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THE SUPERALGEBRA APPROACH TO STUDY OF SUBSPACES OF SKEW-SYMMETRIC ELEMENTS IN FREE ALGEBRAS

Doctoral Thesis

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Contents

1 Introduction

For power-associative algebras and a positive integer n, let $\mathcal{N}il_n$ be the variety of *nil-algebras* of nil-index n defined by the identity $x^n = 0$. The classical Dubnov-Ivanov-Nagata-Higman theorem (see [2, 5], 1943, 1956) states that in characteristic zero every associative nil-algebra of nil-index n is nilpotent of index less or equal $2^n - 1$. It is an interesting problem to find the exact estimate $f(n)$ for the nilpotency index of associative $\mathcal{N}il_n$-algebras. Razmyslov (see [14], 1974) showed that $f(n) \leq n^2$. On the other hand, Kuzmin (see [9], 1975) has proved that $f(n) \geq \frac{n(n+1)}{2}$ and conjectured that the last number gives the exact estimate of the nilpotency index. It is easy to see that Kuzmin's conjecture is true for $n = 2$, and Higman's results imply that it is also true for $n = 3$. Several authors tried to check the conjecture for next small values of n using the computer: Vaughan-Lee (see [29], 1993) confirmed Kuzmin's conjecture for $n = 4$. Shestakov and Zhukavets (see [22], 2004) investigated Kuzmin's conjecture for $n = 5$ and proved that every two-generated superalgebra over a field of characteristic zero in the variety of associative $\mathcal{N}il_5$-superalgebras is nilpotent of index 15.

A weaker condition than the associativity for an algebra is the alternativity: an algebra A is called *alternative* (see [34], 1982) if it satisfies the identities

$$x(xy) = x^2y \quad \text{and} \quad (yx)x = yx^2.$$

An octonion algebra is an example of alternative algebra which is not associative. Recall that by Artin's theorem any two elements of an alternative algebra generate an associative subalgebra. In particular, every alternative algebra is power-associative.

For a nonassociative algebra A two chains of subsets can be inductively defined:

$$\begin{aligned} A^1 &= A, \\ A^{i+1} &= A^iA + A^{i-1}A^2 + \cdots + A^2A^{i-1} + AA^i, \end{aligned}$$

and

$$\begin{aligned} A^{(0)} &= A, \\ A^{(i+1)} &= A^{(i)}A^{(i)}. \end{aligned}$$

An algebra A is called *nilpotent* (*solvable*) if $A^n = 0$ ($A^{(n)} = 0$, respectively) for some positive integer n. Clearly, any nilpotent algebra is solvable. The concepts of solvability and nilpotency are equivalent for associative algebras.

It turns out that, in contrast to the associative case, alternative nil-algebras of bounded index can be non-nilpotent, that is, the Dubnov-Ivanov-Nagata-Higman theorem does not carry over to alternative algebras. This was proved by Dorofeev (see [1], 1960) who constructed an example of a solvable alternative algebra which is not nilpotent. Zhevlakov's theorem (see [34, Theorem 6.3.2]) establishes that in characteristic zero every alternative nil-algebra of index n is solvable of index $\leq \frac{n(n+1)}{2}$. Notice that already Kuzmin's results [9] imply that the solvability index $\geq \log_2 \frac{n(n+1)}{2}$.

It is worth to mention here two more open problems on solvability and nilpotency of alternative algebras. It was Pchelintsev (see [13], 1985) who proved that in characteristics not equal 2 and 3 the square of a solvable alternative algebra is nilpotent without giving any approximation of the nilpotency index. It would be interesting to obtain some estimate for this nilpotency index. Shestakov (see [19], 1989) showed that for any alternative algebra A it holds

$$(A^2)^{g(k)} \subset A^{(k)}, \quad \text{where } g(k) = \tfrac{5^{k-1}+3}{4}.$$

Moreover, till now there were no explicit examples of solvable alternative algebras of arbitrary big index which are not associative.

One way to explore the alternative nil-algebras is the usage of the superalgebras. The superalgebras were successfully used for the study of identities of free algebras and for the development of a structure theory of varieties of algebras. The method, which arose from that, is called the superalgebra technique. If a few authors contributed to the development of this method should be mention, Kemer (see [7, 8], 1984 and 1987), Zelmanov (see [31, 32], 1987 and 1989), Vaughan-Lee (see [29, 30], 1993 and 1998), or Shestakov and Zhukavets (see [22, 23, 24, 25, 26, 27, 28], 2004–2009) must not be forgotten.

Recall, that no base of the free alternative nil-algebra is known. In general, the problem of construction of a base of the free alternative nil-algebra is very difficult, and it seems natural to consider first some special cases. For instance, to construct a base of the subspace of all skew-symmetric elements of the free alternative nil-algebra. In this case, due to papers of Shestakov (see [20], 1999) and Vaughan-Lee (see [30], 1998), the problem is reduced to the free alternative nil-superalgebra on one odd generator, which is easier to deal with.

Aims of the doctoral thesis

The aim of this work is the usage of the superalgebra technique in the study of free algebras. In this wide matters it is focused on the free alternative nil-algebras and on obtaining some corollaries for solvable and nilpotent alternative algebras. The base of the free alternative superalgebra on one odd generator constructed by Shestakov and Zhukavets (see [27], 2007) is used.

As a first step, the finding of a base of the free alternative nil-superalgebra on one odd generator of different nil-index $n \geq 2$ is described. Next, knowing the base, the solvability and the nilpotency can be investigated. The index of solvability of this superalgebra is found and it is confirmed for nil-index $n > 2$ that the superalgebra is not nilpotent.

As an application, the subspace of skew-symmetric elements of the free alternative nil-algebras is described. The problem is solved using the free alternative nil-superalgebra on one odd generator. Further, Grassmann algebra in the variety $\mathcal{A}lt\text{-}\mathcal{N}il_3$ which generalize Dorofeev's example of solvable non-nilpotent alternative algebra (see [1], 1960) is presented. Another application is a construction of an infinite family of solvable alternative nil-algebras of arbitrary big solvability index, using a standard passage to Grassmann envelope over a field of characteristic zero. It should be noted that the research is difficult due to nonassociativity of studied objects.

As a future research, all skew-symmetric central and nuclear elements in alternative algebras will be described.

The thesis is organized as follows.
Section 2: Basic definitions in the theory of nonassociative algebras, free algebras and superalgebras are introduced.
Section 3: This section deals with applications of the superalgebra technique in the study of identities of free algebras and in the study of the structural theory of varieties of (non-associative) algebras.
Section 4: A base of $\mathcal{A}lt\text{-}\mathcal{N}il_n[\emptyset; x]$ – the free alternative nil-superalgebra on one odd generator x of nil-index $n \geq 2$ is constructed in this section, and it is proved that this superalgebra is solvable but, for $n > 2$, it is not nilpotent.
Section 5: Applications of the results obtained in the previous section are presented. Namely, a base of the subspace of skew-symmetric elements of the free alternative nil-algebra on countable set of free generators, a generalization of Dorofeev's example of solvable non-nilpotent alternative algebra, an infinite family of solvable alternative nil-algebras which are not associative of arbitrary big solvability index.
Section 6: A summary of the results and future plans are presented.

2 Algebras, superalgebras and varieties

This chapter presents basic definitions and examples. It is based on the publications [12, 15, 20, 34].

2.1 Algebras and their properties

Let A be a vector space over a field $\mathbb{F}$ with given bilinear mapping (usually called "product")

$$\cdot : A \times A \to A,$$

that means, the distributivity holds for every $x, y, z \in A$ and $\alpha, \beta \in \mathbb{F}$:

$$\begin{aligned} x \cdot (\alpha y + \beta z) &= \alpha(x \cdot y) + \beta(x \cdot z), \\ (\alpha x + \beta y) \cdot z &= \alpha(x \cdot z) + \beta(y \cdot z). \end{aligned}$$

Then A is called an *algebra* over the field $\mathbb{F}$. The product $x \cdot y$ is often abbreviated by juxtaposition xy. Notice that we do not suppose the associativity and commutativity of the mapping $\cdot$.

A *commutative* algebra A_{com} is an algebra with commutative product, that is

$$[x, y] = 0,$$

for any two elements $x, y \in A_{com}$, where $[\ ,\ ]$ is the commutator defined as $[x, y] = xy - yx$. Similarly we define an *associative* algebra A_{assoc} as an algebra with associative product, that is

$$(x, y, z) = 0,$$

for any three elements $x, y, z \in A_{assoc}$, where $(\ ,\ ,\ )$ is the associator defined as $(x, y, z) = (xy)z - x(yz)$. The algebra not necessary commutative, is called *noncommutative*. The algebra not necessary associative, is called *nonassociative*. Replacing the associativity by identities

$$\begin{array}{ll} (x, x, y) = 0, & \text{(left alternativity)} \\ (x, y, y) = 0, & \text{(right alternativity)} \end{array} \tag{1}$$

we get an *alternative* algebra A_{alt}. It was discovered that alternative algebras are "sufficiently near" to associative ones. The essence of this nearness is expressed in Artin's theorem mentioned later.

If A contains a (two-sided) *unit element* 1 satisfying $1 \cdot x = x \cdot 1 = x$ for all $x \in A$, the algebra A is said to be *unital*. As always, the unit element is unique one if exists. For a unital algebra A we will often identify $\mathbb{F}$ with the subalgebra $\mathbb{F} \cdot 1$ of the algebra A. Notice, that we do not require existence of a unit element in A although we always demand a unit scalar in $\mathbb{F}$.

Example 2.1 (Examples of algebras) As a basic example of an algebra we can take a field $\mathbb{F}$ itself as an algebra over $\mathbb{F}$. The complex numbers $\mathbb{C}$ form an algebra over a real numbers $\mathbb{R}$ with usual addition and multiplication of complex numbers. Square matrices $M_n(\mathbb{F})$ over a field $\mathbb{F}$ with matrix addition and multiplication form an algebra over $\mathbb{F}$.

Notice that $\mathbb{C}$ is an associative commutative algebra, while $M_n(\mathbb{F})$ is an associative algebra which is not commutative. $\mathbb{F}$ is an associative and commutative algebra.

2.1.1 Subspaces, homomorphisms etc.

A *subalgebra* B of an algebra A is a subspace closed under multiplication: $BB \subset B$ (i.e. for any $x, y \in B$ the product xy belongs to B). A (two sided) *ideal* I of an algebra A is a subalgebra closed under multiplication by A, i.e.

$$AI + IA \subseteq I.$$

Ideals 0 and A of the algebra A are called *improper* ideals.

Center, nucleus

In the theory of nonassociative algebras we define two subsets of A which do behave associatively: The *nucleus* $N(A)$ (or the *associative center*) of an algebra A is the set of elements $z \in A$ which associate with every pair of elements $a, b \in A$ in the sense that $(z, a, b) = (a, z, b) = (a, b, z) = 0$. That is

$$N(A) = \{z \in A \mid (z, A, A) = (A, z, A) = (A, A, z) = 0\}.$$

The *center* $Z(A)$ of an algebra A is the set of all elements $z \in A$ which commute and associate with all elements in A. That is

$$Z(A) = \{z \in N(A) \mid [z, A] = 0\}.$$

Note that $N(A)$ is an associative and $Z(A)$ is a commutative and associative subalgebra of A. Moreover $Z(A) \subseteq N(A)$.

A *homomorphism* of algebras $\varphi : A \to A'$ is a homomorphism of vector spaces (i.e. a linear mapping) which preserves multiplication,

$$\varphi(xy) = \varphi(x)\varphi(y), \qquad \text{for each } x, y \in A.$$

The set

$$\operatorname{Im} \varphi = \{\varphi(a) \mid a \in A\}$$

is a homomorphic *image* of A and a *kernel* of the homomorphism φ is the set

$$\operatorname{Ker}\varphi = \{a \in A \mid \varphi(a) = 0\}.$$

If φ is injective homomorphism then we say that an algebra A is *imbedded* in A'. A homomorphism of algebras which is bijective is called *isomorphism* (of algebras). An *endomorphism* is a homomorphism of algebras $\varphi: A \to A$.

If I is an ideal of an algebra A then the mapping $A \to A/I$, such that $a \mapsto a + I$, is called a *natural* (or canonical) homomorphism of algebras.

Theorem 2.1 (Fundamental theorem of hom. for algebras)
Let A, A' be algebras. Let I be an ideal of A, $\varphi : A \to A'$ be a homomorphism of algebras and $\pi : A \to A/I$ the natural homomorphism. Then there is a unique homomorphism $\varphi' : A/I \to A'$ such that $\varphi'(a + I) = \varphi(a)$. Furthermore, φ' is an isomorphism if and only if φ is a surjective homomorphism and $\operatorname{Ker}\varphi = I$.

Theorem 2.2 (Isomorphism theorems for algebras)
Let $\varphi : A \to A'$ be an algebra homomorphism, $B \subseteq A$ be a subalgebra of A and $I_2 \subset I_1$ be ideals of A. Then

(i) $\operatorname{Ker}\varphi$ *is an ideal of A,* $\operatorname{Im}\varphi$ *is a subalgebra of A' and*

$$A/\operatorname{Ker}\varphi \simeq \operatorname{Im}\varphi,$$

(ii) $I_1 \cap B$ is an ideal of B, $I_1 + B$ is a subalgebra of A, I_1 is an ideal of $I_1 + B$ and

$$(I_1 + B)/I_1 \simeq B/(I_1 \cap B),$$

(iii) for two ideals $I_2 \subset I_1$ of A, I_1/I_2 is an ideal of A/I_2 and

$$(A/I_2)/(I_1/I_2) \simeq A/I_1.$$

2.1.2 Direct sum and tensor product of algebras

The *external direct sum* $S = A_1 \oplus A_2$ of two algebras A_1 and A_2 is the Cartesian product $A_1 \times A_2$ under the componentwise operations

$$\begin{aligned}
(a_1, a_2) + (b_1, b_2) &= (a_1 + b_1, a_2 + b_2),\\
\alpha(a_1, a_2) &= (\alpha a_1, \alpha a_2),\\
(a_1, a_2)(b_1, b_2) &= (a_1 b_1, a_2 b_2).
\end{aligned}$$

For each $i \in \{1, 2\}$ we have the natural surjection $\pi_i : S \to A_i$ which picks out the i-th coordinate, and the natural injection $\mu_i : A_i \to S$ which maps $a \in A_i$

to the pair (a_1, a_2), where $a_j = \delta_{ij} a$, that is, $\operatorname{Im} \mu_1 = \{(a_1, 0) \mid a_1 \in A_1\}$ and $\operatorname{Im} \mu_2 = \{(0, a_2) \mid a_2 \in A_2\}$. Evidently, $\operatorname{Im} \mu_1 \cong A_1$ and $\operatorname{Im} \mu_2 \cong A_2$.

A vector space A is called the *sum* of two subspaces A_1 and A_2, $A = A_1 + A_2$, if every element of $a \in A$ can be written as $a = a_1 + a_2,\ a_i \in A_i$. If each element of A can be so expressed in only one way, we call A the *internal direct sum.* It is clear that $S = A_1 \oplus A_2$ is in fact the internal direct sum of $\operatorname{Im} \mu_i$, where μ_i are the natural injections.

Let A_1, A_2 be two algebras. Then a tensor product $A_1 \otimes A_2$ of the vector spaces A_1, A_2 has a natural structure of algebra. The multiplication is given by

$$(a_1 \otimes a_2)(a'_1 \otimes a'_2) = a_1 a'_1 \otimes a_2 {a'}_2.$$

2.1.3 Nilpotency and solvability

For an algebra A we define inductively two chains of subsets:

$$\begin{aligned} A^1 &= A, \\ A^{i+1} &= A^i A + A^{i-1} A^2 + \cdots + A^2 A^{i-1} + A A^i, \end{aligned}$$

and

$$\begin{aligned} A^{(0)} &= A, \\ A^{(i+1)} &= A^{(i)} A^{(i)}. \end{aligned}$$

An algebra A is called *nilpotent* (*solvable*) if $A^n = 0$ ($A^{(n)} = 0$, respectively) for some positive integer n. The minimal such n is called the *index of nilpotency* (*index of solvability*, respectively) of the algebra A. Clearly, any nilpotent algebra is solvable. The concepts of solvability and nilpotency are equivalent for associative algebras:

$$A^{(i)} = A^{(i-1)} A^{(i-1)} = A^{2^i}.$$

So if A is associative and solvable of index m then it is nilpotent of index at most 2^m, and conversely, if A is nilpotent of index n then is solvable of index at most $\lceil \log_2 n \rceil$, where $\lceil u \rceil$ means the smallest integer not less then u.

An algebra A is called *power-associative* if every one-generated subalgebra is associative. This is equivalent to defining the powers of a single element $x \in A$ by

$$x^1 = x, \qquad x^{i+1} = x x^i,$$

requiring $x^i x^j = x^{i+j},\ i, j \in \{1, 2, \dots\}$.

A power-associative algebra A is called a *nil-algebra* of index n if each its element is nilpotent of the order n (i.e. the equation $x^n = 0$ holds for every $x \in A$).

Theorem 2.3 (Dubnov-Ivanov-Nagata-Higman, [34])
Let A be an associative nil-algebra of index n without elements of additive order $\leq n$, then A is nilpotent of index $\leq 2^n - 1$.

Note that the bound $2^n - 1$ obtained by this theorem is no exact, since in the case $n = 3$ it is known that every associative nil-algebra is nilpotent of index 6, while $2^3 - 1 = 7$. The question of corresponding exact bound $f(n)$ for the case of an arbitrary n remains open. Razmyslov showed that $f(n) \leq n^2$. On the other hand, Kuzmin has proved that $f(n) \geq \frac{n(n+1)}{2}$ and conjectured, that $f(n) = \frac{n(n+1)}{2}$.

2.1.4 Alternative algebras

Now we introduce some basic facts, mentioned in [34], about the alternative algebras. Recall that an algebra A is *alternative*, if for any two elements $x, y \in A$ relations (1) are satisfied. Clearly, any associative algebra is alternative. We can also describe this algebra using the identities

$$\begin{aligned} (x,z,y) + (z,x,y) = 0, \\ (x,y,z) + (x,z,y) = 0, \end{aligned} \tag{2}$$

which are the linearized forms of the left and right alternativity (1), see the linearization in Example 2.6. From (2) it follows that the associator is a skew-symmetric function of its arguments and in particular,

$$(x,y,x) = 0$$

is also satisfied by every alternative algebra. In a corollary, it also satisfies so called Moufang identities (see [34, Lemma 2.3.7]):

$$\begin{aligned} x(yzy) &= [(xy)z]y, && \text{(right Moufang identity)} \\ (yzy)x &= y[z(yx)], && \text{(left Moufang identity)} \\ (xy)(zx) &= x(yz)x, && \text{(middle Moufang identity)}. \end{aligned}$$

Example 2.2 (Octonions (Cayley numbers)) An octonion algebra $\mathbb{O}$ is an example of alternative algebra which is not associative. It is defined over real numbers and it is neither associative nor commutative. One can choose a base of $\mathbb{O}$ consisting of 1 and seven imaginary units $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ and define a multiplication by the rules:

(i) 1 is the identity element of the algebra $\mathbb{O}$,

(ii) $e_ie_i = -1, e_ie_j = -e_je_i, i \neq j$,

(iii) $e_ie_{i+1} = e_{i+3 \pmod 7}$, and moreover,

(iv) if $e_ie_j = e_k$ then $e_{\pi(i)}e_{\pi(j)} = sgn(\pi)e_{\pi(k)}$ for any $\pi \in Sym(\{i,j,k\})$.

We can see that for example

$$(e_1e_2)e_3 = e_4e_3 = -e_3e_4 = -e_6$$

is not equal

$$e_1(e_2e_3) = e_1e_5 = e_6, \text{ because } e_5e_6 = e_1,$$

that is, $\mathbb{O}$ is not associative.

Fact that alternative algebras are nearly associative is expressed by the following theorem.

Theorem 2.4 (Artin's theorem [34, Theorem 2.3.2])
In an alternative algebra any two elements generate an associative subalgebra.

In particular, every alternative algebra is power-associative.

Contrary to the associative case, the alternative algebras can be solvable, but not nilpotent. This was proved by Dorofeev [1] (or [34, Section 6.2]), who has constructed an example of a solvable alternative algebra which is not nilpotent. We discuss this example in Section 5. It is also known [34, Theorem 13.1.3] that solvability and nilpotency are equivalent for finitely generated alternative algebras.

It is known that alternative nil-algebras can be non-nilpotent, that is, the Dubnov-Ivanov-Nagata-Higman theorem for associative algebras does not carry over to alternative algebras. Zhevlakov estimates the upper bound for index of solvability of alternative nil-algebras.

Theorem 2.5 (Zhevlakov's theorem [34, Theorem 6.3.2])
Let A be an alternative nil-algebra of index n without elements of additive order $\leq n$, then A is solvable of index $\leq \frac{n(n+1)}{2}$.

2.2 Superalgebras and their properties

In general, a *superalgebra* is a $\mathbb{Z}_2$-graded algebra over a field with characteristic not equal two, that means an algebra A which may be written as $A = A_0 + A_1$, subjected to the relation

$$A_iA_j \subseteq A_{i+j}, \qquad i,j \in \mathbb{Z}_2.$$

The subspaces A_0 and A_1 are called the *even* and the *odd* part of the superalgebra A and so are called the elements from A_0 and from A_1, respectively. Furthermore, the element x is called *homogeneous*, if it is either even or odd ($x \in A_0 \cup A_1$), and the symbol $\bar{x}$ is used for a *parity*, i.e. $\bar{x} = 0$ if x is even and $\bar{x} = 1$ if x is odd. Notice that any algebra A has natural structure of a superalgebra $A_0 = A$ and $A_1 = 0$. This superalgebra is called *trivial.* The relation $A_0 A_0 \subseteq A_0$ implies that A_0 is subalgebra of A.

Example 2.3 A basic example of a superalgebra is an algebra of complex numbers $\mathbb{C}$ over $\mathbb{R}$ with standard operations of addition and multiplication with grading $\mathbb{C} = \mathbb{R} + \mathbb{R}\, i$. $\mathbb{R}$ is the even part of the superalgebra $\mathbb{C}$, $\mathbb{R}\, i$ is the odd part. It is easy to check the definition from the products $ab \in \mathbb{R}$, $xy \in \mathbb{R}$ and $ax \in \mathbb{R}\, i$, for any $a, b \in \mathbb{R}$ and $x, y \in \mathbb{R}\, i$.

2.2.1 Subspaces, homomorphisms etc.

A subspace M of the superalgebra $A = A_0 + A_1$ is called a *subsuperspace* (or a graded subspace) if $M = M \cap A_0 + M \cap A_1$, that is,

$$\begin{aligned} M_0 &= M \cap A_0, \\ M_1 &= M \cap A_1, \end{aligned}$$

and M contains the homogeneous components (an even and an odd parts) of all of its elements.

A *subsuperalgebra* of a superalgebra $A = A_0 + A_1$ is any subsuperspace which is a subalgebra of A. Analogously, a subsuperspace of $A = A_0 + A_1$ which is an ideal of A (considered as an algebra) is called an *ideal of a superalgebra A*. A *homogeneous* ideal is an ideal of a superalgebra generated by a set of homogeneous elements.

A *homomorphism* of superalgebras A, B is a homomorphism of algebras which preserves the grading, i.e. $\varphi : A \to B$ such that

$$\varphi(A_i) \subseteq B_i, \qquad i \in \mathbb{Z}_2$$

(the even elements of A maps to even elements of B and the odd elements of A maps to odd elements of B). An *isomorphism* of superalgebras is a bijective homomorphism of superalgebras.

The *direct sum* of superalgebras is constructed as in the ungraded case and the grading is given by

$$\begin{aligned} (A + B)_0 &= A_0 + B_0, \\ (A + B)_1 &= A_1 + B_1. \end{aligned}$$

The *tensor product* $A \otimes B$ of superalgebras $A = A_0 + A_1,\ B = B_0 + B_1$ is the tensor product of algebras A, B with the natural grading

$$\begin{aligned}(A \otimes B)_0 &= (A_0 \otimes B_0) + (A_1 \otimes B_1),\\ (A \otimes B)_1 &= (A_0 \otimes B_1) + (A_1 \otimes B_0).\end{aligned}$$

2.2.2 Grassmann (super)algebra

Let $\mathbb{F}$ be a field of characteristic not equal two. The *Grassmann algebra* G over $\mathbb{F}$ is an associative algebra generated by the elements $1, e_1, e_2, \ldots, e_n, \ldots$ subject to the relation

$$e_i^2 = 0, \qquad e_i e_j = -e_j e_i,\ i \neq j. \tag{3}$$

The products

$$1, e_{i_1} e_{i_2} \cdots e_{i_k}, \quad i_1 < i_2 < \cdots < i_k, \tag{4}$$

form a base of G (we consider 1 as the product of the empty set of the elements e_i).

The Grassmann algebra has a natural structure of a superalgebra over $\mathbb{F}$. Denoting the subspace of products of even length by G_0 and the subspace of products of odd length by G_1 respectively, we get the direct sum of these subspaces and

$$G = G_0 + G_1, \qquad G_i G_j \subseteq G_{i+j}, \quad i, j \in \mathbb{Z}_2.$$

Moreover it satisfies

$$\begin{aligned}gh &= hg, &&\text{for } g \in G_0,\ h \in G\\ gh &= -hg, &&\text{for } g, h \in G_1.\end{aligned}$$

2.3 Free algebras, varieties of algebras and superalgebras

2.3.1 Free algebra

Let us fix the set X. The *free nonassociative algebra* $\mathbb{F}[X]$ over a field $\mathbb{F}$ from the set of generators X is defined by the following universal property: For any algebra A, any mapping $X \to A$ can be uniquely extended to the algebra homomorphism $\mathbb{F}[X] \to A$. The cardinality of the set X is called the *rank* of $\mathbb{F}[X]$.

We can construct the free algebra $\mathbb{F}[X]$, using a set $K[X]$ of nonassociative words of the set X which is defined inductively:

$$x \in K[X], \qquad \forall x \in X,$$
$$x_1x_2,\ x_1(u),\ (v)x_2,\ (u)(v) \in K[X],$$

for $x_1, x_2 \in X,\ u, v \in K[X]$. No other sequences of the elements from X and brackets are not contained in $K[X]$. Let us define the multiplication $\cdot$ on $K[X]$ as

$$x_1 \cdot x_2 = x_1x_2,\ x_1 \cdot u = x_1(u),\ v \cdot x_2 = (v)x_2,\ u \cdot v = (u)(v).$$

Now we consider $\mathbb{F}[X]$ to be a set of formal sums

$$\Big\{ \sum_i \alpha_i u_i \mid \alpha_i \in \mathbb{F}, u_i \in K[X] \Big\}$$

and extend the operation of multiplication defined on $K[X]$ to $\mathbb{F}[X]$ by the rule

$$\Big(\sum_i \alpha_i u_i \Big) \cdot \Big(\sum_j \beta_j v_j \Big) = \sum_{ij} \alpha_i \beta_j (u_i \cdot v_j),$$

where $\alpha_i, \beta_j \in \mathbb{F}$ and $u_i, v_j \in K[X]$. We obtain the free nonassociative algebra $\mathbb{F}[X]$ over a field $\mathbb{F}$ from the set of generators X. The elements of $\mathbb{F}[X]$ are called *nonassociative polynomials* of (noncommutative) variables from the set X.

Polynomials

A nonassociative polynomial of the form

$$\alpha\, v,$$

where $\alpha \in \mathbb{F}$ and $v \in K[X]$ is called a (nonassociative) *monomial.* The length of the word v is called a *degree* of the monomial. The maximum of the degrees of the monomials whose sum forms a polynomial f is called a *degree of the polynomial* f and it is denoted by $deg(f)$.

The monomial $\alpha\, v$ has a *multidegree* $(n_1, \dots, n_k)$ if it contains x_i exactly n_i times, $n_k \neq 0,\ n_j = 0,\ j > k$. A polynomial f is called *homogeneous* if all the monomials have the same multidegree.

Example 2.4 A polynomial $f_1(x, y, z) = (x^2y)z + (xy)(zx)$ is homogeneous since its monomial multidegree is (2,1,1), but a polynomial $f_2(x, y, z) = x^2y + x^2z$ is not homogeneous since its monomial multidegrees are (2,1,0) and (2,0,1).

A homogeneous polynomial is called *multilinear* if it is linear in any of its variable (it is homogeneous of multidegree $(1, 1, \ldots, 1)$). A multilinear polynomial $f(x_1, \ldots, x_k)$ is called *symmetric* if

$$f(x_1, \ldots, x_k) = f(x_{\pi(1)}, \ldots, x_{\pi(k)})$$

for any $\pi \in Sym(k)$, and *skew-symmetric* (or alternating) if

$$f(x_1, \ldots, x_k) = sgn(\pi)\ f(x_{\pi(1)}, \ldots, x_{\pi(k)}).$$

In both cases it is enough to deal only with transpositions $\pi = (i, j) \in Sym(k)$.

Linearization of polynomials

The linearization of the homogeneous polynomials is useful in the study of identities of algebras and in the study of varieties. The relation between a homogeneous polynomial and obtained linearized multilinear polynomial is expressed by the Proposition 2.7 in the next subsection. The process of the linearization is described in detail e.g. in [34, Section 1.5]. It is based on reducing the variable degree of the polynomial, and we describe it now.

Exclude the variable x of degree k from the set X and denote the homogeneous polynomial f by $f(x, X)$. Define a polynomial $g(x, x_1, X)$ as

$$g(x, x_1, X) = f(x + x_1, X) - f(x, X) - f(x_1, X),$$

where $x_1 \notin X$. Then g is a polynomial of degree $k-1$ in the variables x, x_1. Moreover $f(x, X) = \frac{1}{2^k-2}\, g(x, x, X)$. Proceeding by induction on the degree of variables we obtain a multilinear polynomial by the following describe the algorithm.

Step 1: Define $g_1(x, x_1, X) = f(x + x_1, X) - f(x, X) - f(x_1, X)$, where $x_1 \notin X$.
Step 2: For i=2 to k-1 define

$$\begin{aligned} g_i(x, x_1, \ldots, x_i, X) &= g_{i-1}(x + x_i, x_1, \ldots, x_{i-1}, X) \\ &\quad -g_{i-1}(x, x_1, \ldots, x_{i-1}, X) - g_{i-1}(x_i, x_1, \ldots, x_{i-1}, X), \end{aligned}$$

where $x_i \neq x_j,\ j = \{2, \ldots, i-1\}$ and $x_i \notin X$. The polynomial g_k is linear in x and in the new variables $x_1, \ldots, x_{k-1}$.

Step 3: Continue with g_k, apply Step 1 and Step 2 to other variables of degree greater then one. The obtained polynomial is multilinear.

Let us show it on easy examples.

Example 2.5 (Linearization of the polynomial $f(x) = x^2$)

$$\begin{aligned} f(x) &= x^2 \\ g(x,y) &= (x+y)^2 - x^2 - y^2 \\ &= x^2 + xy + yx + y^2 - x^2 - y^2 \\ &= xy + yx. \end{aligned}$$

A polynomial $g(x, y)$ is multilinear. Notice, that in general associative case the polynomial x^n can be linearized as

$$g(x_1, \ldots, x_n) = \sum_{\pi \in Sym\ (n)} x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(n)}$$

in characteristic zero (see for example [22]).

Example 2.6 (Linearization of the left/right alternativity)
Linearization of the polynomial $f(x) = (x, x, y) = x^2y - x(xy)$:

$$\begin{aligned} f(x,y) &= (x,x,y) = x^2y - x(xy) \\ g(x,z,y) &= (x+z)^2y - (x+z)((x+z)y) \\ &\quad -(x^2y - x(xy)) - (z^2y - z(zy)) \\ &= (xz)y + (zx)y - x(zy) - z(xy) \\ &= (x,z,y) + (z,x,y). \end{aligned}$$

Linearization of the polynomial $f(x) = (x, y, y) = (xy)y - xy^2$:

$$\begin{aligned} f(x,y) &= (x,y,y) = (xy)y - xy^2 \\ g(x,y,z) &= (x(y+z))(y+z) - x(y+z)^2 \\ &\quad -((xy)y - xy^2) - ((xz)z - xz^2) \\ &= (xy)z + (xz)y - x(yz) - x(zy) \\ &= (x,y,z) + (x,z,y). \end{aligned}$$

2.3.2 Varieties of algebras

A polynomial $f(x_1, \ldots, x_k) \in \mathbb{F}[X]$ is called an *identity* of the algebra A if $f(a_1, \ldots, a_k) = 0$ for all elements $a_1, \ldots, a_k \in A$. We also say that A satisfies the identity $f(x_1, \ldots, x_k)$.

Let I be a set of polynomials from $\mathbb{F}[X]$. Then the class of all algebras satisfying this set of identities I is called the *variety* of algebras over the field $\mathbb{F}$ defined by the set of identities I. A *subvariety* is a subset of a variety which is itself a variety. Algebras from the variety $\mathcal{V}$ are called shortly $\mathcal{V}$-*algebras*. The variety consisting of only the zero algebra is called *trivial*. The variety is called *homogeneous* if, for every identity f satisfied in the variety $\mathcal{V}$, all the homogeneous components of f are also satisfied in $\mathcal{V}$.

Proposition 2.6 ([34, Corollary 1.4.2]) *Every variety of algebras over an infinite field is homogeneous.*

Proposition 2.7 ([12, Proposition 2.18]) *Over a field of characteristic zero any homogeneous identity is equivalent to a multilinear identity.*

Corollary 2.8 ([12, Corollary 2.19]) *Over a field of characteristic zero any variety can be defined by multilinear identities.*

Example 2.7 The variety of associative algebras $\mathcal{A}ssoc$ is defined by one identity

$$f(x_1, x_2, x_3) = (x_1, x_2, x_3).$$

Example 2.8 The variety of commutative algebras $\mathcal{C}om$ is defined by the identity

$$f(x_1, x_2) = [x_1, x_2].$$

Example 2.9 The variety of alternative algebras $\mathcal{A}lt$ is defined by two identities

$$f(x_1, x_2) = (x_1, x_1, x_2),$$
$$f(x_1, x_2) = (x_1, x_2, x_2).$$

Let us denote by

$$V(S) = \{A \mid \text{an algebra } A \text{ satisfies all the identities from } S\}$$

the variety of algebras defined by the set $S \subset \mathbb{F}[X]$. Notice that $V(S) = V(id\langle S\rangle)$, where $id\langle K\rangle$ denotes the ideal generated by K. Similarly, denote

$$I(A) = \{f \in \mathbb{F}[X] \mid f = 0 \text{ for all } a \in A\}$$

the set of all the identities that are satisfied in A, and

$$I(\mathcal{V}) = \bigcap\{I(A) \mid A \in \mathcal{V}\}.$$

Then $I(A)$ is an ideal of $\mathbb{F}[X]$ and it is called the *ideal of identities* of the algebra A. $I(\mathcal{V})$ is an ideal of identities of the variety $\mathcal{V}$.

Proposition 2.9 ([12, Proposition 2.4]) *Let* $End(\mathbb{F}[X])$ *be a set of all endomorphisms of the algebra* $\mathbb{F}[X]$ *and* A *be an algebra. Then* $\varphi(I(A)) \subseteq I(A)$ *for every* $\varphi \in End(\mathbb{F}[X])$.

An ideal of $\mathbb{F}[X]$ which is invariant under endomorphisms of $\mathbb{F}[X]$ is called a *T-ideal.*

The following correspondence is evident [12, p. 17]:

(i) Let J, J_1, J_2 be ideals of $\mathbb{F}[X]$ and $\mathcal{W}, \mathcal{W}_1, \mathcal{W}_2$ be varieties of algebras. Then

$$\begin{aligned} J_1 \subseteq J_2 &\Rightarrow V(J_1) \supseteq V(J_2), \\ \mathcal{W}_1 \subseteq \mathcal{W}_2 &\Rightarrow I(\mathcal{W}_1) \supseteq I(\mathcal{W}_2), \\ J &\subseteq I(V(J)), \\ \mathcal{W} &\subseteq V(I(\mathcal{W})) \text{ and in a corollary } \mathcal{W} = V(I(\mathcal{W})), \\ VIV &= V, \\ IVI &= I, \end{aligned}$$

and in a corollary

$$I(A) = I(V(I(A))), \text{ for any algebra } A.$$

(ii) $I(\mathcal{W})$ is a T-ideal for every variety $\mathcal{W}$.

(iii) Any T-ideal $J \subset \mathbb{F}[X]$ is an ideal of identities of some algebra.

(iv) $J = I(V(J))$ for any T-ideal $J \subset \mathbb{F}[X]$.

Theorem 2.10 ([12, Theorem 2.11])
The correspondence I *and* $\mathcal{V}$ *are bijective mappings between varieties and T-ideals which invert inclusions.*

Note that any variety of algebras is closed under homomorphisms, subalgebras, and direct products. And to decide whether a class of algebras forms a variety is used the following Birkhoff's (or HSP) theorem.

Theorem 2.11 (Birkhoff's theorem, [12, Theorem 2.3])
A class of algebras $\mathcal{V}$ *form a variety iff* $\mathcal{V}$ *is closed under Homomorphisms, Subalgebras and direct Products (HSP).*

The algebra $\mathcal{V}[X]$ is called a $\mathcal{V}$*-free* (*relatively free* or *free in the variety* $\mathcal{V}$) with the set of generators X, if for any algebra $B \in \mathcal{V}$ every mapping

$$\phi : X \to B \in \mathcal{V}$$

can be uniquely extended to a homomorphism of the algebras

$$\phi_{\mathcal{V}} : \mathcal{V}[X] \to B.$$

The $\mathcal{V}$-free algebra is not free in general but only in the variety $\mathcal{V}$, i.e. it satisfies identities (and their consequences) that defines the variety $\mathcal{V}$. The construction of the $\mathcal{V}$-free algebra is explained by the following theorem:

Theorem 2.12 ([34, Theorem 1.2.2])
Let $\mathcal{V}$ be a nontrivial variety with the system of defining identities I. Then for any set X the natural homomorphism $\mathbb{F}[X] \to \mathbb{F}[X]/I(\mathcal{V})$ is injective and the quotient algebra is free in the variety $\mathcal{V}$ with the free set of generators X. Any two free algebras in $\mathcal{V}$ with equivalent sets of free generators are isomorphic.

In the study of algebras from a specific variety $\mathcal{V}$ we shall often call an element f in the $\mathcal{V}$-free algebra $\mathcal{V}[X]$ an identity of an algebra $A \in \mathcal{V}$, meaning by this that some inverse image (and consequently all the inverse images) in $\mathbb{F}[X]$ of the element f under natural homomorphism of $\mathbb{F}[X]$ onto $\mathcal{V}[X]$ is an identity of the algebra A.

2.3.3 Free alternative algebras

Denote by $I_{\mathcal{A}lt}$ the ideal of the free algebra $\mathbb{F}[X]$ generated by the elements

$$(f_1, f_1, f_2), (f_1, f_2, f_2), \text{ for } f_1, f_2 \in \mathbb{F}[X].$$

This is the ideal of $\mathbb{F}[X]$ generated by the left sides of the identities (1). The *free alternative algebra* $\mathcal{A}lt[X]$ generated by X is the quotient algebra

$$\mathcal{A}lt[X] = \mathbb{F}[X]/I_{\mathcal{A}lt},$$

and the universal property holds: If A is an alternative algebra over $\mathbb{F}$ and $\phi : X \to A$ a mapping, there is a unique homomorphism of algebras $\phi_{\mathcal{A}lt} : \mathcal{A}lt[X] \to A$ such that $\phi_{\mathcal{A}lt}(x) = \phi(x)$ for all $x \in X$.

Denote by $\mathcal{A}lt_n$ the variety of alternative algebras generated by the free alternative algebra $\mathcal{A}lt[X_n]$ from n generators. Evidently

$$\mathcal{A}lt_1 \subseteq \mathcal{A}lt_2 \subseteq \cdots \subseteq \mathcal{A}lt_n \subseteq \cdots \subseteq \mathcal{A}lt \qquad (5)$$

and so is $\mathcal{A}lt = \bigcup_n \mathcal{A}lt_n$. Finding of the the smallest n such that $\mathcal{A}lt_n = \mathcal{A}lt$ was open question for a long time. In 1958 Shirshov in formulated the question on the determination of the exact value of n for the variety of alternative algebras. From Artin's theorem it follows that $n \geq 3$ for this variety. In 1963 Dorofeev proved that $n \geq 4$ and Shestakov in 1977 in [34, Theorem 13.1.2] proved that n is countable. Thus any finitely generated alternative algebra satisfies some identity which is not realized in the free alternative algebra from a countable set of generators. The question on whether there can be equality at some places in the chain of inclusions (5) turned out to be negative. It was proved by Shestakov, and in 1984 Filippov refined this result in [3] for any $n \neq 3$. For $n = 3$ the question is still open. Also, finding some system of defining identities for the varieties $\mathcal{A}lt_m$, $m \geq 3$ stays unsolved.

Finally, we define the *free alternative nil-algebra* $\mathcal{A}lt\text{-}\mathcal{N}il_n[X]$ of nil-index n as a quotient algebra

$$\mathcal{A}lt\text{-}\mathcal{N}il_n[X] = \mathcal{A}lt[X]/I_{Nil_n},$$

where I_{Nil_n} is the ideal of the free alternative algebra $\mathcal{A}lt[X]$ generated by the elements

$$\sum_{\pi \in Sym(n)} (\ldots((f_{\pi(1)} f_{\pi(2)}) f_{\pi(3)}) \cdots) f_{\pi(n)}, \text{ for } f_1, \ldots, f_n \in \mathcal{A}lt[X].$$

See Examples 2.5 and 2.15 for more details.

2.3.4 $\mathcal{V}$-superalgebras

Let $A = A_0 + A_1$ be a superalgebra. The even part of the superalgebra tensor product $G \otimes A$ is an algebra denoted by $G(A)$

$$G(A) = G_0 \otimes A_0 + G_1 \otimes A_1,$$

and it is called the *Grassmann envelope* of A.

Let $\mathcal{V}$ be a variety of algebras over a field $\mathbb{F}$. A superalgebra $A = A_0 + A_1$ is called a *$\mathcal{V}$-superalgebra* if and only if (iff) the Grassmann envelope $G(A)$ belongs to $\mathcal{V}$, i.e. G(A) is a $\mathcal{V}$-algebra. Let us show some interesting properties:

Example 2.10 ($\mathbb{C}$ is not $\mathcal{C}om$-superalgebra)
Recall that complex numbers $\mathbb{C}$ are a commutative algebra. Now we show that $\mathbb{C} = \mathbb{R} + \mathbb{R}\, i$ is not a commutative superalgebra. The superalgebra $\mathbb{C}$ is commutative if the Grassmann envelope $G(\mathbb{C}) = G_0 \otimes \mathbb{R} + G_1 \otimes \mathbb{R}\, i$ is commutative. This means

$$\widetilde{x}\,\widetilde{y} = \widetilde{y}\,\widetilde{x},$$

for any two elements $\widetilde{x},\ \widetilde{y} \in G(A)$, where

$$\begin{aligned}\widetilde{x} &= g_x \otimes x,\\ \widetilde{y} &= g_y \otimes y,\end{aligned}$$

for $g_x, g_y \in G_0 \cup G_1,\ x, y \in \mathbb{R} \cup \mathbb{R}\, i,\ \overline{x} = \overline{g}_x, \overline{y} = \overline{g}_y$. But this identity does not hold for $\widetilde{x},\ \widetilde{y} \in G_1 \otimes \mathbb{R}\, i$:

$$\begin{aligned}\widetilde{x}\,\widetilde{y} &= (g_x \otimes x)(g_y \otimes y) = g_x g_y \otimes xy = -g_y g_x \otimes yx = -(g_y \otimes y)(g_x \otimes x)\\ &= -\widetilde{y}\,\widetilde{x}.\end{aligned}$$

So that $\mathbb{C}$ is not a commutative superalgebra.

Example 2.11 (Grassmann algebra G is $\mathcal{C}om$-superalgebra)
The Grassmann algebra G is not commutative, but it is a commutative superalgebra, since its Grassmann envelope $G(G) = G_0 \otimes G_0 + G_1 \otimes G_1$ is commutative. This is evident from

$$\begin{aligned}\widetilde{x}\,\widetilde{y} &= (g_x \otimes x)(g_y \otimes y) = g_x g_y \otimes xy = (-1)^{\overline{g}_x \overline{g}_y} g_y g_x \otimes (-1)^{\overline{x}\,\overline{y}} yx\\ &= (-1)^{ij+ij} g_y g_x \otimes yx = g_y g_x \otimes yx = (g_y \otimes y)(g_x \otimes x) = \widetilde{y}\,\widetilde{x}\end{aligned}$$

for any $\widetilde{x} = g_x \otimes x,\ \widetilde{y} = g_y \otimes y$, where $x, g_x \in G_i,\ y, g_y \in G_j,\ i, j \in \{0, 1\}$.

Proposition 2.13 *A $\mathbb{Z}_2$-graded algebra $A = A_0 + A_1$ is associative as an algebra if and only if it is associative as a superalgebra.*

Proof: If A is an associative algebra, then $G(A)$ is associative. By the definition, A is an associative superalgebra.

Conversely, let $A = A_0 + A_1$ be an associative superalgebra. Then $G(A)$ is an associative algebra. Denote by

$$\begin{aligned}\widetilde{x}_i &= g_x \otimes x,\\ \widetilde{y}_j &= g_y \otimes y,\\ \widetilde{z}_k &= g_z \otimes z,\end{aligned}$$

where $x, y, z \in A_0 \cup A_1,\ g_x, g_y, g_z \in G_0 \cup G_1$. Then for $\widetilde{x}_i, \widetilde{y}_j, \widetilde{z}_k \in G(A)$ we have

$$(\widetilde{x}_i \widetilde{y}_j)\widetilde{z}_k = \widetilde{x}_i(\widetilde{y}_j \widetilde{z}_k)$$

and it leads to

$$g_x g_y g_z \otimes (xy)z = g_x g_y g_z \otimes x(yz).$$

Therefore $(xy)z = x(yz)$ and A is an associative algebra. □
By this proposition, Grassmann envelope $G(G)$ is associative algebra too, and so is G as a superalgebra.

Corollary 2.14 *Grassmann superalgebra G is associative.* □

Example 2.12 Let A be a superalgebra and $\mathcal{V}$ a variety of associative algebras defined by the identity

$$xyz - zyx = 0. \tag{6}$$

Determine the conditions for a superalgebra A to be a $\mathcal{V}$-superalgebra.

First, associativity of the Grassmann envelope $G(A)$ implies that of A. Furthermore, let $x, y, z \in A_0 \cup A_1$, $g_x, g_y, g_z \in G_0 \cup G_1$, $\overline{x} = \overline{g}_x, \overline{y} = \overline{g}_y$ and $\overline{z} = \overline{g}_z$. Then for $\widetilde{x} = g_x \otimes x$, $\widetilde{y} = g_y \otimes y$, $\widetilde{z} = g_z \otimes z \in G(A)$ we have

$$\begin{aligned} \widetilde{x}\,\widetilde{y}\,\widetilde{z} - \widetilde{z}\,\widetilde{y}\,\widetilde{x} &= g_x g_y g_z \otimes xyz - g_z g_y g_x \otimes zyx = \\ &= g_x g_y g_z \otimes (xyz - (-1)^{\overline{g}_x \overline{g}_y + \overline{g}_x \overline{g}_z + \overline{g}_y \overline{g}_z} zyx) = \\ &= g_x g_y g_z \otimes (xyz - (-1)^{\overline{x}\,\overline{y} + \overline{x}\,\overline{z} + \overline{y}\,\overline{z}} zyx). \end{aligned}$$

Thus $G(A)$ satisfies (6) if and only if A satisfies

$$xyz = (-1)^{\overline{x}\,\overline{y} + \overline{x}\,\overline{z} + \overline{y}\,\overline{z}} zyx. \tag{7}$$

The superalgebra A is a $\mathcal{V}$-superalgebra iff it is associative and satisfies (7).

Now, let us give some examples of identities of $\mathcal{V}$-superalgebras.

Example 2.13 (Identities of $\mathcal{C}om$-superalgebras)
A commutative superalgebra A must satisfy

$$uv = (-1)^{\overline{u}\,\overline{v}} vu,$$

for u, v homogeneous. Let $x, y \in A_0 \cup A_1$ and $g_x, g_y \in G_0 \cup G_1$, $\overline{x} = \overline{g}_x, \overline{y} = \overline{g}_y$. Using the Grassmann envelope, the commutativity $\widetilde{x}\,\widetilde{y} - \widetilde{y}\,\widetilde{x} = 0$ must hold for every $\widetilde{x} = g_x \otimes x, \widetilde{y} = g_y \otimes y \in G(A)$. But in more detail we have

$$\begin{aligned} \widetilde{x}\,\widetilde{y} - \widetilde{y}\,\widetilde{x} &= (g_x \otimes x)(g_y \otimes y) - (g_y \otimes y)(g_x \otimes x) \\ &= g_x g_y \otimes xy - g_y g_x \otimes yx = g_x g_y \otimes xy - (-1)^{\overline{g}_x \overline{g}_y} g_x g_y \otimes yx \\ &= g_x g_y \otimes (xy - (-1)^{\overline{x}\,\overline{y}} yx). \end{aligned}$$

Example 2.14 (Identities of $\mathcal{A}lt$-superalgebras)
We show that an alternative superalgebra A satisfy the identities

$$\begin{aligned} (u, v, w) + (-1)^{\overline{u}\,\overline{v}} (v, u, w) = 0, \\ (u, v, w) + (-1)^{\overline{v}\,\overline{w}} (u, w, v) = 0. \end{aligned} \tag{8}$$

They are obtained from the linearized identities of left and right alternativity (2) using the Grassmann envelope. Let us take $x, y, z \in A_0 \cup A_1$ and $g_x, g_y, g_z \in G_0 \cup G_1$, such that $\overline{x} = \overline{g}_x, \overline{y} = \overline{g}_y$ and $\overline{z} = \overline{g}_z$. Then for every $\widetilde{x} = g_x \otimes x, \widetilde{y} = g_y \otimes y, \widetilde{z} = g_z \otimes z \in G(A)$ we have

$$\begin{aligned}
(\widetilde{x}, \widetilde{y}, \widetilde{z}) + (\widetilde{y}, \widetilde{x}, \widetilde{z}) &= g_x g_y g_z \otimes (x, y, z) + g_y g_x g_z \otimes (y, x, z) = \\
&= g_x g_y g_z \otimes ((x, y, z) + (-1)^{\overline{g}_x \overline{g}_y}(y, x, z)) = \\
&= g_x g_y g_z \otimes ((x, y, z) + (-1)^{\overline{x}\,\overline{y}}(y, x, z))
\end{aligned}$$

$$\begin{aligned}
(\widetilde{x}, \widetilde{y}, \widetilde{z}) + (\widetilde{x}, \widetilde{z}, \widetilde{y}) &= g_x g_y g_z \otimes (x, y, z) + g_x g_z g_y \otimes (x, z, y) = \\
&= g_x g_y g_z \otimes ((x, y, z) + (-1)^{\overline{g}_y \overline{g}_z}(x, z, y)) = \\
&= g_x g_y g_z \otimes ((x, y, z) + (-1)^{\overline{y}\,\overline{z}}(x, z, y)).
\end{aligned}$$

Since $G(A)$ is an alternative algebra, the superalgebra A must satisfy (8).

As we can see, some new type of identities arise.

Superidentities

These new identities we will call *superidentities*. The superidentities can be obtained by the following algorithm: Let $\mathcal{V}$ be a variety of algebras defined by a system of multilinear identities (if not, linearize them first). Passing from $\mathcal{V}$-algebras to $\mathcal{V}$-superalgebras use so called "superization rule" (or Kaplansky's principle). It says, that whenever two odd variables are transposed in the identity, a negative sign is introduced. The superalgebra is then a $\mathcal{V}$-superalgebra iff it satisfies the obtained superidentities. This equivalence comes directly from the definition of $\mathcal{V}$-superalgebras. The superidentity from Example 2.13 is called *supercommutativity*, the identities from Example 2.14 are called *left/right superalternativity*. Let us show more examples.

Example 2.15 (Nil$_n$ superidentity)
Superidentities defining (nonassociative) $\mathcal{N}il_n$-superalgebras are obtained from the linearized form of the identity $x^n = 0$. Due to the fact, that nil-algebras are power-associative, the linearization does not depend on the bracketing. We use the natural lefthandside bracketing of the linearized identity $x^n = 0$:

$$\sum_{\pi \in Sym(n)} (\cdots (x_{\pi(1)} x_{\pi(2)}) x_{\pi(3)}) \cdots) x_{\pi(n)} = 0.$$

Using the superization rule we obtain the superidentity

$$\sum_{\pi \in Sym(n)} \mathrm{sign}_{\mathrm{odd}}(\pi) (\cdots ((u_{\pi(1)} u_{\pi(2)}) u_{\pi(3)}) \cdots) u_{\pi(n)} = 0, \qquad (9)$$

where $\text{sign}_{\text{odd}}(\pi)$ is the sign of the permutation afforded by π on the odd u_i.

Example 2.16 (Alt-Nil$_n$ superidentity)
Using the previous example, the $\mathcal{A}lt$-$\mathcal{N}il_n$-superalgebras are defined by the superidentities (8) and (9).

Associativity and superassociativity are the same (see Proposition 2.13).

It is worth noting that Artin's theorem does not carry over the superalgebras.

Proposition 2.15 (Artin's theorem and superalgebras)
An alternative superalgebra A on one odd generator is not associative.

Two proofs of this Proposition are in Section 3.

2.3.5 Varieties of superalgebras

Now, let us have $Z = X \cup Y,\ X \cap Y = \emptyset$, where X is a set of even generators and Y a set of odd generators. Moreover we define a parity of the generators as $\bar{z} = 0$ if it is even ($\bar{z} = 1$ if it is odd, respectively) for each $z \in Z$. Then we can consider a free algebra $\mathbb{F}[Z]$ as a superalgebra and define a polynomial $f(x_1, \ldots, x_k, y_1, \ldots, y_l) \in \mathbb{F}[Z]$ to be a *graded identity* of a superalgebra A if

$$f(a_1, \ldots, a_k, b_1, \ldots, b_l) = 0$$

for all the elements $a_1, \ldots, a_k \in A_0,\ b_1, \ldots, b_l \in A_1$. Denote by $I_2(A)$ a set of graded identities of a superalgebra A. Then $I_2(A)$ is an *(graded) ideal* of the superalgebra $\mathbb{F}[Z]$.

Proposition 2.16 ([12, Proposition 2.20])
The set $I_2(A)$ is an ideal of the superalgebra $\mathbb{F}[X \cup Y]$.

Observe that ideals of graded identities of superalgebras are also invariant under superalgebra endomorphisms of superalgebra $\mathbb{F}[X \cup Y]$. If I_2 is a graded ideal of $\mathbb{F}[X \cup Y]$ then we can consider a variety of superalgebras $V_2(I)$ defined by I_2 (in fact, any set of elements of $\mathbb{F}[Z]_0 \cup \mathbb{F}[Z]_1$ defines a variety of superalgebras). In the same manner, as for varieties of algebras, there exists a bijective correspondence between graded ideals of superalgebra $\mathbb{F}[X \cup Y]$ and varieties of superalgebras.

Next we study a relation between $I(A)$ and $I_2(A_0 + A_1)$ for a superalgebra $A = A_0 + A_1$. Observe that $I(A)$ is an ideal of $\mathbb{F}[X]$, while $I_2(A_0 + A_1)$ is an ideal of $\mathbb{F}[X \cup Y]$. Let $f = f(x_1, \ldots, x_k) \in \mathbb{F}[X]$ be a multilinear element.

For each subset $I = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$, denote by $f_I(z_1, \dots, z_n)$ a polynomial in $\mathbb{F}[X \cup Y]$, where f_I is equal f, and variables z_i are defined in the following way:

$$z_i = x_i; \text{ if } i \notin I, \qquad z_i = y_i, \text{ if } i \in I.$$

In other words, to obtain $f_I(z_1, \dots, z_n)$ from the polynomial $f(x_1, \dots, x_n)$ we change the variables x_{i_j} by y_{i_j} for $i_j \in I$.

Proposition 2.17 ([12, Lemma 2.21])
Let $A = A_0 + A_1$ be a superalgebra and let $f = f(x_1, \dots, x_k) \in \mathbb{F}[X]$ be a multilinear element. Then $f \in I(A)$ if and only if $f_I \in I_2(A)$, for any subset I of $\{1, \dots, n\}$.

Let $Mult[Z]$ be the subspace of multilinear elements in $\mathbb{F}[Z]$. Each element $f \in Mult[Z]$ can be written in the form

$$f(z_1, \dots, z_n) = \sum_{\pi, u} \alpha(\pi, u)\, u(z_{\pi(1)}, \dots, z_{\pi(n)})$$

where $\alpha(\pi, u) \in \mathbb{F}, \pi \in Sym(n)$ and u is a monomial of degree n. Define a mapping $^* : Mult[Z] \to Mult[Z]$ which maps any $f(z_1, \dots, z_n)$ to

$$f^*(z_1, \dots, z_n) = \sum_{\pi, u} \alpha(\pi, u)\, \mathrm{sign}_{\mathrm{odd}}(\pi) u(z_{\pi(1)}, \dots, z_{\pi(n)})$$

where $\mathrm{sign}_{\mathrm{odd}}(\pi)$ is the sign of the permutation afforded by π on the odd z_i.

Lemma 2.18 ([12, Lemma 2.22])
Let $A = A_0 + A_1$ be a superalgebra.

- *(i) If $f \in \mathbb{F}[X \cup Y]$ is multilinear then $f \in I_2(G(A))$ if and only if $f^* \in I_2(A)$.*
- *(ii) If $f \in \mathbb{F}[X_n]$ is multilinear, $X_n = \{x_1, \dots, x_n\}$, then $f \in I(G(A))$ if and only if $(f_I)^* \in I_2(A)$, for any subset $I \subseteq \{1, \dots, n\}$.*

Corollary 2.19 ([12, Corollary 2.23])
Let $\mathcal{V} = V(f_1, \dots, f_n, \dots)$, where each f_i is multilinear. Then $A = A_0 + A_1$ is a $\mathcal{V}$-superalgebra if and only if for any set I and any index i,

$$(f_i)^*_I \in I_2(A).$$

Notice that for each identity $f(x_1, \dots, x_n)$ we can obtain 2^n graded identities $f^*_I(z_1, \dots, z_n)$ with $I \subseteq \{1, \dots, n\}$:

Example 2.17 (Graded identities of $\mathcal{C}om$-superalgebra)
The commutativity $xy = yx$ leads to the four graded identities:

$$\begin{aligned} x_0y_0 &= y_0x_0, \\ x_0y_1 &= y_1x_0, \quad (10) \\ x_1y_0 &= y_0x_1, \quad (11) \\ x_1y_1 &= -y_1x_1, \end{aligned}$$

for x_0, y_0 are even, x_1, y_1 are odd.

Corollary 2.20 ([12, Corollary 2.24])
Let $\mathcal{V} = V(f_1, \dots, f_n, \dots)$, where each f_i is multilinear. Then a class of all $\mathcal{V}$-superalgebras is a variety of superalgebras. In particular, over a field of characteristic zero, for any variety $\mathcal{V}$ (not necessary defined by multilinear identities), the class of $\mathcal{V}$-superalgebras is also a variety.

Example 2.18 (Variety of Com-superalgebras)
The class of commutative superalgebras is defined by three graded identities derived from the commutativity $ab = ba$:

$$\begin{aligned} ab &= ba, \\ ax &= xa, \\ xy &= -yx, \end{aligned}$$

where the elements a, b are even and x, y are odd elements. (Compare with Example 2.17, the equations (10) and (11) are replaced by one $ax = xa$ here.)

Example 2.19 (Variety of $\mathcal{N}il_3$-superalgebras)
The class of associative nil-superalgebras of nil-index 3 is defined by four graded identities

$$\begin{aligned} abc + bca + cab + acb + bac + cba &= 0, \\ abx + bxa + xab + axb + bax + xba &= 0, \\ axy + xya - yax - ayx + xay - yxa &= 0, \\ xyz + yzx + zxy - xzy - yxz - zyx &= 0, \end{aligned}$$

where a, b, c are even and x, y, z are odd. (Remaining four graded identities coincide with these four.)

3 The superalgebra technique

Superalgebras (or $\mathbb{Z}_2$-graded algebras), naturally occur in physics and the name "super-" came from physics – quantum field theories that involve Fermi and Bose fields. But they were mathematicians, who have used these algebras first in algebraic topology, homology of algebras, calling them graded Lie algebras (today Lie superalgebras). Note that in all articles written before 1974 is used only the name graded Lie algebras. The precise definitions and description of basic algorithms of the superalgebras is dated in the eighties of last century. [10]

Superalgebras can be used to the study of identities of free algebras and to the development of a structure theory of varieties of algebras. The method was first used by Kemer [7] in 1984, who applied it to the study of identities of associative algebras. In the later article (see [8], 1987), he solved the Specht problem by proving that all varieties of associative algebras of characteristic zero are finitely based. In the articles by Zelmanov (see [31, 32], 1987 and 1989), Zelmanov and Shestakov (see [33], 1990) the superalgebra method was developed further and applied to nilpotency problems for nonassociative algebras. [20]

These results of Kemer, Zelmanov or Shestakov are purely "qualitative", without quantitative estimates. The next authors used the superalgebra method just for obtaining exact estimates and computing dimensions of homogeneous component of free algebras (by a computer). Vaughan-Lee (see [29, 30], 1993 and 1998) applied the superalgebra method to decrease the number of variables in finding of the index of the nilpotency of algebras in the variety $\mathcal{N}il_4$. Following his steps, Shestakov and Zhukavets (see [22], 2004) investigated Kuzmin's conjecture for $n = 5$ to confirm it for every two-generated superalgebra over a field of characteristic 0 in the variety of associative $\mathcal{N}il_5$-superalgebras.

In 2003, Shestakov (see [21], 2003) found a base of the free Malcev superalgebra on one odd generator. Then, Shestakov and Zhukavets obtained bases of the universal multiplicative envelope for the free Malcev superalgebra on one odd generator, the Malcev Grassmann algebra (see [23], 2006), and the Poisson Malcev superalgebra related with the free Malcev superalgebra on one odd generator (see [26, 24], 2006).

In a corollary of their study they proved a linear independence of the elements that span the free alternative superalgebra on one odd generator (see [27], 2007), which was a couple of years unproved. They also obtained central skew-symmetric elements in free Malcev and in free alternative algebras [23, 26], they proved that there are no skew-symmetric Malcev s-identities [25], they found a new element of degree 5 from the radical of the free alternative algebra of countable rank and they proved that the square of the radical is not zero [27].

In 2009, using the superalgebra method, they classified all multilinear skew-symmetric identities and central polynomial functions of octonions (see [28]).

The following text is based on [12] and [20]. We will consider only varieties of algebras defined by multilinear identities (which is always satisfied if, for example, a field has zero characteristic), the multilinearity of identities is required for passage from $\mathcal{V}$-algebras to $\mathcal{V}$-superalgebras. Furthermore, while speaking about elements or ideals of superalgebras, we will always assume only homogeneous ones.

Superscalar extension

Recall that for a $\mathcal{V}$-superalgebra A its Grassmann envelope $G(A)$ is a $\mathcal{V}$-algebra. In this section we consider the opposite direction and describe a passage from $\mathcal{V}$-algebras to $\mathcal{V}$-superalgebras, more precisely, for any $\mathcal{V}$-algebra A we show how to construct a $\mathcal{V}$-superalgebra A_G.

Proposition 3.1 *Let $\mathcal{V}$ be a variety of algebras defined by a set of multilinear identities. If B is an associative commutative algebra, then $B \otimes A \in \mathcal{V}$ for any $A \in \mathcal{V}$.*

Proof: Write any identity $f = f(x_1, \dots, x_n)$ satisfied in A as a sum of multilinear monomials

$$f = \sum_i m_i.$$

We prove that f is satisfied in $B \otimes A$. For any $a_1, \dots, a_n \in A,\ b_1, \dots, b_n \in B$ we have

$$\begin{aligned} f(b_1 \otimes a_1, \dots, b_n \otimes a_n) &= \sum_i m_i(b_1 \otimes a_1, \dots, b_n \otimes a_n) \\ &= \sum_i b_1 \cdots b_n \otimes m_i(a_1, \dots, a_n) \\ &= b_1 \cdots b_n \otimes f(a_1, \dots, a_n), \end{aligned}$$

since all multilinear monomials in commutative associative variables of degree n are equal. We can see, that if identity f is satisfied in A, then it is satisfied in $B \otimes A$. □

Corollary 3.2 *Let $\mathcal{V}$ be a variety of algebras defined by a set of multilinear identities and G be a Grassmann superalgebra. Then for any $A \in \mathcal{V}$ the tensor product $A_G = G \otimes A = G_0 \otimes A + G_1 \otimes A$ is a $\mathcal{V}$-superalgebra.*

Proof: The most important condition for this construction is, that G is an associative commutative superalgebra as was shown in Corollary 2.14 and Example 2.11. In this view so is algebra $G(G)$ and it is enough to prove that $G(A_G) = G(G \otimes A)$ is equal $G(G) \otimes A$. We have

$$\begin{aligned} G(A_G) &= G_0 \otimes (G_0 \otimes A) + G_1 \otimes (G_1 \otimes A) \\ &= (G_0 \otimes G_0) \otimes A + (G_1 \otimes G_1) \otimes A \\ &= G(G) \otimes A, \end{aligned}$$

which implies that $G(A_G) \in \mathcal{V}$ for every $A \in \mathcal{V}$ and A_G is a $\mathcal{V}$-superalgebra. □

We have proved that every algebra A in the variety $\mathcal{V}$ can be imbedded into the $\mathcal{V}$-superalgebra A_G by means of the extension of the field of scalars $\mathbb{F}$ to the "domain of superscalars" G. Passage to superscalar extension allow us to reduce the number of variables in the identities that are multilinear skew-symmetric polynomials.

Reduction of the number of variables

It is known that in characteristic zero any multilinear symmetric polynomial function may be obtained by a linearization of a polynomial function of degree n on one variable [20]. Now we investigate in detail the case, when the polynomial is multilinear *skew-symmetric*. Using $G \otimes A$, a number of variables can be reduced to one variable. However, the new variable will not lie in A but in $G \otimes A$. The problem is that in general $G \otimes A$ does not belong to the same variety as A. For instance, if $A = \mathbb{F}$ then $G \otimes \mathbb{F} = G$ is already not commutative. Nevertheless, $G \otimes A$ satisfies certain graded identities related with those of A. More precisely, if

$$f : A \times \cdots \times A \to A$$

is a multilinear skew-symmetric polynomial function of degree n, one can consider $A_G = G \otimes A$ and $f_G : A_G \times \cdots \times A_G \to A_G$ of degree n given by

$$f_G(g_1 \otimes a_1, g_2 \otimes a_2, \ldots, g_n \otimes a_n) = g_1 \cdots g_n \otimes f(a_1, \ldots, a_n),$$

for any homogeneous $g_i \in G$ and $a_i \in A$. If we take an odd element

$$\widetilde{a} = e_1 \otimes a_1 + \cdots + e_n \otimes a_n \in A_G,$$

where $e_1, e_2, \ldots, e_n$ are the odd generators of G, $a_i \in A$, we get

$$\begin{aligned} f_G(\widetilde{a}, \ldots, \widetilde{a}) &= f_G(e_1 \otimes a_1 + \cdots + e_n \otimes a_n, \ldots, \\ &\quad e_1 \otimes a_1 + \cdots + e_n \otimes a_n) \\ &= \sum_{\pi \in Sym(n)} e_{\pi(1)} \cdots e_{\pi(n)} \otimes f(a_{\pi(1)}, \ldots, a_{\pi(n)}) \\ &= \sum_{\pi \in Sym(n)} \mathrm{sign}(\pi) e_1 \cdots e_n \otimes \mathrm{sign}(\pi) f(a_1, \ldots, a_n) \\ &= n! \cdot e_1 \cdots e_n \otimes f(a_1, \ldots, a_n). \end{aligned}$$

Therefore, $f(a_1, a_2, \ldots, a_n) = 0$ if and only if $f_G(\widetilde{a}, \ldots, \widetilde{a}) = 0$ and the identity $f(x_1, x_2, \ldots, x_n) = 0$ is reduced to the identity in one *odd* variable over the superalgebra A_G.

Similar trick works for a *symmetric* polynomial function. It may be reduced to an identity in a single *even* variable

$$\widetilde{a} = e_1 e_2 \otimes a_1 + \cdots + e_{2n-1} e_{2n} \otimes a_n \in A_G.$$

We can also use a passage to superscalar extension to prove of Proposition 2.15.

Proof I: (Proposition 2.15)[Artin's theorem and superalgebras]
We prove that there is an alternative superalgebra on one odd generator that is not associative, i.e. Artin's theorem does not carry over the superalgebras. By the Proposition 3.1 we can imbed a $\mathcal{V}$-algebra A into a $\mathcal{V}$-superalgebra A_G. Take as A an octonion algebra $\mathbb{O}$, with 1 and imaginary units $e_1, \ldots, e_7$. Recall that for example

$$(e_1, e_2, e_3) = -2e_6 \neq 0.$$

$\mathbb{O}_G = G \otimes \mathbb{O} = G_0 \otimes \mathbb{O} + G_1 \otimes \mathbb{O}$ is an alternative superalgebra, since $\mathbb{O}$ is an alternative algebra. If we take an odd generator

$$\widetilde{a} = g_1 \otimes a_1 + g_2 \otimes a_2 + g_3 \otimes a_3$$

for any three elements $a_1, a_2, a_3 \in \mathbb{O}$ and odd generators $g_1, g_2, g_3 \in G$, then this one-generated subsuperalgebra in $\mathbb{O}_G$ is associative iff

$$(\widetilde{a}, \widetilde{a}, \widetilde{a}) = 3! \cdot g_1 g_2 g_3 \otimes (a_1, a_2, a_3) = 0.$$

But for

$$\widetilde{a} = g_1 \otimes e_1 + g_2 \otimes e_2 + g_3 \otimes e_3$$

we get

$$\begin{aligned}(\widetilde{a},\widetilde{a},\widetilde{a}) &= 3!\,\cdot g_1g_2g_3 \otimes (e_1,e_3,e_3) = 3!\,\cdot g_1g_2g_3 \otimes (-2)e_6 \\ &\neq 0\end{aligned}$$

and A is not associative. □

3.1 Applications

In this subsection, we introduce some applications of the superalgebras to the study of identities of free algebras or to development of the structure theory of varieties of algebras. The overview follows.

3.1.1 The subspace of skew-symmetric elements

Let $\mathcal{V}$ be a variety of algebras over a field of characteristic zero. In this subsection we show how the free $\mathcal{V}$-superalgebra on one odd generator can be used to describe the subspace of skew-symmetric elements of the free $\mathcal{V}$-algebra on a countable set of generators.

Let $f = f(x)$ be a homogeneous nonassociative polynomial of degree n on one variable x. It may be written in the form $f(x) = \widetilde{f}(x, x, \ldots, x)$, for a certain multilinear polynomial $\widetilde{f}(x_1, x_2, \ldots, x_n)$. We define the skew-symmetric polynomial *Skew* f as

$$Skew\ f(x_1, x_2, \ldots, x_n) = \sum_{\pi \in Sym(n)} \operatorname{sign}(\pi)\widetilde{f}(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)}). \qquad (12)$$

Let denote by $\mathcal{V}[T;X]$ the free $\mathcal{V}$-superalgebra over a field $\mathbb{F}$ generated by a set T of even generators and by a set X of odd generators. Let $\mathcal{V}[\emptyset;x]$ be the free $\mathcal{V}$-superalgebra on one odd generator x and let $\mathcal{V}[T] = \mathcal{V}[T;\emptyset]$ be the free $\mathcal{V}$-algebra on a countable set of even generators $T = \{t_1, t_2, \ldots, t_n, \ldots\}$. Then $Skew : \mathcal{V}[\emptyset;x] \to \mathcal{V}[T]$ is a linear mapping which maps isomorphically the homogeneous component $\mathcal{V}[\emptyset;x]^{[n]}$ of degree n of $\mathcal{V}[\emptyset;x]$ to the subspace $Skew(\mathcal{V}[T_n])$ of multilinear skew-symmetric elements on $T_n = \{t_1, t_2, \ldots, t_n\}$ of $\mathcal{V}[T]$.

Theorem 3.3 *For a homogeneous polynomial f of degree n, the free $\mathcal{V}$-superalgebra $\mathcal{V}[\emptyset;x]$ on one odd generator x and the free $\mathcal{V}$-algebra $\mathcal{V}[T]$ on a set of even generators $T = \{t_1, t_2, \ldots, t_n, \ldots\}$ holds:*

$$f(x) = 0 \text{ in } \mathcal{V}[\emptyset;x] \text{ if and only if } Skew\, f(t_1, t_2, \ldots, t_n) = 0 \text{ in } \mathcal{V}[T].$$

Proof:
Assume $f(x) = 0$.
Consider in the $\mathcal{V}$-superalgebra $G \otimes \mathcal{V}[T]$ the odd element y:

$$y = e_1 \otimes t_1 + \cdots + e_n \otimes t_n.$$

Since $f(x) = 0$ in the free $\mathcal{V}$-superalgebra,

$$0 = f(y) = e_1 e_2 \cdots e_n \otimes \mathit{Skew}\ f(t_1, \ldots, t_n)$$

and so $\mathit{Skew}\ f(t_1, \ldots, t_n)$ has to be zero.

Now, assume $\mathit{Skew}\ f(t_1, \ldots, t_n) = 0$.
Consider in the Grassmann envelope $G(\mathcal{V}[\emptyset; x])$ of $\mathcal{V}[\emptyset; x]$ the elements $s_1 = e_1 \otimes x, \ldots, s_n = e_n \otimes x_n$. Then we have

$$0 = \mathit{Skew} f(s_1, s_2, \ldots, s_n) = n!\, e_1 e_2 \cdots e_n \otimes f(x)$$

and so $f(x)$ has to be zero too. □

This correspondence was used in [21, 24, 25, 28] or in [27] for a description of the subspace of skew-symmetric elements of the free alternative algebra on a countable set of generators. We will use this correspondence in Section 5.1.

Skew-symmetric elements in alternative algebras

Let $\mathcal{A} = \mathit{Alt}[\emptyset; x]$ be the free alternative superalgebra on one odd generator x. Define by induction

$$x^{[1]} = x,\ \ x^{[i+1]} = [x^{[i]}, x]_s,\ \ i > 0,$$

and denote

$$t = x^{[2]},\ \ z^{[k]} = [x^{[k]}, t],\ \ u^{[k]} = x^{[k]} \circ_s x^{[3]},\ \ k > 1,$$

where $[x, y]_s = xy - (-1)^{\bar{x}\bar{y}} yx$ denotes the *supercommutator* of the homogeneous elements x, y, and by

$$x \circ_s y = xy + (-1)^{\bar{x}\bar{y}} yx$$

denotes their *super-Jordan product.*

The following propositions summarize some results from [24, 25, 27] on the structure of $\mathcal{A}$.

Proposition 3.4

(*i*) *The elements*

$$t^m x^\sigma,\ m+\sigma \geq 1, \quad t^m(x^{[k+2]}x^\sigma),$$
$$t^m(u^{[4k+\varepsilon]}x^\sigma), \quad t^m(z^{[4k+\varepsilon]}x^\sigma), \tag{13}$$

where $k > 0$, $m \geq 0$ are integers; ε, $\sigma \in \{0,1\}$, form a base of the superalgebra $\mathcal{A}$.

(*ii*) *For any integer $k > 0$,*

$$z^{[4k-1]} = z^{[4k-2]} = u^{[4k-1]} = 0, \quad u^{[2]} = 0, \quad u^{[4k+2]} = -tz^{[4k+1]}.$$

(*iii*) *The nucleus (associative center) of $\mathcal{A}$ is equal to the ideal*

$$id_{\mathcal{A}}\langle u^{[k]}, z^{[k]} \mid k > 1\rangle.$$

The center of $\mathcal{A}$ is equal to the vector space

$$vect\langle t^m z^{[k]},\ t^m(2z^{[k]}x - u^{[k]}) \mid m \geq 0,\ k > 2\rangle.$$

□

From now on, we will write the products $t^m u^{[k]} x^\sigma$, $t^m z^{[k]} x^\sigma$, without parenthesis. Moreover, we set $t^0 = 1$ and $t^n = 0$ for $n < 0$.

Proposition 3.5 *The multiplication table on the base elements is given by the following rules: For any $m, n \geq 0$, $k > 2$, ε, $\sigma \in \{0,1\}$, $n + \sigma \geq 1$, we have that*

$$t^m u^{[k]} x^\sigma, \quad t^m z^{[k]} x^\sigma,$$

annihilate all elements of the base (13) except elements of first type. Further,

$$(t^m u^{[k]} x^\varepsilon)(t^n x^\sigma) = \begin{cases} t^{m+n} u^{[k]} x^{\varepsilon+\sigma}, & \varepsilon+\sigma < 2, \\ \frac{1}{2} t^{m+n+1} u^{[k]}, & \varepsilon+\sigma = 2, \end{cases}$$
$$(t^m z^{[k]} x^\varepsilon)(t^n x^\sigma) = \begin{cases} t^{m+n} z^{[k]} x^{\varepsilon+\sigma}, & \varepsilon+\sigma < 2, \\ \frac{1}{2} t^{m+n+1} z^{[k]}, & \varepsilon+\sigma = 2, \end{cases}$$

$$(t^n x)(t^m u^{[k]} x^\varepsilon) = (-1)^{k+1} \begin{cases} t^{m+n}(u^{[k]}x - 2tz^{[k]}), & \varepsilon = 0, \\ t^{m+n+1}(\frac{1}{2}u^{[k]} - 2z^{[k]}x), & \varepsilon = 1, \end{cases}$$

$$(t^n x)(t^m z^{[k]} x^\varepsilon) = (-1)^{k} \begin{cases} t^{m+n}z^{[k]}x, & \varepsilon = 0, \\ \frac{1}{2}t^{m+n+1}z^{[k]}, & \varepsilon = 1. \end{cases}$$

$$\begin{aligned} t^m(t^n x^\sigma) &= t^{m+n}x^\sigma, \\ (t^m x)t^n &= t^{m+n}x - nt^{m+n-1}x^{[3]}, \\ (t^m x)(t^n x) &= \tfrac{1}{2}t^{m+n+1} - nt^{m+n-1}(x^{[3]}x) + \tfrac{m+2n}{3}t^{m+n-1}x^{[4]} \\ &\quad + \tfrac{m(m+2n-1)}{6}t^{m+n-2}z^{[4]}. \end{aligned}$$

$$\begin{aligned} t^m\left(t^n(x^{[k]}x^\varepsilon)\right) &= t^{m+n}(x^{[k]}x^\varepsilon), \\ (t^m x)(t^n x^{[k]}) &= (-1)^k \left(t^{m+n}((x^{[k]}x) - x^{[k+1]}) - t^{m+n-1}(\tfrac{n}{2}u^{[k]} + \tfrac{2m+n}{6}z^{[k+1]})\right), \\ (t^m x)(t^n(x^{[k]}x)) &= (-1)^k \left(\tfrac{1}{2}t^{m+n+1}x^{[k]} + t^{m+n}(\tfrac{2}{3}x^{[k+2]} - x^{[k+1]}x + \tfrac{2m+4n+1}{6}z^{[k]}\right) \\ &- t^{m+n-1}(\tfrac{n}{2}u^{[k]}x - \tfrac{m+2n}{6}u^{[k+1]} + \tfrac{2m+n}{6}z^{[k+1]}x - \tfrac{m}{6}z^{[k+2]})\big). \end{aligned}$$

$$\begin{aligned} (t^n x^{[k]})t^m &= t^{m+n}x^{[k]} + mt^{m+n-1}z^{[k]}, \\ (t^n x^{[k]})(t^m x) &= t^{m+n}(x^{[k]}x) + t^{m+n-1}(mz^{[k]}x + \tfrac{2m+n}{3}z^{[k+1]}), \\ (t^n(x^{[k]}x))t^m &= t^{m+n}(x^{[k]}x) - mt^{m+n-1}(\tfrac{1}{2}u^{[k]} - z^{[k]}x - \tfrac{1}{2}z^{[k+1]}), \\ (t^n(x^{[k]}x))(t^m x) &= \tfrac{1}{2}t^{m+n+1}x^{[k]} + t^{m+n}(\tfrac{1}{3}x^{[k+2]} + \tfrac{7m+2n+2}{6}z^{[k]}) \\ &\quad - t^{m+n-1}(\tfrac{m}{2}u^{[k]}x - \tfrac{2m+n}{6}u^{[k+1]} + \tfrac{m+2n}{6}z^{[k+1]}x \\ &\quad - \tfrac{2m+n}{6}z^{[k+2]}). \end{aligned}$$

$$\begin{aligned} (t^m x^{[i]})(t^n x^{[j]}) &= \tfrac{1}{2}(-1)^{c(j+1)}t^{m+n}(u^{[i+j-3]} + \delta_j tz^{[i+j-4]} - (-1)^j z^{[i+j-2]}), \\ (t^m x^{[i]})(t^n(x^{[j]}x)) &= \tfrac{1}{2}(-1)^{c(j+1)}t^{m+n}(u^{[i+j-3]}x + \delta_j t(z^{[i+j-4]}x) \\ &- (-1)^j z^{[i+j-2]}x - \tfrac{2}{3}z^{[i+j-1]}), \\ (t^m(x^{[i]}x))(t^n x^{[j]}) &= \tfrac{1}{2}(-1)^{c(j)}t^{m+n}(u^{[i+j-3]}x + (-1)^j u^{[i+j-2]} \\ &+ \delta_j t(z^{[i+j-4]}x) + (-1)^j \delta_{j-1}tz^{[i+j-3]} \\ &- (-1)^j z^{[i+j-2]}x - \tfrac{1}{3}z^{[i+j-1]}), \\ (t^m(x^{[i]}x))(t^n(x^{[j]}x)) &= \tfrac{1}{2}(-1)^{c(j)}t^{m+n}(\tfrac{1}{2}tu^{[i+j-3]} + (-1)^j u^{[i+j-2]}x \\ &- \tfrac{1}{3}u^{[i+j-1]} + \tfrac{1}{2}\delta_j t^2 z^{[i+j-4]} - \delta_{j-1}t(z^{[i+j-3]}x) \\ &- (\tfrac{5}{6}\delta_j - \tfrac{1}{2})tz^{[i+j-2]} + \tfrac{1}{3}z^{[i+j-1]}x - \tfrac{1}{3}(-1)^j z^{[i+j]}), \end{aligned}$$

where $c(j) = j(j-1)/2$ *and* $\delta_j = 1 + (-1)^j$. □

Notice that for $k \geq 3$

$$\begin{array}{rcl} x^{[3]}x^{[k]} & = & \frac{1}{2}(-1)^k(u^{[k]} - z^{[k+1]}), \\ x^{[3]}(x^{[k]}x) & = & (-1)^k(\frac{1}{2}u^{[k]}x - \frac{1}{2}z^{[k+1]}x + \frac{1}{3}z^{[k+2]}). \end{array}$$

Moreover, results of [27] also imply

$$\begin{array}{rcl} (t^m, x^{[3]}, x^{[k]}x^{\varepsilon}) & = & 0, \\ (t^m, t^n, v) & = & 0, \end{array}$$

where $\varepsilon \in \{0, 1\}$, $m, n \geq 1$, $k > 2$ and for any element $v \in \mathcal{A}$.

Using this multiplication table, we present another proof of Proposition 2.15.
Proof II: (Proposition 2.15)[Artin's theorem and superalgebras]
From the multiplication table of $\mathcal{A}$ one can easily compute

$$(x, x, x) = \frac{1}{2}x^3 \neq 0.$$

Therefore $\mathcal{A}$ is an alternative superalgebra on one odd generator and it is not associative superalgebra. □

Let $Alt\,[T] = Alt\,[T; \emptyset]$ be the free alternative algebra on a set of even generators T and let $Skew$ be the linear mapping from $\mathcal{A}$ to $Alt[T]$ defined in Subsection 3.1.1. Then $Skew$ maps isomorphically the homogeneous component $\mathcal{A}^{[n]}$ of degree n of $\mathcal{A}$ to the subspace $Skew(Alt[T_n])$ of multilinear skew-symmetric elements on $T_n = \{t_1, t_2, \ldots, t_n\}$ of $Alt\,[T]$.

Theorem 3.6 ([27, Theorem 5.1])
The elements

$$Skew\, f(t_{i_1}, t_{i_2}, \ldots, t_{i_k}),$$

where $f = f(x)$ runs through the set (13), $k = \deg(f)$, $i_1 < i_2 < \cdots < i_k$, form a base of the space $Skew(Alt[T])$ of skew-symmetric elements of $Alt\,[T]$.
□

Corollary 3.7 ([27, Corollary 5.2])
Let $d(n)$ denotes the dimension of the subspace $Skew(Alt[T_n])$. Then $d(1) = d(2) = 1$, $d(3) = 2$, and for $n > 3$

$$d(n) = 2(n-3) + \tfrac{1}{2}(1 + (-1)^{c(n+1)}),$$

where $c(i) = \frac{i(i-1)}{2}$. □

Open problems [27]

It was proved in [23] that the elements $Skew\, z^{[k]}(t_1, \ldots, t_{k+2})$, $k \in \{4n, 4n+1\}$, $k > 4$, are non-zero skew-symmetric central polynomial functions in $Alt\,[T]$. It would be interesting to find all skew-symmetric central and nuclear polynomial functions for alternative algebras. Evidently, they all should be of the type $Skew\, f$, where $f \in Z(\mathcal{A})$ and $f \in N(\mathcal{A})$ (see Proposition 3.4 (iii)) for central and nuclear polynomial functions, respectively. The two problems are still open [27]:

(1) *Describe the elements $n \in N(\mathcal{A})$ for which the corresponding skew-symmetric polynomial function $Skew\, n$ is a nuclear polynomial function, that is, takes values in the nucleus $N(Alt[T])$;*

(2) *Describe the elements $z \in Z(\mathcal{A})$ for which the corresponding skew-symmetric polynomial function $Skew\, z$ is a central polynomial function in $Alt[T]$.*

Note that not every element in $Z(\mathcal{A})$ produces central or nuclear polynomial function. For example, $z^{[4]} \in Z(\mathcal{A})$ but $Skew\, z^{[4]}$ is neither a central nor a nuclear polynomial function in the algebra of octonions $\mathbb{O}$ ($Skew\, t^2$, $Skew\,(x^{[5]} - \frac{1}{2}t^2x)$, $Skew\, u^{[4]}x$ are central in $\mathbb{O}$). Recall, that all the multilinear skew-symmetric identities and central polynomials of $\mathbb{O}$ were classified by Shestakov and Zhukavets (see [28], 2009).

3.1.2 Reduction of dimensions in algebras

Using the representation theory of the symmetric group and Young diagrams, we can write a multilinear identity f of degree n over an algebra A as a sum of multilinear identities f_i, each of them being either symmetric or skew-symmetric in $m_i \geq \sqrt{n}$ arguments. Then, applying the superscalar extension to each f_i, we can reduce f to identities of the same degree n over A_G containing some even or odd variable at least $\sqrt{n}$ times. Therefore, if the superalgebra A_G is, in a certain sense, "generalized nil" of bounded degree, then A is nilpotent.

This application of superalgebras is based on the following theorems: Let $\mathcal{V}[T]$ be the free algebra of countable rank in a variety $\mathcal{V}$ and $Skew\, \mathcal{V}[T_n]$ be a subspace of multilinear elements of degree n in $\mathcal{V}[T]$ on the generators $t_1, \ldots, t_n$. The subspace $Skew\, \mathcal{V}[T_n]$ has a natural structure of a module over the group algebra $\mathbb{F}[Sym(n)]$ of the symmetric group $Sym(n)$. If $I \subseteq \{1, 2, \ldots, n\}$ is a nonempty subset, then we put

$$\varphi_I^+ = \sum_{\pi \in Sym(I)} \pi, \qquad \varphi_I^- = \sum_{\pi \in Sym(I)} \mathrm{sign}(\pi) \cdot \pi.$$

Theorem 3.8 ([30, **Theorem 1**]) *Let $\mathbb{F}$ be a field of characteristic zero and let K be a sum of the dimensions of the irreducible representations of $\mathbb{F}[Sym(n)]$. Then*

$$Skew\,\mathcal{V}[T_n] = \sum_{i=1}^{K} Skew\,\mathcal{V}[T_n]^{(i)},$$

where each $Skew\,\mathcal{V}[T_n]^{(i)}$ has the form

$$Skew\,\mathcal{V}[T_n] \cdot \varphi_{I_1}^{\varepsilon_1} \cdot \varphi_{I_2}^{\varepsilon_2} \cdots \varphi_{I_k}^{\varepsilon_k}$$

for some partition of $\{1, \ldots, n\}$ into disjoint nonempty subsets $I_1, \ldots, I_k$ with $k < \sqrt{2n}$ and some $\varepsilon_1, \ldots, \varepsilon_k = \pm$. □

Let A be an algebra generated by $a_1, a_2, \ldots, a_m$. If w is a product of these generators, then we define the *multiweight* of w as

$$w = (w_1, w_2, \ldots, w_m),$$

where w_i is a number of occurrences of the generator a_i in w, for $i = \{1, 2, \ldots, m\}$.

Theorem 3.9 ([30, **Theorem 2**]) *Let $\mathbb{F}$ be a field of characteristic zero and $I_1, \ldots, I_k$ be some partition of $\{1, \ldots, n\}$ into disjoint nonempty subsets. Let $\varepsilon_1, \ldots, \varepsilon_k = \pm$ and let $n_i = |I_i|$ for $i = 1, \ldots, k$. Then*

$$\dim\left(Skew\,\mathcal{V}[T_n] \cdot \varphi_{I_1}^{\varepsilon_1} \cdot \varphi_{I_2}^{\varepsilon_2} \cdots \varphi_{I_k}^{\varepsilon_k}\right) = \dim U,$$

where U is the multiweight $(n_1, \ldots, n_k)$ component of the free $\mathcal{V}$-superalgebra $\mathcal{V}[z_1, \ldots, z_k]$ of rank k, where, for $i = 1, \ldots, k$, the i-th generator z_i is even if $\varepsilon_i = +$ and is odd if $\varepsilon_i = -$. □

Note that the idea of considering the spaces of multilinear identities as modules over the group $Sym(n)$ and decomposing an identity into irreducible components goes back to the article by Malcev [11]. Hentzel [4] used this idea in his computer programs of proving identities, where he reduced the dimension of problem by passing to irreducible components. Passage to the superalgebras by means of Theorem 3.9 makes this reduction more transparent and easy. It was used by Vaughan-Lee and Shestakov and Zhukavets in confirming Kuzmin's conjecture in special cases.

Recall that Dubnov-Ivanov-Nagata-Higman theorem states that every algebra from $\mathcal{N}il_n$ is nilpotent of degree $2^n - 1$. Kuzmin in [9] showed that the degree cannot be less than $f(n) = \frac{n(n+1)}{2}$ and conjectured that the it is the exact estimate of nilpotency degree for the variety $\mathcal{N}il_n$. It is easy to see that Kuzmin's conjecture is true for $n = 2$, and Higman's results implied that it is also true for $n = 3$.

Vaughan-Lee in [29] did this for $n = 4$, confirming Kuzmin's conjecture in this case. He applied the representation theory of symmetric groups and the superalgebra technique. He reduced the original calculation in 10!-dimensional space to 8 smaller calculations in $\frac{10!}{4!3!2!}$- and $\frac{10!}{4!3!3!}$-dimensional spaces. Shestakov and Zhukavets in [22] investigated Kuzmin's conjecture for $n = 5$ confirming it "only" for every two-generated superalgebra over a field of characteristic zero in the variety of associative $\mathcal{N}il_5$-superalgebras. To confirm the conjecture for $\mathcal{N}il_5$, it is sufficient to prove that any superalgebra from $\mathcal{N}il_5$ on $k \leq 5$ homogeneous generators (even and odd mixed) is nilpotent of degree 15. But already for two generators were computations so huge, that there was no point in increasing the number of them (for example for the multiweight $(5, 5, 5)$ component of 3-generator superalgebra, there must be considered $\frac{15!}{5!5!5!} = 756\ 756$ words – compare with the multiweight $(7, 8)$ component of 2-generator superalgebra with $\frac{15!}{7!8!} = 6\ 435$ words).

3.1.3 Nilpotency problem for a T-ideal

Let A be an algebra and f a noncommutative and nonassociative polynomial. Many problems of the theory of varieties of algebras are reduced to the question as to when the ideal generated by the set $f(A)$ is nilpotent. Under some restrictions on a variety, we can answer this question also in terms of superalgebras.

The superalgebra B is called *prime* (or *simple*) if $B^2 \neq 0$ and if the product of any two nonzero homogeneous ideals of B is not zero. Let A be an algebra, $R_A = \{R_a \mid a \in A\}$ and $L_A = \{L_a \mid a \in A\} \subset \mathrm{End}\ A$ the spaces of right and left multiplications on elements of A and let $M(A)$ be the multiplication algebra of A, i.e. the subalgebra of End A generated by the left and right multiplications on A. The least n such that $M(A) = \sum_{k=1}^{n}(R_A \cup L_A)^k$ is called the *multiplicative length* of A. An ideal I of A is called *strongly nilpotent* of index n, if any product of elements of A that contains at least n factors in I is zero.

Theorem 3.10 (The nilpotency criterion for a T-ideal [20])
Let $\mathcal{V}$ be a variety of algebras over a field $\mathbb{F}$ satisfying the following conditions:

(i) every algebra in $\mathcal{V}$ has finite multiplicative length,

(ii) Every nilpotent ideal of an algebra $A \in \mathcal{V}$ is strongly nilpotent in A.

Then a multilinear element f generates a nilpotent T-ideal of identities in the free algebra of countable rank $\mathcal{V}[X]$ of the variety $\mathcal{V}$ (and consequently, in every algebra of $\mathcal{V}$) if and only if f vanishes identically in the Grassmann envelope $G(A)$ of every prime $\mathcal{V}$-superalgebra A. □

In particular, if there are no prime $\mathcal{V}$-superalgebras then the free algebra is nilpotent.

Condition (i) and (ii) are obviously satisfied in every variety of associative algebras. In case a variety does not satisfy the conditions (i) and (ii), it is still sometimes possible to reduce the problem to subvarieties or certain subalgebras that do satisfy the conditions, see for example [32, 33].

3.1.4 Prime and semiprime varieties

Prime superalgebras relate quite closely to so called prime varieties introduced by Kemer. A variety is *prime*, if its free algebra $\mathcal{V}[X]$ of countable rank is T-prime, i.e. does not contain nonzero T-ideals T_1, T_2 such that

$$T_1 T_2 = 0.$$

The Grassmann envelope of a prime superalgebra generates a prime variety. Similarly, a variety $\mathcal{V}$ is called *semiprime* if the algebra $\mathcal{V}[X]$ is T-semiprime, i.e. does not contain nonzero T-ideals I with $I^2 = 0$. A union of any number of prime varieties is semiprime, conversely, every semiprime variety can be decomposed into a union of prime subvarieties.

Denote by $N(\mathcal{V})$ the least T-ideal I of $\mathcal{V}[X]$ with the property that the quotient algebra $\mathcal{V}[X]/I$ is T-semiprime.

The ideal $N(\mathcal{V})$ is zero iff the variety $\mathcal{V}$ is semiprime. The set of identities $\{f = 0 \mid f \in N(\mathcal{V})\}$ defines the largest semiprime subvariety of $\mathcal{V}$. Clearly, $N(\mathcal{V})$ is contained in the T-ideal of identities of any prime subvariety of $\mathcal{V}$. In particular, it is contained in the intersection of all T-ideals of identities of the Grassmann envelopes of prime $\mathcal{V}$-superalgebras. Moreover, if $\mathcal{V}$ satisfies the condition (i) and (ii) in Theorem 3.10, the reverse inclusion holds too.

Let us denote by $M_{m,k}(\mathbb{F}) = M_0 + M_1$ the matrix superalgebra over $\mathbb{F}$, which is defined as

$$M_0 = \left\{ \begin{smallmatrix} m \\ k \end{smallmatrix} \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\} \qquad M_1 = \left\{ \begin{smallmatrix} m \\ k \end{smallmatrix} \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \right\}.$$

If $k = 0$ then we get the (even) algebra $M_m(\mathbb{F})$. Denote by $G(M_{m,k})$ Grassmann envelope of the superalgebra $M_{m,k}$ and by $M_n(G)$ a matrix algebra over a Grassmann algebra G.

The structure of a variety $\mathcal{V}$ is to a great extend defined by the structure of prime $\mathcal{V}$-superalgebras and their identities. For a varieties of alternative algebras over a field of characteristic zero, the structure theorem holds similar to that for associative algebras:

Theorem 3.11 **([20])** *Let $\mathcal{V}$ be a variety of alternative algebras over a field of characteristic zero $\mathbb{F}$. Then*

(i) the ideal $N(\mathcal{V})$ is solvable,

(ii) the variety $\mathcal{V}$ is semiprime if and only if it is a union of finitely many prime varieties,

(iii) the variety $\mathcal{V}$ is prime if and only if $\mathcal{V}$ is either coincides with the variety of all associative algebras or is generated by one of the algebras $M_n(\mathbb{F})$, $G(M_{m,k}(\mathbb{F}))$, $M_n(G)$ or $\mathbb{O}$. □

The question is still unanswered as to whether any variety of alternative algebras over a field of characteristic zero has a finite base of identities. The affirmative solution is known only for the varieties of a finite basic rank not containing the variety of associative algebras Iltyakov [6, 1999]. In particular, every finite-dimensional alternative algebra over a field of characteristic zero has a finite base of identities.

The question about finiteness of a basic superrank is also open for varieties of alternative algebras.

4 Bases of the free alternative nil-superalgebras on one odd generator of nil index 2, 3 and n

We construct a base of a free alternative nil-superalgebra $\mathcal{B}_n = \mathcal{A}lt\text{-}\mathcal{N}il_n[\emptyset; x]$ of nil-index n, for $n \geq 2$, on one odd generator x in this section. Our results were published in [16, 17, 18]. To construct a base of $\mathcal{B}_n$, for $n \geq 2$, we use the base of the free alternative superalgebra $\mathcal{A} = \mathcal{A}lt[\emptyset; x]$ on one odd generator x constructed in [27]. After that we compute the solvability index of $\mathcal{B}_n$ which is $\lceil \log_2 n \rceil + 1$, for $n \geq 2$. We also prove that $\mathcal{B}_n$ is not nilpotent and that the square of $\mathcal{B}_n$ is nilpotent of index n, for nil-index $n \geq 3$. It was Pchelintsev [13] who proved, that in characteristics $\neq 2, 3$ the square of a solvable alternative algebra is nilpotent without giving any approximation of the nilpotency index. Later Shestakov [19] showed that for any alternative algebra A it holds

$$(A^2)^{f(k)} \subset A^{(k)}, \text{ where } f(k) = \tfrac{5^{k-1} + 3}{4}.$$

It is known from [34], that if A is an alternative algebra over a field of characteristic zero and I_n is a subspace of A spanned by a^n for all $a \in A$, then I_n is an ideal of A. We use a generalization of this statement for superalgebras.

Let $\mathcal{A} = \mathcal{A}[\emptyset; x]$ be the free alternative superalgebra on one odd generator x with the base (13). Let $\mathcal{I}_n$ be a subsuperspace of $\mathcal{A}$ spanned by the elements $W_n(u_1, u_2, \ldots, u_n), u_1, \ldots, u_n \in \mathcal{A}_0 \cup \mathcal{A}_1$, where

$$W_n(u_1, u_2, \ldots, u_n) = \sum_{\sigma \in Sym(n)} \text{sign}_{\text{odd}}(\sigma)(\ldots((u_{\sigma(1)} u_{\sigma(2)}) u_{\sigma(3)}) \cdots) u_{\sigma(n)},$$

then $\mathcal{I}_n$ is an ideal of $\mathcal{A}$. The quotient superalgebra $\mathcal{A}/\mathcal{I}_n$ is the free alternative $\mathcal{N}il_n$-superalgebra on one odd generator x. We will denote it by $\mathcal{B}_n = \mathcal{A}lt\text{-}\mathcal{N}il_n[\emptyset; x]$ and we will maintain the notations from $\mathcal{A}$ for the similar elements of $\mathcal{B}_n$.

Notice that for any odd element $y \in \mathcal{B}_n$ and any $u_i \in \mathcal{B}_n$ it holds that

$$W_n(y, y, u_3, \ldots, u_n) = 0.$$

Moreover, for any permutation $\sigma \in Sym(n)$,

$$W_n(u_{\sigma(1)}, u_{\sigma(2)}, \ldots, u_{\sigma(n)}) = \text{sign}_{\text{odd}}(\sigma) W_n(u_1, u_2, \ldots, u_n).$$

4.1 $\mathcal{A}lt\text{-}\mathcal{N}il_2[\emptyset; x]$

First, we will investigate the case of nil-index $n = 2$. We construct a base of $\mathcal{B}_2 = \mathcal{A}lt\text{-}\mathcal{N}il_2[\emptyset; x]$. After that we compute the solvability index of $\mathcal{B}_2$ which is 2. We also show that $\mathcal{B}_2$ is nilpotent of index 4.

Base of the free alternative $\mathcal{N}il_2$-superalgebra on one odd generator

First we consider an ideal $\mathcal{I}_2 \subset \mathcal{A}$, spanned by the elements $W_2(u, v), u, v \in \mathcal{A}_0 \cup \mathcal{A}_1$, where

$$W_2(u, v) = uv + (-1)^{\overline{u}\,\overline{v}} vu, \tag{14}$$

and construct a base of $\mathcal{I}_2$. Using the multiplication table for $\mathcal{A}$, we obtain from

$$\begin{aligned} W_2(t, t) &= t \cdot t + t \cdot t = 2t^2, \\ W_2(t, x) &= tx + xt = tx + tx - x^{[3]} = 2tx - x^{[3]}, \\ W_2(x, tx) &= x(tx) - (tx)x = \tfrac{1}{2}t^2 - x^{[3]}x + \tfrac{2}{3}x^{[4]} - \tfrac{1}{2}t^2 - \tfrac{1}{3}x^{[4]} \\ &= \tfrac{1}{3}x^{[4]} - x^{[3]}x, \end{aligned}$$

that $t^2, 2tx - x^{[3]}, \frac{1}{3}x^{[4]} - x^{[3]}x \in \mathcal{I}_2$. Using the fact that for every element $i \in \mathcal{I}_2$ also $i \cdot x, x \cdot i \in \mathcal{I}_2$, we get

$$(2tx - x^{[3]}) \cdot x = t^2 + \tfrac{2}{3}x^{[4]} - x^{[3]}x$$

and so $x^{[3]}x, x^{[4]} \in \mathcal{I}_2$. Moreover

$$t^m x^{[\varepsilon]} \in \mathcal{I}_2, \quad m \geq 2, \varepsilon \in \{0, 1\}.$$

Next from the definition of the elements $x^{[k]}, z^{[k]}, u^{[k]},\ k > 3$

$$\begin{aligned} x^{[k+1]} &= x^{[k]}x - (-1)^k x x^{[k]}, \\ z^{[k]} &= x^{[k]}t - t x^{[k]}, \\ u^{[k]} &= x^{[k]}x^{[3]} + (-1)^k x^{[3]}x^{[k]}, \end{aligned}$$

and from $x^{[4]} \in \mathcal{I}_2$ we get

$$\begin{aligned} x^{[5]} = x^{[4]}x - (-1)^k x x^{[4]} &\in \mathcal{I}_2, \\ x^{[5]}x &\in \mathcal{I}_2, \\ x^{[k]}, x^{[k]}x &\in \mathcal{I}_2, k > 3 \end{aligned}$$

$$\begin{aligned} z^{[4]} = x^{[4]}t - t x^{[4]} &\in \mathcal{I}_2, \\ z^{[k]} &\in \mathcal{I}_2,\ k > 3 \end{aligned}$$

$$\begin{aligned} u^{[4]} = x^{[4]}x^{[3]} + (-1)^k x^{[3]}x^{[4]} &\in \mathcal{I}_2, \\ u^{[k]} &\in \mathcal{I}_2,\ k > 3. \end{aligned}$$

Finally we can conclude that for $k > 2,\ m \geq 0,\ \varepsilon \in \{0, 1\}$

$$\begin{aligned} t^{m+2}x^{\varepsilon} &\in \mathcal{I}_2, \\ t^{m+1}x^{[3]},\ t^m(x^{[3]}x),\ t^m(x^{[k+1]}x^{\varepsilon}) &\in \mathcal{I}_2, \\ t^m z^{[k]}x^{\varepsilon} &\in \mathcal{I}_2, \\ t^m u^{[k]}x^{\varepsilon} &\in \mathcal{I}_2. \end{aligned}$$

Observe that for any other base elements u, v values of $W_2(u, v)$ do not bring new elements from $\mathcal{I}_2$. The following proposition is evident:

Proposition 4.1 *The elements*

$$\begin{aligned} 2tx - x^{[3]}, t^{m+2}x^{\varepsilon}, \\ t^{m+1}x^{[3]}, t^m(x^{[3]}x), t^m(x^{[k+3]}x^{\varepsilon}), \\ t^m z^{[4k+\sigma]}x^{\varepsilon}, t^m u^{[4k+\sigma]}x^{\varepsilon}, \end{aligned} \tag{15}$$

where $k > 0,\ m \geq 0,\ \varepsilon, \sigma \in \{0, 1\}$, form a base of $\mathcal{I}_2$. □

Now the superalgebra $\mathcal{B}_2 = \mathcal{A}/\mathcal{I}_2$ is spanned by the elements

$$x,\ t,\ tx \tag{16}$$

all other elements from (13) are zero except

$$x^{[3]} \ = \ 2tx.$$

Moreover, the elements from (16) are linearly independent since any nontrivial linear combination of their preimages does not lie in $\mathcal{I}_2$.

Theorem 4.2 *The superalgebra $\mathcal{B}_2$ has a base (16).* □

The multiplication table of $\mathcal{B}_2$ is:

$$\begin{aligned} x \cdot x &= \tfrac{1}{2}t, \\ t \cdot x &= tx, \\ x \cdot t &= -tx, \\ t \cdot t &= 0, \\ u \cdot tx = tx \cdot u &= 0, \end{aligned}$$

for any u from (16).

Index of solvability of $\mathcal{B}_2$

Since we know a base of the superalgebra $\mathcal{B}_2$ and the multiplication table on base elements, we can compute the index of solvability of $\mathcal{B}_2$ directly.

Corollary 4.3 *The index of solvability of the superalgebra $\mathcal{B}_2$ is 2.*

Proof: By the definition $\mathcal{B}_2^{(0)} = \mathcal{B}_2 = vect\langle x, t, tx\rangle$. Further

$$\begin{aligned} \mathcal{B}_2^{(1)} &= \mathcal{B}_2^{(0)}\mathcal{B}_2^{(0)} = vect\langle t, tx\rangle, \\ \mathcal{B}_2^{(2)} &= \mathcal{B}_2^{(1)}\mathcal{B}_2^{(1)} = 0. \end{aligned}$$

The index of solvability is 2. □

Index of nilpotency of $\mathcal{B}_2$

Corollary 4.4 *The superalgebra $\mathcal{B}_2$ is nilpotent of index 4.*

Proof:

$$\begin{aligned} \left(\mathcal{B}_2\right)^1 &= \mathcal{B}_2 = vect\langle x, t, tx\rangle, \\ \left(\mathcal{B}_2\right)^2 &= \mathcal{B}_2 \cdot \mathcal{B}_2 = vect\langle t, tx\rangle, \\ \left(\mathcal{B}_2\right)^3 &= \left(\mathcal{B}_2\right)^2\mathcal{B}_2 + \mathcal{B}_2\left(\mathcal{B}_2\right)^2 = vect\langle tx\rangle, \\ \left(\mathcal{B}_2\right)^4 &= \left(\mathcal{B}_2\right)^3\mathcal{B}_2 + \left(\mathcal{B}_2\right)^2\left(\mathcal{B}_2\right)^2 + \mathcal{B}_2\left(\mathcal{B}_2\right)^3 = 0. \end{aligned}$$

The superalgebra $\mathcal{B}_2$ is nilpotent and index of nilpotency is 4. □

4.2 $\mathcal{A}lt\text{-}\mathcal{N}il_3[\emptyset; x]$

Here we construct a base of $\mathcal{B}_3 = \mathcal{A}lt\text{-}\mathcal{N}il_3[\emptyset; x]$. After that we compute the solvability index of $\mathcal{B}_3$ which is 3. We also show that $\mathcal{B}_3$ is not nilpotent and $(\mathcal{B}_3)^2$ is nilpotent of index 3.

Base of the free alternative $\mathcal{N}il_3$-superalgebra on one odd generator

First we consider an ideal $\mathcal{I}_3 \subset \mathcal{A}$, spanned by the elements $W_3(u, v, w)$, $u, v, w \in \mathcal{A}_0 \cup \mathcal{A}_1$, where

$$\begin{aligned} W_3(u, v, w) =&(uv)w + (-1)^{\bar{u}(\bar{v}+\bar{w})}(vw)u + (-1)^{\bar{w}(\bar{u}+\bar{v})}(wu)v \\ &+ (-1)^{\bar{v}\bar{w}}(uw)v + (-1)^{\bar{u}\bar{v}}(vu)w + (-1)^{\bar{u}\bar{v}+\bar{u}\bar{w}+\bar{v}\bar{w}}(wv)u, \end{aligned}$$

and construct a base of $\mathcal{I}_3$. Using the multiplication table for $\mathcal{A}$ (see Proposition 3.5), we obtain from

$$\begin{aligned} W_3(x, t, t) &= 2((xt)t + t^2x + (tx)t) \\ &= 2(2(tx)t - x^{[3]}t + t^2x) \\ &= 2(2(t^2x - tx^{[3]}) - tx^{[3]} + t^2x) \\ &= 6(t^2x - tx^{[3]}), \\ W_3(t, t, t) &= 6t^3 \end{aligned}$$

that $t^2x - tx^{[3]}, t^3 \in \mathcal{I}$. Moreover,

$$\begin{aligned} (t^2x - tx^{[3]}) \cdot x &= \tfrac{1}{2}t^3 - 2t(x^{[3]}x) + \tfrac{4}{3}tx^{[4]} + (t(x^{[3]}x) - tx^{[4]} - \tfrac{1}{6}z^{[4]}) \\ &= \tfrac{1}{2}t^3 - t(x^{[3]}x) + \tfrac{1}{3}tx^{[4]} - \tfrac{1}{6}z^{[4]}, \\ x \cdot (t^2x - tx^{[3]}) &= \tfrac{1}{2}t^3 + \tfrac{2}{3}tx^{[4]} + \tfrac{1}{3}z^{[4]} - (t(x^{[3]}x) + \tfrac{1}{3}z^{[4]}) \\ &= \tfrac{1}{2}t^3 - t(x^{[3]}x) + \tfrac{2}{3}tx^{[4]}, \end{aligned}$$

and $t^2x^{[3]}, tx^{[4]} + \frac{1}{2}z^{[4]}, t(x^{[3]}x) + \frac{1}{3}z^{[4]} \in \mathcal{I}$.

Compute for $k \geq 3$

$$\begin{aligned}
W_3(t,t,x^{[k]}) &= 2(t^2x^{[k]} + (tx^{[k]})t + (x^{[k]}t)t) \\
&= 2(t^2x^{[k]} + 2(tx^{[k]})t + z^{[k]}t) \\
&= 2(t^2x^{[k]} + 2(t^2x^{[k]} + tz^{[k]}) + tz^{[k]}) \\
&= 6(t^2x^{[k]} + tz^{[k]}), \\
W_3(t,t,x^{[k]}x) &= 2(t^2(x^{[k]}x) + (t(x^{[k]}x))t + ((x^{[k]}x)t)t) \\
&= 2(t^2(x^{[k]}x) + 2(t(x^{[k]}x))t - (\tfrac{1}{2}u^{[k]} - z^{[k]}x - \tfrac{1}{2}z^{[k+1]})t) \\
&= 6(t^2(x^{[k]}x) - \tfrac{1}{2}tu^{[k]} + tz^{[k]}x + \tfrac{1}{2}tz^{[k+1]}), \\
W_3(x,t,z^{[k]}) &= 6(-1)^k tz^{[k]}x, \\
W_3(x,t,u^{[k]}) &= 6(-1)^{k+1}tu^{[k]}x + 2(-1)^k t^2z^{[k]}.
\end{aligned}$$

Therefore, $t^2x^{[k]} + tz^{[k]}, t^2z^{[k]}, t^2(x^{[k]}x) - \frac{1}{2}tu^{[k]} + \frac{1}{2}tz^{[k+1]}, t^2u^{[k]}, tz^{[k]}x, tu^{[k]}x \in \mathcal{I}_3$. Multiplying by x we get

$$\begin{aligned}
(t^2x^{[k]} + tz^{[k]}) \cdot x &= t^2(x^{[k]}x) + \tfrac{2}{3}tz^{[k+1]} + tz^{[k]}x, \\
x \cdot (t^2x^{[k]} + tz^{[k]}) &= (-1)^k(t^2(x^{[k]}x) - t^2x^{[k+1]} - tu^{[k]} - \tfrac{1}{3}tz^{[k+1]} + tz^{[k]}x),
\end{aligned}$$

and hence $t^2x^{[k+1]}, t^2(x^{[k]}x), tu^{[k]}, tz^{[k+1]} \in \mathcal{I}_3$. Notice that for $k = 3$ we also have $t^2x^{[3]}, tz^{[3]} \in \mathcal{I}_3$.

Continue our computations modulo elements which are already in $\mathcal{I}_3$:

$$\begin{aligned}
W_3(x,t,x^{[k]}) &= (xt + tx)x^{[k]} + (xx^{[k]} + (-1)^k x^{[k]}x)t + (-1)^k(tx^{[k]} + x^{[k]}t)x \\
&= (2tx - x^{[3]})x^{[k]} + (-1)^k(2x^{[k]}x - x^{[k+1]})t + (-1)^k(2tx^{[k]} + z^{[k]})x \\
&= 2(-1)^k(t(x^{[k]}x) - tx^{[k+1]} - \tfrac{1}{3}z^{[k+1]}) - (-1)^k\tfrac{1}{2}(u^{[k]} - z^{[k+1]}) \\
&\quad +2(-1)^k(t(x^{[k]}x) - \tfrac{1}{2}u^{[k]} + z^{[k]}x + \tfrac{1}{2}z^{[k+1]}) - (-1)^k(tx^{[k+1]} + z^{[k+1]}) \\
&\quad +2(-1)^k(t(x^{[k]}x) + \tfrac{1}{3}z^{[k+1]}) + (-1)^k z^{[k]}x \\
&= (-1)^k(6t(x^{[k]}x) - 3tx^{[k+1]} + \tfrac{1}{2}z^{[k+1]} - \tfrac{3}{2}u^{[k]} + 3z^{[k]}x),
\end{aligned}$$

$$\begin{aligned}
W_3(x,t,x^{[k]}x) &= (xt + tx)(x^{[k]}x) + (x(x^{[k]}x) + (-1)^{k+1}(x^{[k]}x)x)t \\
&\quad +(-1)^{k+1}(t(x^{[k]}x) + (x^{[k]}x)t)x \\
&= (2tx - x^{[3]})(x^{[k]}x) + (-1)^k(\tfrac{1}{3}x^{[k+2]} - x^{[k+1]}x - \tfrac{1}{6}z^{[k]})t \\
&\quad +(-1)^{k+1}(2t(x^{[k]}x) - \tfrac{1}{2}u^{[k]} + z^{[k]}x + \tfrac{1}{2}z^{[k+1]})x \\
&= 2(-1)^k(\tfrac{2}{3}tx^{[k+2]} - t(x^{[k+1]}x) + \tfrac{1}{6}u^{[k+1]} - \tfrac{1}{3}z^{[k+1]}x + \tfrac{1}{6}z^{[k+2]}) \\
&\quad -(-1)^k(\tfrac{1}{2}u^{[k]}x - \tfrac{1}{2}z^{[k+1]}x + \tfrac{1}{3}z^{[k+2]}) \\
&\quad +(-1)^k(\tfrac{1}{3}tx^{[k+2]} + \tfrac{1}{3}z^{[k+2]} - t(x^{[k+1]}x) + \tfrac{1}{2}u^{[k+1]} - z^{[k+1]}x - \tfrac{1}{2}z^{[k+2]}) \\
&\quad +(-1)^{k+1}(\tfrac{2}{3}tx^{[k+2]} + \tfrac{1}{3}u^{[k+1]} - \tfrac{2}{3}z^{[k+1]}x + \tfrac{1}{3}z^{[k+2]} - \tfrac{1}{2}u^{[k]}x + \tfrac{1}{2}z^{[k+1]}x) \\
&= (-1)^k(tx^{[k+2]} - 3t(x^{[k+1]}x) + \tfrac{1}{2}u^{[k+1]} - z^{[k+1]}x - \tfrac{1}{2}z^{[k+2]}),
\end{aligned}$$

to obtain $tx^{[k+2]}+\frac{1}{2}z^{[k+2]}+\frac{1}{2}u^{[k+1]}-z^{[k+1]}x, t(x^{[k+1]}x)+\frac{1}{3}z^{[k+2]} \in \mathcal{I}_3$. Moreover,

$$\begin{aligned}
(6t(x^{[k]}x) &- 3tx^{[k+1]} + \tfrac{1}{2}z^{[k+1]} - \tfrac{3}{2}u^{[k]} + 3z^{[k]}x) \cdot x \\
&= 6(\tfrac{1}{3}tx^{[k+2]} + \tfrac{1}{6}u^{[k+1]} - \tfrac{1}{3}z^{[k+1]}x + \tfrac{1}{6}z^{[k+2]}) \\
&\quad -3(t(x^{[k+1]}x) + \tfrac{1}{3}z^{[k+2]}) + \tfrac{1}{2}z^{[k+1]}x - \tfrac{3}{2}u^{[k]}x \\
&= 2tx^{[k+2]} - 3t(x^{[k+1]}x) + u^{[k+1]} - \tfrac{3}{2}z^{[k+1]}x - \tfrac{3}{2}u^{[k]}x, \\
x \cdot (6t(x^{[k]}x) &- 3tx^{[k+1]} + \tfrac{1}{2}z^{[k+1]} - \tfrac{3}{2}u^{[k]} + 3z^{[k]}x) \\
&= 6(-1)^k \left(\tfrac{2}{3}tx^{[k+2]} - t(x^{[k+1]}x) - \tfrac{1}{2}u^{[k]}x + \tfrac{1}{3}u^{[k+1]} - \tfrac{1}{6}z^{[k+1]}x\right) \\
&\quad -3(-1)^{k+1} \left(t(x^{[k+1]}x) - tx^{[k+2]} - \tfrac{1}{2}u^{[k+1]} - \tfrac{1}{6}z^{[k+2]}\right) \\
&\quad +\tfrac{1}{2}(-1)^{k+1}z^{[k+1]}x - \tfrac{3}{2}(-1)^{k+1}u^{[k]}x \\
&= (-1)^k \left(tx^{[k+2]} - 3t(x^{[k+1]}x) - \tfrac{3}{2}u^{[k]}x + \tfrac{1}{2}u^{[k+1]} \right. \\
&\qquad \left. -\tfrac{3}{2}z^{[k+1]}x - \tfrac{1}{2}z^{[k+2]}\right)
\end{aligned}$$

imply that $u^{[k]}x, z^{[k+1]}x \in \mathcal{I}_3$.

Observe that for any other base elements u, v, w values of $W_3(u, v, w)$ do not bring new elements from $\mathcal{I}_3$. Therefore we obtain a base of $\mathcal{I}_3$.

Proposition 4.5 *Elements*

$$\begin{gathered}
t^{m+3}x^{\sigma}, \\
tx^{[3]} - t^2x,\ tx^{[k+3]} + \tfrac{1}{2}z^{[k+3]} + \tfrac{1}{2}u^{[k+2]},\quad t(x^{[k+2]}x) + \tfrac{1}{3}z^{[k+3]}, \\
t^{m+2}(x^{[k+2]}x^{\sigma}), \\
t^m(u^{[4k+\varepsilon]}x^{\sigma}), \qquad m+\sigma \geq 1, \\
t^m(z^{[4k+\varepsilon]}x^{\sigma}), \qquad m+\sigma \geq 1,
\end{gathered} \tag{17}$$

where $k > 0$, $m \geq 0$; ε, $\sigma \in \{0, 1\}$, form a base of $\mathcal{I}_3$. □

Now the superalgebra $\mathcal{B}_3 = \mathcal{A}/\mathcal{I}_3$ is spanned by the elements

$$\begin{gathered}
x,\ t,\ tx,\ t^2,\ t^2x,\ x^{[k]},\ x^{[k]}x,\ k > 2, \\
u^{[4m+\varepsilon]},\ z^{[4m+\varepsilon]},\ m > 0,\ \varepsilon \in \{0, 1\}.
\end{gathered} \tag{18}$$

All other elements from (13) are equal zero, except

$$\begin{aligned}
tx^{[3]} &= t^2x, \\
tx^{[k+1]} &= -\tfrac{1}{2}(u^{[k]} + z^{[k+1]}), \\
t(x^{[k]}x) &= -\tfrac{1}{3}z^{[k+1]}.
\end{aligned}$$

Moreover, the elements from (18) are linearly independent since any non-trivial linear combination of their preimages does not lie in $\mathcal{I}_3$.

Theorem 4.6 *The elements from (18) form a base of the superalgebra $\mathcal{B}_3$.* $\square$

The multiplication table of $\mathcal{B}_3$ is given by that of $\mathcal{A}$ modulo $\mathcal{I}_3$:

$$\begin{aligned}
x\cdot x &= \tfrac{1}{2}\,t\\
t\cdot x &= tx,\\
x\cdot t &= tx - x^{[3]},\\
tx\cdot x &= \tfrac{1}{2}\,t^2 + \tfrac{1}{3}\,x^{[4]},\\
x\cdot tx &= \tfrac{1}{2}\,t^2 - x^{[3]}x + \tfrac{2}{3}\,x^{[4]},\\
\\
t\cdot t &= t^2,\\
t\cdot tx &= t^2x,\\
tx\cdot t &= 0,\\
tx\cdot tx &= -\tfrac{1}{2}\,z^{[4]}.
\end{aligned}$$

Further for $k,i,j\ >\ 2,\ \varepsilon\in\{0,1\}$

$$\begin{aligned}
t\cdot x^{[3]} &= t^2x,\\
t\cdot x^{[k+1]} &= -\tfrac{1}{2}(u^{[k]}+z^{[k+1]}),\\
t\cdot(x^{[k]}x) &= -\tfrac{1}{3}z^{[k+1]},\\
\\
x\cdot x^{[k]} &= (-1)^k\left(x^{[k]}x - x^{[k+1]}\right),\\
x\cdot(x^{[3]}x) &= -\tfrac{1}{2}\,t^2x - \tfrac{2}{3}\,x^{[5]} + x^{[4]}x,\\
x\cdot(x^{[k+1]}x) &= (-1)^{k+1}\left(\tfrac{2}{3}\,x^{[k+3]} - x^{[k+2]}x - \tfrac{1}{4}\,u^{[k]} - \tfrac{1}{12}\,z^{[k+1]}\right),\\
\\
(tx)\cdot x^{[k]} &= (-1)^k\left(\tfrac{1}{2}\,u^{[k]} - \tfrac{1}{6}\,z^{[k+1]}\right),\\
(tx)\cdot(x^{[k]}x) &= (-1)^k\,\tfrac{1}{6}\,\left(z^{[k+2]} - u^{[k+1]}\right).
\end{aligned}$$

$$
\begin{aligned}
x^{[3]} \cdot t &= t^2 x, \\
x^{[k+1]} \cdot t &= \tfrac{1}{2} \left(z^{[k+1]} - u^{[k]} \right), \\
(x^{[k]} x) \cdot t &= \tfrac{1}{6} z^{[k+1]} - \tfrac{1}{2} u^{[k]},
\end{aligned}
$$

$$
\begin{aligned}
x^{[k]} \cdot x &= x^{[k]} x, \\
(x^{[3]} x) \cdot x &= \tfrac{1}{2} t^2 x + \tfrac{1}{3} x^{[k+2]}, \\
(x^{[k+1]} x) \cdot x &= \tfrac{1}{3} x^{[k+3]} - \tfrac{1}{4} u^{[k]} + \tfrac{1}{12} z^{[k+1]},
\end{aligned}
$$

$$
\begin{aligned}
x^{[k]} \cdot (tx) &= \tfrac{1}{3} z^{[k+1]}, \\
(x^{[k]} x) \cdot (tx) &= \tfrac{1}{6} \left(u^{[k+1]} + z^{[k+2]} \right),
\end{aligned}
$$

$$
\begin{aligned}
x^{[i]} \cdot x^{[j]} &= \tfrac{1}{2} (-1)^{c(j+1)} \left(u^{[i+j-3]} - (-1)^j z^{[i+j-2]} \right), \\
x^{[i]} \cdot (x^{[j]} x) &= -\tfrac{1}{3} (-1)^{c(j+1)} z^{[i+j-1]}, \\
(x^{[i]} x) \cdot x^{[j]} &= \tfrac{1}{2} (-1)^{c(j)} \left((-1)^j u^{[i+j-2]} - \tfrac{1}{3} z^{[i+j-1]} \right), \\
(x^{[i]} x) \cdot (x^{[j]} x) &= -\tfrac{1}{6} (-1)^{c(j)} \left(u^{[i+j-1]} + (-1)^j z^{[i+j]} \right),
\end{aligned}
$$

where $c(j) = j(j-1)/2$ and $\delta_j = 1 + (-1)^j$. Elements $t^2, z^{[k]}, u^{[k]}$ annihilate all base elements except

$$
\begin{aligned}
t^2 \cdot x &= t^2 x, \\
x \cdot t^2 &= -t^2 x.
\end{aligned}
$$

Index of solvability of $\mathcal{B}_3$

From Kuzmin's results and Zhevlakov's theorem we know that the index of solvability of alternative nil-algebras of nil-index 3 is between 3 and 6. Since we know a base of the superalgebra $\mathcal{B}_3$ we can easily compute the index of solvability of $\mathcal{B}_3$.

Corollary 4.7 *The index of solvability of* $\mathcal{B}_3$ *is* 3.

Proof: For $k > 2,\ m > 0,\ \varepsilon \in \{0, 1\}$ we get:

$$
\begin{aligned}
\mathcal{B}_3^{(0)} &= \mathcal{B}_3 \\
\mathcal{B}_3^{(1)} &= \mathcal{B}_3 \mathcal{B}_3 \\
&= vect\langle t, tx, t^2, t^2 x, x^{[k]}, x^{[k]} x, u^{[4m+\varepsilon]}, z^{[4m+\varepsilon]} \rangle, \\
\mathcal{B}_3^{(2)} &= \mathcal{B}_3^{(1)} \mathcal{B}_3^{(1)} \\
&= vect\langle t^2, t^2 x, u^{[4m+\varepsilon]}, z^{[4m+\varepsilon]} \rangle, \\
\mathcal{B}_3^{(3)} &= \mathcal{B}_3^{(2)} \mathcal{B}_3^{(2)} = 0.
\end{aligned}
$$

□

Nilpotency of $\mathcal{B}_3$

Corollary 4.8 *$\mathcal{B}_3$ is not nilpotent, moreover $\big(\mathcal{B}_3\big)^2$ is nilpotent of index 3 and $\big(\mathcal{B}_3\big)^m \cdot \big(\mathcal{B}_3\big)^n$ is not zero for any integers $m, n > 0$.*

Proof: First we prove that $\big({\mathcal{B}_3}^2\big)^3 = 0$. For $k > 2,\ m > 0,\ \varepsilon \in \{0, 1\}$ we have

$$\begin{aligned}
\big({\mathcal{B}_3}^2\big)^1 &= {\mathcal{B}_3}^2 = \mathcal{B}_3\mathcal{B}_3 = \mathcal{B}_3^{(1)} \\
&= vect\langle t, tx, t^2, t^2x, x^{[k]}, x^{[k]}x, u^{[4m+\varepsilon]}, z^{[4m+\varepsilon]}\rangle, \\
\big({\mathcal{B}_3}^2\big)^2 &= {\mathcal{B}_3}^2{\mathcal{B}_3}^2 = \mathcal{B}_3^{(2)} \\
&= vect\langle t^2, t^2x, u^{[4m+\varepsilon]}, z^{[4m+\varepsilon]}\rangle, \\
\big({\mathcal{B}_3}^2\big)^3 &= {\mathcal{B}_3}^2\big({\mathcal{B}_3}^2{\mathcal{B}_3}^2\big) + \big({\mathcal{B}_3}^2{\mathcal{B}_3}^2\big){\mathcal{B}_3}^2 = 0.
\end{aligned}$$

To prove that $\big(\mathcal{B}_3\big)^m \neq 0$, it is sufficient to prove that for example $x^{[i]} \in \big(\mathcal{B}_3\big)^i$ for every integer $i > 0$. We prove it by the induction: Obviously $x^{[1]} = x \in \big(\mathcal{B}_3\big)^1$. If $x^{[i]} \in \big(\mathcal{B}_3\big)^i$ for some $i > 1$, then $x^{[i]} \cdot x,\ x \cdot x^{[i]} \in \big(\mathcal{B}_3\big)^{i+1}$, since $\big(\mathcal{B}_3\big)^{i+1} = \big(\mathcal{B}_3\big)^i\mathcal{B}_3 + \cdots + \mathcal{B}_3\big(\mathcal{B}_3\big)^i$. From

$$x^{[i+1]} = x^{[i]} \cdot x - (-1)^i\,(x \cdot x^{[i]})$$

we have that $x^{[i+1]} \in \big(\mathcal{B}_3\big)^{i+1}$. We proved that $x^{[i]} \in \big(\mathcal{B}_3\big)^i$, and $\big(\mathcal{B}_3\big)^i \neq 0$ for all natural $i > 0$. $\mathcal{B}_3$ is not nilpotent.

Now from $x^{[i]} \in \big(\mathcal{B}_3\big)^m, x^{[j]} \in \big(\mathcal{B}_3\big)^n$, $i \geq m,\ j \geq r$, and the multiplication

$$x^{[i]} \cdot x^{[j]} \;=\; \tfrac{1}{2}(-1)^{c(j+1)}(u^{[i+j-3]} - (-1)^j z^{[i+j-2]}),$$

we take i, j large enough, such that $u^{[i+j-3]}$ or $z^{[i+j-2]}$ are not zero (i.e. the indices $i + j - 3$ or $i + j - 2$ must be of the form $4k$ or $4k + 1,\ k > 0$). Such i, j exist, because there are infinitely many elements of the form $x^{[i]} \in \big(\mathcal{B}_3\big)^m$ and $x^{[j]} \in \big(\mathcal{B}_3\big)^n$. We have proved that $\big(\mathcal{B}_3\big)^m\big(\mathcal{B}_3\big)^n \neq 0$. □

4.3 $\mathcal{A}lt\text{-}\mathcal{N}il_n[\emptyset; x]$

Here we construct a base of $\mathcal{B}_n = \mathcal{A}lt\text{-}\mathcal{N}il_n[\emptyset; x]$, for $n > 3$. After that we compute the solvability index of $\mathcal{B}_n$ which is $\lceil \log_2 n \rceil + 1$. We show that $\mathcal{B}_n$ is not nilpotent and $(\mathcal{B}_n)^2$ is nilpotent of index n. All the results are in the correspondence with the case $n = 3$.

Base of the free alternative
$\mathcal{N}il_n$-superalgebra on one odd generator

First we consider an ideal $\mathcal{I}_n \subset \mathcal{A}$, spanned by the elements $W_n(u_1, u_2, \ldots, u_n)$, $u_1, u_2, \ldots, u_n \in \mathcal{A}_0 \cup \mathcal{A}_1$, where

$$W_n(u_1, u_2, \ldots, u_n) = \sum_{\sigma \in Sym(n)} \mathrm{sign}_{\mathrm{odd}}(\sigma)(\ldots((u_{\sigma(1)}u_{\sigma(2)})u_{\sigma(3)})\cdots)u_{\sigma(n)},$$

and construct a base of $\mathcal{I}_n$. Using the multiplication table for $\mathcal{A}$ (see Proposition 3.5), we obtain from

$$\begin{aligned}
W_n(x, t, \ldots, t) &= (n-1)! \sum_{i=0}^{n-1} t^{n-i-1}(xt^i) \\
&= (n-1)! \sum_{i=0}^{n-1} t^{n-i-1}(t^i x - i t^{i-1} x^{[3]}) \\
&= (n-1)! \sum_{i=0}^{n-1} (t^{n-1}x - i t^{n-2} x^{[3]}) \\
&= n! \left(t^{n-1}x - \tfrac{n-1}{2} t^{n-2} x^{[3]}\right), \\
W_n(t, t, \ldots, t) &= n! t^n, \\
W_n(u^{[k]}, t, \ldots, t) &= n!\ t^{n-1} u^{[k]}, \\
W_n(z^{[k]}, t, \ldots, t) &= n!\ t^{n-1} z^{[k]}
\end{aligned}$$

that $t^{n-1}x - \frac{n-1}{2}t^{n-2}x^{[3]}, t^n, t^{n-1}u^{[k]}, t^{n-1}z^{[k]} \in \mathcal{I},\ k > 2$. Moreover,

$$\begin{aligned}
x \cdot \left(t^{n-1}x - \tfrac{n-1}{2}t^{n-2}x^{[3]}\right) &= \tfrac{1}{2}t^n - \tfrac{n-1}{2}(t^{n-2}(x^{[3]}x) - \tfrac{1}{3}t^{n-2}x^{[4]} + \tfrac{n-2}{6}t^{n-3}z^{[4]}), \\
\left(t^{n-1}x - \tfrac{n-1}{2}t^{n-2}x^{[3]}\right) \cdot x &= \tfrac{1}{2}t^n - \tfrac{n-1}{2}(t^{n-2}(x^{[3]}x) - \tfrac{2}{3}t^{n-2}x^{[4]}),
\end{aligned}$$

and $t^{n-1}x^{[3]}, t^{n-2}x^{[4]} + \frac{n-2}{2}t^{n-3}z^{[4]}, t^{n-2}(x^{[3]}x) + \frac{n-2}{3}t^{n-3}z^{[4]} \in \mathcal{I}$.

From the multiplication table (Proposition 3.5) we know that $xt = tx - x^{[3]}$ and $x^{[3]}$ is annihilated by central elements $z^{[k]}$, $k \geq 3$, therefore

$$W_n(z^{[k]}, x, t, \ldots, t) = n!\, t^{n-2} z^{[k]} x.$$

The element $x^{[3]}$ is also annihilated by $u^{[k]}$, $k \geq 3$, and

$$xu^{[k]} = (-1)^{k+1}(u^{[k]}x - 2tz^{[k]}).$$

This implies

$$\begin{aligned} W_n(u^{[k]}, x, t, \dots, t) &= \tfrac{n!}{2}(t^{n-2}u^{[k]}x + (-1)^{k+1}t^{n-2}xu^{[k]}) \\ &= n!\,(t^{n-2}u^{[k]}x - t^{n-1}z^{[k]}). \end{aligned}$$

Compute for $k \geq 3$

$$\begin{aligned} W_n(x^{[k]}, t, \dots, t) &= (n-1)! \sum_{i=0}^{n-1} t^{n-i-1}(x^{[k]}t^i) \\ &= (n-1)! \sum_{i=0}^{n-1} t^{n-i-1}(t^i x^{[k]} + it^{i-1}z^{[k]}) \\ &= n!(t^{n-1}x^{[k]} + \tfrac{n-1}{2}t^{n-2}z^{[k]}), \\ W_n(x^{[k]}x, t, \dots, t) &= (n-1)! \sum_{i=0}^{n-1} t^{n-i-1}((x^{[k]}x)t^i) \\ &= (n-1)! \sum_{i=0}^{n-1} t^{n-i-1}(t^i(x^{[k]}x) - it^{i-1}(\tfrac{1}{2}u^{[k]} - z^{[k]}x - \tfrac{1}{2}z^{[k+1]})) \\ &= n!(t^{n-1}(x^{[k]}x) - \tfrac{n-1}{2}t^{n-2}(\tfrac{1}{2}u^{[k]} - z^{[k]}x - \tfrac{1}{2}z^{[k+1]})). \end{aligned}$$

Therefore $t^{n-1}x^{[k]} + \frac{n-1}{2}t^{n-2}z^{[k]}$, $t^{n-1}(x^{[k]}x) - \frac{n-1}{4}t^{n-2}(u^{[k]} - z^{[k+1]})$, $t^{n-1}z^{[k]}$, $t^{n-1}u^{[k]}$, $t^{n-2}z^{[k]}x$, $t^{n-2}u^{[k]}x \in \mathcal{I}_n$. Multiplying by x we get

$$\begin{aligned} x \cdot (t^{n-1}x^{[k]} + \tfrac{n-1}{2}t^{n-2}z^{[k]}) &= (-1)^k(t^{n-1}(x^{[k]}x - x^{[k+1]}) \\ &\quad - \tfrac{n-1}{2}t^{n-2}(u^{[k]} + \tfrac{1}{3}z^{[k+1]} - z^{[k]}x)), \\ (t^{n-1}x^{[k]} + \tfrac{n-1}{2}t^{n-2}z^{[k]}) \cdot x &= t^{n-1}(x^{[k]}x) + \tfrac{n-1}{2}t^{n-2}(\tfrac{2}{3}z^{[k+1]} + z^{[k]}x), \end{aligned}$$

and $t^{n-1}x^{[k+1]}, t^{n-1}(x^{[k]}x), t^{n-2}z^{[k+1]}, t^{n-2}u^{[k]} \in \mathcal{I}_n$. Notice that for $k = 3$ we also have $t^{n-1}x^{[3]}, t^{n-2}z^{[3]} \in \mathcal{I}_n$.

Recall from [27] that for $k \geq 3$

$$\begin{aligned} x^{[3]}x^{[k]} &= \tfrac{1}{2}(-1)^k(u^{[k]} - z^{[k+1]}), \\ x^{[3]}(x^{[k]}x) &= (-1)^k(\tfrac{1}{2}u^{[k]}x - \tfrac{1}{2}z^{[k+1]}x + \tfrac{1}{3}z^{[k+2]}). \end{aligned}$$

Moreover, results of [27] also imply

$$(t^m, x^{[3]}, x^{[k]}x^{\varepsilon}) = 0,$$

where $\varepsilon \in \{0,1\}$, $m \geq 1$. Now, for m,i,j such that $0 \leq m,i,j \leq n-2$, $m+i+j=n-2$, we get

$$\begin{aligned}
(((t^m x)t^i)x^{[k]})t^j &= ((t^{m+i}x - it^{m+i-1}x^{[3]})x^{[k]})t^j \\
&= (-1)^k(t^{m+i}(x^{[k]}x - x^{[k+1]}) - \tfrac{i}{2}t^{m+i-1}u^{[k]} - \tfrac{2m-i}{6}t^{m+i-1}z^{[k+1]})t^j \\
&= (-1)^k(t^{n-2}(x^{[k]}x - x^{[k+1]}) - t^{n-3}(\tfrac{i+j}{2}u^{[k]} - jz^{[k]}x \\
&\quad + \tfrac{2m-i+3j}{6}z^{[k+1]})), \\
(((t^m x^{[k]})t^i)x)t^j &= ((t^{m+i}x^{[k]} + it^{m+i-1}z^{[k]})x)t^j \\
&= (t^{m+i}(x^{[k]}x) + \tfrac{m+i}{3}t^{m+i-1}z^{[k+1]} + it^{m+i-1}z^{[k]}x)t^j \\
&= t^{n-2}(x^{[k]}x) - t^{n-3}(\tfrac{j}{2}u^{[k]} - (i+j)z^{[k]}x - \tfrac{2m+2i+3j}{6}z^{[k+1]}),
\end{aligned}$$

and

$$\begin{aligned}
(((t^m x)t^i)(x^{[k]}x))t^j &= ((t^{m+i}x - it^{m+i-1}x^{[3]})(x^{[k]}x))t^j \\
&= (-1)^k(\tfrac{1}{2}t^{m+i+1}x^{[k]} + t^{m+i}(\tfrac{2}{3}x^{[k+2]} - x^{[k+1]}x + \tfrac{2m+2i+1}{6}z^{[k]}) \\
&\quad -t^{m+i-1}(\tfrac{i}{2}u^{[k]}x - \tfrac{m+i}{6}u^{[k+1]} + \tfrac{2m-i}{6}z^{[k+1]}x - \tfrac{m-i}{6}z^{[k+2]}))t^j \\
&= (-1)^k(\tfrac{1}{2}t^{n-1}x^{[k]} + t^{n-2}(\tfrac{2}{3}x^{[k+2]} - x^{[k+1]}x + \tfrac{2m+2i+3j+1}{6}z^{[k]}) \\
&\quad -t^{n-3}(\tfrac{i}{2}u^{[k]}x - \tfrac{m+i+3j}{6}u^{[k+1]} + \tfrac{2m-i+6j}{6}z^{[k+1]}x \\
&\quad -\tfrac{m-i+j}{6}z^{[k+2]})), \\
(((t^m(x^{[k]}x))t^i)x)t^j &= ((t^{m+i}(x^{[k]}x) - it^{m+i-1}(\tfrac{1}{2}u^{[k]} - z^{[k]}x - \tfrac{1}{2}z^{[k+1]}))x)t^j \\
&= (\tfrac{1}{2}t^{m+i+1}x^{[k]} + \tfrac{1}{3}t^{m+i}x^{[k+2]} + \tfrac{2m+5i+2}{6}t^i z^{[k]} \\
&\quad -t^{m+i-1}(\tfrac{i}{2}u^{[k]}x - \tfrac{m+i}{6}u^{[k+1]} + \tfrac{2m-i}{6}z^{[k+1]}x - \tfrac{m+i}{6}z^{[k+2]}))t^j \\
&= \tfrac{1}{2}t^{n-1}x^{[k]} + \tfrac{1}{3}t^{n-2}x^{[k+2]} + \tfrac{2m+5i+3j+2}{6}t^{n-2}z^{[k]} \\
&\quad -t^{n-3}(\tfrac{i}{2}u^{[k]}x - \tfrac{m+i}{6}u^{[k+1]} + \tfrac{2m-i}{6}z^{[k+1]}x - \tfrac{m+i+2j}{6}z^{[k+2]}).
\end{aligned}$$

Continue our computations modulo elements which are already in $\mathcal{I}_n$.

$$\begin{aligned}
W_n(x^{[k]}, x, t, \ldots, t) &= (n-2)! \sum_{\substack{0\le m,i,j\le n-2\\ m+i+j=n-2}} ((((t^m x^{[k]})t^i)x)t^j \\
&\quad +(-1)^k(((t^m x)t^i)x^{[k]})t^j) \\
&= (n-2)! \sum_{\substack{0\le m,i,j\le n-2\\ m+i+j=n-2}} (t^{n-2}(2x^{[k]}x - x^{[k+1]}) \\
&\quad -t^{n-3}(\tfrac{i+2j}{2}u^{[k]} - (i+2j)z^{[k]}x - \tfrac{i}{2}z^{[k+1]})) \\
&= \tfrac{n!}{2}(t^{n-2}(2x^{[k]}x - x^{[k+1]}) \\
&\quad -(n-2)t^{n-3}(\tfrac{1}{2}u^{[k]} - z^{[k]}x - \tfrac{1}{6}z^{[k+1]})), \\
W_n(x^{[k]}x, x, t, \ldots, t) &= (n-2)! \sum_{\substack{0\le m,i,j\le n-2\\ m+i+j=n-2}} ((((t^m (x^{[k]}x))t^i)x)t^j \\
&\quad +(-1)^{k+1}(((t^m x)t^i)(x^{[k]}x))t^j) \\
&= (n-2)! \sum_{\substack{0\le m,i,j\le n-2\\ m+i+j=n-2}} (-\tfrac{1}{3}t^{n-2}x^{[k+2]} + t^{n-2}(x^{[k+1]}x) \\
&\quad +t^{n-3}(-\tfrac{j}{2}u^{[k+1]} + jz^{[k+1]}x + \tfrac{2i+j}{6}z^{[k+2]})) \\
&= \tfrac{n!}{2}(-\tfrac{1}{3}t^{n-2}x^{[k+2]} + t^{n-2}(x^{[k+1]}x) \\
&\quad +\tfrac{n-2}{3}t^{n-3}(-\tfrac{1}{2}u^{[k+1]} + z^{[k+1]}x + \tfrac{1}{2}z^{[k+2]})).
\end{aligned}$$

We obtain $t^{n-2}x^{[k+2]} + (n-2)t^{n-3}(\frac{1}{2}u^{[k+1]} - z^{[k+1]}x + \frac{1}{2}z^{[k+2]})$, $t^{n-2}(x^{[k+1]}x) + \frac{n-2}{3}t^{n-3}z^{[k+2]} \in \mathcal{I}_n$. Moreover,

$$\begin{aligned}
W_n(x^{[k]}, x, t, \ldots, t)\cdot x &= \tfrac{2}{3}t^{n-2}x^{[k+2]} - t^{n-2}(x^{[k+1]}x) \\
&\quad +(n-2)t^{n-3}(\tfrac{1}{3}u^{[k+1]} - \tfrac{1}{2}u^{[k]}x - \tfrac{1}{2}z^{[k+1]}x), \\
x\cdot W_n(x^{[k]}, x, t, \ldots, t) &= (-1)^k(t^{n-2}(\tfrac{1}{3}x^{[k+2]} - x^{[k+1]}x) \\
&\quad -(n-2)t^{n-3}(\tfrac{1}{2}u^{[k]}x - \tfrac{1}{6}u^{[k+1]} + \tfrac{1}{2}z^{[k+1]}x + \tfrac{1}{6}z^{[k+2]})).
\end{aligned}$$

imply that $t^{n-3}u^{[k]}x, t^{n-3}z^{[k+1]}x \in \mathcal{I}_n$.

Observe that the values of $W_n(u_1, u_2, \ldots, u_n)$, for any other base elements $u_1, u_2, \ldots, u_n$, do not bring new elements from $\mathcal{I}_n$. Therefore we obtain a base of $\mathcal{I}_n$.

Proposition 4.9 *Elements*

$$\begin{array}{rll} t^{m+n}x^{\sigma}, & & \\ t^{n-1}x - \frac{n-1}{2}t^{n-2}x^{[3]}, & & \\ t^{n-2}x^{[k+3]} + \frac{n-2}{2}t^{n-3}(u^{[k+2]} + z^{[k+3]}), & t^{n-2}(x^{[k+2]}x) + \frac{n-2}{3}t^{n-3}z^{[k+3]}, & \\ t^{m+n-1}(x^{[k+2]}x^{\sigma}), & & \\ t^{m+n-3}(u^{[4k+\varepsilon]}x^{\sigma}), & m+\sigma \geq 1, & \\ t^{m+n-3}(z^{[4k+\varepsilon]}x^{\sigma}), & m+\sigma \geq 1, & (19) \end{array}$$

where $k > 0$, $m \geq 0$; ε, $\sigma \in \{0,1\}$, *form a base of* $\mathcal{I}_n$. □

Now the superalgebra $\mathcal{B}_n = \mathcal{A}/\mathcal{I}_n$ is spanned by the elements

$$\begin{array}{rl} t^i x^{\sigma}, & 0 \leq i < n,\ i+\sigma > 0, \\ t^i(x^{[k]}x^{\sigma}), & 0 \leq i < n-2, \\ t^i u^{[4m+\varepsilon]}x^{\sigma},\ t^i z^{[4m+\varepsilon]}x^{\sigma}, & 0 \leq i < n-2,\ i+\sigma < n-2, \end{array} \qquad (20)$$

where $k > 2$, $m > 0$, $\varepsilon, \sigma \in \{0,1\}$. All other elements from (13) are equal zero, except

$$\begin{array}{rcl} t^{n-1}x & = & \frac{n-1}{2}t^{n-2}x^{[3]}, \\ t^{n-2}x^{[k+3]} & = & -\frac{n-2}{2}t^{n-3}(u^{[k+2]} + z^{[k+3]}), \\ t^{n-2}(x^{[k+2]}x) & = & -\frac{n-2}{3}t^{n-3}z^{[k+3]}. \end{array}$$

Moreover, these elements are linearly independent since any non-trivial linear combination of their preimages does not lie in $\mathcal{I}_n$.

Theorem 4.10 *The elements from* (20) *form a base of the superalgebra* $\mathcal{B}_n$. □

The multiplication table of $\mathcal{B}_n$ is given by that of $\mathcal{A}$ modulo $\mathcal{I}_n$.

Index of solvability of $\mathcal{B}_n$

Corollary 4.11 *The solvability index of* $\mathcal{B}_n$ *is* $\lceil \log_2 n \rceil + 1$ *for* $n > 3$.

Proof: By definition,

$$\mathcal{B}_n^{(0)} = \mathcal{B}_n.$$

For $i > 0$: $\mathcal{B}_n^{(i+1)} = \mathcal{B}_n^{(i)} \cdot \mathcal{B}_n^{(i)}$ is spanned by

$$\begin{aligned} t^j x^\sigma, &\qquad 2^i \le j < n,\ j+\sigma > 0, \\ t^j (x^{[k]} x^\sigma), &\qquad 2^i - 1 \le j < n-2, \\ t^j u^{[4m+\varepsilon]} x^\sigma,\ t^j z^{[4m+\varepsilon]} x^\sigma, &\qquad 2^i - 2 \le j < n-2,\ j+\sigma < n-2, \end{aligned}$$

where $k > 2$, $m > 0$, $\varepsilon, \sigma \in \{0,1\}$. Since $\mathcal{B}_n = \mathcal{A}lt\text{-}\mathcal{N}il_n[\emptyset; x]$ does not contain the element t^n, we can find i such that

$$2^i = n \to i := \lceil \log_2 n \rceil$$

and construct $\mathcal{B}_n^{(\lceil \log_2 n \rceil)}$ that is spanned by

$$\begin{aligned} t^j x^\sigma, &\qquad \lceil n/2 \rceil \le j < n,\ j+\sigma > 0, \\ t^j (x^{[k]} x^\sigma), &\qquad \lceil n/2 \rceil - 1 \le j < n-2, \\ t^j u^{[4m+\varepsilon]} x^\sigma,\ t^j z^{[4m+\varepsilon]} x^\sigma, &\qquad \lceil n/2 \rceil - 2 \le j < n-2,\ j+\sigma < n-2. \end{aligned}$$

where $k > 2$, $m > 0$, $\varepsilon, \sigma \in \{0,1\}$ and

$$\mathcal{B}_n^{(\lceil \log_2 n \rceil + 1)} = \mathcal{B}_n^{(\lceil \log_2 n \rceil)} \cdot \mathcal{B}_n^{(\lceil \log_2 n \rceil)} = 0.$$

We find out that $\mathcal{B}_n^{(\lceil \log_2 n \rceil + 1)} = 0$ (and $\mathcal{B}_n^{(i)} \neq 0$ for all $0 < i \le \lceil \log_2 n \rceil$). □

Nilpotency of $\mathcal{B}_n$

Corollary 4.12 *For $n > 3$, $\mathcal{B}_n$ is not nilpotent, moreover $\big(\mathcal{B}_n\big)^2$ is nilpotent of index n and $\big(\mathcal{B}_n\big)^m \cdot \big(\mathcal{B}_n\big)^r$ is not zero for any integers $m, r > 0$.*

Proof: First we prove that $\big({B_n}^2\big)^n = 0$. From the multiplication table for $\mathcal{B}_n$ (see Appendix 6) we obtain:

- $\big(\mathcal{B}_n\big)^2 = \mathcal{B}_n \mathcal{B}_n = \mathcal{B}_n^{(1)}$ is spanned by

$$\begin{aligned} t^j x^\sigma, &\qquad 1 \le j < n,\ j+\sigma > 0, \\ t^j (x^{[k]} x^\sigma), &\qquad 0 \le j < n-2, \\ t^j u^{[4m+\varepsilon]} x^\sigma,\ t^j z^{[4m+\varepsilon]} x^\sigma, &\qquad 0 \le j < n-2,\ j+\sigma < n-2 \end{aligned}$$

  where $k > 2$, $m > 0$, $\varepsilon, \sigma \in \{0,1\}$ and

$\left(\mathcal{B}_n{}^2\right)^i = \left(\mathcal{B}_n{}^2\right)^{i-1}\mathcal{B}_n{}^2 + \cdots + \mathcal{B}_n{}^2\left(\mathcal{B}_n{}^2\right)^{i-1}$ is spanned by

$$\begin{array}{rl} t^j x^\sigma, & i \le j < n,\ j+\sigma > 0, \\ t^j(x^{[k]}x^\sigma), & i-1 \le j < n-2, \\ t^j u^{[4m+\varepsilon]}x^\sigma,\ t^j z^{[4m+\varepsilon]}x^\sigma, & i-2 \le j < n-2,\ j+\sigma < n-2, \end{array}$$

where $k > 2$, $m > 0$, $\varepsilon, \sigma \in \{0,1\}$. Now it is easy to see that
$\left(\mathcal{B}_n{}^2\right)^{n-1} = \left(\mathcal{B}_n{}^2\right)^{n-2}\mathcal{B}_n{}^2 + \cdots + \mathcal{B}_n{}^2\left(\mathcal{B}_n{}^2\right)^{n-1}$ is spanned by

$$\begin{array}{r} t^{n-1}x^\varepsilon, \\ t^{n-3}u^{[4m+\varepsilon]},\ t^{n-3}z^{[4m+\varepsilon]}, \end{array}$$

where $m > 0$, $\varepsilon, \in \{0,1\}$ and

$$\left(\mathcal{B}_n{}^2\right)^n = 0.$$

- Next we prove that $\left(\mathcal{B}_n\right)^i \neq 0$ for all $i > 0$. First $\left(\mathcal{B}_n\right)^1 = \mathcal{B}_n$. Next observe that
  $\left(\mathcal{B}_n\right)^i = \left(\mathcal{B}_n\right)^{i-1}\mathcal{B}_n + \cdots + \mathcal{B}_n\left(\mathcal{B}_n\right)^{i-1}$, for $3 < i$ is spanned by the elements

  $$\begin{array}{rl} t^m x^\sigma, & i \le 2m+\sigma,\ 0 \le m < n,\ \sigma \in \{0,1\}, \\ t^m\,(x^{[k]}x^\sigma), & i \le 2m+k+\sigma,\ 0 \le m < n-2,\ k > 2,\ \sigma \in \{0,1\} \\ t^m u^{[4r+\varepsilon]}x^\sigma, & i-3 \le 2m+(4r+\varepsilon)+\sigma,\ 0 \le m < n-2, \\ & r > 0,\ \sigma,\varepsilon \in \{0,1\} \\ t^m z^{[4r+\varepsilon]}x^\sigma, & i-2 \le 2m+(4r+\varepsilon)+\sigma,\ 0 \le m < n-2, \\ & r > 0,\ \sigma,\varepsilon \in \{0,1\}. \end{array}$$

  Like in case $n = 3$, it is sufficient to prove for example $x^{[i]} \in \left(\mathcal{B}_n\right)^i$ for all integer $i > 0$. The proof is valid in the same form like in case $n = 3$: Obviously $x^{[1]} = x \in \left(\mathcal{B}_n\right)^1$. If $x^{[i]} \in \left(\mathcal{B}_n\right)^i$ for some $i > 1$, then $x^{[i]}\cdot x,\ x\cdot x^{[i]} \in \left(\mathcal{B}_n\right)^{i+1}$, since $\left(\mathcal{B}_n\right)^{i+1} = \left(\mathcal{B}_n\right)^i\mathcal{B}_n + \cdots + \mathcal{B}_n\left(\mathcal{B}_n\right)^i$. From

  $$x^{[i+1]} = x^{[i]}\cdot x - (-1)^i\,(x\cdot x^{[i]})$$

  we have that $x^{[i+1]} \in \left(\mathcal{B}_n\right)^{i+1}$.

  We have proved that $x^{[i]} \in \left(\mathcal{B}_n\right)^i$ for all natural $i > 0$, and so $\mathcal{B}_n$ is not nilpotent.

- Now from $x^{[i]} \in \big(\mathcal{B}_n\big)^m, x^{[j]} \in \big(\mathcal{B}_n\big)^r$, $i \geq m,\ j \geq r$, and the multiplication

$$x^{[i]} \cdot x^{[j]} \;\; = \;\; \tfrac{1}{2}(-1)^{c(j+1)}(u^{[i+j-3]} - (-1)^j z^{[i+j-2]}),$$

we take i, j large enough, such that $u^{[i+j-3]}$ or $z^{[i+j-2]}$ are not zero (i.e. the indices $i+j-3$ or $i+j-2$ must be of the form $4k$ or $4k+1,\ k > 0$). Such i, j exist, because there are infinitely many elements of the form $x^{[i]} \in \big(\mathcal{B}_n\big)^m$ and $x^{[j]} \in \big(\mathcal{B}_n\big)^r$. We have proved that $\big(\mathcal{B}_n\big)^m \big(\mathcal{B}_n\big)^r \neq 0$. □

5 Applications

5.1 The subspace of skew-symmetric elements of the free alternative nil-algebra

Here we present a base of the subspace of skew-symmetric elements of the free alternative nil-algebra using the base of the superalgebra $\mathcal{B}_n = \mathcal{A}lt\text{-}\mathcal{N}il_n[\emptyset; x]$ constructed in the previous section.

Let $Alt\text{-}Nil_n\,[T] = Alt\text{-}Nil_n\,[T;\emptyset]$ be the free alternative nil-algebra of nil-index n on a set of even generators T and let $Skew$ be the linear mapping from $\mathcal{B}_n$ to $\mathcal{A}lt\text{-}\mathcal{N}il_n[T]$ defined in Subsection 3.1.1. Then $Skew$ maps isomorphically the homogeneous component $\mathcal{B}_n^{[m]}$ of degree m of $\mathcal{B}_n$ to the subspace $Skew(Alt\text{-}Nil_n[T_m])$ of multilinear skew-symmetric elements on $T_m = \{t_1, t_2, \ldots, t_m\}$ of $Alt\text{-}Nil_n\,[T]$.

Theorem 5.1
The elements

$$Skew\, f(t_{i_1}, t_{i_2}, \ldots, t_{i_k}),$$

where $f = f(x)$ runs through the base (20) of $\mathcal{B}_n$, $k = \deg(f)$, $i_1 < i_2 < \cdots < i_k$, form a base of the subspace $Skew(Alt\text{-}Nil_n[T])$ of skew-symmetric elements of $Alt\text{-}Nil_n\,[T]$. □

5.2 Dorofeev's example

Here we present a Grassmann algebra corresponding to $\mathcal{A}lt\text{-}\mathcal{N}il_3[\emptyset; x]$ and show that Dorofeev's example of solvable non-nilpotent alternative algebra (see [1]) is its homomorphic image. We use the definition of $\mathcal{V}$-Grassmann algebra in the variety of algebras $\mathcal{V}$ given in [27], and show that $\mathcal{A}lt\text{-}\mathcal{N}il_3$-Grassmann algebra generalizes Dorofeev's example.

Consider the free $\mathcal{V}$-superalgebra $\mathcal{V}[\emptyset; x]$ on one odd generator x, then its Grassmann envelope $G(\mathcal{V}[\emptyset; x])$ belongs to $\mathcal{V}$. The subalgebra of $G(\mathcal{V}[\emptyset; x])$ generated by the elements

$$e_1 \otimes x,\ e_2 \otimes x, \ldots, e_n \otimes x, \ldots,$$

is called the *$\mathcal{V}$-Grassmann algebra* and is denoted by $G(\mathcal{V})$. The following proposition is evident.

Proposition 5.2 *The $\mathcal{V}$-Grassmann algebra $G(\mathcal{V})$ has a base of the form:*

$$e_\mu \otimes v,\ \ |\mu| = \deg(v),$$

where v runs a (monomial) base of the superalgebra $\mathcal{V}[\emptyset; x]$, $\mu = \{i_1, \dots, i_m\}$, $i_1 < i_2 < \cdots < i_m$, $|\mu| = m$, $e_\mu = e_{i_1} e_{i_2} \cdots e_{i_m} \in G$. □

A base of the $\mathcal{A}lt$-$\mathcal{N}il_3$-Grassmann algebra $B = G(\mathcal{A}lt\text{-}\mathcal{N}il_3)$ is now given by base (18) of the superalgebra $\mathcal{B}_3$.

Dorofeev's example was originally constructed over the ring of integer numbers. We consider the same construction over any field of characteristic zero. Let $T = \{t_1, t_2, \dots, t_n, \dots\}$ be a countable set of symbols, and let R_{t_i} be the operator of "right multiplication" which maps any word v to the word $(v)t_i$. A base of Dorofeev's algebra D consists of the words

$$r_\mu = t_{i_1} R_{t_{i_2}} \cdots R_{t_{i_m}}, \ m > 0,$$
$$s_\mu = (t_{i_1}(t_{i_2} t_{i_3})) R_{t_{i_4}} \cdots R_{t_{i_m}}, \ m > 2,$$

where $\mu = \{i_1, i_2, \dots, i_m\}$, $i_1 < i_2 < \cdots < i_m$.

To simplify the multiplication table, denote by t_μ a base word in general, that is, either r_μ or s_μ. Let $\mu = \{i_1, i_2, \dots, i_m\}$ and $\eta = \{j_1, j_2, \dots, j_n\}$ be two sets of indices. If $\mu \cap \eta = \emptyset$ then denote by $\sigma(\mu, \eta)$ the signature of the permutation $(i_1, i_2, \dots, i_m, j_1, j_2, \dots, j_n)$ and by $\overline{\mu, \eta}$ the set obtained by ordering indices in $\mu \cup \eta$. Otherwise, put $t_{\overline{\mu,\eta}} = 0$.

The multiplication table on base words is given by the rules:

(i) $t_\mu * t_i = \sigma(\mu, i) t_{\overline{\mu,i}}$,

(ii) for $|\mu| = 2$, $t_i * r_\mu = \sigma(i, \mu) s_{\overline{i,\mu}}$,

(iii) for $|\mu| = 3$, $t_i * r_\mu = \sigma(i, \mu)(r_{\overline{i,\mu}} + s_{\overline{i,\mu}})$, and $t_i * s_\mu = -\sigma(i, \mu) r_{\overline{i,\mu}}$,

(iv) for any base word $t_\mu t_j$, $|\mu| \geq 2$, define by induction

$$t_i * (t_\mu t_j) = (t_i * t_\mu + t_\mu * t_i) * t_j,$$

(v) for any base words v_1, v_2, v_3, v_4, put $(v_1 * v_2) * (v_3 * v_4) = 0$.

Now, from (i) we easily obtain that for any base word t_μ it holds

$$(t_\mu * t_i) * t_j = -(t_\mu * t_j) * t_i.$$

Moreover, it was proved in [1] that for any base word t_μ, $|\mu| \geq 2$, it holds

$$\begin{aligned} t_i * (t_\mu * t_j) &= (t_i * t_\mu + t_\mu * t_i) * t_j, \\ t_i * (t_j * t_\mu) &= (t_\mu * t_j) * t_i, \\ (t_i * t_\mu) * t_j &= -(t_j * t_\mu) * t_i. \end{aligned}$$

Construct a homomorphism of vector spaces $\psi : B \to D$ by defining

$$\begin{aligned}
\psi(e_{i_1} \otimes x) &= t_{i_1}, \\
\psi(e_{i_1}e_{i_2} \otimes t) &= [t_{i_1}, t_{i_2}], \\
\psi(e_{i_1}e_{i_2}e_{i_3} \otimes tx) &= [t_{i_1}, t_{i_2}] * t_{i_3}, \\
\psi(e_{i_1}e_{i_2}e_{i_3}e_{i_4} \otimes t^2) &= 0, \\
\psi(e_{i_1}e_{i_2}e_{i_3}e_{i_3}e_{i_5} \otimes t^2x) &= 0, \\
\psi(e_{i_1} \cdots e_{i_k} \otimes x^{[k]}) &= [t_{i_1}, t_{i_2}, \dots, t_{i_k}], \\
\psi(e_{i_1} \cdots e_{i_{k+1}} \otimes x^{[k]}x) &= [t_{i_1}, t_{i_2}, \dots, t_{i_k}] * t_{i_{k+1}}, \\
\psi(e_{i_1} \cdots e_{i_{4n+\varepsilon+3}} \otimes u^{[4n+\varepsilon]}) &= 0, \\
\psi(e_{i_1} \cdots e_{i_{4n+\varepsilon+2}} \otimes z^{[4n+\varepsilon]}) &= 0.
\end{aligned}$$

Here $[t_{i_1}, \dots, t_{i_{k-1}}, t_{i_k}]$ denotes the "long commutator" of elements $t_{i_1}, \dots, t_{i_k}$ which is defined by induction:

$$\begin{aligned}
[t_{i_1}, t_{i_2}] &= t_{i_1} * t_{i_2} - t_{i_2} * t_{i_1}, \\
[t_{i_1}, \dots, t_{i_{k-1}}, t_{i_k}] &= [[t_{i_1}, \dots, t_{i_{k-1}}], t_{i_k}].
\end{aligned}$$

Our objective is to prove, that ψ is a surjective homomorphism of algebras. It is clear that

$$\psi((e_\mu \otimes u)(e_\eta \otimes v)) = \psi(e_\mu \otimes u) * \psi(e_\eta \otimes v) = 0$$

for any elements u, v of the base (18) of degree ≥ 2, $|\mu| = deg(u)$, $|\eta| = deg(v)$. This equality also holds when one of the elements u or v is equal to x and another one is t^2, t^2x, $u^{[4n+\varepsilon]}$ or $z^{[4n+\varepsilon]}$. For the remaining cases let us first compute long commutators in D, and find an explicit expression for the rule (iv).

From the definition and properties of the $*$-multiplication, for any base word t_μ, $|\mu| \geq 2$, we obtain

$$\begin{aligned}
[t_\mu, t_i, t_j] &= (t_\mu * t_i - t_i * t_\mu) * t_j - t_j * (t_\mu * t_i - t_i * t_\mu) \\
&= (t_\mu * t_i) * t_j - (t_i * t_\mu) * t_j \\
&\quad -(t_j * t_\mu + t_\mu * t_j) * t_i + (t_\mu * t_i) * t_j \\
&= 3(t_\mu * t_i) * t_j.
\end{aligned}$$

Therefore, for $\mu = \{i_1, i_2, \dots, i_m\}$, $i_1 < i_2 < \cdots < i_m$,

$$\begin{aligned}
[t_{i_1}, t_{i_2}] &= 2r_\mu, \\
[t_{i_1}, t_{i_2}, t_{i_3}] &= 2(r_\mu - s_\mu), \\
[t_{i_1}, \dots, t_{i_{2k}}] &= 2 \cdot 3^{k-1} r_\mu, \\
[t_{i_1}, \dots, t_{i_{2k+1}}] &= 2 \cdot 3^{k-1}(r_\mu - s_\mu),
\end{aligned}$$

and for $\eta = \{i_1, i_2, \ldots, i_{m-1}\}$,

$$\begin{aligned}
2\cdot 3^{k-1} r_\mu &= [t_{i_1}, \ldots, t_{i_{2k}}] = 2\cdot 3^{k-2}((r_\eta - s_\eta) * t_{i_{2k}} - t_{i_{2k}} * (r_\eta - s_\eta)) \\
&= 2\cdot 3^{k-2}((r_\mu - s_\mu) - t_{i_{2k}} * (r_\eta - s_\eta)), \\
2\cdot 3^{k-1}(r_\mu - s_\mu) &= [t_{i_1}, \ldots, t_{i_{2k+1}}] = 2\cdot 3^{k-1}(r_\eta * t_{i_{2k+1}} - t_{i_{2k+1}} * r_\eta) \\
&= 2\cdot 3^{k-1}(r_\mu - t_{i_{2k+1}} * r_\eta),
\end{aligned}$$

that is,

$$\begin{aligned}
t_{i_{2k}} * (r_\eta - s_\eta) &= -(2r_\mu + s_\mu),\ |\eta| = 2k-1, \\
t_{i_{2k+1}} * (2r_\eta + s_\eta) &= -r_\mu + s_\mu,\ |\eta| = 2k, \\
t_{i_{2k+1}} * r_\eta &= s_\mu,\ |\eta| = 2k, \\
t_{i_{2k+2}} * s_\eta &= r_\mu,\ |\eta| = 2k+1,
\end{aligned}$$

which implies for $k > 1$

$$\begin{aligned}
t_i * r_\mu &= \begin{cases} \sigma(i,\mu) s_{\overline{i,\mu}}, & |\mu| = 2k, \\ \sigma(i,\mu)(r_{\overline{i,\mu}} + s_{\overline{i,\mu}}), & |\mu| = 2k-1, \end{cases} \\
t_i * s_\mu &= \begin{cases} -\sigma(i,\mu)(r_{\overline{i,\mu}} + s_{\overline{i,\mu}}), & |\mu| = 2k, \\ -\sigma(i,\mu) r_{\overline{i,\mu}}, & |\mu| = 2k-1. \end{cases}
\end{aligned}$$

It is easy to see that ψ preserves multiplication and is surjective.

Now we can state the main result of this subsection.

Theorem 5.3 *Dorofeev's algebra is isomorphic to the quotient algebra of the $\mathcal{A}lt$-$\mathcal{N}il_3$-Grassmann algebra B modulo the ideal $(B^2)^2$.* □

5.3 Alternative nil-algebras constructed from $\mathcal{A}lt$-$\mathcal{N}il_n[\emptyset; x]$

Now we construct solvable alternative nil-algebras which are not associative of arbitrary big solvability index. We use a standard passage to Grassmann envelope over a field of characteristic zero and alternative nil-superalgebras $\mathcal{A}lt$-$\mathcal{N}il_n[\emptyset; x]$ on one odd generator x of nil-index $n \geq 3$, constructed in previous section.

Consider $\mathcal{B}_n = \mathcal{A}lt$-$\mathcal{N}il_n[\emptyset; x]$ the free alternative nil-superalgebra on one odd generator of nil-index $n \geq 3$. By the definition, its Grassmann envelope $G(\mathcal{B}_n)$ is an alternative nil-algebra of nil-index n.

Theorem 5.4 *The alternative nil-algebra $G(\mathcal{B}_n)$ of nil-index $n \geq 3$, has a base of the form:*

$$g \otimes v,$$

where v runs the (monomial) base (20) of the superalgebra $\mathcal{B}_n$, $\overline{v} \in \{0,1\}$, and g runs the base (4) of the superalgebra G, and $\overline{g} = \overline{v}$. □

Index of solvability is consistent with that of $\mathcal{B}_n$. By the definition

$$\begin{aligned} G(\mathcal{B}_n)^{(0)} &= G(\mathcal{B}_n) \\ G(\mathcal{B}_n)^{(i)} &= G(\mathcal{B}_n)^{(i-1)} G(\mathcal{B}_n)^{(i-1)} = vect\langle g^{\overline{u}} \otimes u \cdot g^{\overline{v}} \otimes v \\ &\qquad \mid g^{\overline{u}}, g^{\overline{v}} \in (G_0 \cup G_1)^{(i-1)},\ u, v \in B_n^{(i-1)} \rangle \\ &= vect\langle g^{\overline{u}} g^{\overline{v}} \otimes uv \mid g^{\overline{u}}, g^{\overline{v}} \in (G_0 \cup G_1)^{(i-1)},\ u, v \in B_n^{(i-1)} \rangle. \end{aligned}$$

Then

$$G(\mathcal{B}_n)^{(i)} = 0 \quad \text{iff} \quad \mathcal{B}_n^{(i)} = 0$$

for some $i > 0$, i.e. for every nonzero $g^{\overline{u}} g^{\overline{v}}$ and $g^{\overline{u}}, g^{\overline{v}} \in (G_0 \cup G_1)^{(i-1)}$, we have

$$G(B_n)^{(i)} = 0 \Leftrightarrow \text{ all } g^{\overline{u}} g^{\overline{v}} \otimes uv = 0 \Leftrightarrow uv = 0,$$

for all $u, v \in B_n^{(i-1)}$.

Corollary 5.5 *The alternative nil-algebra $G(\mathcal{B}_n)$ of nil-index n is solvable of index*

$$\lceil \log_2 n \rceil + 1,$$

for $n \geq 3$. □

Therefore the solvability index of alternative nil-algebras of nil-index n is $\geq \lceil \log_2 n \rceil + 1$. Notice that already Kuzmin's results [9] imply that this solvability index is $\geq \log_2 \frac{n(n+1)}{2}$.

The nilpotency of $G(\mathcal{B}_n)$ is also consistent with the nilpotency of $\mathcal{B}_n$. Recall that $\mathcal{B}_n$ is not nilpotent for $n \geq 3$. For $G(\mathcal{B}_n)$ we have

$$\begin{aligned} G(\mathcal{B}_n)^1 &= G(\mathcal{B}_n), \\ G(\mathcal{B}_n)^{i+1} &= G(\mathcal{B}_n)^i G(\mathcal{B}_n) + \cdots + G(\mathcal{B}_n) G(\mathcal{B}_n)^i, \end{aligned}$$

and we show that for example $e_1 e_2 \cdots e_{i+1} \otimes x^{[i+1]}$ is nonzero element in $G(\mathcal{B}_n)^{i+1}$, for $i > 0$. Take the elements $e_{i+1} \otimes x \in G(\mathcal{B}_n)$ and $e_1 e_2 \cdots e_i \otimes x^{[i]} \in$

$G(\mathcal{B}_n)^i$, then from

$$e_1e_2\cdots e_i\otimes x^{[i]}\cdot e_{i+1}\otimes x = e_1e_2\cdots e_{i+1}\otimes x^{[i]}\cdot x,$$
$$e_{i+1}\otimes x\cdot e_1e_2\cdots e_i\otimes x^{[i]} = (-1)^i e_1e_2\cdots e_{i+1}\otimes x\cdot x^{[i]},$$

we have that $e_1e_2\cdots e_{i+1}\otimes x^{[i+1]} = e_1e_2\cdots e_{i+1}\otimes(x^{[i]}x-(-1)^i x\cdot x^{[i]})$ is in $G(\mathcal{B}_n)^{i+1}$.

Corollary 5.6 *The alternative nil-algebra $G(\mathcal{B}_n)$ of nil-index n is not nilpotent, for $n\geq 3$.*

6 Summary of the results

The aim of this work was the usage of the superalgebra method in the study of free algebras. In this wide matters, on the free alternative superalgebra on one odd generator, which base is known (see [27]), was focused, and some new applications of this superalgebra were found. The free alternative nil-superalgebra on one odd generator of nil-index $n \geq 2$ was investigated and two basic problems were solved:

1) Finding a base of the free alternative nil-superalgebra on one odd generator of nil-index $n \geq 2$.

2) Computing the solvability index of this superalgebra.

In addition, several applications of this superalgebra were found.

Using the base of the free alternative superalgebra $\mathcal{A} = \mathcal{A}lt[\emptyset; x]$ on one odd generator x constructed in [27], the base of the free alternative nil-superalgebra $\mathcal{B}_n = \mathcal{A}lt$-$\mathcal{N}il_n[\emptyset; x]$ on one odd generator x of nil-index n was constructed. The multiplication table of $\mathcal{B}_n$ was introduced, for $\mathcal{B}_2$ and $\mathcal{B}_3$ were written up, and the solvability index of $\mathcal{B}_n$, for $n \geq 2$ was computed.

It was started with $\mathcal{A}lt$-$\mathcal{N}il_2[\emptyset; x]$ (see [16], 2009). A base of this superalgebra was constructed and it was proved that it is solvable of index 2. It was continued with the case $n = 3$ (see [17], 2010). More precisely, a base of the free alternative nil-superalgebra $\mathcal{A}lt$-$\mathcal{N}il_3[\emptyset; x]$ was constructed and the solvability index of $\mathcal{A}lt$-$\mathcal{N}il_3[\emptyset; x]$ was computed as 3. Last step was a generalization, for $n \geq 3$. A base of $\mathcal{B}_n = \mathcal{A}lt$-$\mathcal{N}il_n[\emptyset; x]$ was constructed and the solvability index was computed as $\lceil \log_2 n \rceil + 1$ (see [18]). It was also shown that, for $n > 2$, $\mathcal{B}_n$ is not nilpotent and ${\mathcal{B}_n}^2$ is nilpotent of index n. It was known from Pchelintsev (see [13], 1985) that ${\mathcal{B}_n}^2$ must be nilpotent. Here the first exactly computed nilpotency index was presented.

The superalgebra $\mathcal{B}_n$ was then used for applications. Due to the result from [23], the $\mathcal{V}$-superalgebra on one odd generator is isomorphic as a vector space to the subspace of all skew-symmetric elements of the free $\mathcal{V}$-algebra on countable many generators. Using the base of the superalgebra $\mathcal{B}_n$, a base of the subspace of skew-symmetric elements of the free alternative nil-algebra on a countable set of generators was described.

A nonassociative Grassmann algebra corresponding to $\mathcal{A}lt$-$\mathcal{N}il_3[\emptyset; x]$ was also presented and it was shown that Dorofeev's example of solvable non-nilpotent alternative algebra is its homomorphic image.

An infinite family of solvable alternative nil-algebras of arbitrary big solvability index was constructed, for $n \geq 3$. The algebras are formed from alternative nil-superalgebras $\mathcal{A}lt\text{-}\mathcal{N}il_n[\emptyset; x]$, using a standard passage to Grassmann envelope over a field of characteristic zero. These algebras are not associative. Till now there were no explicit examples of such algebras.

Further plans

It would be interesting to describe all skew-symmetric central and nuclear elements in alternative algebras. Evidently, they all should be of the type $Skew\, f$, where $f \in N(\mathcal{A})$ and $f \in Z(\mathcal{A})$ for central and nuclear elements, respectively. Recall that not every element in $Z(\mathcal{A})$ produces central or nuclear skew-symmetric element in alternative algebra (see [27], 2007).

Following [23], for the free alternative superalgebra $\mathcal{A} = \mathcal{A}lt[\emptyset; x]$ its universal multiplicative envelope $Mult(\mathcal{A})$ is considered. It can be defined as a subalgebra of the algebra of endomorphisms $End(\mathcal{A}lt[a; x])$ of the free two-generated alternative superalgebra $\mathcal{A}lt[a; x]$, generated by all the operators $L_u, R_u, u \in \mathcal{A}$. In fact, here the parity of the generator a does not matter. Observe that the algebra $Mult(\mathcal{A})$ inherits naturally the superalgebra structure of $\mathcal{A}$: operators L_u and R_u are even(odd) if and only if so is u in $\mathcal{A}$.

Now for a homogeneous element $f \in \mathcal{A}$ holds: $Skew\, f$ is a central skew-symmetric element in the free alternative algebra $Alt[T]$ if and only if $R_f = L_f$ in $Mult(\mathcal{A})$. Therefore the study of operators R_f and L_f, for elements $f \in Z(\mathcal{A})$, is our future plan.

Appendix A

The computing of $\mathcal{B}_n^{(i)}$, $\left(\mathcal{B}_n{}^2\right)^i$ and $(\mathcal{B}_n)^i$

Solvability of $\mathcal{B}_3$

For $k > 2,\ m > 0,\ \varepsilon \in \{0, 1\}$ we have

$$\begin{aligned}
\mathcal{B}_3^{(0)} &= \mathcal{B}_3 \\
\mathcal{B}_3^{(1)} &= \mathcal{B}_3\mathcal{B}_3 \\
&= vect\langle t, tx, t^2, t^2x, x^{[k]}, x^{[k]}x, u^{[4m+\varepsilon]}, z^{[4m+\varepsilon]}\rangle, \\
&\qquad (x^{[3]} \leftarrow x \cdot t,\ x^{[k+1]} \leftarrow x \cdot x^{[k]}) \\
&\qquad (z^{[k]} \leftarrow (x^{[k]}x) \cdot x,\ u^{[k]} \leftarrow (x^{[k]}x) \cdot t) \\
\mathcal{B}_3^{(2)} &= \mathcal{B}_3^{(1)}\mathcal{B}_3^{(1)} \\
&= vect\langle t^2, t^2x, u^{[4m+\varepsilon]}, z^{[4m+\varepsilon]}\rangle, \\
&\qquad (z^{[k]} \leftarrow (tx) \cdot x^{[k]},\ u^{[k]} \leftarrow (x^{[k]}x) \cdot t) \\
\mathcal{B}_3^{(3)} &= \mathcal{B}_3^{(2)}\mathcal{B}_3^{(2)} = 0.
\end{aligned}$$

Nilpotency of square of $\mathcal{B}_3$

For $k > 2,\ m > 0,\ \varepsilon \in \{0, 1\}$ we have

$$\begin{aligned}
\left(\mathcal{B}_3{}^2\right)^1 &= \mathcal{B}_3{}^2 = \mathcal{B}_3\mathcal{B}_3 = \mathcal{B}_3^{(1)} \\
&= vect\langle t, tx, t^2, t^2x, x^{[k]}, x^{[k]}x, u^{[4m+\varepsilon]}, z^{[4m+\varepsilon]}\rangle, \\
\left(\mathcal{B}_3{}^2\right)^2 &= \mathcal{B}_3{}^2\mathcal{B}_3{}^2 = \mathcal{B}_3^{(2)} \\
&= vect\langle t^2, t^2x, u^{[4m+\varepsilon]}, z^{[4m+\varepsilon]}\rangle, \\
\left(\mathcal{B}_3{}^2\right)^3 &= \mathcal{B}_3{}^2(\mathcal{B}_3{}^2\mathcal{B}_3{}^2) + (\mathcal{B}_3{}^2\mathcal{B}_3{}^2)\mathcal{B}_3{}^2 = 0.
\end{aligned}$$

Nilpotency of $\mathcal{B}_3$

$$
\begin{aligned}
\big(\mathcal{B}_3\big)^1 &= \mathcal{B}_3\\
\big(\mathcal{B}_3\big)^2 &= \mathcal{B}_3\mathcal{B}_3\\
&= vect\langle t, tx, t^2, t^2x, x^{[k]}x^{\varepsilon}, u^{[4m+\varepsilon]}, z^{[4m+\varepsilon]} |\ k>2,\ m>0,\ \varepsilon\in\{0,1\}\rangle,\\
\big(\mathcal{B}_3\big)^3 &= \big(\mathcal{B}_3\big)^2\mathcal{B}_3 + \mathcal{B}_3\big(\mathcal{B}_3\big)^2\\
&= vect\langle tx, t^2, t^2x, x^{[k]}x^{\varepsilon}, u^{[4m+\varepsilon]}, z^{[4m+\varepsilon]} |\ k>2,\ m>0,\ \varepsilon\in\{0,1\}\rangle,\\
&\qquad (x^{[3]} \leftarrow x\cdot t)\\
&\qquad (z^{[4]} \leftarrow (x^{[4]}x)\cdot x,\ u^{[4]} \leftarrow (x^{[4]}x)\cdot t)\\
\big(\mathcal{B}_3\big)^4 &= vect\langle t^2, t^2x, x^{[3]}x, x^{[k]}x^{\varepsilon}, u^{[4m+\varepsilon]}, z^{[4m+\varepsilon]} |\ k>3,\ m>0,\ \varepsilon\in\{0,1\}\rangle,\\
&\qquad (x^{[3]} \leftarrow \text{ NA },\quad x^{[3]}x \leftarrow x^{[3]}\cdot x,\ x^{[4]} \leftarrow x\cdot(tx))\\
&\qquad (z^{[4]} \leftarrow (x^{[4]}x)\cdot x,\ u^{[4]} \leftarrow (x^{[4]}x)\cdot t)\\
\big(\mathcal{B}_3\big)^5 &= vect\langle t^2x, x^{[4]}x, x^{[k]}x^{\varepsilon}, u^{[4m+\varepsilon]}, z^{[4m+\varepsilon]} |\ k>4,\ m>0,\ \varepsilon\in\{0,1\}\rangle,\\
&\qquad (x^{[3]}x,\ x^{[4]} \leftarrow \text{ NA },\quad x^{[4]}x \leftarrow x\cdot(x^{[3]}x), x^{[5]} \leftarrow x\cdot(x^{[3]}x))\\
&\qquad (z^{[4]} \leftarrow (x^{[4]}x)\cdot x,\ u^{[4]} \leftarrow (x^{[4]}x)\cdot t)\\
\big(\mathcal{B}_3\big)^6 &= vect\langle x^{[5]}x, x^{[k]}x^{\varepsilon}, u^{[4m+\varepsilon]}, z^{[4m+\varepsilon]} |\ k>5,\ m>0,\ \varepsilon\in\{0,1\}\rangle,\\
&\qquad (x^{[4]}x,\ x^{[5]} \leftarrow \text{ NA },\quad x^{[5]}x \leftarrow x\cdot(x^{[4]}x), x^{[6]} \leftarrow x\cdot(x^{[4]}x))\\
&\qquad (z^{[4]} \leftarrow (x^{[4]}x)\cdot x,\ u^{[4]} \leftarrow (x^{[4]}x)\cdot t)\\
\big(\mathcal{B}_3\big)^7 &= vect\langle x^{[6]}x, x^{[k]}x^{\varepsilon}, u^5, u^{[4m+\varepsilon]}, z^5, z^{[4m+\varepsilon]} |\ k>6,\ m>1,\ \varepsilon\in\{0,1\}\rangle,\\
&\qquad (x^{[5]}x,\ x^{[6]} \leftarrow \text{ NA },\quad x^{[6]}x \leftarrow x\cdot(x^{[5]}x), x^{[7]} \leftarrow x\cdot(x^{[5]}x))\\
&\qquad (z^{[4]},\ u^{[4]} \leftarrow \text{ NA },\qquad z^{[5]} \leftarrow (x^{[5]}x)\cdot x,\ u^{[5]} \leftarrow (x^{[5]}x)\cdot t)\\
\big(\mathcal{B}_3\big)^8 &= vect\langle x^{[7]}x, x^{[k]}x^{\varepsilon}, u^6, u^{[4m+\varepsilon]}, z^6, z^{[4m+\varepsilon]} |\ k>7,\ m>1,\ \varepsilon\in\{0,1\}\rangle,\\
&\qquad (x^{[6]}x,\ x^{[7]} \leftarrow \text{ NA },\quad x^{[7]}x \leftarrow x\cdot(x^{[6]}x), x^{[8]} \leftarrow x\cdot(x^{[6]}x))\\
&\qquad (z^{[5]},\ u^{[5]} \leftarrow \text{ NA },\qquad "z^{[6]}" \leftarrow (x^{[6]}x)\cdot x,\ "u^{[6]}" \leftarrow (x^{[6]}x)\cdot t)
\end{aligned}
$$

so that for $i = 2j+\delta > 1,\ j>0,\ \delta\in\{0,1\}$ we have

$$
\begin{aligned}
\big(\mathcal{B}_3\big)^i &= \big(\mathcal{B}_3\big)^{i-1}\mathcal{B}_3 + \cdots + \mathcal{B}_3\big(\mathcal{B}_3\big)^{i-1}\\
&\overset{i=2j+\delta}{=} vect\langle t^jx^{\delta}, t^lx^{\varepsilon},\ x^{[i-1]}x, x^{[k]}x^{\varepsilon},\ u^{[4m+\varepsilon]}, z^{[4m+1+\varepsilon]}\\
&\qquad\qquad |\ j<l<3,\ k>i-1,\ 4m+\varepsilon\geq i-3,\ \varepsilon\in\{0,1\}\rangle.
\end{aligned}
$$

More exactly $\left(\mathcal{B}_3\right)^i$ is spanned by the elements of the form

$$\begin{array}{rl} t^j x^\delta, t^m x^\varepsilon, & j < m < 3,\ \varepsilon \in \{0,1\}, \\ x^{[i-1]}x, x^{[k]}x^\varepsilon, & k+\varepsilon > i-1,\ \varepsilon \in \{0,1\}, \\ u^{[4m+\varepsilon]} & 4m+\varepsilon \geq i-3 \\ z^{[4m+\varepsilon]}, & 4m+\varepsilon \geq i-2 \end{array}$$

where $i = 2j+\delta > 1,\ j > 0,\ \delta \in \{0,1\}$.

Solvability of $\mathcal{B}_n$

For $k > 2,\ m > 0,\ \varepsilon \in \{0, 1\}$ we have

$$
\begin{aligned}
\mathcal{B}_n^{(0)} &= \mathcal{B}_n \\
\mathcal{B}_n^{(1)} &= \mathcal{B}_n\mathcal{B}_n \\
&= vect\langle t, tx, \ldots, t^{n-1}x,\ \mid x^{[k]}, tx^{[k]}, \ldots, t^{n-3}x^{[k]}, \mid \\
&\quad x^{[k]}x, t(x^{[k]}x), \ldots, t^{n-3}(x^{[k]}x), \mid \\
&\quad u^{[4m+\varepsilon]}, u^{[4m+\varepsilon]}x, \ldots, t^{n-3}u^{[4m+\varepsilon]}, \mid z^{[4m+\varepsilon]}, z^{[4m+\varepsilon]}x, \ldots, t^{n-3}z^{[4m+\varepsilon]}\rangle \\
\mathcal{B}_n^{(2)} &= \mathcal{B}_n^{(1)}\mathcal{B}_n^{(1)} \\
&= vect\langle t^2, t^2x, \ldots, t^{n-1}x,\ \mid tx^{[k]}, t^2x^{[k]}, \ldots, t^{n-3}x^{[k]}, \mid \\
&\quad t(x^{[k]}x), t^2(x^{[k]}x), \ldots, t^{n-3}(x^{[k]}x),\ \mid \\
&\quad u^{[4m+\varepsilon]}, u^{[4m+\varepsilon]}x, \ldots, t^{n-3}u^{[4m+\varepsilon]},\ \mid z^{[4m+\varepsilon]}, z^{[4m+\varepsilon]}x, \ldots, t^{n-3}z^{[4m+\varepsilon]}\rangle \\
\mathcal{B}_n^{(j)} &= \mathcal{B}_n^{(j-1)} \cdot \mathcal{B}_n^{(j-1)} \\
&= vect\langle t^{2^{j-1}}, t^{2^{j-1}+1}x, \ldots, t^{n-1}x,\ \mid t^{2^{j-1}-1}x^{[k]}, \ldots, t^{n-3}x^{[k]}, \mid \\
&\quad t^{2^{j-1}-1}(x^{[k]}x), \ldots, t^{n-3}(x^{[k]}x), \mid \\
&\quad t^{2^{j-1}-2}u^{[4m+\varepsilon]}, t^{2^{j-1}-2}u^{[4m+\varepsilon]}x, \ldots, t^{n-3}u^{[4m+\varepsilon]}, \mid \\
&\quad t^{2^{j-1}-2}z^{[4m+\varepsilon]}, t^{2^{j-1}-2}z^{[4m+\varepsilon]}x, \ldots, t^{n-3}z^{[4m+\varepsilon]}\rangle \\
\mathcal{B}_n^{(j+1)} &= \mathcal{B}_n^{(j)} \cdot \mathcal{B}_n^{(j)} \\
&= vect\langle t^{2^j}, t^{2^j+1}x, \ldots, t^{n-1}x,\ \mid t^{2^j-1}x^{[k]}, \ldots, t^{n-3}x^{[k]}, \mid \\
&\quad t^{2^j-1}(x^{[k]}x), \ldots, t^{n-3}(x^{[k]}x),\ \mid \\
&\quad t^{2^j-2}u^{[4m+\varepsilon]}, t^{2^j-2}u^{[4m+\varepsilon]}x, \ldots, t^{n-3}u^{[4m+\varepsilon]}, \mid \\
&\quad t^{2^j-2}z^{[4m+\varepsilon]}, t^{2^j-2}z^{[4m+\varepsilon]}x, \ldots, t^{n-3}z^{[4m+\varepsilon]}\rangle
\end{aligned}
$$

Nilpotency of square of $\mathcal{B}_n$

For $k > 2,\ m > 0,\ \varepsilon \in \{0, 1\}$ we have

$$
\begin{aligned}
\left(\mathcal{B}_n{}^2\right)^2 &= \mathcal{B}_n\mathcal{B}_n = \mathcal{B}_n^{(1)} \\
&= vect\langle t, tx, t^2, t^2x, \ldots, t^{n-1}x,\ \mid x^{[k]}, x^{[k]}x, \ldots, t^{n-3}x^{[k]}, t^{n-3}(x^{[k]}x), \mid \\
&\qquad u^{[4m+\varepsilon]}, z^{[4m+\varepsilon]} \ldots, t^{n-3}u^{[4m+\varepsilon]}, t^{n-3}z^{[4m+\varepsilon]}\rangle, \\
\left(\mathcal{B}_n{}^2\right)^2 &= \mathcal{B}_n{}^2\mathcal{B}_n{}^2 = \mathcal{B}_n^{(2)} \\
&= vect\langle t^2, t^2x, \ldots, t^{n-1}x,\ \mid tx^{[k]}, t(x^{[k]}x), \ldots, t^{n-3}x^{[k]}, t^{n-3}(x^{[k]}x), \mid \\
&\qquad u^{[k]}, z^{[k]}, \ldots, t^{n-3}u^{[k]}, t^{n-3}z^{[k]}\rangle \\
\left(\mathcal{B}_n{}^2\right)^3 &= \left(\mathcal{B}_n{}^2\right)^2\mathcal{B}_n{}^2 + \mathcal{B}_n{}^2\left(\mathcal{B}_n{}^2\right)^2 \\
&= vect\langle t^3, t^3x, \ldots, t^{n-1}x,\ \mid t^2x^{[k]}, t^2(x^{[k]}x), \ldots, t^{n-3}x^{[k]}, t^{n-3}(x^{[k]}x), \mid \\
&\qquad tu^{[k]}, tz^{[k]}, \ldots, t^{n-3}u^{[k]}, t^{n-3}z^{[k]}\rangle \\
\left(\mathcal{B}_n{}^2\right)^i &= \left(\mathcal{B}_n{}^2\right)^{i-1}\mathcal{B}_n{}^2 + \cdots + \mathcal{B}_n{}^2\left(\mathcal{B}_n{}^2\right)^{i-1} \\
&= vect\langle t^i, t^ix, \ldots, t^{n-1}x,\ \mid t^{i-1}(x^{[k]}x^\sigma), \ldots, t^{n-3}(x^{[k]}x), \mid \\
&\qquad t^{i-2}u^{[4m+\varepsilon]}x^\sigma, t^{i-2}z^{[4m+\varepsilon]}x^\sigma, \ldots, t^{n-3}u^{[4m+\varepsilon]}, t^{n-3}z^{[4m+\varepsilon]}\rangle.
\end{aligned}
$$

Nilpotency of $\mathcal{B}_n$

$$
\begin{array}{rcl}
\big(\mathcal{B}_n\big)^1 & = & \mathcal{B}_n \\
\big(\mathcal{B}_n\big)^2 & = & \mathcal{B}_n\mathcal{B}_n \\
\big(\mathcal{B}_n\big)^3 & = & \big(\mathcal{B}_n\big)^2\mathcal{B}_n + \mathcal{B}_n\big(\mathcal{B}_n\big)^2 \\
 & = & vect\langle tx, t^{m+2}x^{\varepsilon}, \ldots, t^{n-1}x, \\
 & & \qquad x^{[3]}, x^{[3]}x, t^m(x^{[k]}x^{\varepsilon}), t^m u^{[4r+\varepsilon]}x^{\sigma}, t^m z^{[4r+\varepsilon]}x^{\sigma} \\
 & & \qquad \mid k > 2,\ r > 0,\ 0 \le m+\sigma \le n-3,\ \sigma,\varepsilon \in \{0,1\}\rangle, \\
\big(\mathcal{B}_n\big)^4 & = & vect\langle t^2, t^2x, \ldots, t^{n-1}x, \\
 & & \qquad x^{[3]}x, x^{[4]}, t^m(x^{[k]}x^{\varepsilon}), t^m u^{[4r+\varepsilon]}x^{\sigma}, t^m z^{[4r+\varepsilon]}x^{\sigma} \\
 & & \qquad \mid m+k+\varepsilon > 3,\ r > 0,\ 0 \le m+\sigma \le n-3,\ \sigma,\varepsilon \in \{0,1\}\rangle, \\
\big(\mathcal{B}_n\big)^5 & = & vect\langle t^2x, \ldots, t^{n-1}x, \\
 & & \qquad x^{[4]}x, x^{[5]}, tx^{[3]}, t^m(x^{[k]}x^{\varepsilon}), \\
 & & \qquad t^m u^{[4r+\varepsilon]}x^{\sigma}, t^m z^{[4r+\varepsilon]}x^{\sigma} \\
 & & \qquad \mid m+k+\varepsilon > 4,\ r > 0,\ 0 \le m+\sigma \le n-3,\ \sigma,\varepsilon \in \{0,1\}\rangle, \\
\big(\mathcal{B}_n\big)^6 & = & vect\langle t^3, \ldots, t^{n-1}x, \\
 & & \qquad x^{[5]}x, x^{[6]}, t(x^{[3]}x), tx^{[4]}, t^m(x^{[k]}x^{\varepsilon}), \\
 & & \qquad t^m u^{[4r+\varepsilon]}x^{\sigma}, t^m z^{[4r+\varepsilon]}x^{\sigma} \\
 & & \qquad \mid m+k+\varepsilon > 5,\ r > 0,\ 0 \le m+\sigma \le n-3,\ \sigma,\varepsilon \in \{0,1\}\rangle, \\
\big(\mathcal{B}_n\big)^7 & = & vect\langle t^3x, \ldots, t^{n-1}x, \\
 & & \qquad x^{[6]}x, x^{[7]}, t(x^{[4]}x), tx^{[5]}, \\
 & & \qquad t^2x^{[3]}, t^2(x^{[3]}x), t^2x^{[4]}, t^3x^{[3]}, t^m(x^{[k]}x^{\varepsilon}), \\
 & & \qquad t^m u^{[4r+\varepsilon]}x^{\sigma}, \\
 & & \qquad z^{[4]}x, t^m z^{[4r+1+\varepsilon]}x^{\sigma} \\
 & & \qquad \mid m+k+\varepsilon > 6,\ r > 0,\ 0 \le m+\sigma \le n-3,\ \sigma,\varepsilon \in \{0,1\}\rangle, \\
\big(\mathcal{B}_n\big)^8 & = & vect\langle t^4, \ldots, t^{n-1}x, \\
 & & \qquad x^{[7]}x, x^{[8]}, t(x^{[5]}x), tx^{[6]}, \\
 & & \qquad t^2x^{[4]}, t^2(x^{[4]}x), t^2x^{[5]}, \\
 & & \qquad t^3x^{[3]}, t^3(x^{[3]}x), t^3x^{[4]} \\
 & & \qquad t^4x^{[3]}, t^m(x^{[k]}x^{\varepsilon}), \\
 & & \qquad u^{[4]}x, u^{[5]}, t^{m+1}u^{[4+\varepsilon]}x^{\sigma}, t^m u^{[4r+\varepsilon]}x^{\sigma}, \\
 & & \qquad z^{[5]}, z^{[5]}x, t^{m+1}z^{[4+\varepsilon]}x^{\sigma}, t^m z^{[4r+\varepsilon]}x^{\sigma} \\
 & & \qquad \mid m+k+\varepsilon > 7,\ r > 1,\ 0 \le m \le n-3,\ \sigma,\varepsilon \in \{0,1\}\rangle.
\end{array}
$$

Observe, that in each step i is the degree of the elements in $(\mathcal{B}_n)^i$ at least i (It arises that in each step $x \cdot u$ has degree $deg(u)+1$, where u is the minimal degree element in $(\mathcal{B}_n)^{i-1}$). So that for $i = 2j+\delta > 1,\ j > 0,\ \delta \in \{0,1\}$ we have

$$\begin{aligned}
(\mathcal{B}_n)^i &= (\mathcal{B}_n)^{i-1}\mathcal{B}_n + \cdots + \mathcal{B}_n(\mathcal{B}_n)^{i-1} \\
(\mathcal{B}_n)^i &\overset{i=2j+\delta}{=} vect\langle t^j x^\delta, t^{j+1}x^\varepsilon, \ldots, t^{n-1}x, \\
&\qquad x^{[i]}, x^{[i-1]}x, t^m(x^{[k]}x^\sigma), \ldots, t^{n-3}(x^{[k]}x) \\
&\qquad u^{[i-3]}, u^{[i-4]}x, t^m u^{[4r+1+\varepsilon]}x^\sigma, \ldots, t^{n-3}u^{[4r+1+\varepsilon]} \\
&\qquad z^{[i-3]}, z^{[i-3]}x, t^m z^{[4r+\varepsilon]}x^\sigma, \ldots, t^{n-3}z^{[4r+\varepsilon]} \\
&\qquad \mid i \le 2m+k+\sigma,\ i-2 \le 2m+(4r+\varepsilon)+\sigma,\ r>0, \\
&\qquad\ \ 0 \le m < n-2,\ \sigma,\varepsilon \in \{0,1\}\rangle.
\end{aligned}$$

More exactly $(\mathcal{B}_n)^i$ is spanned by the elements of the form

$$\begin{array}{rl}
t^m x^\sigma, & i \le 2m+\sigma,\ 0 \le m < n,\ \sigma \in \{0,1\}, \\
t^m(x^{[k]}x^\sigma), & i \le 2m+k+\sigma,\ 0 \le m < n-2,\ k > 2,\ \sigma \in \{0,1\} \\
t^m u^{[4r+\varepsilon]}x^\sigma, & i-3 \le 2m+(4r+\varepsilon)+\sigma,\ 0 \le m < n-2,\ r > 0,\ \sigma,\varepsilon \in \{0,1\} \\
t^m z^{[4r+\varepsilon]}x^\sigma, & i-2 \le 2m+(4r+\varepsilon)+\sigma,\ 0 \le m < n-2,\ r > 0,\ \sigma,\varepsilon \in \{0,1\}.
\end{array}$$

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