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Contents

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1 Problem Formulation

Many important problems in computer vision as well as in other fields can be formulated using systems of polynomial equations. Examples of problems that require solving complex systems of non-linear polynomial equations are problems of estimating camera geometry, such as relative or absolute camera pose problems. These problems have a broad range of applications, e.g., in structure-from-motion and 3D reconstruction [1, 20, 35, 36], recognition [28, 29], video-based rendering [4], robotics, and augmented reality.

The problem of solving systems of non-linear polynomial equations is a very old problem with many different well-studied solution methods. In general, the methods can be divided into numerical and algebraic methods. In this thesis we focus on algebraic methods. We review two classes of standard algebraic methods, i.e. the Gröbner basis and the resultant based methods. These methods are very useful mathematical methods but since they are general, i.e. they were developed for general systems of polynomial equations, they are usually not sufficiently efficient for systems which appear in computer vision problems.

It is because problems like estimating relative or absolute pose of a camera are usually parts of some large systems, e.g., structure-from-motion pipelines or recognition systems, which require high or even real-time performance. Moreover, the input measurements used for the computation are often contaminated with a large errors. Therefore, these problems have to be solved for many different inputs to find the "best solution", i.e. the solution consistent with as many measurements as possible, e.g., when using in RANSAC-based algorithms [16].

This means that computer vision problems usually require very fast, efficient, and numerically stable solvers which are able to solve many instances of one problem, i.e. many systems of polynomial equations of "one form" only with different "non-degenerate" coefficients, in milliseconds. Unfortunately, this requirement usually cannot be fulfilled by using standard general methods for solving systems of polynomial equations.

Therefore, in recent years various specific algorithms based on algebraic geometry concepts have been proposed to achieve numerical robustness and computational efficiency when solving computer vision problems. The main property of these algorithms is that they use a specific structure of a system of polynomial equations arising from a particular problem to efficiently solve this problem only.

Recently, many efficient specific algorithms for solving various computer vision problems have been proposed [32, 40, 37, 41, 42, 38, 33, 8, 31, 30, 8, 9, 7, 21]. They are mostly based on standard algebraic methods and designed manually for a particular problem. This manual design usually requires

deeper knowledge of algebraic geometry and considerable amount of craft, which makes the process of generation of these specific solvers complex and virtually impenetrable for a non-specialist. Moreover, for some of these problems it is not clear how their solvers were created and therefore non-specialists often use them as black boxes; they are not able to reimplement them, improve them, or create similar solvers for their own new problems.

2 Contributions

This thesis focuses on algebraic methods for solving systems of polynomial equations appearing especially in computer vision problems. Its main goal is to improve algebraic based methods previously used to manually create efficient solvers to some computer vision problems, automate these methods, and apply them to efficiently solve previously unsolved minimal computer vision problems.

The contribution of the proposed thesis can be divided into three main groups:

- **Algebraic methods for solving systems of polynomial equations**
  The first group of contributions of this thesis are modifications of two standard algebraic techniques for solving systems of polynomial equations; the Gröbner basis and the resultant based techniques. These modifications enable the creation of efficient specific solvers for particular problems, i.e. particular systems of polynomial equations.

  The main difference between the proposed specialized methods and general methods is that the specialized methods use the structure of the system of polynomial equations representing a particular problem to design a specific efficient solver for this problem.

  These specialized methods consist of two phases. In the first phase some preprocessing and computations common to all considered instances of the given problem are performed and an efficient specific solver is constructed. For a particular problem this phase needs to be performed only once.

  In the second phase, the efficient specific solver is used to solve concrete instances of the particular problem. This specific efficient solver is not general and solves only systems of polynomial equations of one form, i.e. systems which coefficients always generate the same "path" to the solution. However, the specific solver is faster than a general solver and suitable for applications which appear in computer vision.

In the case of the specialized Gröbner basis method we summarize and extend the Gröbner basis method used to manually create solvers to some previously solved computer vision problems [37, 38, 41, 40]. Thanks to the proposed extensions, e.g., the identification of the form of necessary polynomials, new strategies for generating these polynomials, and new procedures for removing unnecessary polynomials, we are able to create smaller, more efficient, and more stable solvers than the previously manually created solvers [37, 38, 41, 40]. Moreover, all these extensions can be easily automated and therefore the presented specialized Gröbner basis method can be used even by non-experts to solve technical problems leading to systems of polynomial equations.

The second proposed specialized method is based on hidden variable resultants and polynomial eigenvalue problems [3]. In this thesis we propose several strategies for transforming the initial system of polynomial equations to polynomial eigenvalue problem [3] and for reducing the size of this problem. In this way we are able to create efficient and numerically stable specific solvers for many problems appearing in computer vision. Again, this method can be easily automated.

- **Automatic generator of Gröbner basis solvers**
  Since the presented specialized Gröbner basis method requires non-trivial knowledge of algebraic geometry from its user, we propose an automatic method for generating Gröbner basis solvers which could be used even by non-experts to easily solve problems leading to systems of polynomial equations. The input to our solver generator is a system of polynomial equations with a finite number of solutions. The output of our solver generator is a Matlab code that computes solutions to this system for arbitrary "non-degenerate" coefficients. Generating solvers automatically open possibilities for solving more complicated problems that could not be handled manually or solving existing problems in a better and more efficient way. We demonstrate that our automatic generator constructs efficient and numerically stable solvers that are comparable or better than known manually constructed solvers in terms of computational time, space requirements, and numerical stability.

- **Solutions to minimal problems in computer vision**
  To demonstrate the usefulness of the proposed specialized techniques for solving systems of polynomial equations and the usefulness of our automatic generator of Gröbner basis solvers, we provide efficient solutions to several important relative pose problems. In this thesis we describe

solutions to five minimal problems which haven't been solved before:

1. the problem of simultaneous estimation of the fundamental matrix and a common radial distortion parameter for two uncalibrated cameras from eight image point correspondences,

2. the problem of simultaneous estimation of the essential matrix and a common radial distortion parameter for two partially calibrated cameras and six image point correspondences,

3. the problem of simultaneous estimation of the fundamental matrix and two radial distortion parameters for two different uncalibrated cameras from nine image point correspondences,

4. the 6-point relative pose problem for one fully calibrated and one up to focal length calibrated camera, and

5. the problem of estimating epipolar geometry and unknown focal length from images of four points lying on a plane and one off-the-plane point, i.e. the "plane + parallax" problem for cameras with unknown focal length.

Our solutions to these five new minimal problems are based on the specialized Gröbner basis method and are generated using our automatic generator of Gröbner basis solvers. Moreover, for the "8-point radial distortion problem", the "6-point problem for one fully calibrated and one up to focal length calibrated camera" and for the "plane + paralax + focal length" problem, we also propose the "polynomial eigenvalue solutions" based on the specialized resultant based method.

Beside these solutions to previously unsolved minimal problems, we provide new solutions to two well known important relative pose problems:

6. the 5-point relative pose problem for two calibrated cameras, and

7. the 6-point relative pose problem for two cameras with unknown equal focal length.

We show that these two problems lead to polynomial equations that can be solved robustly and efficiently as cubic and quadratic eigenvalue problems. These new solutions are fast, and in the case of the 6-pt solver, also slightly more stable than the existing solution [40]. In the case of the "6-point equal focal length problem", we also propose a new Gröbner basis solution generated by our automatic generator and we show that this solution is slightly more stable and also faster then previous Gröbner basis solutions to this problem [40, 8].

3 State of the Art

Solving systems of polynomial equations

Solving systems of polynomial equations is a classical problem with many applications, e.g., in computer vision, robotics, computer graphics and geometric modeling. This problem has its origin in ancient Greece and China. Therefore it is not surprising that there exists a large number of methods for solving systems of polynomial equations.

We can divide them according to several criteria. One possible and very simple division is the division into numerical and symbolic methods. In this thesis we focus on symbolic (algebraic) methods.

The main idea of symbolic methods for solving systems of polynomial equations is to eliminate variables from the system, and in this way, to reduce the problem to finding the roots of univariate polynomials. These methods have their origins in algebraic geometry and are therefore sometimes called algebraic methods or symbolic elimination methods.

In general, the algorithms for symbolic methods are efficient for smaller systems. For bigger systems, most of the general algorithms based on algebraic methods suffer from accuracy or efficiency problems. It is because these algorithms often require exact or multiple-precision arithmetic, which slows them down considerably.

The symbolic methods may be divided into:

1. resultant methods,

2. Gröbner bases, and

3. Ritt-Wu's methods.

Since these symbolic methods are well studied mathematical methods there are many excellent books and papers available on this topic including theory and applications. An overview of classical symbolic methods can be found in [23, 10, 11]. [34] is a nice survey of these methods with examples of their applications in computer vision.

Implementations of Gröbner bases and resultants algorithms and many algorithms based on their applications are contained in all of the current mathematical software systems like Mathematica, Maple, Magma, Axiom, Derive, Reduce, etc. Also, special software systems exist that are mainly based on the Gröbner bases technique, for example, CoCoA [2], Macaulay [19], Singular [13].

Minimal problems

Many problems in computer vision, especially problems of computing camera geometry, can be formulated using systems of polynomial equations. Such systems of polynomial equations can have an infinite number of solutions, i.e. are under-determined, no solution, i.e. are over-determined, or these systems can have a finite number of solutions.

In the case of problems of computing camera geometry, the number of equations in the system and the corresponding number of solutions depend on the number of geometric constraints and the number of input data (usually 2D-2D, 2D-3D or 3D-3D point or line correspondences) used to formulate the problem. The problems solved from a minimal number of input data and using all the possible geometric constrains that lead to a finite number of solutions are often called "minimal problems" or "minimal cases".

Various minimal problems have been recently studied extensively in computer vision [43, 32, 40, 37, 41, 42, 38, 33, 8, 31, 30]. It is because a smaller number of input data is less likely to contain incorrect inputs and therefore considerably reduces the number of samples needed in RANSAC-based algorithms [16] , which are widely used in many applications.

In this thesis we add several new minimal problem solutions to the family of recently solved minimal problems [16, 32, 33, 33, 40, 37, 18]. All these problems lead to systems of polynomial equations. For some problems, like the perspective three point problem, these systems are not so complicated and the solution to them is known for years and can be found using some manipulations and tricks in a closed form [16]. However, more often these problems lead to very complex systems of non-linear polynomial equations and numerical or symbolic methods based on resultants or Gröbner bases need to be used to solve them.

Macaulay's resultant was, for example, used in [43] to solve the problem of estimating absolute pose of a camera with unknown focal length from four 2D-to-3D correspondences and the problem of estimating absolute pose of a camera with unknown focal length and unknown principal point from five 2D-to-3D correspondences. Sparse resultants were used to solve the well know five point relative pose problem [14]. This five point problem [31], as well as the six point relative pose problem for camera with unknown focal length [30], were solved also using the hidden variable resultant based method [10].

Unfortunately, standard symbolic methods for solving general systems of polynomial equations, like the Buchberger's algorithm for computing Gröbner bases [10], or the previously described standard resultant based solutions [43, 14, 31, 30] may be very inefficient when solving minimal problems in computer

vision.

It is because minimal problems are usually parts of some large systems which require high or even real-time performance, e.g., structure-from-motion pipelines or recognition systems, and due to incorrect inputs and noise need to be solved for many different inputs in RANSAC-like algorithms [16] in milliseconds.

Thus, in recent years, various specific algorithms based on algebraic geometry concepts have been proposed focusing on numerical robustness and computational efficiency when solving minimal problems. The main property of these algorithms is that they use specific properties of a system of polynomial equations arising from a particular problem to efficiently solve this problem only.

Stewénius [37] proposed a method for constructing solvers for problems leading to systems of polynomial equations based on Gröbner bases and multiplication matrices. The resulting solvers are based on facts that in a class of problems the "path" to the Gröbner basis is always the same and that algorithms computing Gröbner bases can be realized using Gauss-Jordan elimination [15]. Based on these facts effective and practical algorithms for solving minimal problems can be designed. Using this method Stewénius et al. solved problems such as the five point relative pose problem [38], the six point relative pose problem for a camera with unknown constant focal length [40], the six point generalized camera problem [41], the nine point problem for estimating para-catadioptric fundamental matrices [18], the minimal problem for infinitesimal camera motion [39] as well as some other minimal problems [37].

A similar Gröbner basis method was used by other authors to solve the 3-point problem for panorama stitching for camera with unknown focal length and radial distortion [7], the absolute pose problem for camera with unknown focal length and radial distortion [21] and the 3-point relative pose problem for camera with known vertical direction [22].

Since for larger systems numerical problems of Gröbner basis solvers may appear [7], various techniques for improving the numerical stability of such solvers in computer vision have been proposed in [8, 9, 6].

4 Specialized algebraic methods for solving systems of polynomial equations

In this thesis we suggest modifications to two general algebraic methods for solving systems of polynomial equations; the Gröbner basis and the resultant based methods, which are suitable for many computer vision problems.

The main difference between the proposed specialized methods and general methods is that the specialized methods use the structure of the system of polynomial equations representing a particular problem to design a specific efficient solver for this problem.

These specialized methods consist of two phases. In the first phase some preprocessing and computations common to all considered instances of the given problem are performed and an efficient specific solver is constructed. For a particular problem this phase needs to be performed only once, therefore, we will call it the "offline phase".

In the second "online phase", the efficient specific solver is used to solve concrete instances of the particular problem.

Specialized Gröbner basis method

Our specialized Gröbner basis method follows and extends ideas of the method proposed and used by Stewenius to manually create Gröbner basis solvers to some computer vision problems [37, 38, 41, 40].

The most important improvements are:

1. identification of the form of the polynomials necessary for constructing a multiplication matrix,

2. several strategies for generating polynomials from the ideal, and

3. new methods for removing unnecessary polynomials and monomials from the found "elimniation template".

Thanks to these improvements we are able to create efficient specific Gröbner basis solvers for particular problems which are usually smaller, more efficient, and more stable than the manually created ones. Moreover, our specialized Gröbner basis method can be easily automated and therefore used even by non-experts to solve technical problems leading to systems of polynomial equations. Next, we describe the basic steps of this specialized Gröbner basis method.

Assume a set S of instances of a particular problem which are all "in the form" of one system of polynomial equations $F = \{f_1 = f_2 = \cdots = f_m = 0\}$. A solver for solving instances belonging to the set S can be created in the following way:

Offline phase

1. Fix a monomial ordering $\succ$ (The graded reverse lexicographical ordering is often good).

2. Find the basis B of the quotient ring $A = \mathbb{C}[x_1, \ldots, x_n]/I$ as the basis which repeatedly appears for several different (usually random) choices of coefficients of input equations that all belong to the set S. Do computations in a suitably chosen finite prime field to speed them up and to avoid numerical problems.

3. Choose a strategy for generating polynomials from the ideal I generated by input polynomials $f_1, \ldots, f_m$.

4. For a suitably chosen monomial $\mathbf{x}^\beta$ and the selected strategy for generating polynomials from the ideal I, find a "path" from the initial polynomials $f_1, \ldots, f_m$ to polynomials q_i that are necessary for constructing the multiplication matrix $\mathtt{M}_{x^\beta}$. Do this by systematically generating higher order polynomials from the initial polynomials using the selected strategy. Again, do computations in a finite prime field and with some "non-degenerate" coefficients from S (usually random).

5. Remove "unnecessary" polynomials, i.e. polynomials that are not necessary for generating polynomials q_i, from all generated polynomials.

6. Detect "unnecessary monomials", i.e. monomials that do not affect the polynomials q_i.

7. The remaining polynomials, together with the way of how they should be eliminated, form the resulting "elimination template" for the "online" solver.

The final "online" solver for solving all systems of polynomial equations "in the form" of the "representing" system F consists of these steps:

Online phase

1. Using the "elimination template" found in the offline phase, generate all necessary polynomials q_i from the initial polynomials with coefficients from $\mathbb{C}$.

2. Construct the multiplication matrix $\mathtt{M}_{\mathbf{x}^\beta}$ from coefficients of polynomials q_i.

3. Find the solutions of the input system of polynomial equations by finding eigenvectors of the multiplication matrix $\mathtt{M}_{\mathbf{x}^\beta}$ or by computing roots of the characteristic polynomial [12, 14] of $\mathtt{M}_{\mathbf{x}^\beta}$.

Specialized resultant based method

The second proposed specialized method is the method based on the hidden variable resultants [10] and the polynomial eigenvalue problems (PEP) [3]. The main idea of this method is the transformation of the initial system of polynomial equations to the polynomial eigenvalue problem [3]

$$\left(\alpha^l \mathtt{C}_l + \alpha^{l-1} \mathtt{C}_{l-1} + \cdots + \alpha \mathtt{C}_1 + \mathtt{C}_0\right) \mathbf{v} = \mathbf{0}, \tag{1}$$

where $\mathbf{v}$ is a vector of monomials in all variables except for a selected variable α and $\mathtt{C}_j$'s are $n \times n$ coefficient matrices. This is done by "hiding" the variable α in the coefficient field and generating polynomials using the modified Macaulay's method for computing resultants. Therefore we also call this specialized resultant based method the polynomial eigenvalue method.

Similarly to the case of the specialized Gröbner basis method this resultant based method can be divided into the "offline" and the "online" phase.

To create efficient robust specific polynomial eigenvalue solvers we propose:

1. several strategies for transforming a system of polynomial equations to a PEP,

2. a method for removing unnecessary polynomials from the polynomial eigenvalue formulation of the problem, and

3. a method for removing zero eigenvalues from the polynomial eigenvalue formulation of the problem.

Thanks to these proposed strategies we are able to create efficient and numerically stable polynomial eigenvalue solvers to many computer vision problems.

5 Automatic generator of minimal problems solvers

In this thesis we propose an automatic generator of Gröbner basis solvers which is based on the proposed specialized Gröbner basis method. This automatic generator searches for an "elimination path" for a given system of polynomial equations F and using this path produces an efficient solver for all instances of the input problem that would lead to the same "elimination path", i.e. all systems of polynomial equations that are "in the form" of the input system F. It is a number of valid paths the generator can find. The final choice of the path is determined by the particular coefficients of F.
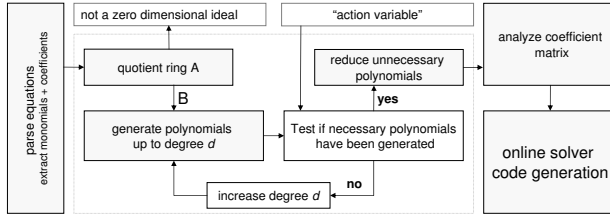
Figure 1: Block diagram of the basic moduls of the automatic generator.

The input to our automatic generator is a system of polynomial equations F that we want to solve, i.e. a system representing considered instances of a particular problem, with concrete coefficients from $\mathbb{Z}_p$ that determine the particular "elimination path". For many problems, the "interesting" solutions can be obtained with almost any, i.e. random, choice of coefficients of F. Therefore, we use random values from $\mathbb{Z}_p$ as default coefficients. The output of the generator is the Matlab or Maple code that returns solutions to all systems of polynomial equations "in the form" of the input system F, with specific coefficients from $\mathbb{C}$. During the online computations, only the resulting solver is called.

This automatic generator can be used even by non-experts to create solvers for new or existing problems and usually results in smaller, more efficient and more stable solvers than the manually created ones. Moreover, generating solvers automatically opens possibilities to solve more complicated problems which could not be handled manually.

Our automatic generator consists of several modules (Figure 1). Since all these modules are independent, they can be further improved or replaced by more efficient implementations.

6 Minimal problems

We demonstrate the usefulness of both the proposed specialized techniques and the automatic generator by providing new efficient solutions to several important relative pose problems, most of which were not solved before.

8-point radial distortion problem

Let us start with the problem is the problem of simultaneous estimation of the fundamental matrix and a common radial distortion parameter, given by the
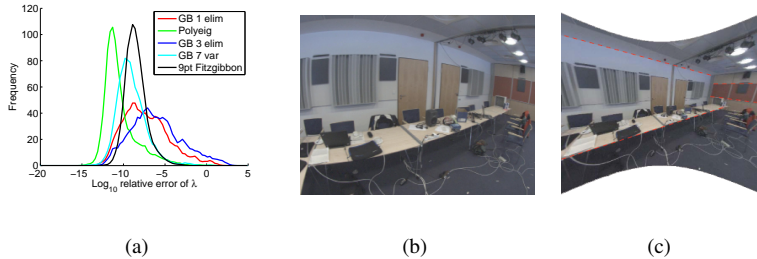
Figure 2: (a) Log_{10} relative error of the radial distortion parameter λ obtained by selecting the real root closest to the ground truth value $\lambda_{th} = -0.2$ for noise free synthetic scene (b) Input image with significant radial distoriton. (c) Corrected image using the presented 8-point algorithm.

division model [17], for two uncalibrated cameras from eight image point correspondences. This problem has 16 solutions and was previously solved only for nine image point correspondences [17].

We propose two formulations of this problem, the first resulting in three polynomial equations in three unknowns and the second in seven equations in seven unknowns, and we present three new minimal solutions to these formulations.

Gröbner basis solution of the "7 in 7" formulation. The first Gröbner basis solution is based on the multiple elimination strategy for generating polynomials from an ideal and even though it results in larger solver than the "3 in 3" formulation it has less critical configurations. The final solver consists of four Gauss-Jordan (G-J) eliminations of 49×119, 154×119, 106×126, and 108×126 matrices and eigenvalue computations of a 16×16 matrix.

Gröbner basis solution of the "3 in 3" formulation. The second Gröbner basis solution is based on the single elimination strategy for generating polynomials from an ideal and was generated using our automatic generator. This solver consists of one G-J elimination a 32×48 matrix and eigenvalue computations of a 16×16 matrix.

Polynomial eigenvalue solution of the "3 in 3" formulation. The polynomial eigenvalue solver for this problem needs to compute the inverse of a 10×10 matrix and eigenvalues of a 29×29 matrix.

The numerical stability of all of the three proposed solvers and the two pre-

viously published solvers [17, 8] can be seen in Figure 2 (a) and the results on real-word photographs with significant radial distortion in Figure 2 (b) and (c).

9-point different radial distortion problem

The second problem that we have solved is the problem of simultaneous estimation of the fundamental matrix and two radial distortion parameters for two uncalibrated cameras with different distortions from nine image point correspondences. This minimal problem has 24 solutions and hasn't been solved before.

The Gröbner basis solver for this problem generated using the proposed automatic generator needs to perform one G-J elimination of a 179×203 matrix and to compute eigenvalues of a 24×24 matrix. This solver is smaller than our previous manually created solver [3] which needs to perform LU decomposition of a 393×390 matrix and eigenvalue computations of a 24×24 matrix.

6-point calibrated radial distortion problem

The third radial distortion problem which we have solved is the problem of simultaneous estimation of the essential matrix and a common radial distortion parameter for two partially calibrated cameras and six image point correspondences. This problem results in a quite complicated system of equations with 52 solutions.

The Gröbner basis solver generated using the proposed automatic generator needs to perform one G-J elimination of a 238×290 and to compute eigenvalues of a 52×52 matrix and is smaller than our previously published manually created solver [3].

5-point relative problem

One of the most important relative pose problems is the 5-point relative pose problem for two calibrated cameras.

In this thesis we show that this problem leads to polynomial equations that can be solved robustly and efficiently as a cubic eigenvalue problem. Moreover, our Gröbner basis solver which uses Danilevskii method [12] for computing the characteristic polynomial is almost as fast as specific manually created and optimized "closed-form" state-of-the-art solution [32] and outperforms the state-ot-the-art Gröbner basis solution [38].
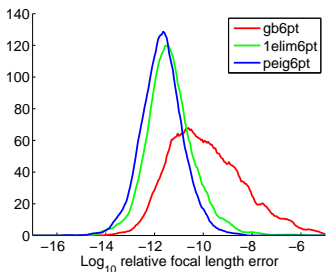
Figure 3: The log_{10} relative focal length error for the 6-point equal focal length problem. Here gb6pt denotes Stewenius Gröbner basis solver [40], 1elim6pt - the Gröbner basis solution with single elimination presented in our thesis and peig6pt our polynomial eigenvalue solver.

6-point equal focal length problem

In the case of the 6-point relative pose problem for two cameras with unknown equal focal length we propose two new solutions.

The Gröbner basis solver generated using our automatic generator needs to perform G-J elimination of a 31×46 matrix and to compute eigenvalues of a 15×15 matrix.

The polynomial eigenvalue solver needs to compute the inverse of one 10×10 matrix and then to compute eigenvalues of a 20×20 matrix.

Both these new solutions are more stable and faster then previous Gröbner basis solutions to this problem [40, 8]. Using the Danilevskii method [12] for computing the characteristic polynomial our Gröbner basis solution gains more than $8 \times$ speed-up over the fastest available Gröbner basis solutions to this problem [40, 8, 6].

Figure 3 shows the numerical stability of the both proposed solvers, the polynomial eigenvalue solver (Blue), and the Gröbner basis solver (Green), compared to the state-of-the-art Gröbner basis solver [40].

6-point one calibrated camera problem

We also propose new efficient and robust minimal solutions to configuration with one completely calibrated camera and one camera with unknown focal length. We show that this solution can cope with the most unpleasant degeneracies and can be effectively used to reconstruct 3D scenes from collections of
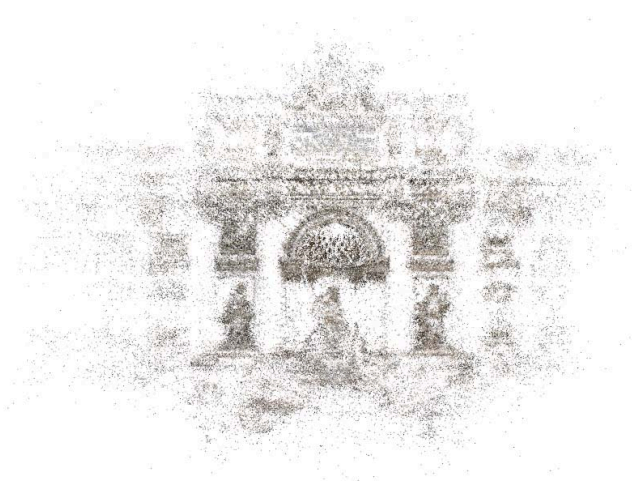
Figure 4: A 3D reconstruction of the Trevi Fountain using our 6-point one calibrated camera problem.

images with a very few (in principle single) images with known focal length(s), e.g., unordered data sets downloaded from the Internet, see Figure 4.

This minimal problem hasn't been solved before and has 9 solutions. Both our minimal solvers for this problem are very simple and efficient.

Our Gröbner basis solver generated using the automatic generator has to perform a single G-J elimination of a 21×30 matrix and to compute eigenvalues of a 9×9 matrix.

The generalized eigenvalue solver for this problem does not perform any elimination; however, it calculates generalized eigenvectors. This involves computation of the inverse of one 10×10 matrix and then eigenvalues of a matrix of the same size.

Plane+parallax problem for cameras with unknown focal length

The final relative pose problem which we have solved is the minimal problem of estimating epipolar geometry and unknown focal length from images of four points lying on a plane and one off-the-plane point. This problem is known as the plane + parallax problem for cameras with unknown focal length and has not been solved before. Such problem is important in situations where a

dominant plane is present in the scene, such as in 3D reconstructions of cities or other man-made objects.

We have once again proposed two efficient robust solutions to this problem. The Gröbner basis solution generated by our automatic generator has to perform a single G-J elimination of a 10×15 matrix and to compute eigenvalues of a 5×5 matrix and the polynomial eigenvalue solution has to compute the inverse of one 4×4 matrix and the eigenvalues of a 8×8 matrix.

7 Conclusion

In this thesis we suggest modifications of two standard algebraic techniques for solving systems of polynomial equations, i.e. the Gröbner basis and the resultant based technique, that are suitable for many computer vision problems.

The proposed specialized methods use the specific structure of a system of polynomial equations representing a particular problem to create an efficient specific solver for solving this problem. Such solver is not general and solves only systems of polynomial equations of one form, i.e. systems "in the form" of the representing system; however, it is faster than general solvers and suitable for applications which appear in computer vision and robotics.

Both presented specialized methods can be easily automated and therefore used even by non-experts to solve technical problems lading to systems of polynomial equations. In this thesis we propose the automatic generator of such efficient specific solvers based on the specialized Gröbner basis method.

We demonstrate the usefulness of the proposed specialized techniques and the automatic generator by providing new efficient and numerical stable solutions to several important relative pose problems, most of which were not solved before. These problems include problems of estimating relative pose and internal parameters of calibrated, partially calibrated (with unknown focal length), or completely uncalibrated perspective or radially distorted cameras observing general scenes or scenes with dominant plane. All these problems can be efficiently used in many applications such as camera localization, structure-from-motion, scene reconstruction, tracking and recognition. The quality of all presented solvers is demonstrated on synthetic and real data.

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Resumé in Czech

Počítačové vidění vyžaduje schopnost efektivně řešit systémy polynomiálních rovnic. Například určování relativní nebo absolutní polohy kamery, lze formulovat jako minimální problémy, tedy je lze řešit z minimálního počtu vstupních dat. Minimální problémy vedou na systémy polynomiálních rovnic s konečným počtem řešení.

Systémy vycházející z minimálních problémů jsou často komplikované a obecné algoritmy k řešení systémů polynomiálních rovnic pro ně vedou na relativně neefektivní řešení. Proto je pro řešení těchto problémů obvykle zapotřebí navrhnout numericky robustní a výpočetně efektivní specifické algoritmy.

V této disertaci navrhujeme modifikace dvou standartních algebraických technik pro řešení systémů polynomialních rovnic a to metody založené na Gröbnerových bázích a metody založené na rezultantech, které jsou vhodné právě pro efektivní řešení mnoha problémů v počítačovém vidění a jiných oblastech.

Základní rozdíl mezi prezentovanými specializovanými metodami a standartními obecnými metodami je, že prezentované specializované metody využívají znalosti struktury systému polynomiálních rovnic, který reprezentuje konkrétní problém k návrhu efektivního a stabilního algoritmu na řešení tohoto problému. Při tomto návrhu se část výpočtů společná pro všechny instance daného problému připraví předem, což ušetří čas při opakovaném řešení systémů s identickou strukturou.

Takto vytvořený algoritmus není obecným algoritmem a řeší jen systémy polynomiálních rovnic jednoho tvaru, avšak je rychlejší než obecné algoritmy a proto je vhodný pro aplikace, které se objevují například v počítačovém vidění.

Obě navržené specializované metody mohou být snadno zautomatizovány a takto používány i neodborníky k řešení problémů vedoucích na systémy polynomiálnich rovnic. V této disertaci prezentujeme automatický generátor efektivních algoritmů založený na modifikované metodě Gröbnerových bázi.

Jako ukázku užitečnosti obou navržených metod a našeho automatického generátoru v této disertaci prezentujeme nová efektivní a numericky stabilní řešení několika velmi důležitých problémů určování relativní polohy kamer. Většina těchto problémů nebyla v minulosti vyřešena. Mezi těmito problémy jsou problémy určování relativní polohy a kalibračních parametrů kalibrovaných, částečně kalibrovaných (s neznámou ohniskovou vzdáleností) nebo kompletně nekalibrovaných perspektivních kamer či kamer s radiálním zkreslením snímajících obecnou scénu nebo scénu s dominující rovinou.

Všechny tyto algoritmy mohou být efektivně použity v aplikacích jako je lokalizace, rekonstrukce 3D scény či rozpoznávání. Kvalita prezentovaných algoritmů je demonstrována experimenty na syntetických i reálných datech.

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