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THE SUPERALGEBRA APPROACH TO STUDY OF SUBSPACES OF SKEW-SYMMETRIC ELEMENTS IN FREE ALGEBRAS

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1 Current Situation Of the Studied Problem

For power-associative algebras and a positive integer n, let $\mathcal{N}il_n$ be the variety of *nil-algebras* of nil-index n defined by the identity $x^n = 0$. The classical Dubnov-Ivanov-Nagata-Higman theorem (see [2, 5], 1943, 1956) states that in characteristic zero every associative nil-algebra of nil-index n is nilpotent of index less or equal $2^n - 1$, but the exact estimate of the nilpotency index of associative $\mathcal{N}il_n$-algebras is unknown. A weaker condition than the associativity for an algebra is the alternativity. It turns out that, in contrast to the associative case, alternative nil-algebras of bounded index can be non-nilpotent, that is, the Dubnov-Ivanov-Nagata-Higman theorem does not carry over to alternative algebras. This was proved by Dorofeev (see [1], 1960).

One way to explore the alternative nil-algebras is the usage of the superalgebras. The superalgebras were successfully used for the study of identities of free algebras and for the development of a structure theory of varieties of algebras. The method, which arose from that, is called the superalgebra technique. Due to the work of Shestakov (see [20], 1999) and Vaughan-Lee (see [30], 1998), the problem of a study of identities of free $\mathcal{V}$-algebras can be solved with the free $\mathcal{V}$-superalgebras. This correspondence can be used to describe the subspace of skew-symmetric elements of the free alternative nil-algebra on a countable set of generators using the free alternative nil-superalgebra on one generator, which is easier to deal with. Recall, that no base of the free alternative nil-algebra is known.

2 Aims of the Doctoral Thesis

The aim of this work is the usage of the superalgebra technique in the study of free algebras. In this wide matters it is focused on the free alternative nil-algebras and on obtaining some corollaries for solvable and nilpotent alternative algebras.

As a first step, the finding of a base of the free alternative nil-superalgebra on one odd generator of different nil-index $n \geq 2$ is described. The index of solvability of this superalgebra is found and it is confirmed for nil-index $n > 2$ that the superalgebra is not nilpotent.

As an application, the subspace of skew-symmetric elements of the free alternative nil-algebras is described. Further, Grassmann algebra in the variety $\mathcal{A}lt$-$\mathcal{N}il_3$ which generalize Dorofeev's example of solvable non-nilpotent alternative algebra (see [1], 1960) is presented. Another application is a construction of an infinite family of solvable alternative nil-algebras of arbitrary big solvability index, using a standard passage to Grassmann envelope over a field of characteristic zero. It should be noted that the research is difficult due to nonassociativity of studied objects.

3 Working Methods

Let $\mathbb{F}$ be a field (or an associative commutative ring with unity). An *algebra* A is a vector space (or a unitary module) endowed with a bilinear multiplication $\cdot : A \cdot A \to A$ that is, distributivity holds for all $x; y; z \in A$, $\alpha, \beta \in \mathbb{F}$:

$$(\alpha x + \beta y)z = \alpha(xz) + \beta(yz)$$
$$x(\alpha y + \beta z) = \alpha(xy) + \beta(xz).$$

Notice that we do not assume associativity or commutativity of multiplication, and existence of unity. *Grassmann algebra* G is an associative algebra on generators $e_1, e_2, \dots e_n, \dots$ which are subject to the relations $e_i^2 = 0$, $e_i e_j = -e_j e_i$. This algebra is neither commutative nor anticommutative. A base of G consists of all monomials $e_{i_1} e_{i_2} \cdots e_{i_n}$ with $i_1 < i_2 < \cdots < i_n$.

A *variety* of algebras is a class of algebras satisfying certain identities. Any *identity* can be seen as an element of the free non-associative algebra (without unity), that is, a non-associative polynomial (without absolute term). A polynomial is called homogeneous of degree n if all its monomials with nonzero coefficients are of the same degree n (we assume that each variable has degree 1). A homogeneous polynomial is called multilinear if it is linear in any of its variable (multihomogeneous of multidegree $(1; 1; 1; \dots; 1)$). A multilinear polynomial $f(x_1, x_2, \dots, x_n)$ is called skew-symmetric if

$$f(x_1, x_2, \dots, x_n) = sgn(\pi) f(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(n)}), \quad \pi \in Sym(n).$$

In characteristic zero, any variety can be defined by multilinear identities. To use a superalgebra technique we will need characteristic zero.

In general, a *superalgebra* means a $\mathbb{Z}_2$-graded algebra, that is an algebra A which may be written as a direct sum of subspaces $A = A_0 + A_1$ subject to the relation

$$A_i A_j \subseteq A_{i+j \pmod 2}.$$

The subspaces A_0 and A_1 are called the even and the odd parts of the superalgebra A and so are called the elements from A_0 and from A_1 respectively. Below all the elements are assumed to be homogeneous, that is, either even or odd. For a homogeneous element $u \in A_i$, $i \in \{0, 1\}$, the symbol $\overline{u} = i$ means its parity.

Grassmann superalgebra $G = G_0 + G_1$ is the Grassmann algebra with the canonical $\mathbb{Z}_2$-grading: G_0 is spanned by the empty word and the products of even length, G_1 is spanned by the products of odd length. For a given variety $\mathcal{V}$ of algebras, superalgebra $A = A_0 + A_1$ is called a *$\mathcal{V}$-superalgebra* if its Grassmann envelope $G(A) = G_0 \otimes A_0 + G_1 \otimes A_1$ belongs to $\mathcal{V}$.

$\mathcal{V}$-superalgebra can be defined by *superidentities*. Notice that to pass from $\mathcal{V}$-algebras to $\mathcal{V}$-superalgebras one has to
1) find an equivalent system of multilinear identities for any identity $\mathcal{V}$,
2) apply to each multilinear identity so called "superization rule" (or "Kaplansky's principle") that whenever two odd variables are transposed a negative sign is introduced.

Proposition 3.1 *Let $\mathcal{V}$ be a variety of algebras. If B is an associative commutative algebra, then $B \otimes A \in \mathcal{V}$ for any $A \in \mathcal{V}$.* □

Corollary 3.2 *Let $\mathcal{V}$ be a variety of algebras defined by a set of multilinear identities and G be a Grassmann superalgebra. Then for any $A \in \mathcal{V}$ the tensor product $A_G = G \otimes A = G_0 \otimes A + G_1 \otimes A$ is a $\mathcal{V}$-superalgebra.* □

Passage to superscalar extension allow us to reduce the number of variables in the identities that are multilinear skew-symmetric polynomial.

Let $\mathcal{V}[T; X]$ denote the free $\mathcal{V}$-superalgebra over a field $\mathbb{F}$ generated by a set T of even generators and a set X of odd generators. We will also use $\mathcal{V}[T]$ for the free algebra $\mathcal{V}[T; \emptyset]$. Consider the free $\mathcal{V}$-superalgebra $\mathcal{V}[\emptyset; x]$ on one odd generator x, and the free $\mathcal{V}$-algebra $\mathcal{V}[T]$ on a set of even generators $T = \{t_1; t_2; \dots t_n; \dots\}$. *Skew* : $\mathcal{V}[\emptyset; x] \to \mathcal{V}[T]$ is a linear mapping which maps isomorphically the homogeneous component $\mathcal{V}[\emptyset; x]^{[n]}$ of degree n of $\mathcal{V}[\emptyset; x]$ to the subspace $\mathcal{V}[T_n]$ of multilinear skew-symmetric elements on $T_n = \{t_1, t_2, \dots, t_n\}$ of $\mathcal{V}[T]$. More exactly:

Theorem 3.3 *For a homogeneous polynomial f of degree n, the free $\mathcal{V}$-superalgebra $\mathcal{V}[\emptyset; x]$ on one odd generator x and the free $\mathcal{V}$-algebra $\mathcal{V}[T]$ on a set of even generators $T = \{t_1, t_2, \dots, t_n, \dots\}$ holds:*

$f(x) = 0$ *in* $\mathcal{V}[\emptyset; x]$ *if and only if* *Skew* $f(t_1, t_2, \dots, t_n) = 0$ *in* $\mathcal{V}[T]$. □
This correspondence was used in [24, 25] or in [27] for a description of the subspace of skew-symmetric elements of the free alternative algebra on a countable set of generators.

Skew-symmetric elements in alternative algebras

Let $\mathcal{A} = \mathcal{A}lt[\emptyset; x]$ be the free alternative superalgebra on one odd generator x. Define by induction

$$x^{[1]} = x,\ x^{[i+1]} = [x^{[i]}, x]_s,\ i > 0,$$

and denote
$$t = x^{[2]},\ z^{[k]} = [x^{[k]}, t],\ u^{[k]} = x^{[k]} \circ_s x^{[3]},\ k > 1.$$

The following propositions summarize some results from [24, 25, 27] on the structure of $\mathcal{A}$.

Proposition 3.4

(i) *The elements*

$$\begin{aligned} t^m x^\sigma,\ m + \sigma \geq 1, \quad & t^m(x^{[k+2]}x^\sigma), \\ t^m(u^{[4k+\varepsilon]}x^\sigma), \quad & t^m(z^{[4k+\varepsilon]}x^\sigma), \end{aligned} \tag{1}$$

where $k > 0$, $m \geq 0$ are integers; ε, $\sigma \in \{0, 1\}$, form a base of the superalgebra $\mathcal{A}$.

(ii) *For any integer $k > 0$,*

$$z^{[4k-1]} = z^{[4k-2]} = u^{[4k-1]} = 0, \quad u^{[2]} = 0, \quad u^{[4k+2]} = -tz^{[4k+1]}.$$

(iii) *The nucleus (associative center) of $\mathcal{A}$ is equal to the ideal*

$$id_{\mathcal{A}}\langle u^{[k]}, z^{[k]} \mid k > 1\rangle.$$

The center of $\mathcal{A}$ is equal to the vector space

$$vect\langle t^m z^{[k]},\ t^m(2z^{[k]}x - u^{[k]}) \mid m \geq 0,\ k > 2\rangle. \qquad \square$$

Let $Alt\,[T] = Alt\,[T; \emptyset]$ be the free alternative algebra on a set of even generators T and $Skew$ a mapping from the homogeneous component $\mathcal{A}^{[n]}$ of degree n of $\mathcal{A}$ to the subspace $Skew(Alt[T_n])$ of multilinear skew-symmetric elements on $T_n = \{t_1, t_2, \ldots, t_n\}$ of $Alt\,[T]$.

Theorem 3.5 ([27, Theorem 5.1])
The elements $Skew\, f(t_{i_1}, t_{i_2}, \ldots, t_{i_k})$, where $f = f(x)$ runs through the set (1), $k = \deg(f)$, $i_1 < i_2 < \cdots < i_k$, form a base of the space $Skew(Alt[T])$ of skew-symmetric elements of $Alt\,[T]$. $\square$

4 Results

We construct a base of the free alternative nil-superalgebra $\mathcal{B}_n = \mathcal{A}lt\text{-}\mathcal{N}il_n[\emptyset; x]$ of nil-index n, for $n \geq 2$, on one odd generator x. To construct a base of $\mathcal{B}_n$, for $n \geq 2$, we use the base of the free alternative superalgebra $\mathcal{A} = \mathcal{A}lt[\emptyset; x]$ on one odd generator x constructed in [27]. After that we compute the solvability index of $\mathcal{B}_n$ which is $\lceil \log_2 n \rceil + 1$ for $n \geq 2$. As a corollary we show for the nil-index $n \geq 3$ that $\mathcal{B}_n$ is not nilpotent and the square of $\mathcal{B}_n$ is nilpotent of index n. We start with nil-index $n = 2$, continue for $n = 3$ a then generalize it.

Let $\mathcal{A}$ be the free alternative superalgebra on one odd generator x with the base (1). Let $\mathcal{I}_n$ be a subsuperspace of $\mathcal{A}$ spanned by the elements $W_n(u_1, u_2, \ldots, u_n), u_1, \ldots, u_n \in \mathcal{A}_0 \cup \mathcal{A}_1$, where

$$W_n(u_1, u_2, \ldots, u_n) = \sum_{\sigma \in Sym(n)} \mathrm{sign}_{\mathrm{odd}}(\sigma)(\ldots((u_{\sigma(1)} u_{\sigma(2)}) u_{\sigma(3)}) \cdots) u_{\sigma(n)},$$

then $\mathcal{I}_n$ is an ideal of $\mathcal{A}$. The quotient superalgebra $\mathcal{A}/\mathcal{I}_n$ is the free alternative $\mathcal{N}il_n$-superalgebra on one odd generator x. We will denote it by $\mathcal{B}_n = \mathcal{B}_n[\emptyset; x]$ and we will maintain the notations from $\mathcal{A}$ for the similar elements of $\mathcal{B}_n$.

Base of the free alternative $\mathcal{N}il_2$-superalgebra on one odd generator

First we consider an ideal $\mathcal{I}_2 \subset \mathcal{A}$, spanned by the elements $W_2(u, v), u, v \in \mathcal{A}_0 \cup \mathcal{A}_1$, where

$$W_2(u, v) = uv + (-1)^{\overline{u}\,\overline{v}} vu, \tag{2}$$

and construct a base of $\mathcal{I}_2$. Using the multiplication table for $\mathcal{A}$, we obtain from

$$\begin{aligned} W_2(t, t) &= t \cdot t + t \cdot t = 2t^2, \\ W_2(t, x) &= tx + xt = tx + tx - x^{[3]} = 2tx - x^{[3]}, \\ W_2(x, tx) &= x(tx) - (tx)x = \tfrac{1}{2}t^2 - x^{[3]}x + \tfrac{2}{3}x^{[4]} - \tfrac{1}{2}t^2 - \tfrac{1}{3}x^{[4]} \\ &= \tfrac{1}{3}x^{[4]} - x^{[3]}x, \end{aligned}$$

that $t^2, 2tx - x^{[3]}, \frac{1}{3}x^{[4]} - x^{[3]}x \in \mathcal{I}_2$. From the definition of the elements $x^{[k]}, z^{[k]}, u^{[k]},\ k > 3$

$$\begin{aligned} x^{[k+1]} &= x^{[k]}x - (-1)^k x x^{[k]}, \\ z^{[k]} &= x^{[k]}t - t x^{[k]}, \\ u^{[k]} &= x^{[k]}x^{[3]} + (-1)^k x^{[3]} x^{[k]}, \end{aligned}$$

and using the fact that for every element $i \in \mathcal{I}_2$ also $i \cdot x, x \cdot i \in \mathcal{I}_2$, we get

$$x^{[k]}, x^{[k]}x, z^{[k]}, u^{[k]} \in \mathcal{I}_2,\ k > 3$$

Observe that for any other base elements u, v values of $W_2(u, v)$ do not bring new elements from $\mathcal{I}_2$.

Proposition 4.1 *The elements*

$$\begin{aligned} 2tx - x^{[3]}, t^{m+2}x^{\varepsilon}, \\ t^{m+1}x^{[3]}, t^{m}(x^{[3]}x), t^{m}(x^{[k+3]}x^{\varepsilon}), \\ t^{m}z^{[4k+\sigma]}x^{\varepsilon}, t^{m}u^{[4k+\sigma]}x^{\varepsilon}, \end{aligned} \tag{3}$$

where $k > 0,\ m \geq 0,\ \varepsilon, \sigma \in \{0, 1\}$, *form a base of* $\mathcal{I}_2$. □

Now the superalgebra $\mathcal{B}_2 = \mathcal{A}/\mathcal{I}_2$ is spanned by the elements

$$x,\ t,\ tx \tag{4}$$

all other elements from (1) are zero except $x^{[3]} = 2tx$. Moreover, the elements from (4) are linearly independent since any non-trivial linear combination of their preimages does not lie in $\mathcal{I}_2$.

Theorem 4.2 *The superalgebra* $\mathcal{B}_2$ *has a base (4).* □

Corollary 4.3 *The index of solvability of the superalgebra* $\mathcal{B}_2$ *is* 2. □

Corollary 4.4 *The superalgebra* $\mathcal{B}_2$ *is nilpotent of index* 4. □

Base of the free alternative $\mathcal{N}il_3$-superalgebra on one odd generator

As in case $n = 2$, we consider an ideal $\mathcal{I}_3 \subset \mathcal{A}$, spanned by the elements $W_3(u, v, w)$, $u, v, w \in \mathcal{A}_0 \cup \mathcal{A}_1$, where

$$\begin{aligned} W_3(u, v, w) =& (uv)w + (-1)^{\bar{u}(\bar{v}+\bar{w})}(vw)u + (-1)^{\bar{w}(\bar{u}+\bar{v})}(wu)v \\ &+ (-1)^{\bar{v}\bar{w}}(uw)v + (-1)^{\bar{u}\bar{v}}(vu)w + (-1)^{\bar{u}\bar{v}+\bar{u}\bar{w}+\bar{v}\bar{w}}(wv)u, \end{aligned}$$

and construct a base of $\mathcal{I}_3$. Using the multiplication table for $\mathcal{A}$, we obtain

$$\begin{aligned} W_3(x, t, t) &= 6t^3 \\ W_3(t, t, x^{[k]}) &= 6(t^2x^{[k]} + tz^{[k]}), \\ W_3(t, t, x^{[k]}x) &= 6(t^2(x^{[k]}x) - \tfrac{1}{2}tu^{[k]} + tz^{[k]}x + \tfrac{1}{2}tz^{[k+1]}), \\ W_3(x, t, z^{[k]}) &= 6(-1)^k tz^{[k]}x, \\ W_3(x, t, u^{[k]}) &= 6(-1)^{k+1}tu^{[k]}x + 2(-1)^k t^2z^{[k]}, \\ W_3(x, t, x^{[k]}) &= (-1)^k(6t(x^{[k]}x) - 3tx^{[k+1]} + \tfrac{1}{2}z^{[k+1]} - \tfrac{3}{2}u^{[k]} + 3z^{[k]}x), \\ W_3(x, t, x^{[k]}x) &= (-1)^k(tx^{[k+2]} - 3t(x^{[k+1]}x) + \tfrac{1}{2}u^{[k+1]} - z^{[k+1]}x - \tfrac{1}{2}z^{[k+2]}). \end{aligned}$$

Using the fact that for every element $W \in \mathcal{I}_3$ also $W \cdot x, x \cdot W \in \mathcal{I}_3$, we get all the elements of $\mathcal{I}_3$ and for any other base elements u, v, w values of $W_3(u, v, w)$ do not bring new elements from $\mathcal{I}_3$. Therefore we obtain a base of $\mathcal{I}_3$.

Proposition 4.5 *Elements*

$$\begin{array}{c} t^{m+3}x^{\sigma}, \\ tx^{[3]} - t^2x,\ tx^{[k+3]} + \frac{1}{2}z^{[k+3]} + \frac{1}{2}u^{[k+2]},\ \ t(x^{[k+2]}x) + \frac{1}{3}z^{[k+3]}, \\ t^{m+2}(x^{[k+2]}x^{\sigma}), \\ t^m(u^{[4k+\varepsilon]}x^{\sigma}), \qquad m+\sigma \geq 1, \\ t^m(z^{[4k+\varepsilon]}x^{\sigma}), \qquad m+\sigma \geq 1, \end{array} \tag{5}$$

where $k > 0$, $m \geq 0$; ε, $\sigma \in \{0, 1\}$, form a base of $\mathcal{I}_3$. □

Now the superalgebra $\mathcal{B}_3 = \mathcal{A}/\mathcal{I}_3$ is spanned by the elements

$$\begin{array}{l} x,\ t,\ tx,\ t^2,\ t^2x,\ x^{[k]},\ x^{[k]}x,\ k > 2, \\ u^{[4m+\varepsilon]},\ z^{[4m+\varepsilon]},\ m > 0,\ \varepsilon \in \{0, 1\}. \end{array} \tag{6}$$

All other elements from (1) are equal zero, except

$$\begin{aligned} tx^{[3]} &= t^2x, \\ tx^{[k+1]} &= -\tfrac{1}{2}(u^{[k]} + z^{[k+1]}), \\ t(x^{[k]}x) &= -\tfrac{1}{3}z^{[k+1]}. \end{aligned}$$

Moreover, the elements from (6) are linearly independent since any non-trivial linear combination of their preimages does not lie in $\mathcal{I}_3$.

Theorem 4.6 *The elements from (6) form a base of the superalgebra $\mathcal{B}_3$.* □

Corollary 4.7 *The index of solvability of $\mathcal{B}_3$ is* 3. □

Corollary 4.8 *$\mathcal{B}_3$ is not nilpotent, moreover $(\mathcal{B}_3)^2$ is nilpotent of index 3 and $(\mathcal{B}_3)^m \cdot (\mathcal{B}_3)^n$ is not zero for any integers $m, n > 0$.* □

Base of the free alternative $\mathcal{N}il_n$-superalgebra on one odd generator

As in previous cases, first we consider an ideal $\mathcal{I}_n \subset \mathcal{A}$, spanned by the elements $W_n(u_1, u_2, \ldots, u_n)$, $u_1, u_2, \ldots, u_n \in \mathcal{A}_0 \cup \mathcal{A}_1$, where

$$W_n(u_1, u_2, \ldots, u_n) = \sum_{\sigma \in Sym(n)} \operatorname{sign}_{\mathrm{odd}}(\sigma)(\ldots((u_{\sigma(1)}u_{\sigma(2)})u_{\sigma(3)})\cdots)u_{\sigma(n)},$$

and construct a base of $\mathcal{I}_n$. Using the multiplication table for $\mathcal{A}$, we obtain

$$
\begin{aligned}
W_n(x,t,\dots,t) &= n!\,\left(t^{n-1}x - \tfrac{n-1}{2}t^{n-2}x^{[3]}\right),\\
W_n(t,t,\dots,t) &= n!\,t^n,\\
W_n(u^{[k]},t,\dots,t) &= n!\ t^{n-1}u^{[k]},\\
W_n(z^{[k]},t,\dots,t) &= n!\ t^{n-1}z^{[k]},\\
W_n(z^{[k]},x,t,\dots,t) &= n!\,t^{n-2}z^{[k]}x,\\
W_n(u^{[k]},x,t,\dots,t) &= n!\,(t^{n-2}u^{[k]}x - t^{n-1}z^{[k]}),\\
W_n(x^{[k]},t,\dots,t) &= n!\,(t^{n-1}x^{[k]} + \tfrac{n-1}{2}t^{n-2}z^{[k]}),\\
W_n(x^{[k]}x,t,\dots,t) &= n!\,(t^{n-1}(x^{[k]}x) - \tfrac{n-1}{2}t^{n-2}(\tfrac{1}{2}u^{[k]} - z^{[k]}x - \tfrac{1}{2}z^{[k+1]})),\\
W_n(x^{[k]},x,t,\dots,t) &= \tfrac{n!}{2}\,(t^{n-2}(2x^{[k]}x - x^{[k+1]})\\
&\quad -(n-2)t^{n-3}(\tfrac{1}{2}u^{[k]} - z^{[k]}x - \tfrac{1}{6}z^{[k+1]})),\\
W_n(x^{[k]}x,x,t,\dots,t) &= \tfrac{n!}{2}\,(-\tfrac{1}{3}t^{n-2}x^{[k+2]} + t^{n-2}(x^{[k+1]}x)\\
&\quad +\tfrac{n-2}{3}t^{n-3}(-\tfrac{1}{2}u^{[k+1]} + z^{[k+1]}x + \tfrac{1}{2}z^{[k+2]})),\\
W_n(x^{[k]},x,t,\dots,t)\cdot x &= (-1)^k(t^{n-2}(\tfrac{1}{3}x^{[k+2]} - x^{[k+1]}x)\\
&\quad -(n-2)t^{n-3}(\tfrac{1}{2}u^{[k]}x - \tfrac{1}{6}u^{[k+1]} + \tfrac{1}{2}z^{[k+1]}x + \tfrac{1}{6}z^{[k+2]})).
\end{aligned}
$$

Proposition 4.9 *Elements*

$$
\begin{array}{cc}
t^{m+n}x^{\sigma}, & \\
t^{n-1}x - \tfrac{n-1}{2}t^{n-2}x^{[3]}, & \\
t^{n-2}x^{[k+3]} + \tfrac{n-2}{2}t^{n-3}(u^{[k+2]} + z^{[k+3]}), & t^{n-2}(x^{[k+2]}x) + \tfrac{n-2}{3}t^{n-3}z^{[k+3]},\\
t^{m+n-1}(x^{[k+2]}x^{\sigma}), & \\
t^{m+n-3}(u^{[4k+\varepsilon]}x^{\sigma}), & m+\sigma \geq 1,\\
t^{m+n-3}(z^{[4k+\varepsilon]}x^{\sigma}), & m+\sigma \geq 1,
\end{array}
\tag{7}
$$

where $k > 0$, $m \geq 0$; ε, $\sigma \in \{0,1\}$, form a base of $\mathcal{I}_n$. □

The superalgebra $\mathcal{B}_n = \mathcal{A}/\mathcal{I}_n$ is spanned by the elements

$$
\begin{array}{rl}
t^i x^{\sigma}, & 0 \leq i < n,\ i+\sigma > 0,\\
t^i(x^{[k]}x^{\sigma}), & 0 \leq i < n-2,\\
t^i u^{[4m+\varepsilon]}x^{\sigma},\ t^i z^{[4m+\varepsilon]}x^{\sigma}, & 0 \leq i < n-2,\ i+\sigma < n-2,
\end{array}
\tag{8}
$$

where $k > 2$, $m > 0$, $\varepsilon, \sigma \in \{0,1\}$. All other elements from (1) are equal zero, except

$$
\begin{aligned}
t^{n-1}x &= \tfrac{n-1}{2}t^{n-2}x^{[3]},\\
t^{n-2}x^{[k+3]} &= -\tfrac{n-2}{2}t^{n-3}(u^{[k+2]}+z^{[k+3]}),\\
t^{n-2}(x^{[k+2]}x) &= -\tfrac{n-2}{3}t^{n-3}z^{[k+3]}.
\end{aligned}
$$

Moreover, these elements are linearly independent since any non-trivial linear combination of their preimages does not lie in $\mathcal{I}_n$.

Theorem 4.10 *The elements from* (8) *form a base of the superalgebra* $\mathcal{B}_n$. □

Corollary 4.11 *The solvability index of* $\mathcal{B}_n$ *is* $\lceil \log_2 n \rceil + 1$ *for* $n > 3$. □

Corollary 4.12 *For* $n > 3$, $\mathcal{B}_n$ *is not nilpotent, moreover* $(\mathcal{B}_n)^2$ *is nilpotent of index* n *and* $(\mathcal{B}_n)^m \cdot (\mathcal{B}_n)^r$ *is not zero for any integers* $m, r > 0$. □

4.1 Application

We present a base of the subspace of skew-symmetric elements of the free alternative nil-algebra using the base of the superalgebra $\mathcal{B}_n = \mathcal{A}lt\text{-}\mathcal{N}il_n[\emptyset; x]$. Let $Alt\text{-}Nil_n\,[T] = Alt\text{-}Nil_n\,[T;\emptyset]$ be the free alternative nil-algebra of nil-index n on a set of even generators T and let $Skew$ be the linear mapping from $\mathcal{B}_n$ to $\mathcal{A}lt\text{-}\mathcal{N}il_n[T]$.

Theorem 4.13
The elements

$$Skew\, f(t_{i_1}, t_{i_2}, \ldots, t_{i_k}),$$

where $f = f(x)$ *runs through the base* (8) *of* $\mathcal{B}_n$, $k = \deg(f)$, $i_1 < i_2 < \cdots < i_k$, *form a base of the subspace* $Skew(Alt\text{-}Nil_n[T])$ *of skew-symmetric elements of* $Alt\text{-}Nil_n\,[T]$. □

We present a Grassmann algebra corresponding to $\mathcal{A}lt\text{-}\mathcal{N}il_3[\emptyset; x]$ and show that the Dorofeev's example of solvable non-nilpotent alternative algebra (see [1]) is its homomorphic image. Consider the free $\mathcal{A}lt\text{-}\mathcal{N}il_3$-superalgebra $\mathcal{A}lt\text{-}\mathcal{N}il_3[\emptyset; x]$ on one odd generator x, then its Grassmann envelope $G(\mathcal{A}lt\text{-}\mathcal{N}il_3[\emptyset; x])$ belongs to $\mathcal{A}lt\text{-}\mathcal{N}il_3$. The subalgebra of $G(\mathcal{A}lt\text{-}\mathcal{N}il_3[\emptyset; x])$ generated by the elements

$$e_1 \otimes x,\ e_2 \otimes x, \ldots, e_n \otimes x, \ldots,$$

is called the $\mathcal{A}lt\text{-}\mathcal{N}il_3$*-Grassmann algebra* and is denoted by $G(\mathcal{A}lt\text{-}\mathcal{N}il_3)$. The following proposition is evident.

Proposition 4.14 *The $\mathcal{A}lt$-$\mathcal{N}il_3$-Grassmann algebra $B = G(\mathcal{A}lt\text{-}\mathcal{N}il_3)$ has a base of the form:*

$$e_\mu \otimes v, \ |\mu| = \deg(v),$$

where v runs a (monomial) base of the superalgebra $\mathcal{A}lt$-$\mathcal{N}il_3[\emptyset; x]$, $\mu = \{i_1, \ldots, i_m\}$, $i_1 < i_2 < \cdots < i_m$, $|\mu| = m$, $e_\mu = e_{i_1} e_{i_2} \cdots e_{i_m} \in G$. □

Dorofeev's example was originally constructed over the ring of integer numbers. We consider the same construction over any field of characteristic zero. A base of Dorofeev's algebra D consists of the words

$$r_\mu = t_{i_1} R_{t_{i_2}} \cdots R_{t_{i_m}}, \ m > 0,$$
$$s_\mu = (t_{i_1}(t_{i_2} t_{i_3})) R_{t_{i_4}} \cdots R_{t_{i_m}}, \ m > 2,$$

where $\mu = \{i_1, i_2, \ldots, i_m\}$, $i_1 < i_2 < \cdots < i_m$.

We construct a surjective homomorphism of vector spaces $\psi : B \to D$ by defining

$$\begin{array}{rcl}
\psi(e_{i_1} \otimes x) & = & t_{i_1}, \\
\psi(e_{i_1} e_{i_2} \otimes t) & = & [t_{i_1}, t_{i_2}], \\
\psi(e_{i_1} e_{i_2} e_{i_3} \otimes tx) & = & [t_{i_1}, t_{i_2}] * t_{i_3}, \\
\psi(e_{i_1} e_{i_2} e_{i_3} e_{i_4} \otimes t^2) & = & 0, \\
\psi(e_{i_1} e_{i_2} e_{i_3} e_{i_3} e_{i_5} \otimes t^2 x) & = & 0, \\
\psi(e_{i_1} \cdots e_{i_k} \otimes x^{[k]}) & = & [t_{i_1}, t_{i_2}, \ldots, t_{i_k}], \\
\psi(e_{i_1} \cdots e_{i_{k+1}} \otimes x^{[k]} x) & = & [t_{i_1}, t_{i_2}, \ldots, t_{i_k}] * t_{i_{k+1}}, \\
\psi(e_{i_1} \cdots e_{i_{4n+\varepsilon+3}} \otimes u^{[4n+\varepsilon]}) & = & 0, \\
\psi(e_{i_1} \cdots e_{i_{4n+\varepsilon+2}} \otimes z^{[4n+\varepsilon]}) & = & 0.
\end{array}$$

Here $[t_{i_1}, \ldots, t_{i_{k-1}}, t_{i_k}]$ denotes the "long commutator" of the elements $t_{i_1}, \ldots, t_{i_k}$ which is defined by induction:

$$\begin{array}{rcl}
[t_{i_1}, t_{i_2}] & = & t_{i_1} * t_{i_2} - t_{i_2} * t_{i_1}, \\
[t_{i_1}, \ldots, t_{i_{k-1}}, t_{i_k}] & = & [[t_{i_1}, \ldots, t_{i_{k-1}}], t_{i_k}].
\end{array}$$

Theorem 4.15 *Dorofeev's algebra is isomorphic to the quotient algebra of the $\mathcal{A}lt$-$\mathcal{N}il_3$-Grassmann algebra B modulo the ideal $(B^2)^2$.* □

We construct solvable alternative nil-algebras which are not associative of arbitrary big solvability index. We use a standard passage to Grassmann envelope over a field of characteristic zero and alternative nil-superalgebras

$\mathcal{B}_n = \mathcal{A}lt\text{-}\mathcal{N}il_n[\emptyset; x]$ on one odd generator x of nil-index $n \geq 3$. By the definition, Grassmann envelope $G(\mathcal{B}_n)$ is an alternative nil-algebra of nil-index n.

Theorem 4.16 *The alternative nil-algebra $G(\mathcal{B}_n)$ of nil-index $n \geq 3$, has a base of the form: $g \otimes v$, where v runs the (monomial) base (8) of the superalgebra $\mathcal{B}_n$, $\overline{v} \in \{0, 1\}$, and g runs the base of the superalgebra G, and $\overline{g} = \overline{v}$.*$\square$

Moreover, The alternative nil-algebra $G(\mathcal{B}_n)$ is not nilpotent, and index of solvability is consistent with that of $\mathcal{B}_n$.

5 Conclusion

The aim of this work was the usage of the superalgebra method in the study of free algebras. In this wide matters, we focused on the free alternative superalgebra on one odd generator, which base is known (see [27]), and we found some new applications of this superalgebra. We investigated the free alternative nil-superalgebra on one odd generator of nil-index $n \geq 2$ and we solved two basic problems:

1) Finding a base of the free alternative nil-superalgebra on one odd generator of nil-index $n \geq 2$.

2) Computing the solvability index of this superalgebra.

In addition, several applications of this superalgebra were found.

We constructed a base of $\mathcal{B}_n = \mathcal{A}lt\text{-}\mathcal{N}il_n[\emptyset; x]$ – the free alternative nil-superalgebra on one odd generator of nil-index n – for $n \geq 2$ and we computed the solvability index as $\lceil \log_2 n \rceil + 1$. We also showed that for $n > 2$, $\mathcal{B}_n$ is not nilpotent and $(\mathcal{B}_n)^2$ is nilpotent of index n. We presented as a first exactly computed bounds of the solvability index. We constructed a base of the subspace of skew-symmetric elements of the free alternative nil-algebra using the base of the superalgebra $\mathcal{B}_n$. We also constructed Grassmann algebra corresponding to $\mathcal{B}_3$ and showed that Dorofeev's example of solvable non-nilpotent alternative algebra is its homomorphic image. Till now there were no explicit examples of nonassociative alternative algebras which are solvable of arbitrarily big solvability index. We introduced solvable alternative nil-algebras which are not associative of arbitrary big solvability index, using the superalgebra $\mathcal{B}_n$ for $n \geq 3$, and the standard passage to Grassmann envelope. Our results were published in [16, 17, 18].

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- J. Scholtzová, N. Zhukavets: *The free alternative nil-superalgebra of index n on one odd generator*. Communications in Algebra. Accepted 2012. [Authorship: 50%]

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6 Summary

In this work we presented a base of the free alternative nil-superalgebra $\mathcal{B}_n = \mathcal{A}lt\text{-}\mathcal{N}il_n[\emptyset; x]$ of nil-index n, for $n \geq 2$, on one odd generator. This superalgebra is solvable of index $\lceil \log_2 n \rceil + 1$ and for $n > 2$ it is not nilpotent. We presented some applications of these results.

Till now there were no explicit examples of nonassociative alternative algebras which are solvable of arbitrarily big solvability index. Using a standard passage to Grassmann envelope over a field of characteristic zero we constructed an infinite family of such algebras. Using the linear mapping, which maps isomorphically the homogeneous component of degree m of the superalgebra $\mathcal{B}_n$ to the subspace of all multilinear skew-symmetric elements of the free $\mathcal{A}lt\text{-}\mathcal{N}il_n$-algebra on m generators, we constructed a base of this subspace.

We also presented a Grassmann algebra in the variety $\mathcal{A}lt\text{-}\mathcal{N}il_3$, and we showed that Dorofeev's example of solvable non-nilpotent alternative algebra (see [1]) is its homomorphic image. Results of our research are published in [16, 17] and [18].

7 Resumé

V této práci představujeme bázi volné alternativní nil-superalgebry $\mathcal{B}_n = \mathcal{A}lt\text{-}\mathcal{N}il_n[\emptyset; x]$ nil-indexu n, pro $n \geq 2$, s jedním lichým generátorem. Tato superalgebra je řešitelná indexu $\lceil \log_2 n \rceil + 1$ a pro $n > 2$ není nilpotentní. Představujeme také některé aplikace tohoto výsledku.

Superalgebry byly použity ke konstrukci dosud neznámého příkladu neasociativních alternativních algeber, které jsou řešitelné libovolně velkého indexu n. Také jsme popsali lineární zobrazení, které izomorfně zobrazí homogenní komponentu stupně m superalgebry $\mathcal{B}_n$ do podprostoru všech antisymetrických prvků volné $\mathcal{A}lt\text{-}\mathcal{N}il_n$-algebry na m generátorech a sestrojena báze tohoto podprostoru.

Prozkoumali jsme také odpovídající Grassmannovu algebru ve varietě $\mathcal{A}lt\text{-}\mathcal{N}il_3$ v souvislosti s klasickým příkladem řešitelné alternativní algebry, která není nilpotentní od Dorofeeva (viz [1]), a ukázali, že tato algebra je homomorfním obrazem této Grassmannovy algebry. Výsledky jsou publikovány v [16, 17] a [18].