Numerical solution of the incompressible flow using a domain decomposition method

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Aims of the work

- The first aim is to develop a numerical method for computational fluid dynamics employing the extension of the multilevel BDDC method towards the nonsymmetric systems arising from the discretization of the Navier-Stokes equations.
- The second aim is to perform missing detailed 3D simulations of the industrial problem of a flow of oil inside the whole moving hydrostatic bearings.

Incompressible stationary Navier-Stokes equations

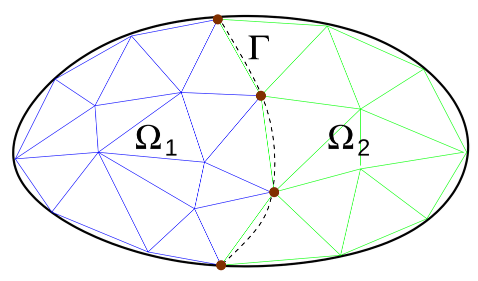
$$\begin{aligned} (\mathrm{u}\cdot\nabla)\mathrm{u} - \nu\Delta\mathrm{u} + \nabla p &= \mathrm{f} && \text{in } \Omega \\ \nabla\cdot\mathrm{u} &= 0 && \text{in } \Omega \\ \mathrm{u} &= \mathrm{g} && \text{on } \Gamma_D \\ -\nu(\nabla\mathrm{u})\mathrm{n} + p\mathrm{n} &= 0 && \text{on } \Gamma_N \end{aligned}$$

Finite element method

- Taylor-Hood Q_2-Q_1 finite elements
- Picard iteration as linearization

$$\begin{bmatrix} \nu A + N(u^k) & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u^{k+1} \\ p^{k+1} \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

Iterative substructuring



Non-overlapping domain decomposition

$$\begin{bmatrix} \nu A_{11} + N_{11} & \nu A_{12} + N_{12} & B_{11}^T & B_{21}^T \\ \nu A_{21} + N_{21} & \nu A_{22} + N_{22} & B_{12}^T & B_{22}^T \\ B_{11} & B_{12} & 0 & 0 \\ B_{21} & B_{22} & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ g_1 \\ g_2 \end{bmatrix}$$

Subscript 1 - interior nodes, subscript 2 - interface nodes

Interface problem

$$S \begin{bmatrix} u_2 \\ p_2 \end{bmatrix} = g \qquad (1)$$

$$S = \begin{bmatrix} \nu A_{22} + N_{22} & B_{22}^T \\ B_{22} & 0 \end{bmatrix} - \begin{bmatrix} \nu A_{21} + N_{21} & B_{12}^T \\ B_{21} & 0 \end{bmatrix} \begin{bmatrix} \nu A_{11} + N_{11} & B_{11}^T \\ B_{11} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \nu A_{12} + N_{12} & B_{21}^T \\ B_{12} & 0 \end{bmatrix}$$

$$g = \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} - \begin{bmatrix} \nu A_{21} + N_{21} & B_{12}^T \\ B_{21} & 0 \end{bmatrix} \begin{bmatrix} \nu A_{11} + N_{11} & B_{11}^T \\ B_{11} & 0 \end{bmatrix}^{-1} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}$$

- problem (1) solved by BiCGstab with the BDDC preconditioner

Multilevel BDDC for nonsymmetric systems

- preconditioner for (1): $M_{BDDC} : r^k \rightarrow u^k$

  residual obtained in the k^{th} iteration: $r^k = g - S \begin{bmatrix} u_2{}^k \\ p_2{}^k \end{bmatrix}$
- preconditioner setup - computing coarse basis functions Ψ_i and Ψ_i^*

$$\begin{bmatrix} S_i & C_i^T \\ C_i & 0 \end{bmatrix} \begin{bmatrix} \Psi_i \\ \Lambda_i \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} \qquad \begin{bmatrix} S_i^T & C_i^T \\ C_i & 0 \end{bmatrix} \begin{bmatrix} \Psi_i^* \\ \Lambda_i \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

- preconditioner action

$$r_i^k = W_i R_i r^k$$

coarse problem

$$S_C = \sum_{i=1}^{N} R_{Ci}^T S_{Ci} R_{Ci}$$

$$r_C^k = \sum_{i=1}^{N} R_{Ci}^T \Psi_i^{*T} r_i^k$$

$S_C u_C = r_C^k$

- 2 levels - direct solver
- **3+ levels - repeat BDDC**

$$u_{Ci} = \Psi_i R_{Ci} u_C$$

subdomain problems

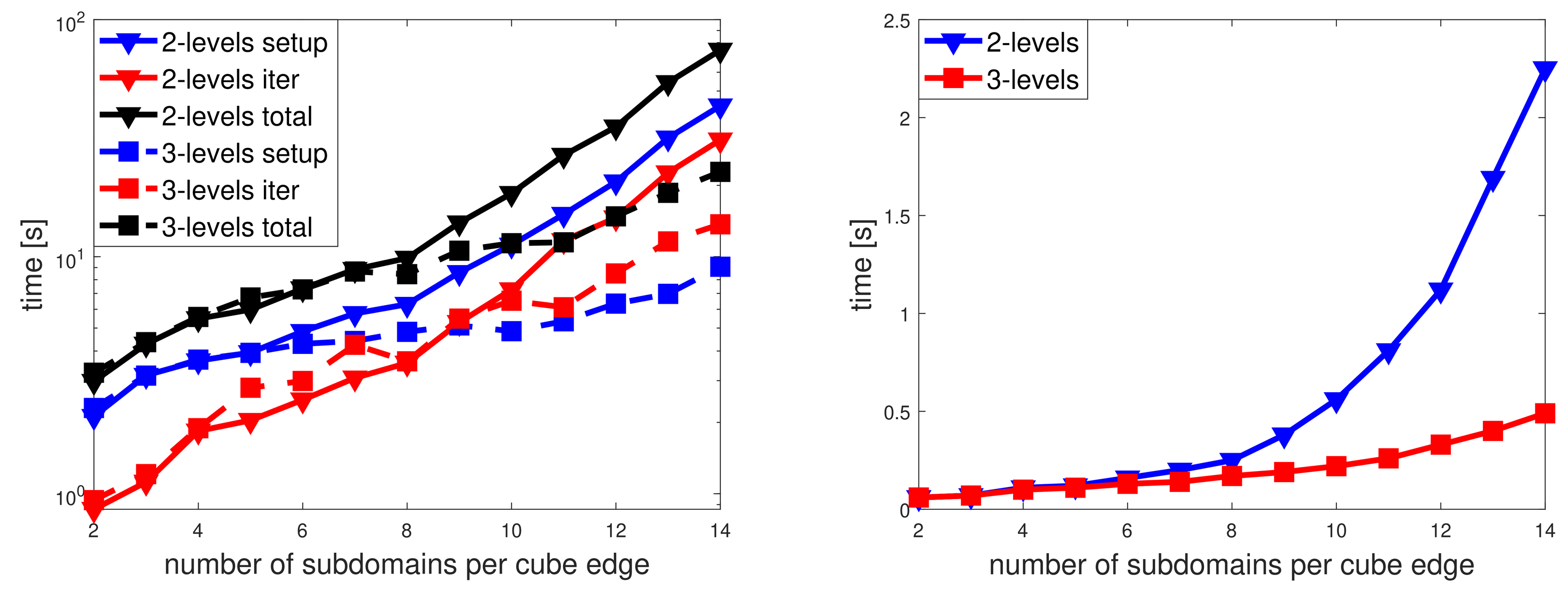
$$\begin{bmatrix} S_i & C_i^T \\ C_i & 0 \end{bmatrix} \begin{bmatrix} u_i \\ \lambda \end{bmatrix} = \begin{bmatrix} r_i^k \\ 0 \end{bmatrix}$$

$$u^k = \sum_{i=1}^{N} R_i^T W_i (u_i + u_{Ci})$$

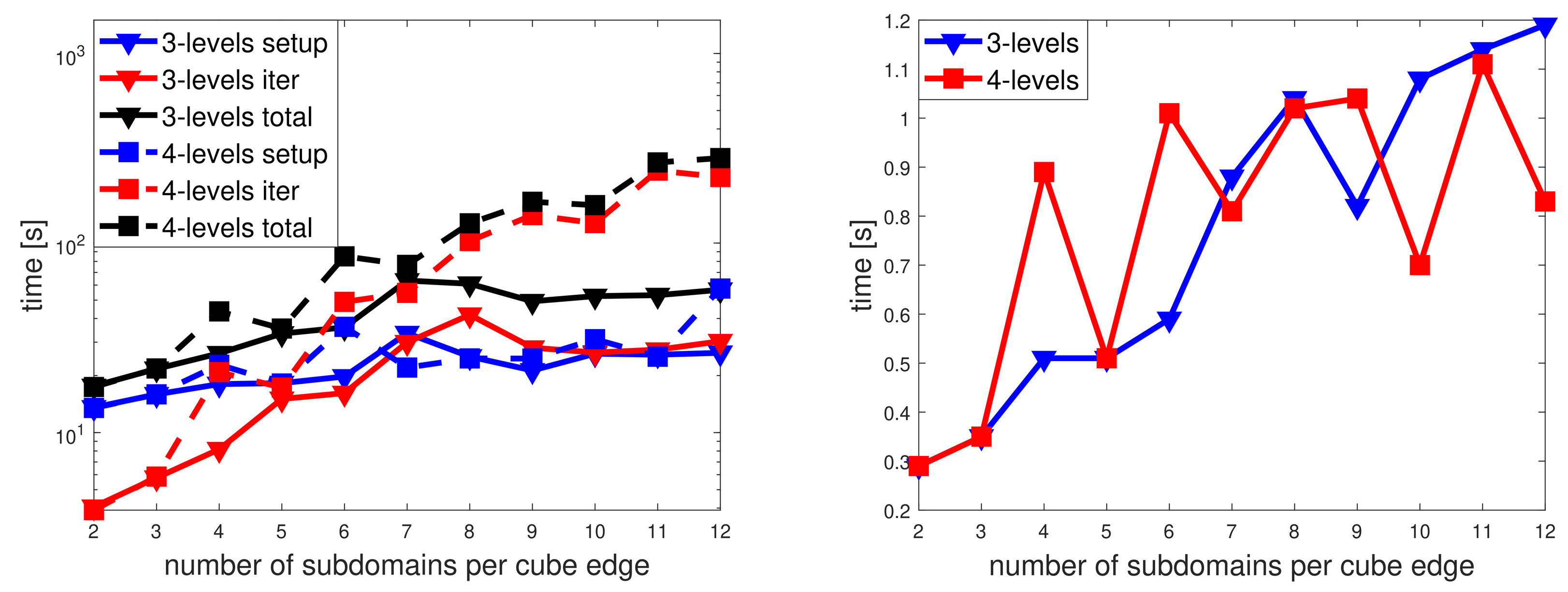
Numerical results

- 3D lid-driven cavity
- flow inside the hydrostatic bearing
- computed on Salomon@IT4Inovations

3-D lid-driven cavity

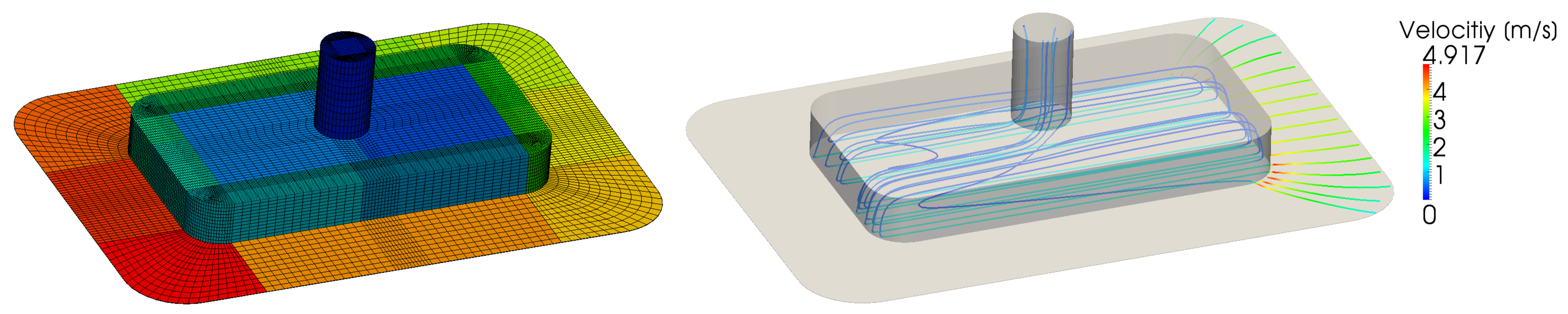


Comparision of 2- and 3-level method. Mean time for setup, mean time for the BiCGstab iterations and mean total time (left), mean time for one iteration (right).



Comparision of 3- and 4-level method. Mean time for setup, mean time for the BiCGstab iterations and mean total time (left), mean time for one iteration (right).

Hydrostatic bearing



Decomposed computational mesh (left) and streamtraces with coloring by the magnitude of velocity (right)

Conclusion

- multilevel extension of BDDC for nonsymmetric systems
- 3-level faster than 2- and 4-level method
- detailed picture of flow of oil inside moving hydrostatic bearing

References

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