) = \min_{u_k} \left[g_k(x_k, u_k) + J_{k+1}^*(x_{k+1}) \right], \tag{1.12}$$

where the cost-to-go function at time k is a function of the state x_k.

The formulation (1.12) is only valid for deterministic systems, where the future state x_{k+1} can be predicted using equation (1.1). For stochastic systems with perfect state information, the criterion is a random variable and therefore it must be reformulated using the expected value $\mathsf{E}[\cdot]$ as

$$J_k^*(x_k) = \min_{u_k} \mathsf{E}\left[g_k(x_k, u_k) + J_{k+1}^*(x_{k+1}) \mid x_k, u_k, \theta \right]. \tag{1.13}$$

If there are no uncertain parameters, this expression can be evaluated using the state prediction model

$$p(x_{k+1}|\theta, x_k, u_k).$$

In the case of imperfect state information and uncertain parameters, the optimal criterion value J_k^* is a function of the data $\mathcal{D}^{k-1}$ rather than directly of the state x_k and the following conditional mean must be used on the right-hand side of (1.13)

$$J_k^*(\mathcal{D}^{k-1}) = \min_{u_k} \mathsf{E}\left[g_k(x_k, u_k) + J_{k+1}^*(\mathcal{D}^k) \mid u_k, \mathcal{D}^{k-1} \right]. \tag{1.14}$$

To evaluate the conditional expected value in this expression, it is necessary to use the following distributions:

- state prediction $p(x_{k+1}|u_k, \mathcal{D}^{k-1})$,
- joint state and parameter estimate $p(\theta, x_k|u_k, \mathcal{D}^{k-1})$.

Expression (1.14) does not explicitly depend on parameters θ, however, the joint state and parameter estimate is necessary for the state prediction, as shown in (1.8).

We will now describe various control strategies based on Bellman equations (1.12) and (1.13), and approaches to modelling the two c.p.d.f.'s above and thus to optimizing the expression (1.14) in case of imperfect state information.

1.2.1 Control of a system with known parameters

Control of a deterministic system

Before moving to more complicated control strategies, let us first show, how the deterministic case fits in the presented framework. Because a perfect state information is available and there are no uncertain parameters, the equation (1.12) can be used. The state develops according to the equation (1.1). In the presented framework, the state prediction c.p.d.f. will then be

$$p(x_{k+1}|\theta, x_k, u_k) = \delta(x_{k+1} - f(\theta, x_k, u_k)),$$

where $\delta(\cdot)$ is a Dirac distribution.

An example of such control strategy is the linear quadratic (LQ) control of a deterministic linear system (1.2) based on minimization of a quadratic cost

$$g_i(x_i, u_i) = x_i^T Q_i x_i + u_i^T R_i u_i, \quad g_N(x_N) = x_N^T Q_N x_N, \tag{1.15}$$

with symmetrical matrices $Q \geq 0$ and $R > 0$.

Control of a stochastic system

This case is more complicated than the previous one. We assume a stochastic system (1.3) and imperfect state information. We have to use both prediction and estimation c.p.d.f.'s, however, without estimating the parameters θ:

- $p(x_{k+1}|\theta, u_k, \mathcal{D}^{k-1})$,
- $p(x_k|\theta, \mathcal{D}^{k-1})$.

An example is the linear quadratic gaussian (LQG) control of a linear stochastic system (1.4), which immediately gives us the prediction c.p.d.f. The cost function is the same as in the previous case, given by (1.15). This case is interesting for the following three reasons.

1. The optimal control is given by the same state feedback as for the LQ control of a deterministic system with equal matrices A and B, with the only difference, that the state x_k is substituted with its conditional mean $\hat{x}_k = \mathsf{E}[x_k|\mathcal{D}^{k-1}]$. This is an example of the so-called *certainty equivalence principle* which says, that random variables in the problem may be substituted by their conditional means.

2. The state estimation c.p.d.f. is given by a Kalman filter, which gives $p(x_k|\theta, \mathcal{D}^{k-1}) \sim \mathcal{N}(\hat{x}_k, P_{x,k})$. This filter can be implemented independently of the controller – thus the estimation and control parts of the strategy are separated. This is called the *separation principle*. It is important that the variances $P_{x,k}$ are independent of control (the state estimate quality cannot be influenced by the control input) and therefore the natural conditions of control really hold. This is a great simplification in the derivation of the LQG controller [10].

3. The value of the LQG cost-to-go function is higher than for the LQ control and the difference is given by extra terms caused by disturbances and by the uncertainty of the state estimation.

1.2.2 Control of a system with uncertain parameters

Certainty equivalent control

We have seen in the previous case that the certainty equivalence principle holds for the state of a linear system when designing the LQG controller. Many other stochastic control strategies use certainty equivalence to simplify calculations or to make these calculations possible at all. In these cases, however, we talk about certainty equivalence (CE) *hypothesis*, as the substitution of random variables by their conditional means is not justified theoretically (and thus leads to suboptimal results), but rather serves as an effective method for simplification. This approach is widely used in adaptive control, see e.g. [24], and is also used as a basis for multiple-step algorithms in this thesis. In this framework, we will have the joint parameter and state estimate in the form

$$p(\theta, x_k|u_k, \mathcal{D}^{k-1}) = p(x_k|\theta, \mathcal{D}^{k-1})\delta(\theta - E[\theta|\mathcal{D}^{k-1}]),$$

where we assume $E[\theta|\mathcal{D}^{k-1}] = E[\theta|u_k, \mathcal{D}^{k-1}]$, i.e. the conditional mean of θ at time k is independent of the input u_k.

Integration with respect to θ then yields the marginal distribution

$$p(x_k|\mathcal{D}^{k-1}) = p(x_k|\hat{\theta}_k, \mathcal{D}^{k-1}),$$

and similarly for the prediction c.p.d.f.

$$p(x_{k+1}|u_k, \mathcal{D}^{k-1}) = \int p(x_{k+1}|\hat{\theta}_k, x_k, u_k)p(x_k|\hat{\theta}_k, \mathcal{D}^{k-1}) \,\mathrm{d}\, x_k,$$

where we have used the notation $\hat{\theta}_k = E[\theta|\mathcal{D}^{k-1}]$.

An example of such strategy can be an LQG controller for a linear system with uncertain parameters. Using certainty equivalence, both the controller and the (extended) Kalman filter are designed as if the parameters were equal to their current estimates (conditional means).

Note 1.5. We have so far only considered one-step predictions that must be used in the Bellman equations (1.12) and (1.14). This could lead to an impression, that similarly to the certainty equivalence of the state, the estimate $\hat{\theta}_k$ can be used in the step k. However, this is not so. Although the control criterion of the LQG controller at time $k = 0$ does not depend on the value of $\hat{x}_k$, $k > 0$, but only on $\hat{x}_0$ and the variances $P_{x,k}$, $k = 0, \ldots, N$ that can be precomputed, it does depend on the parameter values which cannot be precomputed. In other words, the separation principle does not hold here, even if the certainty equivalence is assumed. Because the Bellman equations are solved backwards over the whole control horizon N up to the time $k = 0$ and naturally, the estimate $\hat{\theta}_k$ for $k > 0$ is unknown at time $k = 0$, the estimate $\hat{\theta}_0$ must be used in all steps of the solution.

Note 1.6. Of course, the estimate $\hat{\theta}_k$ becomes available at time k. An adaptive version of the CE controller is possible by redesigning the controller according to the new information at time k. For slowly time-varying parameters, the controller is usually not calculated completely with the new parameters, but it is updated by only one step of the Bellman equation, which is called IST (Iterations Spread in Time) [34].

Cautious control

Cautious control was originally formulated in [17, 18] and further developed in [43, 7, 42] and [8] or later in [53]. It is often used as a basis for adaptive approximate dual algorithms, such as in [19] or [20]. Unlike the CE strategy, the cautious control strategy takes into account the whole c.p.d.f. of the parameters. The problem is that although all the future parameter conditional means and variances are necessary for the controller design, similarly to Note 1.5 they are unknown at time $k = 0$, because both the future conditional mean and future conditional variance of parameters depend on future inputs and outputs. On the other hand, although the same is true for the state estimate, the state conditional mean at time k is not used before the time k, therefore the controller can 'wait' for the estimate. Cautious control deals with the mentioned problems by introducing the following assumptions about the future parameter conditional mean and variance.

1. The future means and variances are substituted by the current ones, i.e.

$$\begin{aligned} E[\theta|\mathcal{D}^{k-1}] &= E[\theta|\mathcal{D}^{-1}], \\ \mathsf{var}[\theta|\mathcal{D}^{k-1}] &= \mathsf{var}[\theta|\mathcal{D}^{-1}], \\ \mathsf{cov}[x_k, \theta|\mathcal{D}^{k-1}] &= \mathsf{cov}[x_0, \theta|\mathcal{D}^{-1}], \end{aligned} \tag{1.16}$$

   for all $k > 0$.

2. The conditional distributions of parameters at time i and j are independent for $i \neq j$. Formally, we need the following

$$\mathsf{cov}\left(\begin{bmatrix} x_k \\ \theta \end{bmatrix}, \mathsf{E}\left(\begin{bmatrix} x_{k+1} \\ \theta \end{bmatrix} \,\middle|\, \mathcal{D}^k\right) \,\middle|\, \mathcal{D}^{k-1}\right) = 0, \tag{1.17}$$

because this expression appears in sequential application of the Bellman equation.

With these two assumptions, the model is equivalent to a model, where the parameters are independent, identically distributed random variables with the first two moments given by (1.16). The expressions (1.16) and (1.17) also contain covariance between the state and parameter, because these cannot be calculated in advance either. However, it depends on the way states and parameters are estimated. If we assume that state and parameters are estimated independently, then we can predict the future variances of the states separately (for example by Kalman filter) and apply the assumptions of cautious control only to the parameter mean and variance. An example of this approach is the cautious controller for an ARX model with uncertain parameters. It is possible to find a state-space model with perfect state information where no state estimation is necessary. The parameter estimate is then given by recursive least squares and it holds that

$$p(\theta|u_0, \mathcal{D}^{-1}) = \mathcal{N}(\hat{\theta}_0, \sigma_e^2 P_0'),$$

where σ_e^2 is the input noise variance and P_0' is the normalized estimate variance matrix at time $k = 0$. More information also in Chapter 2 or in [27].

> **Note 1.7.** If the noise variance σ_e^2 is unknown, it can be substituted by an estimate s^2 with a χ^2 distribution, and the compound c.p.d.f. will have the Student distribution. The Student distribution however converges quickly to the normal distribution and therefore this model is usually not considered.

Another example of cautious control is the cautious LQ controller of ARMAX model derived in Chapter 2. Here the parameter and state estimation are not separated and the form (1.16) is used. Therefore also the future state estimate variances cannot be precomputed.

The assumptions of cautious control make it possible to use stochastic dynamic programming, as the calculation of individual steps of control can be separated, because the assumptions remove the dependence of parameter conditional variance on the inputs. The name 'cautious' indicates that the optimal control in the presence of uncertainty tends to be more careful and thus avoids for example large overshoots if the parameter uncertainty is high. Analysis of properties of cautious LQ controller for an ARMAX model with known MA part (the 'c-parameters') is a substantial part of this thesis and is presented in Chapters 2 and 3. We also discuss some problems of the cautious approach at the end of this chapter in Section 1.4, where some unfavorable properties are shown on a simple example.

Adaptive modification of the cautious algorithm is straightforward – an updated controller can be designed after receiving new data and determining the current conditional c.p.d.f.

Dual adaptive control

If the cautious assumptions are not made, the c.p.d.f. $p(\theta, x_k|u_k, \mathcal{D}^{k-1})$ is a function of the data $\mathcal{D}^{k-1}$. We usually work with quadratic cost functions like (1.15), and therefore the first two moments of the distributions are sufficient for evaluating the criterion. Dual control strategy takes the dependence of future conditional variances on inputs into account. Each control input has then influence on the future variances and thus also on the criterion value. In other words, the dual approach allows to minimize the criterion not only via controlling the future state, but also by decreasing the future conditional variance of parameters. The optimal dual control thus not only aims to fulfill the control objective while taking the parameter uncertainty into account, but also excites the system in such way, that some useful information about the system is gained, and as a result, the uncertainty in the system is lowered in the future, allowing more reliable control.

The concept of dual control was first introduced by Feldbaum in [17]. It is known to be analytically solvable for only very special systems as in [55] or in [4] as it requires solving a complicated Bellman equation [10]. The system described in [4] is a simple integrator with an unknown gain on the input. Numerical solution faces the curse of dimensionality problem, because solving the Bellman equation by stochastic dynamic programming requires iterative computations of the conditional mean and its minimization. In a general case, the complexity of such problem grows exponentially with the dimension.

There exist approximations of the optimal solution based on suboptimal solutions of the original problem, usually using approximate stochastic dynamic programming as in [37, 13], or on problem reformulation as in [21, 19] or [20]. The dual control problem is analysed from the probabilistic point of view in [40] and [39]. An overview of the state-of-the-art methods is given in [62] and [61] and a profound survey in [23] and [19], where an algorithm with dual properties is defined as one that cautiously, but also actively gathers information during the control process, while satisfying the given control performance.

Active adaptive control

In this thesis we propose an approximation of dual adaptive algorithms based on the idea of persistent system excitation [27]. We call such algorithms *active adaptive algorithms*, because they actively collect information about the system via input control and measure the amount of information by the information matrix. The persistent excitation condition requires that the information about the system parameters in the sense of its parameter information matrix is increasing linearly, i.e.

$$P_{t+M}^{-1} - P_t^{-1} \geq \gamma I \tag{1.18}$$

for all t and some given M, where P_k^{-1} denotes the information matrix (the inverse of the variance matrix P_k) after k steps of estimation, γ is a given positive real constant and I denotes the identity matrix of a corresponding dimension. The inequality symbol $>$ ($\geq$) is used in the positive (semi)definiteness meaning, i.e. for two matrices A and B, $A > B (A \geq B)$ means that $A - B$ is a positive (semi)definite matrix. Satisfying the

persistent excitation condition is a necessary precondition for adaptive control algorithms to converge [27].

Similar methods based on so-called input design have been intensively studied recently. An input design methods based on frequency domain description is presented in [26, 33] and [31]. Other input design techniques can be found in [16] and [15] and further in [28] and [29].

The proposed algorithms are based on a constrained MPC control design, that is adjusted such that the persistent excitation condition is satisfied for some γ. The idea of persistent excitation has been used before in algorithms for simultaneous identification and control, such as in [1, 25, 41] or [60]. The presented approach solves the task as a two-phase optimization problem. First, the standard MPC problem is solved and its solution is used to construct a set of admissible perturbations. Second, the perturbation that most increases the information matrix in the sense of (1.18) is searched in the admissible area. This is a modification and generalization of the so called bicriterial approach, introduced in [19], where the control design is also done in two phases. Examples of application of this approach can be found in [20] and [22].

The proposed methods differ from the approach in [19] in two main aspects. First, the cautious controller is not used for the initial control computation, because there might be serious problems regarding the stability and convergence as shown in Chapter 2 and in [5, 6]. This is also a difference from the general definition of dual properties by [19], that requires the dual controller to be cautious. This can be easily eliminated as we show in Section 3.1 and 3.2 that cautious control of ARMAX model can be achieved by CE control by using properly adjusted cost functions. Second, the proposed algorithm predicts the information matrix over more than one step of control. It is shown in Section 5.1 how the multiple-step prediction can significantly improve the parameter tracking performance.

The information matrix prediction is one of the two major problems of the presented methods, as the prediction based on certainty equivalence assumption is used. However, it is confirmed by simulations that such prediction is sufficient. The second problem of this approach is the inherent nonconvexity of the problem formulation that has to be dealt with. The multiple-step algorithm brings more effective parameter estimation compared to the single-step methods, but the price has to be paid in terms of higher computational effort.

One of the proposed methods is based on iterative local approximation of the lowest eigenvalue function by quadratic forms. The term 'lowest eigenvalue function' is used to denote a function that assigns the lowest eigenvalue to a matrix, the elements of which are functions of given variables. In this case, this is the parameter information matrix, which is a function of system inputs. This simplification makes it possible to solve the problem effectively for low-dimensional systems. A conservative partial convexification of this problem is also presented in Section 6.5, thus making the method usable also for higher dimensional systems.

The methods are derived for single-input single-output (SISO) autoregressive models with external input (ARX), but modification for a general ARMAX model with known moving average (MA) parts is possible. Because they are based on perturbation of the con-

trol trajectory generated by an MPC controller and on a simultaneous control and recursive identification, they are based on a very general principle and as such, the modification is available for any controller that is adaptive and any identification algorithm, the accuracy of which can be measured by the information matrix.

1.3 Thesis structure

The first part of this thesis contains derivation and analysis of the cautious controller of a general ARMAX model with known MA part. Chapter 2 contains derivation of the controller and of the simultaneous parameter and state estimator for this model. These results have already been derived in [30, 45] and [59], but the results presented in Chapter 2 are shown in a more compact and understandable form. Chapter 3 then contains a complete analysis of convergence of the so-called cautious Riccati equation that arises from the cautious control problem for ARMAX model. Convergence issues are important when extending the control horizon to infinity to find a steady state controller. These issues have been studied in [5, 6] for scalar systems and systems with a specific structure of uncertainty. The presented analysis is new and covers more general systems. It is also shown that a finite steady state control law exists even in the case of divergence of the cautious Riccati-like equation.

The second part of the thesis proposes novel active adaptive control algorithms. Chapter 4 starts with a single-step algorithm for ARX system based on cautious control. Extensions of this algorithm to multiple step is possible, but has not been studied for inconvenient properties of cautious control. Multiple step active algorithms based on information matrix maximization are presented in Chapter 5. Chapter 6 contains the so-called *ellipsoid* algorithm that is studied in more detail. It also presents a conservative convexification of the algorithm that makes it solvable for higher dimensional systems. Simulations are usually shown at the end of each chapter.

1.4 Problems of cautious control

The bicriterial approach in [19] suggests using cautious control as the initial control u_k^c, with the aim to control more carefully in case the parameter uncertainty is high. The goal of this section is to show problems that arise when using cautious controllers as the primary solutions and thus to justify the use of certainty equivalent controllers. The problems are illustrated on a simple first order system controlled by a cautious modification of the minimum variance controller, but they remain valid for more sophisticated controllers such as the cautious LQ controller presented in Chapter 2.

Let us consider an autoregressive system with external input

$$y_k = ay_{k-1} + bu_k + e_k \tag{1.19}$$

with u_k, y_k and e_k denoting the system input, output and noise, respectively. The noise e_k is assumed to be gaussian white noise with variance σ_e^2. The minimum variance controller

is a controller based on minimization of the criterion

$$u_k^* = \arg\min_{u_k} \mathsf{E}(y_k - r)^2, \tag{1.20}$$

which for this system has the form

$$u_k^* = \frac{a}{b}(\frac{r}{a} - y_{k-1}), \tag{1.21}$$

where r denotes the reference value, see [19, 10]. Let us now consider the case when the system parameters are uncertain. By uncertain it is meant that they are not known exactly, but they remain constant or change slowly in time. However, a cautious modification of the controller (1.20) is gained when the uncertainty of the system parameters is described by the parameter conditional expected values and variances, with $\hat{a}$ and $\hat{b}$ denoting the conditional expected values and σ_a^2, σ_b^2 and σ_{ab} denoting the conditional variance of a, b and the covariance of a and b, respectively. Cautious control thus in fact interprets the uncertainty in a probabilistic way, assuming the parameters to be random variables, identically distributed, independent in time and independent with the system noise. This interpretation is already inconsistent with the uncertainty assumption made above in Section 1.1, which is a conceptual problem of cautious control. Minimization (1.20) then yields the following cautious modification of the minimum variance controller

$$u_k^c = \frac{\hat{b}}{\hat{b}^2 + \sigma_b^2} r - \frac{\hat{a}\hat{b} + \sigma_{ab}}{\hat{b}^2 + \sigma_b^2} y_{k-1}, \tag{1.22}$$

see [19, 10]. Such a controller is not robust (only the overall unit gain of the control loop is assured, but tracking is achieved only for a precise nominal model) and is used only for illustrative purposes. In contrast to the cautious controller, the certainty equivalent (CE) controller has the same form as (1.21), where the actual parameters are substituted with their expected values. Certainty equivalence thus simply assumes the expected values to be correct estimates and the controller is thus designed for the nominal system.

We can see immediately that the control design does not take the uncertainty of the dynamics into account, as it only depends on σ_b^2 and σ_{ab}, so it is not very helpful in case the dynamics is uncertain. Let us next assume that parameter a is known precisely, so the only uncertain parameter is b. The cautious controller has now the following form

$$u_k^* = \frac{a\hat{b}}{\hat{b}^2 + \sigma_b^2}(\frac{r}{a} - y_{k-1}). \tag{1.23}$$

Let us assume a zero reference signal, then the closed loop system is

$$y_k = a(1 - \frac{b\hat{b}}{\hat{b}^2 + \sigma_b^2})y_{k-1} \tag{1.24}$$

and the closed loop eigenvalue is $a(1 - b\hat{b}/(\hat{b}^2 + \sigma_b^2))$. For a nominal system, where $b = \hat{b}$, this value lies in the interval $[0, a)$, depending on the uncertainty σ_b^2 and if $a > 1$, the closed loop

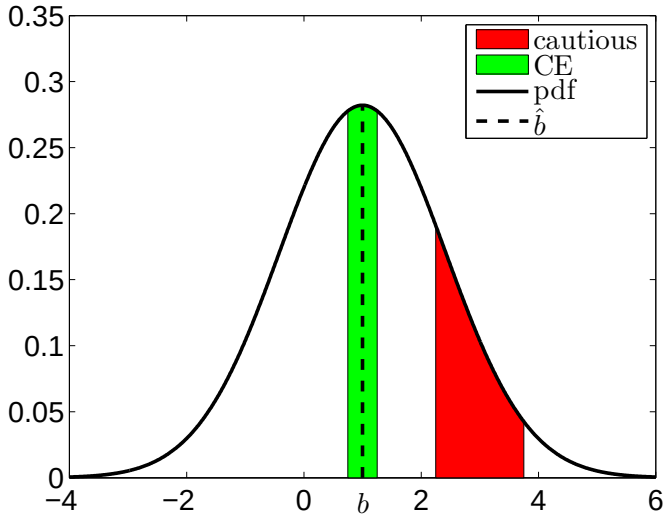


Figure 1.2: Areas of stability for a cautious and CE controller applied to a first order ARX system with $a = 4$ and an uncertain input gain b. The Figure shows the probability density function of the parameter b and the intervals of the actual values of b for which the cautious and CE controller are stable, respectively. The colored areas correspond to probabilities of a closed loop being stable.

may become unstable for the nominal system if σ_b^2 is sufficiently large. Figure 1.2 shows the regions of parameter b for which the closed loop is stable. The depicted situation describes a first order ARX system (1.19) with parameters $a = 4$ and b uncertain with mean $\hat{b} = 1$ and $\sigma_b^2 = 2$. While the nominal system lies in the center of the stability interval of the CE controller, it is clearly not stabilized by the cautious controller. The presented example shows this effect only for unstable systems ($|a| > 1$), but this is not true for more complex systems, where also a stable nominal system can be destabilized by cautious controller.

The stability of the nominal system might not be a crucial requirement for stochastic control, if for example the probability of the system being stable is increased. However, this is also generally not the case. The probability is equal to the area under the probability density function in Figure 1.2. Increasing the variance σ_b^2 moves the stability interval of cautious control further to the right, so eventually, the area gets smaller than the area of the CE controller, as shown in Figure 1.3, where the situation is shown for $a = 1.5$ and $\sigma_b^2 = 100$.

Problems also arise when trying to extend the problem formulation to infinite horizon. It can happen that the criterion value goes to infinity, as pointed out in [5] and [6], where the situation is analyzed for a first order system and a general system with specifically structured uncertainty, respectively. The limit feedback gain, however, converges to a finite value even if the criterion is infinite, so a time invariant control law might be still evaluated.

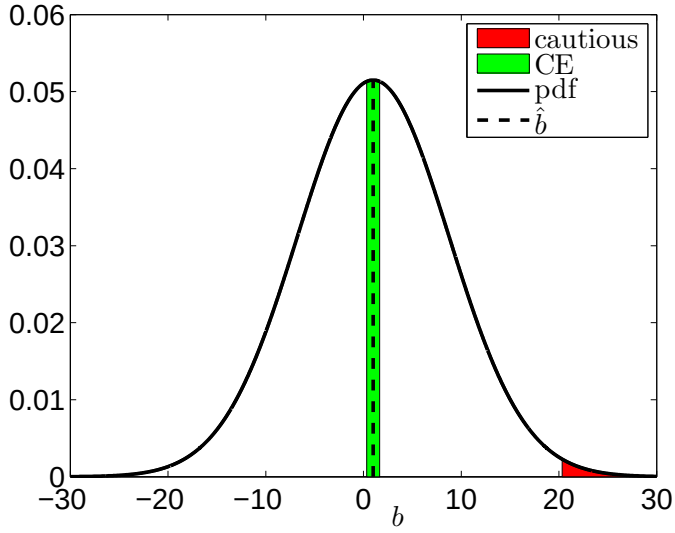


Figure 1.3: Areas of stability for a cautious and CE controller applied to a first order ARX system with $a = 1.5$ and an uncertain input gain b. Similarly to Figure 1.2 the intervals of stability are depicted. It is clearly seen that the probability of cautious control being stable is lower than the probability of the CE stability interval.

The problem is, however, the same as in the previous case and that is the unstable nominal closed loop system and a low probability of closed loop stability.

It was already mentioned in the previous section, that if the system parameters are unknown, they are described as uncertain. Using bayesian approach, parameter uncertainty can be described, using probability densities to express the available knowledge about the parameter values. This is the case in bayesian identification algorithms [12] or in standard recursive least square methods [27]. However, it is important to realize that uncertainty does not necessarily mean randomness. In reality, it is much more likely that the parameters will stay constant or change slowly. The probabilistic description of uncertainty thus does not express the parameters themselves but rather our knowledge about them.

Cautious control is strongly incoherent with this interpretation as it assumes that parameters at different time instants are identically distributed independent random variables. When designing a cautious controller over a horizon of $N > 1$ steps, it is assumed that the parameters have a different value at each step, according to their joint probability density. In such case, the system behavior would depend strongly on the parameter expected values that express the system 'average' behavior. On the other hand, the real system will behave according to the real parameter values, which may differ from the 'average' case.

Another approach to computing the criterion value over steps $2, \ldots, N$ would be to assume that the parameters remain constant over the control horizon, and so their (marginal)

distributions are the same for all $k = 2, \ldots, N$ and are given by the estimate at time $k = 2$. Under this assumption the parameters are no longer independent. The criterion value would then be computed as a mean of the criterion on the whole horizon with respect to the (initial) parameter distribution. Extension of this approach to infinite horizon brings even bigger problems, because as soon as there is a set of parameters with nonzero probability for which the controlled system is unstable, the criterion is infinite. This is indeed the case of the gaussian assumption of parameter distribution.

For illustration, recall the criterion convergence problem mentioned above. If the criterion evaluation is based on the cautious assumption, it may or may not happen that the limit criterion value is finite. If it is finite, it means that the controller works well for the 'average' system, even if there is a nonzero probability of the closed loop system being unstable. In reality, however, the criterion value must always diverge if there is a nonzero probability of an unstable closed loop.

These remarks show the importance of choosing a proper model for parameter uncertainty and that extension to infinite horizon may not be as straightforward as for deterministic systems or systems with only input uncertainties.

Chapter 2

Cautious LQ control of ARMAX model

The first section of this chapter shows the derivation of a simultaneous parameter and state estimator (tracker) for a general ARMAX model under the assumption of perfectly known MA part (c-parameters). The tracker has already been derived in [30, 45], but we propose a simpler method based on a classical Kalman filter design. The second section of the chapter presents derivation of a cautious modification of the linear quadratic (LQ) controller for the ARMAX model, again under the assumption of known c-parameters. Such controller has already been derived in [59] using similar techniques as in this chapter, the presented method is however new due to a more convenient choice of the state-space representation of the ARMAX model, thus leading to simpler and more compact results. The parameter and state estimator forms a counterpart to the cautious LQ controller in the sense that the results of the estimator (estimate of the current state and parameter vector) form a necessary input to the controller, as will be shown in Section 2.2.

The general ARMAX model is described by the equation

$$y_k = \sum_{i=1}^{n} a_i y_{k-i} + \sum_{i=0}^{n} b_i u_{k-i} + \sum_{i=0}^{n} c_i e_{k-i}, \tag{2.1}$$

where y_k, u_k and e_k are the system output, input and input noise at time k, respectively. As mentioned before, the parameters c_i are assumed to be known as well as the observed (directly measurable) inputs and outputs, u_k and y_k, while the parameters a_i and b_i and the input noise e_k are unknown. According to the terminology introduced in Section 1.1, the unknown parameters a_i and b_i are considered uncertain, because they are unknown but probably constant or changing slowly, while the noise e_k is a source of random disturbance in the system and it is assumed to be a gaussian white noise process, i.e. $e_k \sim \mathcal{N}(0, \sigma_e^2)$ and $\mathsf{cov}(e_i, e_j) = 0$ for $i \neq j$.

As described in Section 1.2, a cautious controller is derived under the assumptions that uncertainty is modeled by stochastic methods, particularly that the conditional distri-

butions of the uncertain parameters are assumed to be identical (equal to the conditional distribution at the initial time) and independent with respect to time. The assumption of known parameters c_i ensures that there are no products of random variables in the equation (2.1) and that we will not need higher than second order moments of the distributions. The parameter c_0 is chosen to be equal to 1 to remove the degree of freedom in the representation.

Note that we use two different state space representations of the model (2.1), one for derivation of the tracker/estimator and one for derivation of the controller. The state-space representation is always chosen to best fit the current purpose. When combining the cautious controller with the estimator to construct an adaptive controller, it must be understood that the estimated state is different from the state defining control input – this state must be transformed extra from the available data.

2.1 Simultaneous state estimation and parameter tracking of ARMAX model

This section presents derivation of a parameter tracker and state observer of a general ARMAX model in case the MA part (c-parameters) are known. The presented method uses a standard Kalman filtering theory applied to the following specific state-space representation on the ARMAX model.

$$\begin{aligned} x_{k+1} &= Ax_k + \Gamma e_k, \\ y_k &= C_k x_k + e_k, \end{aligned}$$

with the state vector

$$x_k = [b_0, a_1, \ldots, a_n, b_1, \ldots, b_n, e_{k-1}, \ldots, e_{k-n}]^T,$$

noise matrix

$$\Gamma = \begin{bmatrix} 0_{1,2n+1} & 1 & 0_{1,n-1} \end{bmatrix}^T,$$

time-varying output matrix

$$C_k = \begin{bmatrix} u_k & y_{k-1} & \ldots & y_{k-n} & u_{k-1} & \ldots & u_{k-n} & c_1 & \ldots & c_n \end{bmatrix}$$

and the system matrix

$$A = \begin{bmatrix} I_{2n+1} & 0_{2n+1,n} \\ 0_{n,2n+1} & A_e \end{bmatrix},$$

where $0_{i,j}$ is a zero matrix with i rows and j columns, I_n is an identity matrix of order n, $0_n = 0_{n,n}$ and

$$A_e = \begin{bmatrix} 0_{1,n-1} & 0 \\ I_{n-1} & 0_{n-1,1} \end{bmatrix}$$

is an $n \times n$ matrix. Because the noise e_k appears both in the state and the output equation, the noise variance matrices have the form

$$Q = \sigma_e^2 \Gamma\Gamma^T = \begin{bmatrix} 0_{2n+1} & 0_{2n+1,1} & 0_{2n+1,n-1} \\ 0_{1,2n+1} & \sigma_e^2 & 0_{1,n-1} \\ 0_{n-1,2n+1} & 0_{n-1,1} & 0_{n-1} \end{bmatrix}, \quad R = \sigma_e^2, \quad S = \sigma_e^2 \Gamma = \begin{bmatrix} 0_{2n+1,1} \\ \sigma_e^2 \\ 0_{n-1,1} \end{bmatrix}.$$

2.1.1 The estimator equations

It is now possible to construct a simultaneous state and parameter estimator using the standard Riccati equation for developing the estimate variance matrix and the Kalman gain. Considering an initial state estimate $\hat{x}_0$ and initial estimate variance matrix P_0, the filter equations are given by

$$P_{k+1} = AP_kA^T + Q - \frac{(AP_kC_k^T + S)(C_kP_kA^T + S^T)}{C_kP_kC_k^T + R} \tag{2.2}$$

and

$$\hat{x}_{k+1} = A\hat{x}_k + \frac{AP_kC_k^T + S}{C_kP_kC_k^T + R}(y_k - C_k\hat{x}_k), \tag{2.3}$$

where the denominator $C_kP_kC_k^T + R$ is a scalar. Dividing the equation (2.2) by σ_e^2 yields the more common form for a normalized variance matrix

$$P'_{k+1} = AP'_kA^T + Q' - \frac{(AP'_kC_k^T + S')(C_kP'_kA^T + S'^T)}{C_kP'_kC_k^T + 1} \tag{2.4}$$

and

$$\hat{x}_{k+1} = A\hat{x}_k + \frac{AP'_kC_k^T + S'}{C_kP'_kC_k^T + 1}(y_k - C_k\hat{x}_k), \tag{2.5}$$

where the noise variance matrices are also divided by σ_e^2, i.e.

$$Q' = \Gamma\Gamma^T = \begin{bmatrix} 0_{2n+1} & 0_{2n+1,1} & 0_{2n+1,n-1} \\ 0_{1,2n+1} & 1 & 0_{1,n-1} \\ 0_{n-1,2n+1} & 0_{n-1,1} & 0_{n-1} \end{bmatrix}, \quad S' = \Gamma = \begin{bmatrix} 0_{2n+1,1} \\ 1 \\ 0_{n-1,1} \end{bmatrix}.$$

Note 2.1. The equations above assume that σ_e^2 is known. If this is not the case, it must be estimated from the output errors, for example as

$$s_{k+1}^2 = \frac{k}{k+1}s_k^2 + \frac{1}{k+1}\frac{(y_k - C_k\hat{x}_k)^2}{C_kP_kC_k^T + 1}.$$

The estimate s^2 has a χ^2 distribution and the state estimator then produces estimates with Student distribution, see [45] for details. However, the convergence to normal distribution is fast and normality is assumed in practical cases.

Note 2.2. As the matrix C_k is time-varying, there will be no limit solution of the equation and thus no steady-state filter can be used. There is, however, an interesting connection to the persistent excitation, which we will informally describe. The

part of the system describing the parameter development is uncontrollable from the input noise. The theory of Riccati equations shows, that the corresponding part of the variance matrix P_k should go to zero, provided this part is observable. The observability depends on the time-varying matrix C_k that is formed by past inputs and outputs. Intuitively, this part of the system will be observable, if the matrices C_k are linearly independent, or at least, if for each k there is n_k such that the lines $C_k, \ldots, C_{k_n}$ are linearly independent. This is equivalent to the persistent excitation condition.

On the other hand, the part of the system describing the past noises is controllable and due to time varying C_k will the corresponding part of the variance matrix P_k not converge.

2.1.2 Notation

If we denote

$$a = [a_1, \ldots, a_n]^T, \quad b = [b_1, \ldots, b_n]^T, \quad c = [c_1, \ldots, c_n]^T, \quad \theta = \begin{bmatrix} a^T & b^T & c^T \end{bmatrix}^T, \tag{2.6}$$

it is possible to denote the blocks of the variance matrix P (or P') as

$$P_k = \begin{bmatrix} \sigma_{b_0}^2 & P_{b_0,a} & P_{b_0,b} & P_{b_0,e} \\ P_{a,b_0} & P_a & P_{a,b} & P_{a,e} \\ P_{b,b_0} & P_{b,a} & P_b & P_{b,e} \\ P_{e,b_0} & P_{e,a} & P_{e,b} & P_e \end{bmatrix}, \tag{2.7}$$

where the symbol e represents the past noises $e_{k-i}, i = 1, \ldots, n$ in the state vector x_k. To be precise, we should also use time indexation in the notation of the individual blocks, but we will omit this indexation to keep the notation simple.

2.1.3 ARX model

For an ARX model, the equations reduce to the recursive least squares algorithm, see [27]. The state-space representation of an ARX model for this purpose has the following state and output matrices

$$A = I_{2n+1}, \quad C_k = z_k^T = \begin{bmatrix} u_k & y_{k-1} & \ldots & y_{k-n} & u_{k-1} & \ldots & u_{k-n} \end{bmatrix},$$

where C_k is usually called the regressor and denoted as z_k^T. The state vector is formed only by parameters and is usually denoted as

$$\theta_k = [b_0, a_1, \ldots, a_n, b_1 \ldots, b_n]^T,$$

and the noise variance matrices Q and S are zero matrices and $R = \sigma_e^2$. Substituting the matrices into equations (2.4) and (2.5) yields the standard formulas

$$P'_{k+1} = P'_k - \frac{P'_k z_k z_k^T P'_k}{z_k^T P'_k z_k + 1} \tag{2.8}$$

and

$$\hat{\theta}_{k+1} = \hat{\theta}_k + \frac{P'_k z_k}{z_k^T P'_k z_k + 1}(y_k - z_k^T \hat{\theta}_k). \tag{2.9}$$

Note 2.3. Although it is usual to denote the parameter vector as θ_k as in the description of ARX model above, in this chapter we will prefer to keep the symbol θ according to the Definition (2.6), i.e. without b_0. The reason will be made clear in the next section, where the parameter b_0 plays an important role in the derivation of the cautious controller.

2.2 Cautious control of ARMAX model

For the purpose of derivation of the cautious LQ controller, the following state-space representation of the ARMAX model will be used

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + \Gamma e_k, \\ y_k &= C_k x_k + Du_k + e_k, \end{aligned} \tag{2.10}$$

with the state vector

$$x_k = [y_{k-1}, \ldots, y_{k-n}, u_{k-1}, \ldots, u_{k-n}, e_{k-1}, \ldots, e_{k-n}]^T \tag{2.11}$$

and system matrices

$$\begin{aligned} A &= \begin{bmatrix} a_1 \ldots a_n & b_1 \ldots b_n & c_1 \ldots c_n \\ I_{n-1,n} & 0_{n-1,n} & 0_{n-1,n} \\ 0_n & A_e & 0_n \\ 0_n & 0_n & A_e \end{bmatrix}, \quad B = \begin{bmatrix} b_0 \\ 0_{n-1,1} \\ 1 \\ 0_{2n-1,1} \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 \\ 0_{2n-1,1} \\ 1 \\ 0_{n-1,1} \end{bmatrix}, \\ C &= \begin{bmatrix} a_1 \ldots a_n & b_1 \ldots b_n & c_1 \ldots c_n \end{bmatrix}, \quad D = b_0, \end{aligned} \tag{2.12}$$

where $0_{i,j}$ is a zero matrix with i rows and j columns, $I_{n-1,n} = [I_{n-1}\ 0_{n-1,1}]$, I_n is an identity matrix of order n, $0_n = 0_{n,n}$ and

$$A_e = \begin{bmatrix} 0 & 0 \\ I_{n-1} & 0 \end{bmatrix}.$$

The criterion that is to be minimized is the usual quadratic functional for the LQ problem

$$\mathsf{E}\left[x_N^T Q x_N + \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k \right], \tag{2.13}$$

where the expected value is taken with respect to all uncertainties. The input u_k is a scalar and therefore there is no need to distinguish between u_k and u_k^T, but we will keep the transposed notation to avoid confusion. The symmetrical weighting matrices $Q \geq 0$ and $R > 0$ may also be time varying, in which case the notation Q_k, R_k is necessary.

2.2.1 Results for classical LQ and LQG control

Let us now briefly present the important properties of linear quadratic control. The result of the classical LQ control of a deterministic system (1.2) is that the optimal criterion value at time k (or the *cost-to-go* function) is a quadratic function of the state at time k, i.e.

$$J_k^*(x_k) = x_N^T Q x_N + \sum_{i=k}^{N-1} (x_i^T Q x_i + u_i^{*T} R u_i^*) = x_k^T G_k x_k, \tag{2.14}$$

where G_k is a positive semidefinite symmetrical matrix and generally $G_k, k = 0, \ldots, N-1$ is given by a discrete-time Riccati equation that is solved backwards from an end condition G_N. A limit solution for such equation for $N \to \infty$ exists under the assumptions of stabilizability of the pair (A, B) and observability of the pair (A, Γ_Q), where $\Gamma_Q \Gamma_Q^T = Q$ (see e.g. [63, 10]) The optimal control is then realized by a linear state feedback, i.e.

$$u_k^* = -K_k x_k.$$

The LQ problem for a stochastic system with perfect state information leads to the same control law as for the deterministic system above. The criterion, however, contains an additional absolute term g_k, linearly increasing with time, i.e.

$$J_k^*(x_k) = \mathsf{E}\left[x_N^T Q x_N + \sum_{i=k}^{N-1} (x_i^T Q x_i + u_i^{*T} R u_i^*) \right] = x_k^T G_k x_k + g_k.$$

The term g_k represents the criterion increase due to the noise. In the case of a stochastic system with imperfect state information is the criterion equal to a more general quadratic function

$$\begin{aligned} J_k^*(\mathcal{D}^{k-1}) &= \mathsf{E}\left[x_N^T Q x_N + \sum_{i=k}^{N-1} (x_i^T Q x_i + u_i^{*T} R u_i^*) \mid \mathcal{D}^{k-1} \right] = \\ &= \mathsf{E}(x_k|\mathcal{D}^{k-1})^T G_k \, \mathsf{E}(x_k|\mathcal{D}^{k-1}) + g_k^1 + g_k^2, \end{aligned}$$

where the absolute scalar terms g_k^1 and g_k^2 have the meaning of increase of the criterion due to uncertainty in prediction caused by the noise and due to the uncertainty in state estimation, respectively. The matrix G_k is however given by the same Riccati equation as in the deterministic case and therefore also the convergence conditions for G_k remain the same. Note that the cost-to-go function J_k^* is no longer a function of the state x_k, but rather a function of the available data, i.e. the data $\mathcal{D}^{k-1}$. The state feedback gain is also the same with the only difference that the (unknown) state x_k is substituted with its conditional mean, i.e.

$$u_k^* = -K_k \, \mathsf{E}(x_k|\mathcal{D}^{k-1}),$$

where the conditional mean $\mathsf{E}(x_k|\mathcal{D}^{k-1})$ is the Kalman filter estimate of the current state. The scalar sequences g_k, g_k^1 and g_k^2 depend on G_k and if G_k converges, then g_k has asymptotically a linear growth. This is the reason why a modified criterion should be used for

stochastic systems, e.g.

$$\frac{1}{N}\,\mathsf{E}\left[x_N^T Q x_N + \sum_{k=0}^{N-1}(x_k^T Q x_k + u_k^T R u_k)\right].$$

The division by N solves the problem of linearly increasing absolute terms, but has no impact on the convergence of G_k, therefore we can work with the criterion (2.13). For the three described cases, a sufficient condition for the matrix G_k to converge is stabilizability of the controlled system and observability of the pair (A, Γ_Q), where $\Gamma_Q^T \Gamma_Q = Q$. We will now derive a cautious LQ controller for a linear stochastic system with uncertain parameters. This derivation is the topic of the rest of this chapter. The convergence of G_k is more complicated for such controller, and is in detail discussed in the next chapter.

2.2.2 Bellman equation

Similarly to these LQ problems for a deterministic system, we will show that the criterion (2.13) for the cautious LQ control problem for ARMAX model with uncertain parameters has the form

$$J_k^*(\mathcal{D}^{k-1}) = \mathsf{E}(x_k^T G_k x_k|\mathcal{D}^{k-1}) + 2\gamma_k^T\,\mathsf{E}(x_k|\mathcal{D}^{k-1}) + g_k \tag{2.15}$$

for some matrix G_k, column vector γ_k and a scalar g_k. We will show this by induction using the general scheme of stochastic dynamic programming.

> **Note 2.4.** The first part of the quadratic function in (2.15) is slightly different from the previous cases, where the form $\mathsf{E}(x_k|\mathcal{D}^{k-1})^T G_k\,\mathsf{E}(x_k|\mathcal{D}^{k-1})$ was used. It is easily seen that
>
> $$\begin{aligned}\mathsf{E}(x_k^T G_k x_k|\mathcal{D}^{k-1}) &= \mathsf{E}(x_k|\mathcal{D}^{k-1})^T G_k\,\mathsf{E}(x_k|\mathcal{D}^{k-1}) + \\ &+ \mathsf{E}\left[\left(x_k^T - \mathsf{E}(x_k|\mathcal{D}^{k-1})^T\right) G_k \left(x_k - \mathsf{E}(x_k|\mathcal{D}^{k-1})\right)|\mathcal{D}^{k-1}\right],\end{aligned}$$
>
> where the second term on the right-hand side of the equality is equal to the estimate error weighted by the matrix G_k and is thus independent of x_k. It is then easy to transform the expression (2.15) to a quadratic function in estimates $\mathsf{E}(x_k|\mathcal{D}^{k-1})$ by adding the additional term to the absolute term. The form (2.15) is used for convenience in the following derivation.

The statement (2.15) is obviously true for $k = N$, as from the cost definition (2.13)

$$J_N^*(\mathcal{D}^{N-1}) = \mathsf{E}(x_N^T Q x_N|\mathcal{D}^{N-1}), \tag{2.16}$$

which gives us immediately the terminal conditions $G_N = Q$, $\gamma_N = 0$, $g_N = 0$, where 0 has the meaning of zero vector, scalar (or possibly matrix in the future text) of appropriate dimensions.

For a general time k it holds by the Bellman equation for problems with imperfect state information (see [10])

$$\begin{aligned} J_k^*(\mathcal{D}^{k-1}) &= \min_{u_k} \left[\mathsf{E}\left(x_k^T Q x_k + u_k^T R u_k + J_{k+1}^*(\mathcal{D}^k) \mid u_k, \mathcal{D}^{k-1}\right)\right] = \\ &= \min_{u_k} \left[\mathsf{E}\left(x_k^T Q x_k + u_k^T R u_k + \mathsf{E}(x_{k+1}^T G_{k+1} x_{k+1} | \mathcal{D}^k) \right. \right. + \\ &+ \left. \left. 2\gamma_{k+1}^T \mathsf{E}(x_{k+1}|\mathcal{D}^k) + g_{k+1} \mid u_k, \mathcal{D}^{k-1}\right)\right], \end{aligned} \tag{2.17}$$

where we used the induction hypothesis (2.15). Let us now substitute for x_{k+1} from the system equations (2.10)

$$\begin{aligned} J_k^*(\mathcal{D}^{k-1}) &= \min_{u_k} \left\{\mathsf{E}\left[x_k^T (Q + A^T G_{k+1} A) x_k + e_k^T \Gamma^T G_{k+1} \Gamma e_k \right. \right. + \\ &+ 2\gamma_{k+1}^T A x_k + u_k^T (R + B^T G_{k+1} B) u_k + \\ &+ \left. \left. 2\gamma_{k+1}^T B u_k + 2 x_k^T A^T G_{k+1} B u_k + g_{k+1} \mid u_k, \mathcal{D}^{k-1}\right]\right\}. \end{aligned} \tag{2.18}$$

2.2.3 Notation

To be able to perform the minimization in (2.18) and to close the recursion, we have to evaluate several terms. To simplify the expressions, we will use the following notation for the Kalman filter estimates at time k

$$\mathsf{E}(x_k|\mathcal{D}^{k-1}) = \hat{x}_k, \quad \mathsf{E}(A|\mathcal{D}^{k-1}) = \hat{A}_k, \quad \mathsf{E}(B|\mathcal{D}^{k-1}) = \hat{B}_k. \tag{2.19}$$

Recall that estimates $\hat{A}_k$ and $\hat{B}_k$ are given by the Kalman filter (2.3) from Section 2.1. The state estimate $\hat{x}_k$ is also given by the filter (2.3), as it is only necessary to estimate the past noises $e_{k-i}, i = 1, \ldots, n$.

> **Note 2.5.** The Kalman filter estimates $\hat{A}_k$, $\hat{B}_k$ and $\hat{x}_k$ do not depend on the current input u_k. Therefore it holds
>
> $$\mathsf{E}(x_k|u_k, \mathcal{D}^{k-1}) = \mathsf{E}(x_k|\mathcal{D}^{k-1}) = \hat{x}_k$$
>
> and similarly for $\hat{A}_k$ and $\hat{B}_k$. This will allow us to use the estimates in evaluation of the expression (2.18).

> **Note 2.6.** It is important to realize that the state x_k in this section is defined differently from the state in the previous section, where the parameter and state estimator was derived. This should not lead to any confusion, because x_k in this section always means the state according to the Definition (2.11). Constructing the estimate $\hat{x}_k$ as well as the estimates $\hat{A}_k$ and $\hat{B}_k$ from the estimate (2.3) is straightforward.

To further simplify some expressions that appear in the derivation, we will also introduce

the following notation

$$
\begin{aligned}
P_{b_0,\theta} &= \mathsf{cov}(b_0, \theta^T | \mathcal{D}^{k-1}) = \begin{bmatrix} P_{b_0,a} & P_{b_0,b} & 0_{1,n} \end{bmatrix} = P_{\theta,b_0}^T, \\
P_\theta &= \mathsf{var}(\theta | \mathcal{D}^{k-1}) = \begin{bmatrix} P_a & P_{a,b} & 0_n \\ P_{b,a} & P_b & 0_n \\ 0_n & 0_n & 0_n \end{bmatrix}, \\
P_{b_0,x} &= \mathsf{cov}(b_0, x_k^T | \mathcal{D}^{k-1}) = \begin{bmatrix} 0_{1,n} & 0_{1,n} & P_{b_0,e} \end{bmatrix} = P_{x,b_0}^T, \\
P_{\theta,x} &= \mathsf{cov}(\theta, x_k^T | \mathcal{D}^{k-1}) = \begin{bmatrix} 0_{n,2n} & P_{a,e} \\ 0_{n,2n} & P_{b,e} \\ 0_{n,2n} & 0_n \end{bmatrix} = P_{x,\theta}^T, \\
P_x &= \mathsf{var}(x_k | \mathcal{D}^{k-1}) = \begin{bmatrix} 0_{2n} & 0_n \\ 0_n & P_e \end{bmatrix},
\end{aligned}
\tag{2.20}
$$

where we have used the notation from Definitions (2.7) and (2.6).

Note 2.7. Similarly to the Kalman filter estimates, also the estimate covariances (2.20) were defined as independent of u_k. This is true for the covariance P_x in the case of a system without uncertain parameters and it is a key step in derivation of LQG control. The formula for the estimate covariance (2.2) clearly shows dependence on the data accumulated in the output matrix C_k (here we mean the output matrix of the system representation used for estimation) and therefore, for the covariances in (2.20), the independence of u_k is *assumed* in order to make the evaluation of (2.18) possible. This is the first time where we use the assumptions of cautious control. Taking the influence of u_k on the quality of estimate into account would result in a dual controller – this approach would however lead to analytically unsolvable equations.

Note 2.8. It will also be assumed that G_{k+1}, γ_{k+1} and g_{k+1} are constant, i.e.

$$\mathsf{E}(G_{k+1} | u_k, \mathcal{D}^{k-1}) = G_{k+1}$$

(and similarly for γ_{k+1} and g_{k+1}) and covariances of these terms with any random variables are zero. This is also not generally true, as we will see later, and this assumption is another use of cautious assumptions to simplify the derivation.

We will also denote $G^{i,j}$ the element in the i-th row and j-th column of matrix G and $G^{\cdot,j}$ and $G^{i,\cdot}$ the j-th column and the i-th row of G, respectively. Finally, let us denote

$$\tilde{x}_k = x_k - \hat{x}_k, \quad \tilde{A}_k = A - \hat{A}_k, \quad \tilde{B}_k = B - \hat{B}_k, \quad \tilde{\theta}_k = \theta - \hat{\theta}_k,$$

to simplify the derivation of the expected values.

2.2.4 Optimal control

Using this notation and assumptions we can now write

$$\begin{aligned}
&\mathsf{E}\left[x_k^T A^T G_{k+1} A x_k \mid u_k, \mathcal{D}^{k-1}\right] = \mathsf{E}\left[x_k^T A^T G_{k+1} A x_k \mid \mathcal{D}^{k-1}\right] = \\
&\quad = \mathsf{E}\left[(\hat{x}_k + \tilde{x}_k)^T (\hat{A}_k + \tilde{A}_k)^T G_{k+1} (\hat{A}_k + \tilde{A}_k)(\hat{x}_k + \tilde{x}_k) \mid \mathcal{D}^{k-1}\right] = \\
&\quad = \mathsf{E}\left[x_k^T \hat{A}_k^T G_{k+1} \hat{A}_k x_k \mid \mathcal{D}^{k-1}\right] + \hat{x}_k^T G_{k+1}^{1,1} P_\theta \hat{x}_k + \\
&\quad + 2\,\mathsf{E}\left[\tilde{x}_k^T \hat{A}_k^T G_{k+1} \tilde{A}_k \mid \mathcal{D}^{k-1}\right] \hat{x}_k = \\
&\quad = \mathsf{E}\left[x_k^T (\hat{A}_k^T G_{k+1} \hat{A}_k + G_{k+1}^{1,1} P_\theta) x_k \mid \mathcal{D}^{k-1}\right] - \mathsf{E}\left[\tilde{x}_k^T G_{k+1}^{1,1} P_\theta \tilde{x}_k \mid \mathcal{D}^{k-1}\right] + \\
&\quad + 2 G_{k+1}^{1,\cdot} \hat{A}_k P_{x,\theta} \hat{x}_k,
\end{aligned} \tag{2.21}$$

where the following equalities have been used

$$\begin{aligned}
&\tilde{A}_k \tilde{x}_k = 0, \\
&\mathsf{E}\left[\tilde{x}_k | \mathcal{D}^{k-1}\right] = 0, \quad \mathsf{E}\left[\tilde{A}_k | \mathcal{D}^{k-1}\right] = 0, \\
&\mathsf{E}\left[\tilde{A}_k^T G_{k+1} \tilde{A}_k \mid \mathcal{D}^{k-1}\right] = G_{k+1}^{1,1} P_\theta, \\
&\mathsf{E}\left[\tilde{x}_k^T \hat{A}_k^T G_{k+1} \tilde{A}_k \mid \mathcal{D}^{k-1}\right] \hat{x}_k = \mathrm{tr}\left\{\mathsf{E}\left[\tilde{A}_k \hat{x}_k \tilde{x}_k^T \hat{A}_k^T G_{k+1} \mid \mathcal{D}^{k-1}\right]\right\} = \\
&\quad = \mathsf{E}\left[\tilde{\theta}^T \hat{x}_k \tilde{x}_k^T \hat{A}_k^T G_{k+1}^{\cdot,1} \mid \mathcal{D}^{k-1}\right] = \hat{x}_k^T\, \mathsf{E}\left[\tilde{\theta} \tilde{x}_k^T \mid \mathcal{D}^{k-1}\right] \hat{A}_k^T G_{k+1}^{\cdot,1} = G_{k+1}^{1,\cdot} \hat{A}_k P_{x,\theta} \hat{x}_k,
\end{aligned}$$

with $\mathrm{tr}(\cdot)$ denoting the matrix *trace* operator. Using similar ideas, we can evaluate the rest of the terms from (2.18) to obtain the following expressions

$$\mathsf{E}\left[e_k^T \Gamma^T G_{k+1} \Gamma e_k \mid u_k, \mathcal{D}^{k-1}\right] = \Gamma^T G_{k+1} \Gamma \sigma_e^2, \tag{2.22}$$

$$\mathsf{E}\left[2\gamma_{k+1}^T A x_k \mid u_k, \mathcal{D}^{k-1}\right] = 2\gamma_{k+1}^T \hat{A}_k \hat{x}_k, \tag{2.23}$$

$$\begin{aligned}
&\mathsf{E}\left[u_k^T (R + B^T G_{k+1} B) u_k \mid u_k, \mathcal{D}^{k-1}\right] = \\
&\qquad = u_k^T (R + \hat{B}_k^T G_{k+1} \hat{B}_k + G_{k+1}^{1,1} \sigma_{b_0}^2) u_k,
\end{aligned} \tag{2.24}$$

$$\mathsf{E}\left[2\gamma_{k+1}^T B u_k \mid u_k, \mathcal{D}^{k-1}\right] = 2\gamma_{k+1}^T \hat{B}_k u_k, \tag{2.25}$$

$$\begin{aligned}
&\mathsf{E}\left[2x_k^T A^T G_{k+1} B u_k \mid u_k, \mathcal{D}^{k-1}\right] = \\
&\qquad = 2\hat{x}_k^T (\hat{A}_k^T G_{k+1} \hat{B}_k + G_{k+1}^{1,1} P_{\theta,b_0}) u_k + 2 G_{k+1}^{1,\cdot} \hat{A}_k P_{x,b_0} u_k.
\end{aligned} \tag{2.26}$$

It is now possible to rewrite (2.18) using expressions above

$$\begin{aligned}
J_k^*(\mathcal{D}^{k-1}) \;=\;& \mathsf{E}\left[x_k^T (\hat{A}_k^T G_{k+1} \hat{A}_k + G_{k+1}^{1,1} P_\theta + Q) x_k \mid \mathcal{D}^{k-1}\right] - \\
&- \mathsf{E}\left[\tilde{x}_k^T G_{k+1}^{1,1} P_\theta \tilde{x}_k \mid \mathcal{D}^{k-1}\right] + 2 G_{k+1}^{1,\cdot} \hat{A}_k P_{x,\theta} \hat{x}_k + \\
&+ 2\gamma_{k+1}^T \hat{A}_k \hat{x}_k + \Gamma^T G_{k+1} \Gamma \sigma_e^2 + g_{k+1} + \\
&+ \min_{u_k} \Big[u_k^T (R + \hat{B}_k^T G_{k+1} \hat{B}_k + G_{k+1}^{1,1} \sigma_{b_0}^2) u_k + 2\gamma_{k+1}^T \hat{B}_k u_k + \\
&\qquad + 2\hat{x}_k^T (\hat{A}_k^T G_{k+1} \hat{B}_k + G_{k+1}^{1,1} P_{\theta,b_0}) u_k + 2 G_{k+1}^{1,\cdot} \hat{A}_k P_{x,b_0} u_k \Big].
\end{aligned} \tag{2.27}$$

The first terms have been excluded from minimization according to assumptions of independence on u_k made earlier. Differentiating the terms in the minimization with respect to u_k and setting the derivative equal to zero yields

$$\begin{aligned} 0 &= (R + \hat{B}_k^T G_{k+1} \hat{B}_k + G_{k+1}^{1,1} \sigma_{b_0}^2) u_k + (\hat{B}_k^T G_{k+1} \hat{A}_k + G_{k+1}^{1,1} P_{b_0,\theta}) \hat{x}_k + \\ &+ \hat{B}_k^T \gamma_{k+1} + P_{b_0,x} \hat{A}_k^T G_{k+1}^{\cdot,1}. \end{aligned} \tag{2.28}$$

Assuming $R > 0$ guarantees the existence of $(R + \hat{B}_k^T G_{k+1} \hat{B}_k + G_{k+1}^{1,1} \sigma_{b_0}^2)^{-1}$ and we can write an explicit formula for the optimal control u_k^*

$$\begin{aligned} u_k^* &= -(R + \hat{B}_k^T G_{k+1} \hat{B}_k + G_{k+1}^{1,1} \sigma_{b_0}^2)^{-1} (\hat{B}_k^T G_{k+1} \hat{A}_k + G_{k+1}^{1,1} P_{b_0,\theta}) \hat{x}_k - \\ &- (R + \hat{B}_k^T G_{k+1} \hat{B}_k + G_{k+1}^{1,1} \sigma_{b_0}^2)^{-1} (\hat{B}_k^T \gamma_{k+1} + P_{b_0,x} \hat{A}_k^T G_{k+1}^{\cdot,1}). \end{aligned} \tag{2.29}$$

The first term on the right hand side of the expression (2.29) represents a state feedback from the current state estimate. The second term is independent from the current state and can be seen as a correction of the state feedback according to the covariance $P_{b_0,x}$ between the state and the uncertain input gain b_0. To simplify the expression (2.29) as well as some further considerations, let us define

$$\begin{aligned} K_k &= (R + \hat{B}_k^T G_{k+1} \hat{B}_k + G_{k+1}^{1,1} \sigma_{b_0}^2)^{-1} (\hat{B}_k^T G_{k+1} \hat{A}_k + G_{k+1}^{1,1} P_{b_0,\theta}) \\ L_k &= (R + \hat{B}_k^T G_{k+1} \hat{B}_k + G_{k+1}^{1,1} \sigma_{b_0}^2)^{-1} (\hat{B}_k^T \gamma_{k+1} + P_{b_0,x} \hat{A}_k^T G_{k+1}^{\cdot,1}). \end{aligned} \tag{2.30}$$

The optimal control can be now expressed as

$$u_k^* = -K_k \hat{x}_k - L_k.$$

2.2.5 Optimal cost and recursive equations

Substituting u_k in (2.27) with the optimal control u_k^* from (2.29) yields

$$\begin{aligned} J_k^*(\mathcal{D}^{k-1}) &= \mathsf{E}\left[x_k^T (\hat{A}_k^T G_{k+1} \hat{A}_k + G_{k+1}^{1,1} P_\theta + Q) x_k \mid \mathcal{D}^{k-1} \right] - \\ &- \mathsf{E}\left[\tilde{x}_k^T G_{k+1}^{1,1} P_\theta \tilde{x}_k \mid \mathcal{D}^{k-1} \right] - \\ &- \mathsf{E}\Big[x_k^T (\hat{A}_k^T G_{k+1} \hat{B}_k + G_{k+1}^{1,1} P_{\theta,b_0}) (R + \hat{B}_k^T G_{k+1} \hat{B}_k + G_{k+1}^{1,1} \sigma_{b_0}^2)^{-1} \\ &\qquad (\hat{B}_k^T G_{k+1} \hat{A}_k + G_{k+1}^{1,1} P_{b_0,\theta}) x_k \mid \mathcal{D}^{k-1} \Big] + \\ &+ \mathsf{E}\Big[\tilde{x}_k^T (\hat{A}_k^T G_{k+1} \hat{B}_k + G_{k+1}^{1,1} P_{\theta,b_0}) (R + \hat{B}_k^T G_{k+1} \hat{B}_k + G_{k+1}^{1,1} \sigma_{b_0}^2)^{-1} \\ &\qquad (\hat{B}_k^T G_{k+1} \hat{A}_k + G_{k+1}^{1,1} P_{b_0,\theta}) \tilde{x}_k \mid \mathcal{D}^{k-1} \Big] + \\ &+ 2 \Big[G_{k+1}^{1,\cdot} \hat{A}_k P_{x,\theta} + \gamma_{k+1}^T \hat{A}_k - (\gamma_{k+1}^T \hat{B}_k + G_{k+1}^{1,\cdot} \hat{A}_k P_{x,b_0}) \\ &\qquad (R + \hat{B}_k^T G_{k+1} \hat{B}_k + G_{k+1}^{1,1} \sigma_{b_0}^2)^{-1} (\hat{B}_k^T G_{k+1} \hat{A}_k + G_{k+1}^{1,1} P_{b_0,\theta}) \Big] \hat{x}_k + \\ &+ g_{k+1} + \Gamma^T G_{k+1} \Gamma \sigma_e^2 - \\ &- (\gamma_{k+1}^T \hat{B}_k + G_{k+1}^{1,\cdot} \hat{A}_k P_{x,b_0}) (R + \hat{B}_k^T G_{k+1} \hat{B}_k + G_{k+1}^{1,1} \sigma_{b_0}^2)^{-1} \\ &\qquad (\hat{B}_k^T \gamma_{k+1} + P_{b_0,x} \hat{A}_k^T G_{k+1}^{\cdot,1}). \end{aligned} \tag{2.31}$$

The terms of the form $\mathsf{E}\left[\tilde{x}_k^T(\ldots)\tilde{x}_k \mid \mathcal{D}^{k-1}\right]$ do not depend on x_k, because we can rewrite them as

$$\mathsf{E}\left[\tilde{x}_k^T(\ldots)\tilde{x}_k \mid \mathcal{D}^{k-1}\right] = \mathrm{tr}\left\{(\ldots)\,\mathsf{E}\left[\tilde{x}_k\tilde{x}_k^T \mid \mathcal{D}^{k-1}\right]\right\} = \mathrm{tr}\left\{(\ldots)P_x\right\},$$

where we have assumed the variance matrix P_x to be independent of u_k as well. After reordering the terms, we finally get the main result of this section which is the formula for recursive computation of G_k, γ_k and g_k

$$\begin{aligned}
G_k &= \hat{A}_k^T G_{k+1}\hat{A}_k + G_{k+1}^{1,1}P_\theta + Q - (\hat{A}_k^T G_{k+1}\hat{B}_k + G_{k+1}^{1,1}P_{\theta,b_0}) \qquad (2.32)\\
&\qquad (R + \hat{B}_k^T G_{k+1}\hat{B}_k + G_{k+1}^{1,1}\sigma_{b_0}^2)^{-1}(\hat{B}_k^T G_{k+1}\hat{A}_k + G_{k+1}^{1,1}P_{b_0,\theta}),\\
\gamma_k^T &= G_{k+1}^{1,\cdot}\hat{A}_k P_{x,\theta} + \gamma_{k+1}^T\hat{A}_k - (\gamma_{k+1}^T\hat{B}_k + G_{k+1}^{1,\cdot}\hat{A}_k P_{x,b_0}) \qquad (2.33)\\
&\qquad (R + \hat{B}_k^T G_{k+1}\hat{B}_k + G_{k+1}^{1,1}\sigma_{b_0}^2)^{-1}(\hat{B}_k^T G_{k+1}\hat{A}_k + G_{k+1}^{1,1}P_{b_0,\theta}),\\
g_k &= g_{k+1} + \Gamma^T G_{k+1}\Gamma\sigma_e^2 - \qquad (2.34)\\
&- (\gamma_{k+1}^T\hat{B}_k + G_{k+1}^{1,\cdot}\hat{A}_k P_{x,b_0})(R + \hat{B}_k^T G_{k+1}\hat{B}_k + G_{k+1}^{1,1}\sigma_{b_0}^2)^{-1}\\
&\qquad (\hat{B}_k^T\gamma_{k+1} + P_{b_0,x}\hat{A}_k^T G_{k+1}^{\cdot,1}) +\\
&+ \mathrm{tr}\left\{\left[(\hat{A}_k^T G_{k+1}\hat{B}_k + G_{k+1}^{1,1}P_{\theta,b_0})(R + \hat{B}_k^T G_{k+1}\hat{B}_k + G_{k+1}^{1,1}\sigma_{b_0}^2)^{-1}\right.\right.\\
&\qquad \left.\left.(\hat{B}_k^T G_{k+1}\hat{A}_k + G_{k+1}^{1,1}P_{b_0,\theta}) - G_{k+1}^{1,1}P_\theta\right]P_x\right\}.
\end{aligned}$$

Note 2.9. The recursive formulas (2.32), (2.33) and (2.34) calculate backwards the terms G_k, γ_k and g_k, starting from end conditions G_N, γ_N and g_N. Here it is necessary to assume the independence and identical distribution of the estimates $\hat{A}_i$ and $\hat{B}_i$, $i = k, \ldots, N$. If we did not assume identical distribution, we would know neither the future estimates $\hat{A}_i$ and $\hat{B}_i$, $i = k+1, \ldots, N$ at time k nor the estimate variances. Moreover, without assuming independence, we would not be able to treat the terms G_{k+1}, γ_{k+1} and g_{k+1} as constants in the derivation, as there would be nonzero variances between $\hat{A}_k$ and $\hat{A}_s$, $k < s \leq N$ and so on for other estimates. The assumption of independence and identical distribution allows us to use the current estimates of parameters and their variances and use them to calculate the recursion.

Note 2.10. Unlike the case without uncertain parameters, we cannot even calculate the future state estimate variance P_x. This is normally done by a Kalman filter, which does not depend on future inputs and outputs. The specific structure of the Kalman filter used in the case of ARMAX model with uncertain parameters does not allow this computation, because it depends on the future data. This is however not a significant drawback, because the variance P_x has influence only on the absolute term g_k, which has no impact on the control law.

2.2.6 Cautious Riccati-like equation

The equation (2.32) resembles the standard Riccati equation for a problem without uncertain parameters. The formula for G_k does not depend on γ_{k+1}^T or g_{k+1}, on the contrary, both γ_k^T and g_k depend on G_{k+1}. The limit behavior of equation (2.32) is thus important for the

behavior of the resulting controller and will be studied extensively in Chapter 3. At this point we will define the following terminology.

Definition 2.11. *The discrete-time equation of the form*

$$\begin{aligned} G_k &= A_k^T G_{k+1} A_k + G_{k+1}^{1,1} P_\theta + Q - (A_k^T G_{k+1} B_k + G_{k+1}^{1,1} P_{\theta,b_0}) \\ &\quad (R + B_k^T G_{k+1} B_k + G_{k+1}^{1,1} \sigma_{b_0}^2)^{-1} (B_k^T G_{k+1} A_k + G_{k+1}^{1,1} P_{b_0,\theta}) \end{aligned} \tag{2.35}$$

is called a discrete-time cautious Riccati-like equation.

> **Note 2.12.** The hat notation $\hat{A}$ and $\hat{B}$ has been omitted in Definition 2.11 to simplify notation – the limit behaviour of the equation will be studied in further text and the matrices A_k and B_k can be considered to be constants without loss of generality. On the other hand, the time indexation has been preserved in the general definition, as we will also study the case when the system matrices vary over time. However, for studying the limit behavior, we have to assume that $A_k = A$, $B_k = B$ for some A, B and all $k \in \mathbb{N}$. Therefore we will mostly work with constant system matrices. Working with constant matrices indeed makes good sense for the cautious controller, because we assume $\hat{A}_k = \hat{A}_0$, $\hat{B}_k = \hat{B}_0$ for all $k \in \mathbb{N}$.

2.2.7 ARX model

Cautious control of an ARX model can be viewed as a special case of the previously derived algorithm. However, it brings such significant simplifications that it is useful to point them out.

- The are no parameters c_i in the model, therefore a state-space representation with the following system matrices can be used

$$\begin{aligned} A &= \begin{bmatrix} a_1 \dots a_n & b_1 \dots b_n \\ I_{n-1,n} & 0_{n-1,n} \\ 0_n & A_e \end{bmatrix}, \quad B = \begin{bmatrix} b_0 \\ 0_{n-1,1} \\ 1 \\ 0_{n-1,1} \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 \\ 0_{2n-1,1} \end{bmatrix}, \\ C &= \begin{bmatrix} a_1 \dots a_n & b_1 \dots b_n \end{bmatrix}, \quad D = b_0, \end{aligned} \tag{2.36}$$

  which is a nonminimal state-space representation.

- The state vector in this representation is

$$x_k = [y_{k-1}, \dots, y_{k-n}, u_{k-1}, \dots, u_{k-n}]^T. \tag{2.37}$$

  The state vector contains only past inputs and outputs which means a perfect state information, i.e. $\hat{x}_k = x_k$. In other words, the state vector need not be estimated and the control u_k is calculated from a direct state feedback.

- The covariances P_x, $P_{b_0,x}$ and $P_{\theta,x}$ are all zero, because the state x_k is directly measurable.

- The vector $\gamma_k^T = 0$ for all $k \in \mathbb{N}$. This is a consequence of zero covariances $P_{b_0,x}$ and $P_{\theta,x}$.
- The absolute term is simplified as

$$g_k = g_{k+1} + G_{k+1}^{1,1}\sigma_e^2,$$

  due to zero covariances and $\gamma_k^T = 0$
- Parameter estimate is calculated by simpler formulas (2.8) and (2.9)

It is indeed interesting that the equation for G_k (the discrete-time cautious Riccati-like equation) remains unchanged.

Chapter 3

Convergence of the cautious Riccati-like equation

This chapter presents a criterion for convergence of a discrete-time Riccati-like equation that is associated with the cautious LQ control problem for a general ARMAX model presented in Chapter 2. According to Definition 2.11, the cautious discrete-time Riccati-like equation has the following form

$$\begin{aligned} G_k &= A^T G_{k+1} A + Q + G^{1,1}_{k+1} P_\theta - \\ &- (A^T G_{k+1} B + G^{1,1}_{k+1} P_{\theta,b_0})(B^T G_{k+1} B + R + G^{1,1}_{k+1}\sigma^2_{b_0})^{-1}(B^T G_{k+1} A + G^{1,1}_{k+1} P_{b_0,\theta}), \end{aligned} \tag{3.1}$$

where $G^{1,1}$ denotes the upper-left element of the matrix G (row and column index equal to 1). Because this chapter studies limit behavior of the equation, we assume $A_k = A$ and $B_k = B$ for all $k \in \mathbb{N}$. Note that although the equation was derived as a solution to a specific problem (cautious LQ control problem for a SISO ARMAX model) and a special form of system matrices A and B has been made use of, the presented convergence criterion and its proof are valid for a general discrete-time equation in the form (3.1) with arbitrary system matrices.

To complete the analysis of the limit behavior of the recursive equations defining the cautious LQ controller from Chapter 2, we also show in Section 3.3 the limit behavior of the vector γ_k^T given by (2.33) and the scalar g_k given by (2.34).

Let us remind that the standard Riccati equation describing the deterministic LQ control has the form

$$G_k = A^T G_{k+1} A + Q - (A^T G_{k+1} B)(B^T G_{k+1} B + R)^{-1}(B^T G_{k+1} A), \tag{3.2}$$

that can be seen as a special case of (3.1) for zero covariances of parameter estimates. We will often use the similarity of equations (3.1) and (3.2) in proofs later.

3.1 Scalar equation

Before we formulate and prove the criterion for the matrix equation (3.1), let us first informally examine the problem for a one-dimensional system with two parameters a and b, i.e. for a system

$$x_{k+1} = ax_k + bu_k + e_k, \tag{3.3}$$

with a and b satisfying the assumptions of cautious control. For such system, the equation becomes much simpler, i.e.

$$G_k = Q + G_{k+1}(\hat{a}^2 + \sigma_a^2) - \frac{G_{k+1}^2(\hat{a}\hat{b} + \sigma_{ab})^2}{R + G_{k+1}(\hat{b}^2 + \sigma_b^2)}, \tag{3.4}$$

where the meaning of symbols is as follows:

$$\begin{aligned} \hat{a} &= \mathsf{E}\{a\}, \\ \hat{b} &= \mathsf{E}\{b\}, \\ \sigma_a^2 &= \mathsf{var}\{a\} = \mathsf{E}\{(a-\hat{a})^2\}, \\ \sigma_b^2 &= \mathsf{var}\{b\} = \mathsf{E}\{(b-\hat{b})^2\}, \\ \sigma_{ab} &= \mathsf{cov}\{(a,b)\} = \mathsf{E}\{(a-\hat{a})(b-\hat{b})\}. \end{aligned} \tag{3.5}$$

Let us start with the case when parameters a and b are not uncertain. This is equivalent to all variances being equal to zero and the problem is described by the standard Riccati equation

$$G_k = Q + G_{k+1}a^2 - \frac{G_{k+1}^2 a^2 b^2}{R + G_{k+1}b^2}, \tag{3.6}$$

which is always convergent (if $b \neq 0$). The convergence can be derived by the following analysis of (3.6). The equation expresses one step of a solution of a discrete-time equation. It is convenient to rewrite the discrete-time equation (3.6) to a general form

$$G_k = f(G_{k+1}), \tag{3.7}$$

where $f(G_{k+1})$ represents the right-hand side of (3.6). If such equation converges to some $G > 0$, then the function $f(G_{k+1})$ has a fixed point in G. The conditions for uniqueness and existence of the fixed point G, as well as convergence of the discrete-time equation to this point is given by the Banach fixed-point theorem. Because the function $f(G_{k+1})$ is increasing, the convergence to the unique fixed point G is ensured if and only if the function $f(G_{k+1})$ asymptotically follows a line (for $G_{k+1} \to \infty$) that intersects with a function

$$G_k = G_{k+1}.$$

This happens if the tangent of the asymptotical line is smaller than one. The tangent of the asymptotical line is calculated as

$$\alpha = \lim_{G_{k+1}\to\infty} \frac{f(G_{k+1})}{G_{k+1}} = 0,$$

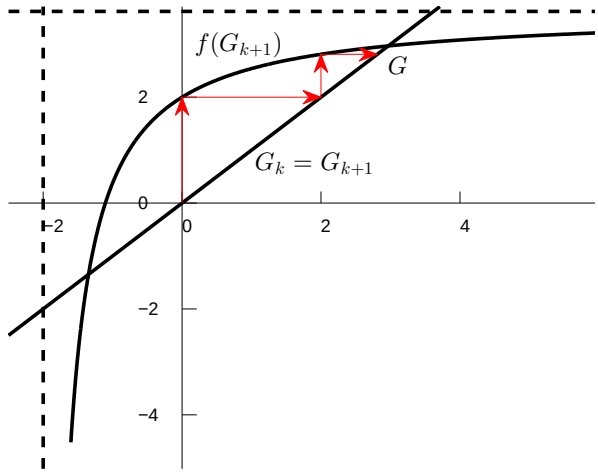


Figure 3.1: Dependence of G_k on G_{k+1} for standard discrete time Riccati equation. The dashed lines represent the asymptotes.

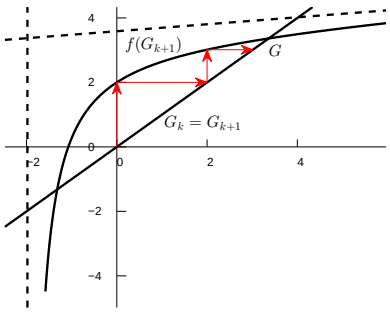
and the offset of the asymptote is

$$\beta = \lim_{G_{k+1}\to\infty} [f(G_{k+1}) - \alpha G_{k+1}] = Q + R\frac{a^2}{b^2}.$$

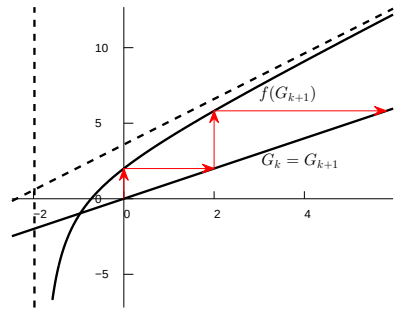
The tangent $\alpha = 0$ implies that the asymptote is a horizontal line and thus the fixed point G always exists. The situation is depicted in Figure 3.1 for a system with parameters $a = 0.9$ and $b = 1$. The function f has two asymptotes that are represented by the dashed lines. The horizontal asymptote calculated above is a function $y = \beta$, the vertical asymptote intersects the horizontal axis at $G_{k+1} = -R/b^2$. The figure also shows graphically, how the iterations of the equation converge to the fixed point G from the zero initial condition, however, the iteration would be analogous for any nonnegative initial condition. The fixed point G is then the intersection of function f (the right-hand side of (3.6) as a function of G_{k+1}) and the identity function $G_k = G_{k+1}$.

Let us now return to the case with uncertain parameters a and b and the corresponding scalar cautious Riccati-like equation (3.4). In [5] an analysis of solution of this Riccati-like equation is given and a condition for convergence of a cautious discrete-time Riccati-like equation is formulated for the scalar case. Later, in [6], another condition for a more general higher order system with a special structure of uncertainty is given. Such systems are, however, not a subject of this text. The condition based on [5] is

$$\frac{\sigma_a^2\sigma_b^2 + \hat{b}^2\sigma_a^2 + \hat{a}^2\sigma_b^2 - 2\hat{a}\hat{b}\sigma_{ab} - \sigma_{ab}^2}{\hat{b}^2 + \sigma_b^2} < 1 \tag{3.8}$$



(a) Convergent case: $\sigma_a^2 = 0.1$.



(b) Divergent case: $\sigma_a^2 = 1.5$.

Figure 3.2: Dependence of G_k on G_{k+1} for a discrete time cautious Riccati-like equation (3.4). The dashed lines represent the asymptotes.

and can be derived by calculating the asymptote tangent for the right-hand side of (3.4), i.e.

$$\alpha = \lim_{G_{k+1}\to\infty} \frac{f(G_{k+1})}{G_{k+1}} = (\hat{a}^2 + \sigma_a^2) - \frac{(\hat{a}\hat{b} + \sigma_{ab})^2}{\hat{b}^2 + \sigma_b^2},$$

which is equal to the left-hand side in condition (3.8). If the tangent is smaller than one, then there will exist a fixed point G. The offset of the asymptote is

$$\beta = \lim_{G_{k+1}\to\infty} [f(G_{k+1}) - \alpha G_{k+1}] = Q + R\frac{(\hat{a}\hat{b} + \sigma_{ab})^2}{(\hat{b}^2 + \sigma_b^2)^2}$$

and the situation is depicted in Figure 3.2 for a system with the following parameters: $\hat{a} = 0.9$, $\hat{b} = 1$, $\sigma_{ab} = 0$ and $\sigma_b^2 = 10^-2$. The variance σ_a^2 varies and the concrete values are stated in the figure descriptions. The function f has two asymptotes that are represented by the dashed lines. One asymptote is a function $y = \alpha \cdot G_{k+1} + \beta$, the vertical asymptote intersects the horizontal axis at $G_{k+1} = -R/(\hat{b}^2 + \sigma_b^2)$. The Figure 3.2(b) shows that if the condition (3.8) is not satisfied, then it holds for all $G > 0$ that $f(G) > G$, i.e. the sequence G_k is increasing and $G_k \to \infty$ as $N \to \infty$ (the equation is solved backwards from N).

The feedback gain K_k is given by the formula

$$K_k = -\frac{G_{k+1}(\hat{a}\hat{b} + \sigma_{ab})}{R + G_{k+1}(\hat{b}^2 + \sigma_b^2)}. \tag{3.9}$$

Taking the limit for $N \to \infty$ in the equation (3.9) yields either

$$K = \lim_{N\to\infty} K_k = -\frac{G(\hat{a}\hat{b} + \sigma_{ab})}{R + G(\hat{b}^2 + \sigma_b^2)}, \tag{3.10}$$

if the limit $G = \lim_{N\to\infty} G_k$ exists, or

$$K = \lim_{N\to\infty} K_k = -\frac{\hat{a}\hat{b} + \sigma_{ab}}{\hat{b}^2 + \sigma_b^2}, \tag{3.11}$$

if $G_k \to \infty$ and thus the feedback gain is convergent even if the criterion itself is not. The case of divergence be studied in more detail in Section 3.2, where also stability issues are addressed.

Let us now make the following consideration that will help us derive the criterion for convergence of the matrix equation (3.1). If G is the fixed point of the function $f(G_k)$, then it satisfies the (algebraic) equation

$$G = Q + G(\hat{a}^2 + \sigma_a^2) - \frac{G^2(\hat{a}\hat{b} + \sigma_{ab})^2}{R + G(\hat{b}^2 + \sigma_b^2)}. \tag{3.12}$$

Dividing the equation by G, we get

$$1 = \frac{Q}{G} + \hat{a}^2 + \sigma_a^2 - \frac{(\hat{a}\hat{b} + \sigma_{ab})^2}{\frac{R}{G} + \hat{b}^2 + \sigma_b^2}. \tag{3.13}$$

The right-hand side is given by the control problem up to the unknown variable G, so the existence of the fixed point can be interpreted as existence of such finite G, that the equation (3.13) holds. If we see the right-hand side of the equation as a function of G, then it is continuous, decreasing as $G \to \infty$ and goes to infinity as $G \to 0$. The fixed point will exist, if and only if the right-hand side falls below 1 as $G \to \infty$, i.e. if and only if

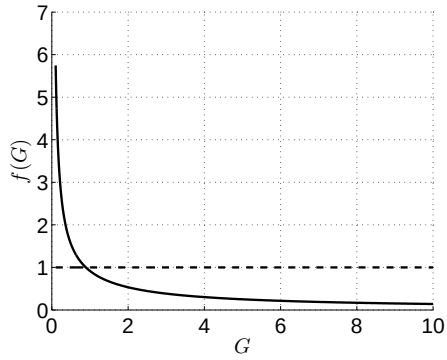
$$\hat{a}^2 + \sigma_a^2 - \frac{(\hat{a}\hat{b} + \sigma_{ab})^2}{\hat{b}^2 + \sigma_b^2} < 1. \tag{3.14}$$

Condition (3.14) is equivalent to the condition (3.8), but the idea used to derive its form is more general and will be used in the further text. Figures 3.3(a) and 3.3(b) show the right-hand side of (3.13) as a function $f(G)$ for a convergent and divergent case, respectively. Figures 3.4 and 3.5 show the corresponding sequences G_k and K_k for the two different cases.

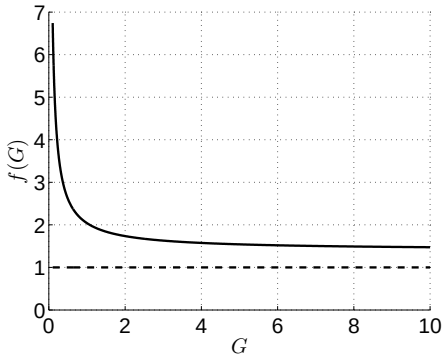
3.2 Matrix equation

In this section, we will focus on limit behavior of the sequence G_k given by the equation (3.1). Compared to the scalar case, it will be more difficult to prove the convergence, but the principle showed in derivation of the condition (3.14) for the scalar equation will be used with only mild modification. Unlike in the study of the scalar case in the previous subsection, we will use a formal approach to prove the convergence criterion.

Let us start with some changes in notation that will be more convenient for description of limit behavior. The nature of the control problem results in a backward recursion in the equation (3.1), i.e. G_k is computed from G_{k+1} with an initial condition G_N. Such

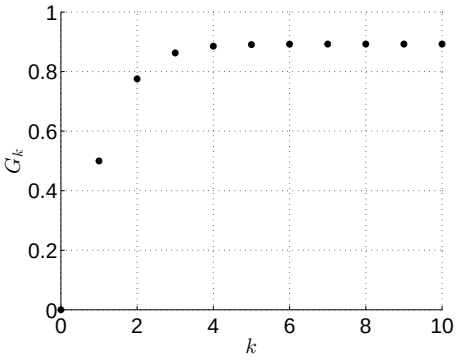


(a) Convergent case.

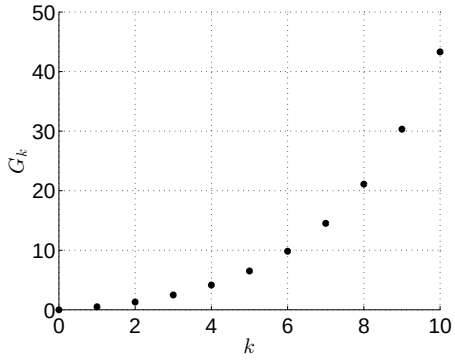


(b) Divergent case.

Figure 3.3: The right-hand side of equation (3.13) as a function $f(G)$. In the convergent case is the condition (3.14) satisfied, $f(G) < 1$ for $G > 0.9$ approximately. In the divergent case the function $f(G)$ stays approximately above the value 1.5.

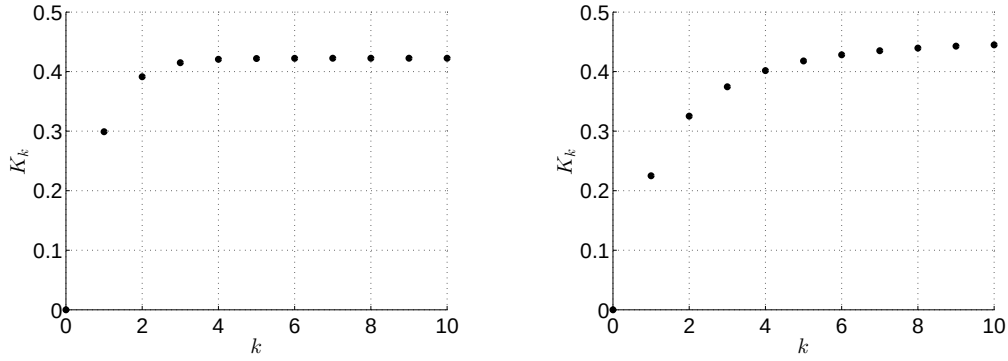


(a) Convergent case.



(b) Divergent case.

Figure 3.4: The sequence G_k given by the cautious discrete-time Riccati-like equation (3.4) for the case of convergence and divergence.



(a) Convergent case.

(b) Divergent case.

Figure 3.5: The sequence K_k given by the equation (3.9) for the case of convergence and divergence. K_k converges even if the corresponding sequence G_k is divergent.

notation is not convenient, as studying the limit behavior means studying the term G_0 as $N \to \infty$. That is why we prefer to reverse the indexation in the equations, obtaining the following recursion.

Definition 3.1. *A forward discrete-time cautious Riccati-like equation is an equation in the following form*

$$\begin{aligned} G_{k+1} &= A^T G_k A + Q + G_k^{1,1} P_\theta - \\ &- (A^T G_k B + G_k^{1,1} P_{\theta,b_0})(B^T G_k B + R + G_k^{1,1}\sigma_{b_0}^2)^{-1}(B^T G_k A + G_k^{1,1} P_{b_0,\theta}), \end{aligned} \tag{3.15}$$

with an initial condition G_0.

In this notation, it holds that the feedback gain matrix is expressed as

$$K_{k+1} = (B^T G_k B + R + G_k^{1,1}\sigma_{b_0}^2)^{-1}(B^T G_k A + G_k^{1,1} P_{b_0,\theta}). \tag{3.16}$$

Note 3.2. Sometimes we will need to use the equation (3.15) in a form with time-varying system matrices A, B and weighting matrices Q, R. In such case the equation has the following form

$$\begin{aligned} G_{k+1} &= A_{k+1}^T G_k A_{k+1} + Q_{k+1} + G_k^{1,1} P_\theta - \\ &- (A_{k+1}^T G_k B_{k+1} + G_k^{1,1} P_{\theta,b_0})(B_{k+1}^T G_k B_{k+1} + R_{k+1} + G_k^{1,1}\sigma_{b_0}^2)^{-1} \\ &\quad (B_{k+1}^T G_k A_{k+1} + G_k^{1,1} P_{b_0,\theta}). \end{aligned}$$

We show this form because it might not be straightforward, that the matrices appear in the equation with time index $k+1$.

Note 3.3. The cautious discrete-time Riccati-like equation resembles the standard Riccati equation for a deterministic LQ problem. Assume a standard Riccati equation with the same system matrices and constant weighting matrices $Q' = Q + \xi P_\theta$, $S' = \xi P_{\theta,b_0}$ and $R' = R + \xi\sigma_{b_0}^2$, where $\xi = G_{N-1}^{1,1}$ with $G_{N-1}^{1,1}$ taken from the cautious Riccati-like equation at a fixed time $N-1$. Then the matrix G_N from the cautious Riccati-like equation will be equal to the matrix H_N from the standard Riccati equation. We will use this equality in some proofs later. The standard Riccati equation has a nice interpretation of a criterion value, as it holds (in the new forward notation) that the optimal criterion value for an N-step deterministic LQ control problem is

$$\begin{aligned} J_0^*(x_0) &= x_0^T H_N x_0 = \\ &= \begin{bmatrix} x_0^T & u_0^T \end{bmatrix} \begin{bmatrix} Q + H_{N-1}^{1,1} P_\theta & H_{N-1}^{1,1} P_{\theta,b_0} \\ H_{N-1}^{1,1} P_{b_0,\theta} & R + H_{N-1}^{1,1}\sigma_{b_0}^2 \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} + x_1^T H_{N-1} x_1. \end{aligned} \tag{3.17}$$

This equality is implied by the Bellman equation for the cost-to-go function (1.12) and the result for deterministic LQ control (2.14). The analogous result for cautious controller is more complicated and inconvenient to use in proofs. However, due to the similarity of cautious and standard Riccati equation in one step, the equality above holds also for the solution of the cautious Riccati-like equation G_k.

3.2.1 Criterion for convergence

Definition 3.4 (Algebraic cautious Riccati-like equation)**.** *The algebraic cautious Riccati-like equation is a matrix equation in the following form*

$$\begin{aligned} G &= A^T G A + Q + G^{1,1} P_\theta - \\ &- (A^T G B + G^{1,1} P_{\theta,b_0})(R + B^T G B + G^{1,1}\sigma_{b_0}^2)^{-1}(B^T G A + G^{1,1} P_{b_0,\theta}), \end{aligned} \tag{3.18}$$

where the usual notation is used. The solution of an algebraic cautious Riccati-like equation is a matrix G that satisfies the equation (3.18)*.*

We will next prove the criterion for existence and uniqueness of a solution to an algebraic cautious Riccati-like equation (3.18) and then prove convergence of a sequence G_k given by the discrete-time equation (3.15) to this unique solution. Later we will also examine the divergent behavior of G_k and the behavior of the feedback gain matrix. It will be shown that even for divergent sequence G_k, the feedback gain matrix K_k converges to a limit solution that corresponds to a limit solution for modified system matrices A and B. Thus we will show that the class of all cautious LQ controllers for a general SISO ARMAX model extends the class of all certainty equivalent LQ controllers for the same model. The extension, however, contains also controllers that are not stabilizing the nominal system, i.e. for the system defined by expected values of system parameters.

Lemma 3.5. *Let $G_{1,0}$, $G_{2,0}$ be two symmetrical positive semidefinite matrices such that $G_{1,0} \geq G_{2,0}$. Let $G_{1,k}$ be the sequence given by the cautious Riccati-like equation* (3.15) *with weighting matrices $Q_{1,k}$, $R_{1,k}$ and initial condition $G_{1,0}$, i.e. the weighting matrices need*

not be constant, but may vary with each step k. Similarly, let $G_{2,k}$ be the sequence given by the cautious Riccati-like equation (3.15) *with weighting matrices $Q_{2,k}$, $R_{2,k}$ and initial condition $G_{2,0}$. Assume $Q_{1,k} \geq Q_{2,k}$ and $R_{1,k} \geq R_{2,k}$ Then for each $k \in \mathbb{N}$ $G_{1,k} \geq G_{2,k}$.*

Proof. We will use induction to prove the lemma. For $k = 0$ the lemma is true by assumption. Now let us assume that the lemma is true for $k \in \mathbb{N}$. The following holds by the analogy with the standard Riccati equation described in Note 3.3

$$x_0^T G_{1,k+1} x_0 = \begin{bmatrix} x_0^T & u_{1,0}^T \end{bmatrix} \begin{bmatrix} Q_{1,k+1} + G_{1,k}^{1,1} P_\theta & G_{1,k}^{1,1} P_{\theta,b_0} \\ G_{1,k}^{1,1} P_{b_0,\theta} & R_{1,k+1} + G_{1,k}^{1,1} \sigma_{b_0}^2 \end{bmatrix} \begin{bmatrix} x_0 \\ u_{1,0} \end{bmatrix} + x_{1,1}^T G_{1,k} x_{1,1}$$

and

$$x_0^T G_{2,k+1} x_0 = \begin{bmatrix} x_0^T & u_{2,0}^T \end{bmatrix} \begin{bmatrix} Q_{2,k+1} + G_{2,k}^{1,1} P_\theta & G_{2,k}^{1,1} P_{\theta,b_0} \\ G_{2,k}^{1,1} P_{b_0,\theta} & R_{2,k+1} + G_{2,k}^{1,1} \sigma_{b_0}^2 \end{bmatrix} \begin{bmatrix} x_0 \\ u_{2,0} \end{bmatrix} + x_{2,1}^T G_{2,k} x_{2,1},$$

where $u_{i,k}$ and $x_{i,k}$ are the optimal control and state sequences for the problem with initial matrix $G_{i,0}$ and weighting matrices $Q_{i,k}$, $R_{i,k}$. By assumption $G_{1,k} \geq G_{2,k}$, therefore also $G_{1,k}^{1,1} \geq G_{2,k}^{1,1}$ and consequently

$$\begin{bmatrix} Q_{1,k+1} + G_{1,k}^{1,1} P_\theta & G_{1,k}^{1,1} P_{\theta,b_0} \\ G_{1,k}^{1,1} P_{b_0,\theta} & R_{1,k+1} + G_{1,k}^{1,1} \sigma_{b_0}^2 \end{bmatrix} \geq \begin{bmatrix} Q_{2,k+1} + G_{2,k}^{1,1} P_\theta & G_{2,k}^{1,1} P_{\theta,b_0} \\ G_{2,k}^{1,1} P_{b_0,\theta} & R_{2,k+1} + G_{2,k}^{1,1} \sigma_{b_0}^2 \end{bmatrix}.$$

These inequalities imply that

$$\begin{aligned} x_0^T G_{1,k+1} x_0 &= \begin{bmatrix} x_0^T & u_{1,0}^T \end{bmatrix} \begin{bmatrix} Q_{1,k+1} + G_{1,k}^{1,1} P_\theta & G_{1,k}^{1,1} P_{\theta,b_0} \\ G_{1,k}^{1,1} P_{b_0,\theta} & R_{1,k+1} + G_{1,k}^{1,1} \sigma_{b_0}^2 \end{bmatrix} \begin{bmatrix} x_0 \\ u_{1,0} \end{bmatrix} + x_{1,1}^T G_{1,k} x_{1,1} \\ &\geq \begin{bmatrix} x_0^T & u_{1,0}^T \end{bmatrix} \begin{bmatrix} Q_{2,k+1} + G_{2,k}^{1,1} P_\theta & G_{2,k}^{1,1} P_{\theta,b_0} \\ G_{2,k}^{1,1} P_{b_0,\theta} & R_{2,k+1} + G_{2,k}^{1,1} \sigma_{b_0}^2 \end{bmatrix} \begin{bmatrix} x_0 \\ u_{1,0} \end{bmatrix} + x_{1,1}^T G_{2,k} x_{1,1}, \end{aligned}$$

where we used the optimal sequences $u_{1,k}$ and $x_{1,k}$ for both equations. This sequence is indeed suboptimal for the second equation and therefore

$$\begin{aligned} &\begin{bmatrix} x_0^T & u_{1,0}^T \end{bmatrix} \begin{bmatrix} Q_{2,k+1} + G_{2,k}^{1,1} P_\theta & G_{2,k}^{1,1} P_{\theta,b_0} \\ G_{2,k}^{1,1} P_{b_0,\theta} & R_{2,k+1} + G_{2,k}^{1,1} \sigma_{b_0}^2 \end{bmatrix} \begin{bmatrix} x_0 \\ u_{1,0} \end{bmatrix} + x_{1,1}^T G_{2,k} x_{1,1} \geq \\ \geq &\begin{bmatrix} x_0^T & u_{2,0}^T \end{bmatrix} \begin{bmatrix} Q_{2,k+1} + G_{2,k}^{1,1} P_\theta & G_{2,k}^{1,1} P_{\theta,b_0} \\ G_{2,k}^{1,1} P_{b_0,\theta} & R_{2,k+1} + G_{2,k}^{1,1} \sigma_{b_0}^2 \end{bmatrix} \begin{bmatrix} x_0 \\ u_{2,0} \end{bmatrix} + x_{2,1}^T G_{2,k} x_{2,1}. \end{aligned}$$

The two inequalities imply

$$x_0^T G_{1,k+1} x_0 \geq x_0^T G_{2,k+1} x_0, \quad \forall x_0 \in \mathbb{R}^n,$$

which completes the proof. □

Corollary 3.6. *Let $G_k, k \in \mathbb{N}$ be a matrix sequence given by the discrete-time cautious Riccati-like equation* (3.15)*. If for some $k' \in \mathbb{N}$, $G_{k'} \leq G_{k'+1}$ ($G_{k'} \geq G_{k'+1}$) then $G_k \leq G_{k+1}$ ($G_k \geq G_{k+1}$) for every $k \geq k'$.*

Proof. To prove the corollary, use the previous lemma for $G_{1,0} = G_{k'+1}$ and $G_{2,0} = G_{k'}$. □

Note 3.7. The statements of Lemma 3.5 and Corollary 3.6 are also true for a cautious Riccati-like equation with constant weighting matrices and also for a standard Riccati equation, that can be seen as a special case of equation (3.15) with a zero variance matrix P. The proof of Lemma 3.5 and Corollary 3.6 will then remain unmodified, as there were no conditions on the variance matrix in Lemma 3.5.

Lemma 3.8. *Let A and B be system matrices, (A, B) stabilizable. Let $Q_\alpha \geq 0$ and $R_\alpha > 0$ be weighting matrices and G_α be the solution of the corresponding algebraic Riccati equation. Let $Q_\beta \geq 0$ and $R_\beta > 0$ be weighting matrices such that*

$$Q_\beta \geq Q_\alpha, \quad R_\beta \geq R_\alpha, \tag{3.19}$$

and G_β be the solution of the corresponding algebraic Riccati equation. Finally assume that the pairs (A, Γ_{Q_α}), $\Gamma_{Q_\alpha}^T \Gamma_{Q_\alpha} = Q_\alpha$ and (A, Γ_{Q_β}), $\Gamma_{Q_\beta}^T \Gamma_{Q_\beta} = Q_\beta$ are observable. Then

$$G_\beta \geq G_\alpha.$$

Let $Q_n \geq 0$, $R_n > 0$ be sequences of weighting matrices such that

$$Q_n \xrightarrow{n\to\infty} Q, \quad R_n \xrightarrow{n\to\infty} R, \tag{3.20}$$

where $Q \geq 0$, $R > 0$ are weighting matrices, (A, Γ_Q), $\Gamma_Q^T \Gamma_Q = Q$ observable. Then for the solutions of the corresponding problems, G_n, G

$$G_n \xrightarrow{n\to\infty} G.$$

Proof. According to the Note 3.7, the Lemma 3.5 is true for a standard Riccati equation with constant matrices Q_α, R_α and Q_β, R_β, respectively. It is known from the theory of Riccati equations [10], that the stabilizability and observability assumptions above ensure the existence of unique solutions G_α and G_β of the corresponding algebraic Riccati equations. It is also true that the sequence given by the discrete-time equation converges to the unique solution for each positive semidefinite initial condition G_0, i.e.

$$G_\alpha = \lim_{k\to\infty} G_{\alpha,k}$$

and

$$G_\beta = \lim_{k\to\infty} G_{\beta,k}.$$

Let us assume an arbitrary initial condition $G_0 \geq 0$. Then by Lemma 3.5, it holds that $G_{\alpha,k} \geq G_{\beta,k}$ for each $k \in \mathbb{N}$ and therefore also $G_\alpha \geq G_\beta$.

To prove the second part, first note that $G > 0$ due to the assumption of observability. The assumptions (3.20) imply that for each $0 < \varepsilon$ there is n_ε such that $\forall n \geq n_\varepsilon$ we have $Q - \varepsilon Q \leq Q_n \leq Q + \varepsilon I$ and $R - \varepsilon R \leq R_n \leq R + \varepsilon I$, where I is an identity matrix. The choice of asymmetrical bounds will be explained later in Note 3.9. Let us denote $G_{+\varepsilon}$

and $G_{-\varepsilon}$ the solution of the algebraic equation for matrices $Q - \varepsilon Q$, $R - \varepsilon R$ and $Q + \varepsilon I$, $R + \varepsilon I$, respectively. By the first part of this lemma, the following holds:

$$G_{-\varepsilon} \leq G_n \leq G_{+\varepsilon} \quad \forall n \geq n_\varepsilon.$$

Note that the control problem with matrices $Q + \varepsilon I$, $R + \varepsilon I$ is well defined as both matrices are positive definite. For the lower bound, let us assume $\varepsilon < 1$ so that $Q - \varepsilon Q \geq 0$ and $R - \varepsilon R > 0$ and the control problem is well defined. By the first part of this lemma, as $\varepsilon \to 0$, the solution $G_{-\varepsilon}$ is increasing and $G_{+\varepsilon}$ decreasing. The solutions are also bounded form above (below) by G, which is also implied by the first part of this lemma. There must therefore exist limits

$$G_{-\varepsilon} \xrightarrow{\varepsilon \to 0} G_-$$

and

$$G_{+\varepsilon} \xrightarrow{\varepsilon \to 0} G_+.$$

Taking a limit for $\varepsilon \to 0$ of both sides of the algebraic equation with matrices $Q - \varepsilon Q$ and $R - \varepsilon R$ shows that the matrix G_- must satisfy the algebraic equation for matrices Q and R. By the uniqueness of the solution, $G_- = G$ and using the same technique, also $G_+ = G$. This clearly implies that

$$G_n \xrightarrow{n \to \infty} G,$$

which completes the proof. □

> **Note 3.9.** Note that it was necessary to construct the two bounds differently. We could not use a sequence $Q + \varepsilon Q$ for the upper bound, as Q is not necessarily positive definite and therefore the sequence would not be a general upper bound. On the other hand, a sequence $Q - \varepsilon I$ might be indefinite for each $\varepsilon > 0$ if Q is not positive definite.

Lemma 3.10. *Let Q_n, R_n be weighting matrices of the classical LQ control problem for each $n \in \mathbb{N}$ such that*

$$Q_n = \alpha_n Q, \quad R_n = \alpha_n R \tag{3.21}$$

for some $Q \geq 0$, $R > 0$, (A, Γ_Q), $\Gamma_Q^T \Gamma_Q = Q$ observable, and a sequence $\alpha_n \in \mathbb{R}$, $\alpha_n \xrightarrow{n \to \infty} \infty$. Then the following is true for solutions of the corresponding algebraic Riccati equation:

$$G_n = \alpha_n G, \quad \lambda_{min}(G_n) \xrightarrow{n \to \infty} \infty, \tag{3.22}$$

where $\lambda_{min}(G_n)$ denotes the minimum eigenvalue of matrix G_n.

Proof. We will use the fact that for the problem with matrices Q, R, the solution $G > 0$

due to assumed observability. The criterion value for each n is

$$\begin{aligned} x_0^T G x_0 &= \sum_{k=0}^{\infty} x_k^{*T} Q_n x_k^* + u_k^{*T} R_n u_k^* = \\ &= \sum_{k=0}^{\infty} x_k^{*T} \alpha_n Q x_k^* + u_k^{*T} \alpha_n R u_k^* = \\ &= \alpha_n \sum_{k=0}^{\infty} x_k^{*T} Q x_k^* + u_k^{*T} R u_k^* = \alpha_n x_0^T G x_0. \end{aligned} \tag{3.23}$$

As $G > 0$, it must hold that $\lambda_{min}(G) > 0$ and therefore $\lambda_{min}(G_n) = \alpha_n \lambda_{min}(G) \xrightarrow{n\to\infty} \infty$. □

Theorem 3.11 (Existence of a solution to the algebraic cautious Riccati-like equation)**.** *Assume an algebraic cautious Riccati-like equation* (3.18)*, where*

$$Q \geq 0, \quad R > 0, \quad \sigma_{b_0}^2 > 0,$$

the pair (A, B) *is stabilizable and the pairs* (A, Γ_Q) *and* (A, Γ_{P_θ}) *are observable,* $\Gamma_Q^T \Gamma_Q = Q$*,* $\Gamma_{P_\theta}^T \Gamma_{P_\theta} = P_\theta$*. A positive semidefinite solution* G_a *to an algebraic cautious Riccati-like equation* (3.18) *exists if and only if the unique positive semidefinite solution* G_* *of an (ordinary) algebraic Riccati equation*

$$G = A^T G A + P_\theta - (A^T G B + P_{\theta,b_0})(B^T G B + \sigma_{b_0}^2)^{-1}(B^T G A + P_{b_0,\theta}) \tag{3.24}$$

satisfies the condition

$$G_*^{1,1} < 1. \tag{3.25}$$

Moreover, if the positive semidefinite solution G_a *exists, then* $G_a > 0$ *and it is unique in the set of all positive semidefinite matrices.*

> **Note 3.12.** It is important that the inequality in (3.25) is strict. Also note that similarly to the equation (3.13), the equation (3.24) can be derived by dividing both sides of the algebraic equation (3.18) by $G_{1,1}$ and then neglecting the terms containing matrices Q and R.

Proof of Theorem 3.11. The equation (3.24) has a form of a standard Riccati equation with weighting matrices $Q = P_\theta$, $R = \sigma_{b_0}^2$ and $S = P_{\theta,b_0}$. The assumption of the theorem gives us observability of the pair (A, Γ_{P_θ}). Together with the assumed stabilizability of (A, B), we get the existence, uniqueness and positive definiteness of the solution G_*, as shown in [10], and therefore also $G_*^{1,1} > 0$.

If the solution G_* in the theorem satisfies $G_*^{1,1} < 1$, then there exists ε such that the solution $G_{*,\varepsilon}$ of the equation

$$G = A^T G A + \varepsilon Q + P_\theta - (A^T G B + P_{\theta,b_0})(B^T G B + \varepsilon R + \sigma_{b_0}^2)^{-1}(B^T G A + P_{b_0,\theta}) \tag{3.26}$$

satisfies $G^{1,1}_{*,\varepsilon} = 1$. This is a corollary of Lemma 3.8 and Lemma 3.10 as we assume the pair (A, Γ_Q) to be observable. Dividing the equation (3.26) by ε will give the original equation (3.18) and its solution $G_a = G_{*,\varepsilon}/\varepsilon$. This also explains why the inequality (3.25) must be strict - we need to have $\varepsilon > 0$ to be able to find the solution.

Conversely, assume there exists a solution G_a of the equation (3.18). Observability of (A, Γ_Q) and positive definiteness of the variance matrix P yields $G^{1,1}_a > 0$. Then dividing the equation by $G^{1,1}_a$ gives an algebraic equation in a form (3.26) where $\varepsilon = 1/G^{1,1}_a$ and the upper-left element of its solution equals 1. Using Lemma 3.8 and Lemma 3.10, the solution G_* of (3.24) will satisfy the condition (3.25). □

Lemma 3.13. *Let $M, A_k, k \in \mathbb{N}$ be symmetrical matrices, $A_i \leq A_j$ whenever $i < j$, $M \geq A_k$ for all $k \in \mathbb{N}$. Then there exists a symmetrical matrix $A = \lim_{k\to\infty} A_k$.*

Proof. The Lemma is true for sequences of real numbers. Using vector $x_1 = [1\ 0\ \cdots\ 0]^T$, we get $x_1^T A_k x_1 = A^{1,1}_k$ and similarly $x_i^T A_k x_i = A^{i,i}_k$, where x_i is a vector with 1 in the i-th position and zeros elsewhere. The sequences $A^{i,i}_k$ are monotonous by the assumption of the lemma and thus the limit exists for all diagonal elements. Existence of the remaining limits can be proved by using vectors with 1 in the corresponding places and the fact that diagonal elements converge, e.g. vector $[1\ 1\ 0\ \cdots\ 0]$ for the element $A^{1,2}_k = A^{2,1}_k$ and so on. □

Lemma 3.14. *Let $\alpha > 1$ is a real number, G is a solution to an algebraic cautious Riccati-like equation with matrices A, B, Q, R and P. Then*

$$\begin{aligned} \alpha G &\geq A^T \alpha G A + Q + \alpha G^{1,1} P_\theta - \\ &- (A^T \alpha G B + \alpha G^{1,1} P_{\theta,b_0})(B^T \alpha G B + R + \alpha G^{1,1} \sigma^2_{b_0})^{-1}(B^T \alpha G A + \alpha G^{1,1} P_{b_0,\theta}). \end{aligned} \tag{3.27}$$

Proof. Multiplying the algebraic cautious Riccati-like equation by α yields the following equality

$$\begin{aligned} \alpha G &= A^T \alpha G A + \alpha Q + \alpha G^{1,1} P_\theta - \\ &- (A^T \alpha G B + \alpha G^{1,1} P_{\theta,b_0})(B^T \alpha G B + \alpha R + \alpha G^{1,1} \sigma^2_{b_0})^{-1}(B^T \alpha G A + \alpha G^{1,1} P_{b_0,\theta}). \end{aligned} \tag{3.28}$$

Because $\alpha > 1$, then also $\alpha Q \geq Q$ and $\alpha R \geq R$ and thus the right-hand side of (3.28) is greater than the right-hand side of (3.27). □

Theorem 3.15 (Convergence of the discrete-time cautious Riccati-like equation)**.** *Let G_k be a matrix sequence given by the discrete-time cautious Riccati-like equation for some initial condition $G_0 \geq 0$, stabilizable pair of system matrices (A, B), and weighting matrices $Q \geq 0$ and $R > 0$, such that the pair (A, Γ_Q) is observable, $\Gamma_Q^T \Gamma_Q = Q$. Let the positive semidefinite solution G of the algebraic cautious Riccati-like equation exist. Then*

$$G_k \xrightarrow{k\to\infty} G.$$

Proof. Let us divide the proof into three parts. We will denote $G_k(G_0)$ the k-th element of the sequence given by the discrete-time cautious Riccati-like equation with an initial condition G_0. The system (A, B) is stabilizable and the pair (A, Γ_Q) observable, and thus the algebraic solution G is positive definite.

1. Let $G_0 = 0$. Then $G_1 = Q$ and $Q > G_0$. It follows from Corollary 3.6 that the sequence $G_k(0)$ is monotonous. The sequence $G_k(G)$ is a constant sequence $G_k = G$ for every $k \geq 0$. As $G > 0$, $G_k(0) \leq G$ for every $k \geq 0$ by Lemma 3.5. $G_k(0)$ is therefore an increasing, bounded from above sequence, and as such has a limit according to lemma 3.13. This limit must satisfy the algebraic equation and be positive definite. As the positive definite solution od the algebraic equation is unique, the sequence $G_k(0)$ converges to the algebraic solution G.

2. Let $\alpha > 1$ be an arbitrary real number. Using αG as initial condition yields $G_1(\alpha G) < \alpha G$ according to lemma 3.14. The sequence $G_k(\alpha G)$ is a decreasing sequence by Lemma 3.5. Similarly to step 1, $\alpha G > G$ implies $G_k(\alpha G) > G$ for every $k \geq 0$. The sequence is thus bounded from below and therefor convergent. Due to uniqueness of the positive definite algebraic solution G, the limit must be G.

3. Let $G_0 \geq 0$ be an arbitrary initial condition. Then there exists $\alpha > 1$ such that $\alpha G > G_0$. Lemma 3.5 yields that $G_k(0) < G_k(G_0) < G_k(\alpha G)$ for every $k \geq 0$. As we have shown that both $G_k(0) \xrightarrow{k\to\infty} G$ and $G_k(\alpha G) \xrightarrow{k\to\infty} G$, it holds that $G_k(G_0) \xrightarrow{k\to\infty} G$.

□

3.2.2 Divergent equation

Lemma 3.16. *Let A, B be system matrices and Q, R weighting matrices. Let $\alpha_k > 0$ and $\beta_k > 0$ $k \in \mathbb{N}$ be two real sequences, $\alpha_k \geq \beta_k$. Let $G_{\alpha,0}$, $G_{\beta,0}$ be two symmetrical positive semidefinite matrices such that $G_{\alpha,0} \geq G_{\beta,0}$. Let $G_{\alpha,k}$ be the sequence given by the cautious Riccati-like equation* (3.15) *with weighting matrices Q, R, system matrices $\alpha_k A$, $\alpha_k B$ and initial condition $G_{\alpha,0}$, i.e. the system matrices need not be constant, but may vary with factor α_k with each step k. Similarly, let $G_{\beta,k}$ be the sequence given by the cautious Riccati-like equation* (3.15) *with weighting matrices Q, R, system matrices $\beta_k A$, $\beta_k B$ and initial condition $G_{\beta,0}$. Then for each $k \in \mathbb{N}$ $G_{\alpha,k} \geq G_{\beta,k}$.*

Proof. The proof is similar to the proof of Lemma 3.5. We will again use induction to prove the lemma. For $k = 0$ the lemma is true by assumption. Now let us assume that the lemma is true for $k \in \mathbb{N}$. The following holds by the analogy described in Note 3.3

$$x_0^T G_{\alpha,k+1} x_0 = \begin{bmatrix} x_0^T & u_{\alpha,0}^T \end{bmatrix} \begin{bmatrix} Q + G_{\alpha,k}^{1,1} P_\theta & G_{\alpha,k}^{1,1} P_{\theta,b_0} \\ G_{\alpha,k}^{1,1} P_{b_0,\theta} & R + G_{\alpha,k}^{1,1} \sigma_{b_0}^2 \end{bmatrix} \begin{bmatrix} x_0 \\ u_{\alpha,0} \end{bmatrix} + x_{\alpha,1}^T G_{\alpha,k} x_{\alpha,1},$$

and

$$x_0^T G_{\beta,k+1} x_0 = \begin{bmatrix} x_0^T & u_{\beta,0}^T \end{bmatrix} \begin{bmatrix} Q + G_{\beta,k}^{1,1} P_\theta & G_{\beta,k}^{1,1} P_{\theta,b_0} \\ G_{\beta,k}^{1,1} P_{b_0,\theta} & R + G_{\beta,k}^{1,1} \sigma_{b_0}^2 \end{bmatrix} \begin{bmatrix} x_0 \\ u_{\beta,0} \end{bmatrix} + x_{\beta,1}^T G_{\beta,k} x_{\beta,1},$$

where $u_{\alpha,0}$ and $x_{\alpha,1}$ are the optimal control and state for the problem with initial matrix $G_{\alpha,0}$ and system matrices $\alpha_{k+1} A$, $\alpha_{k+1} B$ (and the same for β_k). By assumption $G_{\alpha,k} \geq G_{\beta,k}$,

therefore also $G^{1,1}_{\alpha,k} \geq G^{1,1}_{\beta,k}$ and consequently

$$\begin{bmatrix} Q + G^{1,1}_{\alpha,k} P_\theta & G^{1,1}_{\alpha,k} P_{\theta,b_0} \\ G^{1,1}_{\alpha,k} P_{b_0,\theta} & R + G^{1,1}_{\alpha,k} \sigma^2_{b_0} \end{bmatrix} \geq \begin{bmatrix} Q + G^{1,1}_{\beta,k} P_\theta & G^{1,1}_{\beta,k} P_{\theta,b_0} \\ G^{1,1}_{\beta,k} P_{b_0,\theta} & R + G^{1,1}_{\beta,k} \sigma^2_{b_0} \end{bmatrix}. \tag{3.29}$$

Due to the modified indexation introduced at the beginning of this subsection, the optimal control $u_{\alpha,0}$ is given by a state feedback control law with index $k+1$, i.e. $u_{\alpha,0} = -K_{\alpha,k+1} x_0$, where $K_{\alpha,k+1}$ is a matrix given by the the equation (3.16). Similarly, the state $x_{\alpha,1} = \alpha_{k+1} A x_0 + \alpha_{k+1} B u_{\alpha,0}$. Using the control law $K_{\alpha,k+1}$ for the system $\beta_{k+1}A$, $\beta_{k+1}B$ will result in the control $u_{\alpha,0}$ and the state $\frac{\beta_{k+1}}{\alpha_{k+1}} x_{\alpha,1}$. The inequality (3.29) and the fact that $\beta_{k+1} \leq \alpha_{k+1}$ imply that

$$\begin{aligned} x_0^T G_{\alpha,k+1} x_0 &= \begin{bmatrix} x_0^T & u_{\alpha,0}^T \end{bmatrix} \begin{bmatrix} Q + G^{1,1}_{\alpha,k} P_\theta & G^{1,1}_{\alpha,k} P_{\theta,b_0} \\ G^{1,1}_{\alpha,k} P_{b_0,\theta} & R + G^{1,1}_{\alpha,k} \sigma^2_{b_0} \end{bmatrix} \begin{bmatrix} x_0 \\ u_{\alpha,0} \end{bmatrix} + x_{\alpha,1}^T G_{\alpha,k} x_{\alpha,1} \geq \\ &\geq \begin{bmatrix} x_0^T & u_{\alpha,0}^T \end{bmatrix} \begin{bmatrix} Q + G^{1,1}_{\beta,k} P_\theta & G^{1,1}_{\beta,k} P_{\theta,b_0} \\ G^{1,1}_{\beta,k} P_{b_0,\theta} & R + G^{1,1}_{\beta,k} \sigma^2_{b_0} \end{bmatrix} \begin{bmatrix} x_0 \\ u_{\alpha,0} \end{bmatrix} + \frac{\beta_{k+1}}{\alpha_{k+1}} x_{\alpha,1}^T G_{\beta,k} \frac{\beta_{k+1}}{\alpha_{k+1}} x_{\alpha,1}, \end{aligned}$$

where we used the same control law for both equations, as described above. Such control law is indeed suboptimal for the second equation and therefore

$$\begin{aligned} & \begin{bmatrix} x_0^T & u_{\alpha,0}^T \end{bmatrix} \begin{bmatrix} Q + G^{1,1}_{\beta,k} P_\theta & G^{1,1}_{\beta,k} P_{\theta,b_0} \\ G^{1,1}_{\beta,k} P_{b_0,\theta} & R + G^{1,1}_{\beta,k} \sigma^2_{b_0} \end{bmatrix} \begin{bmatrix} x_0 \\ u_{\alpha,0} \end{bmatrix} + \frac{\beta_{k+1}}{\alpha_{k+1}} x_{\alpha,1}^T G_{\beta,k} \frac{\beta_{k+1}}{\alpha_{k+1}} x_{\alpha,1} \geq \\ \geq & \begin{bmatrix} x_0^T & u_{\beta,0}^T \end{bmatrix} \begin{bmatrix} Q + G^{1,1}_{\beta,k} P_\theta & G^{1,1}_{\beta,k} P_{\theta,b_0} \\ G^{1,1}_{\beta,k} P_{b_0,\theta} & R + G^{1,1}_{\beta,k} \sigma^2_{b_0} \end{bmatrix} \begin{bmatrix} x_0 \\ u_{\beta,0} \end{bmatrix} + x_{\beta,1}^T G_{\beta,k} x_{\beta,1}. \end{aligned}$$

The two inequalities imply

$$x_0^T G_{1,k+1} x_0 \geq x_0^T G_{2,k+1} x_0, \quad \forall x_0 \in \mathbb{R}^n,$$

which completes the proof. □

> **Note 3.17.** The statements of Lemma 3.16 are also true for a cautious Riccati-like equation with constant α and β and also for a standard Riccati equation, that can be seen as a special case of equation (3.15) with a zero variance matrix P. The proof of Lemma 3.16 will then remain unmodified, as there were no conditions on the variance matrix in Lemma 3.16.

Lemma 3.18. *Let (A, B) be a stabilizable pair of system matrices and $Q \geq 0$, $R > 0$ weighting matrices, such that the pair (A, Γ_Q) is observable, $\Gamma_Q^T \Gamma_Q = Q$. Let $\alpha \geq \beta > 0$. Let G_α and G_β be the solution of an algebraic Riccati equation with weighting matrices Q and R and system matrices αA, αB and βA, βB, respectively. Then $G_\alpha \geq G_\beta$. Let $\alpha_n > 0$, $n \in \mathbb{N}$ be a real sequence, $\alpha_n \xrightarrow{k\to\infty} 1$. Let G_n be a solution of an algebraic Riccati equation with weighting matrices Q, R and system matrices $\alpha_n A$, $\alpha_n B$. Then $G_n \xrightarrow{k\to\infty} G$, where G is a solution of an algebraic Riccati equation for weighting matrices Q, R and system matrices A, B.*

Proof. According to the Note 3.17, the Lemma 3.16 is true for a standard Riccati equation. It is known from the theory of Riccati equations [10], that the stabilizability and observability assumptions above ensure the existence of unique solutions G_α and G_β of the corresponding algebraic Riccati equations. It is also true that the sequence given by the discrete-time equation converges to the unique solution for each positive semidefinite initial condition G_0, i.e.

$$G_\alpha = \lim_{k\to\infty} G_{\alpha,k}$$

and

$$G_\beta = \lim_{k\to\infty} G_{\beta,k}.$$

Let us assume an arbitrary initial condition $G_0 \geq 0$. Then by Lemma 3.16, it holds that $G_{\alpha,k} \geq G_{\beta,k}$ for each $k \in \mathbb{N}$ and therefore also $G_\alpha \geq G_\beta$.

To prove the second part, first note that $G > 0$ due to the assumption of observability. We can easily construct two sequences $\beta_n > 0$ and $\gamma_n > 0$ such that $\forall n \in \mathbb{N}$ we have $\beta_n \leq \alpha_n \leq \gamma_n$, $\beta_n \xrightarrow{k\to\infty} 1$, $\gamma_n \xrightarrow{k\to\infty} 1$, β_n nondecreasing, γ_n nonincreasing. (Such sequences can be for example $\beta_n = \inf_{i\geq n} \alpha_i$ and $\gamma_n = \sup_{i\geq n} \alpha_i$). Let us denote $G_{\beta,n}$ and $G_{\gamma,n}$ the solution of the algebraic equation for matrices $\beta_n A$, $\beta_n B$ and $\gamma_n A$, $\gamma_n B$, respectively. By the first part of the Lemma, the following holds:

$$G_{\beta,n} \leq G_n \leq G_{\gamma,n} \quad \forall n \in \mathbb{N}.$$

By the first part of this lemma, as $n \to \infty$, the solution $G_{\beta,n}$ is increasing and $G_{\gamma,n}$ decreasing. The solutions are also bounded form above (below) by G, which is also implied by the first part of this lemma. There must therefore exist limits

$$G_{\beta,n} \xrightarrow{n\to\infty} G_\beta$$

and

$$G_{\gamma,n} \xrightarrow{n\to\infty} G_\gamma.$$

Taking a limit for $n \to \infty$ of both sides of the algebraic equation with matrices $\beta_n A$, $\beta_n B$ and $\gamma_n A$, $\gamma_n B$ shows that the matrix G_β must satisfy the algebraic equation for matrices A and B. By the uniqueness of the solution, $G_\beta = G$ and using the same technique, also $G_\gamma = G$. This clearly implies that

$$G_n \xrightarrow{n\to\infty} G,$$

which completes the proof. □

Lemma 3.19. *Let (A, B) be a stabilizable pair of system matrices, $Q_k \geq 0$, $R_k > 0$ weighting matrices, (A, Γ_{Q_k}) observable, $\Gamma_{Q_k}^T \Gamma_{Q_k} = Q_k$ for every $k \in \mathbb{N}$. Let $Q_k \xrightarrow{k\to\infty} Q \geq 0$, $R_k \xrightarrow{k\to\infty} R > 0$, (A, Γ_Q) observable, $\Gamma_Q^T \Gamma_Q = Q$. Let $0 < \alpha_k \in \mathbb{R}$, $\alpha_k \xrightarrow{k\to\infty} 1$. Let us denote $A_k = \alpha_k A$, $B_k = \alpha_k B$. Let G_k be a sequence of matrices given by the discrete-time Riccati equation*

$$G_{k+1} = A_{k+1}^T G_k A_{k+1} + Q_{k+1} - (A_{k+1}^T G_k B_{k+1})(B_{k+1}^T G_k B_{k+1} + R_{k+1})^{-1}(B_{k+1}^T G_k A_{k+1}). \tag{3.30}$$

Then

$$G_k \xrightarrow{k\to\infty} G,$$

where G is a solution of the algebraic Riccati equation for system matrices A, B and weighting matrices Q, R.

Proof. Similarly to the proofs of Lemma 3.8 and Lemma 3.18, let us define sequences $0 < \beta_k \leq \alpha_k \leq \gamma_k \forall k \in \mathbb{N}$, such that $\beta_k \xrightarrow{k\to\infty} 1$ and $\gamma_k \xrightarrow{k\to\infty} 1$, β_k nondecreasing, γ_k nonincreasing. Let us also define $0 \leq Q_k^- \leq Q_k \leq Q_k^+$ and $0 < R_k^- \leq R_k \leq R_k^+$, such that $Q_k^- \xrightarrow{k\to\infty} Q$, $Q_k^+ \xrightarrow{k\to\infty} Q$, $R_k^- \xrightarrow{k\to\infty} R$, $R_k^+ \xrightarrow{k\to\infty} R$, where Q_k^- and R_k^- nondecreasing, Q_k^+ and R_k^+ nonincreasing. Let us denote G_k^- the sequence given by a discrete-time Riccati equation with system matrices $\beta_k A$, $\beta_k B$ and weighting matrices Q_k^-, R_k^-. Let us denote G_k^+ the sequence given by a discrete-time Riccati equation with system matrices $\gamma_k A$, $\gamma_k B$ and weighting matrices Q_k^+, R_k^+. By Lemma 3.8 nad Lemma 3.18, the sequence G_k^- is nondecreasing bounded by G from above and G_k^+ nonincreasing, bounded by G from below. This implies the existence of limits

$$G^- = \lim_{k\to\infty} G_k^-$$

and

$$G^+ = \lim_{k\to\infty} G_k^+.$$

The limits must satisfy the equation taken to limits on both sides and by the uniqueness of solution of the algebraic Riccati equation, it must hold that $G^- = G^+ = G$. As $G_k^- \leq G_k \leq G_k^+$ for every $k \in \mathbb{N}$, it also holds that

$$\lim_{k\to\infty} G_k = G,$$

which completes the proof. □

We will now study the behavior of the discrete-time cautious Riccati-like equation in the case that no positive semidefinite solution of the algebraic equation exists. We will first show, that the solution is unbounded and we will then study the divergent behavior in more detail. In particular, it will be shown, that the divergence is 'uniform' in the sense that although all elements of the matrix G_k go to infinity, there is a matrix $\overline{G}$ such that $G_k/G_k^{1,1} \xrightarrow{k\to\infty} \overline{G}$. As a consequence, the feedback gain converges to a limit matrix and as such gives a time-invariant control law.

Theorem 3.20. *Let G_k be a matrix sequence given by the discrete-time cautious Riccati-like equation for some initial condition $G_0 \geq 0$, stabilizable pair of system matrices A, B and weighting matrices $Q \geq 0$ and $R > 0$. Assume that no algebraic solution of the algebraic cautious Riccati-like equation exists. Then*

$$G_k^{1,1} \xrightarrow{k\to\infty} \infty.$$

Proof. Assume the sequence $G_k(0)$. This sequence is increasing by Lemma 3.5 and the fact that $G_1(0) = Q > 0$. We will first show that the sequence is not bounded from above. Assume that $G_k(0)$ is bounded from above, then it is a bounded monotonous sequence and as such has a limit that must satisfy the algebraic equation. This is a contradiction, as such matrix does not exist by assumption.

We have shown that $G_k(0)$ si unbounded, in particular, there exist $1 \leq i, j \leq n$ such that $G_k^{i,j}(0) \xrightarrow{k\to\infty} \infty$. This, however, does not ensure $G_k^{1,1}(0) \xrightarrow{k\to\infty} \infty$. We will prove this again by contradiction. Let us assume, that $G_k^{1,1}(0) \xrightarrow{k\to\infty} L \in \mathbb{R}$ ($G_k^{1,1}(0)$ is monotonous, and thus either convergent or divergent to infinity). If we substitute L for $G_k^{1,1}$ in the cautious equation (3.15), we get a standard Riccati equation with constant weighting matrices. This equation has an algebraic solution G^L and also defines a sequence $G_k^L(0) \xrightarrow{k\to\infty} G^L$. According to Lemma 3.5 and Note 3.7, the sequence is monotonous and bounded from above by G^L.

Let us now assume the sequence $G_k(0)$ given by the original equation. As $G_k^{1,1}(0) < L$ for each $k \geq 0$, then also $G_k(0) < G_k^L(0)$ by Lemma 3.5 and therefore $G_k(0)$ is bounded from above. This is a contradiction with assumptions. □

Let us now consider the case, when the sequence G_k given by the cautious discrete-time Riccati-like equation does not converge. According to Theorem 3.15, this happens if and only if the corresponding algebraic equation has no positive semidefinite solution, which is equivalent to the condition (3.25) in Theorem 3.11. Let us now assume, that $G_*^{1,1} = 1$. We can define a modified equation, where we divide both sides of the equation (3.15) by the number $G_k^{1,1}$, i.e.

$$\begin{aligned} \frac{G_{k+1}}{G_k^{1,1}} &= A^T \frac{G_k}{G_k^{1,1}} A + \frac{Q}{G_k^{1,1}} + P_\theta \\ &- (A^T \frac{G_k}{G_k^{1,1}} B + P_{\theta,b_0})(B^T \frac{G_k}{G_k^{1,1}} B + \frac{R}{G_k^{1,1}} + \sigma_{b_0}^2)^{-1}(B^T \frac{G_k}{G_k^{1,1}} A + P_{b_0,\theta}). \end{aligned} \tag{3.31}$$

We can do this without loss of generality, as we have proven that $G_k^{1,1} > 0$ for $k \geq k_0$ for some finite k_0. If $k_0 > 0$, we can formally put $G_0 = G_{k_0}$ and start the solution from this point. If we denote $\overline{G_{k+1}} = \frac{G_{k+1}}{G_k^{1,1}}$ and define $G_{-1}^{1,1} = 1$, then the equation (3.31) will have the following form

$$\begin{aligned} \overline{G_{k+1}} &= A^T \frac{\overline{G_k}}{\overline{G_k}^{1,1}} A + \frac{Q}{\prod_{i=0}^{k} \overline{G_i}^{1,1}} + P_\theta \\ &- (A^T \frac{\overline{G_k}}{\overline{G_k}^{1,1}} B + P_{\theta,b_0})(B^T \frac{\overline{G_k}}{\overline{G_k}^{1,1}} B + \frac{R}{\prod_{i=0}^{k} \overline{G_i}^{1,1}} + \sigma_{b_0}^2)^{-1}(B^T \frac{\overline{G_k}}{\overline{G_k}^{1,1}} A + P_{b_0,\theta}). \end{aligned} \tag{3.32}$$

Note that we can always return to the original solution by a simple mapping $G_{k+1} = \overline{G_{k+1}} G_k^{1,1} = \overline{G_{k+1}} \prod_{i=0}^{k} \overline{G_i}^{1,1}$, as in this notation it holds that $G_k^{1,1} = \prod_{i=0}^{k} \overline{G_i}^{1,1}$. The equa-

tion (3.32) thus describes the behavior of a normalized solution $\overline{G_k}$. We have shown that $G_k^{1,1} \xrightarrow{k\to\infty} \infty$. If we show that $\overline{G_k}^{1,1} \xrightarrow{k\to\infty} 1$, we will get the equation in a form (3.30) and we will be able to use Lemma 3.19.

Theorem 3.21. *Let A, B be system matrices and Q, R weighting matrices. Let there be no solution to the algebraic equation* (3.18) *and let $G_*^{1,1} = 1$, where $G_*^{1,1}$ is defined in Theorem 3.11. Let G_k be a sequence given by the cautious discrete-time Riccati-like equation with matrices A, B, Q and R. Then*

$$\frac{G_k}{G_k^{1,1}} \xrightarrow{k\to\infty} G_*, \tag{3.33}$$

where G_* is given in Theorem 3.11

Proof. We will first prove the theorem for a nondecreasing sequence G_k given by the discrete-time equation (3.15). Let us define G_k^* as a solution to an algebraic equation

$$\begin{aligned} G &= A^TGA + Q + G_k^{1,1}P_\theta \\ &- (A^TGB + G_k^{1,1}P_{\theta,b_0})(B^TGB + R + G_k^{1,1}\sigma_{b_0}^2)^{-1}(B^TGA + G_k^{1,1}P_{b_0,\theta}). \end{aligned} \tag{3.34}$$

Then $\frac{G_k^*}{G_k^{1,1}}$ solves the equation

$$\begin{aligned} G &= A^TGA + \frac{Q}{G_k^{1,1}} + P_\theta \\ &- (A^TGB + P_{\theta,b_0})(B^TGB + \frac{R}{G_k^{1,1}} + \sigma_{b_0}^2)^{-1}(B^TGA + P_{b_0,\theta}), \end{aligned} \tag{3.35}$$

and therefore, by Lemma 3.8, $G_* \le \frac{G_k^*}{G_k^{1,1}}$. As G_k is assumed to be nondecreasing, it also holds that $G_{k+1} \ge G_k$ and $G_k^* \ge G_{k+1}$. The second inequality follows from the following. If we take G_k as an initial condition for the equation (3.15), then the right-hand side of the equation will be the same as in (3.34). After one iteration, we get the matrix G_{k+1}, which by assumption is greater that G_k. If we continue solving the equation with the right-hand side given by (3.34), i.e. if we do not update the factor $G_k^{1,1}$ to $G_{k+1}^{1,1}$, we will continue with a nondecreasing sequence (Corollary 3.6) to the limit solution G_k^*, which implies the inequality $G_k^* \ge G_{k+1}$. Dividing both inequalities by $G_k^{1,1} > 0$ yields

$$\frac{G_k^*}{G_k^{1,1}} \ge \frac{G_{k+1}}{G_k^{1,1}} \ge \frac{G_k}{G_k^{1,1}}.$$

The leftmost expression converges from above to G_* by Lemma 3.8 and by assumption, $G_*^{1,1} = 1$. The $(1,1)$-element (upper-left element) of the rightmost expression is equal to 1 for each $k \in \mathbb{N}$. Finally, the expression in the middle is by definition equal to $\overline{G_{k+1}}$. Taking the inequality to the limit as $k \to \infty$ yields $\overline{G_{k+1}^{1,1}} \xrightarrow{k\to\infty} 1$. By Lemma 3.19 is the proof complete for a nondecreasing sequence G_k.

For a general sequence G_k, let us consider two nondecreasing sequences G_k^- and G_k^+ such that $G_k^- \le G_k \le G_k^+$. Let us choose $G_0^- = 0$ and $G_0^+ = G_m^- \ge G_0$. Such m exists

for every $G_0 \geq 0$ as it is implied by Theorem 3.20 and Lemmas 3.8 and 3.10 that for zero initial condition, the minimum eigenvalue of G_k^- goes to infinity as $k \to \infty$. The number m is fixed for each G_0 and therefore it holds

$$G_k^{+,1,1} = \prod_{i=k+1}^{k+m} \overline{G_i}^{1,1} G_k^{-,1,1},$$

where the finite product goes to 1 as $k \to \infty$. Let us examine the inequality

$$\frac{G_{k+1}^-}{G_k^{-,1,1}} \leq \frac{G_{k+1}}{G_k^{-,1,1}} \leq \frac{G_{k+1}^+}{G_k^{-,1,1}}$$

and the inequality

$$\frac{G_{k+1}^-}{G_k^{+,1,1}} \leq \frac{G_{k+1}}{G_k^{+,1,1}} \leq \frac{G_{k+1}^+}{G_k^{+,1,1}}.$$

The $(1,1)$-elements of leftmost and rightmost expressions in both inequalities converge to 1, therefore it also holds for the middle. Finally, note that

$$\frac{G_{k+1}}{G_k^{-,1,1}} \geq \frac{G_{k+1}}{G_k^{1,1}} \geq \frac{G_{k+1}}{G_k^{+,1,1}},$$

and therefore $\overline{G_k}^{1,1} \xrightarrow{k\to\infty} 1$ for an arbitrary initial condition G_0. □

In the case $G_*^{1,1} > 1$, the Theorem 3.21 does not hold, but it can be used to derive a form of 'normalized convergence' similar to (3.33). Let us consider the following algebraic equation

$$G = \varepsilon^2 \left[A^T G A + P_\theta - (A^T G B + P_{\theta,b_0})(B^T G B + \sigma_{b_0}^2)^{-1}(B^T G A + P_{b_0,\theta}) \right], \tag{3.36}$$

which corresponds to the equation (3.24) for a system εA, εB and variances $\varepsilon^2 P_\theta$, $\varepsilon^2 P_{\theta,b_0}$ and $\varepsilon^2 \sigma_{b_0}^2$. The factor ε is chosen such that the solution G_ε of the algebraic equation (3.36) satisfies $G_\varepsilon^{1,1} = 1$. By Lemmas 3.8 and 3.10, there always exists such a factor $0 < \varepsilon < 1$. Let us denote G_k^ε the sequence given by the cautious discrete-time Riccati-like equation

$$\begin{aligned} G_{k+1} &= \varepsilon^2 \Big[A^T G_k A + Q + G_k^{1,1} P_\theta - \\ &- (A^T G_k B + G_k^{1,1} P_{\theta,b_0})(B^T G_k B + R + G_k^{1,1} \sigma_{b_0}^2)^{-1}(B^T G_k A + G_k^{1,1} P_{b_0,\theta}) \Big], \end{aligned} \tag{3.37}$$

derived from the equation (3.15) by using scaled matrices εA, εB, $\varepsilon^2 Q$, $\varepsilon^2 R$, $\varepsilon^2 P_\theta$, $\varepsilon^2 P_{\theta,b_0}$ and $\varepsilon^2 \sigma_{b_0}^2$. Due to the scaling, the sequence G_k given by the original equation (3.15) will be given by the mapping

$$G_k = \varepsilon^{-2k} G_k^\varepsilon.$$

According to Theorem 3.21 above, the normalized sequence $\overline{G_k^\varepsilon}$ given by a modified discrete-time equation (3.32) with scaled matrices εA, εB, $\varepsilon^2 Q$, $\varepsilon^2 R$, $\varepsilon^2 P_\theta$, $\varepsilon^2 P_{\theta,b_0}$ and $\varepsilon^2 \sigma_{b_0}^2$ will converge to G_ε. As it holds that

$$G_{k+1}^\varepsilon = \overline{G_{k+1}^\varepsilon} G_k^{\varepsilon,1,1} = \overline{G_{k+1}^\varepsilon} \prod_{i=0}^{k} \overline{G_i^{\varepsilon,1,1}},$$

the mapping from the normalized sequence $\overline{G^{\varepsilon}_{k+1}}$ to the original sequence G_{k+1} is given by

$$G_{k+1} = \varepsilon^{-2(k+1)}\overline{G^{\varepsilon}_{k+1}}G^{\varepsilon,1,1}_{k} = \varepsilon^{-2(k+1)}\overline{G^{\varepsilon}_{k+1}}\prod_{i=0}^{k}\overline{G^{\varepsilon,1,1}_{i}}.$$

Let us now use these equalities to write

$$\frac{G_{k+1}}{G^{1,1}_{k+1}} = \frac{\varepsilon^{-2(k+1)}\overline{G^{\varepsilon}_{k+1}}G^{\varepsilon,1,1}_{k}}{G^{1,1}_{k+1}} = \frac{\varepsilon^{-2(k+1)}\overline{G^{\varepsilon}_{k+1}}G^{\varepsilon,1,1}_{k}}{\varepsilon^{-2(k+1)}G^{\varepsilon,1,1}_{k+1}} = \frac{\overline{G^{\varepsilon}_{k+1}}}{\overline{G^{\varepsilon,1,1}_{k+1}}}.$$

We can now express the limit

$$\lim_{k\to\infty}\frac{G_{k+1}}{G^{1,1}_{k+1}} = \lim_{k\to\infty}\frac{\overline{G^{\varepsilon}_{k+1}}}{\overline{G^{\varepsilon,1,1}_{k+1}}} = G_{\varepsilon}, \tag{3.38}$$

where the last equality follows from the previous text. We have now proven the last theorem of this section.

Theorem 3.22. *Let A, B be system matrices and Q, R weighting matrices. Let there be no solution to the algebraic equation* (3.18) *and let $G^{1,1}_{*} > 1$, where $G^{1,1}_{*}$ is defined in Theorem 3.11. Let G_k be a sequence given by the cautious discrete-time Riccati-like equation with matrices A, B, Q and R. Then*

$$\frac{G_k}{G^{1,1}_{k}} \xrightarrow{k\to\infty} G_{\varepsilon}, \tag{3.39}$$

where G_{ε} a solution of the equation (3.36)*, where the factor ε is chosen such that $G^{1,1}_{\varepsilon} = 1$.*

Proof. The proof follows from equation (3.38) and the considerations in the preceding text. □

3.3 The limit cautious controller

The feedback gain matrix is given by the equation (3.16). Using the theorems from the previous section, we can examine the limit behavior of the feedback gain also in the divergent case

$$\begin{aligned}\lim_{k\to\infty}K_{k+1} &= \lim_{k\to\infty}(B^TG_kB + R + G^{1,1}_{k}\sigma^2_{b_0})^{-1}(B^TG_kA + G^{1,1}_{k}P_{b_0,\theta}) = \\ &= \lim_{k\to\infty}(B^T\frac{G_k}{G^{1,1}_{k}}B + \frac{R}{G^{1,1}_{k}} + \sigma^2_{b_0})^{-1}(B^T\frac{G_k}{G^{1,1}_{k}}A + P_{b_0,\theta}).\end{aligned} \tag{3.40}$$

The expression on the right-hand side of the equation (3.40) converges to a constant finite matrix independent of convergence of the corresponding sequence G_k, which follows from Theorems 3.21 and 3.22. In the case described by Theorem 3.21 ($G^{1,1}_{*} = 1$), the limit controller is equivalent to the LQ controller designed for the nominal system and weighting matrices P_{θ}, P_{θ,b_0} and $\sigma^2_{b_0}$. In this case the equation (3.40) yields

$$K = (B^TG_*B + \sigma^2_{b_0})^{-1}(B^TG_*A + P_{b_0,\theta}).$$

In the case described by Theorem 3.22 ($G_*^{1,1} > 1$), the limit controller is equivalent to the LQ controller for a scaled system εA, εB with scaled weighting matrices given by $\varepsilon^2 P_\theta$, $\varepsilon^2 P_{\theta,b_0}$ and $\varepsilon^2 \sigma_{b_0}^2$ (the scaling factor will not change the result in (3.40)). In this case the equation (3.40) takes the form

$$K = (B^T G_\varepsilon B + \sigma_{b_0}^2)^{-1}(B^T G_\varepsilon A + P_{b_0,\theta}).$$

The resulting feedback gain K ensures stability of the closed-loop matrix $\varepsilon(A - BK)$. Depending on the factor $0 < \varepsilon < 1$, the limit controller may cause an unstable closed loop for the nominal system, i.e. the matrix $(A - BK)$ may have eigenvalues outside the unit circle.

The cautious Riccati-like equation analyzed in the previous sections describes the development of the matrix G_k which is just one of the three terms defining the cautious LQ controller of an ARMAX model, the other two being γ_k^T and g_k. Recursive expressions for calculating G_k, γ_k^T and g_k are given by equations (2.32), (2.33) and (2.34) in Chapter 2, respectively. The limit behavior of γ_k^T and g_k will now be studied. Similarly to the cautious Riccati-like equation, this only makes sense if $\hat{A}_k = A$ and $\hat{B}_k = B$ for some A, B and all $k \in \mathbb{N}$. We will first rewrite the equations (2.33) and (2.34) using the notation (2.30), assuming constant system matrices and reversing the time indexation similarly to Definition 3.1

$$\begin{aligned}
\gamma_{k+1}^T &= G_k^{1,\cdot} A(P_{x,\theta} - P_{x,b_0} K_{k+1}) + \gamma_k^T (A - BK_{k+1}), && (3.41)\\
g_{k+1} &= g_k + \Gamma^T G_k \Gamma \sigma_e^2 - L_{k+1}^T (R + B^T G_k B + G_k^{1,1}\sigma_{b_0}^2) L_{k+1} + && (3.42)\\
&+ \operatorname{tr}\left\{ \left[K_{k+1}^T (R + B^T G_k B + G_k^{1,1}\sigma_{b_0}^2) K_{k+1} - G_k^{1,1} P_\theta \right] P_x \right\},
\end{aligned}$$

where

$$\begin{aligned}
K_{k+1} &= (R + \hat{B}^T G_k \hat{B} + G_k^{1,1}\sigma_{b_0}^2)^{-1}(\hat{B}^T G_k \hat{A} + G_k^{1,1} P_{b_0,\theta}), && (3.43)\\
L_{k+1} &= (R + \hat{B}^T G_k \hat{B} + G_k^{1,1}\sigma_{b_0}^2)^{-1}(\hat{B}^T \gamma_k + P_{b_0,x} \hat{A}^T G_k^{\cdot,1}).
\end{aligned}$$

We showed in Section 3.2 that under some very general conditions is the equation (2.35) from Definition 2.11 either convergent (Theorem 3.15) or uniformly divergent (Theorems 3.21 and 3.22). Thus we have either

$$G_k \xrightarrow{k\to\infty} G, \tag{3.44}$$

where G is a finite, symmetrical, positive semidefinite matrix, or

$$\frac{G_k}{G_k^{1,1}} \xrightarrow{k\to\infty} G_\varepsilon, \quad G_k^{1,1} \xrightarrow{k\to\infty} \infty. \tag{3.45}$$

We will assume in the further text that one of the conditions (3.44) and (3.45) holds. Both conditions directly imply the convergence of the feedback gain K_k, i.e.

$$K_k \xrightarrow{k\to\infty} K.$$

This is straightforward in the convergent case and in the divergent case it is shown in equation (3.40). The main goal of this section is to show the limit behavior of γ_k^T.

Note 3.23. The matrix G_ε is defined in Theorem 3.22 together with $0 < \varepsilon < 1$. We will extend the definition also for $\varepsilon = 1$, in which case $G_\varepsilon = G_*$ from Theorem 3.21. Considering $0 < \varepsilon \leq 1$ will thus also include the case described in Theorem 3.21.

Theorem 3.24. *Assume that the sequence G_k is given by the cautious discrete-time Riccati-like equation* (3.15) *and that the pair (A, B) is stabilizable. Then*

(i) If the condition (3.44) *is satisfied, then*

$$\gamma_k^T \xrightarrow{k\to\infty} \gamma^T < \infty, \tag{3.46}$$

where γ^T is a solution of a linear equation

$$\gamma^T(I - A + BK) = G^{1,\cdot}A(P_{x,\theta} - P_{x,b_0}K). \tag{3.47}$$

(ii) If the condition (3.45) *is satisfied, then*

$$\frac{\gamma_k^T}{G_k^{1,1}} \xrightarrow{k\to\infty} \gamma_\varepsilon^T < \infty, \tag{3.48}$$

where γ_ε^T is a solution of a linear equation

$$\xi^T\left(I - \varepsilon^2(A - BK)\right) = G_\varepsilon^{1,\cdot}\varepsilon^2 A(P_{x,\theta} - P_{x,b_0}K) \tag{3.49}$$

in variable ξ^T.

Proof. (i) The development of γ_k^T is described by the equation (3.41), which is an equation of a discrete-time linear time-variant system with system matrix $A - BK_{k+1}$ and input $G_k^{1,\cdot}A(P_{x,\theta} - P_{x,b_0}K_{k+1})$. Because $K_k \to K$, there is such $k_0 > 0$ that the system matrix is stable for all $k > k_0$. The vector γ_k^T thus converges to the system equilibrium given by the equation (3.47). Such solution is unique, because the matrix $I - A + BK$ is regular, if $A - BK$ is stable.

(ii) Let us divide both sides of the equation (3.41) by $G_k^{1,1}$ and multiply them by ε^2 to obtain

$$\frac{\gamma_{k+1}^T}{G_{k+1}^{1,1}}\frac{G_{k+1}^{1,1}}{G_k^{1,1}}\varepsilon^2 = \frac{G_k^{1,\cdot}}{G_k^{1,1}}\varepsilon^2 A(P_{x,\theta} - P_{x,b_0}K_{k+1}) + \frac{\gamma_k^T}{G_k^{1,1}}\varepsilon^2(A - BK_{k+1}).$$

The factor ε is defined in Theorem 3.22 and is uniquely given by the cautious Riccati-like equation producing the sequence G_k. We can see this modified equation as a linear system in variable $\frac{\gamma_k^T}{G_k^{1,1}}$. Such system is stable, because matrix $\varepsilon(A - BK)$ is guaranteed to be stable and $0 < \varepsilon \leq 1$ (the case $\varepsilon = 1$ is described in Note 3.23). Therefore the system will converge to the stable equilibrium. According to the considerations in Theorem 3.22, it holds that

$$\frac{G_{k+1}^{1,1}}{G_k^{1,1}}\varepsilon^2 \xrightarrow{k\to\infty} 1.$$

Taking the limit $k \to \infty$ on both sides of the equation and defining

$$\xi^T = \lim_{k\to\infty} \frac{\gamma_k^T}{G_k^{1,1}}$$

yields the equation (3.49). The solution is unique for the same reason as in the proof of the first part.

□

The previous theorem says that the vector γ_k^T converges if and only if G_k is convergent. The possible divergence is 'uniform' in the sense of Theorem 3.24. The theorem also makes it possible to derive the convergence of L_k by considerations similar to those in (3.40).

The convergence of G_k thus implies convergence of γ_k^T and also the linear growth of term g_k, as can be seen from equation (3.42). This justifies the use of theoretically correct division by N in the cost function. On the other hand, divergence of G_k implies divergence of γ_k and g_k and all these divergences are exponential. The optimal control law, however, remains finite, because it is given by the state feedback K_k and the absolute term L_k that have both been shown to converge although G_k diverges.

Chapter 4

Single-step active adaptive control

In this chapter, we will derive a single-step ahead active adaptive controller based on modification of cautious control. Cautious control was introduced as a concept in Chapter 1 and a cautious controller for ARMAX model was presented in Chapter 2. The single-step active adaptive controller assumes, that the control input may influence the knowledge about system parameters in the future, thus reducing the uncertainty and improving the overall control performance. Such controller is an example of an approximate dual controller, because it minimizes the control criterion, taking into account the impact on uncertainty.

First a general idea of modification of the cautious LQ controller to a single-step active adaptive controller will be shown. Then we will present application of this scheme to an ARX system with some simulations and remarks. The purpose of this chapter is to introduce the idea of active control on an example and at the same time point out some difficulties that arise when cautious control is used. An extension of the idea of the presented single-step controller to multiple-step is also proposed. This extension is however not studied in this thesis, because of problems with cautious control convergence.

4.1 Controller based on cautious strategy

The drawback of the cautious control strategy is the fact that the parameter uncertainty is assumed to remain constant during the control process. Such model of parameter uncertainty is easy to deal with but it is also unrealistic. In fact, the parameter vector is an unknown but constant vector that can be estimated using statistical methods. As usual, we will denote $\hat{(\cdot)}_k$ the (parameter) conditional mean at time k and $P_{(\cdot),k}$ the (parameter) conditional variance matrix at time k. The variance matrix is indexed by time k, because it is assumed that it may change over time. The $(\cdot)$ notation is used here, where the dot represents the concrete parameters being estimated, such as θ or b_0.

It is possible to use the parameter estimate for computing the cautious control strategy in each step, but such approach expects the uncertainty to remain the same during the whole process. In reality, we can expect some improvement of the knowledge about the system, expressed by the change of the variance matrix $P_{(\cdot),k}$. This approach leads to a strategy that actively improves the knowledge about the system.

In the case of cautious control of ARMAX model, the optimal criterion value at time k is a function of the state x_k given by (2.15). The matrices G_k, γ_k^T and g_k of the quadratic form are given by (2.32), (2.33) and (2.34), respectively. The resulting matrices G_k, γ_k^T and g_k depend on the variances of uncertain parameters and so does the optimal criterion value $J_k^*(x_k)$. In the case of cautious controller, we assumed the variances to be constant for all $k = 1, \ldots, N$. However, it is possible to take into account the influence of the input u_k on variance matrices $P_{(\cdot),k+1}$ at time $k+1$, i.e. $P_{(\cdot),k+1} = P_{(\cdot),k+1}(u_k)$. Thus we have to consider the matrices G_{k+1}, γ_k^T and g_k to be functions of u_k, i.e. $G_{k+1} = G_{k+1}(u_k)$ etc. Consequently the criterion J_{k+1}^* depends on u_k not only via the state prediction, but also via matrices $G_{k+1}(u_k)$ etc. Therefore we will write $J_{k+1}^*(\mathcal{D}^{k-1}, u_k, y_k)$ to emphasize the dependence. If we assume this influence only one step ahead and the rest of the criterion (from time $k+1$ to N) is estimated by the cautious control criterion, the Bellman equation (2.17) still holds, but it will now take the form

$$J_k^*(\mathcal{D}^{k-1}) = \min_{u_k} \mathsf{E}\left[x_k^T Q x_k + u_k^T R u_k + J_{k+1}^*(\mathcal{D}^{k-1}, u_k, y_k) | u_k, \mathcal{D}^{k-1}\right]. \tag{4.1}$$

So in the single-step active control strategy, we assume that the variance matrix of the parameter estimate is changed only after the current step of control and that cautious strategy based on the last estimate will be applied on the rest of the horizon. Under this assumption, it is possible to evaluate the criterion value $J_{k+1}^*(\mathcal{D}^{k-1}, u_k, y_k)$, provided we know, how the input u_k changes the covariance $P_{(\cdot),k}$ in the next step, i.e. the transition from $P_{(\cdot),k}$ to $P_{(\cdot),k+1}$ is expressed as a function of u_k. This influence is known for the ARMAX model of a stochastic system, where the parameters can be estimated using the Kalman filter (2.2). The minimization does not lead to a closed form functions of the state x_k, as in the case of cautious controller and must be performed numerically. The minimized expression is a continuous function of u_k, so standard optimization methods can be used for minimizing the criterion. The function need not be convex and therefore only a local optimum is expected to be found. Because the optimal input for cautious control strategy is known, the optimization can be started at this point. This guarantees that the locally optimal solution of (4.1) is not worse than the cautious control.

It is important to remind that this approach is based on finite horizon optimization and therefore its disadvantage is that the criterion matrix G_k must be computed iteratively and so must be the optimization for finding u_k^* in each step. It would seem reasonable to extend the problem to infinite horizon and find a limit solution as a solution of an algebraic cautious Riccati-like equation. However, this approach may lead to problems because the limit solution of the equation is not guaranteed to exist. This issue was thoroughly studied in Chapter 3.

Generalization of this method to multiple-step can be done by using the matrix

G_{k+M} to estimate the criterion at time $k+M$, where M is the number of steps in which the decrease of variance is considered. The task is then to find the minimizers $u_k, \ldots, u_{k+M-1}$ for the nonconvex multidimensional optimization problem

$$\begin{aligned} J_k^*(\mathcal{D}^{k-1}) &= \min_{u_k,\ldots,u_{k+M-1}} \mathsf{E}\left[\sum_{i=k}^{k+M-1} \left(x_i^T Q x_i + u_i^T R u_i\right) + \right. \\ &+ \left. J_{k+M}^*(\mathcal{D}^{k+M-1}; u_k, \ldots, u_{k+M-1}) | \mathcal{D}^{k-1}, u_k, \ldots, u_{k+M-1}\right]. \end{aligned} \tag{4.2}$$

This is a partially predictive formulation, because the expression on the right-hand side is conditioned by measurements until $k-1$, therefore all states x_{k+1} to x_{k+M} must be predicted. The minimum must be found numerically, taking the cautious optimal control as a starting point. Note that also the cautious optimal control is predicted, as it is based on predictions $\mathsf{E}[x_{k+i}|\mathcal{D}^{k-1}, u_k, \ldots, u_{k+i-1}]$ for $i = 0, \ldots, M$.

4.2 Simulations

In this section we present simulation experiments performed for the discrete integrator with unknown gain on the input

$$y_k = y_{k-1} + b u_k + e_k. \tag{4.3}$$

The following values are assumed

$$\widehat{b} = 10, \quad \sigma_e^2 = 1, \quad \sigma_b^2 \in [1, 10^5]. \tag{4.4}$$

The further settings are used: the number of steps for computing the criterion (the control horizon) $N = 10$, the weighting factor $R = 1$ and the initial condition $y(0) = 10$.

Figure 4.1 shows the optimal criterion values computed for the system (4.3) on the horizon of $N = 10$ steps. It compares the values of criterion for cautious and active control strategy. It can be seen that the highest difference is around the variance $\sigma_b^2 = 4 \cdot 10^3$ and that for high uncertainty, the benefit of using active strategy disappears.

This can be explained by analyzing the shape of the curve describing the cautious strategy. The value of criterion for the active strategy is composed of the cost of the first step and the cost of the cautious strategy over steps $2, \ldots, N$. In step 2, uncertainty is decreased to a value given by the Kalman filter (2.8). The benefit of using the active strategy is thus given mainly by the difference of the cautious strategy criterion value at points given by uncertainty in step 1 and 2. This difference is greatest in the area where the curve is rising. On the other hand, for high uncertainties, the curve remains almost constant and the benefit is negligible.

For the next experiment, both control strategies (cautious and active) are applied on the system (4.3) in the following way. In each step of control k, the optimal input sequence over the whole horizon $N = 10$ is computed. Then only the first input u_k^* is applied and after the real output y_k is measured, the knowledge about parameter b is updated in terms

of expressions (2.9) and (2.8). In the next step $k+1$ the new control sequence is computed based on this improved knowledge and again, only the first input u^*_{k+1} is applied. This makes it possible to see that the active strategy takes the identification process into account. The control process is simulated for $K = 10$ steps of control.

Figures 4.2 and 4.5 show an example of the control process with $b = 25$. The graph shows that the active strategy starts with a greater (absolute) value of control signal than the cautious one. This leads to faster parameter identification as well as faster decrease of the uncertainty expressed by σ_b^2, as can be seen in Figure 4.3 and Figure 4.6. This shows how the controller looks for an optimal tradeoff between identification and control in the first step.

Figure 4.4 shows the dependence of the real (measured) criterion value on the real gain b. The values of parameter b are chosen in the interval $[-80, 100]$. Other settings are chosen as in (4.4), with $\sigma_b^2 = 10^3$. To reduce the influence of the noise e on the result, the control process is simulated 10 times for each parameter value from interval $[-80, 100]$ and the average of the criterion is taken.

It can be seen that the active strategy brings better results if the initial estimate is not far from the real parameter value. This can be explained by the fact that the first control is more aggressive in order to excite the system and enable faster identification. If such input is applied to a system that is far from the estimate assumed by the controller, the unexpectedly high output in the first step followed by a necessarily high input in the second step may cause a rise of the criterion. This is shown in Figure 4.7 and Figure 4.8, where the control process is simulated for a real parameter value $b = -25$.

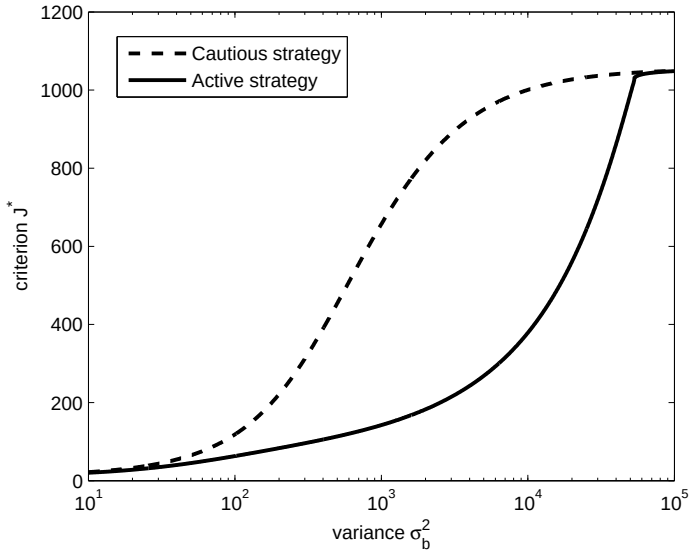


Figure 4.1: The dependence of the optimal criterion value J^* on the parameter variance σ_b^2 for cautious and active control strategy.

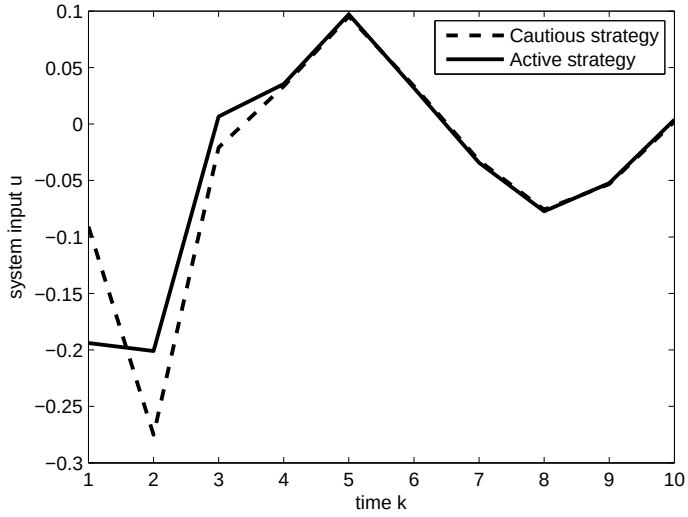


Figure 4.2: Input to a controlled process with $b = 25$, while $\widehat{b} = 10$ and $\sigma_b^2 = 10^3$ is assumed.

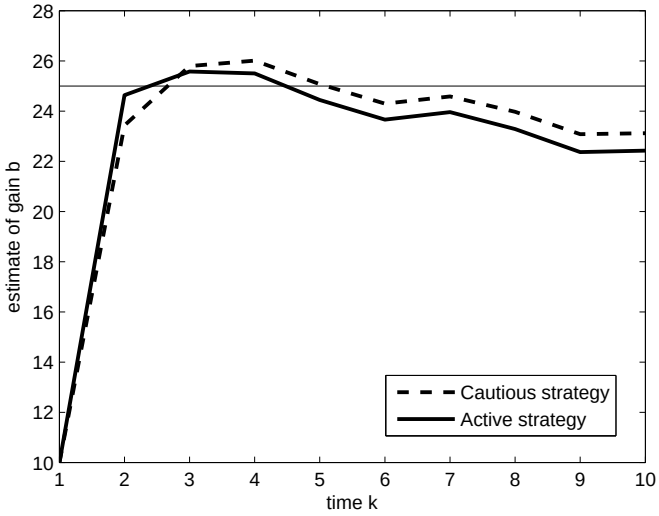


Figure 4.3: The estimate of the system gain for $b = 25$, while $\widehat{b} = 10$ and $\sigma_b^2 = 10^3$ is assumed.

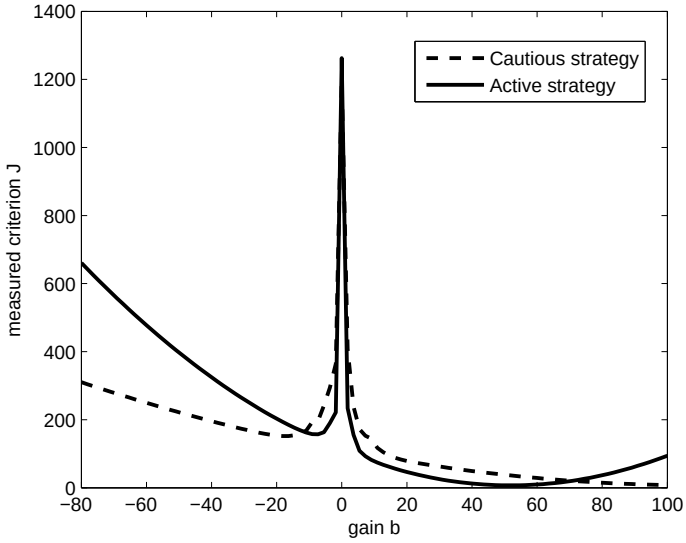


Figure 4.4: Measured criterion values according to the real value of gain b, while $\widehat{b} = 10$ and $\sigma_b^2 = 10^3$ is assumed.

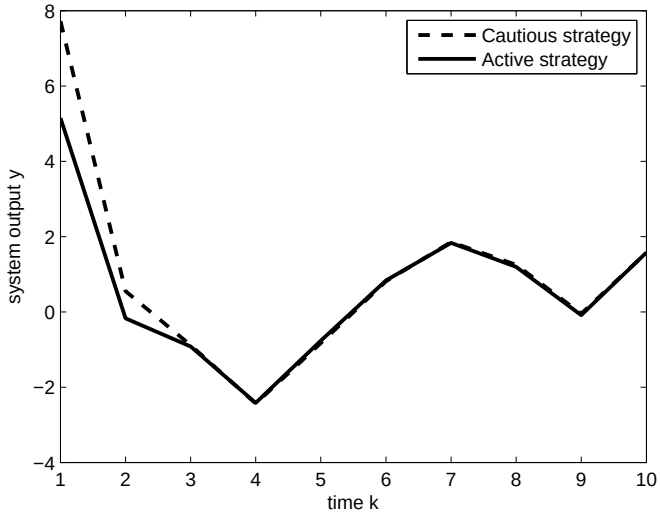


Figure 4.5: Output of a controlled process with $b = 25$, while $\widehat{b} = 10$ and $\sigma_b^2 = 10^3$ is assumed.

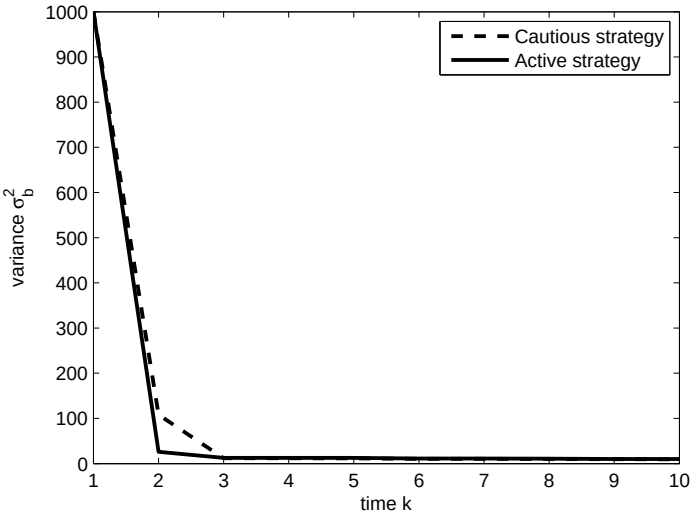


Figure 4.6: The variance σ_b^2 of the estimate of the gain b with $b = 25$, while $\widehat{b} = 10$ and $\sigma_b^2 = 10^3$ is assumed.

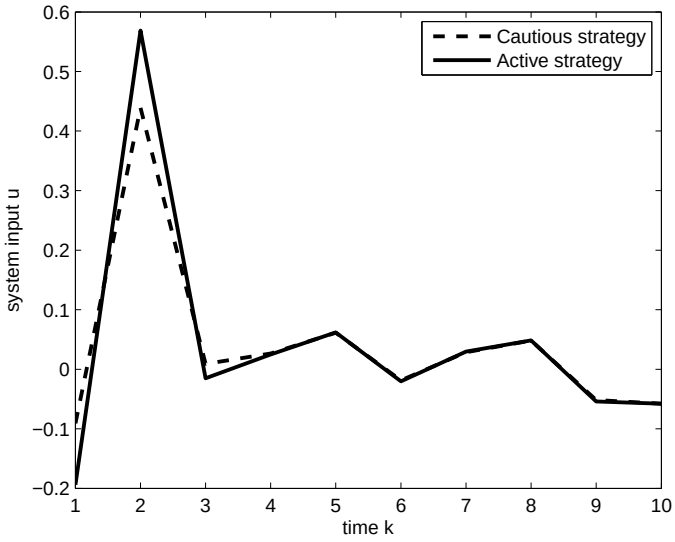


Figure 4.7: Input of a controlled process with $b = -25$, while $\widehat{b} = 10$ and $\sigma_b^2 = 10^3$ is assumed.

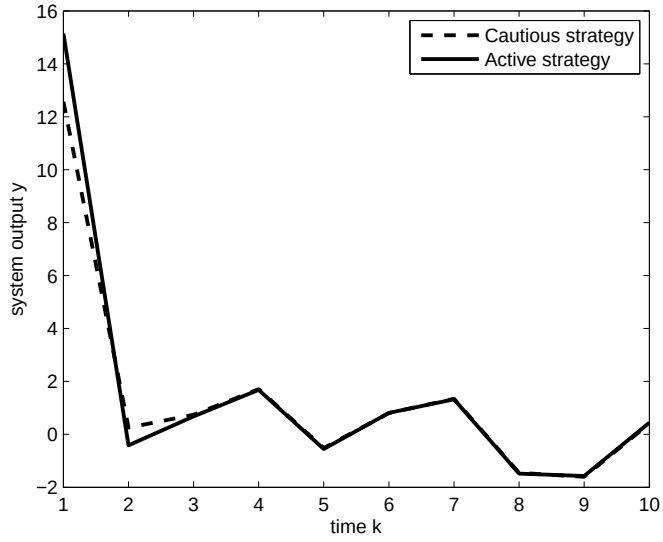


Figure 4.8: Output of a controlled process with $b = -25$, while $\widehat{b} = 10$ and $\sigma_b^2 = 10^3$ is assumed.

Chapter 5

Multiple-step active adaptive control

This chapter presents the multiple-step active adaptive strategy based on maximizing the lowest eigenvalue of the predicted information matrix, by which the persistent excitation of the system is reached. Persistent excitation is a sufficient condition for the least squares identification algorithm to converge. The chapter starts with analyzing the benefit of the multiple-step approach and shows it on a simple example. The formal definition of the approach is given next as a nonconvex optimization problem, continuing with three concrete approximate solutions of the original task. Simulations are provided at the end of the chapter.

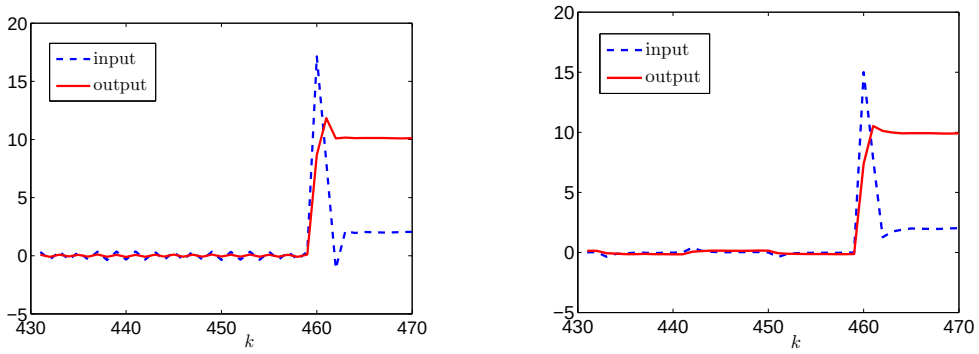
5.1 Benefit of the multiple step approach

Most methods used to approximate the dual control strategy use just a single step ahead looking approach. In this section we will show that it is important to look more steps ahead when designing an excitation signal. The problem with single-step ahead design is illustrated on a simple first order example controlled by a cautious modification of the minimum variance controller.

Let us assume the simple ARX system in (1.19) and the cautious minimum variance controller (1.22). The single-step bicriterial active approach in [19] suggests that the control input is augmented by a predefined value Δu so that the absolute value of the input is increased, i.e.

$$u_k^a = u_k^c + \text{sign}(u_k^c)\Delta u, \tag{5.1}$$

where u_k^c is the optimal control from (1.22). The signum function is used to indicate the sign of the optimal control u_k^c and thus decide, whether Δu should be added or subtracted, in order to increase the absolute value of the sum u_k^a. The reason is that, while the optimality of the solution is violated in a similar way no matter what the sign of the perturbation



(a) Single-step control.

(b) Two-step control.

Figure 5.1: An example of the single-step and two-step control of an uncertain first order ARX system. A step in the reference signal occurs at time $k = 460$.

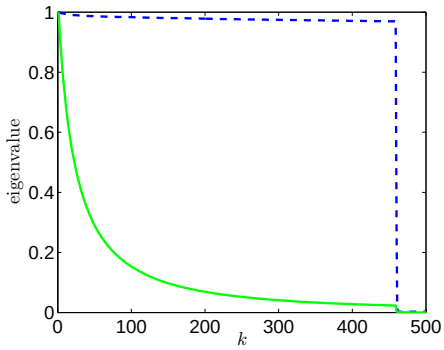
is, the absolute value of the input is on the other hand crucial for efficient identification. For more detail see [19]. The two-step approach, on the other hand, calculates the biggest information gain, measured by the lowest eigenvalue of the information matrix, after two steps of control, when both steps may be altered by at most $\pm\Delta u$. This leads to a different excitation strategy as shown later.

Simulation of both different strategies were done for a simple system

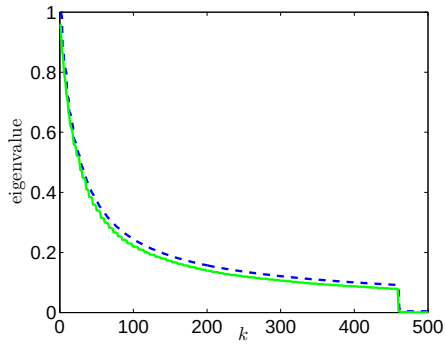
$$y_k = 0.9y_{k-1} + 0.5u_k + e_k, \tag{5.2}$$

with noise variance $\sigma_e^2 = 10^{-4}$. Initial parameter estimates were chosen $\hat{a} = 0.1$, $\hat{b} = 0.8$ with covariances of elements $\sigma_b^2 = \sigma_a^2 = 1$ and $\sigma_{ab} = 0$. Figures 5.1(a) and 5.1(b) show the response to a reference signal with a step from 0 to 10 occurring at time $k = 460$. Figure 5.1(a) shows the simulation when the single-step ahead active control is used, while Figure 5.1(b) shows the same situation for the two-step ahead active control. Figures 5.2(a) and 5.2(b) show the eigenvalues of the parameter estimate variance matrix when the single-step and two-step ahead active control is used, respectively. Finally, Figures 5.3(a) and 5.3(b) show the parameter estimates when the single-step and two-step ahead active control is used, respectively.

The parameter eigenvalues in Figures 5.2(a) and 5.2(b) show that in the case of the single-step controller, the parameter space is not excited uniformly and there is a direction (vector) in which the uncertainty is only little improved. In this case parameter estimates in Figure 5.3(a) do not converge to the true values, which is caused by the shape of the input, that is not persistently exciting, as shown in the detailed Figure 5.4. The responses in Figures 5.1(a) and 5.1(b) show that after the step change of the reference has occurred, the single-step ahead controller has a worse performance in terms of the overshoot.

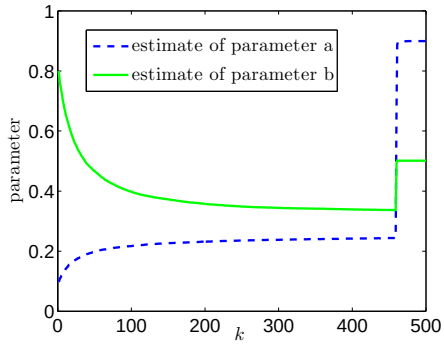


(a) Single-step control.

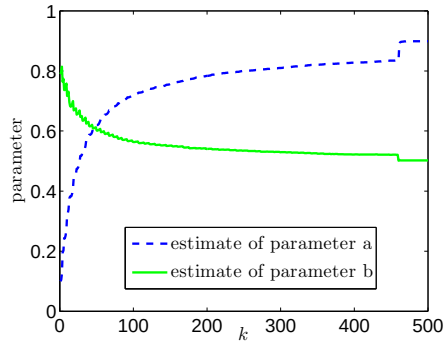


(b) Two-step control.

Figure 5.2: Eigenvalues of the parameter estimate variance matrix when the single-step and two-step control is applied. For the single-step control, one eigenvalue is decreased rapidly, while the other remains almost unchanged, indicating that there is a direction in the parameter space, in which only little information is gathered. For the two-step control, both eigenvalues are decreased uniformly. The change is slower than the fastest change in the single-step case, but information is gathered in all directions.



(a) Single-step control.



(b) Two-step control.

Figure 5.3: Results of the parameter identification process for the single-step and two-step controller. For the single-step control, the parameters do not converge to the actual values, as the persistent excitation condition is not satisfied. For the two-step control, the persistent excitation condition is satisfied, so the parameters converge to their actual values.

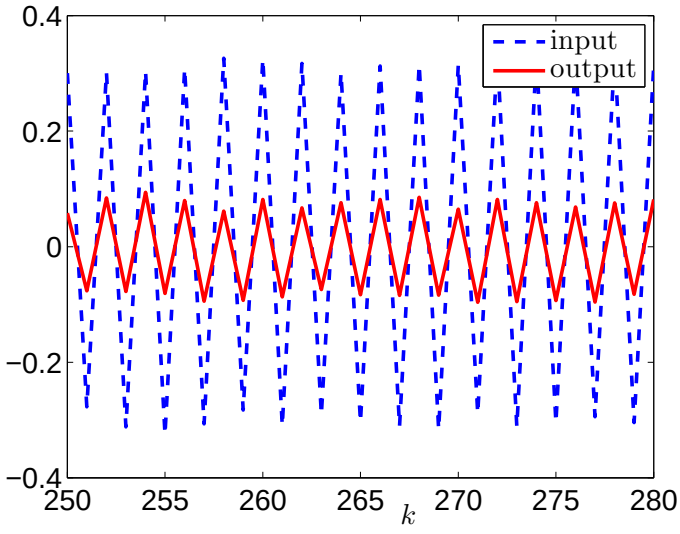


Figure 5.4: A detail of Figure 5.1(a) between the times $k_1 = 250$ and $k_2 = 280$.

A detailed look at the excitation signal from Figures 5.1(a) and 5.1(b) is in Figures 5.4, 5.5 and 5.6, where we can see a significant difference. The single-step controller is getting as much information as possible about the gain by changing signs rapidly. The two-step controller, on the other hand, makes a fast change to improve the knowledge about the gain and then keeps unchanged for several steps, to identify also the dynamics. Figure 5.5 shows the excitation between the time $k_1 = 50$ and $k_2 = 100$, where the parameter estimate is still quite far from the truth. That is why the system output stays in negative values most of the time. The situation between $k_1 = 250$ and $k_2 = 300$ is shown in Figure 5.6, where the system output is already equally distributed around the zero value.

The single-step effect when information is not gathered about all parameters uniformly is even more significant in case of higher order systems. It is possible to tell in advance, which parameters need to be identified well and which do not have much influence on the quality of control. Such approach based on the criterion sensitivity to parameters is used for example in [32]. However, the algorithms proposed in this thesis aim to excite the system uniformly and thus to minimize the risk of any parameter being poorly identified.

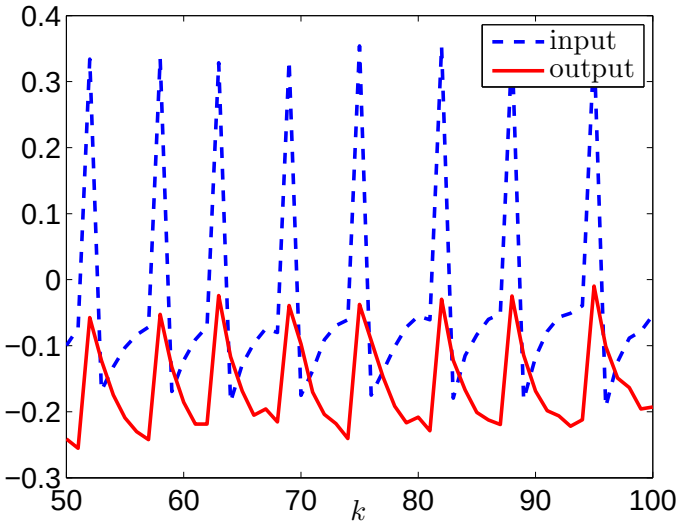


Figure 5.5: A detail of Figure 5.1(b) between the times $k_1 = 50$ and $k_2 = 100$.

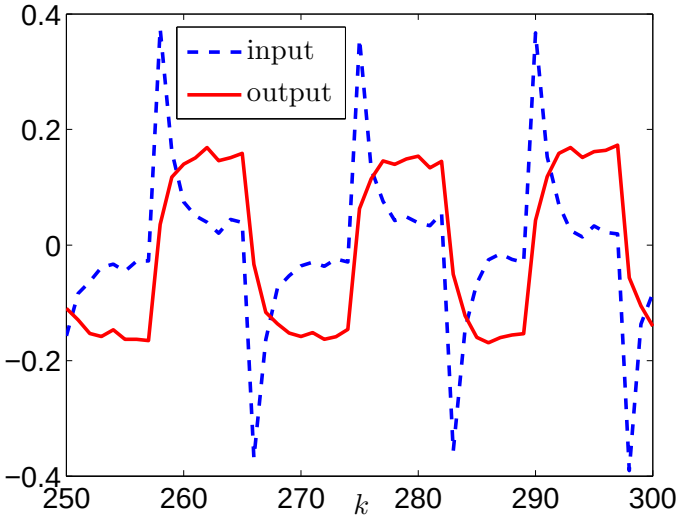


Figure 5.6: A detail of Figure 5.1(b) between the times $k_1 = 250$ and $k_2 = 300$.

5.2 Problem formulation and definitions

The presented algorithms are derived using the state-space descriptions of a SISO linear stochastic discrete-time system [3], in an innovation form

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + \Gamma e_k, \\ y_k &= Cx_k + Du_k + e_k, \end{aligned} \tag{5.3}$$

where A, B, C, D and Γ are system matrices of proper dimensions, u_k, y_k and x_k are the system input, output and state, respectively, and e_k is a gaussian white noise sequence with a zero mean and constant finite variance. Let us next consider that the system matrices depend on some vector of parameters θ that is uncertain. Uncertain is used here to express that it is unknown, possibly described by a probability distribution, but constant or slowly changing in time. Because the persistent excitation condition (1.18) is easily defined for ARX systems, we will consider that the system of interest is in an ARX form

$$y_k = -\sum_{i=1}^{n} a_i y_{k-i} + \sum_{i=0}^{n} b_i u_{k-i} + e_k = z_k^T \theta + e_k, \tag{5.4}$$

where z_k is the system regressor at time k defined for a SISO ARX system as $z_k = [u_k, -y_{k-1}, u_{k-1}, \ldots, -y_{k-n}, u_{k-n}]^T$, where n is the system order. The uncertain parameters θ are now simply the parameters of the model, i.e. a_i and b_i from (5.4). Note that unlike in Chapter 2, θ includes the parameter b_0 for convenience of notation.

> **Note 5.1.** The persistent excitation condition for ARX model can be also used for ARMAX model with known c-parameters, that was used in Chapter 2. A variance matrix update for the estimator is given in equation (2.2). This matrix cannot be easily inverted to obtain the information matrix update as it is in the case of ARX model. However, the only interesting part of the variance (or in fact information) matrix is the part corresponding to parameters a_i and b_i, because the accuracy of the estimate of the past noises e_{k-i} is limited. Maximizing the information matrix of an ARX system is thus a reasonable approach also for ARMAX models.

We will work with a nonminimal state-space representation of this model

$$\begin{aligned} A &= \begin{bmatrix} a_1 & b_1 & \cdots & b_{n-1} & a_n & b_n \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} b_0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \\ C &= \begin{bmatrix} a_1 & b_1 & \cdots & b_{n-1} & a_n & b_n \end{bmatrix}, \quad D = \begin{bmatrix} b_0 \end{bmatrix}, \\ \Gamma &= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}^T. \end{aligned} \tag{5.5}$$

In a minimal representation, each state is formed by a linear combination of the input and output at time k. The representation (5.5) resembles the controllable canonical form built

of 2x2 matrix blocks, but is in fact uncontrollable. This corresponds to the fact, that the form (5.5) is nonminimal and each block can thus be replaced by one state of the minimal representation. However, the system is stabilizable with n uncontrollable eigenvalues that are equal to zero, which enables its use for control design. The state vector in representation (5.5) is formed by previous inputs and outputs, $x_k = [-y_{k-1}, u_{k-1}, \ldots, -y_{k-n}, u_{k-n}]^T$ and is directly measurable.

The uncertainty is described in a probabilistic way, i.e. the parameter vector is described by the conditional mean and variance matrix at time k

$$\hat{\theta}_k = \mathsf{E}\left[\theta|\mathcal{D}^{k-1}\right], \quad P_{\theta,k} = \mathsf{var}\left[\theta|\mathcal{D}^{k-1}\right], \tag{5.6}$$

where the symbol $\mathcal{D}^{k-1}$ is used to denote available data up to time $k-1$ as defined in (1.5) and the parameters are considered to be stochastically independent on the noise e_k. As the controller design is formulated for a fixed time k, we will consider $k = 0$ for simplicity of notation, and thus omit the time indexation in (5.6), defining $\hat{\theta} = \hat{\theta}_0$ and $P_\theta = P_{\theta,0}$.

It was said in the introductory section that the algorithm is based on a bicriterial approach, i.e. in the first step an initial control sequence $U_c^* = [u_1^c, u_2^c, \ldots, u_N^c]^T$ is found as a result of any existing control algorithm and in the second step, a control sequence $U^* = [u_1^*, u_2^*, \ldots, u_N^*]^T$ maximizing the lowest eigenvalue of the information matrix is found in a feasible set around the initial sequence U_c^*. Here, N is the control horizon. The feasibility set is defined as some neighborhood of U_c^* in which the altered control does not change the overall performance significantly.

The presented algorithm uses MPC control for the initial solution U_c^*. In such case, it is usual to define the performance criterion as an expected value of the quadratic cost

$$J^{mpc} = \mathsf{E}\left[\sum_{k=1}^{N}\left(\rho u_k^2 + y_k^2\right) \mid \mathcal{D}^0, u_1, \ldots, u_N\right] = \sum_{k=1}^{N}\left(\rho u_k^2 + \mathsf{E}\left[y_k^2 \mid \mathcal{D}^0, u_1, \ldots, u_k\right]\right), \tag{5.7}$$

where ρ is a positive real weighting constant and the expectation is taken with respect both to the parameter uncertainty and the noise e_k. The MPC algorithm searches the N dimensional space of input sequences to find the optimal one, so in this formulation all inputs u_k are deterministic variables, which justifies the second equality in (5.7).

The expectations of outputs in (5.7) are difficult to calculate as the formula for open loop output prediction contains multiplications of parameters θ and thus also higher-order moments of parameter distribution. Therefore, as explained in Section 1.4, we will not consider parameter uncertainty in the controller design, but rather use the certainty equivalence (CE) approach, i.e. parameter mean values will be used as if they were the actual ones. The predictive c.p.d.f. is

$$\begin{aligned} p(y_k \mid \mathcal{D}^0, u_1, \ldots, u_k) &= \int p(y_k \mid \mathcal{D}^0, u_1, \ldots, u_k, \theta) p(\theta \mid \mathcal{D}^0, u_1, \ldots, u_k)\, \mathrm{d}\,\theta \\ &= p(y_k \mid \mathcal{D}^0, u_1, \ldots, u_k, \hat{\theta}), \end{aligned}$$

where the following CE assumption was used

$$p(\theta \mid \mathcal{D}^0, u_1, \ldots, u_k) = \delta(\theta - \hat{\theta}).$$

The CE criterion will be defined as

$$J^{ce} = \sum_{k=1}^{N} \left(\rho u_k^2 + \mathsf{E}\left[y_k^2 \mid \mathcal{D}^0, u_1, \ldots, u_k, \hat{\theta} \right] \right), \tag{5.8}$$

which makes the open loop predictions simple. The notation $\hat{x}$ will be used for the conditional expectation $\mathsf{E}[x|\theta = \hat{\theta}]$, i.e. for objects that generally depend on θ, in which θ is substituted by $\hat{\theta}$, e.g. the matrix $\hat{A} = A(\hat{\theta})$.

It can be shown that the problem (5.8) is equivalent to the MPC problem for a deterministic system ($e_k = 0$), with the only difference in the criterion value, which for a stochastic system has a linearly increasing additive term J^s that does not depend on control, i.e.

$$J^{ce} = \sum_{k=1}^{N} \left(\rho u_k^2 + \hat{y}_k^2 \right) + J^s. \tag{5.9}$$

The optimal control U_c^* minimizing the cost (5.8) is thus equivalent to the optimal control minimizing the cost

$$J = \sum_{k=1}^{N} \left(\rho u_k^2 + \hat{y}_k^2 \right). \tag{5.10}$$

The problem can now be formulated as the following minimization problem for a SISO system

$$\begin{aligned} U_c^* \quad &= \quad \arg\min_{U_1^N} J = \arg\min_{U_1^N} \sum_{k=1}^{N} \left(\rho u_k^2 + \hat{y}_k^2 \right), \\ \text{s. t.} \quad & \hat{x}_{k+1} = \hat{A}\hat{x}_k + \hat{B}u_k, \\ & \hat{y}_k = \hat{C}\hat{x}_k + \hat{D}u_k, \\ & |u_k| \leq u_{max}, \\ & \hat{x}_1 = \xi, \end{aligned} \tag{5.11}$$

where u_{max} is a hard constraint on inputs, ξ is an initial condition for the problem and $U_1^N = [u_1, \ldots, u_N]^T$ is an input sequence. The initial control U_c^* is thus a result of a quadratic optimization problem.

For the case of a variety of recursive identification algorithms (e.g. for ARX identification by recursive least squares) the information matrix yields the formula [2]

$$P_M^{-1} - P_0^{-1} = \sum_{k=1}^{M} \left(z_k z_k^T \right). \tag{5.12}$$

As the future regressors are unknown at time $t = 0$, the future value of the information matrix cannot be computed exactly and for a given control sequence U_1^N with $N \geq M$ it will be estimated as

$$\hat{P}_M^{-1} = P_0^{-1} + \sum_{k=1}^{M} \left(\hat{z}_k \hat{z}_k^T \right), \tag{5.13}$$

where $\hat{z}_k$ is the system regressor, where unknown future outputs y_k for $k > t$ are substituted by their predictions $\hat{y}_k$. An exception to this is the finite impulse response (FIR) model, for which $a_i = 0$ and the regressor consists of inputs only. Note that the prediction horizon N in (5.11) is different from (greater than) M which expresses the number of steps after which the matrix $\hat{P}_M^{-1}$ is evaluated and which will be called the excitation horizon. That is why in the second step, only M first inputs are involved in optimization and the following $N - M$ steps remain unchanged.

Let us now introduce some useful notation. It holds that $\sum_{k=1}^{M} \left(\hat{z}_k \hat{z}_k^T\right) = \hat{Z}_M^T \hat{Z}_M$, where $\hat{Z}_M = [\hat{z}_1, \ldots, \hat{z}_M]^T$. The transposed predicted regressors $\hat{z}_k, k = 1, \ldots, M$ form the rows of the matrix Z_M and can be expressed as a linear function of the initial state x_1 and an input vector $U_1^M = [u_1, \ldots, u_M]^T$ as

$$\hat{z}_k = F_k \begin{bmatrix} x_1 \\ U_1^M \end{bmatrix}, \quad k = 1, \ldots, M. \tag{5.14}$$

The predicted information matrix $\hat{P}_M^{-1}$ is thus a function of the system matrices $\hat{A}, \hat{B}, \hat{C}$ and $\hat{D}$, the vector x_1 and the input vector U_1^M. As the only variable here is the input vector U_1^M, we will denote $\hat{P}_M^{-1} : \mathbb{R}^M \to \mathbb{R}^{(2n+1)\times(2n+1)}$ the matrix-valued function

$$\hat{P}_M^{-1}(U_1^M) = P_0^{-1} + \sum_{k=1}^{M} \left\{ F_k \begin{bmatrix} x_1 \\ U_1^M \end{bmatrix} \begin{bmatrix} x_1^T & (U_1^M)^T \end{bmatrix} F_k^T \right\}, \tag{5.15}$$

the value of which is a symmetric, positive semidefinite matrix, that can only be positive definite, if $M \geq 2n + 1$, i.e. greater than the length of the regressor. Similarly, the columns of Z_M are formed by shifted inputs and outputs, particularly $[u_1, \ldots, u_M]^T$ to $[u_{-n+1}, \ldots, u_{M-n}]^T$ and $[y_0, \ldots, \hat{y}_{M-1}]^T$ to $[y_{-n+1}, \ldots, \hat{y}_{M-n}]^T$. Let us denote the k-th column of Z_M as $w_k, k = 1, \ldots, 2n + 1$. Then w_k can be expressed by

$$w_k = K_k \begin{bmatrix} x_1 \\ U_1^M \end{bmatrix}, \quad k = 1, \ldots, 2n + 1, \tag{5.16}$$

where K_k is a matrix of appropriate dimensions. The vector of output predictions $\hat{Y}_1^N = [\hat{y}_1, \ldots, \hat{y}_N]^T$ can be expressed as

$$\hat{Y} = G \begin{bmatrix} x_1 \\ U_1^N \end{bmatrix} = G \begin{bmatrix} x_1 \\ U_1^M \\ U_{M+1}^N \end{bmatrix}, \tag{5.17}$$

where G is a matrix of corresponding dimensions, x_1 is the initial state of the system and U_{M+1}^N is the part of the input sequence that is not changed in the second step, i.e. $U_{M+1}^N = [u_{M+1}, \ldots, u_N]$. Using this notation (5.17), the MPC criterion in (5.11) is expressed as

$$J = \begin{bmatrix} x_1^T & (U_1^M)^T & (U_{M+1}^N)^T \end{bmatrix} H \begin{bmatrix} x_1 \\ U_1^M \\ U_{M+1}^N \end{bmatrix}, \tag{5.18}$$

where

$$H = G^T G + \begin{bmatrix} 0 & 0 \\ 0 & \rho I \end{bmatrix}.$$

As mentioned above, the excitation horizon M is generally shorter than the control horizon N so the final control U^* differs from the MPC control only in the first M steps. In fact, N should be significantly greater than M so that the control criterion can take into account the future impact of identification procedure on the control quality. Let us decompose the input sequences by the following notation

$$U^* = \begin{bmatrix} (U^*)_1^M \\ (U^*)_{M+1}^N \end{bmatrix} \qquad U_c^* = \begin{bmatrix} (U_c^*)_1^M \\ (U_c^*)_{M+1}^N \end{bmatrix}. \tag{5.19}$$

It is now possible to formulate the problem as

$$\begin{aligned} (U^*)_{M+1}^N &= (U_c^*)_{M+1}^N, && (5.20) \\ (U^*)_1^M &= \arg\max \gamma, \\ \text{s. t.} \quad & U_1^M \in \mathcal{U}, \\ & |u_k| \le u_{max}, \\ & \hat{P}_M^{-1}(U_1^M) - P_0^{-1} \ge \gamma I. && (5.21) \end{aligned}$$

To describe the set $\mathcal{U}$, let us denote the optimal MPC criterion value $J^* = J(U_c^*)$. In this case, it is natural to define the feasibility set as such neighborhood, in which the MPC quadratic criterion does not change more than specified, i.e.

$$\mathcal{U} = \left\{ U_1^M \in \mathbb{R}^M : J\left(\begin{bmatrix} U_1^M \\ (U_c^*)_{M+1}^N \end{bmatrix} \right) < J^* + \Delta J \right\} \tag{5.22}$$

for a given maximum criterion change ΔJ. Note that such feasible set (not taking into account the hard constraint) is an M-dimensional ellipsoid in $\mathbb{R}^M$. This is clear from the criterion formulation in (5.18). The choice of ΔJ is a part of the controller design. One possibility is to design a time-varying ΔJ depending on the information matrix, so that ΔJ is large if P_0^{-1} is small and vice versa. Such choice leads to faster parameter identification when the uncertainty is high and as soon as the information is gained, the perturbation is decreased. However, ΔJ should be bounded from above, to guarantee that the perturbations are also bounded.

> **Note 5.2.** The condition (5.21) is expressed for the information matrix $\hat{P}_M^{-1}(U_1^M)$. This means that the influence of inputs U_1^M is measured at time M. However, the influence could also be evaluated later, e.g. on the control horizon N as $\hat{P}_N^{-1}(U_1^M)$ or generally at any time $M \le k \le N$ as $\hat{P}_k^{-1}(U_1^M)$. The time k may be then viewed as additional tuning parameter of the controller design.

Because the information matrix (5.13) consists of quadratic and bilinear terms, the problem is nonconvex in control inputs, as demonstrated in Figure 5.7, which shows the lowest eigenvalue of the information matrix of a second order ARX system after two steps of

control as a function of the two inputs u_1 and u_2. This is a difference from simple single-step approaches where the solution always lies on the constraints ([19]) and is a reason for using numerical methods.

5.3 Multiple-step algorithms

In the previous section, the problem was formulated as a nonconvex problem. The nonconvexity introduced by (5.21) can be handled in several ways. This section presents three different methods to solve the problem (5.20).

5.3.1 Rank 1 algorithm

The Rank 1 algorithm is based on a convex relaxation of the problem and concentrating all nonconvexity into a rank constraint. Using the Definition (5.15), (5.21) is rewritten as

$$\sum_{k=1}^{M} F_k \begin{bmatrix} x_1 \\ U_1^M \end{bmatrix} \begin{bmatrix} x_1^T & (U_1^M)^T \end{bmatrix} F_k^T > \gamma I, \tag{5.23}$$

or in a simplified form

$$\sum_{k=1}^{M} F_k U_X F_k^T > \gamma I, \tag{5.24}$$

using the notation

$$\begin{bmatrix} x_1 \\ U_1^M \end{bmatrix} \begin{bmatrix} x_1^T & (U_1^M)^T \end{bmatrix} = \begin{bmatrix} x_1 x_1^T & x_1 (U_1^M)^T \\ U_1^M x_1^T & U_1^M (U_1^M)^T \end{bmatrix} = U_X. \tag{5.25}$$

The matrix U_X consists of constant terms $x_1 x_1^T$, terms $x_1 (U_1^M)^T$ and $U_1^M x_1^T$ linear in U_1^M, and the term $U_1^M (U_1^M)^T$ quadratic in U_1^M. The quadratic term makes the problem (5.24) unsolvable as an LMI directly, and therefore the following reformulation is used

$$U_{X2} = \begin{bmatrix} x_1 x_1^T & x_1 (U_1^M)^T \\ U_1^M x_1^T & U_q \end{bmatrix}, \tag{5.26}$$

$$\text{s. t.} \qquad \operatorname{rank}(U_{X2}) = 1, \tag{5.27}$$

where U_q is a general positive semidefinite matrix, replacing the quadratic term $U_1^M (U_1^M)^T$. All nonconvexity is now concentrated in the rank constraint (5.27) and dropping this constraint the task can be solved as a normal LMI problem ([11]) in more variables, known also as Schor's relaxation ([58, 36]). Expressing the criterion as a Schur complement ([9]) this

relaxation makes it possible to solve the original problem as a rank constrained LMI

$$
\begin{aligned}
(U^*)_{M+1}^N &= (U_c^*)_{M+1}^N, \\
(U_1^M)^* &= \arg\max_{U_1^M} \gamma, \\
\text{s.t.} \quad & \begin{bmatrix} J^* + \Delta J & \begin{bmatrix} x_1^T & (U_1^N)^T \end{bmatrix} G^T & (U_1^N)^T \\ G \begin{bmatrix} x_1 \\ U_1^N \end{bmatrix} & I & 0 \\ U_1^N & 0 & \frac{1}{\rho} I \end{bmatrix} \geq 0, \\
& |u_k| < u_{max}, \\
& \sum_{k=1}^{M} F_k U_{X2} F_k^T > \gamma I, \\
& \text{rank } U_{X2} = 1.
\end{aligned} \tag{5.28}
$$

5.3.2 Gershgorin circle algorithm

This algorithm is based on eigenvalue approximation in terms of Gershgorin circles ([9]). For a real matrix A with entries a_{ij} define $R_i = \sum_{j \neq i} |a_{ij}|$, i.e. the sum of absolute values of elements of the i-th row without the diagonal element. Then each eigenvalue lies in at least one of the Gershgorin circles defined as intervals $[a_{ii} - R_i; a_{ii} + R_i]$ for every i. This idea can be used to create constraints on the elements of the matrix $\hat{P}_M^{-1} - P_0^{-1}$. If the diagonal elements a_{ii} are greater than some γ_1 and the nondiagonal sum less than γ_2, then the lowest eigenvalue must be greater than $\gamma_1 - \gamma_2$.

Let us now formulate the above idea as an optimization problem. The first part is formed similarly to (5.20)

$$
\begin{aligned}
(U^*)_{M+1}^N &= (U_c^*)_{M+1}^N, \\
(U^*)_1^M &= \arg\max \gamma_1 - \gamma_2, \\
\text{s.t.} \quad & U_1^M \in \mathcal{U}, \\
& |u_k| \leq u_{max},
\end{aligned} \tag{5.29}
$$

with the set $\mathcal{U}$ given by (5.22). The additional constraint (5.21) is replaced by conditions imposed on the elements a_{ij} of the information matrix increase $\hat{P}_M^{-1} - P_0^{-1} = Z_M Z_M^T$. Using

the fact that $a_{ij} = w_i w_j^T$ and notation (5.16), it is necessary to ensure that

$$\begin{aligned}
b_{ij} &> \begin{bmatrix} x_1^T & (U_1^M)^T \end{bmatrix} K_i^T K_j \begin{bmatrix} x_1 \\ U_1^M \end{bmatrix}, \forall i,j = 1,\ldots,2n+1, i<j, \qquad (5.30)\\
b_{ij} &> -\begin{bmatrix} x_1^T & (U_1^M)^T \end{bmatrix} K_i^T K_j \begin{bmatrix} x_1 \\ U_1^M \end{bmatrix}, \forall i,j = 1,\ldots,2n+1, i<j,\\
b_{ij} &= b_{ji},\\
\gamma_2 &> \sum_{j\neq i} b_{ij}, \forall i = 1,\ldots,2n+1,\\
\gamma_1 &< \begin{bmatrix} x_1^T & (U_1^M)^T \end{bmatrix} K_i^T K_i \begin{bmatrix} x_1 \\ U_1^M \end{bmatrix}, \forall i = 1,\ldots,2n+1.
\end{aligned}$$

where b_{ij} are artificial variables that have the meaning of absolute values of a_{ij}. Because the matrix $\hat{P}_M^{-1}$ is symmetrical, the first two constraints in are only required for $i < j$.

5.3.3 Orthogonal regressors algorithm

This algorithm is based on the idea, that the regressors shape the information ellipsoid, that is the ellipsoid $x^T(\hat{P}_M^{-1} - P_0^{-1})^{-1}x = 1$. The eigenvalues of $\hat{P}_M^{-1} - P_0^{-1}$ correspond to the ellipsoid radii. Therefore similarly to the previous algorithm, it is necessary to ensure that the regressors' norms $|z_i| > \gamma_1$ and that the regressors are 'as much orthogonal as possible', meaning that for all $i \neq j$, $z_i^T z_j < \gamma_2$. The problem starts the same as (5.29), with the following constraints

$$\begin{aligned}
b_{ij} &> \begin{bmatrix} x_1^T & (U_1^N)^T \end{bmatrix} F_i^T F_j \begin{bmatrix} x_1 \\ U_1^N \end{bmatrix}, \quad \forall i,j = 1,\ldots,M, \quad i<j, \qquad (5.31)\\
b_{ij} &> -\begin{bmatrix} x_1^T & (U_1^N)^T \end{bmatrix} F_i^T F_j \begin{bmatrix} x_1 \\ U_1^N \end{bmatrix}, \quad \forall i,j = 1,\ldots,M, \quad i<j,\\
b_{ij} &< \gamma_2, \quad \forall i,j = 1,\ldots,M, \quad i<j,\\
\gamma_1 &< \begin{bmatrix} x_1^T & (U_1^N)^T \end{bmatrix} F_i^T F_i \begin{bmatrix} x_1 \\ U_1^N \end{bmatrix}, \quad \forall i = 1,\ldots,M.
\end{aligned}$$

The structure of the problem is similar to the previous one, the difference is in the problem dimension. While the number of constraints is $\frac{(2n+1)(2n)}{2}$ and the dimension of the vectors is M in the Gershgorin algorithm, in this case it is the reverse, i.e. the dimension of regressors is $2n+1$ and the number of constraints is $\frac{(M)(M-1)}{2}$. This implies that in this case, M should be equal to $2n+1$, as the number of regressors should not be higher than their dimension.

5.4 Simulations

Simulations of the previously proposed algorithms are shown in this section. The following ARX system was used

$$y_k = 1.64y_{k-1} - 0.67y_{k-2} + 0.2u_k + 0.22u_{k-1} - 0.12u_{k-2} + e_k, \qquad (5.32)$$

which is obtained by discretization of a system $1/(s+1)^2$ with a sampling period $T_s = 0.2s$ and modified in order to have $b_0 \neq 0$. The system is controlled to zero from the initial state $x_0 = [10, 0, 0, 0]^T$. Note that the nonminimal representation (5.5) is used, so the system order is 4. The control was designed for $N = 30$, $M = 5$, $r = 1$ and $\Delta J = 0.1J^*$. Figures 5.8(a) and 5.9(a) show the inputs and outputs of a control process for optimal MPC controller and all three designed controllers, respectively. For comparison, Figures 5.8(b) and 5.9(b) show the results for the ellipsoid algorithm presented in the next section. Figure 5.10 shows the development of the variance matrix in the sense of its greatest eigenvalue. The best results in terms of minimizing the maximum eigenvalue of the variance matrix were achieved by the Rank 1 algorithm, which obviously outperformed all the other methods, including the ellipsoid algorithm from the next section. However, this is rather a coincidence than a general rule, because all the algorithms solve nonconvex problems and the performance of individual algorithms depends on the controlled system, initial conditions for optimization, etc. On the other hand, the ellipsoid algorithm is more sophisticated and therefore it would probably beat the other algorithms on average.

The Rank 1 algorithm was solved by YALMIP ([38]) in MATLAB, with help of the LMIRANK solver ([44]). As the solver only searches for feasible points, the algorithm was run sequentially with γ varying according to the interval bisection method to find the maximum information. Both the Gershgorin and the regressor algorithm were solved by the MATLAB standard function FMINCON.

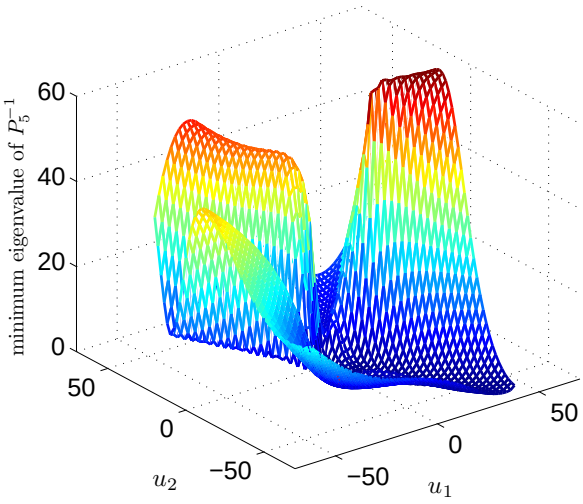
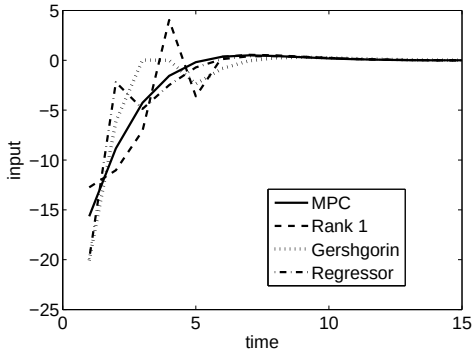
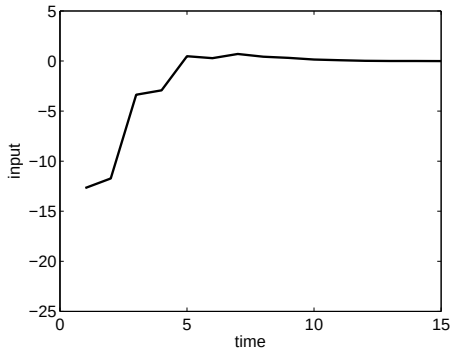


Figure 5.7: The lowest eigenvalue of the predicted information matrix after $M = 5$ steps of control as a function of the two first inputs u_1 and u_2 around the optimal MPC solution for the ARX system (5.32).

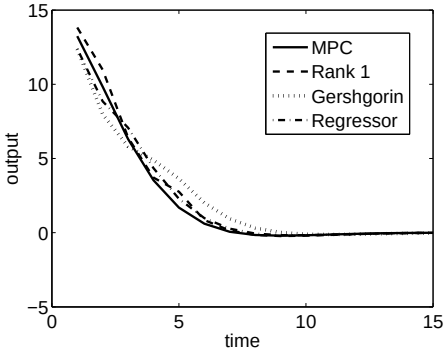


(a) The three algorithms proposed in this section compared to MPC.

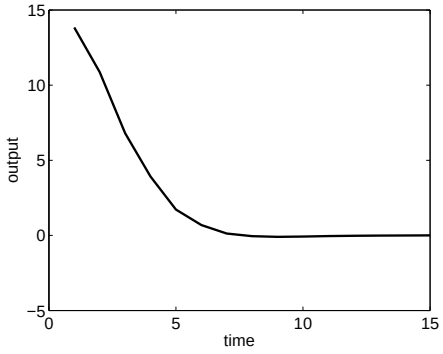


(b) Ellipsoid algorithm.

Figure 5.8: The control input designed by classical MPC and modifications by all proposed algorithms for excitation horizon $M = 5$.

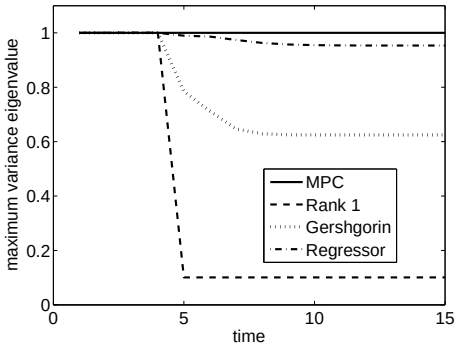


(a) The three algorithms proposed in this section compared to MPC.

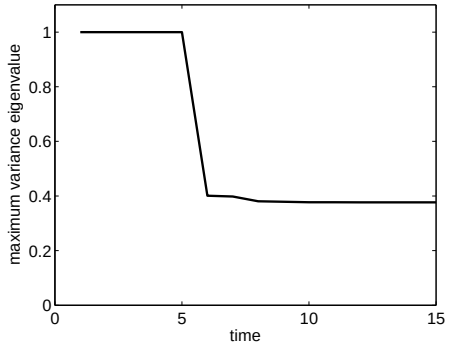


(b) Ellipsoid algorithm.

Figure 5.9: The output of a system controlled by classical MPC and modifications by all proposed algorithms for excitation horizon $M = 5$.



(a) The three algorithms proposed in this section compared to MPC.



(b) Ellipsoid algorithm.

Figure 5.10: The maximum eigenvalue of the estimate variance matrix for control designed by classical MPC and modifications by all three proposed algorithms for excitation horizon $M = 5$.

Chapter 6

The ellipsoid algorithm

In the previous chapter, the problem of MPC control with information matrix maximization was formulated as a nonconvex problem, where the nonconvexity is introduced by the constraint (5.21). This chapter presents a more advanced method to solve the problem (5.20) than the simple algorithms introduced in the previous section. The algorithm uses a uniform approximation of the minimum eigenvalue function by an upper bound, formed by the minimum of specific quadratic forms. In this chapter, U denotes an M-dimensional vector, the usual notation U_1^M will be omitted.

6.1 Derivation of the algorithm

Recall that the nonconvex constraint in (5.20) is in the form

$$\hat{P}_M^{-1}(U) - P_0^{-1} \geq \gamma I, \tag{6.1}$$

which is equivalent to requiring that the minimum eigenvalue of $\hat{P}_M^{-1}(U) - P_0^{-1}$ be greater than γ. If we denote $m(U)$ the function that maps an input sequence U to the minimum eigenvalue of the information matrix increase $\hat{P}_M^{-1}(U) - P_0^{-1}$, we can write the problem (5.20) as

$$\begin{aligned} U^* &= \arg\max m(U), \\ \text{s.t.} \quad & U \in \mathcal{U}, \\ & |u_k| \leq u_{max}, \end{aligned} \tag{6.2}$$

where $\mathcal{U}$ denotes the feasible set given by the criterion relaxation (5.22). The ellipsoid algorithm is based on approximation of the function $m(U)$ by an upper bound constructed as a minimum of a set of quadratic forms. The construction of the algorithm will be shown intuitively, see Section 6.4 for a more formal approach.

Using basic eigenvalue properties, the function $m(U)$ can be expressed as

$$m(U) = \min_{|v|=1} v^T(\hat{P}_M^{-1} - P_0^{-1})v, \tag{6.3}$$

i.e. the minimum value of the quadratic form on the set of unit vectors v (unit ball). Using the trace operator and substituting (5.15) for $\hat{P}_M^{-1}(U)$ we get an equivalent expression

$$m(U) = \min_{|v|=1} \begin{bmatrix} x_1^T & U^T \end{bmatrix} \sum_{k=1}^{M} \left(F_k^T v v^T F_k\right) \begin{bmatrix} x_1 \\ U \end{bmatrix}, \tag{6.4}$$

see Theorem 6.2 for details.

The expression (6.4) shows the minimum eigenvalue function as a minimum of quadratic forms in U. For each v, (6.4) defines a matrix

$$W(v) = \sum_{k=1}^{M} \left\{F_k^T v v^T F_k\right\}.$$

The set $v : |v| = 1$ is uncountable, but it is compact and the function $m(U)$ has a limited growth (it is Lipschitz, see proof of Lemma 6.4), which makes it possible to approximate the function $m(U)$ with a given precision by taking the minimum over a finite subset of unit vectors, i.e. the minimum of a finite set of quadratic forms given by matrices $W(v_1), \ldots, W(v_K)$. Maximizing the approximation is still a nonconvex task, but it is computationally more convenient than the original task, as it requires evaluation of the minimum of a finite number of quadratic forms compared to evaluation of the minimum eigenvalue.

The approach proposed in this algorithm is based on covering the feasible set $\mathcal{U}$ by a sufficiently dense net of points $\mathcal{U}'$. In each step k, the minimum of the given $(k-1)$ quadratic forms at each of these points is evaluated (let us denote the minimum $m'(U)$ as an approximation of $m(U)$ at step k) and the maximum of these values is found at a point U_k^*. Next, a new quadratic form is added to the set, that best approximates the function around the maximizer U_k^*. Such form is found in the following way: The minimum eigenvalue λ_k and the corresponding eigenvector v_k of $\hat{P}_M^{-1}(U_k^*) - P_0^{-1}$ are computed. The form defined by the matrix $W(v_k)$ is added to the current set of quadratic forms. It holds (see Lemma 6.3) that the value of the form given by $W(v_k)$ at the point U_k^* is equal to $\lambda_k = m(U_k^*)$, so the approximation at U_k^* is exact. The algorithm will finish in a finite number of steps, when no other improvement of the precision can be done. This is identified by $U_k^* = U_{k-1}^*$, meaning that no new form would be added to the set.

Note, that we are constructing an upper bound for the original function, therefore the approximation can only be lowered by adding new quadratic forms. Therefore, if at some point U, the approximation $m'(U)$ is lower than already achieved value $m(U_k^*) = \lambda_k$, such point U can be removed from the set, because it will never be the optimal point.

The algorithm also has a straightforward geometric interpretation of searching for such feasible control $U \in \mathcal{U}$ that lies outside of a union of ellipsoids. The situation is illustrated in Figure 6.1 for a two-dimensional problem. The dark contour restricts a region of admissible control $\mathcal{U}$, so the searched point must lie inside the dark contour. On the other hand, the lighter contours correspond to quadratic forms $W(v)$ for some v and the requirement is that the quadratic form given by $W(v)$ be greater than γ for the variable U, so the point U must lie outside the union of all such ellipsoids. Figure 6.2 shows a two-

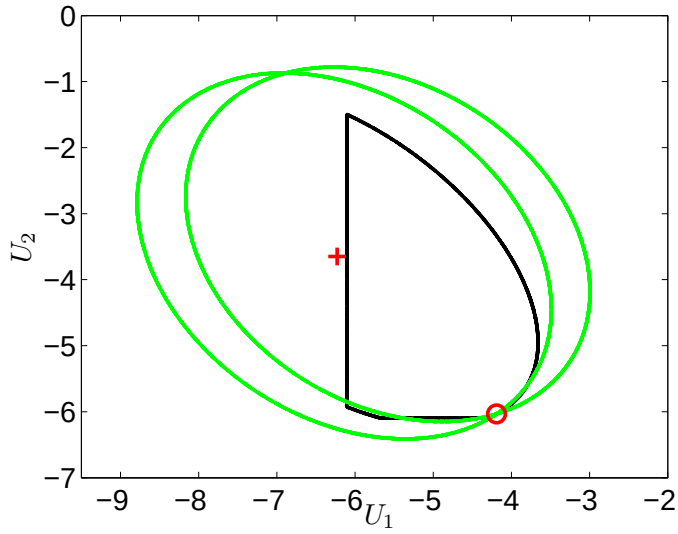


Figure 6.1: Geometric interpretation of the algorithm. The black contour marks the boundary of $\mathcal{U}$, originally an ellipse with a center marked by the red +. The green contours represent two different quadratic forms corresponding to two different directions in parameter space, which are to be excited. The optimal solution lies inside the black contour and outside all grey contours and is marked by a red circle.

dimensional region $\mathcal{U}$ and how the points are removed from the set during the algorithm, based on the above note.

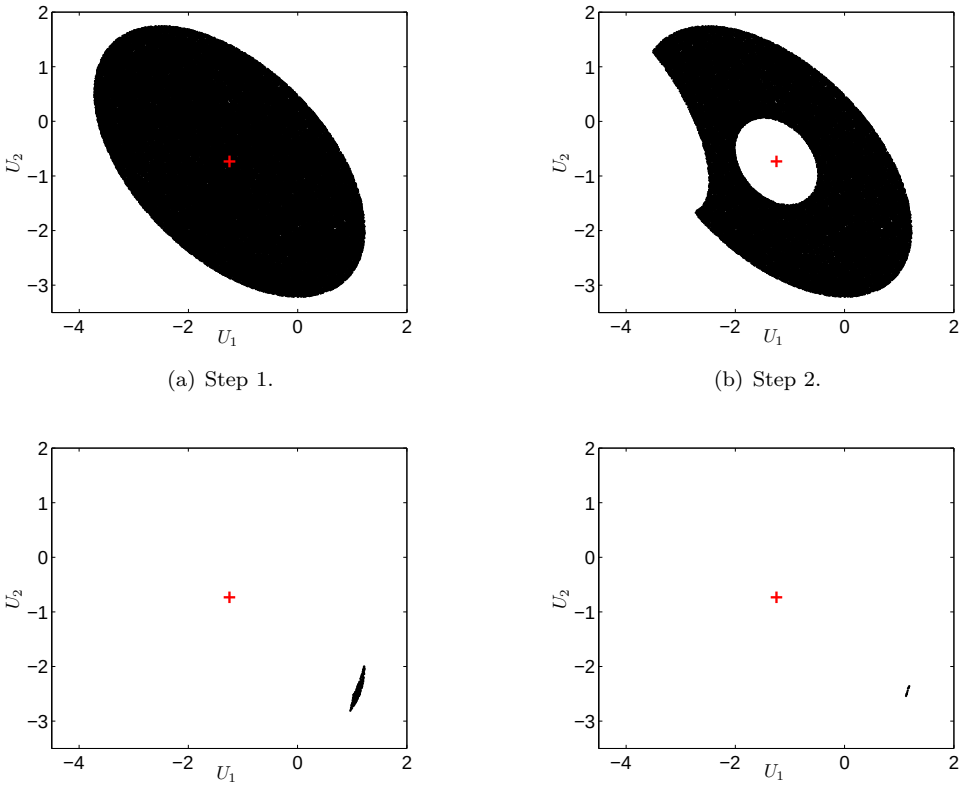
It is shown in detail in Section 6.4 that it is possible to approximate the function $m(U)$ with an arbitrary precision. The algorithm starts with an empty set of quadratic forms and the initial point U_c^*, which is the initial optimal (MPC) control.

6.1.1 Algorithm

The inputs to the algorithm are the system matrices $\hat{A}, \hat{B}, \hat{C}$ and $\hat{D}$, the MPC parameter ρ, the admissible criterion perturbance ΔJ, the initial condition x_1, the control horizon N, the excitation horizon M (greater or equal to the number of system parameters), the hard constraint on inputs u_{max} and the required precision ε. The outputs are the control sequence U^* and the information gain γ. The variables that are changed iteratively are marked with an accent, such as m', U' etc.

1. Solve the original MPC task (5.11).

2. Estimate the Lipschitz constant L as described in the proof of Lemma 6.4 and find a corresponding k such that $\max_{U \in \mathcal{U}}(m_k(U) - m(U)) < \varepsilon$.



(a) Step 1.

(b) Step 2.

(c) Step 3.

(d) Step 4.

Figure 6.2: Restriction of the set $\mathcal{U}$ of admissible input perturbations during the algorithm. The original shape is an ellipsoid with a center marked by the red +, from which areas are removed in each step of the algorithm.

3. Construct the set $\mathcal{U}' = \mathcal{U}_k$ from Lemma 6.5 and choose an initial point U' as the solution of the MPC task in step 1.

4. Initialize $m'(U) = \infty$ for all $U \in \mathcal{U}'$.

5. Compute $v_{U'}$ and for all $U \in \mathcal{U}'$ do:
   if
   $$\begin{bmatrix} x_1^T & U^T \end{bmatrix} W(v_{U'}) \begin{bmatrix} x_1 \\ U \end{bmatrix} < m'(U),$$
   then set
   $$m'(U) = \begin{bmatrix} x_1^T & U^T \end{bmatrix} W(v_{U'}) \begin{bmatrix} x_1 \\ U \end{bmatrix}.$$

6. Remove all $U \in \mathcal{U}'$ for which $m'(U) < m'(U') = m(U')$.

7. If
   $$\arg\max_{U \in \mathcal{U}'} m'(U) \neq U',$$
   set $U' = \arg\max_{U \in \mathcal{U}'} m'(U)$ and go to step 5.

8. Set $U^* = U'$ and finish.

Note 6.1. The proposed algorithm was derived using the constraint (5.21), i.e. $\hat{P}_M^{-1}(U_1^M) - P_0^{-1} \geq \gamma I$. In some cases, if the eigenvalues of the initial information matrix P_0^{-1} are different, it may be useful to define this constraint for the information matrix itself instead of its increase, i.e.

$$\hat{P}_M^{-1}(U_1^M) \geq \gamma I. \tag{6.5}$$

The algorithm is then slightly modified, because the minimum eigenvalue function has to be redefined as

$$m(U) = \min_{|v|=1} \left\{ v^T P_0^{-1} v + \begin{bmatrix} x_1^T & U^T \end{bmatrix} \sum_{k=1}^{M} \left(F_k^T v v^T F_k \right) \begin{bmatrix} x_1 \\ U \end{bmatrix} \right\},$$

and consequently the quadratic function in the step 5 of the algorithm above is changed to

$$v_{U'}^T P_0^{-1} v_{U'} + \begin{bmatrix} x_1^T & U^T \end{bmatrix} W(v_{U'}) \begin{bmatrix} x_1 \\ U \end{bmatrix}.$$

The term $v_{U'}^T P_0^{-1} v_{U'}$ is constant for a fixed $v_{U'}$ and therefore only changes the size of the ellipsoids in the geometric interpretation in Figure 6.1 without affecting their shape.

6.2 Simulations

The algorithm was tested on the following second order ARX system with oscillatory behavior

$$y_k = 0.91y_{k-1} - 0.67y_{k-2} + 0.35u_{k-1} + 0.4u_{k-2} + e_k,$$

with $\sigma_e^2 = 10^{-6}$. First, an MPC controller was designed with the weighting parameter $\rho = 0.5$. The perturbation was then searched in a region $J(U) \leq J^* + \Delta J$ with $\Delta J = 0.01$. The control horizon for the MPC controller was 20 steps and as the system has 5 parameters ($b_0 = 0$), the excitation horizon $M = 5$. The control was applied as a receding horizon, i.e. in each step the whole control sequence was computed, but only the current input was used and a new sequence was computed after the measurement and data update.

The initial guess of the parameters was

$$\begin{aligned} b_0 &= 0.5 \\ a_1 &= -0.1 \\ b_1 &= 0.1 \\ a_2 &= 0.1 \\ b_2 &= 1 \end{aligned} \tag{6.6}$$

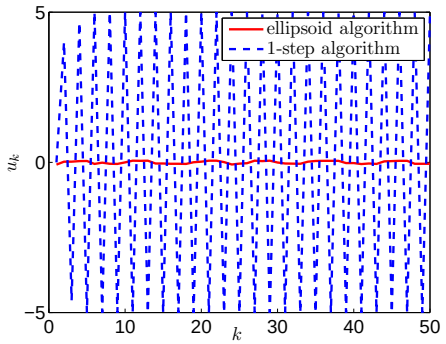
and the initial parameter uncertainty was given by a variance matrix $P_0^{-1} = I$ (identity matrix). After each measurement the initial guess was updated by the recursive least square algorithm and the control in the next step was already based on the updated parameter values. This algorithm was compared to a single-step approach, which is also based on MPC as the initial control, but uses only information about the parameter variance matrix after one step in the future. This algorithm perturbates the initial control by the following rule

$$U^* = U_C^* + \text{sign}(U_C^*)\Delta U, \tag{6.7}$$

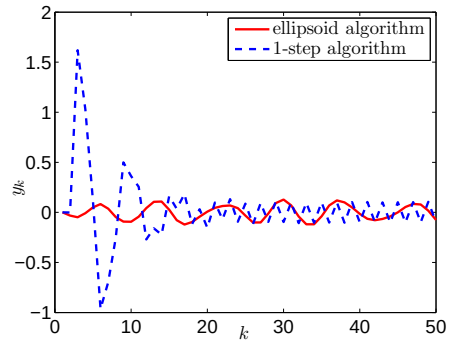
where ΔU is a predescribed value. Both algorithms were supposed to keep the system output at a zero level and at the same time slightly perturb the control input to identify the system parameters, starting with values (6.6).

For the first simulation, the perturbation ΔU in the single-step algorithm was chosen $\Delta U = 4$, so that the output variance is comparable to the multiple-step case. Figures 6.3(a), 6.3(c) and 6.3(b), 6.3(d) show the input and output, respectively, during simulation with the proposed dual controller compared to the MPC single-step solution. Figures 6.5(a) and 6.5(c) show how the eigenvalues of the parameter estimate variance matrix change during the control process for the dual and the single-step MPC-based controller, respectively. Finally, Figures 6.5(b) and 6.5(d) show the development of the parameter estimates for both cases. The parameters were estimated by the classical recursive least square method.

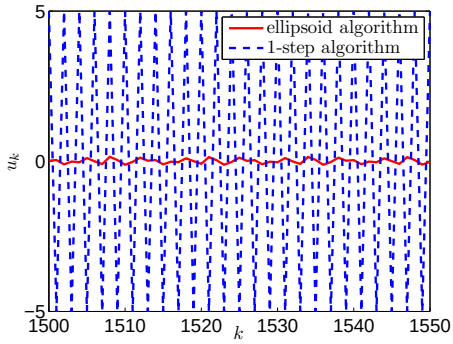
The experiment shows that the single-step algorithm is much more aggressive with an immediate effect on parameter estimation. The multiple-step algorithm, on the other hand, needs a longer time interval to achieve the same precision, but keeps improving identification



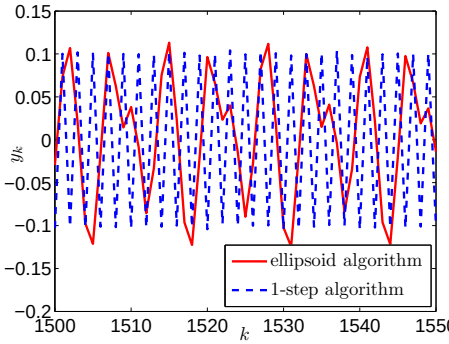
(a) Control input until the time $k = 50$.



(b) Output of the system until the time $k = 50$.



(c) Control input between the times $k_1 = 1500$ and $k_2 = 1550$.



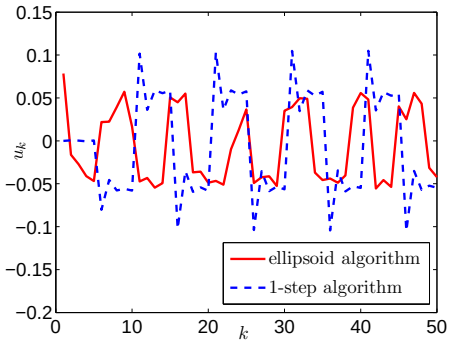
(d) Output of the system between the times $k_1 = 1500$ and $k_2 = 1550$.

Figure 6.3: Comparison of the control input and system output for the single-step and multiple-step controller, where the single-step controller changes the excitation ΔU in each step. The controllers were tuned to achieve a comparable output perturbation.

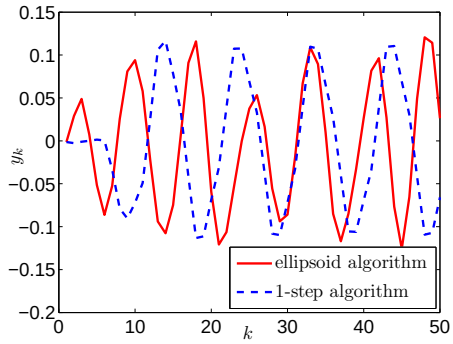
for the whole simulation period. Also the overall criterion value differs significantly, $J = 17.8$ for the multiple-step algorithm and $J = 2.5 \cdot 10^4$ for the single-step algorithm.

For the second experiment, we adjusted the single-step algorithm so that the input perturbation can be changed only every 5th step. This change would excite the system at lower frequencies, with a possibly better effect on identification. The perturbation was chosen $\Delta U = 0.08$, again so that the output variance is comparable to the multiple-step algorithm.

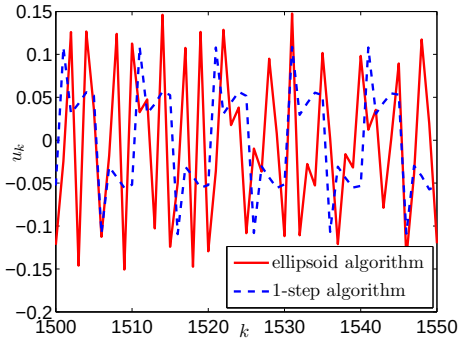
Similarly to the previous case, Figures 6.4(a), 6.4(c) and 6.4(b), 6.4(d) show the input and output, respectively, during simulation with the proposed dual controller compared to the MPC single-step solution. Figures 6.5(a) and 6.5(e) show how the eigenvalues of the parameter estimate variance matrix change during the control process for the dual and the



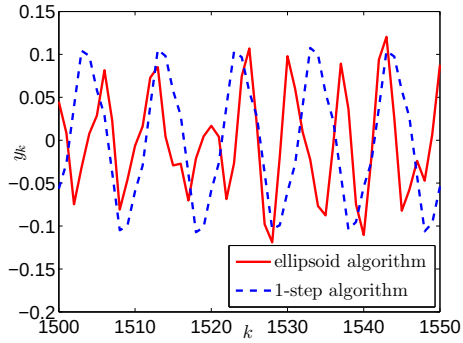
(a) Control input of the multiple-step controller compared to the single-step controller until the time $k = 50$.



(b) Output of the system controlled by the multiple-step controller and the single-step controller until the time $k = 50$.



(c) Control input of the multiple-step controller compared to the single-step controller between the times $k_1 = 1500$ and $k_2 = 1550$.

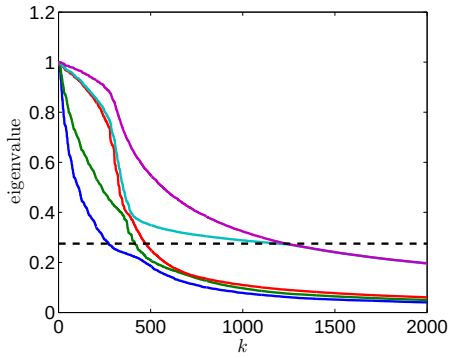


(d) Output of the system controlled by the multiple-step controller and the single-step controller between the times $k_1 = 1500$ and $k_2 = 1550$.

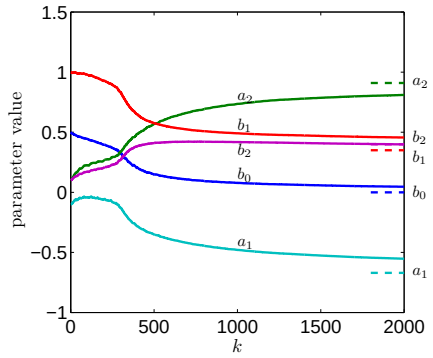
Figure 6.4: Comparison of the control input and system output for the single-step and multiple-step controller, where the single-step controller changes the excitation ΔU once in 5 steps. The controllers were tuned to achieve a comparable output perturbation.

single-step MPC-based controller, respectively. Finally, Figures 6.5(b) and 6.5(f) show the development of the parameter estimates for both cases.

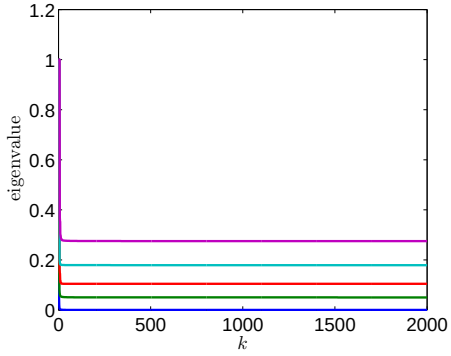
The second experiment shows that the lower frequency perturbation excites the system output more than the high frequency signal. The criterion is comparable, $J = 17.6$ for the multiple-step case and $J = 14.6$ in the modified single-step case. However, as seen from the eigenvalue responses, the single-step algorithm is significantly less effective in terms of improving parameter precision.



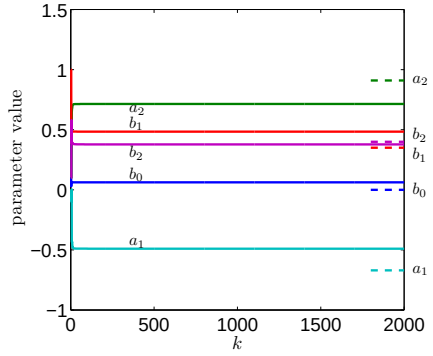
(a) Eigenvalues for the multiple-step controller. The dashed line represents the maximum eigenvalue of the single-step algorithm in Figure 6.5(c).



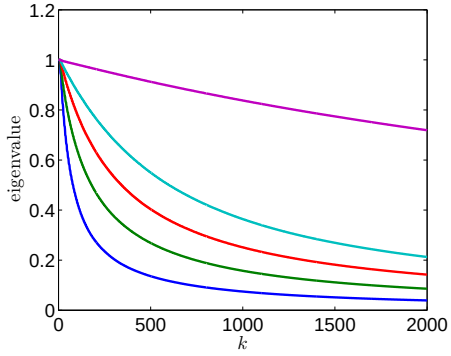
(b) Parameter estimates for the multiple-step controller.



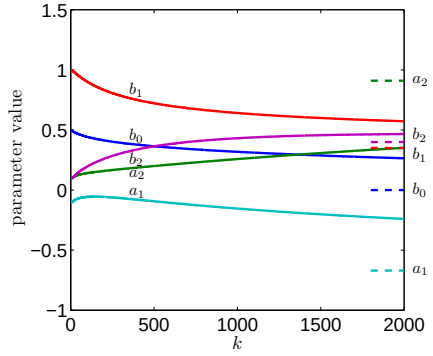
(c) Eigenvalues for the single-step controller. The excitation ΔU is changed in each step.



(d) Parameter estimates for the single-step controller. The excitation ΔU is changed in each step.



(e) Eigenvalues for the single-step controller. The excitation ΔU is changed every 5th step.



(f) Parameter estimates for the single-step controller. ΔU is changed every 5th step.

Figure 6.5: Comparison of the development of parameter estimates and the eigenvalues of the parameter estimate variance matrix for the single-step and multiple-step controller.

6.3 Properties of the algorithm

This section contains remarks on complexity and stability of the ellipsoid algorithm.

6.3.1 Complexity

The covering net of points is a subset of $\mathbb{R}^M$. To ensure that the error is no larger than a specified ε, the distance between adjacent points must be lower than a fixed δ_ε. Thus the number of points grows exponentially with the system order. However, this is caused by the nonconvexity of the problem and the number of points may be lowered in case locally optimal solutions are sufficient. The major advantage of the algorithm is that evaluation of a quadratic form at a given point is computationally much faster than computing the lowest eigenvalue function. The quadratic forms are also useful for error estimation, because they are second order approximations of the lowest eigenvalue function at a given point.

6.3.2 Stability

The stability of a system controlled by the proposed controller in the usual (Lyapunov) sense can be guaranteed for the nominal system, i.e. the system for which $\hat{a_i} = a_i$ and $\hat{b_i} = b_i$ for all $i = 1 \dots n$. This follows from the stability of the MPC controller [51, 52], which gives us stability of the nominal system controlled by the MPC control input U_c^*, if there is no limitation on the input signal. In case of constrained input, stability is achieved only for a certain set $\mathcal{X}$ of initial states, which is a known limitation. Let us assume the unconstrained case first. Let us also assume that the necessary conditions for MPC stability, such as control horizon length or sufficient weight on terminal state, are satisfied.

The control U^* can be viewed as a sum of the MPC control U_c^* and a perturbation ΔU^*. The fact that the criterion relaxation ΔJ is bounded implies that ΔU^* is bounded. It is important that ΔJ be bounded uniformly with respect to time, so that ΔU^* is also uniformly bounded.

The MPC control U_c^* has a stabilizing effect on the nominal system. In a linear system, the effect of U^* on the nominal system can be viewed as an effect of the input perturbation ΔU^* on a nominal system already stabilized by U_c^*. As a result, the effect of the optimal control U^* is an effect of a bounded input sequence on a stable system. A stable system is also BIBO (Bounded input, bounded output) stable, therefore the closed loop is stable for the nominal system.

Note that if U^* satisfies the input constraints imposed by the MPC problem (5.11), the idea given above also holds for a constrained problem. However, as in the standard MPC problem, it must be assured that the system state does not leave the set $\mathcal{X}$. This can be achieved by a suitable choice of ΔU^*.

6.4 Formal derivation of the algorithm

This section contains a formal proof of convergence of the ellipsoid algorithm. The proof is based on constructing a sequence of approximations and showing that this sequence converges to the original function. As usual in this chapter, U denotes an M-dimensional vector and the usual notation U_1^M will be omitted.

6.4.1 Expressing the minimum eigenvalue by quadratic forms

The information matrix maximization in (5.20) is equivalently expressed as maximization of its lowest eigenvalue. Let us define a function $\tilde{m} : \mathbb{R}^{n\times n} \to \mathbb{R}$, that assigns a matrix its lowest eigenvalue, i.e. $\tilde{m} : A \mapsto \lambda_m$, where $\lambda_m = \min\{\lambda \in \mathbb{R} : \exists 0 \neq v \in \mathbb{R}^n, Av = \lambda v\}$. Such function is correctly defined, as a real symmetric matrix of order n has exactly n real eigenvalues. The problem is then to find

$$U^* = \arg\max_{U\in\mathcal{U}} \tilde{m}(P_M^{-1}(U) - P_0^{-1}) \stackrel{\text{def}}{=} \arg\max_U m(U), \tag{6.8}$$

where $\mathcal{U}$ denotes the (without loss of generality compact) admissible region of U defined by the constraints in (5.20).

Theorem 6.2. *The function m defined in* (6.8) *above is continuous in U and the following holds:*

$$m(U) = \min_{v\in S^{n-1}} \begin{bmatrix} x_1^T & U^T \end{bmatrix} \sum_{k=1}^{M} \left(F_k^T v v^T F_k\right) \begin{bmatrix} x_1 \\ U \end{bmatrix}, \tag{6.9}$$

where $S^{n-1} = \{v \in \mathbb{R}^n : \|v\| = 1\}$ denotes the unit sphere in $\mathbb{R}^n$ (the Hausdorff dimension of which is $n-1$).

Proof. The right hand side of (6.9) is well defined, as it is a continuous function in v and the minimum exists on the compact set S^{n-1}. By definition, $m(U)$ is the lowest eigenvalue of the matrix

$$P_M^{-1}(U) - P_0^{-1} = \sum_{k=1}^{M} \left\{ F_k \begin{bmatrix} x_1 \\ U \end{bmatrix} \begin{bmatrix} x_1^T & U^T \end{bmatrix} F_k^T \right\},$$

so, using the properties of eigenvectors,

$$m(U) = \min_{v\in S^{n-1}} v^T \sum_{k=1}^{M} \left\{ F_k \begin{bmatrix} x_1 \\ U \end{bmatrix} \begin{bmatrix} x_1^T & U^T \end{bmatrix} F_k^T \right\} v,$$

where the minimum is achieved for the eigenvector belonging to the lowest eigenvalue. Then, using the matrix trace properties

$$\begin{aligned} & v^T \textstyle\sum_{k=1}^{M} \left\{ F_k \begin{bmatrix} x_1 \\ U \end{bmatrix} \begin{bmatrix} x_1^T & U^T \end{bmatrix} F_k^T \right\} v \quad = \\ = \quad & \begin{bmatrix} x_1^T & U^T \end{bmatrix} \textstyle\sum_{k=1}^{M} \left(F_k^T v v^T F_k\right) \begin{bmatrix} x_1 \\ U \end{bmatrix}. \end{aligned}$$

Finally, the function $m(U)$ is continuous, because it is a minimum of (quadratic) continuous functions of U. □

Theorem 6.2 says that the minimum eigenvalue of the information matrix (5.15) can be expressed as a minimum of quadratic forms in U parametrized by v. Let us denote

$$W(v) = \sum_{k=1}^{M} \left(F_k^T v v^T F_k \right), \tag{6.10}$$

for simplification of further expressions.

6.4.2 Approximation by finite sets of functions

The idea is now to approximate the function $m(U)$ with a sufficient precision by minimizing (6.9) over a finite subset V of the unit sphere S^{n-1}. Let us first notice, that due to the continuity of m and separability of the (compact) unit sphere S^{n-1}, there exists a dense countable subset $S' \subset S^{n-1}$ such that

$$\begin{aligned} \min_{v \in S^{n-1}} \begin{bmatrix} x_1^T & U^T \end{bmatrix} W(v) \begin{bmatrix} x_1 \\ U \end{bmatrix} \quad &= \\ = \quad \inf_{v \in S'} \begin{bmatrix} x_1^T & U^T \end{bmatrix} W(v) \begin{bmatrix} x_1 \\ U \end{bmatrix}. \end{aligned} \tag{6.11}$$

Our goal now is to construct a monotone sequence of finite subsets $\{V_k\}_{k \in \mathbb{N}}, V_k \subset S^{n-1}, V_k \subset V_{k+1}$ in such way that

$$\lim_{k \to \infty} \min_{v \in V_k} \begin{bmatrix} x_1^T & U^T \end{bmatrix} W(v) \begin{bmatrix} x_1 \\ U \end{bmatrix} = m(U), U \in \mathcal{U}, \tag{6.12}$$

where the limit is uniform (i.e. in supremum metrics) on the compact set $\mathcal{U} \subset \mathbb{R}^N$. Let us first summarize some important properties of the function $m(U)$.

Lemma 6.3. *For every fixed x_1 and every $U \in \mathcal{U}$ there is a $v_U \in S^{n-1}$ such that*

$$\begin{bmatrix} x_1^T & U^T \end{bmatrix} W(v_U) \begin{bmatrix} x_1 \\ U \end{bmatrix} = m(U).$$

Proof. The searched $v_U \in S^{n-1}$ is the eigenvector corresponding to the lowest eigenvalue of the information matrix increase $P_M^{-1}(U) - P_0^{-1}$. This is a direct corollary of the trace properties used in the proof of Theorem 6.2 □

Lemma 6.4. *Let*

$$m_V(U) = \min_{v \in V \subset S^{n-1}} \begin{bmatrix} x_1^T & U^T \end{bmatrix} W(v) \begin{bmatrix} x_1 \\ U \end{bmatrix},$$

where the minimum is taken over any closed nonempty subset V of the unit sphere S^{n-1}. Then the function $m_V(U)$ is Lipschitz with a constant L independent of the set V, i.e. $|m_V(U_1) - m_V(U_2)| < L\|U_1 - U_2\|$.

Proof. The function $m_V(U)$ is well defined, since any closed subset of a compact set is a compact set, hence the minimum exists. It holds that

$$\begin{aligned} & m_V(U_1) - m_V(U_2) = \\ = & \begin{bmatrix} x_1^T & U_1^T \end{bmatrix} W(v_1) \begin{bmatrix} x_1 \\ U_1 \end{bmatrix} - \\ - & \begin{bmatrix} x_1^T & U_2^T \end{bmatrix} W(v_2) \begin{bmatrix} x_1 \\ U_2 \end{bmatrix} \leq \\ \leq & \begin{bmatrix} x_1^T & U_1^T \end{bmatrix} W(v_2) \begin{bmatrix} x_1 \\ U_1 \end{bmatrix} - \\ - & \begin{bmatrix} x_1^T & U_2^T \end{bmatrix} W(v_2) \begin{bmatrix} x_1 \\ U_2 \end{bmatrix}, \end{aligned}$$

where $v_1, v_2 \in V$ are the minimizers at points U_1, U_2. The last expression is a difference between two values of the same quadratic form in two different points. Symmetrically, we get the inequality for absolute values. Therefore we have from the multidimensional mean value theorem that L is at most the maximal norm of the all the gradients, where the maximum is taken over the feasible set for each quadratic form and then over all unit vectors v, i.e.

$$L \leq \max_{v \in S^{n-1}} \max_{U \in \mathcal{U}} 2\|W_{12}(v)x_1 + W_{22}U\|^2. \tag{6.13}$$

The maximum is well defined, as both $\mathcal{U}$ and S^{n-1} are compact sets and x_1 is fixed. The formal proof is left out for brevity. □

Lemma 6.5. *Let $\{\mathcal{U}_k\}_{k\in\mathbb{N}}$ be a sequence of finite subsets of $\mathcal{U}$, $\mathcal{U}_k \subset \mathcal{U}_{k+1}$, such that for every $U \in \mathcal{U}$ there exists $U' \in \mathcal{U}_k$ such that $\|U - U'\| \leq \frac{1}{k}$. Let $V_k = \{v_U : U \in \mathcal{U}_k\}$ be a set of all eigenvectors corresponding to the minimum eigenvalues of matrices $P_M^{-1}(U) - P_0^{-1}$ for all $U \in \mathcal{U}_k$. Then the sequence*

$$m_k(U) = m_{V_k}(U) = \min_{v \in V_k} \begin{bmatrix} x_1^T & U^T \end{bmatrix} W(v) \begin{bmatrix} x_1 \\ U \end{bmatrix}$$

converges pointwise to $m(U)$.

Proof. The set $\mathcal{U}_k$ exists for every $k \in \mathbb{N}$ from compactness of $\mathcal{U}$. It holds that $m_k(U) = m(U)$ for every $U \in \mathcal{U}_k$. Let $U \in \mathcal{U}$ be an arbitrary point. Then there exists a sequence $U_1, U_2, \ldots \to U, U_k \in \mathcal{U}_k$. Let $\varepsilon > 0$. Then $|m_k(U) - m(U)| \leq |m_k(U) - m_k(U_k)| + |m_k(U_k) - m(U_k)| + |m(U_k) - m(U)| = |m_k(U) - m_k(U_k)| + |m(U_k) - m(U)|$ because $|m_k(U_k) - m(U_k)| = 0$. From the Lipschitz property from Lemma 6.4, $|m_k(U) - m_k(U_k)| \leq L\|U - U_k\|$, where it is important that L is independent of k. Now k can be chosen so that $L\|U - U_k\| < \varepsilon/2$ and $|m(U_k) - m(U)| < \varepsilon/2$ as m is a continuous function. □

Lemma 6.6. *The sequence of functions $\{m_k\}_{k\in\mathbb{N}}$ from Lemma 6.5 converges uniformly to m.*

Proof. The sequence $\{m_k\}_{k\in\mathbb{N}}$ is a decreasing sequence of continuous functions converging pointwise to a continuous function m on a compact set $\mathcal{U}$. Uniform convergence is a direct corollary of Dini's theorem. □

Theorem 6.7. *Let $\varepsilon > 0$. Then the ellipsoid algorithm converges to a point $U' \in \mathcal{U}$ in a finite number of steps and $m(U*) - m(U') < \varepsilon$.*

Proof. The algorithm operates on a finite set of points and must finish, if the point U' does not change at the end of the iteration. Thus the algorithm must finish in a finite number of steps. The points that were removed from the set $\mathcal{U}'$ in step 6 can not be candidates for the maximum U', as for such U it holds that $m(U) \leq m'(U) < m'(U') = m(U')$ The solution U' is a maximum of the approximate function m' and at the same time $m'(U') = m(U')$. Therefore $m(U^*) < m'(U') + \varepsilon$ from the construction of the initial $\mathcal{U}'$. □

6.5 Approximation by outer ellipsoid

The ellipsoid algorithm presented in this chapter has one major disadvantage – the amount of points needed to guarantee a given precision grows exponentially with the dimension. Considering a second order ARX system, we already have 5 parameters to be estimated and thus also at least a 5-dimensional problem. Generally, the minimum dimension for an n-th order system is $2n+1$ and the algorithm is thus practically usable only for lower order systems.

In this section, we present a method for transforming the optimization problem into a one-dimensional conservative approximation of the problem. The idea is based on the geometric ellipsoid interpretation of the algorithm. The interpretation was shown in Section 6.1, where the Figure 6.1 shows the geometry for a 2-dimensional example. Each quadratic function given by a matrix $W(v)$ represents the direction v in the parameter space, and the value of the quadratic function at a given point U determines the improvement of information in the corresponding direction v. The curves of constant values γ of the quadratic forms are ellipsoids, shown in Figure 6.1. The ellipsoid algorithm 'inflates' all ellipsoids uniformly until the last feasible point outside the union of uniformly inflated ellipsoids is found.

The proposed modification is based on finding an outer ellipsoid for the union of ellipsoids and 'inflating' only this one ellipsoid instead. Such simplification is conservative and transforms the 'inflating' to a one-dimensional problem. Finding the outer ellipsoid itself is a convex positive semidefinite programming task.

6.5.1 Minimum-volume outer ellipsoid

Assume we already have a set of K vectors $v_1, \ldots, v_K$ and corresponding matrices $W(v_i), i = 1, \ldots, K$. Each constant $\gamma \in \mathbb{R}$ defines a set of K ellipsoids in the parameter space, given by equations

$$T_i(U) = \begin{bmatrix} x_1^T & U^T \end{bmatrix} W(v_i) \begin{bmatrix} x_1 \\ U \end{bmatrix} - \gamma \leq 0, \quad i = 1, \ldots, K. \tag{6.14}$$

The outer minimum-volume ellipsoid for the union of ellipsoid (6.14) can be defined by an equation

$$T_0(U) \leq 0, \tag{6.15}$$

where T_0 is a quadratic function in variable U, which is unique up to a positive scaling factor, because the inequality (6.15) is not affected by multiplication by a positive real constant.

The minimum-volume outer ellipsoid for a set of ellipsoids is found as an LMI optimization task, which is in detail described in [11]. Let us first introduce a general quadratic function in variable U using the following notation

$$T(U) = U^T A U + 2U^T b + c, \tag{6.16}$$

where $A = A^T > 0$ is a square matrix, b is a column vector and $c \in \mathbb{R}$. An inequality $T(U) \leq 0$ then defines an ellipsoid as a set all U for which this inequality is true. Completing the square gives an alternative expression $T(U) = (U+A^{-1}b)^T A(U+A^{-1}b) - b^T A^{-1} b + c \leq 0$, which shows that the set $T(U) \leq 0$ is nonempty if and only if

$$b^T A^{-1} b - c > 0, \tag{6.17}$$

therefore we will only take into account those quadratic functions that satisfy this inequality. To transform quadratic functions in (6.14) into the form (6.16), let us first rewrite the matrix $W(v_i)$ as

$$W(v_i) = \begin{bmatrix} W_1(v_i) & W_2(v_i) \\ W_2^T(v_i) & W_3(v_i) \end{bmatrix}, \tag{6.18}$$

where the dimensions of the blocks correspond to dimensions of vectors x_1 and U. The inequalities (6.14) can then be written as

$$x_1^T W_1(v_i) x_1 + U^T W_2^T(v_i) x_1 + x_1^T W_2(v_i) U + U^T W_3(v_i) U - \gamma \leq 0, \quad i = 1, \ldots, K, \tag{6.19}$$

which corresponds to the notation (6.16) with matrices given by the following transformations

$$\begin{aligned} A_i &= W_3(v_i), \\ b_i &= W_2^T(v_i) x_1, \\ c_i &= x_1^T W_1(v_i) x_1 - \gamma. \end{aligned} \tag{6.20}$$

Using the notation (6.16), the minimum-volume outer ellipsoid is given by the inequality $T_0(U) = U^T A_0 U + 2U^T b_0 + c_0 \leq 0$. The unknown variables A_0, b_0 and c_0 are computed by solving the following LMI optimization task

$$\begin{aligned} \min \quad & \log\det A_0^{-1} \\ \text{s.t.} \quad & A_0 > 0, \tau_1 \geq 0, \ldots, \tau_K \geq 0, \\ & \begin{bmatrix} A_0 & b_0 & 0 \\ b_0^T & -1 & b_0^T \\ 0 & b_0 & -A_0 \end{bmatrix} - \tau_i \begin{bmatrix} A_i & b_i & 0 \\ b_i^T & c_i & 0 \\ 0 & 0 & 0 \end{bmatrix} \leq 0, \quad i = 1, \ldots, K, \end{aligned} \tag{6.21}$$

where $c_0 = b_0^T A_0^{-1} b_0 - 1$ to get rid of the extra degree of freedom in the representation. Note that depending on γ, the condition (6.17) might not be satisfied for some i. Therefore the condition (6.17) should be checked prior to solving the optimization task and those quadratic forms not satisfying the condition should be excluded from the constraints. An example of the minimum-volume outer ellipsoid is shown in Figure 6.6(a).

Having found the minimum-volume outer ellipsoid, it is possible to define an approximate solution to the problem (5.20) as maximization of the quadratic function $T_0(U)$ defining the outer ellipsoid, within the feasible set $\mathcal{U}$. Such maximization is a one-dimensional convex task solvable for example by the Newton method or the interval bisection method, as will be shown in the next subsection. A drawback of this method is that hard constraints on inputs $|u_k| \leq u_{max}$ cannot be used.

Maximizing the quadratic function $T_0(U)$ given by the outer ellipsoid within the feasible set $\mathcal{U}$ means finding the lowest possible $\delta \in \mathbb{R}$ such that the feasible set $\mathcal{U}$ is still a subset of an ellipsoid given by $T_0(U) \leq \delta$, as depicted in Figures 6.6(b) and 6.6(c). The problem is, that this new bigger (or smaller for $\delta < 0$) ellipsoid $T_0(U) \leq \delta$ is not the minimum-volume ellipsoid for ellipsoids $T_i \leq \delta$, as shown in Figure 6.6(d). Therefore it is desirable to find such γ, for which the value δ is zero, or in practice $|\delta| \leq \varepsilon$ for some $\varepsilon > 0$. Such γ can be again found e.g. by using the interval bisection method. The initial interval for γ is $(-\infty, \infty)$, so it is convenient to work with some substitution, for example $\gamma = \tan(x)$ with $x \in (-\pi/2, \pi/2)$.

6.5.2 Quadratic programming with one quadratic constraint

The control U' is the point for which the function T_0 is maximal within the feasible set $\mathcal{U}$. After this point is found, the algorithm proceeds analogously to the idea presented in Section 6.1. The algorithm finds such direction $v_{U'}$ in the parameter space that is least excited by the control U'. A new ellipsoid associated with this direction (given by the matrix $W(v_{U'})$ is added to the set (6.14) and the algorithm repeats for the updated set. The algorithm ends when no new direction $v_{U'}$ can be found.

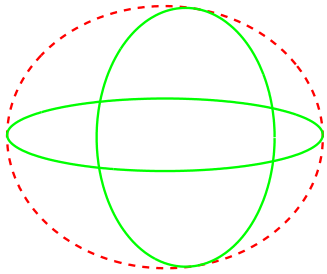
Maximizing the quadratic function $T_0(U)$ that defines the outer ellipsoid within the feasible set (U) is in fact a quadratic programming task that can be written as

$$\begin{aligned} \max \quad & T_0(U) = U^T A_0 U + 2U^T b_0 + c_0 && (6.22) \\ \text{s.t.} \quad & U \in \mathcal{U}, && (6.23) \end{aligned}$$

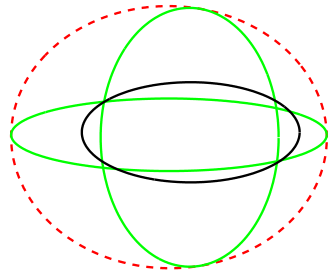
where the constraint $U \in \mathcal{U}$ is in fact a quadratic constraint in variable U that can be rewritten to the general form (6.16) as

$$U^T H U + 2U^T f + g \leq 0$$

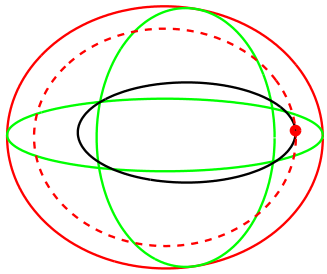
for some positive definite matrix H, vector f and real constant g. We will show one method of solving this problem, more information about the topic can be found in [14] and [35]. It is



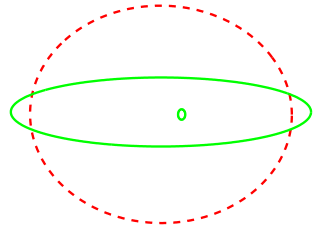
(a) Two ellipsoids (green) and the outer ellipsoid (red).



(b) The black contour represents the boundary of the feasible set $\mathcal{U}$.

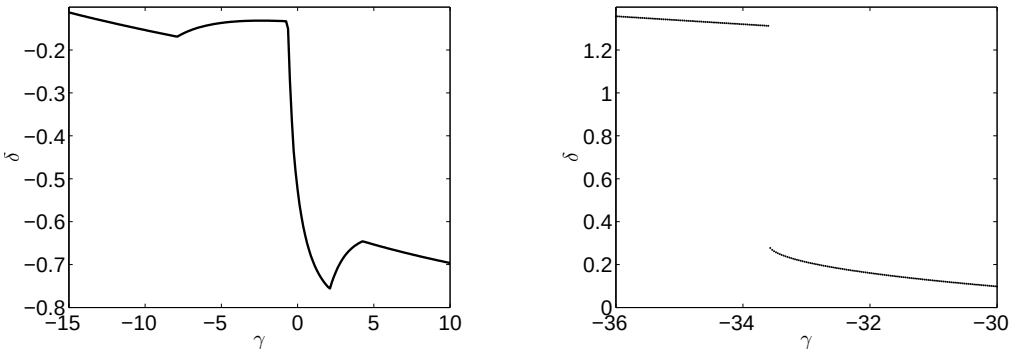


(c) The shrunken outer ellipsoid is showed by the dashed red contour and the point U' is marked by the red circle.



(d) The situation after shrinking the green ellipsoids by the same factor as the outer ellipsoid was shrunken in Figure 6.6(c).

Figure 6.6: Graphical representation of the outer ellipsoid approximate algorithm. Two ellipsoids are first covered by an outer ellipsoid, which is then shrunken to find the point U' in the feasible set $\mathcal{U}$. It can be directly seen, that the point U' is suboptimal, as the optimal solution would lie on the intersection of both green ellipsoids. Finally, it is showed, that after shrinking all ellipsoids by the same factor, the shrunken outer ellipsoid is no longer an outer ellipsoid for the shrunken ones.



(a) Continuous, but nonconvex dependence.

(b) Discontinuity in the dependence.

Figure 6.7: An example of dependence between δ and γ for a set of 5 ellipsoids.

convenient to transform the problem to a new system of coordinates, in which the constraint has the form

$$U^T U \leq 1, \tag{6.24}$$

i.e. the maximization of T_0 takes place on a unit ball. This can be always done using an appropriate affine transformation. To avoid introducing new variables, let us simply assume the constraint in the form (6.24), keeping in mind, that the function T_0 must also change accordingly after the transformation.

Note that the maximum of a quadratic function with a positive definite matrix always lies on the boundary of the feasible set. Therefore we can even assume the constraint to be

$$U^T U = 1, \tag{6.25}$$

which is convenient in further considerations. Let us now define the function

$$f(U, \alpha) = -\alpha(U^T A_0 U + 2U^T b_0) + U^T U, \tag{6.26}$$

where

$$\alpha \in \left(0, \frac{1}{\lambda_{max}(A_0)}\right).$$

Such alpha ensures positive definiteness of the function f and the unconstrained minimum is

$$U'(\alpha) = (I - \alpha A_0)^{-1} \alpha \cdot b_0. \tag{6.27}$$

We can now find the solution of the task (6.22) by finding such α, for which the unconstrained minimum (6.27) satisfies

$$U'(\alpha)^T U'(\alpha) = 1,$$

which can be found by any appropriate one-dimensional optimization method, such as interval bisection or Newton method. The maximum value is then

$$\delta = T_0(U')$$

and can be generally both positive and negative, depending on the original size of the ellipsoid given by T_0.

6.5.3 Algorithm

The inputs to the algorithm are the system matrices $\hat{A}, \hat{B}, \hat{C}$ and $\hat{D}$, the MPC parameter ρ, the admissible criterion perturbance ΔJ, the initial condition x_1, the control horizon N, the excitation horizon M (greater or equal to the number of system parameters) and the precision ε. The outputs is the optimal control sequence $U*$ and the information gain γ. The algorithm starts with an empty set of vectors $\mathcal{V}$, $k = 1$, $\delta = 0$ and $\gamma = 0$.

1. Solve the original MPC task (5.11)
2. Choose an initial point U' as the solution of the MPC task in step 1, find the corresponding vector $v_1 = v(U')$ and add it to the (empty) set $\mathcal{V}$.
3. For all vectors v_i from the set $\mathcal{V}$ create an ellipsoid representation (6.20) using the current value γ.
4. Repeat iterations by interval bisection in γ:
   (a) Check condition (6.17) for all ellipsoids in set $\mathcal{V}$. Find the outer ellipsoid by solving (6.21), considering only those ellipsoids satisfying (6.17).
   (b) Find U' and δ by solving the task (6.22).
   (c) Update γ according to δ and adjust the ellipsoid representation (6.20) according to the new γ.

   until $|\delta| < \varepsilon$.
5. Set $k = k + 1$.
6. Take the input U' from the last iteration of solving (6.22) in step 4 and find the corresponding vector $v_k = v(U')$.
7. If $v_k \notin \mathcal{V}$, then add v_k to $\mathcal{V}$ and go to step 3.
8. Set $U^* = U'$ and finish.

Note 6.8. Similarly to the original algorithm, an initial information matrix P_0 may be considered in the algorithm. This would lead to a modification of c_i in the representation (6.20) to

$$c_i = x_1^T W_1(v_i) x_1 - \gamma + v_i^T P_0^{-1} v_i.$$

The additional term $v_i^T P_0^{-1} v_i$ is constant for a fixed v_i and thus only changed the size of the associated ellipsoid, not its shape.

6.5.4 Properties of the algorithm

The main structure of the algorithm is the same as for the original algorithm presented in Section 6.1. In each iteration of the main loop a control perturbation is found and consequently such direction in the parameter space, that is least excited by this control perturbation. The proposed approximation takes place in the subtask of finding the control perturbation for each step, i.e. in step 4 of the algorithm above.

The advantage of the outer ellipsoid approximation is the transformation of the multidimensional nonconvex optimization problem into a sequence of iterations, each iteration involving one multidimensional convex problem (finding the outer ellipsoid) and one one-dimensional convex problem (maximizing the quadratic function given by the outer ellipsoid within the feasible set). Although the number of iterations may be large, the transformation makes the problem feasible more or less independent of the problem dimension. Finding the control perturbation U' in step 4 is in fact still a nonsmooth optimization problem in variable γ. An example of a continuous nonconvex dependence of δ on γ is shown in Figure 6.7(a), another example of even discontinuous dependence is in Figure 6.7(b). The dependence is nonsmooth, because it may happen that the condition (6.17) is satisfied for one ellipsoid at some point γ_0, but is not satisfied for all $\gamma < \gamma_0$. In such case, the task is qualitatively changed at the point γ_0, because this ellipsoid is added to (removed from) the set (6.14) and thus the shape and position of the outer ellipsoid may change stepwise.

A disadvantage of the approximation is that there is currently no usable upper bound for the optimal excitation. In other words, there is no reasonable estimate, of how suboptimal we are at the current point.

Chapter 7

Conclusions

The goal of this thesis was to analyze existing stochastically optimal control strategies and to develop new active adaptive strategies for linear discrete-time systems with uncertain parameters as computationally feasible approximations of dual control. Bayesian methods were chosen as a framework for describing time-domain system models as well as different stochastically optimal control strategies. The bayesian approach helped us understand the problems connected with stochastic control in a systematic way and served as a basis for the consequent considerations. It was particularly important to understand the correct interpretation of probabilistic description of uncertainty in different cases, both as a description of i.i.d. random variables (e.g. noise) and as a description of a progressively accumulating subjective knowledge about a constant unknown parameter. Our first results in the area of stochastic control were published in [48].

After the introductory chapter, the thesis starts with the first result, which is a novel derivation of the cautious LQ control strategy for the ARMAX model with uncertain parameters and known MA part and a novel derivation of a simultaneous parameter tracker and state estimator for this system. The derivation of the control strategy uses a specific state-space representation that leads to simpler expressions containing only the first two moments of probability distributions that describe the uncertainty in the ARMAX model. These moments are determined by the simultaneous parameter tracker and state estimator, which forms a counterpart to the controller by supplying it with the necessary information. Unlike a general linear stochastic model with uncertain parameters, the special structure of the ARMAX model makes derivation of the controller and estimator possible. On the other hand, the ARMAX model is general enough to describe a variety of systems, in fact it is equivalent to the system state-space representation in the observer canonical form.

The derived cautious control strategy is calculated as a feedback control law that is an affine function of the current state estimate. The control law is calculated via a cautious Riccati-like equation that, unlike the standard Riccati equation of the deterministic LQ problem, does not always have a limit solution, if extended to infinite control horizon. The next part of the thesis presents a necessary and sufficient condition for the cautious Riccati-

like equation to converge to a limit solution. It also presents an analysis of the feedback control law in case of divergence of the equation. It is shown that the control law always converges to a limit solution. This convergence analysis is the newest result of the thesis and it is currently being prepared for publication.

The next result of the thesis are new active adaptive control algorithms, starting with a single-step ahead algorithm based on the cautious strategy and continues with multiple-step strategies based on perturbations of the certainty equivalence controller. The benefit of the multiple-step approach is discussed and showed by simulations. Also the inconvenience of cautious control is discussed, pointing out problems with stability and convergence of cautious control. These issues are also discussed in the publication [50]. Three multiple-step active adaptive strategies are then presented, all based on maximizing the minimum eigenvalue of the parameter estimate information matrix. The single step algorithm was published in [49], the multiple-step algorithms in [46]

Finally, an iterative ellipsoid algorithm is proposed that solves the problem of maximizing the lowest eigenvalue of the parameter estimate information matrix of an ARX system with uncertain parameters. Information matrix maximization corresponds to persistent excitation and thus effective system identification, so the algorithm gives a computationally feasible approximation of dual control. The method is based on a bicriterial approach, where in the first step, the optimal control is found using MPC problem formulation, and in the second step, such perturbation of the optimal control sequence is searched, that maximally increases the information matrix lowest eigenvalue. This ellipsoid algorithm is the fourth result of the thesis and was published in [47].

Simulations also show that the multiple step algorithms tend to excite the system uniformly even if there is no reference signal, which pays off in the future during the control process, when the reference signal changes. The ellipsoid algorithm was proved to converge to the optimal solution and to stabilize the nominal system. As the problem is nonconvex, finding a globally optimal solution with a given precision requires a large number of iterations. The nonconvexity of the problem makes the ellipsoid algorithm usable only for low-dimensional systems. This issue is addressed by proposing a conservative modification that transforms the problem to a one-dimensional one and makes it practically insensitive to problem dimension at the cost of suboptimality.

The goals of the thesis were accomplished by the described four main results. The last mentioned conservative modification of the ellipsoid algorithm can be a topic for future research. The thesis contains the basic idea, which should be at least tested on simulations and if succesful, it could be further used in practical applications. Theoretical issues such as suboptimality of the solution or discontinuities in the optimized function should also be addressed. The other proposed multiple-step algorithms can be also further studied, especially in terms of optimization techniques for their solution.

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Publications

Publications related to the thesis

Journals with impact factor

J. Rathouský, J. and V. Havlena. MPC-based approximate dual controller by information matrix maximization. *International Journal of Adaptive Control and Signal Processing*, 27(11):974–999, 2013.
Contribution: 70 %
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- E. Zacekova, S. Privara, M. Pcolka, Persistent excitation condition within the dual control framework. *Journal of Process Control*, 23(9):1270–1280, 2013.

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