Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Generalized geometry and Palatini formalism

Zobecněná geometrie a Palatiniho formalismus

Master's Thesis

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Abstrakt: Strunovou efektivní akci a jí příslušející pohybové rovnice lze přeformulovat pomocí konexí na Courantově algebroidu. Cílem práce je pro tuto přeformulovanou akci vytvořit obdobu Palatiniho formalismu z obecné teorie relativity. Nejprve jsou detailně zavedeny veškeré koncepty nutné pro pochopení obecné teorie konexí na Courantových algebroidech. Je představen konkrétní Courantův algebroid asociovaný se zobecněnou geometrií. Následně jsou konexe na tomto Courantově algebroidu využity pro reformulaci strunové efektivní akce. Na závěr je vybudován Palatiniho přístup ke Courant-Einstein-Hilbertově akci, zobecnění strunové efektivní akce na úroveň obecného Courantova algebroidu, jehož výsledkem jsou pohybové rovnice svazující dohromady zobecněnou metriku a konexi na Courantově algebroidu.

Klíčová slova: Courant-Einstein-Hilbertova akce, Levi-Civitovy konexe na Courantových algebroidech, Palatiniho formalismus, strunová efektivní akce, zobecněná geometrie

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Abstract: A string effective action and the corresponding equations of motion can be reformulated in the language of Courant algebroid connections. The aim of the thesis is to develop an analogue of the Palatini formalism from general relativity for the reformulated string effective action. First, the necessary and detailed introduction into the general theory of Courant algebroid connections is presented. The Courant algebroid associated with the generalized geometry is introduced, Courant algebroid connections on it are then used for the string effective action reformulation. Finally, the Palatini approach to Courant-Einstein-Hilbert action, a generalization of the string effective action to the level of general Courant algebroids, is devised. It results into equations of motion binding together a generalized metric and a Courant algebroid connection.

Keywords: Courant-Einstein-Hilbert action, generalized geometry, Levi-Civita Courant algebroid connections, Palatini formalism, string effective action

Contents

Introduction

In 1925, Albert Einstein devised a so called Palatini approach to the Einstein-Hilbert action, the confusing nomenclature is explained in [1]. It is a variational formulation of the general theory of relativity that, among other things, justifies the choice of the Levi-Civita affine connection for a mathematical formulation of the physical theory. Ninety-seven years later, we are standing in front of somewhat similar problem. Authors in [2] suggested a reformulation of the string effective action in terms of Courant algebroid connections, they are using Levi-Civita Courant algebroid connections, however, they lack a robust argument for it. In this thesis, we will build on their work and aim to invent an analogue of the Palatini approach for the reformulated string effective action to justify their more or less artificial-looking approach.

Chapter 1 is concerned with the general theory of Courant algebroid connections. We will start by giving the very definition of a Courant algebroid itself and show some of its basic properties. Using the inherent structure of Courant algebroids, we will generalize ordinary vector bundle connections to so called Courant algebroid connections and then we will use them for the construction of Courant algebroid versions of the torsion and the Riemann tensor. Besides the Courant algebroid connection, yet another additional structure on Courant algebroids will be introduced, namely a generalized metric. In the last section of this chapter, we will combine both of these additional structures into the concept crucial for this thesis, the concept of Levi-Civita Courant algebroid connections.

Chapter 2 deals with the specific example of a Courant algebroid, namely with the generalized tangent bundle, which is the Whitney sum of the tangent and the cotangent bundle, endowed with an appropriate structure. This particular Courant algerboid is closely related to the mathematical field called generalized geometry. At the level of this special Courant algebroid, we will successively and in detail examine all concepts introduced in the previous chapter.

The two chapters containing purely mathematical preliminaries will be followed by two chapters that bring some physics into play. In Chapter 3, we will start by stating the string effective action in an ordinary form. Then, we will show how it together with the corresponding equations of motion can be reformulated in terms of Courant algebroid connections on the specific Courant algebroid from Chapter 2. Chapter 4 will summarize the original Palatini approach to the Einstein-Hilbert action from the general relativity in an elegant geometrical fashion.

The final Chapter 5 is a pinnacle of the whole thesis. We will devise not only the Palatini approach to the string effective action associated with the Courant algebroid from Chapter 2, what was the original intention, but we will go even further. In particular, we will invent the Palatini approach to some kind of generalization of the reformulated string effective action to the level of a general Courant algebroid.

Chapter 1

Courant algebroids

Generalized geometry, the first term contained in the title of this thesis, is a modification of the standard differential geometry of smooth manifolds. The modification consists of two steps. Firstly, the tangent bundle is replaced by the Whitney sum of the tangent and the cotangent bundle, a so called generalized tangent bundle. Secondly, the standard Lie bracket of vector fields is replaced by the H-twisted Dorfman bracket of generalized tangent bundle sections. The generalized tangent bundle equipped with the aforementioned bracket is a specific example of a vector bundle, which admits the structure of Courant algebroid [3, 4, 5]. It is advantageous and more insightful to introduce general theory of Courant algebroids and then apply its results to this for us important example. Most of the ideas contained in this chapter have been adopted from [2, 6].

1.1 Basic concepts

Definition 1.1. Suppose $E \xrightarrow{\pi} M$ is a vector bundle. A symmetric $C^\infty(M)$-bilinear map $h : \Gamma(E) \times \Gamma(E) \to C^\infty(M)$ is called a **fiber-wise metric** on E if and only if the induced map $\psi \mapsto h(\psi, .)$ is a $C^\infty(M)$-module isomorphism of $\Gamma(E)$ and $\Gamma(E^*)$.

Notation 1.2. A fiber-wise metric is apparently an extension of the notion of a (semi-)Riemannian metric to an arbitrary vector bundle. For this reason, we will adopt the notation commonly used in (semi-)Riemannian geometry. In particular, the $C^\infty(M)$-module isomorphism $\psi \mapsto h(\psi, .)$ will be denoted as $\flat_h$, its inverse as $\sharp_h$, and we also define an **inverse fiber-wise metric** $h^{-1} : \Gamma(E^*) \times \Gamma(E^*) \to C^\infty(M)$ as

$$h^{-1}(\mathcal{A}, \mathcal{B}) := h(\sharp_h \mathcal{A}, \sharp_h \mathcal{B}), \tag{1.1}$$

for all $\mathcal{A}$, $\mathcal{B} \in \Gamma(E^*)$.

Remark 1.3. Consider two vector bundles $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M$. Then each $C^\infty(M)$-linear map $\Phi : \Gamma(E) \to \Gamma(E')$ uniquely determines for all $p \in M$ an $\mathbb{R}$-linear map $\Phi_p : E_p \to E'_p$ such that the equality

$$(\Phi(\psi))(p) = \Phi_p(\psi(p)) \tag{1.2}$$

is satisfied for all $\psi \in \Gamma(E)$, for a proof see [7, Proposition 7.25]. Since the set $C^\infty(M)$ can be considered as the set of smooth sections of the trivial vector bundle $M \times \mathbb{R} \xrightarrow{\pi_1} M$, every fiber-wise metric h induces for all $p \in M$ a symmetric $\mathbb{R}$-bilinear form $h_p : E_p \times E_p \to \mathbb{R}$.

Moreover, h_p is non-degenerate, this easily follows from the fact that $\flat_h$ is an isomorphism. One could be interested if there is some relation between the signatures of h_p in two distinct points. In fact, an important claim holds; if the base manifold M is connected, the signature of h_p is constant for all $p \in M$.[1] Consequently, it is possible to unambiguously define the signature and the definiteness of an arbitrary fiber-wise metric on each connected component of the base manifold.

Definition 1.4. Let $E \overset{\pi}{\to} M$ be a vector bundle, $\rho : \Gamma(E) \to \Gamma(TM)$ a $C^\infty(M)$-linear map called the **anchor**, $[.\,,.]_E : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ an $\mathbb{R}$-bilinear map called the **Courant bracket** and g_E a fiber-wise metric on E called the **Courant metric**. Then the 4-tuple $(E \overset{\pi}{\to} M, \rho, [.\,,.]_E, g_E)$ is called a **Courant algebroid** if and only if it meets the following requirements:

(I) For all ψ_1, $\psi_2 \in \Gamma(E)$ and all $f \in C^\infty(M)$ there holds

$$[\psi_1, f\psi_2]_E = (\rho(\psi_1)f)\psi_2 + f[\psi_1, \psi_2]_E. \tag{1.3}$$

(II) For all ψ_1, ψ_2, $\psi_3 \in \Gamma(E)$ there holds

$$[\psi_1, [\psi_2, \psi_3]_E]_E = [[\psi_1, \psi_2]_E, \psi_3]_E + [\psi_2, [\psi_1, \psi_3]_E]_E. \tag{1.4}$$

(III) For all ψ_1, ψ_2, $\psi_3 \in \Gamma(E)$ there holds

$$\rho(\psi_1)g_E(\psi_2, \psi_3) = g_E([\psi_1, \psi_2]_E, \psi_3) + g_E(\psi_2, [\psi_1, \psi_3]_E). \tag{1.5}$$

(IV) For all ψ_1, $\psi_2 \in \Gamma(E)$ there holds

$$g_E([\psi_1, \psi_1]_E, \psi_2) = \frac{1}{2}\rho(\psi_2)g_E(\psi_1, \psi_1). \tag{1.6}$$

Remark 1.5. The structure of a Courant algebroid fits in a more general concept of a Leibniz algebroid, the 3-tuple $(E \overset{\pi}{\to} M, \rho, [.\,,.]_E)$, which satisfies the first two axioms of the definition above.

The first two characteristic properties of a Courant algebroid are analogous to the properties possessed by the Lie bracket of vector fields, however, they are not exactly the same. Since we are not capable of acting on a smooth function by a smooth section of a general vector bundle, we have to employ the anchor in (1.3). The second property looks very similar to the Jacobi identity and for the skew-symmetric Courant bracket it is actually equivalent to it. However, in general the skew-symmetry is not guaranteed, and moreover, it is usually not possessed. The absence of skew-symmetry leads to another complication, namely that we do not immediately know what is the behaviour of the Courant bracket with respect to the multiplication by a smooth function in the first (left) input. As we will show, it is not such a big deal, because the rule can be determined by using the first and the fourth axiom of a Courant algebroid. The third axiom relates all three additional structures together within the identity, which reminds us of the vanishing of the Lie derivative of metric from (semi-)Riemannian geometry. The fourth requirement allows us to express the symmetric part of the Courant bracket in terms of the anchor and the Courant metric.

[1] For a detailed discussion see [8].

Example 1.6 (Courant algebroids over a point)**.** For the case of 0-dimensional base manifolds, vector bundles simply become vector spaces, and in particular the tangent bundles over such manifolds are the trivial ones, so the anchor is necessarily the zero map. Therefore, an arbitrary Courant algerborid over the point-like manifold can be considered as a vector space $\mathfrak{g}$ endowed with an $\mathbb{R}$-bilinear map $[.\,,.]_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and a metric tensor $\langle .\,,. \rangle_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. The first axiom of a Courant algebroid is redundant, since it reduces to the requirement of $\mathbb{R}$-linearity of $[.\,,.]_{\mathfrak{g}}$ in the second argument, which is already ensured. It follows from the non-degeneracy of $\langle .\,,. \rangle_{\mathfrak{g}}$ and from the triviality of the anchor that the fourth axiom is equivalent to the fact that $[.\,,.]_{\mathfrak{g}}$ is skew-symmetric. The second axiom then becomes the Jacobi identity, hence $(\mathfrak{g}, [.\,,.]_{\mathfrak{g}})$ is a Lie algebra. The only remaining axiom is the third one, which takes the form

$$\langle [v_1, v_2]_{\mathfrak{g}}, v_3 \rangle_{\mathfrak{g}} + \langle v_2, [v_1, v_3]_{\mathfrak{g}} \rangle_{\mathfrak{g}} = 0, \tag{1.7}$$

for all $v_1,\ v_2,\ v_3 \in \mathfrak{g}$. However, the Lie algebra $(\mathfrak{g}, [.\,,.]_{\mathfrak{g}})$ equipped with a metric tensor $\langle .\,,. \rangle_{\mathfrak{g}}$ such that the relation above holds is precisely what is called a *quadratic Lie algebra*. In order to summarize it, Courant algebroids over a point are exactly the same thing as quadratic Lie algebras.

Example 1.7 (Tangent bundle as Courant algebroid)**.** As we have already suggested in of the paragraph above, the 3-tuple $(TM \overset{\pi}{\to} M, \mathrm{Id}_{\Gamma(TM)}, [.\,,.])$, where M is an arbitrary manifold and $[.\,,.]$ is the standard Lie bracket of vector fields, forms a Leibniz algebroid. Consider that we have also a (semi-)Riemannian metric g at our disposal. It is now natural to ask if the 4-tuple $(TM \overset{\pi}{\to} M, \mathrm{Id}_{\Gamma(TM)}, [.\,,.], g)$ satisfies also the third and the fourth Courant algebroid axiom. These can be for this particular case equivalently stated as $\pounds_X g = 0$ and $g(X, X) =$ *const.*, for all $X \in \Gamma(TM)$, respectively. Apparently, both of them are not generally true, thus $(TM \overset{\pi}{\to} M, \mathrm{Id}_{\Gamma(TM)}, [.\,,.], g)$ does not form a Courant algebroid in general.

Let us now show some basic properties of Courant algebroids.

Proposition 1.8. *Let $(E \overset{\pi}{\to} M, \rho, [.\,,.]_E, g_E)$ be a Courant algebroid. Then*

$$\rho([\psi_1, \psi_2]_E) = [\rho(\psi_1), \rho(\psi_2)] \tag{1.8}$$

holds for all $\psi_1,\ \psi_2 \in \Gamma(E)$.

Proof. Consider arbitrary sections $\psi_1,\ \psi_2,\ \psi_3 \in \Gamma(E)$ and arbitrary $f \in C^\infty(M)$. Then the second Courant algebroid axiom (1.4) says:

$$[\psi_1, [\psi_2, f\psi_3]_E]_E = [[\psi_1, \psi_2]_E, f\psi_3]_E + [\psi_2, [\psi_1, f\psi_3]_E]_E. \tag{1.9}$$

By using the first axiom (1.3) repeatedly, we obtain:

$$\begin{aligned}
&[\psi_1, [\psi_2, f\psi_3]_E]_E \\
&\qquad = [\psi_1, (\rho(\psi_2)f)\psi_3 + f[\psi_2, \psi_3]_E]_E \\
&\qquad = (\rho(\psi_1)\rho(\psi_2)f)\psi_3 + (\rho(\psi_2)f)[\psi_1, \psi_3]_E + (\rho(\psi_1)f)[\psi_2, \psi_3]_E + f[\psi_1, [\psi_2, \psi_3]_E]_E
\end{aligned} \tag{1.10}$$

for the left-hand side of (1.9), and

$$\begin{aligned}
&[[\psi_1, \psi_2]_E, f\psi_3]_E + [\psi_2, [\psi_1, f\psi_3]_E]_E \\
&\qquad = (\rho([\psi_1, \psi_2]_E)f)\psi_3 + f[[\psi_1, \psi_2]_E, \psi_3]_E + [\psi_2, (\rho(\psi_1)f)\psi_3 + f[\psi_1, \psi_3]_E]_E \\
&\qquad = (\rho([\psi_1, \psi_2]_E)f)\psi_3 + f[[\psi_1, \psi_2]_E, \psi_3]_E + (\rho(\psi_2)\rho(\psi_1)f)\psi_3 + (\rho(\psi_1)f)[\psi_2, \psi_3]_E \\
&\qquad\quad + (\rho(\psi_2)f)[\psi_1, \psi_3]_E + f[\psi_2, [\psi_1, \psi_3]_E]_E
\end{aligned} \tag{1.11}$$

for the right-hand side. Putting equations (1.10) and (1.11) together and using the axiom (1.4) results into

$$\begin{aligned} & (\rho([\psi_1,\psi_2]_E)f)\psi_3 = (\rho(\psi_1)\rho(\psi_2)f)\psi_3 - (\rho(\psi_2)\rho(\psi_1)f)\psi_3 = ([\rho(\psi_1),\rho(\psi_2)]f)\psi_3, \\ \Leftrightarrow \quad & \rho([\psi_1,\psi_2]_E) = [\rho(\psi_1),\rho(\psi_2)]. \end{aligned}$$

□

Proposition 1.9. *Consider a vector bundle $E \xrightarrow{\pi} M$ equipped with a $C^\infty(M)$-linear map $\rho:\ \Gamma(E)\to\Gamma(TM)$, an $\mathbb{R}$-bilinear map $[.,.]_E:\Gamma(E)\times\Gamma(E)\to\Gamma(E)$ and a fiber-wise metric g_E. Then the fourth Courant algebroid axiom (1.6) can be equivalently stated as*

$$[\psi_1,\psi_2]_E + [\psi_2,\psi_1]_E = \mathcal{D}\, g_E(\psi_1,\psi_2), \tag{1.12}$$

for all ψ_1, $\psi_2\in\Gamma(E)$, where $\mathcal{D}: C^\infty(M)\to\Gamma(E)$ is an $\mathbb{R}$-linear map defined uniquely by the formula

$$g_E(\mathcal{D} f,\psi) = \rho(\psi)f, \tag{1.13}$$

for all $\psi\in\Gamma(E)$ and $f\in C^\infty(M)$.

Proof. Firstly, assume that (1.6) holds. For arbitrary ψ_1, $\psi_2\in\Gamma(E)$, one easily sees that

$$g_E(2[\psi_1,\psi_1]_E,\psi_2) \underset{(1.6)}{=} \rho(\psi_2)g_E(\psi_1,\psi_1) \underset{(1.13)}{=} g_E(\mathcal{D}\, g_E(\psi_1,\psi_1),\psi_2). \tag{1.14}$$

It follows from the fact that g_E is a fiber-wise metric that for all $\psi\in\Gamma(E)$ there holds

$$2[\psi,\psi]_E = \mathcal{D}\, g_E(\psi,\psi). \tag{1.15}$$

Choosing $\psi=\psi_1+\psi_2$ results in

$$\begin{aligned} 2[\psi_1+\psi_2,\psi_1+\psi_2]_E &= \mathcal{D}\, g_E(\psi_1+\psi_2,\psi_1+\psi_2) \\ &= \mathcal{D}\, g_E(\psi_1,\psi_1) + \mathcal{D}\, g_E(\psi_2,\psi_2) + 2\,\mathcal{D}\, g_E(\psi_1,\psi_2), \end{aligned} \tag{1.16}$$

while on the other hand an $\mathbb{R}$-bilinearity of $[.,.]_E$ implies that

$$2[\psi_1+\psi_2,\psi_1+\psi_2]_E = 2[\psi_1,\psi_1]_E + 2[\psi_2,\psi_2]_E + 2([\psi_1,\psi_2]_E + [\psi_2,\psi_1]_E). \tag{1.17}$$

By combining these two and using (1.15) twice, we obtain exactly (1.12). Conversely, assume that (1.12) holds then especially for all $\psi_1\in\Gamma(E)$, we have

$$[\psi_1,\psi_1]_E = \frac{1}{2}\mathcal{D}\, g_E(\psi_1,\psi_1). \tag{1.18}$$

Acting on the equation above by $g_E(.,\psi_2)\in\Gamma(E^*)$ for all $\psi_2\in\Gamma(E)$ gives

$$g_E([\psi_1,\psi_1]_E,\psi_2) = \frac{1}{2}g_E(\mathcal{D}\, g_E(\psi_1,\psi_1),\psi_2). \tag{1.19}$$

Finally, using the definition of map $\mathcal{D}$ leads precisely to (1.6). □

Let us now recall a notation from the abstract algebra, whose usefulness will be immediately illustrated in the consecutive proposition.

Notation 1.10. Suppose A and B are modules over a ring R, and $\rho : A \to B$ is an R-linear map. We define a map $\rho^T : B^* \to A^*$ as

$$(\rho^T(\beta))(a) := \beta(\rho(a)), \tag{1.20}$$

for all $\beta \in B^*$ and $a \in A$. Note that ρ^T is apparently R-linear.

Proposition 1.11. *Suppose $(E \overset{\pi}{\to} M, \rho, [.,.]_E, g_E)$ is a Courant algebroid. The $\mathbb{R}$-linear map $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ introduced in the previous proposition can be equivalently defined as*[2]

$$\mathcal{D} = \sharp_E \circ \rho^T \circ \mathrm{d}\,. \tag{1.21}$$

Proof. The proof is just straightforward handling with the particular objects.

$$\begin{aligned}
\mathcal{D} = \sharp_E \circ \rho^T \circ \mathrm{d} \quad &\Leftrightarrow \quad \forall f \in C^\infty(M) & \mathcal{D} f &= (\sharp_E \circ \rho^T \circ \mathrm{d}) f \\
&\Leftrightarrow \quad \forall f \in C^\infty(M) & \flat_E(\mathcal{D} f) &= \rho^T(\mathrm{d} f) \\
&\Leftrightarrow \quad \forall f \in C^\infty(M),\ \forall \psi \in \Gamma(E) & g_E(\mathcal{D} f, \psi) &= (\rho^T(\mathrm{d} f))(\psi) \\
& & &= \mathrm{d} f(\rho(\psi)) \\
& & &= \rho(\psi) f.
\end{aligned}$$

□

While the axiom (1.3) determines a behaviour of the Courant bracket with respect to the multiplication by a smooth function in the second (right) argument, the analogous behaviour for the left input is so far unknown. Equipped with the map $\mathcal{D}$, we are able to express the desired formula.

Proposition 1.12. *Let $(E \overset{\pi}{\to} M, \rho, [.,.]_E, g_E)$ be a Courant algebroid. Then the relation*

$$[f\psi_1, \psi_2]_E = f[\psi_1, \psi_2]_E - (\rho(\psi_2) f)\psi_1 + g_E(\psi_1, \psi_2)\, \mathcal{D} f \tag{1.22}$$

is satisfied for all ψ_1, $\psi_2 \in \Gamma(E)$ and $f \in C^\infty(M)$.

Proof. Since for all f, $g \in C^\infty(M)$ there holds $\mathrm{d}(fg) = g\,\mathrm{d} f + f\,\mathrm{d} g$, we also have

$$\mathcal{D}(fg) = g\, \mathcal{D} f + \ f\, \mathcal{D} g. \tag{1.23}$$

Hence, it follows for all ψ_1, $\psi_2 \in \Gamma(E)$ and $f \in C^\infty(M)$ that

$$\begin{aligned}
\mathcal{D}(f g_E(\psi_1, \psi_2)) &= \ g_E(\psi_1, \psi_2)\, \mathcal{D} f + f\, \mathcal{D}\, g_E(\psi_1, \psi_2) \\
&\underset{(1.12)}{=} g_E(\psi_1, \psi_2)\, \mathcal{D} f + f[\psi_1, \psi_2]_E + f[\psi_2, \psi_1]_E,
\end{aligned} \tag{1.24}$$

while on the other hand, we have the following:

$$\begin{aligned}
\mathcal{D}(f g_E(\psi_1, \psi_2)) &= \ \mathcal{D}\, g_E(f\psi_1, \psi_2) \underset{(1.12)}{=} [f\psi_1, \psi_2]_E + [\psi_2, f\psi_1]_E \\
&\underset{(1.3)}{=} [f\psi_1, \psi_2]_E + (\rho(\psi_2) f)\psi_1 + f[\psi_2, \psi_1].
\end{aligned} \tag{1.25}$$

Comparing both (1.24) and (1.25) results exactly in the formula to be proven. □

[2] For the case of Courant metric, notation introduced in 1.2 will be in the whole text simplified as $\flat_E := \flat_{g_E}$ and $\sharp_E := \sharp_{g_E}$.

The last property of Courant algebroids listed in this section describes the composition rules for the map $\mathcal{D}$ and the additional structures of a Courant algebroid; the anchor, the Courant bracket and the Courant metric.

Proposition 1.13. *Let $(E \overset{\pi}{\to} M, \rho, [.\,,.]_E, g_E)$ be a Courant algebroid. The composition rules*

$$\rho \circ \mathcal{D} = 0, \qquad g_E(\mathcal{D} f, \mathcal{D} h) = 0, \qquad [\mathcal{D} f, \psi]_E = 0 \tag{1.26}$$

are satisfied for all f, $h \in C^\infty(M)$ and $\psi \in \Gamma(E)$.

Proof. Acting by the anchor on (1.12) and using linearity of ρ together with (1.8) leads to

$$[\rho(\psi_1), \rho(\psi_2)] + [\rho(\psi_2), \rho(\psi_1)] = \rho(\mathcal{D}\, g_E(\psi_1, \psi_2)), \tag{1.27}$$

for all ψ_1, $\psi_2 \in \Gamma(E)$. Since the Lie bracket of vector fields is skew-symmetric, the left-hand side vanishes identically. Therefore, for all $f \in C^\infty(M)$ there holds

$$0 = \rho(\mathcal{D}\, g_E(f\psi_1, \psi_2)) = \rho(\mathcal{D}(f g_E(\psi_1, \psi_2)) = g_E(\psi_1, \psi_2)\rho(\mathcal{D} f) + f\rho(\mathcal{D}\, g_E(\psi_1, \psi_2)). \tag{1.28}$$

As we have already proven, the second term on the right-hand side vanishes, thus we get

$$g_E(\psi_1, \psi_2)\rho(\mathcal{D} f) = 0. \tag{1.29}$$

Since this holds for arbitrary sections ψ_1 and ψ_2 and the fiber-wise metric in non-degenerate, necessarily $\rho(\mathcal{D} f) = 0$ for all $f \in C^\infty(M)$, that is $\rho \circ \mathcal{D} = 0$. The composition rule for the Courant metric follows directly from the definition of map $\mathcal{D}$, see (1.13), and the equality just proven. To prove the last formula, we first show that the equation

$$[\mathcal{D}\, g_E(\psi_1, \psi_2), \psi]_E = 0, \tag{1.30}$$

is valid for all ψ_1, ψ_2, $\psi \in \Gamma(E)$. It can be proven in the following way, consider arbitrary sections ψ_1, ψ_2, $\psi \in \Gamma(E)$ and act by $[.\,, \psi]_E$ on (1.12), it yields

$$\begin{aligned}
[\mathcal{D}\, g_E(\psi_1, \psi_2), \psi]_E &= [[\psi_1, \psi_2]_E, \psi]_E + [[\psi_2, \psi_1], \psi]_E \\
&\underset{(1.4)}{=} [\psi_1, [\psi_2, \psi]_E]_E - [\psi_2, [\psi_1, \psi]_E]_E + [\psi_2, [\psi_1, \psi]_E]_E - [\psi_1, [\psi_2, \psi]_E]_E \\
&= 0.
\end{aligned} \tag{1.31}$$

Hence, for all $f \in C^\infty(M)$ there holds

$$[\mathcal{D}\, g_E(f\psi_1, \psi_2), \psi]_E = 0, \tag{1.32}$$

while on the other hand, the same expression can be expanded as follows:

$$\begin{aligned}
&[\mathcal{D}\, g_E(f\psi_1, \psi_2), \psi]_E \\
&\quad = [\mathcal{D}(f g_E(\psi_1, \psi_2)), \psi]_E = [g_E(\psi_1, \psi_2)\, \mathcal{D} f, \psi]_E + [f\, \mathcal{D}\, g_E(\psi_1, \psi_2), \psi]_E \\
&\quad \underset{(1.22)}{=} g_E(\psi_1, \psi_2)[\mathcal{D} f, \psi]_E - (\rho(\psi) g_E(\psi_1, \psi_2))\, \mathcal{D} f + g_E(\mathcal{D} f, \psi)\, \mathcal{D}\, g_E(\psi_1, \psi_2) \\
&\qquad + f[\mathcal{D}\, g_E(\psi_1, \psi_2), \psi]_E - (\rho(\psi) f)\, \mathcal{D}\, g_E(\psi_1, \psi_2) + g_E(\mathcal{D}\, g_E(\psi_1, \psi_2), \psi)\, \mathcal{D} f \\
&\quad \underset{(1.13)}{=} g_E(\psi_1, \psi_2)[\mathcal{D} f, \psi]_E + f[\mathcal{D}\, g_E(\psi_1, \psi_2), \psi]_E \\
&\quad \underset{(1.30)}{=} g_E(\psi_1, \psi_2)[\mathcal{D} f, \psi]_E
\end{aligned} \tag{1.33}$$

By putting equations (1.32) and (1.33) together and realizing that g_E is non-degenerate, we obtain the desired composition rule. □

As for every algebraic structure, it is convenient to have a criterion for saying, that two Courant algebroids are in principle the same, that is isomorphic. This criterion is formalized by the following definition.

Definition 1.14. Let $(E \xrightarrow{\pi} M, \rho, [.,.]_E, g_E)$ and $(E' \xrightarrow{\pi'} M, \rho', [.,.]_{E'}, g_{E'})$ be two Courant algebroids over the same base manifold and $\mathcal{F} : \Gamma(E) \to \Gamma(E')$ be a $C^\infty(M)$-linear map. We say that $\mathcal{F}$ is a **Courant algebroid morphism** if and only if the following requirements are met:

$$\rho = \rho' \circ \mathcal{F}, \qquad \mathcal{F}[\psi_1, \psi_2]_E = [\mathcal{F}\psi_1, \mathcal{F}\psi_2]_{E'}, \qquad g_E(\psi_1, \psi_2) = g_{E'}(\mathcal{F}\psi_1, \mathcal{F}\psi_2) \tag{1.34}$$

for all ψ_1, $\psi_2 \in \Gamma(E)$, that is $\mathcal{F}$ preserves all Courant algebroid structures. In addition, if $\mathcal{F}$ is bijective, it is called a **Courant algebroid isomorphism**.

Remark 1.15. It can be easily checked that if $\mathcal{F}$ is a Courant algebroid isomorphism, then also $\mathcal{F}^{-1}$ is a Courant algebroid isomorphism.

1.2 Courant algebroid connections

The structure of Courant algebroids allows one to act on smooth functions by a section of a Courant algebroid, it is naturally provided through the anchor. Consequently, we can introduce Courant algebroid connections, which are a generalization of affine connections well-known from the standard differential geometry.

Definition 1.16. Suppose $(E \xrightarrow{\pi} M, \rho, [.,.]_E, g_E)$ is a Courant algebroid. An $\mathbb{R}$-bilinear map $\nabla : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ satisfying

(I) $\nabla_{f\psi_1}\psi_2 = f\nabla_{\psi_1}\psi_2$,

(II) $\nabla_{\psi_1}(f\psi_2) = (\rho(\psi_1)f)\psi_2 + f\nabla_{\psi_1}\psi_2$,

(III) $\rho(\psi_1)g_E(\psi_2, \psi_3) = g_E(\nabla_{\psi_1}\psi_2, \psi_3) + g_E(\psi_2, \nabla_{\psi_1}\psi_3)$,

for all ψ_1, ψ_2, $\psi_3 \in \Gamma(E)$ and $f \in C^\infty(M)$, is called a **Courant algebroid connection**. In the formulas above, we denoted the induced endomorphism $\nabla(\psi, .) : \Gamma(E) \to \Gamma(E)$ as $\nabla_\psi := \nabla(\psi, .)$, for all $\psi \in \Gamma(E)$, this notation will be used across the whole thesis.

Remark 1.17. Apart from the first two properties, which correspond to a straightforward generalization of affine connections, there is also the third one, an a priori assumption that Courant algebroid connection is compatible with the Courant metric g_E.

It is worthy to ask if there is some Courant algebroid connection on every Courant algebroid. It can be shown, see [9, Proposition 2.17], that every vector bundle $E \xrightarrow{\pi} M$ equipped with a fiber-wise metric g_E admits a vector bundle connection compatible with g_E, that is a vector bundle connection $\nabla' : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ satisfying

$$Xg_E(\psi_1, \psi_2) = g_E(\nabla'_X\psi_1, \psi_2) + g_E(\psi_1, \nabla'_X\psi_2) \tag{1.35}$$

for all $X \in \Gamma(TM)$ and ψ_1, $\psi_2 \in \Gamma(E)$. For a given vector bundle connection ∇' compatible with g_E on a Courant algebroid $(E \xrightarrow{\pi} M, \rho, [.,.]_E, g_E)$, we can define an $\mathbb{R}$-bilinear map $\nabla : \ \Gamma(E) \times \ \Gamma(E) \to \Gamma(E)$ as

$$\nabla_{\psi_1}\psi_2 := \nabla'_{\rho(\psi_1)}\psi_2, \tag{1.36}$$

for all ψ_1, $\psi_2 \in \Gamma(E)$. It is clear that ∇ meets all the requirements to be a Courant algebroid connection. Therefore, the existence of a Courant algebroid connection on an arbitrary Courant algebroid has been just proven.

We have already introduced a Courant algebroid isomorphism in the previous section. It is convenient to extend this concept even to Courant algebroids equipped with a Courant algebroid connection.

Definition 1.18. Let $(E \overset{\pi}{\to} M, \rho, [.,.]_E, g_E)$ and $(E' \overset{\pi'}{\to} M, \rho', [.,.]_{E'}, g_{E'})$ be two Courant algebroids equipped with Courant algebroid connections ∇ and ∇' respectively. Courant algebroid isomorphism $\mathcal{F} : \Gamma(E) \to \Gamma(E')$ is called a **connection preserving Courant algebroid isomorphism** if and only if it relates their Courant algebroid connections as

$$\mathcal{F}(\nabla_{\psi_1}\psi_2) = \nabla'_{\mathcal{F}\psi_1}\mathcal{F}\psi_2, \tag{1.37}$$

for all ψ_1, $\psi_2 \in \Gamma(E)$.

Remark 1.19. One could be curious if there is some Courant algebroid connection ∇' on E' which would satisfy the relation (1.37). In other words; if we have a Corant algebroid connection ∇ on E, does the formula

$$\nabla'_{\psi'_1}\psi'_2 = \mathcal{F}(\nabla_{\mathcal{F}^{-1}\psi'_1}\mathcal{F}^{-1}\psi'_2), \tag{1.38}$$

for all ψ'_1, ψ'_2, define a Courant algebroid connection on E'? It is pretty straightforward to check that it is always true.

Definition 1.20. Consider a Courant algebroid $(E \overset{\pi}{\to} M, \rho, [.,.]_E, g_E)$ endowed with a Courant algebroid connection ∇. We define **covariant divergences** $\operatorname{div}_\nabla : \Gamma(E) \to C^\infty(M)$ and $\operatorname{div}_\nabla : \Gamma(E^*) \to C^\infty(M)$ as[3]

$$\operatorname{div}_\nabla \psi := \xi^\mu(\nabla_{\xi_\mu}\psi), \qquad \operatorname{div}_\nabla \mathcal{A} := (\nabla_{\xi_\mu}\mathcal{A})(\sharp_E\, \xi^\mu), \tag{1.39}$$

for all $\psi \in \Gamma(E)$ and all $\mathcal{A} \in \Gamma(E^*)$ respectively, where $\{\xi_\mu\}_{\mu=1}^{\mathrm{Rank}(E)}$ is an arbitrary local frame of E over some $U \subseteq M$, and $\{\xi^\mu\}_{\mu=1}^{\mathrm{Rank}(E)}$ is the corresponding dual one.

Notation 1.21. The previous definition is not entirely correct, one should add restriction symbols in the following way:

$$(\operatorname{div}_\nabla \psi)|_U := \xi^\mu(\nabla|_{U\xi_\mu}\, \psi|_U), \qquad (\operatorname{div}_\nabla \mathcal{A})|_U := (\nabla|_{U\xi_\mu}\, \mathcal{A}|_U)(\sharp_E\,|_U\, \xi^\mu). \tag{1.40}$$

Since there is always a local frame over some neighbourhood of an arbitrary point of the base manifold, it clearly defines a smooth function on the whole base manifold. Apparently, expressions without restrictions are cleaner and more readable, precisely for this reason, we will quietly assume that all object are always appropriately restricted and we will thus usually omit all the restriction symbols in the remaining part of this thesis. Moreover, we will freely and without any special emphasizing use $\{\xi_\mu\}_{\mu=1}^{\mathrm{Rank}(E)}$ as an arbitrary local frame of E over some $U \subseteq M$.

Remark 1.22. Note that the two types of divergences are related to each other as

$$\operatorname{div}_\nabla \psi = \operatorname{div}_\nabla \flat_E\, \psi, \qquad \operatorname{div}_\nabla \mathcal{A} = \operatorname{div}_\nabla \sharp_E\, \mathcal{A}, \tag{1.41}$$

[3]Action of covariant derivative associated with a Courant algebroid connection is extended to a tensor fields of an arbitrary rank in the exactly same way as it is done in the case of ordinary affine connections.

for all $\psi \in \Gamma(E)$ and $\mathcal{A} \in \Gamma(E^*)$. It follows immediately from the fact that $\nabla_\psi \circ \flat_E = \flat_E \circ \nabla_\psi$, for all $\psi \in \Gamma(E)$, which is an easy consequence of the third axiom of a Courant algebroid connection and can be proven as

$$\begin{aligned}(\flat_E \nabla_{\psi_1}\psi_2)(\psi_3) &= g_E(\nabla_{\psi_1}\psi_2, \psi_3) = \rho(\psi_1) g_E(\psi_2, \psi_3) - g_E(\psi_2, \nabla_{\psi_1}\psi_3) \\ &= \rho(\psi_1)(\flat_E \psi_2)(\psi_3) - (\flat_E \psi_2)(\nabla_{\psi_1}\psi_3) = (\nabla_{\psi_1} \flat_E \psi_2)(\psi_3),\end{aligned} \tag{1.42}$$

for all $\psi_1,\ \psi_2,\ \psi_3 \in \Gamma(E)$.

As we will see, there arise a lot of invariants preserving under the Courant algebroid ismorphisms. First of them are the just introduced divergences.

Proposition 1.23. *Let* $(E \stackrel{\pi}{\to} M, \rho, [.,.]_E, g_E)$ *and* $(E' \stackrel{\pi'}{\to} M, \rho', [.,.]_{E'}, g_{E'})$ *be two Courant algebroids equipped with Courant algebroid connections* ∇ *and* ∇' *respectively. Moreover, assume that there is a connection preserving Courant algebroid isomorphism* $\mathcal{F} : \Gamma(E) \to \Gamma(E')$ *between them. Then the corresponding covariant divergences are related as*

$$\operatorname{div}_\nabla = \operatorname{div}_{\nabla'} \circ \mathcal{F}, \qquad\qquad \operatorname{div}_\nabla = \operatorname{div}_{\nabla'} \circ (\mathcal{F}^{-1})^T. \tag{1.43}$$

Proof. As $\mathcal{F}$ is a $C^\infty(M)$-module isomorphism and $\{\xi_\mu\}_{\mu=1}^{\mathrm{Rank}(E)}$ is a local frame of E, one easily sees that $\{\mathcal{F}\xi_\mu\}_{\mu=1}^{\mathrm{Rank}(E)}$ is a local frame of E', moreover, $\{(\mathcal{F}^{-1})^T \xi^\mu\}_{\mu=1}^{\mathrm{Rank}(E)}$ is apparently the corresponding dual one. Therefore, the definition of connection preserving Courant algebroid isomorphism implies

$$\operatorname{div}_\nabla \psi = \xi^\mu(\nabla_{\xi_\mu}\psi) = \xi^\mu(\mathcal{F}^{-1}\nabla'_{\mathcal{F}\xi_\mu}\mathcal{F}\psi) = ((\mathcal{F}^{-1})^T\xi^\mu)(\nabla'_{\mathcal{F}\xi_\mu}\mathcal{F}\psi) = \operatorname{div}_{\nabla'} \mathcal{F}\psi, \tag{1.44}$$

for all $\psi \in \Gamma(E)$, this proves the first identity. To prove the second one, realize that the Courant metric is invariant under Courant algebroid isomorphisms, so for all $\psi_1,\ \psi_2 \in \Gamma(E)$ we can perform the following:

$$(\flat_E \psi_1)(\psi_2) = g_E(\psi_1, \psi_2) = g_{E'}(\mathcal{F}\psi_1, \mathcal{F}\psi_2) = (\flat_{E'} \mathcal{F}\psi_1)(\mathcal{F}\psi_2) = (\mathcal{F}^T \flat_{E'} \mathcal{F}\psi_1)(\psi_2), \tag{1.45}$$

that means $\flat_E = \mathcal{F}^T \flat_{E'} \mathcal{F}$, or equivalently

$$\sharp_E = \mathcal{F}^{-1} \sharp_{E'} (\mathcal{F}^{-1})^T. \tag{1.46}$$

By employing the relation for $\sharp_E$ into (1.41), we obtain

$$\operatorname{div}_\nabla \mathcal{A} = \operatorname{div}_\nabla \sharp_E \mathcal{A} = \operatorname{div}_\nabla \mathcal{F}^{-1} \sharp_{E'} (\mathcal{F}^{-1})^T \mathcal{A} = \operatorname{div}_{\nabla'} \sharp_{E'} (\mathcal{F}^{-1})^T \mathcal{A} = \operatorname{div}_{\nabla'} (\mathcal{F}^{-1})^T \mathcal{A}, \tag{1.47}$$

for all $\mathcal{A} \in \Gamma(E^*)$. □

1.2.1 Torsion

With Courant algebroid connections in our hands, we are able to speak about Courant algebroid alternative to a torsion tensor. Naively, we would define a torsion operator as an $\mathbb{R}$-bilinear map $T_0 : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ by the following formula:

$$T_0(\psi_1, \psi_2) := \nabla_{\psi_1}\psi_2 - \nabla_{\psi_2}\psi_1 - [\psi_1, \psi_2]_E, \tag{1.48}$$

for all $\psi_1,\ \psi_2 \in \Gamma(E)$, where $(E \stackrel{\pi}{\to} M, \rho, [.,.]_E, g_E)$ is a Courant algebroid equipped with a Courant algebroid connection ∇. One immediately sees that unlike the ordinary torsion operator,

T_0 is not skew-symmetric. It would not be such a problem, however, for all $\psi_1, \psi_2 \in \Gamma(E)$ and $f \in C^\infty(M)$ there holds

$$T_0(f\psi_1, \psi_2) = \nabla_{f\psi_1}\psi_2 - \nabla_{\psi_2}(f\psi_1) - [f\psi_1, \psi_2]_E = fT_0(\psi_1, \psi_2) - g_E(\psi_1, \psi_2)\, \mathcal{D} f, \tag{1.49}$$

where we have used axioms of a Courant algebroid connection and the relation (1.22). It means that T_0 is not $C^\infty(M)$-bilinear, which is already a serious issue, because it does not define a tensor field on E. Let us try to fix it, consider a new torsion operator $T : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ defined for all $\psi_1, \psi_2 \in \Gamma(E)$ as

$$T(\psi_1, \psi_2) = T_0(\psi_1, \psi_2) + \Theta(\psi_1, \psi_2), \tag{1.50}$$

where $\Theta : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ is an unspecified $\mathbb{R}$-bilinear map satisfying

$$\Theta(\psi_1, f\psi_2) = f\Theta(\psi_1, \psi_2), \tag{1.51}$$
$$\Theta(f\psi_1, \psi_2) = f\Theta(\psi_1, \psi_2) + g_E(\psi_1, \psi_2)\, \mathcal{D} f, \tag{1.52}$$
$$\Theta(\psi_1, \psi_1) = [\psi_1, \psi_1]_E, \tag{1.53}$$

for all $\psi_1, \psi_2 \in \Gamma(E)$ and $f \in C^\infty(M)$. Such T is apparently $C^\infty(M)$-bilinear, and moreover, it is skew-symmetric thanks to the last equation. Now, the task is to find the most general form of Θ such that all three equations above are satisfied. Since $\flat_E$ is an isomorphism, the equation (1.53) holds if and only if

$$g_E(\Theta(\psi_1, \psi_1), \psi_2) = g_E([\psi_1, \psi_1]_E, \psi_2),$$

for all $\psi_1, \psi_2 \in \Gamma(E)$. Using (1.12), (1.13) and compatibility of ∇ with the g_E consecutively, it can be further rewritten as follows:

$$g_E(\Theta(\psi_1, \psi_1), \psi_2) = \frac{1}{2} g_E(\mathcal{D}\, g_E(\psi_1, \psi_1), \psi_2) = \frac{1}{2}\rho(\psi_2) g_E(\psi_1, \psi_1) = g_E(\nabla_{\psi_2}\psi_1, \psi_1). \tag{1.54}$$

By choosing $\psi_2 = \sharp_E\, \xi^\mu$ for all $\mu \in \{1, \ldots, \mathrm{Rank}(E)\}$, one finds that equation (1.54) is satisfied if and only if

$$\Theta(\psi_1, \psi_1) = g_E(\nabla_{\sharp_E\, \xi^\mu}\psi_1, \psi_1)\xi_\mu. \tag{1.55}$$

It follows from the linearity that

$$\Theta_S(\psi_1, \psi_2) := \frac{1}{2}(\Theta(\psi_1, \psi_2) + \Theta(\psi_2, \psi_1)) = \frac{1}{2}(g_E(\nabla_{\sharp_E\, \xi^\mu}\psi_1, \psi_2)\xi_\mu + g_E(\psi_1, \nabla_{\sharp_E\, \xi^\mu}\psi_2)\xi_\mu), \tag{1.56}$$

for all $\psi_1, \psi_2 \in \Gamma(E)$. Therefore, the equation (1.53) uniquely determines the symmetric part of the map Θ. As the set of all skew-symmetric $\mathbb{R}$-bilinear maps forms a vector space, the skew-symmetric part can be cast for all $\psi_1, \psi_2 \in \Gamma(E)$ in the form

$$\begin{aligned}\Theta_A(\psi_1, \psi_2) &:= \frac{1}{2}(\Theta(\psi_1, \psi_2) - \Theta(\psi_2, \psi_1)) \\ &= \frac{1}{2}(g_E(\nabla_{\sharp_E\, \xi^\mu}\psi_1, \psi_2)\xi_\mu - g_E(\psi_1, \nabla_{\sharp_E\, \xi^\mu}\psi_2)\xi_\mu) + A(\psi_1, \psi_2),\end{aligned} \tag{1.57}$$

where $A : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ is an unspecified skew-symmetric $\mathbb{R}$-bilinear map. One easily finds that the equations (1.51) and (1.52) are then equivalent to imposing the $C^\infty(M)$-bilinearity condition on a map A. Therefore, the most general solution of the given set of equations takes the form

$$\Theta(\psi_1, \psi_2) = g_E(\nabla_{\sharp_E\, \xi^\mu}\psi_1, \psi_2)\xi_\mu + A(\psi_1, \psi_2), \tag{1.58}$$

for all $\psi_1, \psi_2 \in \Gamma(E)$ and for an arbitrary skew-symmetric $C^\infty(M)$-bilinear map A. In order to make a Courant algebroid version of a torsion operator as simple as possible, we set $A := 0$.

Notation 1.24. In the previous paragraph, we have used the symbols Θ_S and Θ_A for the complete symmetrization and complete skew-symmetrization of the $\mathbb{R}$-bilinear map Θ respectively. We will continue to use the notation in the remaining part of the thesis also for tensor fields of an arbitrary rank. For example, if we take $F \in \mathcal{T}^0_k(E)$, $k \in \mathbb{N}$, we have

$$F_S(\psi_1, \ldots, \psi_k) := \frac{1}{k!} \sum_{\sigma \in S_k} F(\psi_{\sigma(1)}, \ldots, \psi_{\sigma(k)}), \tag{1.59}$$

$$F_A(\psi_1, \ldots, \psi_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn} \sigma \, F(\psi_{\sigma(1)}, \ldots, \psi_{\sigma(k)}), \tag{1.60}$$

for all $\psi_1, \ldots, \psi_k \in \Gamma(E)$. The symbol S_k denotes the symmetric group of order k.

Finally, the considerations above lead to the following formal definition of a torsion operator. The same definition was proposed earlier in [10].

Definition 1.25. Let $(E \overset{\pi}{\to} M, \rho, [. , .]_E, g_E)$ be a Courant algebroid equipped with a Courant algebroid connection ∇. The **torsion operator** $T : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ is defined as

$$T(\psi_1, \psi_2) := \nabla_{\psi_1}\psi_2 - \nabla_{\psi_2}\psi_1 - [\psi_1, \psi_2]_E + g_E(\nabla_{\sharp_E \xi^\mu}\psi_1, \psi_2)\xi_\mu, \tag{1.61}$$

for all $\psi_1, \psi_2 \in \Gamma(E)$.

Theorem 1.26 (Torsion tensor)**.** *Let $(E \overset{\pi}{\to} M, \rho, [. , .]_E, g_E)$ be a Courant algebroid equipped with a Courant algebroid connection ∇. Then the* ***torsion tensor*** *$T : \Gamma(E) \times \Gamma(E) \times \Gamma(E) \to C^\infty(M)$ defined through the torsion operator as*

$$T(\psi_1, \psi_2, \psi_3) := g_E(T(\psi_1, \psi_2), \psi_3), \tag{1.62}$$

for all $\psi_1, \psi_2, \psi_3 \in \Gamma(E)$, can be expressed as

$$T(\psi_1, \psi_2, \psi_3) = g_E(\nabla_{\psi_1}\psi_2 - \nabla_{\psi_2}\psi_1 - [\psi_1, \psi_2]_E, \psi_3) + g_E(\nabla_{\psi_3}\psi_1, \psi_2). \tag{1.63}$$

Moreover, it is $C^\infty(M)$-linear in all three inputs and it is completely skew-symmetric, in other words $T \in \Omega^3(E)$.

Proof. Firstly, let us prove that formula (1.63) holds. For arbitrary $\psi_1, \psi_2, \psi_3 \in \Gamma(E)$ we have

$$\begin{aligned} g_E(\nabla_{\sharp_E \xi^\mu}\psi_1, \psi_2) g_E(\xi_\mu, \psi_3) &= g_E(\nabla_{g_E(\sharp_E \xi^\mu, \sharp_E \xi^\nu)\xi_\nu}\psi_1, \psi_2) g_E(\xi_\mu, \psi_3) \\ &= g_E(\nabla_{\xi_\nu}\psi_1, \psi_2) g_E(\sharp_E \xi^\nu, \psi_3) \\ &= g_E(\nabla_{\psi_3}\psi_1, \psi_2). \end{aligned} \tag{1.64}$$

This result together with the definition of the torsion operator and the torsion tensor proves (1.63). Almost everything else has been already proven in the introductory paragraph to this subsection. It remains to prove that the torsion tensor is $C^\infty(M)$-linear in the third input and that it is completely skew-symmetric. The former is trivial, let us check the latter. Since we have already proven the skew-symmetry of the torsion operator, it is sufficient to show the skew-symmetry in the last two inputs. For arbitrary $\psi_1, \psi_2 \in \Gamma(E)$ we have

$$\begin{aligned} T(\psi_1, \psi_2, \psi_2) &= g_E(\nabla_{\psi_1}\psi_2 - \nabla_{\psi_2}\psi_1 - [\psi_1, \psi_2]_E, \psi_2) + g_E(\nabla_{\psi_2}\psi_1, \psi_2) \\ &= g_E(\nabla_{\psi_1}\psi_2, \psi_2) - g_E([\psi_1, \psi_2]_E, \psi_2) \\ &= \frac{1}{2}\rho(\psi_1) g_E(\psi_2, \psi_2) - \frac{1}{2}\rho(\psi_1) g_E(\psi_2, \psi_2) = 0, \end{aligned} \tag{1.65}$$

where the third axiom of a Courant algebroid connection and also the third axiom of a Courant algebroid itself have been used in the third step. ∎

Remark 1.27. Note that the Courant algebroid version of torsion tensor is defined directly in a fully covariant fashion, unlike an ordinary torsion tensor, which is originally defined as a tensor of the type $\binom{1}{2}$. Another difference is that an ordinary torsion tensor is skew-symmetric only in the first two arguments.

1.2.2 Curvature

Besides the torsion tensor, there is also another prominent tensor fully determined by an affine connection in standard differential geometry, it is the Riemann curvature tensor or just the Riemann tensor. In this subsection its Courant algebroid equivalent will be established.

Characteristic properties of the ordinary Riemann tensor corresponding to the Levi-Civita affine connection are its symmetries, which among all other things allow one to define the Ricci tensor and the Ricci scalar in an unambiguous way. For this reason, we would like to a Courant algebroid version of the Riemann tensor to also possess these symmetries. Consider a Courant algebroid $(E \overset{\pi}{\to} M, \rho, [.,.]_E, g_E)$ equipped with a Courant algebroid connection ∇. As well as in the case of torsion, it is convenient to introduce the Riemann tensor as a fully covariant tensor field. Let us start again naively and define the Riemann tensor as an $\mathbb{R}$-multilinear map $R_0 : \Gamma(E) \times \Gamma(E) \times \Gamma(E) \times \Gamma(E) \to C^\infty(M)$ in the following way:

$$R_0(\psi_1, \psi_2, \psi_3, \psi_4) := g_E(\nabla_{\psi_3}\nabla_{\psi_4}\psi_2 - \nabla_{\psi_4}\nabla_{\psi_3}\psi_2 - \nabla_{[\psi_3,\psi_4]_E}\psi_2, \psi_1), \tag{1.66}$$

for all ψ_1, ψ_2, ψ_3, $\psi_4 \in \Gamma(E)$. Let us check the $C^\infty(M)$-linearity in all four inputs.

(I) In the first input, it is a direct consequence of the fact that g_E is $C^\infty(M)$-bilinear.

(II) For all ψ_1, ψ_2, ψ_3, $\psi_4 \in \Gamma(E)$ and for all $f \in C^\infty(M)$ there holds

$$\begin{aligned}
&R_0(\psi_1, f\psi_2, \psi_3, \psi_4) \\
&= g_E(\nabla_{\psi_3}\nabla_{\psi_4} f\psi_2 - \nabla_{\psi_4}\nabla_{\psi_3} f\psi_2 - \nabla_{[\psi_3,\psi_4]_E} f\psi_2, \psi_1) \\
&= fR_0(\psi_1, \psi_2, \psi_3, \psi_4) + g_E\Big((\rho(\psi_3)\rho(\psi_4)f)\psi_2 + (\rho(\psi_4)f)\nabla_{\psi_3}\psi_2 + (\rho(\psi_3)f)\nabla_{\psi_4}\psi_2 \\
&\quad - (\rho(\psi_4)\rho(\psi_3)f)\psi_2 - (\rho(\psi_3)f)\nabla_{\psi_4}\psi_2 - (\rho(\psi_4)f)\nabla_{\psi_3}\psi_2 - (\rho([\psi_3,\psi_4]_E)f)\psi_2, \psi_1\Big) \\
&= fR_0(\psi_1, \psi_2, \psi_3, \psi_4) + g_E\Big(([\rho(\psi_3), \rho(\psi_4)]f)\psi_2 - (\rho([\psi_3,\psi_4]_E)f)\psi_2, \psi_1\Big) \\
&= fR_0(\psi_1, \psi_2, \psi_3, \psi_4),
\end{aligned} \tag{1.67}$$

where the second axiom of a Courant algebroid connection has been used multiple times in the second step, and then we have used (1.8) in the last step.

(III) For all ψ_1, ψ_2, ψ_3, $\psi_4 \in \Gamma(E)$ and for all $f \in C^\infty(M)$ there holds

$$\begin{aligned}
&R_0(\psi_1, \psi_2, f\psi_3, \psi_4) \\
&= g_E(\nabla_{f\psi_3}\nabla_{\psi_4}\psi_2 - \nabla_{\psi_4}\nabla_{f\psi_3}\psi_2 - \nabla_{[f\psi_3,\psi_4]_E}\psi_2, \psi_1) \\
&= fR_0(\psi_1, \psi_2, \psi_3, \psi_4) \\
&\quad + g_E\Big(- (\rho(\psi_4)f)\nabla_{\psi_3}\psi_2 + (\rho(\psi_4)f)\nabla_{\psi_3}\psi_2 - g_E(\psi_3, \psi_4)\nabla_{\mathcal{D}f}\psi_2, \psi_1\Big) \\
&= fR_0(\psi_1, \psi_2, \psi_3, \psi_4) - g_E(\psi_3, \psi_4) g_E(\nabla_{\mathcal{D}f}\psi_2, \psi_1),
\end{aligned} \tag{1.68}$$

where we have used the first and the second axiom of a Courant alegebroid connection together with (1.22).

(IV) If we compare the relations (1.3) and (1.22), we see that that the term responsible for spoiling the $C^\infty(M)$-linearity in the third input of R_0 is missing in (1.3). Therefore, R_0 is $C^\infty(M)$-linear in the fourth input.

We have just shown that R_0 is $C^\infty(M)$-linear in all inputs except for the third one, therefore we have to modify it to get an appropriate definition of the Courant algebroid version of the Riemann tensor. Before we do that, let us first investigate if R_0 at least possesses the proper symmetries.

(A) The first of them is the skew-symmetry in the last two arguments, it is possessed by the ordinary Riemann tensor corresponding to an arbitrary affine connection. In the case of R_0, for all ψ, ψ_1, ψ_2 there holds

$$R_0(\psi_1, \psi_2, \psi, \psi) = -g_E(\nabla_{[\psi,\psi]_E}\psi_2, \psi_1), \tag{1.69}$$

which can be equivalently expressed by using (1.12) as

$$R_0(\psi_1, \psi_2, \psi, \psi) = -\frac{1}{2} g_E(\nabla_{\mathcal{D}\, g_E(\psi,\psi)}\psi_2, \psi_1), \tag{1.70}$$

hence R_0 lacks the skew-symmetry in the last two inputs.

(B) Another one is the skew-symmetry in the first two inputs, which does not hold generally in the standard differential geometry, but we have to assume an affine connection compatible with the metric to ensure it. Let us check it for R_0, take arbitrary ψ, ψ_3, $\psi_4 \in \Gamma(E)$ and expand $R_0(\psi, \psi, \psi_3, \psi_4)$ as follows:

$$\begin{aligned}
&R_0(\psi, \psi, \psi_3, \psi_4)\\
&= g_E(\nabla_{\psi_3}\nabla_{\psi_4}\psi - \nabla_{\psi_4}\nabla_{\psi_3}\psi - \nabla_{[\psi_3,\psi_4]_E}\psi, \psi)\\
&= \rho(\psi_3) g_E(\nabla_{\psi_4}\psi, \psi) - g_E(\nabla_{\psi_4}\psi, \nabla_{\psi_3}\psi) - \rho(\psi_4) g_E(\nabla_{\psi_3}\psi, \psi) + g_E(\nabla_{\psi_3}\psi, \nabla_{\psi_4}\psi)\\
&\quad - \rho([\psi_3, \psi_4]_E) g_E(\psi, \psi) + g_E(\psi, \nabla_{[\psi_3,\psi_4]_E}\psi)\\
&= [\rho(\psi_3), \rho(\psi_4)] g_E(\psi, \psi) - \rho([\psi_3, \psi_4]_E) g_E(\psi, \psi) - \rho(\psi_3) g_E(\psi, \nabla_{\psi_4}\psi)\\
&\quad + \rho(\psi_4) g_E(\psi, \nabla_{\psi_3}\psi) + g_E(\psi, \nabla_{[\psi_3,\psi_4]_E}\psi)\\
&= - g_E(\nabla_{\psi_3}\psi, \nabla_{\psi_4}\psi) - g_E(\psi, \nabla_{\psi_3}\nabla_{\psi_4}\psi) + g_E(\nabla_{\psi_4}\psi, \nabla_{\psi_3}\psi)\\
&\quad + g_E(\psi, \nabla_{\psi_4}\nabla_{\psi_3}\psi) + g_E(\psi, \nabla_{[\psi_3,\psi_4]_E}\psi)\\
&= - R_0(\psi, \psi, \psi_3, \psi_4),
\end{aligned} \tag{1.71}$$

which implies

$$R_0(\psi, \psi, \psi_3, \psi_4) = 0. \tag{1.72}$$

Therefore, R_0 is skew-symmetric in the first two arguments. During the derivation, the compatibility of ∇ with g_E has been used multiple times, and in the fourth step, we have also used (1.8).

(C) The two already discussed symmetries would already be sufficient for the unambiguous definition of the Ricci tensor, however, the ordinary Riemann tensor corresponding to the Levi-Civita affine connection possesses yet another symmetry, which makes the Ricci tensor symmetric in both of its inputs. It is symmetry with respect to the interchange of the first pair of inputs for the second pair of them. Since $R_0(\psi, \psi, \psi_1, \psi_2) \neq R_0(\psi_1, \psi_2, \psi, \psi)$, it is clear that R_0 does not have this symmetry.

Unlike the torsion tensor, there is not such a straightforward way how to fix the issues of R_0 outlined in the previous paragraph. Therefore we just state the definition of a Courant algebroid version of the Riemann tensor, and then check if it meets all requirements mentioned above. The definition we use was proposed in the paper [2, Definition 4.8].

Definition 1.28. Consider a Courant algebroid $(E \xrightarrow{\pi} M, \rho, [.\,,.]_E, g_E)$ equipped with a Courant algebroid connection ∇. Then the **Riemann tensor** $R : \Gamma(E)\times\Gamma(E)\times\Gamma(E)\times\Gamma(E) \to C^\infty(M)$ is an $\mathbb{R}$-multilinear map defined as

$$R(\psi_1,\psi_2,\psi_3,\psi_4) = \frac{1}{2}\Big(R_0(\psi_1,\psi_2,\psi_3,\psi_4) + R_0(\psi_4,\psi_3,\psi_2,\psi_1) + g_E(\nabla_{\xi_\mu}\psi_3,\psi_4)g_E(\nabla_{\sharp_E\,\xi^\mu}\psi_2,\psi_1)\Big) \tag{1.73}$$

for all ψ_1, ψ_2, ψ_3, $\psi_4 \in \Gamma(E)$.

Theorem 1.29 (Riemann tensor)**.** *Let $(E \xrightarrow{\pi} M, \rho, [.\,,.]_E, g_E)$ be a Courant algebroid equipped with a Courant algebroid connection ∇. Then the Riemann tensor is $C^\infty(M)$-linear in all inputs, that is $R \in \mathcal{T}^0_4(E)$. Moreover, it possesses the following symmetries:*

$$R(\psi_1,\psi_2,\psi_3,\psi_4) = -\,R(\psi_1,\psi_2,\psi_4,\psi_3), \tag{1.74}$$
$$R(\psi_1,\psi_2,\psi_3,\psi_4) = -\,R(\psi_2,\psi_1,\psi_3,\psi_4), \tag{1.75}$$
$$R(\psi_1,\psi_2,\psi_3,\psi_4) = R(\psi_3,\psi_4,\psi_1,\psi_2), \tag{1.76}$$

for all ψ_1, ψ_2, ψ_3, $\psi_4 \in \Gamma(E)$.

Proof. Let us begin with the symmetries. Take arbitrary ψ, ψ_1, $\psi_2 \in \Gamma(E)$, for the third term on the right-hand side of (1.73) we have

$$\begin{aligned} g_E(\nabla_{\xi_\mu}\psi,\psi)g_E(\nabla_{\sharp_E\,\xi^\mu}\psi_2,\psi_1) &= \frac{1}{2}(\rho(\xi_\mu)g_E(\psi,\psi))g_E(\sharp_E\,\xi^\mu,\sharp_E\,\xi^\nu)g_E(\nabla_{\xi_\nu}\psi_2,\psi_1) \\ &= \frac{1}{2}g_E(\mathcal{D}\,g_E(\psi,\psi),\sharp_E\,\xi^\nu)g_E(\nabla_{\xi_\nu}\psi_2,\psi_1) \\ &= \frac{1}{2}g_E(\nabla_{\mathcal{D}\,g_E(\psi,\psi)}\psi_2,\psi_1), \end{aligned} \tag{1.77}$$

where we have used the identity $\sharp_E\,\xi^\mu = g_E(\sharp_E\,\xi^\mu,\sharp_E\,\xi^\nu)\xi_\nu$, the definition of map $\mathcal{D}$, and also the axioms of a Courant algebroid connection are used multiple times. This result together with the equations (1.70) and (1.72) implies

$$R(\psi_1,\psi_2,\psi,\psi) = 0, \tag{1.78}$$

hence R is skew-symmetric in the second pair of arguments. One would proceed in the exactly same way to show the skew-symmetry in the first two arguments. Due to the identity $\sharp_E\,\xi^\mu = g_E(\sharp_E\,\xi^\mu,\sharp_E\,\xi^\nu)\xi_\nu$, it is clear that the equality

$$R(\psi_1,\psi_2,\psi_3,\psi_4) = R(\psi_4,\psi_3,\psi_2,\psi_1) \tag{1.79}$$

holds for all ψ_1, ψ_2, ψ_3, $\psi_4 \in \Gamma(E)$. This result together with the already proven symmetries (1.74) and (1.75) gives us the remaining symmetry (1.76). As we have already shown that R_0 is $C^\infty(M)$-linear in all inputs except for the third one, it is apparent that R is $C^\infty(M)$-linear in the first and the last argument. Since symmetries (1.74) and (1.75) are possessed by the Riemann tensor, it is enough for the $C^\infty(M)$-linearity also in the second and the third input. $\square$

Although the definition of the Riemann tensor R may look rather clumsy at first glance, it is indeed $C^\infty(M)$-linear in all inputs and it also possesses all characteristic symmetries. Another argument in favour of our choice of the Riemann tensor can be stated, it satisfies the analogue of the algebraic Bianchi identity.

Proposition 1.30 (Algebraic Bianchi identity)**.** *Let $(E \overset{\pi}{\to} M, \rho, [.,.]_E, g_E)$ be a Courant algebroid equipped with a Courant algebroid connection ∇. Then the corresponding Riemann tensor R and the torsion tensor T satisfy the identity*

$$\begin{aligned} R(\psi_1,\psi_2,\psi_3,\psi_4) + cyc(\psi_2,\psi_3,\psi_4) = \frac{1}{2}\Big(&(\nabla_{\psi_2}T)(\psi_3,\psi_4,\psi_1) + T(\psi_1, T(\psi_2,\psi_3),\psi_4) \\ &+ cyc(\psi_2,\psi_3,\psi_4) - (\nabla_{\psi_1}T)(\psi_2,\psi_3,\psi_4)\Big), \end{aligned} \tag{1.80}$$

for all ψ_1, ψ_2, ψ_3, $\psi_4 \in \Gamma(E)$. In particular, if ∇ is torsion-free, one has

$$R(\psi_1,\psi_2,\psi_3,\psi_4) + cyc(\psi_2,\psi_3,\psi_4) = 0. \tag{1.81}$$

Proof. The proof is based on long technical calculations, see [2, Theorem 4.13]. □

We hope that aforementioned arguments are enough to convince the reader to accept the definition of a Courant algebroid equivalent of the ordinary Riemann tensor. Now let us define the already announced Courant algebroid version of the Ricci tensor and the Ricci scalar analogously as it is done in (semi-)Riemannian geometry.

Definition 1.31. Suppose $(E \overset{\pi}{\to} M, \rho, [.,.]_E, g_E)$ is a Courant algebroid equipped with a Courant algebroid connection ∇. By the **Ricci tensor** $\mathrm{Ric} \in \mathcal{T}_2^0(E)$ is understood the contraction of the Riemann tensor R in the first and the third argument with respect to the Courant metric g_E, that is

$$\mathrm{Ric}(\psi_1,\psi_2) := R(\sharp_E\,\xi^\mu, \psi_1, \xi_\mu, \psi_2), \tag{1.82}$$

for all ψ_1, $\psi_2 \in \Gamma(E)$. The **Courant-Ricci scalar** $\mathcal{R}_E \in C^\infty(M)$ is defined as the contracted Ricci tensor with respect to the Courant metric g_E, that is

$$\mathcal{R}_E := \mathrm{Ric}(\sharp_E\,\xi^\nu, \xi_\nu) \equiv R(\sharp_E\,\xi^\mu, \sharp_E\,\xi^\nu, \xi_\mu, \xi_\nu). \tag{1.83}$$

To finish this section, we examine a behaviour of the torsion and curvature under the connection preserving Courant algebroid isomorphisms.

Proposition 1.32. *Let $(E \overset{\pi}{\to} M, \rho, [.,.]_E, g_E)$ and $(E' \overset{\pi'}{\to} M, \rho', [.,.]_{E'}, g_{E'})$ be two Courant algebroids equipped with Courant algebroid connections ∇ and ∇' respectively. Moreover, assume that there is a connection preserving Courant algebroid isomorphism $\mathcal{F} : \Gamma(E) \to \Gamma(E')$ between them. Then the corresponding torsion and curvature tensors are related as*

$$T_\nabla = \mathcal{F}^* T_{\nabla'}, \qquad R_\nabla = \mathcal{F}^* R_{\nabla'}, \qquad \mathrm{Ric}_\nabla = \mathcal{F}^*\,\mathrm{Ric}_{\nabla'}, \qquad \mathcal{R}_E = \mathcal{R}_{E'}. \tag{1.84}$$

Proof. Using the relations (1.34) and (1.37) yields

$$\begin{aligned} R_{0\nabla}(\psi_1,\psi_2,\psi_3,\psi_4) =& g_E(\nabla_{\psi_3}\nabla_{\psi_4}\psi_2 - \nabla_{\psi_4}\nabla_{\psi_3}\psi_2 - \nabla_{[\psi_3,\psi_4]_E}\psi_2, \psi_1) \\ =& g_{E'}\Big(\mathcal{F}(\nabla_{\psi_3}\nabla_{\psi_4}\psi_2 - \nabla_{\psi_4}\nabla_{\psi_3}\psi_2 - \nabla_{[\psi_3,\psi_4]_E}\psi_2), \mathcal{F}\psi_1\Big) \\ =& g_{E'}\Big(\nabla'_{\mathcal{F}\psi_3}(\mathcal{F}\nabla_{\psi_4}\psi_2) - \nabla'_{\mathcal{F}\psi_4}(\mathcal{F}\nabla_{\psi_3}\psi_2) - \nabla'_{\mathcal{F}[\psi_3,\psi_4]_E}\mathcal{F}\psi_2, \mathcal{F}\psi_1\Big) \\ =& g_{E'}\Big(\nabla'_{\mathcal{F}\psi_3}\nabla'_{\mathcal{F}\psi_4}\mathcal{F}\psi_2 - \nabla'_{\mathcal{F}\psi_4}\nabla'_{\mathcal{F}\psi_3}\mathcal{F}\psi_2 - \nabla'_{[\mathcal{F}\psi_3,\mathcal{F}\psi_4]_{E'}}\mathcal{F}\psi_2, \mathcal{F}\psi_1\Big) \\ =& R_{0\nabla'}(\mathcal{F}\psi_1, \mathcal{F}\psi_2, \mathcal{F}\psi_3, \mathcal{F}\psi_4), \end{aligned} \tag{1.85}$$

for all ψ_1, ψ_2, ψ_3, $\psi_4 \in \Gamma(E)$. By using the relation (1.46) and the definition of a connection preserving Courant algebroid isomorphism, we obtain

$$\begin{aligned} g_E(\nabla_{\xi_\mu}\psi_3, \psi_4) g_E(\nabla_{\sharp_E \xi^\mu}\psi_2, \psi_1) &= g_{E'}(\mathcal{F}(\nabla_{\xi_\mu}\psi_3), \mathcal{F}\psi_4) g_{E'}(\mathcal{F}(\nabla_{\sharp_E \xi^\mu}\psi_2), \mathcal{F}\psi_1) \\ &= g_{E'}(\nabla'_{\mathcal{F}\xi_\mu}\mathcal{F}\psi_3, \mathcal{F}\psi_4) g_{E'}(\nabla'_{\sharp_{E'}(\mathcal{F}^{-1})^T\xi^\mu}\mathcal{F}\psi_2, \mathcal{F}\psi_1), \end{aligned} \tag{1.86}$$

for all ψ_1, ψ_2, ψ_3, $\psi_4 \in \Gamma(E)$. Therefore, combining (1.85) and (1.86) implies that the equation

$$R_\nabla(\psi_1, \psi_2, \psi_3, \psi_4) = R_{\nabla'}(\mathcal{F}\psi_1, \mathcal{F}\psi_2, \mathcal{F}\psi_3, \mathcal{F}\psi_4) \tag{1.87}$$

holds for all ψ_1, ψ_2, ψ_3, $\psi_4 \in \Gamma(E)$, which is exactly what is meant by the notation $R_\nabla = \mathcal{F}^* R_{\nabla'}$. Consequently, also the relations for the Ricci tensors and the Courant-Ricci scalars hold. The proof for the torsion tensors can be carried out in similar way as it has been done for the Riemann tensors. □

1.3 Generalized metric

In the previous section, Courant algebroid alternatives to well-known notions from the standard differential geometry have been introduced. In this section, we will deal with a so called generalized metric, whose alternative is not commonly used in standard (semi-)Riemannian geometry. Therefore, it may look rather artificially at first glance, but as we will see later, it is a very useful concept.

Definition 1.33. Let $E \xrightarrow{\pi} M$ be a vector bundle equipped with a fiber-wise metric g_E. By a **generalized metric** on E is understood a $C^\infty(M)$-module automorphism $\tau : \Gamma(E) \to \Gamma(E)$ such that $\tau^2 = \mathrm{Id}_{\Gamma(E)}$, and furthermore, the map $g_E(.\,, \tau\,.) \in \mathcal{T}_2^0(E)$ is a positive definite fiber-wise metric.

Remark 1.34. Since $g_E(.\,, \tau\,.)$ is a fiber-wise metric, it is first of all symmetric, hence a generalized metric is symmetric with respect to g_E. The symmetry together with the involutivity of τ implies also the orthogonality of a generalized metric with respect to g_E.

Let us formulate yet another and in general different definition of the generalized metric.

Definition 1.35. Let $E \xrightarrow{\pi} M$ be a vector bundle equipped with a fiber-wise metric g_E. Then any maximal positive definite subbundle $V_+ \subseteq E$ is called a **generalized metric** on E.

Remark 1.36. By a positive definite subbundle is understood a subbundle $V_+ \subseteq E$ such that $g_{Ep}\big|_{V_{+p} \times V_{+p}}$ is postive definite, for all $p \in M$. As we discussed in the remark 1.3, the signature of g_{Ep} is constant if the base manifold is connected. To avoid an unnecessary discussion, we will quite often assume that manifolds are connected. The fact that a subbundle is maximal means that there is not any positive subbundle with a higher rank.

As it is strange to have two different definitions of a generalized metric, let us point out the following theorem.

Theorem 1.37. *Under the assumption of a connected base manifold stated in the remark 1.36, there is a one-to-one correspondence between both definitions of a generalized metric.*

Lemma 1.38. *Consider a finite-dimensional real vector space V and a vector space endomorphism A satisfying $A^2 = \mathrm{Id}_V$. Then A is diagonalizable, and moreover, its spectrum is a subset of the set $\{-1, 1\}$.*

Proof of the lemma. The second part is trivial, since $A^2 = \mathrm{Id}_V$, A is a two-sided inverse of itself, hence A is an automorphism. Therefore, $\lambda \in \mathbb{R}$ solves the $Ax = \lambda x$ for some $x \in V$ if and only if it solves the $x = \lambda^2 x$, from this it follows that each eigenvalue of A must be either -1 or 1. Apparently, $\{\mathrm{Id}_V, A\}$ is a representation of the finite abelian group $(\mathbb{Z}_2, +)$ on V. It is an easy consequence of the Schur's lemma that every finite-dimensional representation of the finite abelian group consists of diagonalizable operators, hence A is diagonalizable. □

Proof of the theorem. Firstly, let us take an arbitrary generalized metric τ in the sense of the definition 1.33. Since $\tau : \Gamma(E) \to \Gamma(E)$ is a $C^\infty(M)$-module isomorphism, it induces a vector space isomorphism $\tau_p : E_p \to E_p$, for all $p \in M$, defined as $\tau_p v = (\tau\psi)(p)$, for all $v \in E_p$, where $\psi \in \Gamma(E)$ is an arbitrary section satisfying $\psi(p) = v$, such a section always exist see [11, Lemma 10.12]. Let us take an arbitrary point $p \in M$. It is clear from the definition of τ_p that $\tau_p^2 = \mathrm{Id}_{E_p}$. Then from the previous lemma, it follows that τ_p is diagonalizable and its spectrum is a subset of the $\{-1, 1\}$, hence there is a vector space decomposition

$$E_p = V_{+p} \oplus V_{-p}, \tag{1.88}$$

where V_{+p} and V_{-p} denote eigenspaces corresponding to the eigenvalues 1 and -1 respectively. Since $g_E(.\,, \tau\,.)$ is a positive definite fiber-wise metric on E, $g_{Ep}(.\,, \tau_p\,.)$ is a positive definite form on E_p, hence also $g_{Ep}|_{V_{+p}\times V_{+p}}$ is a positive definite form, because we have $\tau_p|_{V_{+p}} = \mathrm{Id}_{V_{+p}}$. The assumption of a connected base manifold ensures that $\dim(V_{+p})$ is the same for all $p \in M$, hence $V_+ := \bigsqcup_{p\in M} V_{+p}$ is a disjoint union of the vector spaces with a constant dimension. Since $V_{+p} = \mathrm{Ker}(\tau_p - \mathrm{Id}_{E_p})$ for all $p \in M$, $V_+ = \mathrm{Ker}(\tau - \mathrm{Id}_{\Gamma(E)})$ and $\tau - \mathrm{Id}_{\Gamma(E)}$ is apparently a $C^\infty(M)$-module morphism. Therefore, it follows from [11, Theorem 10.34] that V_+ forms a subbundle of E. As $\tau_p|_{V_{-p}} = -\mathrm{Id}_{V_{-p}}$ and $g_{Ep}(.\,, \tau_p\,.)$ is a positive definite form on E_p for all $p \in M$, $g_{Ep}|_{V_{-p}\times V_{-p}}$ is a negative definite form. Hence, there is no non-zero vector in V_{-p}, which could be added to V_{+p} to increase the dimension and at the same time to not spoil the positive definiteness. Consequently, the subbundle V_+ constructed in the manner just demonstrated is clearly maximal. Thus, we have just proven that for each generalized metric in the sense of the definition 1.33, we are able to unambiguously find a generalized metric in the sense of the definition 1.35.

Conversely, let us assume that V_+ is a generalized metric in the sense of the definition 1.35 and denote

$$V_- := \bigsqcup_{p\in M} V_{+p}^{\perp} \subseteq E, \tag{1.89}$$

where $V_{+p}^{\perp}$ is an orthogonal complement to V_{+p} with respect to g_{Ep}, for all $p \in M$. We would like V_- to form a vector subbundle of E. The idea is to use *Local frame criterion for subbundles*, see [11, Lemma 10.32]. It follows from the V_+ being a subbundle that V_{+p} has the same dimension, for all $p \in M$, hence also $V_{+p}^{\perp}$ has this property, thus the first assumption is satisfied. For all $p \in M$ there always exists an orthonormal[4] local frame $\{\xi_\mu\}_{\mu=1}^{\mathrm{Rank}(E)}$ of E over some neighbourhood $U \subseteq M$ of the point p, for a proof see [9, Lemma 2.14]. Hence, there are indices $\mu_1, \cdots, \mu_{\dim(V_{+p}^{\perp})} \in \{1, \cdots, \mathrm{Rank}(E)\}$ such that $(\xi_{\mu_1}(p), \cdots, \xi_{\mu_{\dim(V_{+p}^{\perp})}}(p))$ forms a basis for the $V_{+p}^{\perp}$. As $g_E|_U$ maps to the set of smooth, hence continuous, functions on U,

[4] With respect to the fiber-wise metric g_E.

$(\xi_{\mu_1}(q), \cdots, \xi_{\mu_{\dim(V_{+p}^{\perp})}}(q))$ forms a basis even for the $V_{+q}^{\perp}$, for all $q \in U$, thus also the second requirement is met. Consequently, V_- forms a subbundle of E. For all $p \in M$ the decomposition $E_p = V_{+p} \oplus V_{-p}$ induces a vector space isomorphism $\Phi_p : E_p \to V_{+p} \oplus V_{-p} : x \mapsto v + u$, where $v \in V_{+p}$ and $u \in V_{-p}$ are uniquely determined by the equality $x = v + u$. By the condition $\Phi|_{E_p} = \Phi_p$, $C^\infty(M)$-module isomorphism $\Phi : \Gamma(E) \to \Gamma(E)$ is defined. Apparently, $\operatorname{Im}\Phi = V_+ \oplus V_- := \bigsqcup_{p\in M}(V_{+p} \oplus V_{-p})$, and hence, it follows from the [11, Theorem 10.34] that $V_+ \oplus V_- \subseteq E$ is a vector subbundle of E. Furthermore, since Φ is an isomorphism, there holds

$$E = V_+ \oplus V_-. \tag{1.90}$$

Therefore, each section $\psi \in \Gamma(E)$ can be uniquely decomposed as $\psi = \psi_+ + \psi_-$, where $\psi_+ \in \Gamma(V_+)$ and $\psi_- \in \Gamma(V_-)$. Now, we are able to define an automorphism $\tau : \Gamma(E) \to \Gamma(E)$ as follows:

$$\tau\psi = \tau(\psi_+ + \psi_-) := \psi_+ - \psi_-, \tag{1.91}$$

for all $\psi \in \Gamma(E)$. Apparently, the identity $\tau^2 = \mathrm{Id}_{\Gamma(E)}$ is true. It remains to check if $g_E(\,.\,, \tau\,.\,)$ is a positive definite fiber-wise metric. For all $p \in M$ and all $u \in V_{-p}$, $u \neq 0$, we claim that

$$g_{E_p}(u, u) < 0. \tag{1.92}$$

To show that it is true, let us take an arbitrary point p and an arbitrary orthonormal basis for E_p, whose existence is well-known, see for example [12, Proposition 2.63]. The maximality of V_+ together with the connected manifold assumption implies that those basis vectors, whose norm squared is equal to -1, form a basis for the V_{-p}, hence (1.92) holds. Finally, for all $\psi \in \Gamma(E)$ there holds

$$g_E(\psi, \tau\psi) = g_E(\psi_+ + \psi_-, \tau(\psi_+ + \psi_-)) = g_E(\psi_+, \psi_+) - g_E(\psi_-, \psi_-), \tag{1.93}$$

where we used that $V_{-p} = V_{+p}^{\perp}$ for all $p \in M$ and also the definition of the map τ. Since (1.92) holds and V_+ is a positive definite subbundle, (1.93) implies that $g_E(\,.\,, \tau\,.\,)$ is positive definite fiber-wise metric. □

Remark 1.39. The generalized metric is apparently a non-trivial concept if and only if the fiber-wise metric is indefinite. For a positive- or a negative-definite one, the generalized metric is determined uniquely as $V_+ = E$ or $V_+ = 0$ respectively. In the indefinite case, a generalized metric always exists, however, there are infinitely many of them. Indeed, observe that the fiber decomposition $E_p = V_{+p} \oplus V_{+p}^{\perp}$ is clearly not unique in any point $p \in M$. Notice that the choice of the generalized metric induces a unique vector bundle decomposition $E = V_+ \oplus V_-$, where $\oplus$ is the Whitney sum and V_- is characterized by the condition

$$g_E(\psi_+, \psi_-) = 0, \tag{1.94}$$

for all $\psi_+ \in \Gamma(V_+)$ and $\psi_- \in \Gamma(V_-)$.

One feels there is still something missing, it is some motivation for the name of the notion generalized metric. This missing piece is given by the following and already the third definition of a generalized metric.

Definition 1.40. Let $E \xrightarrow{\pi} M$ be a vector bundle equipped with a fiber-wise metric g_E. We say that a positive definite fiber-wise metric G is a **generalized metric** on E if and only if for all ψ_1, $\psi_2 \in \Gamma(E)$ there holds

$$g_E^{-1}(\flat_G\, \psi_1, \flat_G\, \psi_2) = g_E(\psi_1, \psi_2). \tag{1.95}$$

Probably, it is not a big surprise that the following theorem is true.

Theorem 1.41. *Assuming a connected base manifold, all three definitions of generalized metric are mutually equivalent in the same sense as in the theorem 1.37.*

Proof. As we have already proven the equivalence between the definitions 1.33 and 1.35, it remains to prove that there is an equivalence between the definitions 1.40 and 1.33. Let us have a generalized metric τ in the sense of the definition 1.33 and impose

$$G(\psi_1, \psi_2) := g_E(\psi_1, \tau\psi_2), \tag{1.96}$$

for all $\psi_1,\ \psi_2 \in \Gamma(E)$. As G is a positive definite fiber-wise metric directly by the definition of τ, it remains to prove that the relation (1.95) holds. For all $\psi_1,\ \psi_2 \in \Gamma(E)$, we obtain

$$(\flat_G\, \psi_1)(\psi_2) = G(\psi_2, \psi_1) = g_E(\psi_2, \tau\psi_1) = (\flat_E\, \tau\psi_1)(\psi_2), \tag{1.97}$$

hence $\flat_G = \flat_E\, \tau$, and therefore

$$g_E^{-1}(\flat_G\, \psi_1, \flat_G\, \psi_2) = g_E^{-1}(\flat_E\, \tau\psi_1, \flat_E\, \tau\psi_2) = g_E(\tau\psi_1, \tau\psi_2) = g_E(\psi_1, \psi_2), \tag{1.98}$$

for all $\psi_1,\ \psi_2 \in \Gamma(E)$. In the last step, the orthogonality of τ is used. Conversely, assume that G is a positive definite fiber-wise metric satisfying the relation (1.95) and impose $\tau := \sharp_E\, \flat_G$. For all $\psi_1,\ \psi_2 \in \Gamma(E)$ there holds

$$g_E(\psi_1, \tau\psi_2) = g_E(\psi_1, \sharp_E\, \flat_G\, \psi_2) = (\flat_G\, \psi_2)(\psi_1) = G(\psi_1, \psi_2). \tag{1.99}$$

Therefore, it remains to show that $\tau^2 = \mathrm{Id}_{\Gamma(E)}$. As $\tau^2 = \sharp_E\, \flat_G\, \sharp_E\, \flat_G$, it is sufficient to show that $\flat_G\, \sharp_E\, \flat_G = \flat_E$. For an arbitrary ψ_1 and $\psi_2 \in \Gamma(E)$, one obtains

$$\begin{aligned}(\flat_E\, \psi_1)(\psi_2) &= g_E(\psi_1, \psi_2) = g_E^{-1}(\flat_G\, \psi_1, \flat_G\, \psi_2) = g_E(\sharp_E\, \flat_G\, \psi_1, \sharp_E\, \flat_G\, \psi_2) \\ &= (\flat_G\, \psi_2)(\sharp_E\, \flat_G\, \psi_1) = G(\psi_2, \sharp_E\, \flat_G\, \psi_1) = (\flat_G\, \sharp_E\, \flat_G\, \psi_1)(\psi_2),\end{aligned} \tag{1.100}$$

so indeed $\flat_G\, \sharp_E\, \flat_G = \flat_E$, and hence $\tau^2 = \mathrm{Id}_{\Gamma(E)}$. The fact that τ is an automorphism follows from $\tau^2 = \mathrm{Id}_{\Gamma(E)}$, because it apparently implies that τ has a two-sided inverse. ☐

Notation 1.42. Since all three definitions are equivalent, the name generalized metric will be used interchangeably for all three kinds of objects without any hesitation.

Remark 1.43. As we have already mentioned, by choosing a generalized metric one induces a vector bundle decomposition $E = V_+ \oplus V_-$, consequently one can visualise each tensor field $A \in \mathcal{T}_2^0(E)$ as a 2x2 matrix in the following way:

$$A = \begin{pmatrix} A|_{\Gamma(V_+)\times\Gamma(V_+)} & A|_{\Gamma(V_+)\times\Gamma(V_-)} \\ A|_{\Gamma(V_-)\times\Gamma(V_+)} & A|_{\Gamma(V_-)\times\Gamma(V_-)} \end{pmatrix}. \tag{1.101}$$

Especially, one can easily realize that g_E and G can be represented as

$$g_E = \begin{pmatrix} g_E^+ & 0 \\ 0 & g_E^- \end{pmatrix}, \qquad G = \begin{pmatrix} g_E^+ & 0 \\ 0 & -g_E^- \end{pmatrix}, \tag{1.102}$$

where $g_E^\pm := g_E|_{\Gamma(V_\pm)\times\Gamma(V_\pm)}$.

It was already aforementioned that if fiber-wise metric is indefinite, generalized metric is not given uniquely. The following proposition shows the way how another generalized metrics can be generated out of one specified.

Proposition 1.44. *Let $E \xrightarrow{\pi} M$ be a vector bundle equipped with a fiber-wise metric g_E and let G be a generalized metric on E. Then each $C^\infty(M)$-module automorphism $P : \Gamma(E) \to \Gamma(E)$, which is orthogonal with respect to g_E, defines a new generalized metric G_P as*

$$G_P(\psi_1, \psi_2) := G(P\psi_1, P\psi_2), \tag{1.103}$$

for all ψ_1, $\psi_2 \in \Gamma(E)$. Moreover, if $V_\pm$ and τ corresponds to G, then $V_{\pm P} := P^{-1}V_\pm$ and $\tau_P := P^{-1}\tau P$ corresponds to G_P.

Proof. Take an arbitrary orthogonal automorphism P, for all ψ_1, $\psi_2 \in \Gamma(E)$ there holds

$$(\flat_{G_P} \psi_1)\psi_2 = G_P(\psi_1, \psi_2) = G(P\psi_1, P\psi_2) = (\flat_G P\psi_1)(P\psi_2) = (P^T \flat_G P\psi_1)\psi_2, \tag{1.104}$$

hence $\flat_{G_P} = P^T \flat_G P$. Since $\flat_{G_P}$ is a composition of isomorphisms, it is also an isomorphism. Apparently, G_P is $C^\infty(M)$-bilinear and symmetric, thus it is a fiber-wise metric. It remains to check if it possesses the property (1.95), or equivalently[5] if $\flat_{G_P} \sharp_E \flat_{G_P} = \flat_E$ holds. Using the relations $P \sharp_E P^T = \sharp_E$ and $P^T \flat_E P = \flat_E$, which can be easily derived from the orthogonality of the automorphism P, yields

$$\flat_{G_P} \sharp_E \flat_{G_P} = p^T \flat_G P \sharp_E P^T \flat_G P = P^T \flat_G \sharp_E \flat_G P = P^T \flat_E P = \flat_E . \tag{1.105}$$

Let us move to the second part. For a given G_P, the corresponding τ_P is determined as $\tau_P = \sharp_E \flat_{G_P}$. In the former part of the proof, we have discovered the equality $\flat_{G_P} = P^T \flat_G P$. By putting these things together with the relation $P \sharp_E P^T = \sharp_E$, which comes from the orthogonality of P, we obtain

$$\tau_P = \sharp_E \flat_{G_P} = P^{-1} \sharp_E (P^T)^{-1} P^T \flat_G P = P^{-1} \sharp_E \flat_G P = P^{-1}\tau P. \tag{1.106}$$

It remains to check the transformation relations for $V_\pm$. As we have already proven the relation for τ_P and we know how $V_\pm$ and τ are related from the proof of the theorem 1.37, we can write

$$\begin{aligned} V_{\pm P} &= \mathrm{Ker}(\tau_P \mp \mathrm{Id}_{\Gamma(E)}) = \{\psi \in \Gamma(E) \mid P^{-1}\tau P\psi \mp \psi = 0\} \\ &= \{\psi \in \Gamma(E) \mid P^{-1}(\tau P\psi \mp P\psi) = 0\} = \{\psi \in \Gamma(E) \mid (\tau \mp \mathrm{Id}_{\Gamma(E)})P\psi = 0\} \\ &= P^{-1}\,\mathrm{Ker}(\tau \mp \mathrm{Id}_{\Gamma(E)}) = P^{-1}V_\pm, \end{aligned} \tag{1.107}$$

where we have used the fact that P is an automorphism in the fourth step. □

As the reader has probably noticed, we have been not working in the framework of Courant algebroids since the beginning of this section. Instead, we were assuming only a vector bundle equipped with a fiber-wise metric, what is actually a more general set-up. However, it is now time for a return to the world ruled by Courant algebroids, where, of course, all the previous results also apply. Similarly to a Courant algebroid connection, a generalized metric is also an additional structure on Courant algebroids. Therefore, we extend the concept of a Courant algebroid isomorphism even for it.

[5]See (1.100).

Definition 1.45. Let $(E \overset{\pi}{\to} M, \rho, [.,.]_E, g_E)$ and $(E' \overset{\pi'}{\to} M, \rho', [.,.]_{E'}, g_{E'})$ be two Courant algebroids equipped with generalized metrics G and G' respectively. A Courant algebroid isomorphism $\mathcal{F} : \Gamma(E) \to \Gamma(E')$ is called a **metric preserving Courant algebroid isomorphism** if and only if it relates their generalized metrics as

$$G(\psi_1, \psi_2) = G'(\mathcal{F}\psi_1, \mathcal{F}\psi_2), \tag{1.108}$$

for all ψ_1, $\psi_2 \in \Gamma(E)$.

Remark 1.46. If we modify the proof of proposition 1.44 by replacing P with $\mathcal{F}^{-1}$, we find out that $G'(\mathcal{F}.,\mathcal{F}.)$ actually defines a generalized metric on E', thus the previous definition makes sense. Furthermore, (1.108) can be equivalently stated as ${V'}_\pm = \mathcal{F}V_\pm$ or $\tau' = \mathcal{F}\tau\mathcal{F}^{-1}$.

Let us now investigate how Courant algebroid connections and generalized metrics interplay together. In particular, we will introduce two invariants of connection and metric preserving Courant algebroid isomorphisms, which will play crucial role in the main part of this thesis.

Definition 1.47. Suppose $(E \overset{\pi}{\to} M, \rho, [.,.]_E, g_E)$ is a Courant algebroid equipped with a Courant algebroid connection ∇ and a generalized metric G. The G-**Ricci scalar** $\mathcal{R}_G \in C^\infty(M)$ is defined as

$$\mathcal{R}_G := \operatorname{Ric}(\sharp_G \xi^\nu, \xi_\nu) \equiv R(\sharp_E \xi^\mu, \sharp_G \xi^\nu, \xi_\mu, \xi_\nu). \tag{1.109}$$

Proposition 1.48. *Let $(E \overset{\pi}{\to} M, \rho, [.,.]_E, g_E)$ and $(E' \overset{\pi'}{\to} M, \rho', [.,.]_{E'}, g_{E'})$ be two Courant algebroids equipped with Courant algebroid connections ∇ and ∇'; and generalized metrics G and G' respectively. Moreover, assume that there is a connection and metric preserving Courant algebroid isomorphism $\mathcal{F} : \Gamma(E) \to \Gamma(E')$ between them. Then*

$$\mathcal{R}_G = \mathcal{R}_{G'}. \tag{1.110}$$

Proof. Since $\mathcal{F}$ is a connection preserving Courant algebroid isomorphism, $\mathcal{F}^* \operatorname{Ric}_{\nabla'} = \operatorname{Ric}_\nabla$, see proposition 1.32. From the fact that $\mathcal{F}$ is even a metric preserving Courant algebroid isomorphism, it follows that $\mathcal{F}^T \flat_{G'} \mathcal{F} = \flat_G$, or equivalently $\mathcal{F}^{-1} \sharp_{G'} (\mathcal{F}^{-1})^T = \sharp_G$. Putting these together yields

$$\begin{aligned}\mathcal{R}_G &= \operatorname{Ric}_\nabla(\sharp_G \xi^\mu, \xi_\mu) = \operatorname{Ric}_\nabla(\mathcal{F}^{-1}\sharp_{G'} (\mathcal{F}^{-1})^T \xi^\mu, \xi_\mu) = \operatorname{Ric}_{\nabla'}(\sharp_{G'} (\mathcal{F}^{-1})^T \xi^\mu, \mathcal{F}\xi_\mu) \\ &= \mathcal{R}_{G'}.\end{aligned} \tag{1.111}$$

The last step was already discussed within the proof of proposition 1.23. □

Definition 1.49. Let $(E \overset{\pi}{\to} M, \rho, [.,.]_E, g_E)$ be a Courant algebroid equipped with a Courant algebroid connection ∇ and a generalized metric V_+. We say that ∇ is **Ricci compatible** with V_+ if and only if

$$\operatorname{Ric}|_{\Gamma(V_+)\times\Gamma(V_-)} = 0. \tag{1.112}$$

Proposition 1.50. *Let $(E \overset{\pi}{\to} M, \rho, [.,.]_E, g_E)$ and $(E' \overset{\pi'}{\to} M, \rho', [.,.]_{E'}, g_{E'})$ be two Courant algebroids equipped with Courant algebroid connections ∇ and ∇'; and generalized metrics V_+ and ${V'}_+$ respectively. Moreover, assume that there is a connection and metric preserving Courant algebroid isomorphism $\mathcal{F} : \Gamma(E) \to \Gamma(E')$ between them. Then ∇ is Ricci compatible with V_+ if and only if ∇' is Ricci compatible with ${V'}_+$.*

Proof. Firstly, the fact that $\mathcal{F}$ is a connection preserving Courant algebroid isomorphism ensures that $\mathcal{F}^* \operatorname{Ric}_{\nabla'} = \operatorname{Ric}_{\nabla}$. Hence, for all $\psi_+ \in \Gamma(V_+)$ and $\psi_- \in \Gamma(V_-)$

$$\operatorname{Ric}_{\nabla}(\psi_+, \psi_-) = \operatorname{Ric}_{\nabla'}(\mathcal{F}\psi_+, \mathcal{F}\psi_-). \tag{1.113}$$

Secondly, as $\mathcal{F}$ is even a metric preserving Courant algebroid isomorphism, it follows from the remark 1.46 that generalized metrics are related as $V'_\pm = \mathcal{F}V_\pm$, which completes the proof. □

1.4 Levi-Civita Courant algebroid connections

In (semi-)Riemannian geometry, there is a prominent affine connection called the Levi-Civita affine connection, which is unambiguously defined by two characteristic properties, the torsion-freeness and the metric compatibility. For a Courant algebroid equipped with a Courant algebroid connection and a generalized metric, it is thus natural to think about a Courant algebroid alternative to the Levi-Civita affine connection. It is pretty clear how thw torsion-freeness should be modified. However, as we have stated three formally different definitions of a generalized metric, the metric compatibility can be formulated in various ways.

Proposition 1.51. *Let* $(E \xrightarrow{\pi} M, \rho, [.,.]_E, g_E)$ *be a Courant algebroid equipped with a Courant algebroid connection* ∇ *and a generalized metric* G*. Then the following statements are equivalent:*

(I) For all ψ_1*,* ψ_2*,* $\psi_3 \in \Gamma(E)$ *there holds*

$$\rho(\psi_1)G(\psi_2, \psi_3) = G(\nabla_{\psi_1}\psi_2, \psi_3) + G(\psi_2, \nabla_{\psi_1}\psi_3). \tag{1.114}$$

(II) For all $\psi \in \Gamma(E)$ *there holds*

$$\nabla_\psi \circ \tau = \tau \circ \nabla_\psi. \tag{1.115}$$

(III) For all $\psi \in \Gamma(E)$ *there holds*

$$\nabla_\psi(\Gamma(V_+)) \subseteq \Gamma(V_+). \tag{1.116}$$

Proof. Let us begin with the implication *(I)* ⇒ *(II)*. Since G and τ are related by (1.96), *(I)* is equivalent to that the equation

$$\rho(\psi_1)g_E(\psi_2, \tau\psi_3) = g_E(\nabla_{\psi_1}\psi_2, \tau\psi_3) + g_E(\psi_2, \tau\nabla_{\psi_1}\psi_3), \tag{1.117}$$

is valid for all ψ_1, ψ_2, $\psi_3 \in \Gamma(E)$. If we add and subtract the term $g_E(\psi_2, \nabla_{\psi_1}\tau\psi_3)$ to the right-hand side and use the third axiom of a Courant algebroid connection, we obtain

$$g_E(\psi_2, \tau\nabla_{\psi_1}\psi_3 - \nabla_{\psi_1}\tau\psi_3) = 0, \tag{1.118}$$

for all ψ_1, ψ_2, $\psi_3 \in \Gamma(E)$. Hence and from the fact that $\flat_E$ is an isomorphism, the desired identity follows.

To prove the implication *(II)* ⇒ *(III)*, realize that $\psi \in \Gamma(V_+)$ if and only if $\tau\psi = \psi$. Since we are assuming that τ and covariant derivative commutes, it is a trivial observation that the condition *(III)* is fulfilled.

It remains to prove that the implication *(III)* ⇒ *(I)* holds. First of all, we will need to prove that also $\nabla_\psi(\Gamma(V_-)) \subseteq \Gamma(V_-)$ is true for all $\psi \in \Gamma(E)$. Take an arbitrary $\psi \in \Gamma(E)$ and

$\psi'' \in \nabla_\psi(\Gamma(V_-))$, hence there is $\psi' \in \Gamma(V_-)$ such that $\psi'' = \nabla_\psi \psi'$. For all $\psi_+ \in \Gamma(V_+)$, the term $g_E(\psi'', \psi_+)$ can be expanded as follows:

$$g_E(\psi'', \psi_+) = g_E(\nabla_\psi \psi', \psi_+) = \rho(\psi) g_E(\psi', \psi_+) - g_E(\psi', \nabla_\psi \psi_+) = 0, \tag{1.119}$$

which means that $\psi'' \in \Gamma(V_-)$. With this result in our minds and by using the compatibility of ∇ with g_E, we can proceed for all ψ_1, ψ_2, $\psi_3 \in \Gamma(E)$ in the following way:

$$\begin{aligned}
&\rho(\psi_1)G(\psi_2,\psi_3) - G(\nabla_{\psi_1}\psi_2,\psi_3) - G(\psi_2,\nabla_{\psi_1}\psi_3)\\
&\quad = \rho(\psi_1)g_E(\psi_2,\tau\psi_3) - g_E(\nabla_{\psi_1}\psi_2,\tau\psi_3) - g_E(\psi_2,\tau\nabla_{\psi_1}\psi_3)\\
&\quad = \rho(\psi_1)g_E(\psi_2,\psi_{3+}-\psi_{3-}) - g_E(\nabla_{\psi_1}\psi_2,\psi_{3+}-\psi_{3-}) - g_E(\psi_2,\tau\nabla_{\psi_1}(\psi_{3+}+\psi_{3-}))\\
&\quad = \rho(\psi_1)g_E(\psi_2,\psi_{3+}) - g_E(\nabla_{\psi_1}\psi_2,\psi_{3+}) - g_E(\psi_2,\nabla_{\psi_1}\psi_{3+})\\
&\qquad - \rho(\psi_1)g_E(\psi_2,\psi_{3-}) + g_E(\nabla_{\psi_1}\psi_2,\psi_{3-}) + g_E(\psi_2,\nabla_{\psi_1}\psi_{3-}) = 0,
\end{aligned} \tag{1.120}$$

which is exactly the formula to be proven. □

Definition 1.52. Let all assumptions of the previous proposition hold. We say that a Courant algebroid connection ∇ is **compatible with the generalized metric** G if and only if one of the conditions (1.114), (1.115) or (1.116) is satisfied.

Proposition 1.53. *Let $(E \overset{\pi}{\to} M, \rho, [.,.]_E, g_E)$ be a Courant algebroid. For an arbitrary generalized metric V_+, there is a Courant algebroid connection on E compatible with such V_+.*

Proof. Take an arbitrary generalized metric V_+. The idea of the proof is not very complicated, we simply construct a connection satisfying appropriate requirements. We have already discussed, see the paragraph under the remark 1.17, that on every vector bundle equipped with a fiber-wise metric, there is a vector bundle connection compatible with the fiber-wise metric. Since $(V_\pm \overset{\pi|_{V_\pm}}{\to} M, g_E^\pm)$ can be clearly considered as this kind of vector bundles, we have some vector bundle connections $\nabla^\pm$ compatible with $g_E^\pm$ on $V_\pm$ at our disposal. Next, impose

$$\nabla_{\psi_1}\psi_2 := {\nabla^+}_{\rho(\psi_1)}\psi_{2+} + {\nabla^-}_{\rho(\psi_1)}\psi_{2-}, \tag{1.121}$$

for all ψ_1, $\psi_2 \in \Gamma(E)$. We claim that ∇ is a Courant algebroid connection compatible with V_+. It can be easily checked that ∇ is $\mathbb{R}$-bilinear and that it satisfies the first two axioms of a Courant algebroid connection. It thus remains to show the compatibility with the Courant metric g_E and the generalized metric V_+. The latter is trivial, since the condition (1.116) is apparently satisfied. To prove the former, take arbitrary ψ_1, ψ_2, $\psi_3 \in \Gamma(E)$ and proceed as follows:

$$\begin{aligned}
&g_E(\nabla_{\psi_1}\psi_2,\psi_3) + g_E(\psi_2,\nabla_{\psi_1}\psi_3)\\
&\quad = g_E({\nabla^+}_{\rho(\psi_1)}\psi_{2+} + {\nabla^-}_{\rho(\psi_1)}\psi_{2-},\psi_3) + g_E(\psi_2,{\nabla^+}_{\rho(\psi_1)}\psi_{3+} + {\nabla^-}_{\rho(\psi_1)}\psi_{3-})\\
&\quad = g_E^+({\nabla^+}_{\rho(\psi_1)}\psi_{2+},\psi_{3+}) + g_E^-({\nabla^-}_{\rho(\psi_1)}\psi_{2-},\psi_{3-}) + g_E^+(\psi_{2+},{\nabla^+}_{\rho(\psi_1)}\psi_{3+})\\
&\qquad + g_E^-(\psi_{2-},{\nabla^-}_{\rho(\psi_1)}\psi_{3-})\\
&\quad = \rho(\psi_1)g_E^+(\psi_{2+},\psi_{3+}) + \rho(\psi_1)g_E^-(\psi_{2-},\psi_{3-})\\
&\quad = \rho(\psi_1)\Big(g_E(\psi_{2+},\psi_{3+}) + g_E(\psi_{2-},\psi_{3-}) + g_E(\psi_{2+},\psi_{3-}) + g_E(\psi_{2-},\psi_{3+})\Big)\\
&\quad = \rho(\psi_1)g_E(\psi_2,\psi_3).
\end{aligned} \tag{1.122}$$

We have used the mutual orthogonality of $\Gamma(V_+)$ and $\Gamma(V_-)$ with respect to g_E in the second and fourth step; and the compatibility of $\nabla^\pm$ with $g_E^\pm$ in the third step. □

Definition 1.54. Consider a Courant algebroid $(E \overset{\pi}{\to} M, \rho, [.,.]_E, g_E)$ equipped with a Courant algebroid connection ∇ and a generalized metric G. We say that ∇ is a **Levi-Civita Courant algebroid connection** on E with respect to the generalized metric G if and only if it is torsion-free, that is $T = 0$, and at the same time it is compatible with G.

One could be curious about the existence of this kind of Courant algebroid connection. The answer is that Levi-Civita Courant algebroid connection always exists, moreover, we will show how to construct it out of the given generalized metric compatible Courant algebroid connection, whose existence is guaranteed by the proposition 1.53.

Theorem 1.55 (Existence of Levi-Civita Courant algebroid connection)**.** *Consider a Courant algebroid $(E \overset{\pi}{\to} M, \rho, [.,.]_E, g_E)$ equipped with a generalized metric G. Then there is a Levi-Civita Courant algebroid connection on E with respect to G.*

Lemma 1.56. *Consider a Courant algebroid $(E \overset{\pi}{\to} M, \rho, [.,.]_E, g_E)$ and a fixed Courant algebroid connection ∇^0 on E. Then for every Courant algebroid connection ∇ on E there is $C \in \Omega^1(E) \otimes \Omega^2(E)$ such that the relation*

$$\nabla_{\psi_1}\psi_2 = \nabla^0_{\psi_1}\psi_2 + \sharp_E\, C(\psi_1, \psi_2, .) \tag{1.123}$$

is satisfied, for all $\psi_1,\ \psi_2 \in \Gamma(E)$.

Proof of the lemma. Assume two arbitrary Courant algebroid connections ∇ and ∇^0 and denote their difference contracted by the Courant metric as C, that is

$$C(\psi_1, \psi_2, \psi_3) := g_E(\nabla_{\psi_1}\psi_2 - \nabla^0_{\psi_1}\psi_2, \psi_3), \tag{1.124}$$

for all $\psi_1,\ \psi_2,\ \psi_3 \in \Gamma(E)$. Apparently, it is $C^\infty(M)$-linear in the first and third input. Let us check the $C^\infty(M)$-linearity in the second argument, for arbitrary $\psi_1,\ \psi_2,\ \psi_3 \in \Gamma(E)$ and $f \in C^\infty(M)$ there holds

$$\begin{aligned} C(\psi_1, f\psi_2, \psi_3) &= g_E(\nabla_{\psi_1}(f\psi_2) - \nabla^0_{\psi_1}(f\psi_2), \psi_3) \\ &= g_E((\rho(\psi_1)f)\psi_2 + f\nabla_{\psi_1}\psi_2 - (\rho(\psi_1)f)\psi_2 - f\nabla^0_{\psi_1}\psi_2, \psi_3) \\ &= f g_E(\nabla_{\psi_1}\psi_2 - \nabla^0_{\psi_1}\psi_2, \psi_3) \\ &= fC(\psi_1, \psi_2, \psi_3). \end{aligned} \tag{1.125}$$

As opposed to a Courant algebroid connection itself, the difference of two Courant algebroid connections contracted by a fiber-wise metric is $C^\infty(M)$-linear in all three inputs, that is $C \in \mathcal{T}^0_3(E)$. The first two Courant algebroid connection axioms apparently do not impose any other constraints on C, however, there is also the third axiom, compatibility with the Courant metric. It says that for all $\psi_1,\ \psi_2,\ \psi_3 \in \Gamma(E)$ there holds

$$\begin{aligned} 0 &= \rho(\psi_1)g_E(\psi_2, \psi_3) - g_E(\nabla_{\psi_1}\psi_2, \psi_3) - g_E(\psi_2, \nabla_{\psi_1}\psi_3) \\ &= \rho(\psi_1)g_E(\psi_2, \psi_3) - g_E(\nabla^0_{\psi_1}\psi_2, \psi_3) - C(\psi_1, \psi_2, \psi_3) - g_E(\psi_2, \nabla^0_{\psi_1}\psi_3) - C(\psi_1, \psi_3, \psi_2) \\ &= -\, C(\psi_1, \psi_2, \psi_3) - C(\psi_1, \psi_3, \psi_2), \end{aligned} \tag{1.126}$$

hence C is necessarily skew-symmetric in the last two inputs, that is $C \in \Omega^1(E) \otimes \Omega^2(E)$. Since $\flat_E$ is a $C^\infty(M)$-module isomorphism, one can write

$$\nabla_{\psi_1}\psi_2 = \nabla^0_{\psi_1}\psi_2 + \sharp_E\, C(\psi_1, \psi_2, .), \tag{1.127}$$

for all $\psi_1,\ \psi_2 \in \Gamma(E)$. □

Proof of the theorem. As we have already proven, see 1.53, there is a Courant algebroid connection compatible with the generalized metric G, assume that we have one and denote it as ∇^0. The lemma implies that every other Courant algebroid connection (thus also even a Levi-Civita one) has to be necessarily of the form (1.123). Let us now investigate, what conditions should be imposed on $C \in \Omega^1(E) \otimes \Omega^2(E)$ to ensure that ∇ is already Levi-Civita.

Firstly, ∇ is compatible with G if and only if

$$\begin{aligned} 0 &= \rho(\psi_1)G(\psi_2, \psi_3) - G(\nabla_{\psi_1}\psi_2, \psi_3) - G(\psi_2, \nabla_{\psi_1}\psi_3) \\ &= \rho(\psi_1)G(\psi_2, \psi_3) - G(\nabla^0_{\psi_1}\psi_2, \psi_3) - G(\sharp_E\, C(\psi_1, \psi_2, .), \psi_3) - G(\psi_2, \nabla^0_{\psi_1}\psi_3) \\ &\quad - G(\psi_2, \sharp_E\, C(\psi_1, \psi_3, .)) \\ &= -\, g_E(\sharp_E\, C(\psi_1, \psi_2, .), \tau\psi_3) - g_E(\tau\psi_2, \sharp_E\, C(\psi_1, \psi_3, .)) \\ &= -\, C(\psi_1, \psi_2, \tau\psi_3) - C(\psi_1, \psi_3, \tau\psi_2), \end{aligned} \tag{1.128}$$

is satisfied for all ψ_1, ψ_2, $\psi_3 \in \Gamma(E)$. During the derivation, we have used compatibility of ∇^0 with G and relation between G and τ in the third step. We claim that the condition (1.128) is satisfied if and only if the condition

$$C|_{\Gamma(E)\times\Gamma(V_+)\times\Gamma(V_-)} = 0 \tag{1.129}$$

holds. Assume that (1.128) holds and take arbitrary $\psi \in \Gamma(E)$, $\psi_+ \in \Gamma(V_+)$ and $\psi_- \in \Gamma(V_-)$, it yields

$$\begin{aligned} 0 = -C(\psi, \psi_+, \tau\psi_-) - C(\psi, \psi_-, \tau\psi_+) \qquad &\Leftrightarrow \qquad C(\psi, \psi_+, \psi_-) = -C(\psi, \psi_+, \psi_-) \\ &\Leftrightarrow \qquad C(\psi, \psi_+, \psi_-) = 0, \end{aligned}$$

hence $C|_{\Gamma(E)\times\Gamma(V_+)\times\Gamma(V_-)} = 0$. Conversely, assume that (1.129) is satisfied and take arbitrary ψ_1, ψ_2, $\psi_3 \in \Gamma(E)$, the relation between $V_\pm$ and τ implies the following:

$$\begin{aligned} C(\psi_1, \psi_2, \tau\psi_3) &= C(\psi_1, \psi_{2+} + \psi_{2-}, \psi_{3+} - \psi_{3-}) = C(\psi_1, \psi_{2+}, \psi_{3+}) - C(\psi_1, \psi_{2-}, \psi_{3-}) \\ &= C(\psi_1, \psi_{2+}, \psi_{3+}) - C(\psi_1, \psi_{2-}, \psi_{3-}) + C(\psi_1, \psi_{2+}, \psi_{3-}) - C(\psi_1, \psi_{2-}, \psi_{3+}) \\ &= C(\psi_1, \tau\psi_2, \psi_3) = -C(\psi_1, \psi_3, \tau\psi_2), \end{aligned} \tag{1.130}$$

besides (1.129) we have also used the skew-symmetry of C in the last two inputs. Therefore, ∇ is compatible with G if and only if (1.129) is satisfied.

Next, examine the condition for the torsion-freeness of ∇. Take arbitrary ψ_1, ψ_2, $\psi_3 \in \Gamma(E)$, then $T_\nabla(\psi_1, \psi_2, \psi_3) = 0$ if and only if

$$\begin{aligned} 0 &= g_E(\nabla_{\psi_1}\psi_2 - \nabla_{\psi_2}\psi_1 - [\psi_1, \psi_2]_E, \psi_3) + g_E(\nabla_{\psi_3}\psi_1, \psi_2) \\ &= T_{\nabla^0}(\psi_1, \psi_2, \psi_3) + C(\psi_1, \psi_2, \psi_3) - C(\psi_2, \psi_1, \psi_3) + C(\psi_3, \psi_1, \psi_2), \end{aligned} \tag{1.131}$$

or equivalently

$$C(\psi_1, \psi_2, \psi_3) + cyc(\psi_1, \psi_2, \psi_3) = -T_{\nabla^0}(\psi_1, \psi_2, \psi_3). \tag{1.132}$$

Finally, it remains to find an explicit $C \in \Omega^1(E)\otimes\Omega^2(E)$, which satisfies (1.129) and (1.132). As we have the decomposition $E = V_+ \oplus V_-$ at our disposal, any tensor field of the rank 3 is fully determined by eight of its restrictions. While the compatibility with the generalized metric

(1.129) together with the skew-symmetry in the last two inputs say that four of these restrictions vanish identically, the torsion-freeness (1.132) implies that

$$C(\psi_{1\pm},\psi_{2\mp},\psi_{3\mp})+C(\psi_{2\mp},\psi_{3\mp},\psi_{1\pm})+C(\psi_{3\mp},\psi_{1\pm},\psi_{2\mp})=-T_{\nabla^0}(\psi_{1\pm},\psi_{2\mp},\psi_{3\mp}), \tag{1.133}$$
$$C(\psi_{1\pm},\psi_{2\pm},\psi_{3\pm})+C(\psi_{2\pm},\psi_{3\pm},\psi_{1\pm})+C(\psi_{3\pm},\psi_{1\pm},\psi_{2\pm})=-T_{\nabla^0}(\psi_{1\pm},\psi_{2\pm},\psi_{3\pm}), \tag{1.134}$$

for all $\psi_{1\pm}$, $\psi_{2\pm}$, $\psi_{3\pm}\in\Gamma(V_\pm)$. The second and the third term on the left-hand side of the first equation vanishes identically because of (1.129), hence

$$C|_{\Gamma(V_\pm)\times\Gamma(V_\mp)\times\Gamma(V_\mp)}=-\,T_{\nabla^0}|_{\Gamma(V_\pm)\times\Gamma(V_\mp)\times\Gamma(V_\mp)}\,. \tag{1.135}$$

It follows from the complete skew-symmetry of the torsion tensor that the second equation (1.134) can be equivalently written as

$$\begin{aligned}C(\psi_{1\pm},\psi_{2\pm},\psi_{3\pm})&+cyc(\psi_{1\pm},\psi_{2\pm},\psi_{3\pm})\\&=-\frac{1}{3}\Big(T_{\nabla^0}(\psi_{1\pm},\psi_{2\pm},\psi_{3\pm})+cyc(\psi_{1\pm},\psi_{2\pm},\psi_{3\pm})\Big),\end{aligned} \tag{1.136}$$

which can be apparently solved by imposing

$$C|_{\Gamma(V_\pm)\times\Gamma(V_\pm)\times\Gamma(V_\pm)}:=-\frac{1}{3}\,T_{\nabla^0}|_{\Gamma(V_\pm)\times\Gamma(V_\pm)\times\Gamma(V_\pm)}\,. \tag{1.137}$$

Putting (1.129), (1.135) and (1.137) together leads to the following choice for C:

$$\begin{aligned}C(\psi_1,\psi_2,\psi_3):=&-\frac{1}{3}(T_{\nabla^0}(\psi_{1+},\psi_{2+},\psi_{3+})+T_{\nabla^0}(\psi_{1-},\psi_{2-},\psi_{3-}))\\&-T_{\nabla^0}(\psi_{1+},\psi_{2-},\psi_{3-})-T_{\nabla^0}(\psi_{1-},\psi_{2+},\psi_{3+}),\end{aligned} \tag{1.138}$$

for all ψ_1, ψ_2, $\psi_3\in\Gamma(E)$. The tensor field C defined in this way is apparently $C^\infty(M)$-linear in all three arguments, moreover, it is skew-symmetric in the last two inputs, that is $C\in\Omega^1(E)\otimes\Omega^2(E)$. Therefore, an $\mathbb{R}$-bilinear map $\nabla:\Gamma(E)\times\Gamma(E)\to\Gamma(E)$ defined for all ψ_1, $\psi_2\in\Gamma(E)$ as

$$\nabla_{\psi_1}\psi_2=\nabla^0_{\psi_1}\psi_2+\sharp_E\,C(\psi_1,\psi_2,.), \tag{1.139}$$

where C is given by the (1.138) and ∇^0 is an arbitrary Courant algebroid connection compatible with G, is a Levi-Civita Courant algebroid connection with respect to G. □

Remark 1.57. Note that the equation (1.138) can be even simplified. Since ∇^0 is generalized metric compatible, for all $\psi\in\Gamma(E)$ there holds $\nabla^0_\psi(\Gamma(V_\pm))\subseteq\Gamma(V_\pm)$, hence

$$\begin{aligned}T_{\nabla^0}(\psi_{1\pm},\psi_{2\mp},\psi_{3\mp})&=g_E(\nabla^0_{\psi_{1\pm}}\psi_{2\mp}-\nabla^0_{\psi_{2\mp}}\psi_{1\pm}-[\psi_{1\pm},\psi_{2\mp}]_E,\psi_{3\mp})+g_E(\nabla^0_{\psi_{3\mp}}\psi_{1\pm},\psi_{2\mp})\\&=g_E(\nabla^0_{\psi_{1\pm}}\psi_{2\mp}-[\psi_{1\pm},\psi_{2\mp}]_E,\psi_{3\mp}).\end{aligned} \tag{1.140}$$

Therefore, (1.138) can be equivalently expressed as

$$\begin{aligned}C(\psi_1,\psi_2,\psi_3)=&-\frac{1}{3}(T_{\nabla^0}(\psi_{1+},\psi_{2+},\psi_{3+})+T_{\nabla^0}(\psi_{1-},\psi_{2-},\psi_{3-}))\\&+g_E([\psi_{1+},\psi_{2-}]_E-\nabla^0_{\psi_{1+}}\psi_{2-},\psi_{3-})\\&+g_E([\psi_{1-},\psi_{2+}]_E-\nabla^0_{\psi_{1-}}\psi_{2+},\psi_{3+}).\end{aligned} \tag{1.141}$$

A Levi-Civita Courant algebroid connection constructed out of ∇^0 by this particular choice of C is called the **minimal Levi-Civita Courant algebroid connection** corresponding to ∇^0.

Remark 1.58. Thanks to the lemma 1.56, we can easily determine how covariant divergences of two arbitrary Courant algebroid connections are related, one immediately sees that there holds

$$\operatorname{div}_{\nabla} = \operatorname{div}_{\nabla^0} - \mathcal{C}, \tag{1.142}$$

where $\mathcal{C} := C(\sharp_E \, \xi^\mu, \xi_\mu, .) \in \Omega^1(E)$.

Once we have solved the question of the Levi-Civita Courant algebroid connection existence, another question arises. Is it determined uniquely by the generalized metric, as it is in the case of standard (semi-)Riemannian geometry? The answer is no, except for certain special cases.

Theorem 1.59 (Non-uniqueness of Levi-Civita Courant algebroid connection)**.** *Consider a Courant algebroid* $(E \overset{\pi}{\to} M, \rho, [.\,,.]_E, g_E)$ *equipped with a generalized metric* G *and denote the set of all Levi-Civita Courant algebroid connections on* E *as* $\mathrm{LC}(E, G)$*. Then* $\mathrm{LC}(E, G)$ *is an affine space, whose associated vector space forms a module of sections of a vector bundle of the rank*

$$\frac{1}{3}p(p^2 - 1) + \frac{1}{3}q(q^2 - 1), \tag{1.143}$$

where (p, q) *is the signature of the Courant metric* g_E*.*

Proof. The previous theorem says that $\mathrm{LC}(E, G) \neq \emptyset$, so take an arbitrary $\nabla^0 \in \mathrm{LC}(E, G)$. The lemma 1.56 implies that any other Courant algebroid connection ∇ can be written in the form (1.123) for some $C \in \Omega^1(E) \otimes \Omega^2(E)$. Since we are interested only in Levi-Civita Courant algebroid connections, there are some additional constraints on C. The compatibility with generalized metric G implies

$$C|_{\Gamma(E)\times\Gamma(V_+)\times\Gamma(V_-)} = 0, \tag{1.144}$$

as we have shown during the proof of theorem 1.55. Since ∇^0 is now not only generalized metric compatible but even Levi-Civita, the torsion-freeness imposes the following condition:

$$C(\psi_1, \psi_2, \psi_3) + cyc(\psi_1, \psi_2, \psi_3) = 0, \tag{1.145}$$

for all ψ_1, ψ_2, $\psi_3 \in \Gamma(E)$, see the equation (1.132). The property (1.145) is under the assumption $C \in \Omega^1(E) \otimes \Omega^2(E)$ equivalent to $C_A = 0$. Assume that we have a $C \in \Omega^1(E) \otimes \Omega^2(E)$ satisfying both (1.144) and (1.145), then for all $\psi_{1-} \in \Gamma(V_-)$ and for all ψ_{2+}, $\psi_{3+} \in \Gamma(V_+)$ there holds

$$C(\psi_{1-}, \psi_{2+}, \psi_{3+}) + cyc(\psi_{1-}, \psi_{2+}, \psi_{3+}) = 0, \tag{1.146}$$

hence it follows from (1.144) that $C|_{\Gamma(V_-)\times\Gamma(V_+)\times\Gamma(V_+)} = 0$. Analogously, one can derive that $C|_{\Gamma(V_+)\times\Gamma(V_-)\times\Gamma(V_-)} = 0$. Therefore, there are only two non-trivial restrictions of C and those are $C|_{\Gamma(V_\pm)\times\Gamma(V_\pm)\times\Gamma(V_\pm)}$. Each of these restrictions is an element of the $C^\infty(M)$-module $\Omega^1(V_\pm) \otimes \Omega^2(V_\pm)$, and moreover, since their cyclic permutations or equivalently its complete skew-symmetrizations vanish, we have to exclude all completely skew-symmetric elements, that is elements of $\Omega^3(V_\pm)$. Consequently, every $C \in \Omega^1(E) \otimes \Omega^2(E)$ satisfying conditions (1.144) and (1.145) can be considered as an element of the $C^\infty(M)$-module

$$(\Omega^1(V_+) \otimes \Omega^2(V_+)) \big/ \Omega^3(V_+) \bigoplus (\Omega^1(V_-) \otimes \Omega^2(V_-)) \big/ \Omega^3(V_-). \tag{1.147}$$

It is a trivial task to show the converse. Therefore, it has been just proven that if we have some fixed $\nabla^0 \in \mathrm{LC}(E, G)$, every other $\nabla \in \mathrm{LC}(E, G)$ can be for all ψ_1, $\psi_2 \in \Gamma(E)$ expressed as

$$\nabla_{\psi_1}\psi_2 = \nabla^0_{\psi_1}\psi_2 + \sharp_E \, C(\psi_1, \psi_2, .), \tag{1.148}$$

where C is an arbitrary element of the $C^\infty(M)$-module (1.147), hence $\mathrm{LC}(E,G)$ is an affine space. To complete the proof it remains to compute the rank of the associated module of sections of the vector bundle. The fact that (p,q) is the signature of g_E implies that $\mathrm{Rank}(V_+) = p$ and $\mathrm{Rank}(V_-) = q$, the rest is just straightforward calculation. □

Remark 1.60. It is now clear that a Levi-Civita Courant algebroid connection is not uniquely determined by the generalized metric in general, moreover, there are infinitely many of them. An exception is the case $p, q \in \{0,1\}$, in which there is exactly one Levi-Civita Courant algebroid connection.

Notation 1.61. As a tensor field $C \in \Omega^1(E) \otimes \Omega^2(E)$ defined by the lemma 1.56, will arise a lot in the remaining part of this thesis, we introduce the following notation:

(I) $C \in \Omega^1(E) \otimes \Omega^2(E)$ denotes a difference between a Levi-Civita Courant algebroid connection and a Courant algebroid connection compatible with a generalized metric, it means C possesses the additional properties (1.129) and (1.132). Moreover, $\mathcal{C} \in \Omega^1(E)$ denotes the partial trace of C, that is $\mathcal{C}(\psi) := C(\sharp_E\, \xi^\mu, \xi_\mu, \psi)$ for all $\psi \in \Gamma(E)$.

(II) $K \in \Omega^1(E) \otimes \Omega^2(E)$ denotes a difference between two Levi-Civita Courant algebroid connections, it means K possesses the additional properties (1.129) and (1.145), or equivalently

$$K = K^+ + K^-, \tag{1.149}$$

where $K^\pm := K|_{\Gamma(V_\pm)\times\Gamma(V_\pm)\times\Gamma(V_\pm)} \in \Omega^1(V_\pm) \otimes \Omega^2(V_\pm)$ and $K^\pm_A = 0$. Analogously, $\mathcal{K} \in \Omega^1(E)$ denotes the partial trace[6] of K.

Remark 1.62. A similar discussion of the concept of Levi-Civita Courant algebroid connections appeared earlier in [13, 14].

The following series of propositions deals with the transformation rules for the Riemann tensor and its contractions in terms of C and K, whose usefulness will be appreciated later.

Proposition 1.63. *Let $(E \xrightarrow{\pi} M, \rho, [.\,,.]_E, g_E)$ be a Courant algebroid equipped with a generalized metric G and let ∇^0 be a Courant algebroid connection compatible with G. If $\nabla \in \mathrm{LC}(E,G)$ is related to ∇^0 as*

$$\nabla_{\psi_1}\psi_2 = \nabla^0_{\psi_1}\psi_2 + \sharp_E\, C(\psi_1,\psi_2,.), \tag{1.150}$$

for all ψ_1, $\psi_2 \in \Gamma(E)$, then the corresponding Riemann tensors are related as

$$\begin{aligned} R_\nabla(\psi_1,\psi_2,\psi_3,\psi_4) = {}& R_{\nabla^0}(\psi_1,\psi_2,\psi_3,\psi_4) + \frac{1}{2}\Big((\nabla^0{}_{\psi_2}C)(\psi_1,\psi_3,\psi_4) - (\nabla^0{}_{\psi_1}C)(\psi_2,\psi_3,\psi_4) \\ &+ (\nabla^0{}_{\psi_3}C)(\psi_4,\psi_2,\psi_1) - (\nabla^0{}_{\psi_4}C)(\psi_3,\psi_2,\psi_1) + C(\psi_3,\sharp_E\, C(\psi_4,\psi_2,.),\psi_1) \\ &- C(\psi_4,\sharp_E\, C(\psi_3,\psi_2,.),\psi_1) + C(\psi_2,\sharp_E\, C(\psi_1,\psi_3,.),\psi_4) \\ &- C(\psi_1,\sharp_E\, C(\psi_2,\psi_3,.),\psi_4) + C(\sharp_E\, C(.\,,\psi_3,\psi_4),\psi_2,\psi_1) \\ &+ C(T_{\nabla^0}(\psi_2,\psi_1),\psi_3,\psi_4) + C(T_{\nabla^0}(\psi_3,\psi_4),\psi_2,\psi_1)\Big), \end{aligned} \tag{1.151}$$

[6] Note that, since C and K are skew-symmetric in the last two inputs, there is precisely one (up to an overall sign) non-trivial partial trace.

for all ψ_1, ψ_2, ψ_3, $\psi_4 \in \Gamma(E)$. In particular, if ∇^0 is also torsion-free, thus $\nabla^0 \in \mathrm{LC}(E, G)$, one has

$$\begin{aligned} R_\nabla(\psi_1,\psi_2,\psi_3,\psi_4) = R_{\nabla^0}(\psi_1,\psi_2,\psi_3,\psi_4) + \frac{1}{2}\Big(& ({\nabla^0}_{\psi_2}K)(\psi_1,\psi_3,\psi_4) - ({\nabla^0}_{\psi_1}K)(\psi_2,\psi_3,\psi_4) \\ & + ({\nabla^0}_{\psi_3}K)(\psi_4,\psi_2,\psi_1) - ({\nabla^0}_{\psi_4}K)(\psi_3,\psi_2,\psi_1) + K(\psi_3, \sharp_E\, K(\psi_4,\psi_2,.),\psi_1) \\ & - K(\psi_4, \sharp_E\, K(\psi_3,\psi_2,.),\psi_1) + K(\psi_2, \sharp_E\, K(\psi_1,\psi_3,.),\psi_4) \\ & - K(\psi_1, \sharp_E\, K(\psi_2,\psi_3,.),\psi_4) + K(\sharp_E\, K(.\,,\psi_3,\psi_4),\psi_2,\psi_1)\Big). \end{aligned} \tag{1.152}$$

Proof. The proof is based on straightforward calculations. □

Right before we state the proposition describing the transformation of the Ricci tensor, we state a useful lemma. It says that taking the partial trace with respect to a fiber-wise metric commutes with a tensorial covariant derivative if the fiber-wise metric is compatible with the respective Courant algebroid connection.

Lemma 1.64. *Consider a Courant algebroid $(E \xrightarrow{\pi} M, \rho, [.\,,.]_E, g_E)$, a fiber-wise metric h on E and a Courant algebroid connection ∇ on E compatible with h, that is*

$$\rho(\psi_1)h(\psi_2,\psi_3) = h(\nabla_{\psi_1}\psi_2,\psi_3) + h(\psi_2,\nabla_{\psi_1}\psi_3), \tag{1.153}$$

for all ψ_1, ψ_2, $\psi_3 \in \Gamma(E)$. Then for an arbitrary $A \in \mathcal{T}^0_k(E)$, $k \geq 2$, and for all ψ, $\psi_1, \ldots, \psi_k \in \Gamma(E)$ there holds

$$(\nabla_\psi A)(\sharp_h\,\xi^\mu,\xi_\mu,\psi_1,\ldots,\psi_k) = (\nabla_\psi \tilde{A})(\psi_1,\ldots,\psi_k), \tag{1.154}$$

where $\tilde{A}(\psi_1,\ldots,\psi_k) := A(\sharp_h\,\xi^\mu,\xi_\mu,\psi_1,\ldots,\psi_k)$.

Proof. Taking arbitrary $(k+1)$-tuple of sections ψ, ψ_1, …, $\psi_k \in \Gamma(E)$, one sees

$$\begin{aligned} &(\nabla_\psi A)(\sharp_h\,\xi^\mu,\xi_\mu,\psi_1,\ldots,\psi_k) - (\nabla_\psi\tilde{A})(\psi_1,\ldots,\psi_k) \\ &\qquad = -A(\nabla_\psi\sharp_h\,\xi^\mu,\xi_\mu,\psi_1,\ldots,\psi_k) - A(\sharp_h\,\xi^\mu,\nabla_\psi\xi_\mu,\psi_1,\ldots,\psi_k). \end{aligned} \tag{1.155}$$

It follows easily from the compatibility of ∇ with h that the identity

$$\sharp_h \circ \nabla_\psi = \nabla_\psi \circ \sharp_h \tag{1.156}$$

is satisfied for all $\psi \in \Gamma(E)$. Using this, one can express components of the section $\nabla_\psi\sharp_h\,\xi^\mu$ for an arbitrary $\psi \in \Gamma(E)$ with respect to local frame $\{\sharp_h\,\xi^\mu\}_{\mu=1}^{\mathrm{Rank}(E)}$ of E as

$$\begin{aligned} (\flat_h\,\xi_\nu)(\nabla_\psi\sharp_h\,\xi^\mu) &= h(\xi_\nu,\sharp_h\,\nabla_\psi\xi^\mu) = (\nabla_\psi\xi^\mu)(\xi_\nu) = \rho(\psi)\delta^\mu_\nu - \xi^\mu(\nabla_\psi\xi_\nu) \\ &= -\xi^\mu(\nabla_\psi\xi_\nu), \end{aligned} \tag{1.157}$$

hence

$$\nabla_\psi\sharp_h\,\xi^\mu = -\xi^\mu(\nabla_\psi\xi_\nu)\sharp_h\,\xi^\nu. \tag{1.158}$$

Employing (1.156) and (1.158) into (1.155) leads to

$$\begin{aligned} &(\nabla_\psi A)(\sharp_h\,\xi^\mu,\xi_\mu,\psi_1,\ldots,\psi_k) - (\nabla_\psi\tilde{A})(\psi_1,\ldots,\psi_k) \\ &\quad = \xi^\mu(\nabla_\psi\xi_\nu)A(\sharp_h\,\xi^\nu,\xi_\mu,\psi_1,\ldots,\psi_k) - \xi^\nu(\nabla_\psi\xi_\mu)A(\sharp_h\,\xi^\mu,\xi_\nu,\psi_1,\ldots,\psi_k) = 0. \end{aligned} \tag{1.159}$$

□

Proposition 1.65. *Let all assumptions of the proposition 1.63 hold. Then the corresponding Ricci tensors are related as*

$$\begin{aligned}\mathrm{Ric}_{\nabla}(\psi_1,\psi_2) = \mathrm{Ric}_{\nabla^0}(\psi_1,\psi_2) + \frac{1}{2}\Big(&(\nabla^0{}_{\psi_1}\mathcal{C})(\psi_2) + (\nabla^0{}_{\psi_2}\mathcal{C})(\psi_1) + (\nabla^0{}_{\xi_\mu}C)(\psi_1,\psi_2,\sharp_E\,\xi^\mu) \\ &+ (\nabla^0{}_{\xi_\mu}C)(\psi_2,\psi_1,\sharp_E\,\xi^\mu) - \mathcal{C}(\xi_\mu)\big(C(\psi_1,\psi_2,\sharp_E\,\xi^\mu) + C(\psi_2,\psi_1,\sharp_E\,\xi^\mu)\big) \\ &- C(\psi_2,\sharp_E\,C(\xi_\mu,\psi_1,.),\sharp_E\,\xi^\mu) - C(\sharp_E\,\xi^\mu,\sharp_E\,C(\psi_1,\xi_\mu,.),\psi_2) \\ &+ C(\sharp_E\,C(.\,,\xi_\mu,\psi_2),\psi_1,\sharp_E\,\xi^\mu) + C(T_{\nabla^0}(\psi_1,\sharp_E\,\xi^\mu),\xi_\mu,\psi_2) \\ &+ C(T_{\nabla^0}(\xi_\mu,\psi_2),\psi_1,\sharp_E\,\xi^\mu)\Big), \end{aligned} \tag{1.160}$$

for all ψ_1, $\psi_2 \in \Gamma(E)$. In particular, if ∇^0 is also torsion-free, thus $\nabla^0 \in \mathrm{LC}(E,G)$, one has

$$\begin{aligned}\mathrm{Ric}_{\nabla}(\psi_1,\psi_2) = \mathrm{Ric}_{\nabla^0}(\psi_1,\psi_2) + \frac{1}{2}\Big(&(\nabla^0{}_{\psi_1}\mathcal{K})(\psi_2) + (\nabla^0{}_{\psi_2}\mathcal{K})(\psi_1) + (\nabla^0{}_{\xi_\mu}K)(\psi_1,\psi_2,\sharp_E\,\xi^\mu) \\ &+ (\nabla^0{}_{\xi_\mu}K)(\psi_2,\psi_1,\sharp_E\,\xi^\mu) - \mathcal{K}(\xi_\mu)\big(K(\psi_1,\psi_2,\sharp_E\,\xi^\mu) + K(\psi_2,\psi_1,\sharp_E\,\xi^\mu)\big) \\ &- K(\psi_2,\sharp_E\,K(\xi_\mu,\psi_1,.),\sharp_E\,\xi^\mu) - K(\sharp_E\,\xi^\mu,\sharp_E\,K(\psi_1,\xi_\mu,.),\psi_2) \\ &+ K(\sharp_E\,K(.\,,\xi_\mu,\psi_2),\psi_1,\sharp_E\,\xi^\mu)\Big). \end{aligned} \tag{1.161}$$

Proof. The proof is based on the use of the previous lemma and straightforward calculations. □

Notation 1.66. Before we state the transformation rule for the Courant-Ricci scalar, let us introduce some notation, which will be useful here but not only here. If we have a vector bundle $E \overset{\pi}{\to} M$ equipped with a fiber-wise metric h, we are able to introduce a $C^\infty(M)$-bilinear map $(.\,,.)_h :\ \mathcal{T}^0_k(E) \times \mathcal{T}^0_k(E) \to C^\infty(M)$ as

$$(F,L)_h := \frac{1}{k!}F(\sharp_h\,\xi^{\mu_1},\cdots,\sharp_h\,\xi^{\mu_k})L(\xi_{\mu_1},\cdots,\xi_{\mu_k}), \tag{1.162}$$

for all F, $L \in \mathcal{T}^0_k(E)$ and all $k \in \mathbb{N}_0$. If we have even a Courant algebroid instead of an ordinary vector bundle at our disposal, we impose $(.\,,.)_{g_E} := (.\,,.)_E$, where g_E is the Courant metric of the given Courant algebroid.

Proposition 1.67. *Let all assumptions of the proposition 1.63 hold. Then the corresponding Courant-Ricci scalars are related as*

$$\mathcal{R}^{\nabla}_E = \mathcal{R}^{\nabla^0}_E + 2\,\mathrm{div}_{\nabla^0}\,\mathcal{C} - g_E^{-1}(\mathcal{C},\mathcal{C}) - 3(C,T_{\nabla^0})_E. \tag{1.163}$$

In particular, if ∇^0 is also torsion-free, thus $\nabla^0 \in \mathrm{LC}(E,G)$, one has

$$\mathcal{R}^{\nabla}_E = \mathcal{R}^{\nabla^0}_E + 2\,\mathrm{div}_{\nabla^0}\,\mathcal{K} - g_E^{-1}(\mathcal{K},\mathcal{K}). \tag{1.164}$$

Proof. It follows from the lemma 1.64 and the proposition 1.65 that

$$\begin{aligned}\mathcal{R}^{\nabla}_E =& \mathcal{R}^{\nabla^0}_E + \frac{1}{2}\Big(4(\nabla^0{}_{\xi_\mu}\mathcal{C})(\sharp_E\,\xi^\mu) - 2\mathcal{C}(\xi_\mu)\mathcal{C}(\sharp_E\,\xi^\mu) - 2C(\xi_\nu,\sharp_E\,C(\xi_\mu,\sharp_E\,\xi^\nu,.),\sharp_E\,\xi^\mu) \\ &+ C(\sharp_E\,C(.\,,\xi_\mu,\xi_\nu),\sharp_E\,\xi^\nu,\sharp_E\,\xi^\mu) - 2C(T_{\nabla^0}(\sharp_E\,\xi^\mu,\sharp_E\,\xi^\nu),\xi_\mu,\xi_\nu)\Big) \\ =& \mathcal{R}^{\nabla^0}_E + 2\,\mathrm{div}_{\nabla^0}\,\mathcal{C} - g_E^{-1}(\mathcal{C},\mathcal{C}) + \frac{1}{2}\Big(C(\sharp_E\,C(.\,,\xi_\mu,\xi_\nu),\sharp_E\,\xi^\nu,\sharp_E\,\xi^\mu) \\ &- 2C(\xi_\nu,\sharp_E\,C(\xi_\mu,\sharp_E\,\xi^\nu,.),\sharp_E\,\xi^\mu) - 2C(T_{\nabla^0}(\sharp_E\,\xi^\mu,\sharp_E\,\xi^\nu),\xi_\mu,\xi_\nu)\Big), \end{aligned} \tag{1.165}$$

Note that $\sharp_E C(\xi_\mu, \sharp_E \xi^\nu, .) = C(\xi_\mu, \sharp_E \xi^\nu, \sharp_E \xi^\kappa)\xi_\kappa$ and expand the first two terms in parentheses as follows:

$$
\begin{aligned}
&C(\sharp_E C(.\,, \xi_\mu, \xi_\nu), \sharp_E \xi^\nu, \sharp_E \xi^\mu) - 2C(\xi_\nu, \sharp_E C(\xi_\mu, \sharp_E \xi^\nu, .), \sharp_E \xi^\mu) \\
&= C(\sharp_E \xi^\kappa, \xi_\mu, \xi_\nu) C(\xi_\kappa, \sharp_E \xi^\nu, \sharp_E \xi^\mu) - 2C(\xi_\mu, \sharp_E \xi^\nu, \sharp_E \xi^\kappa) C(\xi_\nu, \xi_\kappa, \sharp_E \xi^\mu) \\
&= C(\sharp_E \xi^\kappa, \sharp_E \xi^\mu, \sharp_E \xi^\nu)\Big(C(\xi_\kappa, \xi_\nu, \xi_\mu) - 2C(\xi_\mu, \xi_\nu, \xi_\kappa)\Big) \\
&= C(\sharp_E \xi^\kappa, \sharp_E \xi^\mu, \sharp_E \xi^\nu)\Big(C(\xi_\kappa, \xi_\nu, \xi_\mu) + C(\xi_\mu, \xi_\kappa, \xi_\nu) + C(\xi_\mu, \xi_\kappa, \xi_\nu)\Big) \\
&= C(\sharp_E \xi^\kappa, \sharp_E \xi^\mu, \sharp_E \xi^\nu)\Big(- C(\xi_\nu, \xi_\mu, \xi_\kappa) - T_{\nabla^0}(\xi_\kappa, \xi_\nu, \xi_\mu) + C(\xi_\mu, \xi_\kappa, \xi_\nu)\Big) \\
&= C(\sharp_E \xi^\kappa, \sharp_E \xi^\mu, \sharp_E \xi^\nu) T_{\nabla^0}(\xi_\kappa, \xi_\nu, \xi_\mu) = 6(C, T_{\nabla^0})_E,
\end{aligned}
\tag{1.166}
$$

where we have used (1.132) in the fourth step. The fifth equality follows from the fact that C's in the parentheses are symmetric in μ and ν, while the C in front of the parentheses is skew-symmetric in those indices. Next, realize that $T_{\nabla^0}(\sharp_E \xi^\mu, \sharp_E \xi^\nu) = \sharp_E T_{\nabla^0}(\sharp_E \xi^\mu, \sharp_E \xi^\nu, .) = T_{\nabla^0}(\sharp_E \xi^\mu, \sharp_E \xi^\nu, \sharp_E \xi^\kappa)\xi_\kappa$ and proceed for the last term in parentheses in the following way

$$
C(T_{\nabla^0}(\sharp_E \xi^\mu, \sharp_E \xi^\nu), \xi_\mu, \xi_\nu) = T_{\nabla^0}(\sharp_E \xi^\mu, \sharp_E \xi^\nu, \sharp_E \xi^\kappa) C(\xi_\kappa, \xi_\mu, \xi_\nu) = 6(C, T_{\nabla^0})_E. \tag{1.167}
$$

By putting all results together we obtain the first formula to be proven. The second formula easily comes from the first one by imposing $T_{\nabla^0} = 0$. □

Notation 1.68. Consider a Courant algebroid $(E \xrightarrow{\pi} M, \rho, [.\,,.]_E, g_E)$ endowed with a generalized metric $V_+ \subseteq E$. For an arbitrary tensor field $F \in \mathcal{T}^0_k(E)$, $k \in \mathbb{N}$, we impose

$$
F^\pm := F|_{\Gamma(V_\pm)^k}\,. \tag{1.168}
$$

Assume that we have also a Courant algebroid connection ∇ on E at our disposal. Then, we define $\mathbb{R}$-linear maps $\operatorname{div}^\pm_\nabla : \Gamma(E^*) \to C^\infty(M)$ for all $\mathcal{A} \in \Gamma(E^*)$ as

$$
\operatorname{div}^\pm_\nabla \mathcal{A} := (\nabla_{\xi^\pm_a} \mathcal{A})(\sharp_E \xi^{\pm a}), \tag{1.169}
$$

where $\{\xi^\pm_a\}_{a=1}^{\operatorname{Rank}(V_\pm)}$ are arbitrary local frames of $V_\pm$.

Proposition 1.69. *Let all assumptions of the proposition 1.63 hold. Then the corresponding G-Ricci scalars are related as*

$$
\begin{aligned}
\mathcal{R}^\nabla_G = {} & \mathcal{R}^{\nabla^0}_G + 2\operatorname{div}^+_{\nabla^0} \mathcal{C} - 2\operatorname{div}^-_{\nabla^0} \mathcal{C} - (g^+_E)^{-1}(\mathcal{C}^+, \mathcal{C}^+) + (g^-_E)^{-1}(\mathcal{C}^-, \mathcal{C}^-) \\
& + C(\xi^+_a, \xi^-_b, \xi^-_c) T_{\nabla^0}(\sharp_E \xi^{+a}, \sharp_E \xi^{-b}, \sharp_E \xi^{-c}) - C(\xi^-_a, \xi^+_b, \xi^+_c) T_{\nabla^0}(\sharp_E \xi^{-a}, \sharp_E \xi^{+b}, \sharp_E \xi^{+c}) \\
& + 6(C^-, T^-_{\nabla^0})_{g^-_E} - 6(C^+, T^+_{\nabla^0})_{g^+_E},
\end{aligned}
\tag{1.170}
$$

where $\{\xi^+_a\}_{a=1}^{\operatorname{Rank}(V_+)} \cup \{\xi^-_a\}_{a=1}^{\operatorname{Rank}(V_-)}$ is an arbitrary local frame of E adapted to the decomposition $E = V_+ \oplus V_-$. In particular, if ∇^0 is also torsion-free, thus $\nabla^0 \in \operatorname{LC}(E, G)$, one has

$$
\mathcal{R}^\nabla_G = \mathcal{R}^{\nabla^0}_G + 2\operatorname{div}^+_{\nabla^0} \mathcal{K} - 2\operatorname{div}^-_{\nabla^0} \mathcal{K} - (g^+_E)^{-1}(\mathcal{K}^+, \mathcal{K}^+) + (g^-_E)^{-1}(\mathcal{K}^-, \mathcal{K}^-). \tag{1.171}
$$

Lemma 1.70. *Consider a vector bundle $E \xrightarrow{\pi} M$ equipped with a fiber-wise metric g_E, a generalized metric V_+ on E and an orthonormal local frame $\{\xi_\mu\}_{\mu=1}^{\operatorname{Rank}(E)}$ of E over $U \subseteq M$ adapted to the decomposition $E = V_+ \oplus V_-$. Then, $\xi_\mu \in \Gamma_U(V_\pm)$ if and only if $\sharp_E \xi^\mu \in \Gamma_U(V_\pm)$, for all $\mu \in \{1, \cdots, \operatorname{Rank}(E)\}$.*

Proof of the lemma. Take an arbitrary $\mu \in \{1, \cdots, \mathrm{Rank}(E)\}$. It follows from the definition of $V_\pm$ that $\sharp_E\, \xi^\mu \in \Gamma_U(V_\pm)$ if and only if

$$0 = g_E(\sharp_E\, \xi^\mu, \psi) = \xi^\mu(\psi), \tag{1.172}$$

for all $\psi \in \Gamma_U(V_\pm)$. Any $\psi \in \Gamma_U(E)$ can be expressed as $\psi = \psi^\nu \xi_\nu$, where $\psi^\nu \in C^\infty(U)$ for all $\nu \in \{1, \cdots, \mathrm{Rank}(E)\}$. Therefore, $\sharp_E\, \xi^\mu \in \Gamma_U(V_\pm)$ if and only if $\psi^\mu = 0$ for all $\psi \in \Gamma_U(V_\mp)$. As the local frame is adapted to the decomposition $E = V_+ \oplus V_-$, it is equivalent to $\xi_\mu \in \Gamma_U(V_\pm)$. □

Proof of the proposition. Starting point is again the lemma 1.64 and the proposition 1.65. Let us work (without loss of generality) with a local frame $\{\xi_\mu\}_{\mu=1}^{\mathrm{Rank}(E)}$ adapted to the decomposition $E = V_+ \oplus V_-$ and denote the part of the local frame corresponding to the V_+ and V_- as $\{\xi_a^+\}_{a=1}^{\mathrm{Rank}(V_+)}$ and $\{\xi_a^-\}_{a=1}^{\mathrm{Rank}(V_-)}$ respectively. One immediately sees that

$$\begin{aligned}
\mathcal{R}_G^{\nabla} = {}&\mathcal{R}_G^{\nabla^0} + ({\nabla^0}_{\xi_\nu}\mathcal{C})(\sharp_G\, \xi^\nu) + ({\nabla^0}_{\xi_\mu} C)(\sharp_G\, \xi^\nu, \xi_\nu, \sharp_E\, \xi^\mu) - \mathcal{C}(\xi_\mu) C(\sharp_G\, \xi^\nu, \xi_\nu, \sharp_E\, \xi^\mu) \\
&+ \frac{1}{2}\Big(- C(\xi_\nu, \sharp_E\, C(\xi_\mu, \sharp_G\, \xi^\nu, .), \sharp_E\, \xi^\mu) - C(\sharp_E\, \xi^\mu, \sharp_E\, C(\sharp_G\, \xi^\nu, \xi_\mu, .), \xi_\nu) \\
&+ C(\sharp_E\, C(., \xi_\mu, \xi_\nu), \sharp_G\, \xi^\nu, \sharp_E\, \xi^\mu) - 2C(T_{\nabla^0}(\sharp_E\, \xi^\mu, \sharp_G\, \xi^\nu), \xi_\mu, \xi_\nu)\Big).
\end{aligned} \tag{1.173}$$

Take the second and the third term of the right-hand side and expand them as follows:

$$\begin{aligned}
&({\nabla^0}_{\xi_\nu}\mathcal{C})(\sharp_G\, \xi^\nu) + ({\nabla^0}_{\xi_\mu} C)(\sharp_G\, \xi^\nu, \xi_\nu, \sharp_E\, \xi^\mu) \\
&\quad = ({\nabla^0}_{\xi_a^+}\mathcal{C})(\sharp_G\, \xi^{+a}) + ({\nabla^0}_{\xi_a^-}\mathcal{C})(\sharp_G\, \xi^{-a}) + ({\nabla^0}_{\xi_b^+} C)(\sharp_G\, \xi^{+a}, \xi_a^+, \sharp_E\, \xi^{+b}) \\
&\qquad + ({\nabla^0}_{\xi_b^-} C)(\sharp_G\, \xi^{-a}, \xi_a^-, \sharp_E\, \xi^{-b}) \\
&\quad = ({\nabla^0}_{\xi_a^+}\mathcal{C})(\sharp_E\, \xi^{+a}) - ({\nabla^0}_{\xi_a^-}\mathcal{C})(\sharp_E\, \xi^{-a}) + ({\nabla^0}_{\xi_b^+} C)(\sharp_E\, \xi^{+a}, \xi_a^+, \sharp_E\, \xi^{+b}) \\
&\qquad - ({\nabla^0}_{\xi_b^-} C)(\sharp_E\, \xi^{-a}, \xi_a^-, \sharp_E\, \xi^{-b}) \\
&\quad = \mathrm{div}^+_{\nabla^0}\mathcal{C} - \mathrm{div}^-_{\nabla^0}\mathcal{C} + ({\nabla^0}_{\xi_b^+} C)(\sharp_E\, \xi^\mu, \xi_\mu, \sharp_E\, \xi^{+b}) - ({\nabla^0}_{\xi_b^-} C)(\sharp_E\, \xi^\mu, \xi_\mu, \sharp_E\, \xi^{-b}) \\
&\quad = 2\,\mathrm{div}^+_{\nabla^0}\mathcal{C} - 2\,\mathrm{div}^-_{\nabla^0}\mathcal{C},
\end{aligned} \tag{1.174}$$

During the procedure, we have used (1.129), (1.116) and the lemmas 1.64, 1.70 in the first and the third step; and the relation $\sharp_G = \tau \sharp_E$ in the second step. Similarly, one can deal with the fourth term on the right-hand side of (1.173):

$$\begin{aligned}
\mathcal{C}(\xi_\mu) C(\sharp_G\, \xi^\nu, \xi_\nu, \sharp_E\, \xi^\mu) &= \mathcal{C}(\xi_a^+) C(\sharp_G\, \xi^{+b}, \xi_b^+, \sharp_E\, \xi^{+a}) + \mathcal{C}(\xi_a^-) C(\sharp_G\, \xi^{-b}, \xi_b^-, \sharp_E\, \xi^{-a}) \\
&= \mathcal{C}(\xi_a^+) C(\sharp_E\, \xi^{+b}, \xi_b^+, \sharp_E\, \xi^{+a}) - \mathcal{C}(\xi_a^-) C(\sharp_E\, \xi^{-b}, \xi_b^-, \sharp_E\, \xi^{-a}) \\
&= \mathcal{C}(\xi_a^+) C(\sharp_E\, \xi^\mu, \xi_\mu, \sharp_E\, \xi^{+a}) - \mathcal{C}(\xi_a^-) C(\sharp_E\, \xi^\mu, \xi_\mu, \sharp_E\, \xi^{-a}) \\
&= \mathcal{C}(\xi_a^+)\mathcal{C}(\sharp_E\, \xi^{+a}) - \mathcal{C}(\xi_a^-)\mathcal{C}(\sharp_E\, \xi^{-a}) \\
&= (g_E^+)^{-1}(\mathcal{C}^+, \mathcal{C}^+) - (g_E^-)^{-1}(\mathcal{C}^-, \mathcal{C}^-).
\end{aligned} \tag{1.175}$$

By following the approach outlined in the proof of proposition 1.67 and adding some extra tricks already used within this proof, one obtains the following result for the terms in the parentheses on the right-hand side of (1.173):

$$\begin{aligned}
&2C(\xi_a^+, \xi_b^-, \xi_c^-) T_{\nabla^0}(\sharp_E\, \xi^{+a}, \sharp_E\, \xi^{-b}, \sharp_E\, \xi^{-c}) - 2C(\xi_a^-, \xi_b^+, \xi_c^+) T_{\nabla^0}(\sharp_E\, \xi^{-a}, \sharp_E\, \xi^{+b}, \sharp_E\, \xi^{+c}) \\
&+ 12(C^-, T^-_{\nabla^0})_{g_E^-} - 12(C^+, T^+_{\nabla^0})_{g_E^+},
\end{aligned} \tag{1.176}$$

Putting all these together, one obtains the first formula to be proven. The second formula arises by imposing $T_{\nabla^0} = 0$. □

As usual, we conclude this section and also the whole chapter with a proposition about Courant algebroid ismorphisms.

Proposition 1.71. *Let* $(E \overset{\pi}{\to} M, \rho, [.,.]_E, g_E)$ *and* $(E' \overset{\pi'}{\to} M, \rho', [.,.]_{E'}, g_{E'})$ *be two Courant algebroids equipped with Courant algebroid connections* ∇ *and* ∇'*; and generalized metrics* G *and* G' *respectively. Moreover, assume that there is a connection and metric preserving Courant algebroid isomorphism* $\mathcal{F} : \Gamma(E) \to \Gamma(E')$ *between them. Then* $\nabla \in \mathrm{LC}(E, G)$ *if and only if* $\nabla' \in \mathrm{LC}(E', G')$*.*

Proof. It follows directly from the proposition 1.32 that ∇ is torsion-free if and only if ∇' is torsion-free. Let us investigate the compatibility with the generalized metrics, ∇ is compatible with G if and only if

$$\nabla_{\psi_1} \tau \psi_2 = \tau \nabla_{\psi_1} \psi_2, \tag{1.177}$$

for all ψ_1, $\psi_2 \in \Gamma(E)$. Since $\mathcal{F}$ is a $C^\infty(M)$-module isomorphism, it is equivalent to the equality $\mathcal{F}(\nabla_{\psi_1} \tau \psi_2) = \mathcal{F}(\tau \nabla_{\psi_1} \psi_2)$, which can be due to the metric preserving property ($\tau' = \mathcal{F} \tau \mathcal{F}^{-1}$) and also the connection preserving property equivalently rewritten as

$$\nabla'_{\mathcal{F}\psi_1} (\tau' \mathcal{F} \psi_2) = \tau' \nabla'_{\mathcal{F}\psi_1} \mathcal{F} \psi_2. \tag{1.178}$$

Since $\mathcal{F}$ is a $C^\infty(M)$-module isomorphism, it means that ∇' is compatible with G', which completes the proof. □

Chapter 2

Generalized geometry

The previous chapter was concerned with abstract Courant algebroids and its main purpose was to build a robust mathematical framework, which will be now applied to a particular example associated with the generalized geometry [15, 16]. As well as the previous chapter, also this one is based on the publications [2, 6].

2.1 Generalized tangent bundle as Courant algebroid

Example 2.1 (Generalized tangent bundle)**.** For an arbitrary smooth manifold M, we can consider a so called **generalized tangent bundle** $E = \mathbb{T}M := TM \oplus T^*M$, the Whitney sum of the tangent and the cotangent bundle of M, thus $\operatorname{Rank}(\mathbb{T}M) = 2\dim(M)$. The $C^\infty(M)$-module of its sections $\Gamma(E)$ consists of ordered pairs (X, α), where X is a smooth vector field and $\alpha \in \Omega^1(M)$. It is not difficult to realize how elements of $\Gamma(E^*) \equiv \Omega^1(E)$ look like, additivity of an arbitrary $\mathcal{A} \in \Gamma(E^*)$ yields

$$\mathcal{A}((X,\alpha)) = \mathcal{A}((X,0) + (0,\alpha)) = \mathcal{A}((X,0)) + \mathcal{A}((0,\alpha)), \tag{2.1}$$

for all $(X, \alpha) \in \Gamma(E)$, hence $E^* \simeq T^*M \oplus TM$. Therefore, any element of $\Gamma(E^*)$ can be considered as an ordered pair (β, Y), where $\beta \in \Omega^1(M)$ and $Y \in \Gamma(TM)$, which acts on an arbitrary $(X, \alpha) \in \Gamma(E)$ as

$$(\beta, Y)((X,\alpha)) = \beta(X) + \alpha(Y). \tag{2.2}$$

It has been already mentioned that the generalized tangent bundle admits a structure of a Courant algebroid, let us see how this structure may look like. The anchor and the Courant metric are implemented in a pretty natural way, the former is simply the projection onto the first component of an ordered pair, and the latter is defined through imposing that $\flat_E$ is the canonical isomorphism $(X, \alpha) \mapsto (\alpha, X)$. Explicitly,

$$\rho((X,\alpha)) := X, \tag{2.3}$$

$$g_E((X,\alpha),(Y,\beta)) := \beta(X) + \alpha(Y), \tag{2.4}$$

for all (X, α), $(Y, \beta) \in \Gamma(E)$. Note that since $\sharp_E$ is the inverse to $\flat_E$, it is clearly given as $\sharp_E : (\alpha, X) \mapsto (X, \alpha)$, so both $\flat_E$ and $\sharp_E$ simply swap the components of ordered pairs. It remains to specify the Courant bracket, an appropriate choice is the H-**Dorfman bracket** $[.\,,.]^H_D$ defined for all (X, α), $(Y, \beta) \in \Gamma(E)$ as

$$[(X,\alpha),(Y,\beta)]^H_D = ([X,Y], \pounds_X\beta - \mathrm{i}_Y \,\mathrm{d}\,\alpha - H(X,Y,.)), \tag{2.5}$$

where H is some fixed but arbitrary 3-form on M. One should now check if $(E \xrightarrow{\pi} M, \rho, [.,.]_D^H, g_E)$ given in this way indeed meets all the requirements to be a Courant algebroid. Firstly, ρ and g_E are clearly $C^\infty(M)$-linear in all their inputs, furthermore g_E is symmetric and $\flat_E$ is a $C^\infty(M)$-module isomorphism for sure. Since all objects contained in the definition of $[.,.]_D^H$ are $\mathbb{R}$-linear in all their inputs, also the H-twisted Dorfman bracket is $\mathbb{R}$-bilinear. It remains to prove that all four Courant algebroid axioms are satisfied. After a rather straightforward but long procedure, see [9, Example 1.17], one finds out that $(E \xrightarrow{\pi} M, \rho, [.,.]_D^H, g_E)$ defined above is a Courant algebroid if and only if $H \in \Omega^3(E)$ is closed.

Notation 2.2. The Courant algebroid introduced in the previous example is crucial for this thesis. We will work solely with it throughout the whole chapter, consequently $(E \xrightarrow{\pi} M, \rho, [.,.]_D^H, g_E)$ will always and without any special emphasizing denote the Courant algebroid described in the previous example.

Example 2.3 (Local frames of generalized tangent bundle)**.** Consider a Courant algebroid $(E \xrightarrow{\pi} M, \rho, [.,.]_D^H, g_E)$ and an arbitrary local frame $\{e_j\}_{j=1}^{\dim(M)}$ of TM over some $U \subseteq M$ and denote $\{e^j\}_{j=1}^{\dim(M)}$ the corresponding dual one, it is easy to realize that $\{(e_j,0),(0,e^j)\}_{j=1}^{\dim(M)}$ forms a local frame of $\mathbb{T}M$ over $U \subseteq M$. The name **local frame of the standard type** will refer to this kind of local frame, which can be apparently constructed over a neighbourhood of an arbitrary point $p \in M$. Therefore, we can always and without loss of generality choose this one to work with. Although it is of course possible to determine the signature of the Courant metric g_E introduced in the example 2.1 from any local frame, the easiest way is to do it from an orthonormal one. Unfortunately, a local frame of the standard type is not orthonormal, since we have

$$g_E((e_j,0),(e_j,0)) = 0, \qquad g_E((e^j,0),(e^j,0)) = 0, \tag{2.6}$$

for all $j \in \{1, \cdots, \dim(M)\}$. However, if we denote

$$e_j^+ := \frac{1}{\sqrt{2}}(e_j, e^j), \qquad e_j^- := \frac{1}{\sqrt{2}}(e_j, -e^j), \tag{2.7}$$

for all $j \in \{1, \cdots, \dim(M)\}$, one immediately sees

$$g_E(e_j^+, e_j^+) = \frac{1}{2}(e^j(e_j) + e^j(e_j)) = 1, \qquad g_E(e_j^-, e_j^-) = \frac{1}{2}(-e^j(e_j) - e^j(e_j)) = -1. \tag{2.8}$$

Clearly, $\{e_j^+, e_j^-\}_{j=1}^{\dim(M)}$ is a local frame, since it can be expressed through the corresponding local frame of the standard type and vice versa, hence the signature of g_E is $(\dim(M), \dim(M))$. This follows from the fact that such orthonormal local frame can be constructed over a neighbourhood of an arbitrary point of the base manifold.

Notation 2.4. In the same way as in the case of $\{\xi_\mu\}_{\mu=1}^{\mathrm{Rank}(E)}$, we will use $\{e_j\}_{j=1}^{\dim(M)}$ as an arbitrary local frame of TM without any special emphasizing throughout the whole thesis. Note that we are distinguishing three types of indices, Greek (μ, ν, κ, $\cdots$), Latin from the alphabet beginnings (a, b, c, $\cdots$) and Latin from the middle of the alphabet (j, k, l, $\cdots$). They are associated with a whole vector bundle, a generalized metric and a tangent bundle respectively.

Remark 2.5. We have just shown that the signature of g_E is constant even without the assumption of a connected base manifold. Especially, it means that all three definitions of generalized metric stated in Section 1.3 are equivalent in the framework of $(\mathbb{T}M, g_E)$ for an arbitrary base manifold M.

Example 2.6 (Map $\mathcal{D}$)**.** In Section 1.1, we introduced a map $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ and also we proved the identity $\mathcal{D} = \sharp_E \circ \rho^T \circ \mathrm{d}$ for it. Let us find out how $\mathcal{D}$ exactly looks like in the case of $(E \xrightarrow{\pi} M, \rho, [.\,,.]^H_D, g_E)$. For all $X \in \Gamma(TM)$ and all $\alpha,\ \beta \in \Omega^1(M)$, the definition of map $\rho^T : \Omega^1(M) \to\ \Omega^1(E)$ implies

$$(\rho^T\beta)(X,\alpha) = \beta(\rho((X,\alpha))) = \beta(X) = (\beta, 0)((X,\alpha)), \tag{2.9}$$

hence $\rho^T\beta = (\beta, 0)$, for all $\beta \in \Omega^1(M)$. Since we have already discussed how $\sharp_E$ acts on the element of $\Omega^1(E)$, see example 2.1, we are able to determine $\mathcal{D}$ as

$$\mathcal{D}\, f = (0, \mathrm{d}\, f), \tag{2.10}$$

for all $f \in C^\infty(M)$.

Note that there is not just one Courant algebroid $(E \xrightarrow{\pi} M, \rho, [.\,,.]^H_D, g_E)$, but there is a whole class of them parametrized by a closed 3-form H. The following proposition shows that this parametrization is actually even rougher.

Proposition 2.7. *For all $B \in \Omega^2(M)$, there is an Courant algebroid isomorphism e^B between $(E \xrightarrow{\pi} M, \rho, [.\,,.]^H_D, g_E)$ and $(E \xrightarrow{\pi} M, \rho, [.\,,.]^{H+\mathrm{d}\,B}_D, g_E)$ defined as*

$$\mathrm{e}^B(X,\alpha) = (X, \alpha + B(X,.)), \tag{2.11}$$

for all $(X,\alpha) \in \Gamma(E)$.

Proof. First of all, one has to show that e^B is a $C^\infty(M)$-module automorphism, which is in fact very simple when you realize that e^{-B} is two-sided inverse of e^B. It remains to show that e^B possesses all the properties stated in (1.34). The relation for ρ is satisfied trivially that leaves us with two identities to prove. One easily finds that for all (X,α), $(Y,\beta) \in \Gamma(E)$ there holds

$$\begin{aligned} g_E(\mathrm{e}^B(X,\alpha), \mathrm{e}^B(Y,\beta)) &= g_E((X, \alpha + B(X,.)), (Y, \beta + B(Y,.))) \\ &= \beta(X) + B(Y,X) + \alpha(Y) + B(X,Y) = \beta(X) + \alpha(Y) \\ &= g_E((X,\alpha), (Y,\beta)) \end{aligned} \tag{2.12}$$

To show that relation between the Courant brackets holds, it needs a little bit more care. Let us take arbitrary (X,α), $(Y,\beta) \in \Gamma(E)$ and expand the expression $[\mathrm{e}^B(X,\alpha), \mathrm{e}^B(Y,\beta)]^{H+\mathrm{d}\,B}_D$ as follows:

$$\begin{aligned} &[\mathrm{e}^B(X,\alpha), \mathrm{e}^B(Y,\beta)]^{H+\mathrm{d}\,B}_D \\ &\quad= [(X, \alpha + B(X,.)), (Y, \beta + B(Y,.))]^{H+\mathrm{d}\,B}_D \\ &\quad= \Big([X,Y], \pounds_X(\beta + B(Y,.)) - \mathrm{i}_Y\,\mathrm{d}(\alpha + B(X,.)) - H(X,Y,.) - (\mathrm{d}\,B)(X,Y,.)\Big) \\ &\quad= \Big([X,Y], \pounds_X\beta - \mathrm{i}_Y\,\mathrm{d}\,\alpha - H(X,Y,.) + (\pounds_X\mathrm{i}_Y - \mathrm{i}_Y\,\mathrm{d}\,\mathrm{i}_X - \mathrm{i}_Y\mathrm{i}_X\,\mathrm{d})B\Big) \\ &\quad= \Big([X,Y], \pounds_X\beta - \mathrm{i}_Y\,\mathrm{d}\,\alpha - H(X,Y,.) + (\pounds_X\mathrm{i}_Y - \mathrm{i}_Y\pounds_X)B\Big) \\ &\quad= \Big([X,Y], \pounds_X\beta - \mathrm{i}_Y\,\mathrm{d}\,\alpha - H(X,Y,.) + B([X,Y],.)\Big) \\ &\quad= \mathrm{e}^B[(X,\alpha), (Y,\beta)]^H_D. \end{aligned} \tag{2.13}$$

The formulas $\pounds_X =\ \mathrm{d} \circ \mathrm{i}_X + \mathrm{i}_X \circ \mathrm{d}$ and $\mathrm{i}_{[X,Y]} = \pounds_X \circ \mathrm{i}_Y - \mathrm{i}_Y \circ \pounds_X$ have been used in the fourth and the fifth step respectively. □

Remark 2.8. We have just shown that the set of all Courant algebroids $(E \xrightarrow{\pi} M, \rho, [.\,,.]^H_D, g_E)$ can be decomposed into disjoint equivalence classes, each class being represented by an element of the third de Rham cohomology group $H^3_{dR}(M)$.

2.2 Generalized metric

In this section, we will investigate the concept of a generalized metric on the generalized tangent bundle. We start by introducing the crucial property of each generalized metric on $\mathbb{T}M$.

Theorem 2.9. *Consider the generalized tangent bundle $\mathbb{T}M$ over a smooth manifold M and the fiber-wise metric g_E on $\mathbb{T}M$ defined in the example 2.1. Then there is a one-to-one correspondence between generalized metrics on $\mathbb{T}M$ and ordered pairs (g, B), where g is a Riemannian metric on M and $B \in \Omega^2(M)$.*

Proof. Let us start with a generalized metric $V_+ \subseteq \mathbb{T}M$. Every generalized metric is a positive definite subbundle and therefore $g_E(\psi, \psi) \neq 0$, for all $\psi \in \Gamma(V_+)$, $\psi \neq 0$. As for all $\alpha \in \Gamma(T^*M)$ there holds

$$g_E((0, \alpha), (0, \alpha)) = 0, \tag{2.14}$$

necessarily $\Gamma(V_+) \cap (\{0\} \oplus \Gamma(T^*M)) = \{(0, 0)\}$. This together with the fact that $\Gamma(V_+)$ is closed with respect to $C^\infty(M)$-linear combinations implies that $\Gamma(V_+)$ is a graph of a $C^\infty(M)$-module morphism $\Phi : \Gamma(W) \to \Gamma(T^*M)$, where $W \subseteq TM$ is a subbundle. Taking an arbitrary point $p \in M$ we have

$$g_{Ep}((v, 0), (v, 0)) = 0, \qquad g_{Ep}((0, a), (0, a)) = 0, \tag{2.15}$$

for all $v \in T_pM$ and all $a \in T_p^*M$, hence

$$V_{+p} \cap (T_pM \oplus \{0\}) = (0, 0), \qquad V_{+p} \cap (\{0\} \oplus T_p^*M) = (0, 0). \tag{2.16}$$

It now follows from the fact that V_{+p} is closed with respect to $\mathbb{R}$-linear combinations that $\Phi_p : W_p \to T_p^*M$ is an vector space monomorphism. As both of the vector spaces are finite-dimensional Φ_p is even an isomorphism, so $W_p = T_pM$ since $\dim(T_pM) = \dim(T_p^*M)$, that is $W = TM$. In summary, $\Gamma(V_+)$ is a graph of a $C^\infty(M)$-module isomorphism $\Phi : \Gamma(TM) \to \Gamma(T^*M)$. Let us proceed further, one can define $F \in \mathcal{T}_2^0(M)$ as

$$F(X, Y) := (\Phi(X))(Y), \tag{2.17}$$

for all X, $Y \in \Gamma(TM)$ and denote the symmetric and the skew-symmetric part of F as $g := F_S$ and $B := F_A$ respectively. So, $B \in \Omega^2(M)$ has been just discovered, and it therefore remains to find a Riemannian metric on M. An obvious candidate is g, it is symmetric by the definition, thus the last missing ingredients are to show that g induces an $C^\infty(M)$-module isomorphism and that it is positive definite. Take arbitrary ψ_1, $\psi_2 \in \Gamma(V_+)$, we have already found out that there are $X, Y \in \Gamma(TM)$ such that $\psi_1 = (X, \Phi(X)) \equiv (X, g(X, .) + B(X, .))$ and $\psi_2 = (Y, g(Y, .) + B(Y, .))$. One easily finds

$$\begin{aligned}(\flat_E\, (X, \Phi(X)))((Y, \Phi(Y))) &= g_E((X, g(X, .) + B(X, .)), (Y, g(Y, .) + B(Y, .))) = 2g(X, Y) \\ &= 2(\flat_g\, X)(Y),\end{aligned} \tag{2.18}$$

hence $\flat_g = \frac{1}{2}\, \phi^T \circ \flat_E \circ \phi$, where $\phi : X \mapsto (X, \Phi(X))$. Since Φ is a $C^\infty(M)$-module isomorphism, the map $\phi : \Gamma(TM) \to \Gamma(V_+)$ is one too. So, $\flat_g$ is a composition of $C^\infty(M)$-module isomorphisms, thus it is an isomorphism itself. By choosing $Y := X$ in (2.18), we obtain

$$g_E((X, g(X, .) + B(X, .)), (X, g(X, .) + B(X, .))) = 2g(X, X), \tag{2.19}$$

for all $X \in \Gamma(TM)$. As V_+ is a positive subbundle, for all $p \in M$ and all $v \in T_pM$, $v \neq 0$ there holds

$$g_p(v,v) = \frac{1}{2} g_{Ep}((v, g_p(v,.) + B_p(v,.)), (v, g_p(v,.) + B_p(v,.))) > 0. \tag{2.20}$$

Conversely, take an arbitrary pair of Riemannian metric and 2-form on M, and denote it as (g, B). In the correspondence with the previous, we impose

$$V_+ := \bigsqcup_{p \in M} \{(v, g_p(v,.) + B_p(v,.)) \mid v \in T_pM\}. \tag{2.21}$$

Apparently, it satisfies *Local frame criterion for subbundles*, see [11, Lemma 10.32], and consequently it is a subbundle of $\mathbb{T}M$. Clearly the equality (2.19) holds, and since g is positive definite, V_+ is a positive subbundle. Furthermore, it is easy to see that $\mathrm{Rank}(V_+) = \dim(M)$, which means due to the signature of g_E that V_+ is even maximal. In conclusion, V_+ is a generalized metric on $\mathbb{T}M$. □

Remark 2.10. For the exactly same reasons as in the case of V_+, also $\Gamma(V_-)$ is a graph of a $C^\infty(M)$-module isomorphism, that is

$$V_- = \bigsqcup_{p \in M} \{(v, \tilde{g}_p(v,.) + \tilde{B}_p(v,.)) \mid v \in T_pM\}, \tag{2.22}$$

where $\tilde{g} \in \mathcal{T}_2^0(M)$ is symmetric and $\tilde{B} \in \Omega^2(M)$. As $g_E|_{\Gamma(V_+)\times\Gamma(V_-)} = 0$, the equation

$$\begin{aligned} 0 &= g_E((X, g(X,.) + B(X,.), (Y, \tilde{g}(Y,.) + \tilde{B}(Y,.))) \\ &= \tilde{g}(Y,X) + \tilde{B}(Y,X) + g(X,Y) + B(X,Y) \end{aligned} \tag{2.23}$$

is valid for all X, $Y \in \Gamma(TM)$. It can be decomposed into a symmetric and a skew-symmetric part, they say $\tilde{g} = -g$ and $\tilde{B} = B$ respectively. In conclusion, if V_+ corresponds to (g, B), V_- is given as

$$V_- = \bigsqcup_{p \in M} \{(v, -g_p(v,.) + B_p(v,.)) \mid v \in T_pM\}. \tag{2.24}$$

For the future reference, denote the $C^\infty(M)$-module isomorphisms spanning $\Gamma(V_\pm)$ as

$$\Phi_+(X) := (X, \flat_g X + B(X,.)), \qquad \Phi_-(X) := (X, -\flat_g X + B(X,.)), \tag{2.25}$$

for all $X \in \Gamma(TM)$.

Notation 2.11. One immediately sees that for a fixed Riemannian metric g there is a distinguished generalized metric, it is the one corresponding to the choice $B = 0$. This particular generalized metric is called a **minimal generalized metric** on $(\mathbb{T}M, g_E)$ corresponding to g and is denoted as V_+^g, possibly τ^g or G^g.

We have just discovered that V_+ and (g, B) are related through (2.21), let us see how the other objects representing a generalized metric can be expressed in terms of (g, B).

Notation 2.12. Realize that generalized tangent bundle is defined as the Whitney sum of two vector bundles. Consequently, we can visualize any tensor field $A \in \mathcal{T}_2^0(\mathbb{T}M)$ as a matrix with respect to this decomposition, analogously as we did it in the remark 1.43 for a vector bundle decomposition induced by a generalized metric. Moreover, note that every vector space endomorphism $\Phi : \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ can be due to additivity decomposed into four components, namely for all $(X, \alpha) \in \Gamma(E)$ one can write

$$\begin{aligned}\Phi(X,\alpha) =&\Phi((X,0)+(0,\alpha)) = \Phi(X,0)+\Phi(0,\alpha)\\ =&\pi_1\Phi(X,0)+\pi_2\Phi(X,0)+\pi_1\Phi(0,\alpha)+\pi_2\Phi(0,\alpha),\end{aligned} \tag{2.26}$$

where $\pi_1(\equiv \rho)$ and π_2 are the projections onto the first and the second component respectively. Therefore, not only fully covariant tensor fields of the rank two, but also vector space endomorphisms can be represented as matrices. The action of such endomorphism can be expressed as

$$\Phi(X,\alpha) = \begin{pmatrix} \pi_1\Phi(.\,,0) & \pi_1\Phi(0,.) \\ \pi_2\Phi(.\,,0) & \pi_2\Phi(0,.) \end{pmatrix} \begin{pmatrix} X \\ \alpha \end{pmatrix}, \tag{2.27}$$

for all $(X, \alpha) \in \Gamma(E)$.

Proposition 2.13. *Let all assumptions of the previous theorem hold, and furthermore, let V_+ be a generalized metric on $\mathbb{T}M$ associated with (g, B), a pair of a Riemannian metric and a 2-form on M. Then the corresponding representations of the generalized metric τ and G are related to (g, B) as*[7]

$$\tau = \begin{pmatrix} -\sharp_g B(\star,.) & \sharp_g \\ \flat_g - B(\sharp_g B(\star,.),.) & B(\sharp_g \star,.) \end{pmatrix}, \quad G = \begin{pmatrix} g+g^{-1}(B(\star,.),B(\star,.)) & g^{-1}(B(\star,.),\star) \\ g^{-1}(\star,B(\star,.)) & g^{-1} \end{pmatrix}, \tag{2.28}$$

where the symbol $\star$ marks the input. Especially, if V_+ is a minimal generalized metric corresponding to g, one has

$$\tau^g = \begin{pmatrix} 0 & \sharp_g \\ \flat_g & 0 \end{pmatrix}, \qquad G^g = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}. \tag{2.29}$$

Lemma 2.14. *Let all assumptions of the proposition hold. The projectors $\pi_\pm : \Gamma(\mathbb{T}M) \to \Gamma(V_\pm)$ can be explicitly expressed as*

$$\pi_\pm(X,\alpha) = \frac{1}{2}\Phi_\pm(X \pm \sharp_g \alpha \mp \sharp_g B(X,.)), \tag{2.30}$$

for all $(X, \alpha) \in \Gamma(\mathbb{T}M)$.

Proof of the lemma. Take an arbitrary $(X, \alpha) \in \Gamma(\mathbb{T}M)$. Since we have the decomposition $\mathbb{T}M = V_+ \oplus V_-$ and $\Phi_\pm : \Gamma(TM) \to \Gamma(V_\pm)$ are isomorphisms, there exist $X_\pm \in \Gamma(TM)$ such that

$$\begin{aligned}(X,\alpha) &= \Phi_+(X_+)+\Phi_-(X_-)\\ &= (X_+, \flat_g X_+ + B(X_+,.)) + (X_-, -\flat_g X_- + B(X_-,.)\\ &= (X_+ + X_-, \flat_g (X_+ - X_-) + B(X_+ + X_-,.),\end{aligned} \tag{2.31}$$

hence $X_+ + X_- = X$, and at the same time $\flat_g (X_+ - X_-) + B(X_+ + X_-,.) = \alpha$. The latter condition can be equivalently rewritten under the assumption of the former condition as

$$\begin{aligned} & X_+ - X_- = \sharp_g \alpha - \sharp_g B(X,.),\\ \Leftrightarrow \quad & \mp X \pm 2X_\pm = \sharp_g \alpha - \sharp_g B(X,.),\\ \Leftrightarrow \quad & X_\pm = \frac{1}{2}(X \pm \sharp_g \alpha \mp \sharp_g B(X,.)),\end{aligned} \tag{2.32}$$

Apparently, there holds $\pi_\pm(X,\alpha) = \Phi_\pm(X_\pm)$, which concludes the proof. □

[7] Matrices are considered with respect to the decomposition $\mathbb{T}M = TM \oplus T^*M$.

Proof of the proposition. It follows from the previous lemma and from the relation between τ and V_+ that

$$
\begin{aligned}
&\tau(X,\alpha)\\
&=\pi_+(X,\alpha)-\pi_-(X,\alpha)\\
&=\frac{1}{2}\Phi_+(X+\sharp_g\,\alpha-\sharp_g\,B(X,.))-\frac{1}{2}\Phi_-(X-\sharp_g\,\alpha+\sharp_g\,B(X,.))\\
&=\frac{1}{2}\Big(X+\sharp_g\,\alpha-\sharp_g\,B(X,.),\flat_g\,X+\alpha-B(X,.)+B(X,.)+B(\sharp_g\,\alpha,.)-B(\sharp_g\,B(X,.),.)\Big)\\
&\quad-\frac{1}{2}\Big(X-\sharp_g\,\alpha+\sharp_g\,B(X,.),-\flat_g\,X+\alpha-B(X,.)+B(X,.)-B(\sharp_g\,\alpha,.)+B(\sharp_g\,B(X,.),.)\Big)\\
&=(-\sharp_g\,B(X,.)+\sharp_g\,\alpha,\flat_g\,X-B(\sharp_g\,B(X,.),.)+B(\sharp_g\,\alpha,.)),
\end{aligned}
\tag{2.33}
$$

hence

$$
\tau=\begin{pmatrix}-\sharp_g\,B(\star,.) & \sharp_g\\ \flat_g\,-B(\sharp_g\,B(\star,.),.) & B(\sharp_g\,\star,.)\end{pmatrix}. \tag{2.34}
$$

To find a matrix for G, one simply uses the relation $G(.\,,.)=g_E(.\,,\tau\,.)$. Take an arbitrary (X,α), $(Y,\beta)\in\Gamma(\mathbb{T}M)$ and proceed as follows:

$$
\begin{aligned}
&G((X,\alpha),(Y,\beta))\\
&=g_E((X,\alpha),\tau(Y,\beta))\\
&=g_E\Big((X,\alpha),(-\sharp_g\,B(Y,.)+\sharp_g\,\beta,\flat_g\,Y-B(\sharp_g\,B(Y,.),.)+B(\sharp_g\,\beta,.)\Big)\\
&=g(X,Y)-B(\sharp_g\,B(Y,.),X)+B(\sharp_g\,\beta,X)-\alpha(\sharp_g\,B(Y,.))+\alpha(\sharp_g\,\beta)\\
&=g(X,Y)-g^{-1}(B(X,.),B(Y,.))+g^{-1}(B(X,.),\beta)+g^{-1}(\alpha,B(Y,.))+g^{-1}(\alpha,\beta),
\end{aligned}
\tag{2.35}
$$

hence

$$
G=\begin{pmatrix}g+g^{-1}(B(\star,.),B(\star,.)) & g^{-1}(B(\star,.),\star)\\ g^{-1}(\star,B(\star,.)) & g^{-1}\end{pmatrix}. \tag{2.36}
$$

The matrices for the minimal generalized metric corresponding to g arise by imposing $B=0$ in (2.34) and (2.36). □

Remark 2.15. Consider a Courant algebroid $(E\overset{\pi}{\to}M,\rho,[.\,,.]_D^H,g_E)$ equipped with an arbitrary generalized metric V_+ associated with (g,B), one easily finds $V_+=\mathrm{e}^B\,V_+^g$, that is e^B is not only a Courant algebroid isomorphism but it is even a metric preserving Courant algebroid isomorphism between $(E\overset{\pi}{\to}M,\rho,[.\,,.]_D^H,g_E)$ and $(E\overset{\pi}{\to}M,\rho,[.\,,.]_D^{H+\mathrm{d}\,B},g_E)$ equipped with V_+^g and V_+ respectively.

2.3 Levi-Civita Courant algebroid connections

In the previous chapter, we have analyzed the features of generalized metrics on our prominent Courant algebroid. In this chapter, we will continue with the interpretation of results acquired on the level of general Courant algebroids, namely we will concern ourselves with Levi-Civita Courant algebroid connections on $(E\overset{\pi}{\to}M,\rho,[.\,,.]_D^H,g_E)$. Our first intention will be to describe the whole set $\mathrm{LC}(\mathbb{T}M,G^g)$ for an arbitrary Riemannian metric g on M.

In Section 1.4, a general procedure for constructing Levi-Civita Courant algebroid connections was invented. Its input data is some fixed generalized metric compatible Courant algebroid connection, the following example will provide us one of those.

Example 2.16 (Natural Courant algebroid connection compatible with a minimal generalized metric)**.** Consider now the minimal generalized metric G^g corresponding to a Riemannian metric g on M and examine properties of the map $\nabla^g : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ defined as

$$\nabla^g_{(X,\alpha)} = \begin{pmatrix} \nabla^{LC,g}_X & 0 \\ 0 & \nabla^{LC,g}_X \end{pmatrix}, \tag{2.37}$$

for all $(X, \alpha) \in \Gamma(E)$, the symbol $\nabla^{LC,g}$ denotes the unique Levi-Civita affine connection on the Riemannian manifold (M, g). First, let us check whether ∇^g is even a Courant algebroid connection, as it is indicated. For sure it is $\mathbb{R}$-bilinear, because $\nabla^{LC,g}$ is $\mathbb{R}$-bilinear, so let us deal only with the Courant algebroid connection axioms. The validity of the first axiom is pretty obvious, the second one can be checked as follows:

$$\begin{aligned} \nabla^g_{(X,\alpha)}(f(Y,\beta)) &= (\nabla^{LC,g}_X(fY), \nabla^{LC,g}_X(f\beta)) = ((Xf)Y + f\nabla^{LC,g}Y, (Xf)\beta + f\nabla^{LC,g}\beta) \\ &= (\rho((X,\alpha))f)(Y,\beta) + f\nabla^g_{(X,\alpha)}(Y,\beta), \end{aligned} \tag{2.38}$$

for all (X, α), $(Y, \beta) \in \Gamma(E)$ and $f \in C^\infty(M)$. To prove the compatibility of ∇^g with the Courant metric, take arbitrary (X, α), (Y, β), $(Z, \gamma) \in \Gamma(E)$ and make a few straightforward steps, namely

$$\begin{aligned} \rho((X,\alpha))g_E((Y,\beta),(Z,\gamma)) &= X\gamma(Y) + X\beta(Z) \\ &= (\nabla^{LC,g}_X\gamma)(Y) + \gamma(\nabla^{LC,g}_X Y) + (\nabla^{LC,g}_X\beta)(Z) + \beta(\nabla^{LC,g}_X Z) \\ &= g_E(\nabla^g_{(X,\alpha)}(Y,\beta),(Z,\gamma)) + g_E((Y,\beta),\nabla^g_{(X,\alpha)}(Z,\gamma)). \end{aligned} \tag{2.39}$$

Indeed, ∇^g is a Courant algebroid connection. We claim that ∇^g is even compatible with the minimal generalized metric G^g. This follows from that the following series of equalities:

$$\begin{aligned} \rho((X,\alpha))G^g((Y,\beta),(Z,\gamma)) &= Xg(Y,Z) + Xg^{-1}(\beta,\gamma) \\ &= g(\nabla^{LC,g}_X Y, Z) + g(Y, \nabla^{LC,g}_X Z) + g^{-1}(\nabla^{LC,g}_X\beta,\gamma) + g^{-1}(\beta,\nabla^{LC,g}_X\gamma) \\ &= G^g(\nabla^g_{(X,\alpha)}(Y,\beta),(Z,\gamma)) + G^g((Y,\beta),\nabla^g_{(X,\alpha)}(Z,\gamma)) \end{aligned} \tag{2.40}$$

is satisfied for all (X, α), (Y, β), $(Z, \gamma) \in \Gamma(E)$. During the procedure, we have used the proposition 2.13 twice.

Remark 2.17. Note that we have used the fact that $\nabla^{LC,g}$ is compatible with the g exclusively to prove the compatibility of ∇^g with the G^g, and furthermore, the torsion-freeness of $\nabla^{LC,g}$ has not been used at all. In particular, it means that any affine connection defines a Courant algebroid connection on $(E \overset{\pi}{\to} M, \rho, [.,.]^H_D, g_E)$ in the way of (2.37), and also that any affine connection compatible with a metric induces a Courant algebroid connection compatible with the minimal generalized metric corresponding to the metric.

Example 2.18 (Torsion and curvature of ∇^g)**.** We have not calculated any torsion or Riemann tensor explicitly yet. It was mainly because we did not have any specific Courant algebroid connection in our hands. However, this has just changed. Take an arbitrary Riemannian metric

g on M and for all (X,α), (Y,β), $(Z,\gamma)\in\Gamma(E)$ proceed as follows:

$$\begin{aligned}
&T_{\nabla^g}((X,\alpha),(Y,\beta),(Z,\gamma))\\
&\quad= g_E(\nabla^g_{(X,\alpha)}(Y,\beta)-\nabla^g_{(Y,\beta)}(X,\alpha)-[(X,\alpha),(Y,\beta)]^H_D,(Z,\gamma))+g_E(\nabla^g_{(Z,\gamma)}(X,\alpha),(Y,\beta))\\
&\quad= g_E\Big((\nabla^{LC,g}_X Y-\nabla^{LC,g}_Y X-[X,Y],\nabla^{LC,g}_X\beta-\nabla^{LC,g}_Y\alpha-\pounds_X\beta+\mathrm{i}_Y\,\mathrm{d}\,\alpha+H(X,Y,.)),(Z,\gamma)\Big)\\
&\qquad+g_E((\nabla^{LC,g}_Z X,\nabla^{LC,g}_Z\alpha),(Y,\beta))\\
&\quad= T^{LC,g}(\gamma,X,Y)+H(X,Y,Z)+(\nabla^{LC,g}_X\beta)(Z)-(\pounds_X\beta)(Z)+\beta(\nabla^{LC,g}_Z X)\\
&\qquad-(\nabla^{LC,g}_Y\alpha)(Z)+(\mathrm{d}\,\alpha)(Y,Z)+(\nabla^{LC,g}_Z\alpha)(Y).
\end{aligned}\tag{2.41}$$

The terms on the right-hand side containing β can be rewritten as

$$\begin{aligned}
&(\nabla^{LC,g}_X\beta)(Z)-(\pounds_X\beta)(Z)+\beta(\nabla^{LC,g}_Z X)\\
&\quad= X\beta(Z)-\beta(\nabla^{LC,g}_X Z)-X\beta(Z)+\beta([X,Z])+\beta(\nabla^{LC,g}_Z X)=T^{LC,g}(\beta,Z,X),
\end{aligned}\tag{2.42}$$

and similarly for the terms with α, using the formula $(\mathrm{d}\,\alpha)(Y,Z)=Y\alpha(Z)-Z\alpha(Y)-\alpha([Y,Z])$ results into

$$(\mathrm{d}\,\alpha)(Y,Z)+(\nabla^{LC,g}_Z\alpha)(Y)-(\nabla^{LC,g}_Y\alpha)(Z)=T^{LC,g}(\alpha,Y,Z).\tag{2.43}$$

Since the Levi-Civita affine connection is torsion free, we obtain

$$T_{\nabla^g}((X,\alpha),(Y,\beta),(Z,\gamma))=H(X,Y,Z),\qquad\Leftrightarrow\qquad T_{\nabla^g}=\rho^*H.\tag{2.44}$$

Let us now have a look at the Riemann tensor. For all (X,α), (Y,β), (Z,γ), $(V,\omega)\in\Gamma(E)$ there holds

$$\begin{aligned}
&R_{0\nabla^g}((X,\alpha),(Y,\beta),(Z,\gamma),(V,\omega))\\
&= g_E(\nabla^g_{(Z,\gamma)}\nabla^g_{(V,\omega)}(Y,\beta)-\nabla^g_{(V,\omega)}\nabla^g_{(Z,\gamma)}(Y,\beta)-\nabla^g_{[(Z,\gamma),(V,\omega)]^H_D}(Y,\beta),(X,\alpha))\\
&= g_E\Big(((\nabla^{LC,g}_Z\nabla^{LC,g}_V-\nabla^{LC,g}_V\nabla^{LC,g}_Z-\nabla^{LC,g}_{[Z,V]})Y,(\nabla^{LC,g}_Z\nabla^{LC,g}_V-\nabla^{LC,g}_V\nabla^{LC,g}_Z-\nabla^{LC,g}_{[Z,V]})\beta),(X,\alpha)\Big)\\
&= R^{LC,g}(\alpha,Y,Z,V)+(\nabla^{LC,g}_Z\nabla^{LC,g}_V\beta)(X)-(\nabla^{LC,g}_V\nabla^{LC,g}_Z\beta)(X)-(\nabla^{LC,g}_{[Z,V]}\beta)(X).
\end{aligned}\tag{2.45}$$

The terms on the right-hand side containing β can be further rewritten as

$$\begin{aligned}
&(\nabla^{LC,g}_Z\nabla^{LC,g}_V\beta)(X)-(\nabla^{LC,g}_V\nabla^{LC,g}_Z\beta)(X)-(\nabla^{LC,g}_{[Z,V]}\beta)(X)\\
&\quad= Z(\nabla^{LC,g}_V\beta)(X)-(\nabla^{LC,g}_V\beta)(\nabla^{LC,g}_Z X)-V(\nabla^{LC,g}_Z\beta)(X)+(\nabla^{LC,g}_Z\beta)(\nabla^{LC,g}_V X)\\
&\qquad-[Z,V]\beta(X)+\beta(\nabla^{LC,g}_{[Z,V]}X)\\
&\quad= ZV\beta(X)-Z\beta(\nabla^{LC,g}_V X)-V\beta(\nabla^{LC,g}_Z X)+\beta(\nabla^{LC,g}_V\nabla^{LC,g}_Z X)-VZ\beta(X)\\
&\qquad+V\beta(\nabla^{LC,g}_Z X)+Z\beta(\nabla^{LC,g}_V X)-\beta(\nabla^{LC,g}_Z\nabla^{LC,g}_V X)-[Z,V]\beta(X)+\beta(\nabla^{LC,g}_{[Z,V]}X)\\
&\quad= \beta(\nabla^{LC,g}_V\nabla^{LC,g}_Z X)-\beta(\nabla^{LC,g}_Z\nabla^{LC,g}_V X)-\beta(\nabla^{LC,g}_{[V,Z]}X)\\
&\quad= R^{LC,g}(\beta,X,V,Z),
\end{aligned}\tag{2.46}$$

and therefore

$$R_{0\nabla^g}((X,\alpha),(Y,\beta),(Z,\gamma),(V,\omega))=R^{LC,g}(\alpha,Y,Z,V)+R^{LC,g}(\beta,X,V,Z).\tag{2.47}$$

Besides this, there is also one term to be determined in order to express the whole Riemann tensor. By choosing an arbitrary local frame of the standard type, one immediately sees that this term vanishes identically, and therefore the Riemann tensor R_{∇^g} can be expressed as

$$\begin{aligned} R_{\nabla^g}((X,\alpha),(Y,\beta),(Z,\gamma),(V,\omega)) = \frac{1}{2}\Big(& R^{LC,g}(\alpha,Y,Z,V) + R^{LC,g}(\beta,X,V,Z) \\ & + R^{LC,g}(\gamma,V,X,Y) + R^{LC,g}(\omega,Z,Y,X)\Big), \end{aligned} \tag{2.48}$$

for all (X,α), (Y,β), (Z,γ), $(V,\omega) \in \Gamma(E)$. We can continue and compute the Ricci tensor Ric_{∇^g}. Taking arbitrary (X,α) and $(Y,\beta) \in \Gamma(E)$, choosing an arbitrary local frame of the standard type for taking trace lead to

$$\begin{aligned} \mathrm{Ric}_{\nabla^g}((X,\alpha),(Y,\beta)) &= R_{\nabla^g}(\sharp_E\,\xi^\mu,(X,\alpha),\xi_\mu,(Y,\beta)) \\ &= R_{\nabla^g}((0,e^j),(X,\alpha),(e_j,0),(Y,\beta)) + R_{\nabla^g}((e_k,0),(X,\alpha),(0,e^k),(Y,\beta)) \\ &= \frac{1}{2}(R^{LC,g}(e^j,X,e_j,Y) + R^{LC,g}(e^k,Y,e_k,X)) \\ &= \frac{1}{2}(\mathrm{Ric}^{LC,g}(X,Y) + \mathrm{Ric}^{LC,g}(Y,X)). \end{aligned} \tag{2.49}$$

Since $\mathrm{Ric}^{LC,g}$ is symmetric, we have

$$\mathrm{Ric}_{\nabla^g}((X,\alpha),(Y,\beta)) = \mathrm{Ric}^{LC,g}(X,Y), \qquad \Leftrightarrow \qquad \mathrm{Ric}_{\nabla^g} = \rho^*\,\mathrm{Ric}^{LC,g}\,. \tag{2.50}$$

There are two yet undetermined curvature objects, the Courant-Ricci scalar and G^g-Ricci scalar. For the former, one easily finds

$$\mathcal{R}_E^{\nabla^g} = \mathrm{Ric}_{\nabla^g}(\sharp_E\,\xi^\mu,\xi_\mu) = \mathrm{Ric}_{\nabla^g}((0,e^j),(e_j,0)) + \mathrm{Ric}_{\nabla^g}((e_k,0),(0,e^k)) = 0, \tag{2.51}$$

whereas for the latter, we obtain

$$\begin{aligned} \mathcal{R}_{G^g}^{\nabla^g} &= \mathrm{Ric}_{\nabla^g}(\sharp_{G^g}\,\xi^\mu,\xi_\mu) = \mathrm{Ric}_{\nabla^g}(\tau^g\sharp_E\,\xi^\mu,\xi_\mu) \\ &= \mathrm{Ric}_{\nabla^g}(\tau^g(0,e^j),(e_j,0)) + \mathrm{Ric}_{\nabla^g}(\tau^g(e_k,0),(0,e^k)) \\ &= \mathrm{Ric}_{\nabla^g}((\sharp_g\,e^j,0),(e_j,0)) + \mathrm{Ric}_{\nabla^g}((0,\flat_g\,e_k),(0,e^k)) = \mathrm{Ric}^{LC,g}(\sharp_g\,e^j,e_j) = \mathcal{R}^{LC,g}. \end{aligned} \tag{2.52}$$

We have used the general identity $\sharp_G = \tau\sharp_E$, see proof of the theorem 1.41, in the second step and the matrix representation of τ^g, see the proposition 2.13, in the fourth step.

Remark 2.19. It is good to realize that if we start from an arbitrary affine connection ∇^{AC} and not the Levi-Civita one, we would obtain the associated torsion tensor T in the form

$$T((X,\alpha),(Y,\beta),(Z,\gamma)) = H(X,Y,Z) + T^{AC}(\alpha,Y,Z) + T^{AC}(\beta,Z,X) + T^{AC}(\gamma,X,Y), \tag{2.53}$$

for all (X,α), (Y,β), $(Z,\gamma) \in \Gamma(E)$. The associated Riemann tensor, Courant-Ricci scalar and G^g-Ricci scalar would take exactly the same form as the quantities derived above, but all the ordinary curvature tensor fields would be related to ∇^{AC} instead of $\nabla^{LC,g}$. The Ricci tensor would be of the form (2.49), as Ric^{AC} is not necessarily symmetric.

Let us now proceed further with the construction of a Levi-Civita Courant algebroid connection. As we already have a Courant algebroid connection compatible with G^g at our disposal, we are able to derive the minimal Levi-Civita Courant algebroid connection corresponding to it.

Notation 2.20. Note that covariant divergences can be naturally used even on the level of ordinary affine connection, namely for a given metric g and an affine connection ∇^{AC} on M, we define

$$\operatorname{div}_{\nabla^{AC}} X := e^j(\nabla^{AC}_{e_j} X), \qquad \operatorname{div}^g_{\nabla^{AC}} \alpha := (\nabla^{AC}_{e_j}\alpha)(\sharp_g e^j), \tag{2.54}$$

for all $X \in \Gamma(TM)$ and all $\alpha \in \Omega^1(M)$, especially we state $\operatorname{div}^g := \operatorname{div}_{\nabla^{LC,g}}$ and $\operatorname{div}^g := \operatorname{div}^g_{\nabla^{LC,g}}$.

Example 2.21 (Minimal Levi-Civita Courant algebroid connection corresponding to ∇^g)**.** To obtain such connection, it is necessarily to compute the tensor field C given by (1.138). For arbitrary (X,α), (Y,β), $(Z,\gamma) \in \Gamma(E)$, one sees

$$\begin{aligned}
&C((X,\alpha),(Y,\beta),(Z,\gamma))\\
&\quad= -\frac{1}{3}(T_{\nabla^g}(\pi_+(X,\alpha),\pi_+(Y,\beta),\pi_+(Z,\gamma)) + T_{\nabla^g}(\pi_-(X,\alpha),\pi_-(Y,\beta),\pi_-(Z,\gamma))\\
&\qquad - T_{\nabla^g}(\pi_+(X,\alpha),\pi_-(Y,\beta),\pi_-(Z,\gamma)) - T_{\nabla^g}(\pi_-(X,\alpha),\pi_+(Y,\beta),\pi_+(Z,\gamma))\\
&\quad= -\frac{1}{24}\Big(H(X+\sharp_g\alpha, Y+\sharp_g\beta, Z+\sharp_g\gamma) + H(X-\sharp_g\alpha, Y-\sharp_g\beta, Z-\sharp_g\gamma)\\
&\qquad + 3H(X+\sharp_g\alpha, Y-\sharp_g\beta, Z-\sharp_g\gamma) + 3H(X-\sharp_g\alpha, Y+\sharp_g\beta, Z+\sharp_g\gamma)\Big)\\
&\quad= -\frac{1}{3}H(X,Y,Z) - \frac{1}{3}H(X,\sharp_g\beta,\sharp_g\gamma) + \frac{1}{6}H(\sharp_g\alpha,Y,\sharp_g\gamma) + \frac{1}{6}H(\sharp_g\alpha,\sharp_g\beta,Z),
\end{aligned} \tag{2.55}$$

the equation (2.44) and the lemma 2.14 have been used simultaneously in the second step. To get an explicit form of the minimal Levi-Civita connection ∇^0 corresponding to ∇^g, we have to cast the smooth section $\sharp_E C((X,\alpha),(Y,\beta),.)$ in the form of an ordered pair. It can be done by solving the equation

$$0 = g_E((V,\omega) - \sharp_E C((X,\alpha),(Y,\beta),.),(Z,\gamma)) \tag{2.56}$$

for (V,ω), where (X,α), (Y,β), $(Z,\gamma) \in \Gamma(E)$ are arbitrary. By employing (2.55), one finds that the equation above is equivalent to

$$\begin{aligned}
&\gamma(V) + \omega(Z)\\
&\quad= C((X,\alpha),(Y,\beta),(Z,\gamma))\\
&\quad= -\frac{1}{3}H(X,Y,Z) - \frac{1}{3}H(X,\sharp_g\beta,\sharp_g\gamma) + \frac{1}{6}H(\sharp_g\alpha,Y,\sharp_g\gamma) + \frac{1}{6}H(\sharp_g\alpha,\sharp_g\beta,Z)\\
&\quad= \gamma(-\frac{1}{3}\sharp_g H(X,\sharp_g\beta,.) + \frac{1}{6}\sharp_g H(\sharp_g\alpha,Y,.)) + (-\frac{1}{3}H(X,Y,.) + \frac{1}{6}H(\sharp_g\alpha,\sharp_g\beta,.))(Z),
\end{aligned} \tag{2.57}$$

hence

$$\begin{aligned}
&\sharp_E C((X,\alpha),(Y,\beta),.)\\
&\qquad= \Big(-\frac{1}{3}\sharp_g H(X,\sharp_g\beta,.) + \frac{1}{6}\sharp_g H(\sharp_g\alpha,Y,.), -\frac{1}{3}H(X,Y,.) + \frac{1}{6}H(\sharp_g\alpha,\sharp_g\beta,.)\Big).
\end{aligned} \tag{2.58}$$

Finally, ∇^0 can be written as

$$\begin{aligned}
\nabla^0{}_{(X,\alpha)}(Y,\beta) = \nabla^g_{(X,\alpha)}(Y,\beta) + \Big(&-\frac{1}{3}\sharp_g H(X,\sharp_g\beta,.) + \frac{1}{6}\sharp_g H(\sharp_g\alpha,Y,.),\\
&-\frac{1}{3}H(X,Y,.) + \frac{1}{6}H(\sharp_g\alpha,\sharp_g\beta,.)\Big),
\end{aligned} \tag{2.59}$$

or in the matrix representation as

$$\nabla^0{}_{(X,\alpha)} = \begin{pmatrix} \nabla^{LC,g}_X + \frac{1}{6}\sharp_g H(\sharp_g \alpha, \star, .) & -\frac{1}{3}\sharp_g H(X, \sharp_g \star, .) \\ -\frac{1}{3}H(X, \star, .) & \nabla^{LC,g}_X + \frac{1}{6}H(\sharp_g \alpha, \sharp_g \star, .) \end{pmatrix}, \tag{2.60}$$

for all (X, α), $(Y, \beta) \in \Gamma(E)$. At the end of the example, we derive the covariant divergence corresponding to ∇^0. It is actually sufficient to calculate div_{∇^g} and $\mathcal{C}$, see (1.142). For all $(X, \alpha) \in \Gamma(E)$, one finds

$$\begin{aligned}\mathcal{C}((X,\alpha)) &= C(\sharp_E \xi^\mu, \xi_\mu, (X,\alpha)) \\ &= C((0,e^j),(e_j,0),(X,\alpha)) + C((e_k,0),(0,e^k),(X,\alpha)) \\ &= -\frac{1}{6}H(\sharp_g e^j, e_j, \sharp_g \alpha) = 0. \end{aligned}\tag{2.61}$$

It vanishes, since H is completely skew-symmetric. While $\mathcal{C} = 0$, we find the following for div_{∇^g}:

$$\mathrm{div}_{\nabla^g}(X,\alpha) = \xi^\mu(\nabla^g_{\xi_\mu}(X,\alpha)) = (e^j,0)(\nabla^g_{(e_j,0)}(X,\alpha)) = e^j(\nabla^{LC,g}_{e_j} X) = \mathrm{div}^g X. \tag{2.62}$$

Putting these together, we arrive at

$$\mathrm{div}_{\nabla^0} = \mathrm{div}_{\nabla^g} = \mathrm{div}^g \circ \rho. \tag{2.63}$$

Thanks to the theorem 1.59, any element ∇ of the set $\mathrm{LC}(\mathbb{T}M, G^g)$ can be for all (X, α) $(Y, \beta) \in \Gamma(E)$ expressed as

$$\nabla_{(X,\alpha)}(Y,\beta) = \nabla^0_{(X,\alpha)}(Y,\beta) + \sharp_E K((X,\alpha),(Y,\beta),.), \tag{2.64}$$

where ∇^0 is the minimal Levi-Civita Courant algebroid connection corresponding to ∇^g known explicitly from the previous example and K is a tensor field possessing appropriate properties, see 1.61. So, the task is accomplished, we have just described the whole set $\mathrm{LC}(\mathbb{T}M, G^g)$. However, another additional step can be made, we can "simplify" the parametrization of this set. What exactly is meant under the "simplification" is described by the following lemma.

Lemma 2.22. *Consider the generalized tangent bundle $\mathbb{T}M$ over a smooth manifold M, the fiber-wise metric g_E on $\mathbb{T}M$ defined in the example 2.1 and a generalized metric τ. Then for any $K \in \Omega^1(E) \otimes \Omega^2(E)$ satisfying*

$$K(\psi_1, \psi_2, \tau^g \psi_3) - K(\psi_1, \tau^g \psi_2, \psi_3) = 0, \tag{2.65}$$

$$K(\psi_1, \psi_2, \psi_3) + cyc(\psi_1, \psi_2, \psi_3) = 0, \tag{2.66}$$

for all ψ_1, ψ_2, $\psi_3 \in \Gamma(E)$, there is a unique ordered pair (W, J) of ordinary tensor fields from the set $\mathcal{T}^0_3(M)$ such that the equality holds

$$\begin{aligned} &K((X,\alpha),(Y,\beta),(Z,\gamma)) \\ &\quad = W(\sharp_g \alpha, Y, Z) + W(X, \sharp_g \beta, Z) + W(X, Y, \sharp_g \gamma) + W(\sharp_g \alpha, \sharp_g \beta, \sharp_g \gamma) \\ &\qquad + J(X, \sharp_g \beta, \sharp_g \gamma) + J(\sharp_g \alpha, Y, \sharp_g \gamma) + J(\sharp_g \alpha, \sharp_g \beta, Z) + J(X, Y, Z), \end{aligned}\tag{2.67}$$

for all (X, α), (Y, β), $(Z, \gamma) \in \Gamma(E)$. Moreover, both of the tensor fields W and J are skew-symmetric in the last two inputs and their complete skew-symmetrizations vanish identically.

Proof. Take an arbitrary $K \in \Omega^1(E) \otimes \Omega^2(E)$ satisfying the properties (2.65) and (2.66); and an arbitrary $(X,\alpha) \in \Gamma(E)$, and define a $C^\infty(M)$-module endomorphism $\hat{K}_{(X,\alpha)} : \Gamma(E) \to \Gamma(E)$ as

$$\hat{K}_{(X,\alpha)}(Y,\beta) := \sharp_E\, K((X,\alpha),(Y,\beta),.), \tag{2.68}$$

for all $(Y,\beta) \in \Gamma(E)$. Apparently, $\hat{K}_{(X,\alpha)} = \hat{K}_{(X,0)} + \hat{K}_{(0,\alpha)}$, and therefore we can represent it as

$$\hat{K}_{(X,\alpha)} = \begin{pmatrix} L_X + \Lambda_\alpha & N_X + \Xi_\alpha \\ S_X + \Sigma_\alpha & T_X + \Theta_\alpha \end{pmatrix}, \tag{2.69}$$

where all eight objects contained in the matrix are $C^\infty(M)$-linear maps between the appropriate $C^\infty(M)$-modules. Let us see what constraints are imposed on them by the properties of K. First of all, K is skew-symmetric in the last two inputs, that is

$$\begin{aligned} 0 &= K((X,\alpha),(Y,\beta),(Z,\gamma)) + K((X,\alpha),(Z,\gamma),(Y,\beta)) \\ &= g_E(\hat{K}_{(X,\alpha)}(Y,\beta),(Z,\gamma)) + g_E(\hat{K}_{(X,\alpha)}(Z,\gamma),(Y,\beta)) \\ &= g_E((L_XY + \Lambda_\alpha Y + N_X\beta + \Xi_\alpha\beta, S_XY + \Sigma_\alpha Y + T_X\beta + \Theta_\alpha\beta),(Z,\gamma)) \\ &\quad + g_E((L_XZ + \Lambda_\alpha Z + N_X\gamma + \Xi_\alpha\gamma, S_XZ + \Sigma_\alpha Z + T_X\gamma + \Theta_\alpha\gamma),(Y,\beta)), \end{aligned} \tag{2.70}$$

for all (X,α), (Y,β), $(Z,\gamma) \in \Gamma(E)$. One can check that it is equivalent to the set of relations

$$\begin{aligned} T_X &= -\,L_X^T, & \gamma(N_X\beta) &= -\,\beta(N_X\gamma), & (S_XY)(Z) &= -\,(S_XZ)(Y), \\ \Theta_\alpha &= -\,\Lambda_\alpha^T, & \gamma(\Xi_\alpha\beta) &= -\,\beta(\Xi_\alpha\gamma), & (\Sigma_\alpha Y)(Z) &= -\,(\Sigma_\alpha Z)(Y). \end{aligned} \tag{2.71}$$

In the exactly same way, one finds that (2.65) is under the assumption that (2.71) holds equivalent to the set of relations

$$L_X^T = -\,\flat_g\, L_X \sharp_g\,, \qquad \Lambda_\alpha^T = -\,\flat_g\, \Lambda_\alpha \sharp_g\,, \qquad S_X = \flat_g\, N_X \flat_g\,, \qquad \Sigma_\alpha = \flat_g\, \Xi_\alpha \flat_g\,, \tag{2.72}$$

hence

$$\hat{K}_{(X,\alpha)} = \begin{pmatrix} L_X + \Lambda_\alpha & N_X + \Xi_\alpha \\ \flat_g\,(N_X + \Xi_\alpha)\flat_g & \flat_g\,(L_X + \Lambda_\alpha)\sharp_g \end{pmatrix}. \tag{2.73}$$

Considering $\hat{K}_{(X,\alpha)}$ in this form yields

$$\begin{aligned} &K((X,\alpha),(Y,\beta),(Z,\gamma)) \\ &= g_E(\hat{K}_{(X,\alpha)}(Y,\beta),(Z,\gamma)) \\ &= g_E((L_XY + \Lambda_\alpha Y + N_X\beta + \Xi_\alpha\beta, \flat_g\, N_X\flat_g\, Y + \flat_g\, \Xi_\alpha\flat_g\, Y + \flat_g\, L_X\sharp_g\, \beta + \flat_g\, \Lambda_\alpha\sharp_g\, \beta),(Z,\gamma)) \\ &= \gamma(L_XY) + \gamma(\Lambda_\alpha Y) + \gamma(N_X\beta) + \gamma(\Xi_\alpha\beta) + (\flat_g\, N_X\flat_g\, Y)(Z) + (\flat_g\, \Xi_\alpha\flat_g\, Y)(Z) \\ &\quad + (\flat_g\, L_X\sharp_g\, \beta)(Z) + (\flat_g\, \Lambda_\alpha\sharp_g\, \beta)(Z), \end{aligned} \tag{2.74}$$

for all (X,α), (Y,β), $(Z,\gamma) \in \Gamma(E)$. Denoting

$$W(X,Y,Z) := g(L_XY,Z), \qquad J(X,Y,Z) := g(\Lambda_{\flat_g X}Y,Z), \tag{2.75}$$

for all X, Y, $Z \in \Gamma(TM)$ leads to

$$\begin{aligned} &K((X,\alpha),(Y,\beta),(Z,\gamma)) \\ &\quad = W(X,Y,\sharp_g\,\gamma) + W(X,\sharp_g\,\beta,Z) + g(\Xi_\alpha\beta,\sharp_g\,\gamma) + g(\Xi_\alpha\flat_g\, Y,Z) \\ &\qquad + J(\sharp_g\,\alpha,Y,\sharp_g\,\gamma) + J(\sharp_g\,\alpha,\sharp_g\,\beta,Z) + g(N_X\beta,\sharp_g\,\gamma) + g(N_X\flat_g\, Y,Z). \end{aligned} \tag{2.76}$$

By comparing with (2.67), one sees that the last missing piece of the puzzle is to show that the relations

$$\Xi_\alpha \circ \flat_g = L_{\sharp_g \alpha}, \qquad N_X = \Lambda_{\flat_g X} \circ \sharp_g \tag{2.77}$$

hold. To do that, we need to employ the property (2.66) in the following form:

$$K(\tau^g(X,\alpha),(X,\alpha),(Y,\beta)) + cyc(\tau^g(X,\alpha),(X,\alpha),(Y,\beta)) = 0, \tag{2.78}$$

for all (X,α), $(Y,\beta) \in \Gamma(E)$. Realize that the skew-symmetry in the last two inputs of K says

$$K((Y,\beta),\tau^g(X,\alpha),(X,\alpha)) = -K((Y,\beta),(X,\alpha),\tau^g(X,\alpha)), \tag{2.79}$$

whereas (2.65) implies

$$K((Y,\beta),\tau^g(X,\alpha),(X,\alpha)) = K((Y,\beta),(X,\alpha),\tau^g(X,\alpha)). \tag{2.80}$$

That is $K((Y,\beta),\tau^g(X,\alpha),(X,\alpha)) = 0$, and therefore we can write (2.78) equivalently as

$$0 = K((\sharp_g \alpha, \flat_g X),(X,\alpha),(Y,\beta)) - K((X,\alpha),(\sharp_g \alpha, \flat_g X),(Y,\beta)). \tag{2.81}$$

For $\hat{K}_{(X,\alpha)}$ in the form (2.73), it says

$$\begin{aligned} 0 = {} & g(N_{\sharp_g \alpha}\flat_g X - \Lambda_\alpha X + \Xi_{\flat_g X}\flat_g X - L_X X + L_{\sharp_g \alpha}\sharp_g \alpha - \Xi_\alpha \alpha + \Lambda_{\flat_g X}\sharp_g \alpha - N_X \alpha, Y) \\ & + \beta(L_{\sharp_g \alpha} X - \Xi_\alpha \flat_g X + \Lambda_{\flat_g X} X - N_X \flat_g X + N_{\sharp_g \alpha}\alpha - \Lambda_\alpha \sharp_g \alpha + \Xi_{\flat_g X}\alpha - L_X \sharp_g \alpha). \end{aligned} \tag{2.82}$$

In particular, choosing $X = Y = 0$ and $X = 0$, $\beta = 0$ lead to

$$0 = N_{\sharp_g \alpha}\alpha - \Lambda_\alpha \sharp_g \alpha, \qquad 0 = \Xi_\alpha \alpha - L_{\sharp_g \alpha}\sharp_g \alpha, \tag{2.83}$$

for all $\alpha \in \Omega^1(M)$ respectively. By putting these back into (2.82) and imposing $\beta = 0$ and $Y = 0$, we obtain

$$0 = N_{\sharp_g \alpha}\flat_g X - \Lambda_\alpha X + \Lambda_{\flat_g X}\sharp_g \alpha - N_X \alpha, \quad 0 = L_{\sharp_g \alpha} X - \Xi_\alpha \flat_g X + \Xi_{\flat_g X}\alpha - L_X \sharp_g \alpha, \tag{2.84}$$

for all $\alpha \in \Omega^1(M)$ and $X \in \Gamma(TM)$ respectively. Thanks to the linearity, the relations (2.83) can be expanded as follows:

$$\begin{aligned} 0 &= N_{\sharp_g (\alpha + \flat_g X)}(\alpha + \flat_g X) - \Lambda_{\alpha + \flat_g X}\sharp_g (\alpha + \flat_g X) \\ &= N_X \alpha + N_{\sharp_g \alpha}\flat_g X - \Lambda_\alpha X - \Lambda_{\flat_g X}\sharp_g \alpha, \end{aligned} \tag{2.85}$$

$$\begin{aligned} 0 &= \Xi_{\alpha + \flat_g X}(\alpha + \flat_g X) - L_{\sharp_g (\alpha + \flat_g X)}\sharp_g (\alpha + \flat_g X) \\ &= \Xi_\alpha \flat_g X + \Xi_{\flat_g X}\alpha - L_{\sharp_g \alpha} X - L_X \sharp_g \alpha. \end{aligned} \tag{2.86}$$

Combining these with (2.84) results into the desired relations

$$N_X \alpha = \Lambda_{\flat_g X}\sharp_g \alpha, \qquad \Xi_\alpha \flat_g X = L_{\sharp_g \alpha} X, \tag{2.87}$$

thus we have just found W and J such that (2.67) holds. This proves the existence. To prove the uniqueness, it remains to realize that any pair (W, J) satisfying (2.67) can be expressed as

$$W(X,Y,Z) = K((0,\flat_g X),(0,\flat_g Y),(0,\flat_g Z)), \quad J(X,Y,Z) = K((X,0),(Y,0),(Z,0)), \tag{2.88}$$

for all X, Y, $Z \in \Gamma(TM)$. It is immediately seen if we choose $X = Y = Z = 0$ or $\alpha = \beta = \gamma = 0$ for (2.67). The last thing to prove is that W and J are skew-symmetric in the last two inputs and that their complete skew-symmetrizations vanish identically. Considering W and J in the form (2.88), it follows directly from that K possesses those properties. □

Remark 2.23. We have just shown that set $\mathrm{LC}(\mathbb{T}M, G^g)$ can be parametrized by a pair of ordinary manifold tensor fields instead of one tensor field on $\mathbb{T}M$, thus the aforementioned "simplification" means that parametrization can be reformulated in the terms of more fundamental objects. Note that the correspondence between K and (W, J) is one-to-one.

Corollary 2.24. *Let all assumptions and notation of the previous lemma hold and impose*

$$\mathcal{W} := W(\sharp_g e^j, e_j, .), \qquad \mathcal{J} := J(\sharp_g e^j, e_j, .). \tag{2.89}$$

Then the identities

$$W(X,Y,Z) = K((0, \flat_g X), (0, \flat_g Y), (0, \flat_g Z)) \tag{2.90}$$

$$J(X,Y,Z) = K((X,0),(Y,0),(Z,0)) \tag{2.91}$$

$$\mathcal{K} = 2(\mathcal{W}, \sharp_g \mathcal{J}), \tag{2.92}$$

are satisfied for all X, Y, $Z \in \Gamma(TM)$.

Proof. The validity of first two identities have been already shown at the end of the proof of the previous lemma. To prove the third one, take an arbitrary $(X, \alpha) \in \Gamma(E)$ and proceed as follows:

$$\begin{aligned}\mathcal{K}((X,\alpha)) &= K(\sharp_E \xi^\mu, \xi_\mu, (X,\alpha)) = K((0,e^j),(e_j,0),(X,\alpha)) + K((e_k,0),(0,e^k),(X,\alpha)) \\ &= W(\sharp_g e^j, e_j, X) + J(\sharp_g e^j, e_j, \sharp_g \alpha) + W(e_k, \sharp_g e^k, X) + J(e_k, \sharp_g e^k, \sharp_g \alpha) \\ &= 2\mathcal{W}(X) + 2\mathcal{J}(\sharp_g \alpha),\end{aligned} \tag{2.93}$$

hence $\mathcal{K} = 2(\mathcal{W}, \sharp_g \mathcal{J})$. The relation (2.67) has been used in the third step. □

In the next, we will calculate both of the Ricci tensor contractions associated with an arbitrary $\nabla \in \mathrm{LC}(\mathbb{T}M, G^g)$. Especially, the result for G^g-Ricci scalar will play a crucial role in the next chapter.

Theorem 2.25. *Consider a Courant algebroid $(E \overset{\pi}{\to} M, \rho, [.,.]_D^H, g_E)$ and a Riemannian metric g. Moreover, let $\nabla^0 \in \mathrm{LC}(\mathbb{T}M, G^g)$ be the minimal Levi-Civita Courant algebroid connection corresponding to ∇^g defined in the example 2.16. If $\nabla \in \mathrm{LC}(\mathbb{T}M, G^g)$ is related to ∇^0 as*

$$\nabla_{(X,\alpha)}(Y,\beta) = \nabla^0_{(X,\alpha)}(Y,\beta) + \sharp_E K((X,\alpha),(Y,\beta),.), \tag{2.94}$$

for all (X,α), $(Y,\beta) \in \Gamma(E)$, then the corresponding Courant-Ricci scalar and G^g-Ricci scalar can be expressed as

$$\mathcal{R}_E^\nabla = 4\,\mathrm{div}^g(\mathcal{J}) - 8g^{-1}(\mathcal{W}, \mathcal{J}), \tag{2.95}$$

$$\mathcal{R}_{G^g}^\nabla = \mathcal{R}^{LC,g} - \frac{1}{2}(H,H)_g + 4\,\mathrm{div}^g(\mathcal{W}) - 4g^{-1}(\mathcal{W},\mathcal{W}) - 4g^{-1}(\mathcal{J},\mathcal{J}), \tag{2.96}$$

where W, $J \in \mathcal{T}_3^0(M)$ are uniquely given by K trough the lemma 2.22.

Lemma 2.26. *Let all assumptions of the theorem hold. Then the Ricci tensor corresponding to ∇^0 can be expressed as*

$$\begin{aligned}\mathrm{Ric}_{\nabla^0}((X,\alpha),(Y,\beta)) = \mathrm{Ric}^{LC,g}(X,Y) &- \frac{1}{4}((\nabla^{LC,g}_{e_j} H)(X, \sharp_g \beta, \sharp_g e^j) - (\nabla^{LC,g}_{e_j} H)(\sharp_g \alpha, Y, \sharp_g e^j)) \\ &- \frac{1}{3}(\mathrm{i}_X H, \mathrm{i}_Y H)_g + \frac{1}{6}(\mathrm{i}_{\sharp_g \alpha} H, \mathrm{i}_{\sharp_g \beta} H)_g,\end{aligned} \tag{2.97}$$

for all (X,α), $(Y,\beta) \in \Gamma(E)$. *Moreover, the corresponding Courant-Ricci scalar and* G^g*-Ricci scalar take the form*

$$\mathcal{R}_E^{\nabla^0} = 0, \qquad \mathcal{R}_{G^g}^{\nabla^0} = \mathcal{R}^{LC,g} - \frac{1}{2}(H,H)_g. \tag{2.98}$$

Proof of the lemma. The first part of the proof is a simple use of the proposition 1.65, we just need to compute all the terms on the right-hand side of (1.160) for the particular choice of C given by (2.55). In our case, the first term is Ric_{∇^g}, which can be due to (2.50) cast as

$$\mathrm{Ric}_{\nabla^g} = \rho^* \,\mathrm{Ric}^{LC,g}\,. \tag{2.99}$$

The second, the third and the sixth term is proportional to $\mathcal{C}$. As (2.61) says $\mathcal{C} = 0$, all three terms vanish. The fourth and the fifth term are both of the same type, the former can be expanded as follows:

$$\begin{aligned}
&(\nabla^g_{\xi_\mu} C)((X,\alpha),(Y,\beta),\sharp_E\,\xi^\mu) \\
&\quad= (\nabla^g_{(e_j,0)} C)((X,\alpha),(Y,\beta),(0,e^j)) + (\nabla^g_{(0,e^k)} C)((X,\alpha),(Y,\beta),(e_k,0)) \\
&\quad= e_j C((X,\alpha),(Y,\beta),(0,e^j)) - C((\nabla^{LC,g}_{e_j} X, \nabla^{LC,g}_{e_j}\alpha),(Y,\beta),(0,e^j)) \\
&\qquad - C((X,\alpha),(\nabla^{LC,g}_{e_j} Y, \nabla^{LC,g}_{e_j}\beta),(0,e^j)) - C((X,\alpha),(Y,\beta),(0,\nabla^{LC,g}_{e_j} e^j)) \\
&\quad= \frac{1}{6}(\nabla^{LC,g}_{e_j} H)(\sharp_g\,\alpha, Y, \sharp_g\, e^j) - \frac{1}{3}(\nabla^{LC,g}_{e_j} H)(X, \sharp_g\,\beta, \sharp_g\, e^j),
\end{aligned} \tag{2.100}$$

for all (X,α), $(Y,\beta) \in \Gamma(E)$ and the latter is obtained by swapping (X,α) with (Y,β). Let us now move to the quadratic terms in C, for all (X,α), $(Y,\beta) \in \Gamma(E)$ there holds

$$\begin{aligned}
&C(\sharp_E\, C(.\,,\xi_\mu,(Y,\beta)),(X,\alpha),\sharp_E\,\xi^\mu) \\
&\quad= C(\sharp_E\,\xi^\nu,\xi_\mu,(Y,\beta))C(\xi_\nu,(X,\alpha),\sharp_E\,\xi^\mu) \\
&\quad= C((0,e^j),(e_k,0),(Y,\beta))C((e_j,0),(X,\alpha),(0,e^k)) + C((e_l,0),(e_k,0),(Y,\beta)) \\
&\qquad \cdot C((0,e^l),(X,\alpha),(0,e^k)) + C((0,e^j),(0,e^m),(Y,\beta))C((e_j,0),(X,\alpha),(e_m,0)) \\
&\qquad + C((e_l,0),(0,e^m),(Y,\beta))C((0,e^l),(X,\alpha),(e_m,0)) \\
&\quad= -\frac{1}{18}H(\sharp_g\,e^j, e_k, \sharp_g\,\beta)H(e_j,\sharp_g\,\alpha,\sharp_g\,e^k) - \frac{1}{18}H(e_l,e_k,Y)H(\sharp_g\,e^l, X, \sharp_g\,e^k) \\
&\qquad - \frac{1}{18}H(\sharp_g\,e^j,\sharp_g\,e^m, Y)H(e_j,X,e_m) - \frac{1}{18}H(e_l,\sharp_g\,e^m,\sharp_g\,\beta)H(\sharp_g\,e^l,\sharp_g\,\alpha,e_m) \\
&\quad= -\frac{1}{9}(\mathrm{i}_{\sharp_g\alpha}H)(\sharp_g\,e^j,\sharp_g\,e^k)(\mathrm{i}_{\sharp_g\beta}H)(e_j,e_k) - \frac{1}{9}(\mathrm{i}_X H)(\sharp_g\,e^l,\sharp_g\,e^k)(\mathrm{i}_Y H)(e_l,e_k) \\
&\quad= -\frac{2}{9}(\mathrm{i}_{\sharp_g\alpha}H,\mathrm{i}_{\sharp_g\beta}H)_g - \frac{2}{9}(\mathrm{i}_X H,\mathrm{i}_Y H)_g.
\end{aligned} \tag{2.101}$$

The two remaining quadratic terms can be calculated in the exactly same manner. Moreover, both of them are mutually the same up to the order of the sections (X,α) and (Y,β); and each of them leads precisely to

$$\frac{1}{18}(\mathrm{i}_{\sharp_g\alpha}H,\mathrm{i}_{\sharp_g\beta}H)_g - \frac{1}{9}(\mathrm{i}_X H,\mathrm{i}_Y H)_g. \tag{2.102}$$

It remains to deal with the last two terms. For the first of them, the equations (2.44) and (2.55) yield

$$\begin{aligned} C(T_{\nabla^g}((X,\alpha),\sharp_E\,\xi^\mu),\xi_\mu,(Y,\beta)) &= T_{\nabla^g}((X,\alpha),\sharp_E\,\xi^\mu,\sharp_E\,\xi^\nu)C(\xi_\nu,\xi_\mu,(Y,\beta)) \\ &= T_{\nabla^g}((X,\alpha),(e_j,0),(e_k,0))C((0,e^k),(0,e^j),(Y,\beta)) \\ &= \frac{1}{6}H(X,e_j,e_k)H(\sharp_g\,e^k,\sharp_g\,e^j,Y) \\ &= -\frac{1}{3}(\mathrm{i}_X H,\mathrm{i}_Y H)_g, \end{aligned} \tag{2.103}$$

for all (X,α), $(Y,\beta)\in\Gamma(E)$. Note that we have omitted three terms in the second step, it is because they vanish identically thanks to (2.44). Since the second term with the torsion differs only in the order of the inputs (X,α) and (Y,β) and the expression above is symmetric in those, it must yield the same result. Putting (2.99), (2.61), (2.100), (2.101), (2.102) and (2.103) together results into the dseired formula for the Ricci tensor. Let us now have a look on the part concerning with the Ricci scalars. As both $\mathcal{R}_E^{\nabla^g}$ and $\mathcal{C}$ vanish, see (2.51) and (2.61), the formula (1.163) says that

$$\mathcal{R}_E^{\nabla^0} = -3(C,T_{\nabla^0})_E = -\frac{1}{2}C((0,e^j),(0,e^k),(0,e^l))H(e_j,e_k,e_l) = 0. \tag{2.104}$$

By using the result for Ric_{∇^0} and following the approach for derivation of $\mathcal{R}_{G^g}^{\nabla^g}$, see (2.52), we obtain

$$\begin{aligned} \mathcal{R}_{G^g}^{\nabla^0} &= \mathrm{Ric}_{\nabla^0}(\sharp_{G^g}\,\xi^\mu,\xi_\mu) = \mathrm{Ric}_{\nabla^0}(\tau^g\sharp_E\,\xi^\mu,\xi_\mu) \\ &= \mathrm{Ric}_{\nabla^0}((\sharp_g\,e^j,0),(e_j,0)) + \mathrm{Ric}_{\nabla^0}((0,\flat_g\,e_k),(0,e^k)) \\ &= \mathcal{R}^{LC,g} - \frac{1}{3}(\mathrm{i}_{\sharp_g e^j}H,\mathrm{i}_{e_j}H)_g + \frac{1}{6}(\mathrm{i}_{e_k}H,\mathrm{i}_{\sharp_g e^k}H)_g \\ &= \mathcal{R}^{LC,g} - \frac{1}{2}(H,H)_g. \end{aligned} \tag{2.105}$$

□

Proof of the theorem. The cornerstones of this proof are the propositions 1.67 and 1.69. The Courant-Ricci scalar $\mathcal{R}_E^{\nabla^0}$ is already known from the lemma, so let us have a look on the expressions containing $\mathcal{K} = 2(\mathcal{W},\sharp_g\,\mathcal{J})$, see (2.92). The formula for the covariant divergence of the minimal Levi-Civita connection (2.63) together with (1.41) implies

$$\mathrm{div}_{\nabla^0}\,\mathcal{K} = 2\,\mathrm{div}^g\,\mathcal{J}. \tag{2.106}$$

It remains to determine the last term on the right-hand side of (1.164), one easily finds

$$g_E^{-1}(\mathcal{K},\mathcal{K}) = g_E(\sharp_E\,\mathcal{K},\sharp_E\,\mathcal{K}) = 4g_E((\sharp_g\,\mathcal{J},\mathcal{W}),(\sharp_g\,\mathcal{J},\mathcal{W})) = 8\mathcal{W}(\sharp_g\,\mathcal{J}) = 8g^{-1}(\mathcal{W},\mathcal{J}). \tag{2.107}$$

By employing (2.98), (2.106) and (2.107) into the equation (1.164), we obtain the relation to be proven. To prove the second formula, first realize that $\Gamma(V_\pm^g) = \{(X,\pm\flat_g\,X) \mid X\in\Gamma(TM)\}$, hence $\{(e_j,\pm\flat_g\,e_j)\}_{j=1}^{\dim(M)}$ is apparently a local frame of $V_\pm^g$, and $\{(\frac{1}{2}e^j,\pm\frac{1}{2}\sharp_g\,e^j)\}_{j=1}^{\dim(M)}$ is the corresponding dual one. Without loss of generality, we will work exclusively with the local frames of this form. The important ingredients for calculating the divergences are

$$\begin{aligned} \nabla^0_{(e_j,\pm\flat_g\,e_j)}(\pm\sharp_g\,e^j,e^j) &= \begin{pmatrix} \nabla^{LC,g}_{e_j} \pm \frac{1}{6}\sharp_g\,H(e_j,\star,.) & -\frac{1}{3}\sharp_g\,H(e_j,\sharp_g\,\star,.) \\ -\frac{1}{3}H(e_j,\star,.) & \nabla^{LC,g}_{e_j} \pm \frac{1}{6}H(e_j,\sharp_g\,\star,.) \end{pmatrix} \begin{pmatrix} \pm\sharp_g\,e^j \\ e^j \end{pmatrix} \\ &= (\pm\nabla^{LC,g}_{e_j}\sharp_g\,e^j,\nabla^{LC,g}_{e_j}e^j), \end{aligned} \tag{2.108}$$

where vanishing of the partial trace of $H \in \Omega^3(M)$ has been used. Considering this leads to

$$\begin{aligned}\operatorname{div}^{\pm}_{\nabla^0} \mathcal{K} &= \frac{1}{2}(\nabla^0_{(e_j, \pm\flat_g e_j)}\mathcal{K})(\sharp_E (e^j, \pm\sharp_g e^j)) = \frac{1}{2} e_j \mathcal{K}((\pm\sharp_g e^j, e^j)) - \frac{1}{2}\mathcal{K}(\nabla^0_{(e_j, \pm\flat_g e_j)}(\pm\sharp_g e^j, e^j)) \\ &= e_j(\pm\mathcal{W}(\sharp_g e^j) + e^j(\sharp_g \mathcal{J})) \mp \mathcal{W}(\nabla^{LC,g}_{e_j}\sharp_g e^j) - (\nabla^{LC,g}_{e_j} e^j)(\sharp_g \mathcal{J}) \\ &= \pm (\nabla^{LC,g}_{e_j}\mathcal{W})(\sharp_g e^j) + e^j(\nabla^{LC,g}_{e_j}\sharp_g \mathcal{J}) = \pm \operatorname{div}^g \mathcal{W} + \operatorname{div}^g \mathcal{J}. \end{aligned} \tag{2.109}$$

It remains to compute the terms $(g_E^{\pm})^{-1}(\mathcal{K}^{\pm}, \mathcal{K}^{\pm})$, one easily derives that

$$\begin{aligned}(g_E^{\pm})^{-1}(\mathcal{K}^{\pm}, \mathcal{K}^{\pm}) &= \mathcal{K}^{\pm}(\sharp_E \mathcal{K}^{\pm}) = \mathcal{K}^{\pm}(\xi^{\pm}_a)\mathcal{K}^{\pm}(\sharp_E \xi^{\pm a}) = \frac{1}{2}\mathcal{K}^{\pm}((e_j, \pm\flat_g e_j))\mathcal{K}^{\pm}(\sharp_E (e^j, \pm\sharp_g e^j)) \\ &= \frac{1}{2}\mathcal{K}((e_j, \pm\flat_g e_j))\mathcal{K}((\pm\sharp_g e^j, e^j)) = 2(\mathcal{W}(e_j) \pm \mathcal{J}(e_j))(\pm\mathcal{W}(\sharp_g e^j) + \mathcal{J}(\sharp_g e^j)) \\ &= \pm 2g^{-1}(\mathcal{W}, \mathcal{W}) + 4g^{-1}(\mathcal{W}, \mathcal{J}) \pm 2g^{-1}(\mathcal{J}, \mathcal{J}) \end{aligned} \tag{2.110}$$

Employing (2.98), (2.109) and (2.110) into (1.171) yields the formula for the G^g-Ricci scalar. □

Another result, which we will appreciate later, is the criterion for the Ricci compatibility of an arbitrary $\nabla \in \operatorname{LC}(\mathbb{T}M, G^g)$ with the G^g.

Notation 2.27. Right before we state the respective theorem, we recall the commonly used notation from the Riemannian geometry. Consider an oriented Riemannian manifold (M, g, o), then for all $k \in \{1, \cdots, \dim(M)\}$ one can introduce a map $\delta_g : \Omega^k(M) \to \Omega^{k-1}(M)$, a so called **codifferential**, as

$$\delta_g := (-1)^k \star^{-1}_{g,o} \mathrm{d} \star_{g,o}, \tag{2.111}$$

where $\star_{g,o}$ is the *Hodge star operator* associated with g and o. Moreover, one can show that the identity

$$(\delta_g \omega)(X_1, \cdots, X_{k-1}) = -(\nabla^{LC,g}_{e_j}\omega)(\sharp_g e^j, X_1, \cdots, X_{k-1}) \tag{2.112}$$

is satisfied for all $\omega \in \Omega^k(M)$ and all $X_1, \cdots, X_{k-1} \in \Gamma(TM)$. In particular, for $k = 1$ one has $\delta_g = -\operatorname{div}^g$. From the equation above we see that we do not actually need an oriented manifold to have the codifferential defined.

Theorem 2.28. *Let all assumptions of the theorem 2.25 hold. Then ∇ is Ricci compatible with G^g if and only if*

$$\begin{aligned} 0 = \operatorname{Ric}^{LC,g}(X, Y) &- \frac{1}{2}(\delta_g H)(X, Y) - \frac{1}{2}(\mathrm{i}_X H, \mathrm{i}_Y H)_g - H(X, Y, \sharp_g \mathcal{W}) \\ &+ (\nabla^{LC,g}_X \mathcal{W})(Y) + (\nabla^{LC,g}_Y \mathcal{W})(X) + (\nabla^{LC,g}_Y \mathcal{J})(X) - (\nabla^{LC,g}_X \mathcal{J})(Y), \end{aligned} \tag{2.113}$$

for all X, $Y \in \Gamma(TM)$.

Lemma 2.29. *Let all assumptions of the theorem 2.25 hold. Then ∇^0 is Ricci compatible with G^g if and only if*

$$0 = \operatorname{Ric}^{LC,g}(X, Y) - \frac{1}{2}(\delta_g H)(X, Y) - \frac{1}{2}(H, H)_g, \tag{2.114}$$

for all X, $Y \in \Gamma(TM)$.

Proof of the lemma. As $\Gamma(V_\pm^g) = \{(X, \pm\flat_g X) \mid X \in \Gamma(TM)\}$, ∇^0 is Ricci compatible with G^g if and only if

$$0 = \mathrm{Ric}_{\nabla^0}((X, \flat_g X), (Y, -\flat_g Y)), \tag{2.115}$$

for all X, $Y \in \Gamma(TM)$. Thanks to (2.112) and the lemma 2.26, the right hand-side can be expanded as follows:

$$\begin{aligned}\mathrm{Ric}_{\nabla^0}((X, \flat_g), (Y, -\flat_g Y)) &= \mathrm{Ric}^{LG,g}(X, Y) + \frac{1}{4}((\nabla^{LC,g}_{e_j} H)(X, Y, \sharp_g e^j)) + (\nabla^{LC,g}_{e_j} H)(X, Y, \sharp_g e^j)) \\ &\quad - \frac{1}{3}(\mathrm{i}_X H, \mathrm{i}_Y H)_g - \frac{1}{6}(\mathrm{i}_X H, \mathrm{i}_Y H)_g \\ &= \mathrm{Ric}^{LC,g}(X, Y) - \frac{1}{2}(\delta_g H)(X, Y) - \frac{1}{2}(\mathrm{i}_X H, \mathrm{i}_Y H)_g,\end{aligned} \tag{2.116}$$

and thus the proof is complete. □

Proof of the theorem. We start similarly as in the lemma, ∇ is Ricci compatible with G^g if and only if

$$0 = \mathrm{Ric}_{\nabla}((X, \flat_g X), (Y, -\flat_g Y)), \tag{2.117}$$

for all X, $Y \in \Gamma(TM)$. Combining the property (1.149) possessed by K with the lemma 1.70 and (1.116) leads through (1.161) to the following equivalent formulation of the Ricci compatibility:

$$\begin{aligned}0 &= \mathrm{Ric}_{\nabla^0}((X, \flat_g X), (Y, -\flat_g Y)) \\ &\quad + \frac{1}{2}((\nabla^0{}_{(X, \flat_g X)}\mathcal{K})(Y, -\flat_g Y) + (\nabla^0{}_{(Y, -\flat_g Y)}\mathcal{K})((X, \flat_g X))),\end{aligned} \tag{2.118}$$

for all X, $Y \in \Gamma(TM)$. To be able to make the next step, we need to calculate the covariant derivative $\nabla^0{}_{(X, \pm\flat_g X)}(Y, \mp\flat_g Y)$. One immediately sees

$$\begin{aligned}\nabla^0{}_{(X, \pm\flat_g X)}(Y, \mp\flat_g Y) &= \begin{pmatrix}\nabla^{LC,g}_X \pm \frac{1}{6}\sharp_g H(X, \star, .) & -\frac{1}{3}\sharp_g H(X, \sharp_g \star, .) \\ -\frac{1}{3}H(X, \star, .) & \nabla^{LC,g}_X \pm \frac{1}{6}H(X, \sharp_g \star, .)\end{pmatrix}\begin{pmatrix}Y \\ \mp\flat_g Y\end{pmatrix} \\ &= (\nabla^{LC,g}_X Y \pm \frac{1}{2}\sharp_g H(X, Y, .), \mp\nabla^{LC,g}_X \flat_g Y - \frac{1}{2}H(X, Y, .))\end{aligned} \tag{2.119}$$

By employing this together with the lemma into (2.118), we obtain

$$\begin{aligned}0 &= \mathrm{Ric}^{LC,g}(X, Y) - \frac{1}{2}(\delta_g H)(X, Y) - \frac{1}{2}(\mathrm{i}_X H, \mathrm{i}_Y H)_g \\ &\quad + \frac{1}{2}\Big(X\mathcal{K}((Y, -\flat_g Y)) - \mathcal{K}((\nabla^{LC,g}_X Y + \frac{1}{2}\sharp_g H(X, Y, .), -\nabla^{LC,g}_X \flat_g Y - \frac{1}{2}H(X, Y, .))) \\ &\quad + Y\mathcal{K}((X, \flat_g X)) - \mathcal{K}((\nabla^{LC,g}_Y X - \frac{1}{2}\sharp_g H(Y, X, .), \nabla^{LC,g}_Y \flat_g X - \frac{1}{2}H(Y, X, .))\Big) \\ &= \mathrm{Ric}^{LC,g}(X, Y) - \frac{1}{2}(\delta_g H)(X, Y) - \frac{1}{2}(\mathrm{i}_X H, \mathrm{i}_Y H)_g + X\mathcal{W}(Y) - X\mathcal{J}(Y) - \mathcal{W}(\nabla^{LC,g}_X Y) \\ &\quad + \mathcal{J}(\nabla^{LC,g}_X Y) + Y\mathcal{W}(X) + Y\mathcal{J}(X) - \mathcal{W}(\nabla^{LC,g}_Y X) - \mathcal{J}(\nabla^{LC,g}_Y X) - \mathcal{W}(\sharp_g H(X, Y, .)) \\ &= \mathrm{Ric}^{LC,g}(X, Y) - \frac{1}{2}(\delta_g H)(X, Y) - \frac{1}{2}(\mathrm{i}_X H, \mathrm{i}_Y H)_g - H(X, Y, \sharp_g \mathcal{W}) \\ &\quad + (\nabla^{LC,g}_X \mathcal{W})(Y) + (\nabla^{LC,g}_Y \mathcal{W})(X) + (\nabla^{LC,g}_Y \mathcal{J})(X) - (\nabla^{LC,g}_X \mathcal{J})(Y),\end{aligned} \tag{2.120}$$

which is exactly what should have been proven. □

Corollary 2.30. *Let all assumptions of the theorem 2.25 hold. Then ∇ is Ricci compatible with G^g if and only if the pair of equations*

$$0 = \mathrm{Ric}^{LC,g}(X,Y) - \frac{1}{2}(\mathrm{i}_X H, \mathrm{i}_Y H)_g + (\nabla^{LC,g}_X \mathcal{W})(Y) + (\nabla^{LC,g}_Y \mathcal{W})(X), \tag{2.121}$$

$$0 = H(X,Y,\sharp_g \mathcal{W}) + \frac{1}{2}(\delta_g H)(X,Y) + (\nabla^{LC,g}_X \mathcal{J})(Y) - (\nabla^{LC,g}_Y \mathcal{J})(X) \tag{2.122}$$

is satisfied for all X, $Y \in \Gamma(TM)$.

Proof. It follows immediately by decomposing (2.113) into the symmetric and skew-symmetric part. □

As you have probably noticed, we were concerned exclusively with the simplest case of generalized metric, a minimal one. In the general case, the set $\mathrm{LC}(\mathbb{T}M, G)$ is so far completely unknown. However, there is a trick that makes the task of describing an arbitrary Levi-Civita Courant algebroid connection trivial. Consider a Courant algebroid $(E \overset{\pi}{\to} M, \rho, [.,.]^H_D, g_E)$ equiped with a generalized metric G and denote the corresponding Riemannian metric and 2-form on M as g and B respectively. Then, thanks to the proposition 2.7 and the remark 2.15, e^B is a metric preserving Courant algebroid isomorphism mapping $(E \overset{\pi}{\to} M, \rho, [.,.]^{H-\mathrm{d}B}_D, g_E)$ endowed with G^g to $(E \overset{\pi}{\to} M, \rho, [.,.]^H_D, g_E)$ endowed with G. Consequently, see 1.18 and 1.71, any Levi-Civita Courant algebroid connection $\hat{\nabla} \in \mathrm{LC}(\mathbb{T}M, G)$ on $(E \overset{\pi}{\to} M, \rho, [.,.]^H_D, g_E)$ can be for all (X, α), $(Y, \beta) \in \Gamma(E)$ expressed as

$$\hat{\nabla}_{(X,\alpha)}(Y,\beta) = \mathrm{e}^B(\nabla_{\mathrm{e}^{-B}(X,\alpha)}\, \mathrm{e}^{-B}(Y,\beta)), \tag{2.123}$$

where $\nabla \in \mathrm{LC}(\mathbb{T}M, G^g)$ is some Levi-Civita Courant algebroid connection on the Courant algebroid $(E \overset{\pi}{\to} M, \rho, [.,.]^{H-\mathrm{d}B}_D, g_E)$ with respect to G^g. Note, that all such ∇ are available by the previous procedure, since we can clearly replace H with $H - \mathrm{d}B$ in all the relations above. Therefore, we have just described all Levi-Civita Courant algebroid connections on $(E \overset{\pi}{\to} M, \rho, [.,.]^H_D, g_E)$ with respect to an arbitrary generalized metric. Moreover, if $\hat{\nabla}$ and ∇ are related as (2.123), we are able to easily compute its curvature tensors. Now, we state the most important result of the whole chapter in the form of corollary.

Remark 2.31. We would like to emphasize that by choosing a 4-tuple of ordinary manifold tensor fields (g, B, W, J) with the suitable properties, one uniquely determines any Levi-Civita connection on $(E \overset{\pi}{\to} M, \rho, [.,.]^H_D, g_E)$ with respect to the appropriate generalized metric.

Corollary 2.32. *Consider a Courant algebroid $(E \overset{\pi}{\to} M, \rho, [.,.]^H_D, g_E)$ equipped with a generalized metric G associated with a pair of Riemannian metric and 2-form (g, B) on M; and with $\hat{\nabla} \in \mathrm{LC}(\mathbb{T}M, G)$. Then the corresponding Courant-Ricci scalar and G-Ricci scalar can be expressed as*

$$\mathcal{R}^{\hat{\nabla}}_E = 4\,\mathrm{div}^g(\mathcal{J}) - 8g^{-1}(\mathcal{W}, \mathcal{J}), \tag{2.124}$$

$$\mathcal{R}^{\hat{\nabla}}_G = \mathcal{R}^{LC,g} - \frac{1}{2}(H - \mathrm{d}B, H - \mathrm{d}B)_g + 4\,\mathrm{div}^g(\mathcal{W}) - 4g^{-1}(\mathcal{W}, \mathcal{W}) - 4g^{-1}(\mathcal{J}, \mathcal{J}). \tag{2.125}$$

Moreover, $\hat{\nabla}$ is Ricci compatible with G if and only if the pair of equations

$$0 = \mathrm{Ric}^{LC,g}(X,Y) - \frac{1}{2}(\mathrm{i}_X(H - \mathrm{d}B), \mathrm{i}_Y(H - \mathrm{d}B))_g + (\nabla^{LC,g}_X \mathcal{W})(Y) + (\nabla^{LC,g}_Y \mathcal{W})(X), \tag{2.126}$$

$$0 = (H - \mathrm{d}B)(X,Y,\sharp_g \mathcal{W}) + \frac{1}{2}(\delta_g(H - \mathrm{d}B))(X,Y) - (\nabla^{LC,g}_Y \mathcal{J})(X) + (\nabla^{LC,g}_X \mathcal{J})(Y) \tag{2.127}$$

is satisfied for all X, $Y \in \Gamma(TM)$. Tensor fields W, $J \in \mathcal{T}^0_3(M)$ are given uniquely by (2.123).

Proof. The proof is based on the discussion in the paragraph above. By combining (2.95) with 1.32, one immediately obtains the relation for the Courant-Ricci scalar. Similarly, using (2.96) and 1.48 leads to the G-Ricci scalar formula. The part concerning with Ricci compatibility follows from the corollary 2.30 combined with 1.50. □

Chapter 3

String effective action

Although the previous chapters have already provided us many interesting and maybe surprising results, the true miracle[8] is yet to come. We will show that the *string effective action*, see [17], and the associated equations of motion can be expressed in the language of Courant algebroids. Moreover, this Courant algebroid formulation seems very natural in the sense that it reminds us of the *Einstein-Hilbert action* and the *vacuum Einstein field equation* from general relativity.

3.1 In classical fashion

The **string effective action** is an action functional arising from string theory, which describes a motion of the bosonic string in the target oriented manifold M of dimension equal to 26. If we omit all overall constants, it is given as

$$S[g, B, \phi] := \int_M \mathrm{e}^{-2\phi}(\mathcal{R}^{LC,g} - \frac{1}{2}(\mathrm{d}\, B, \mathrm{d}\, B)_g + 4(\mathrm{d}\, \phi, \mathrm{d}\, \phi)_g) \operatorname{Vol}_g . \tag{3.1}$$

As you can see it depends on three background fields, namely a Riemannian[9] metric g on M, $B \in \Omega^2(M)$ called the **B-field** (also known as the **Kalb-Ramond field**) and $\phi \in C^\infty(M)$ called the **dilaton**. The symbol Vol_g denotes the *canonical volume form* corresponding to g. Let us make a slight generalization, we will suppose that for some fixed closed 3-form H, the 3-form $H' := H - \mathrm{d}\, B$ appears instead of $\mathrm{d}\, B$ in the string effective action formula. It is clear that for a special choice $H = 0$ it reduces to the original case.

Notation 3.1. We introduce the **Lapalace-Beltrami operator**, $\Delta_g : C^\infty(M) \to C^\infty(M)$, as

$$\Delta_g := \operatorname{div}^g \circ \mathrm{d} . \tag{3.2}$$

Note that $\Delta_g = -\delta_g \circ \mathrm{d}$.

Theorem 3.2 (Equations of motion for string effective action)**.** *Let M be an oriented manifold and $H \in \Omega^3(M)$ an arbitrary closed 3-form on M. Then (g, B, ϕ), where g is a Riemannian metric on M, $B \in \Omega^2(M)$ and $\phi \in C^\infty(M)$, is an extremal of the action functional*

$$S[g, B, \phi] = \int_M \mathrm{e}^{-2\phi}(\mathcal{R}^{LC,g} - \frac{1}{2}(H', H')_g + 4(\mathrm{d}\, \phi, \mathrm{d}\, \phi)_g) \operatorname{Vol}_g, \tag{3.3}$$

[8] It was discovered by authors in [2].

[9] In physics g is not Riemannian but Lorentzian. However, Lorentzian metric is more complicated to handle with. So, in order to make all steps clearer, we will be happy with just a Riemannian one.

where $H' = H - \mathrm{d}\, B$, if and only if

$$0 = \mathcal{R}^{LC,g} - \frac{1}{2}(H', H')_g + 4\Delta_g \phi - 4(\mathrm{d}\,\phi, \mathrm{d}\,\phi)_g, \tag{3.4}$$

$$0 = \frac{1}{2}(\delta_g H')(X, Y) + H'(X, Y, \sharp_g\, \mathrm{d}\,\phi), \tag{3.5}$$

$$0 = \mathrm{Ric}^{LC,g}(X, Y) - \frac{1}{2}(\mathrm{i}_X H', \mathrm{i}_Y H')_g + (\nabla^{LC,g}_X \mathrm{d}\,\phi)(Y) + (\nabla^{LC,g}_Y \mathrm{d}\,\phi)(X), \tag{3.6}$$

for all X, $Y \in \Gamma(TM)$.

Proof. The proof is based on the variations of the action functional with respect to g, B and ϕ. It actually means to perform several pages long calculations. Since this result is already known and it is not main purpose of this thesis, we omit the proof. □

Remark 3.3. The set of equations (3.4), (3.5) and (3.6) is sometimes called the **supergravity equations** or the **SUGRA equations** for short.

3.2 In Courant algebroid fashion

In this section, we reformulate the string effective action and the corresponding equations of motion in terms of Courant algebroids, namely in terms of the one introduced in Chapter 2.

Notation 3.4. In order to make expressions more readable, we the adopt index notation commonly used in the physical literature. Suppose $E \xrightarrow{\pi} M$ is a vector bundle equipped with a fiber-wise metric h. For an arbitrary $A \in \mathcal{T}^0_k(E)$, $k \in \mathbb{N}$, we impose

$$A_{\mu_1 \ldots \mu_k} := A(\xi_{\mu_1}, \ldots, \xi_{\mu_k}), \qquad A^{\mu_1 \ldots \mu_k} := A(\sharp_h\, \xi^{\mu_1}, \ldots, \sharp_h\, \xi^{\mu_k}), \tag{3.7}$$

where $\{\xi_\mu\}_{\mu=}^{\mathrm{Rank}(E)}$ denotes an arbitrary local frame of E.

Lemma 3.5. *Let (M, g, o) be an oriented Riemannian manifold and $k \in \{0, \ldots, \dim(M) - 1\}$. Then for all $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k+1}(M)$, the identity*

$$\int_M (\mathrm{d}\,\alpha, \beta)_g \,\mathrm{Vol}_g = \int_M (\alpha, \delta_g \beta)_g \,\mathrm{Vol}_g + \int_{\partial M} \alpha \wedge \star_{g,o} \beta \tag{3.8}$$

holds.

Proof. First of all, denote $n := \dim(M)$, choose an arbitrary $k \in \{0, \ldots, n\}$, take arbitrary α, $\beta \in \ \Omega^k(M)$ and expand $\alpha \wedge \star_{g,o} \beta$ in the way

$$\begin{aligned}
\alpha \wedge \star_{g,o} \beta &= \left(\frac{1}{k!} \alpha_{j_1 \ldots j_k} e^{j_1} \wedge \cdots \wedge e^{j_k} \right) \wedge \left(\frac{1}{(n-k)!k!} \beta^{l_1 \ldots l_k} \varepsilon_{l_1 \ldots l_k m_1 \ldots m_{n-k}} e^{m_1} \wedge \cdots \wedge e^{m_{n-k}} \right) \\
&= \frac{1}{(n-k)!k!k!} \alpha_{j_1 \ldots j_k} \beta^{l_1 \ldots l_k} \varepsilon_{l_1 \ldots l_k m_1 \ldots m_{n-k}} \varepsilon^{j_1 \ldots j_k m_1 \ldots m_{n-k}} e^1 \wedge \cdots \wedge e^n \\
&= \frac{1}{k!k!} \alpha_{j_1 \ldots j_k} \beta^{l_1 \ldots l_k} \delta^{j_1 \ldots j_k}_{l_1 \ldots l_k} e^1 \wedge \cdots \wedge e^n \\
&= \frac{1}{k!} \alpha_{j_1 \ldots j_k} \beta^{j_1 \ldots j_k} e^1 \wedge \cdots \wedge e^n \\
&= (\alpha, \beta)_g \,\mathrm{Vol}_g \,.
\end{aligned} \tag{3.9}$$

We have used the fact that one can always and without loss of generality choose a right-handed orthonormal local frame to work with. Considering the just proven identity one can for all $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k+1}$ proceed as follows:

$$\begin{aligned} \mathrm{d}(\alpha \wedge \star_{g,o}\beta) &= \mathrm{d}\,\alpha \wedge \star_{g,o}\beta + (-1)^k \alpha \wedge \mathrm{d}\star_{g,o}\beta = (\mathrm{d}\,\alpha, \beta)_g \operatorname{Vol}_g + \alpha \wedge \star_{g,o}((-1)^k \star_{g,o}^{-1} \mathrm{d}\star_{g,o})\beta \\ &= (\mathrm{d}\,\alpha, \beta)_g \operatorname{Vol}_g + \alpha \wedge \star_{g,o}\delta_g \beta = (\mathrm{d}\,\alpha, \beta)_g \operatorname{Vol}_g + (\alpha, \delta_g\beta)_g \operatorname{Vol}_g . \end{aligned} \tag{3.10}$$

The rest follows immediately from the *Stokes' theorem.* □

Theorem 3.6 (String effective action in terms of Courant algebroids)**.** *Let M be an oriented manifold with $\dim(M) \neq 1$ and let $(E \overset{\pi}{\to} M, \rho, [.\,,.]_D^H, g_E)$ be a Courant algebroid defined in the example 2.1. Then for a given 3-tuple (g, B, ϕ) consisting of a Riemannian metric g, 2-form B and a smooth function ϕ on M, there is a generalized metric G and a Levi-Civita Courant algebroid connection $\nabla \in$ LC(E, G) on E such that the string effective action given by the formula* (3.3) *can be cast as*

$$S[g, B, \phi] = \int_M \mathrm{e}^{-2\phi}\, \mathcal{R}_G^\nabla \operatorname{Vol}_g . \tag{3.11}$$

Proof. Note that the string effective action (3.3) is a functional, which needs solely an oriented manifold M and a closed 3-form for its living. These two objects are given by choosing a Courant algebroid $(E \overset{\pi}{\to} M, \rho, [.\,,.]_D^H, g_E)$ over an oriented base manifold. Next, take arbitrary Riemannian metric g on M, $B \in \Omega^2(M)$ and $\phi \in C^\infty(M)$, that is an arbitrary input of the action functional. The pair (g, B) uniquely determines a generalized metric G on E, see theorem 2.9. By choosing an arbitrary $\nabla \in \mathrm{LC}(E, G)$, it follows from (2.125) that

$$S[g, B, \phi] = \int_M \mathrm{e}^{-2\phi}(\mathcal{R}_G^\nabla - 4\operatorname{div}^g(\mathcal{W}) + 4g^{-1}(\mathcal{W}, \mathcal{W}) + 4g^{-1}(\mathcal{J}, \mathcal{J}) + 4(\mathrm{d}\,\phi, \mathrm{d}\,\phi)_g) \operatorname{Vol}_g . \tag{3.12}$$

It means that Levi-Civita Courant algebroid connection cannot be chosen arbitrarily, but we have to impose some constraints on it. Since the only remaining free object, which is not bound to the Courant algebroid yet, is the smooth function ϕ, we should be looking for a relation between ϕ and a pair of tensor fields (W, J), the last two yet unspecified parameters of the Levi-Civita Courant algebroid connection. The lemma 3.5 implies that for all $f \in C^\infty(M)$ there holds

$$\begin{aligned} \int_M \mathrm{e}^{-2f}\, \Delta_g f \operatorname{Vol}_g &= -\int_M (\mathrm{e}^{-2f}, \delta_g\, \mathrm{d}\, f)_g \operatorname{Vol}_g \\ &= 2\int_M \mathrm{e}^{-2f}(\mathrm{d}\, f, \mathrm{d}\, f)_g \operatorname{Vol}_g + \int_{\partial M} \mathrm{e}^{-2f} \star_{g,o} \mathrm{d}\, f. \end{aligned} \tag{3.13}$$

Therefore, choosing $\mathcal{J} := 0$ and $\mathcal{W} := \mathrm{d}\,\phi$ leads precisely to[10]

$$S[g, B, \phi] = \int_M \mathrm{e}^{-2\phi}\, \mathcal{R}_G^\nabla \operatorname{Vol}_g . \tag{3.14}$$

The last thing to be done is to find out if such a choice for $\mathcal{W}$ and $\mathcal{J}$ is possible. In other words, if there are tensor fields W, J possessing the appropriate properties, which are compatible with

[10] Under the physically relevant assumption that $\mathrm{d}\,\phi$ vanishes identically on the boundary of M.

the choice $\mathcal{J} = 0$ and $\mathcal{W} = \mathrm{d}\,\phi$. One immediately sees that $J = 0$ certainly meets all the requirements. The choice for W is not so obvious, however, one can easily check that

$$W(X,Y,Z) := \frac{1}{\dim(M)-1}(g(X,Y)(\mathrm{d}\,\phi)(Z) - g(X,Z)(\mathrm{d}\,\phi)(Y)), \tag{3.15}$$

for all X, Y, $Z \in \Gamma(TM)$, is a suitable one. □

We have just arrived at the pretty interesting result, the reformulated string effective action looks very similar to the *Einstein-Hilbert action* from general relativity. It indicates that Courant algebroid connections form a natural mathematical framework for the field theory associated with the respective action. We hope that the next arguments and results will convince the reader that it is not just a coincidence, but that there is a true natural bound between Courant algebroids and the respective physical theory.

Since the relation between a general Levi-Civita Courant algebroid connection and tensor fields (W, J) is rather complicated, we would like to rephrase the constraint on the connections arising from the previous proof in a clearer way.

Proposition 3.7. *Let $(E \xrightarrow{\pi} M, \rho, [.,.]_D^H, g_E)$ be a Courant algebroid defined in the example 2.1 endowed with a generalized metric G and a Levi-Civita Courant algebroid connection ∇ and let $\phi \in C^\infty(M)$ be a smooth function. Then the equality*

$$\rho(\psi)\phi = \frac{1}{2}(\mathrm{div}^g\,\rho(\psi) - \mathrm{div}_\nabla\,\psi), \tag{3.16}$$

holds for all $\psi \in \Gamma(E)$ if and only if

$$\mathcal{J} = 0, \qquad \mathcal{W} = \mathrm{d}\,\phi. \tag{3.17}$$

Proof. First, assume that (3.16) is satisfied. For an arbitrary $X \in \Gamma(TM)$, using (2.92) and (1.142) consecutively gives

$$2\mathcal{J}(X) = \mathcal{K}((0, \flat_g X)) = \mathrm{div}_{\nabla^0}(0, \flat_g X) - \mathrm{div}_{\tilde{\nabla}}(0, \flat_g X), \tag{3.18}$$

where $\tilde{\nabla}$ is a Levi-Civita Courant algebroid connection on $(E \xrightarrow{\pi} M, \rho, [.,.]_D^{H-\mathrm{d}\,B}, g_E)$ with respect to G^g given uniquely by (2.123). The identity (2.63) together with 1.23 then implies

$$2\mathcal{J}(X) = -\,\mathrm{div}_\nabla\,\mathrm{e}^B(0, \flat_g X) = -\,\mathrm{div}_\nabla(0, \flat_g X) = 0, \tag{3.19}$$

where (3.16) has been used in the last step. Analogously, for all $X \in \Gamma(TM)$ there holds

$$\begin{aligned}\mathrm{d}\,\phi(X) = X\phi = \rho((X,0))\phi &= \frac{1}{2}(\mathrm{div}^g X - \mathrm{div}_\nabla(X,0)) = \frac{1}{2}(\mathrm{div}^g X - \mathrm{div}_{\tilde{\nabla}}\,\mathrm{e}^{-B}(X,0)) \\ &= \frac{1}{2}(\mathrm{div}^g X - \mathrm{div}_{\nabla^0}(X, -B(X,.)) + \mathcal{K}((X, -B(X,.)))) = \mathcal{W}(X) - B(X, \sharp_g \mathcal{J}) \\ &= \mathcal{W}(X).\end{aligned} \tag{3.20}$$

The last equality follows from the already proven equality $\mathcal{J} = 0$. Conversely, realize that

$$\begin{aligned}\rho((X,\alpha))\phi = (\mathrm{d}\,\phi)(X) &= \mathcal{W}(X) + (\alpha - B(X,.))(\sharp_g \mathcal{J}) = \frac{1}{2}\mathcal{K}(\mathrm{e}^{-B}(X,\alpha)) \\ &= \frac{1}{2}(\mathrm{div}^g\,\rho((X,\alpha)) - \mathrm{div}_{\nabla^0}\,\mathrm{e}^{-B}(X,\alpha) + \mathcal{K}(\mathrm{e}^{-B}(X,\alpha))) \\ &= \frac{1}{2}(\mathrm{div}^g\,\rho((X,\alpha)) - \mathrm{div}_{\tilde{\nabla}}\,\mathrm{e}^{-B}(X,\alpha)) \\ &= \frac{1}{2}(\mathrm{div}^g\,\rho((X,\alpha)) - \mathrm{div}_\nabla(X,\alpha))\end{aligned} \tag{3.21}$$

is satisfied for all $(X, \alpha) \in \Gamma(E)$. It follows simply by a modification the steps of the derivation (3.20) and taking them in the reverse order. □

Theorem 3.8 (Equations of motion for string effective action in terms of Courant algebroids)**.** *Let* M *be an oriented manifold and* $(E \overset{\pi}{\to} M, \rho, [.,.]_D^H, g_E)$ *be a Courant algebroid defined in the example 2.1. Then* (g, B, ϕ) *satisfies equations of motion associated with the string effective action* (3.4)*,* (3.5) *and* (3.6) *if and only if*

$$\mathcal{R}_G^{\nabla} = 0, \qquad \mathrm{Ric}_{\nabla}|_{\Gamma(V_+)\times\Gamma(V_-)} = 0, \tag{3.22}$$

where G *is a generalized metric on* E *related to* (g, B) *and* $\nabla \in \mathrm{LC}(E, G)$ *is related to* ϕ *by the formula* (3.16)*.*

Proof. It follows immediately from 2.32. Indeed, by employing $\mathcal{J} = 0$ and $\mathcal{W} = \mathrm{d}\,\phi$, equations (2.125), (2.127) and (2.126) precisely become the equations of motion for dilaton, B-field and Riemannian metric respectively. □

This result provides another support for our belief that Courant algebroids are suitable for describing the field theory associated with the string effective action. It is because the reformulated equations of motion remind us of *the vacuum Einstein field equation without cosmological constant* from general relativity, $\mathrm{Ric}^{LC,g} = 0$. There are some differences, the second equation of (3.22) says that not the whole Ricci tensor vanishes, but only “one half of it”, whereas the ordinary manifold version of the first equation $\mathcal{R}^{LC,g} = 0$ is the consequence of the vanishing the ordinary Ricci tensor, unlike in the case of (3.22), where both equations of motion are mutually independent.

Chapter 4

Palatini approach to Einstein-Hilbert action

Right before we get to the final part of this thesis, let us take a little detour, we will present the famous *Palatini approach to the Einsten-Hilbert action.* Instead of following the approach provided by standard general relativity books, we will take a more insightful way motivated by the paper [18].

Theorem 4.1 (Palatini approach to Einstein-Hilbert action)**.** *Let M be an oriented smooth manifold with $\dim(M) \neq 2$. Then a (semi-)Riemannian metric g and an affine connection ∇ on M extremalize the Einstein-Hilbert action*

$$S[g,\nabla] := \int_M \mathcal{R}^{\nabla,g}\,\mathrm{Vol}_g \tag{4.1}$$

if and only if there is $\omega \in \Omega^1(M)$ such that the equations

$$0 = (\mathrm{Ric}^{\nabla})_S(X,Y) - \frac{1}{2}\mathcal{R}^{\nabla,g} g(X,Y), \tag{4.2}$$

$$\nabla_X Y = \nabla_X^{LC,g} Y + \omega(X)Y, \tag{4.3}$$

are satisfied for all X, $Y \in \Gamma(TM)$.

Lemma 4.2. *Suppose (M,g) is a (semi-)Riemannian manifold and $h \in \mathcal{T}^0_2(M)$ is a symmetric tensor field with a compact support. Then there is a real number $\epsilon > 0$ such that $g' := g + \epsilon h$ remains a (semi-)Riemannian metric with the same signature as g, and furthermore, that there holds*

$$\sharp_{g'} = \sharp_g - \epsilon \sharp_g h(\sharp_g \star, .) + O(\epsilon^2). \tag{4.4}$$

Proof of the lemma 4.2. The proof is rather technical and it is not crucial for the main part of the thesis, thus we omit it. □

Lemma 4.3. *For all $A \in GL(n,\mathbb{R})$ and all j, $k \in \{1,\dots n\}$ there holds $\frac{\partial \det(A)}{\partial A_{jk}} = \det(A)A^{-1}_{kj}$.*

Proof of the lemma 4.3. By using the *Laplace expansion along the j-th row*, we get

$$\begin{aligned}\frac{\partial \det(A)}{\partial A_{jk}} &= \sum_{l=1}^{n} (-1)^{j+l} \underbrace{\frac{\partial A_{jl}}{\partial A_{jk}}}_{\delta^l_k} \det(A^{(j,l)}) + \sum_{l=1}^{n} (-1)^{j+l} A_{jl} \underbrace{\frac{\partial \det(A^{(j,l)})}{\partial A_{jk}}}_{0} \\ &= (-1)^{j+k}\det(A^{(j,k)}) = (\mathrm{adj}A)_{kj} = \det(A)A^{-1}_{kj},\end{aligned} \tag{4.5}$$

$A^{(j,l)}$ denotes a matrix that results from A by removing the j-th row and the l-th column. □

Proof of the theorem. The idea of the proof is simple, we just perform variations of the action with respect to a metric and an affine connection. For the variation with respect to g, assume a symmetric and compactly supported tensor field $h \in \mathcal{T}_2^0(M)$ and a real number $\epsilon > 0$ small enough to ensure that $g' := g + \epsilon h$ is still a (semi-)Riemannian metric with the same signature as g and that (4.4) holds, its existence is guaranteed by the lemma 4.2. Since g and ∇ are mutually independent, the Ricci scalar varies simply as

$$\begin{aligned}\mathcal{R}^{\nabla,g'} &= \mathrm{Ric}^{\nabla}(\sharp_{g'} e^j, e_j) = \mathrm{Ric}^{\nabla}(\sharp_g e^j, e_j) - \epsilon\, \mathrm{Ric}^{\nabla}(\sharp_g h(\sharp_g e^j, .), e_j) + O(\epsilon^2)\\ &= \mathcal{R}^{\nabla,g} - \epsilon h^{jk} (\mathrm{Ric}^{\nabla})_{(jk)} + O(\epsilon^2)\end{aligned} \tag{4.6}$$

under the transformation $g \mapsto g + \epsilon h$. The second object depending on the metric is the canonical volume form, choosing a right-handed local frame leads to

$$\mathrm{Vol}_{g'} = \sqrt{|\det(g')|}\, e^1 \wedge \cdots \wedge e^{\dim(M)} = \sqrt{|\det(g+\epsilon h)|}\, e^1 \wedge \cdots \wedge e^{\dim(M)}, \tag{4.7}$$

so we need to determine a transformation rule for $\det(g)$. One can consider $\det(g + \epsilon h)$ as a real function of a real variable, and furthermore, since g and h are smooth tensor fields, it is even a smooth function. Therefore, *Taylor's theorem* can be used to express $\det(g + \epsilon h)$ as

$$\det(g + \epsilon h) = \det(g) + \left.\frac{\partial \det(g')}{\partial g'_{jk}}\right|_{\epsilon=0} h_{jk}\epsilon + O(\epsilon^2). \tag{4.8}$$

By using the lemma 4.3, formula (4.4) and the symmetry of g'^{-1}, one finds

$$\begin{aligned}\frac{\partial \det(g')}{\partial g'_{jk}} &= \det(g')(g'^{-1})^{jk} = \det(g')g'(\sharp_{g'} e^j, \sharp_{g'} e^k) = \det(g')g(\sharp_g e^j, \sharp_g e^k) + O(\epsilon)\\ &= \det(g)g^{jk} + O(\epsilon),\end{aligned} \tag{4.9}$$

hence

$$\det(g + \epsilon h) = \det(g) + 2\epsilon \det(g)(h, g)_g + O(\epsilon^2). \tag{4.10}$$

The Taylor expansion then yields

$$\sqrt{|\det(g + \epsilon h)|} = \sqrt{|\det(g)|}\sqrt{1 + 2\epsilon(h, g)_g + O(\epsilon^2)} = \sqrt{|\det(g)|}(1 + \epsilon(h, g)_g) + O(\epsilon^2). \tag{4.11}$$

Finally, we see that

$$\mathrm{Vol}_{g'} = (1 + \epsilon(h, g)_g)\,\mathrm{Vol}_g + O(\epsilon^2). \tag{4.12}$$

By employing (4.6) and (4.12), we obtain

$$\begin{aligned}S[g + \epsilon h, \nabla] &= \int_M \mathcal{R}^{\nabla,g'}\,\mathrm{Vol}_{g'}\\ &= \int_M (\mathcal{R}^{\nabla,g} - \epsilon h^{jk}(\mathrm{Ric}^{\nabla})_{(jk)} + O(\epsilon^2))(1 + \frac{1}{2}\epsilon h^{jk} g_{jk} + O(\epsilon^2))\,\mathrm{Vol}_g\\ &= S[g, \nabla] - \epsilon \int_M h^{jk}((\mathrm{Ric}^{\nabla})_{(jk)} - \frac{1}{2}\mathcal{R}^{\nabla,g} g_{jk})\,\mathrm{Vol}_g + O(\epsilon^2).\end{aligned} \tag{4.13}$$

Therefore, for the variation with respect to a metric, it follows that

$$\delta^g S[g,\nabla] \equiv \lim_{\epsilon\to 0^+} \frac{S[g+\epsilon h,\nabla]-S[g,\nabla]}{\epsilon} = -\int_M h^{jk}((\mathrm{Ric}^\nabla)_{(jk)} - \frac{1}{2}\mathcal{R}^{\nabla,g} g_{jk}) \,\mathrm{Vol}_g\,. \tag{4.14}$$

Consequently, *Fundamental lemma of calculus of variations* states that $\delta^g S[g,\nabla]=0$ if and only if

$$(\mathrm{Ric}^\nabla)_S - \frac{1}{2}\mathcal{R}^{\nabla,g} g = 0. \tag{4.15}$$

Let us now deal with the variation with respect to an affine connection. One easily realizes that any affine connection ∇ on (M,g) can be for all X, $Y\in\Gamma(TM)$ expressed as

$$\nabla_X Y = \nabla^{LC,g}_X Y + \sharp_g L(X,Y,.), \tag{4.16}$$

where $L\in\mathcal{T}^0_3(M)$.[11] Note that L does not possess any other additional properties, unlike a difference tensor field of two Courant algebroid connections. Note that an affine connection on (M,g) is fully characterized by the choice of a tensor field L, thus it is convenient and without loss of generality to perform the variation with respect to a tensor field L instead of the variation with respect to an affine connection. Suppose $\epsilon>0$ is a real number and $N\in\mathcal{T}^0_3(M)$ is a tensor field vanishing identically on the boundary of M, moreover, denote

$$\nabla'_X Y := \nabla_X Y + \epsilon\sharp_g N(X,Y,.) \equiv \nabla^{LC,g}_X Y + \sharp_g L(X,Y,.) + \epsilon\sharp_g N(X,Y,.). \tag{4.17}$$

In order to discover how $\mathcal{R}^{\nabla,g}$ varies under the transformation $\nabla\mapsto\nabla'$, we have to start with the Riemann tensor

$$\begin{aligned}
&R^{\nabla'}(\alpha,X,Y,Z)\\
&\quad=\alpha(\nabla'_Y\nabla'_Z X - \nabla'_Z\nabla'_Y X - \nabla'_{[Y,Z]}X)\\
&\quad=R^\nabla(\alpha,X,Y,Z) + \epsilon\alpha\Big(\nabla_Y\sharp_g N(Z,X,.) + \sharp_g N(Y,\nabla_Z X,.) - \nabla_Z\sharp_g N(Y,X,.)\\
&\qquad - \sharp_g N(Z,\nabla_Y X,.) - \sharp_g N([Y,Z],X,.)\Big) + O(\epsilon^2)\\
&\quad=R^\nabla(\alpha,X,Y,Z) + \epsilon(\alpha\circ\sharp_g)\Big(\nabla^{LC,g}_Y N(Z,X,.) - N(Z,\nabla^{LC,g}_Y X,.) - N(\nabla^{LC,g}_Y Z,X,.)\\
&\qquad - \nabla^{LC,g}_Z N(Y,X,.) + N(Y,\nabla^{LC,g}_Z X,.) + N(\nabla^{LC,g}_Z Y,X,.) + L(Y,\sharp_g N(Z,X,.),.)\\
&\qquad + N(Y,\sharp_g L(Z,X,.),.) - L(Z,\sharp_g N(Y,X,.),.) - N(Z,\sharp_g L(Y,X,.),.)\Big) + O(\epsilon^2)\\
&\quad=R^\nabla(\alpha,X,Y,Z) + \epsilon\Big((\nabla^{LC,g}_Y N)(Z,X,\sharp_g\alpha) - (\nabla^{LC,g}_Z N)(Y,X,\sharp_g\alpha)\\
&\qquad + N(Z,X,\sharp_g e^l)L(Y,e_l,\sharp_g\alpha) + N(Y,\sharp_g e^l,\sharp_g\alpha)L(Z,X,e_l) - N(Y,X,\sharp_g e^l)L(Z,e_l,\sharp_g\alpha)\\
&\qquad - N(Z,\sharp_g e^l,\sharp_g\alpha)L(Y,X,e_l)\Big) + O(\epsilon^2),
\end{aligned} \tag{4.18}$$

for all $\alpha\in\Omega^1(M)$ and all X, Y, $Z\in\Gamma(TM)$. Note that we have used the compatibility with the metric and the torsion-freeness of the Levi-Civita affine connection in the third step. If we denote $\mathcal{N}_1 := N(\sharp_g e^j, e_j, .)$ and $\mathcal{N}_2 := N(\sharp_g e^j, ., e_j)$, the Ricci scalars are then related as

$$\begin{aligned}
\mathcal{R}^{\nabla',g} &= R^{\nabla'}(e^j,\sharp_g e^k, e_j, e_k)\\
&= \mathcal{R}^{\nabla,g} + \epsilon(\mathrm{div}^g(\mathcal{N}_1-\mathcal{N}_2) + N^k{}_k{}^l L^j{}_{lj} + N^{jl}{}_j L^k{}_{kl} - N^{jkl}L_{klj} - N^{klj}L_{jkl})) + O(\epsilon^2)\\
&= \mathcal{R}^{\nabla,g} + \epsilon(\delta_g(\mathcal{N}_2-\mathcal{N}_1) + N^{jkl}(g_{jk}L^m{}_{lm} + g_{jl}L^m{}_{mk} - L_{klj} - L_{ljk})) + O(\epsilon^2).
\end{aligned} \tag{4.19}$$

[11] The proof is a simple modification of the first part of the proof of the lemma 1.56.

Combining lemma 3.5 and $N|_{\partial M} = 0$ yields

$$\int_M \delta_g(\mathcal{N}_2 - \mathcal{N}_1)\,\mathrm{Vol}_g = -\int_{\partial M} \star_{g,o}(\mathcal{N}_2 - \mathcal{N}_1) = 0, \tag{4.20}$$

and therefore

$$S[g,\nabla'] = S[g,\nabla] + \epsilon \int_M N^{jkl}(g_{jk}L^m{}_{lm} + g_{jl}L^m{}_{mk} - L_{klj} - L_{ljk}))\,\mathrm{Vol}_g + O(\epsilon^2). \tag{4.21}$$

Fundamental lemma of calculus of variations then says that $\delta^\nabla S[g,\nabla] = 0$ if and only if for all X, Y, $Z \in \Gamma(TM)$ there holds

$$0 = g(X,Y)\mathcal{L}_2(Z) + g(X,Z)\mathcal{L}_1(Y) - L(Y,Z,X) - L(Z,X,Y), \tag{4.22}$$

where $\mathcal{L}_1 := L(\sharp_g e^j, e_j, .)$ and $\mathcal{L}_2 := L(\sharp_g e^j, ., e_j)$.

In order to conclude the proof, we have to find a most general solution of the equation (4.22) for L. Writing the equation three times with X, Y, Z cyclically permuted, we obtain

$$0 = g(X,Y)\mathcal{L}_2(Z) + g(X,Z)\mathcal{L}_1(Y) - L(Y,Z,X) - L(Z,X,Y), \tag{4.23}$$
$$0 = g(Y,Z)\mathcal{L}_2(X) + g(Y,X)\mathcal{L}_1(Z) - L(Z,X,Y) - L(X,Y,Z), \tag{4.24}$$
$$0 = g(Z,X)\mathcal{L}_2(Y) + g(Z,Y)\mathcal{L}_1(X) - L(X,Y,Z) - L(Y,Z,X). \tag{4.25}$$

Subtracting the second and the third equation from the first one yields

$$\begin{aligned} &2L(X,Y,Z) \\ &\quad = g(X,Y)(\mathcal{L}_1(Z) - \mathcal{L}_2(Z)) + g(Y,Z)(\mathcal{L}_1(X) + \mathcal{L}_2(X)) - g(Z,X)(\mathcal{L}_1(Y) - \mathcal{L}_2(Y)). \end{aligned} \tag{4.26}$$

Taking the partial trace in the first two inputs results into

$$2\mathcal{L}_1(Z) = \dim(M)(\mathcal{L}_1(Z) - \mathcal{L}_2(Z)) + \mathcal{L}_1(Z) + \mathcal{L}_2(Z) - \mathcal{L}_1(Z) + \mathcal{L}_2(Z), \tag{4.27}$$
$$\Leftrightarrow \quad 0 = (\dim(M) - 2)\mathcal{L}_1(Z) - (\dim(M) - 2)\mathcal{L}_2(Z), \tag{4.28}$$

which is equivalent to $\mathcal{L}_1 = \mathcal{L}_2$ for $\dim(M) \neq 2$. It follows immediately by employing this into (4.26) that $L(X,Y,Z) = \mathcal{L}_1(X)g(Y,Z)$ for all X, Y, $Z \in \Gamma(TM)$. In other words, there exists $\omega \in \Omega^1(M)$ such that $L = \omega \otimes g$, so we have just proven that $\delta^\nabla S[g,\nabla] = 0$ implies (4.3). The converse can be easily checked by plugging $L = \omega \otimes g$ into the equation (4.22). □

One sees that the Palatini approach to the Einstein-Hilbert action does not determine an affine connection uniquely, instead it says that there is a freedom of choice provided through a 1-form ω. However, as the next proposition states, the choice of the 1-form is irrelevant for the physics.

Proposition 4.4. *Let (M,g) be a (semi-)Riemannian manifold equipped with an affine connection ∇ of the form* (4.3) *for some $\omega \in \Omega^1(M)$. Then the general relativity dynamics is invariant with respect to the choice of the* 1*-form.*

Proof. By the invariance of the general relativity dynamics is understood that the vacuum Einstein field equation remains the same for all $\omega \in \Omega^1(M)$, and moreover, the property of being a geodesic does not depend on the choice of the 1-form. To see how the field equation is varied, let us calculate the Riemann tensor and its contractions associated with an affine connection ∇ given by (4.3). One easily finds that for all $\alpha \in \Omega^1(M)$ and all X, Y, $Z \in \Gamma(TM)$ there holds

$$R^{\nabla}(\alpha, X, Y, Z) = R^{LC,g}(\alpha, X, Y, Z) + ((\nabla_Y^{LC,g}\omega)(Z) - (\nabla_Z^{LC,g}\omega)(Y))\alpha(X), \tag{4.29}$$
$$\mathrm{Ric}^{\nabla}(X, Y) = \mathrm{Ric}^{LC,g}(X, Y) + (\nabla_X^{LC,g}\omega)(Y) - (\nabla_Y^{LC,g}\omega)(X), \tag{4.30}$$
$$\mathcal{R}^{\nabla} = \mathcal{R}^{LC,g}, \tag{4.31}$$

hence for any $\omega \in \Omega^1(M)$, we have

$$(\mathrm{Ric}^{\nabla})_S - \frac{1}{2}\mathcal{R}^{\nabla,g} g = (\mathrm{Ric}^{LC,g})_S - \frac{1}{2}\mathcal{R}^{LC,g} g. \tag{4.32}$$

Secondly, a curve γ on M is a geodesic with respect to ∇ if and only if there is $f \in C^\infty(M)$ such that

$$\nabla_{\dot\gamma}\dot\gamma = f\dot\gamma. \tag{4.33}$$

Since ∇ is given by (4.3), it is can be equivalently expressed as

$$\nabla_{\dot\gamma}^{LC,g}\dot\gamma = (f - \omega(\dot\gamma))\dot\gamma, \tag{4.34}$$

that is γ is a geodesic with respect to the Levi-Civita affine connection, thus the property of being a geodesic apparently does not depend on the choice of $\omega \in \Omega^1(M)$. □

The Palatini approach together with the previous proposition provide a robust argument for choosing the Levi-Civita affine connection for the general relativity, since the case of $\omega = 0$ is the simplest one, and moreover, the physics does not depend on the choice.

To conclude this chapter, we would like to point out an interesting proposition.

Proposition 4.5. *Let (M, g) be a (semi-)Riemannian manifold and $\omega \in \Omega^1(M)$. Then it is enough to impose either the torsion-freeness or the metric compatibility with g on affine connection of the form* (4.3) *to ensure that it is already the Levi-Civita one.*

Proof. Assume that ∇ of the form (4.3) is torsion-free, that is

$$0 = T^{\nabla}(\alpha, X, Y) = \alpha(\nabla_X Y - \nabla_Y X - [X, Y]) = \omega(X)\alpha(Y) - \omega(Y)\alpha(X), \tag{4.35}$$

for all $\alpha \in \Omega^1(M)$ and X, $Y \in \Gamma(TM)$. Choosing an arbitrary $j, k \in \{1, \dots, \dim(M)\}$, $j \neq k$ and imposing $\alpha := e^k$, $Y := e_k$, $X := e_j$ yields $\omega(e_j) = 0$. Since it can be done for all j and all local frames, we have $\omega = 0$, thus $\nabla = \nabla^{LC,g}$. On the other hand, assuming that ∇ is compatible with the metric g leads for all X, Y, $Z \in \Gamma(TM)$ to

$$0 = X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = -2\omega(X)g(Y, Z). \tag{4.36}$$

As Y and Z can be for sure chosen in the way that $g(Y, Z) \neq 0$, the equation above is equivalent to $\omega = 0$, and therefore $\nabla = \nabla^{LC,g}$. □

Chapter 5

Palatini approach to Courant-Einstein-Hilbert action

In light of the two last chapters, one might be curious if there is something like the Palatini approach to Einstein-Hilbert action also for the string effective action of the form (3.11), which would justify our choice of the Courant algebroid connection in the proof of the theorem 3.6. During the research, we have found out that it is possible to go even further. As you will see, we devised the Palatini formalism not only for the Courant algebroid associated with the generalized geometry (widely discussed in Chapter 2), but even for a general Courant algebroid.

First of all, let us clarify what action functional will be at our point of interest. For an arbitrary Courant algebroid $(E \xrightarrow{\pi} M, \rho, [.,.]_E, g_E)$, we define **Courant-Einstein-Hilbert action** as the functional

$$S[G, \nabla, \mathrm{Vol}] := \int_M \mathcal{R}_G^{\nabla} \, \mathrm{Vol}, \tag{5.1}$$

which depends on three inputs, a generalized metric G on E, a Courant algebroid connection ∇ on E and a volume form[12] Vol on M. Moreover, in contrast to the action (3.11), where the Courant algebroid connection is a priori related to the generalized metric and the volume form by the Levi-Civita condition and (3.16) respectively, we assume no a priori relations between the inputs of the Courant-Einstein-Hilbert action. Therefore, it is a more general concept even on the level of the specific Courant algebroid from Chapter 2.

Right before we state the most important theorem of the whole thesis, we point out several lemmas, which will be useful for the variation of the Courant-Einstein-Hilbert action with respect to a Courant algebroid connection.

Lemma 5.1. *Suppose M is an oriented connected smooth manifold. For any volume form* Vol *on M there is a Riemannian metric g on M such that either* $\mathrm{Vol} = \mathrm{Vol}_g$ *or* $\mathrm{Vol} = -\mathrm{Vol}_g$.

Proof. Choose an arbitrary auxiliary Riemannian metric g_0 on M and denote $n := \dim(M)$. Since $\mathrm{Rank}(\Omega^n(M)) = 1$, for any volume form Vol there is a unique nowhere vanishing smooth function $h \in C^\infty(M)$ such that

$$\mathrm{Vol} = h \, \mathrm{Vol}_{g_0} . \tag{5.2}$$

[12] By a volume form on M is understood a nowhere vanishing $\dim(M)$-form on M.

Taking an arbitrary right-handed local frame $\{e_j\}_{j=1}^n$ of TM, one finds

$$\begin{aligned}\mathrm{Vol} &= h\,\mathrm{Vol}_{g_0}\\ &= h\sqrt{\det(g_0)}\,e^1\wedge\cdots\wedge e^n\\ &= \operatorname{sgn} h\sqrt{h^2\det(g_0)}\,e^1\wedge\cdots\wedge e^n\\ &= \operatorname{sgn} h\sqrt{(h^{\frac{2}{n}})^n\det(g_0)}\,e^1\wedge\cdots\wedge e^n\\ &= \operatorname{sgn} h\sqrt{\det(h^{\frac{2}{n}}g_0)}\,e^1\wedge\cdots\wedge e^n.\end{aligned} \tag{5.3}$$

In order to avoid an unnecessary discussion, we have restricted ourselves to connected manifolds in the above calculation, thus $\operatorname{sgn} h$ is defined properly because a smooth non-vanishing function clearly cannot swap the sign on a connected manifold. As $g := h^{\frac{2}{n}} g_0$ is certainly a Riemannian metric on M, the proof is complete. □

Remark 5.2. Note that a Riemannian metric g from the above lemma is not determined uniquely by Vol. If we take some other auxiliary metric g_0' at the beginning, it induces a unique smooth nowhere vanishing function $h' \in C^\infty(M)$ such that the equality

$$\mathrm{Vol} = h'\,\mathrm{Vol}_{g_0'} \tag{5.4}$$

holds. For any point $p \in M$ there are right-handed orthonormal local frames of M with respect to g_0 and g_0' over some neighbourhoods $U \subseteq M$ and $U' \subseteq M$ of p, let us denote some of these as $\{e_j\}_{j=1}^n$ and $\{e_j'\}_{j=1}^n$ respectively. Since both of them are local frames over $U \cap U'$, there is a smooth map $\mathbb{A} : U \cap U' \to GL(n, \mathbb{R})$ relating both of the local frames as

$$e_j' = \mathbb{A}^k{}_j e_k, \tag{5.5}$$

for all $j \in \{1, \dots, n\}$. Therefore, it follows from (5.4) that

$$\begin{aligned}he^1\wedge\cdots\wedge e^n &= \mathrm{Vol}\\ &= h'e'^1\wedge\cdots\wedge e'^n\\ &= h'\mathbb{A}^{-1}{}^1{}_{j_1}\ldots\mathbb{A}^{-1}{}^n{}_{j_n}e^{j_1}\wedge\cdots\wedge e^{j_n}\\ &= h'\mathbb{A}^{-1}{}^1{}_{j_1}\ldots\mathbb{A}^{-1}{}^n{}_{j_n}\varepsilon^{j_1\ldots j_n}e^1\wedge\cdots\wedge e^n\\ &= h'\det(\mathbb{A}^{-1})e^1\wedge\cdots\wedge e^n,\end{aligned} \tag{5.6}$$

hence

$$h = \frac{1}{\det(\mathbb{A})}h'. \tag{5.7}$$

By using (5.5), one obtains

$$g_0(e_j, e_k) = \delta_{jk} = g_0'(e_j', e_k') = \mathbb{A}^l{}_j\mathbb{A}^m{}_k g_0'(e_l, e_m). \tag{5.8}$$

Therefore,

$$g(e_j, e_k) = h^{\frac{2}{n}}g_0(e_j, e_k) = \frac{1}{\det(\mathbb{A})^{\frac{2}{n}}}h'^{\frac{2}{n}}\mathbb{A}^l{}_j\mathbb{A}^m{}_k g_0'(e_l, e_m) = \frac{\mathbb{A}^l{}_j\mathbb{A}^m{}_k}{\det(\mathbb{A})^{\frac{2}{n}}}g'(e_l, e_m), \tag{5.9}$$

which in general means $g \neq g'$.

Lemma 5.3. *Let* $(E \overset{\pi}{\to} M, \rho, [.,.]_E, g_E)$ *be a Courant algebroid with an oriented connected base manifold and let* G *be a generalized metric on it. If the signature* (p,q) *of the Courant metric* g_E *satisfies* $p \neq 1$ *and at the same time* $q \neq 1$*, then for any volume form* Vol *there is* $\nabla \in \mathrm{LC}(E,G)$ *and a Riemannian metric* g *on* M *satisfying the relation*

$$\int_M f \operatorname{div}_\nabla \psi \operatorname{Vol} = \pm \int_{\partial M} f \star_{g,o} \flat_g \rho(\psi) - \int_M \rho(\psi) f \operatorname{Vol}, \tag{5.10}$$

for all $f \in C^\infty(M)$.[13]

Proof. Take an arbitrary volume form Vol on M. Thanks to the previous lemma there is a Riemannian metric g such that $\mathrm{Vol} = \pm \mathrm{Vol}_g$. Moreover, assume that there is $\nabla \in \mathrm{LC}(E,G)$, the divergence of which is given as

$$\operatorname{div}_\nabla = \operatorname{div}^g \circ \rho. \tag{5.11}$$

Then for all $f \in C^\infty(M)$ there holds

$$\begin{aligned}
\int_M f \operatorname{div}_\nabla \psi \operatorname{Vol} &= \int_M (\operatorname{div}_\nabla(f\psi) - \rho(\psi) f) \operatorname{Vol} \\
&= \pm \int_M \operatorname{div}^g(\rho(f\psi)) \operatorname{Vol}_g - \int_M \rho(\psi) f \operatorname{Vol} \\
&= \mp \int_M (1, \delta_g \flat_g \rho(f\psi))_g \operatorname{Vol}_g - \int_M \rho(\psi) f \operatorname{Vol} \\
&= \pm \int_{\partial M} f \star_{g,o} \flat_g \rho(\psi) - \int_M \rho(\psi) f \operatorname{Vol}.
\end{aligned} \tag{5.12}$$

In the first step, the second axiom of a Courant algebroid connection is used. The last equality follows from the lemma 3.5. The main issue is now to prove that there actually is a Levi-Civita Courant algebroid connection $\nabla \in \mathrm{LC}(E,G)$ satisfying (5.11). As we already know that $\mathrm{LC}(E,G) \neq \emptyset$, see theorem 1.55, one can take an arbitrary $\nabla^{LC} \in \mathrm{LC}(E,G)$. Moreover, we know, see theorem 1.59, that any other $\nabla' \in \mathrm{LC}(E,G)$, hence also the potentially existing one satisfying (5.11), is related to ∇^{LC} through some $K \in \Omega^1(E) \otimes \Omega^2(E)$ possessing certain additional properties. The task is now to find a suitable K. It follows from (1.142) that condition (5.11) already unambiguously determines the partial trace of K as

$$\mathcal{K} = \operatorname{div}_{\nabla^{LC}} - \operatorname{div}^g \circ \rho. \tag{5.13}$$

The first question is if such $\mathcal{K}$ is at least $C^\infty(M)$-linear, take an arbitrary $f \in C^\infty(M)$ and proceed as follows:

$$\mathcal{K}(f\psi) = \operatorname{div}_{\nabla^{LC}}(f\psi) - \operatorname{div}^g \rho(f\psi) = f\mathcal{K}(\psi) + \rho(\psi) f - (e_j f) e^j(\rho(\psi)) = f\mathcal{K}(\psi). \tag{5.14}$$

Therefore, it makes sense to continue searching for the desired K. Relatively speaking, the obvious candidate is

$$\begin{aligned}
K(\psi_1, \psi_2, \psi_3) := \frac{1}{p-1} \Big(& g_E(\psi_{1+}, \psi_{2+})(\operatorname{div}_{\nabla^{LC}} \psi_{3+} - \operatorname{div}^g \rho(\psi_{3+})) \\
& - g_E(\psi_{1+}, \psi_{3+})(\operatorname{div}_{\nabla^{LC}} \psi_{2+} - \operatorname{div}^g \rho(\psi_{2+})) \Big) \\
& + \frac{1}{q-1} \Big(g_E(\psi_{1-}, \psi_{2-})(\operatorname{div}_{\nabla^{LC}} \psi_{3-} - \operatorname{div}^g \rho(\psi_{3-})) \\
& - g_E(\psi_{1-}, \psi_{3-})(\operatorname{div}_{\nabla^{LC}} \psi_{2-} - \operatorname{div}^g \rho(\psi_{2-})) \Big),
\end{aligned} \tag{5.15}$$

[13]The ambiguity of the sign in front of the integral over the ∂M means that the relation is satisfied with either a plus or a minus.

for all ψ_1, ψ_2, $\psi_3 \in \Gamma(E)$. Apparently, K is skew-symmetric in the last two inputs and all of six "mixed" restrictions of K vanishes identically. Using the Courant metric symmetry one easily finds that the cyclic permutations of $K|_{\Gamma(V_\pm)\times\Gamma(V_\pm)\times\Gamma(V_\pm)}$ vanish. Therefore, an $\mathbb{R}$-bilinear map $\nabla : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ defined as

$$\nabla_{\psi_1}\psi_2 = \nabla^{LC}_{\psi_1}\psi_2 + \sharp_E\, K(\psi_1, \psi_2, .), \tag{5.16}$$

for all ψ_1, $\psi_2 \in \Gamma(E)$, is indeed a Levi-Civita Courant algebroid connection with respect to G. The last thing to check is whether the partial trace of our specific K meets the requirement (5.13). This is easily seen if one chooses a local frame of E adapted to the generalized metric G for taking the partial trace of K and uses the $C^\infty(M)$-linearity of $(\operatorname{div}_{\nabla^{LC}} - \operatorname{div}^g \circ\rho)$ proven by (5.14). □

Lemma 5.4. *Consider an oriented smooth manifold M. Then for an arbitrary volume form* Vol *and for any pair of Riemannian metrics g and g' assigned to* Vol *by lemma 5.1 there holds*

$$\operatorname{div}^g = \operatorname{div}^{g'}. \tag{5.17}$$

Proof. First of all, note that for an arbitrary Riemannian metric m on M and an arbitrary local frame $\{\tilde{e}_j\}_{j=1}^{\dim(M)}$ of TM it follows easily from the *Koszul formula* that

$$\operatorname{div}^m X = \tilde{e}^j(\nabla^{LC,m}_{\tilde{e}_j} X) = \tilde{e}^j([\tilde{e}_j, X]) - \frac{1}{2} m(\tilde{e}_j, \tilde{e}_k)(X m(\sharp_m \tilde{e}^j, \sharp_m \tilde{e}^k)) \tag{5.18}$$

holds for all $X \in \Gamma(TM)$. Especially, if the local frame is orthonormal with respect to m, one has

$$\operatorname{div}^m X = \tilde{e}^j([\tilde{e}_j, X]), \tag{5.19}$$

since $m(\sharp_m \tilde{e}^j, \sharp_m \tilde{e}^k) = \delta^{jk}$. In the next, let us use the same notation as in the remark 5.2. As $g'(\sharp_{g'} e^j, \sharp_{g'} e^k)$ are components of the inverse matrix to the matrix with components $g'(e_j, e_k)$, which satisfies (5.9), one arrives at

$$\begin{aligned}
&g'(e_j, e_k) X g'(\sharp_{g'} e^j, \sharp_{g'} e^k)\\
&\quad= \det(\mathbb{A})^{\frac{2}{n}} \mathbb{A}^{-1l}{}_j \mathbb{A}^{-1m}{}_k g(e_l, e_m)(X(\frac{1}{\det(\mathbb{A})^{\frac{2}{n}}} \mathbb{A}^j{}_s g(\sharp_g e^s, \sharp_g e^r)\mathbb{A}^k{}_r))\\
&\quad= h^{\frac{2}{n}} \det(\mathbb{A})^{\frac{2}{n}} \mathbb{A}^{-1l}{}_j \mathbb{A}^{-1}{}_{lk}(X(\frac{1}{h^{\frac{2}{n}} \det(\mathbb{A})^{\frac{2}{n}}} \mathbb{A}^j{}_s \mathbb{A}^{ks}))\\
&\quad= (X\mathbb{A}^j{}_s)\mathbb{A}^{-1s}{}_j + (X\mathbb{A}^{ks})\mathbb{A}^{-1}{}_{sk} + n(-\frac{2}{n}\frac{(Xh)}{h} - \frac{2}{n}\frac{1}{\det(\mathbb{A})}\frac{\partial \det(\mathbb{A})}{\partial \mathbb{A}_{jk}}(X\mathbb{A}_{jk}))\\
&\quad= 2((X\mathbb{A}_{jk})(\mathbb{A}^{-1kj} - \frac{1}{\det(\mathbb{A})}\frac{\partial \det(\mathbb{A})}{\partial \mathbb{A}_{jk}}) - \frac{1}{h}(\mathrm{d}\,h)(X))\\
&\quad= -2(\mathrm{d}\log|h|)(X),
\end{aligned} \tag{5.20}$$

where we have used the lemma 4.3 in the last step. Therefore,

$$\operatorname{div}^{g'} X = \operatorname{div}^{g_0}(X) + (\mathrm{d}\log|h|)(X), \tag{5.21}$$

while on the other hand

$$\begin{aligned}
\operatorname{div}^g X - \operatorname{div}^{g_0}(X) &= -\frac{1}{2} g(e_j, e_k) X g(\sharp_g e^j, \sharp_g e^k) = -\frac{1}{2} h^{\frac{2}{n}} \delta_{jk}(X(\frac{1}{h^{\frac{2}{n}}}\delta^{jk})) = \frac{1}{h}(Xh)\\
&= (\mathrm{d}\log|h|)(X),
\end{aligned} \tag{5.22}$$

for all $X \in \Gamma(TM)$. □

Remark 5.5. It is good to realize that lemma 5.1 together with the lemma 5.4 say that any volume form Vol uniquely determines a divergence div^g.

It is finally the time to state the theorem.

Theorem 5.6 (Palatini approach to Courant-Einstein-Hilbert action)**.** *Consider a Courant algebroid* $(E \xrightarrow{\pi} M, \rho, [.,.]_E, g_E)$ *over an oriented connected smooth manifold* M *with the Courant metric* g_E*, whose signature* (p,q) *satisfies* $p \neq 1$ *and at the same time* $q \neq 1$*. Then a generalized metric* G *on* E*, a Courant algebroid connection* ∇ *on* E *and a volume form* Vol *on* M *extremalize the Courant-Einstein-Hilbert action*

$$S[G, \nabla, \mathrm{Vol}] := \int_M \mathcal{R}_G^{\nabla} \, \mathrm{Vol}, \tag{5.23}$$

if and only if the G-Ricci scalar $\mathcal{R}_G^{\nabla}$ *vanishes identically,* ∇ *is Ricci compatible with* G *and Levi-Civita with respect to* G*, and moreover,*

$$\mathrm{div}_{\nabla} = \mathrm{div}^g \circ \rho, \tag{5.24}$$

where div^g *is uniquely determined by* Vol*.*

Lemma 5.7. *Suppose* M *is a smooth manifold,* Vol *is an arbitrary volume form on it and* $f \in C^\infty(M)$ *is a compactly supported smooth function. Then there is a real number* $\epsilon > 0$ *such that* $(1+\epsilon f)\,\mathrm{Vol}$ *is a volume form.*

Proof of the lemma 5.7. Volume form is a nowhere vanishing $\dim(M)$-form on M, so we have to find $\epsilon \in \mathbb{R}^+$ small enough for ensuring $((1+\epsilon f)\,\mathrm{Vol})|_p \neq 0$, for all $p \in M$. For any $p \in M \smallsetminus \mathrm{supp}\, f$ we can choose ϵ arbitrarily, thus it remains to investigate only the subset $\mathrm{supp}\, f \subseteq M$. Take an arbitrary $p \in \mathrm{supp}\, f$ and denote

$$\epsilon_p := \frac{1}{|f(p)|+1}. \tag{5.25}$$

Apparently, a smooth function $(1+\epsilon_p f) \in C^\infty(M)$ is not vanishing in p. Since it is smooth, hence also continuous, there is a neighbourhood $U_p \subseteq M$ of the point p such that $((1+\epsilon_p f)\,\mathrm{Vol})|_{U_p}$ is nowhere vanishing form on U_p. Therefore, we have the open cover $\{U_p \cap \mathrm{supp}\, f\}_{p \in \mathrm{supp}\, f}$ of the compact set $\mathrm{supp}\, f$ and the collection of real strictly positive numbers $\{\epsilon_p\}_{p \in \mathrm{supp}\, f}$ such that $((1+\epsilon_p f)\,\mathrm{Vol})|_{U_p \cap \mathrm{supp}\, f} \neq 0$, for all $p \in \mathrm{supp}\, f$. The compactness of $\mathrm{supp}\, f$ implies that there is a finite collection of points $\{p_j\}_{j=1}^N \subseteq \mathrm{supp}\, f$, $N \in \mathbb{N}$, such that $\{U_{p_j} \cap \mathrm{supp}\, f\}_{j=1}^N$ is a cover of $\mathrm{supp}\, f$. Imposing

$$\epsilon := \min_{j \in \{1,\dots,N\}} \epsilon_{p_j} \tag{5.26}$$

concludes the proof. $\square$

Lemma 5.8. *Let* $(E \xrightarrow{\pi} M, \rho, [.,.]_E, g_E)$ *be a Courant algebroid equipped with a generalized metric* $V_+ \subseteq E$*. Then for any other generalized metric* $V'_+ \subseteq E$ *there is a* $C^\infty(M)$*-module morphism* $\Psi_+ : \Gamma(V_+) \to \Gamma(V_-)$ *such that*

$$\Gamma(V'_+) = \{\psi + \Psi_+ \psi \,|\, \psi \in \Gamma(V_+)\}. \tag{5.27}$$

On the other hand, for any compactly supported $C^\infty(M)$*-module morphism* $\Psi_+ : \Gamma(V_+) \to \Gamma(V_-)$*, there is a real number* $\epsilon > 0$ *such that*

$$V'_+ := \bigsqcup_{p \in M} \{v + \epsilon \Psi_{+p} v \,|\, v \in V_{+p}\} \tag{5.28}$$

is a generalized metric on E, and furthermore, that there holds

$$\tau' = \tau + 2\epsilon \begin{pmatrix} 0 & \sharp_E \Psi_+^T \flat_E \\ \Psi_+ & 0 \end{pmatrix} + O(\epsilon^2), \tag{5.29}$$

where the matrix is considered with respect to the decomposition $E = V_+ \oplus V_-$.

Proof of the lemma 5.8. Assume that we have an arbitrary pair of generalized metrics V_+ and V_+'. As V_+' and V_- is a positive and a negative definite subbundle respectively, one sees that $\Gamma(V_-) \cap \Gamma(V_+') = \{0\}$. Hence for all $\psi \in \Gamma(V_+')$, $\psi \neq 0$, there are $\psi_+ \in \Gamma(V_+)$, $\psi_+ \neq 0$, and $\psi_- \in \Gamma(V_-)$ such that $\psi = \psi_+ + \psi_-$. Moreover, due to the fact that $\Gamma(V_+')$ is closed with respect to the $C^\infty(M)$-linear combinations, ψ_- is determined uniquely by the choice of ψ_+. Therefore, there is a $C^\infty(M)$-module morphism $\Psi_+ : \Gamma(V_+) \to \Gamma(V_-)$ such that

$$\Gamma(V_+') = \{\psi + \Psi_+\psi \,|\, \psi \in W \subseteq \Gamma(V_+)\}, \tag{5.30}$$

where W is a $C^\infty(M)$-submodule of $\Gamma(V_+)$. The maximality of V_+' implies $\mathrm{Rank}(V_+) = \mathrm{Rank}(V_+')$, thus $W = \Gamma(V_+)$. In the same way, one can show that there is $\Psi_- : \Gamma(V_-) \to \Gamma(V_+)$ such that

$$\Gamma(V_-') = \{\psi + \Psi_-\psi \,|\, \psi \in \Gamma(V_-)\}. \tag{5.31}$$

Moreover, the map Ψ_- can be expressed in the terms of Ψ_+. It follows easily from the defining relation $V_-' = V'^{\perp}_+$ that any pair of $\psi_\pm \in \Gamma(V_\pm)$ satisfies

$$\begin{aligned} 0 &= g_E(\psi_+ + \Psi_+\psi_+, \psi_- + \Psi_-\psi_-) = g_E(\Psi_+\psi_+, \psi_-) + g_E(\psi_+, \Psi_-\psi_-) \\ &= (\flat_E\,\psi_-)(\Psi_+\psi_+) + (\flat_E\,\psi_+)(\Psi_-\psi_-) = (\Psi_+^T\flat_E\,\psi_-)(\psi_+) + (\flat_E\,\psi_+)(\Psi_-\psi_-) \\ &= (\flat_E\,\psi_+)((\sharp_E\,\Psi_+^T\flat_E + \Psi_-)\psi_-). \end{aligned} \tag{5.32}$$

Since $\psi_\pm \in \Gamma(V_\pm)$ were arbitrary, the relation

$$\Psi_- = -\sharp_E\,\Psi_+^T\flat_E \tag{5.33}$$

is true.

To prove the second part of the lemma, take an arbitrary generalized metric $V_+ \subseteq E$ and a compactly supported $C^\infty(M)$-module morphism $\Psi_+ : \Gamma(V_+) \to \Gamma(V_-)$. It follows easily from *Local frame criterion for subbundles*, see [11, Lemma 10.32], that

$$V_+' := \bigsqcup_{p \in M} \{v + \epsilon\Psi_{+p}v \,|\, v \in V_{+p}\} \tag{5.34}$$

is a subbundle of E, for any $\epsilon \in \mathbb{R}^+$. Apparently $\mathrm{Rank}(V_+) = \mathrm{Rank}(V_+')$, thus it remains to find out how to choose a real number $\epsilon > 0$ in order to ensure V_+' is positive definite. For all $p \in \mathrm{supp}\,\Psi_+$ and all $v \in V_{+p}$ there holds

$$g_{Ep}(v + \epsilon\Psi_{+p}v, v + \epsilon\Psi_{+p}v) = g_{Ep}(v, v) + \epsilon^2 g_{Ep}(\Psi_{+p}v, \Psi_{+p}v) \tag{5.35}$$

because of $V_{-p} = V_{+p}^{\perp}$ with respect to g_{Ep}. One immediately sees that for any $v \in \mathrm{Ker}\,\Psi_{+p}$, $v \neq 0$, the inequality

$$g_{Ep}(v + \epsilon\Psi_{+p}v, v + \epsilon\Psi_{+p}v) = g_{Ep}(v, v) > 0 \tag{5.36}$$

is true for all $\epsilon \in \mathbb{R}^+$. On the other hand, if we take $v \in V_{+p} \setminus \operatorname{Ker} \Psi_{+p}$,

$$g_{Ep}(v + \epsilon \Psi_{+p} v, v + \epsilon \Psi_{+p} v) > 0 \qquad \Leftrightarrow \qquad \epsilon < \frac{\sqrt{g_{Ep}(v,v)}}{\sqrt{-g_{Ep}(\Psi_{+p} v, \Psi_{+p} v)}}. \tag{5.37}$$

Note that the maps $\|.\|_p^{\pm} : V_{\pm p} \to \mathbb{R}_0^+$ defined for all $v \in V_{\pm p}$ as

$$\|v\|_p^{\pm} := \sqrt{\pm g_{Ep}(v,v)} \tag{5.38}$$

represent norms on the respective vector spaces, thus for all $v \in V_{+p}$ one has

$$\left\|\Psi_{+p} v\right\|_p^{-} \leq \left\|\Psi_{+p}\right\| \|v\|_p^{+}, \tag{5.39}$$

where $\|.\|$ denotes the operator norm. Consequently, the condition

$$\epsilon < \frac{1}{\left\|\Psi_{+p}\right\|} \tag{5.40}$$

already ensures the positive definiteness of $g_{Ep}\big|_{V'_{+p} \times V'_{+p}}$ because $\frac{1}{\|\Psi_{+p}\|} \leq \frac{\|v\|_p^+}{\|\Psi_{+p} v\|_p^-}$. Since the operator norm is a continuous map, the function $f : M \to \mathbb{R}^+$ introduced as

$$f(p) := \frac{1}{\left\|\Psi_{+p}\right\| + 1}, \tag{5.41}$$

for all $p \in M$, is also continuous. The fact that Ψ_+ is compactly supported then implies

$$\epsilon_1 := \inf_{p \in M} f(p) = \min\{1, \inf_{p \in \operatorname{supp} \Psi_+} f(p)\} > 0, \tag{5.42}$$

hence indeed $V'_+ := \bigsqcup_{p \in M} \{v + \epsilon_1 \Psi_{+p} v \,|\, v \in V_{+p}\}$ is a generalized metric on E.

Consider the generalized metric V'_+ constructed in the previous paragraph, it induces a vector bundle decomposition $E = V'_+ \oplus V'_-$, hence for any $\psi_+ \in \Gamma(V_+)$ there exist a unique $\phi_\pm \in \Gamma(V_\pm)$ such that

$$\psi_+ = \phi_+ + \epsilon \Psi_+ \phi_+ + \phi_- + \epsilon \Psi_- \phi_-, \tag{5.43}$$

where $\Psi_- := -\sharp_E \, \Psi_+^T \flat_E$, see (5.33). It follows from the unambiguity of the section decomposition with respect to $E = V_+ \oplus V_-$ that

$$\psi_+ = \phi_+ + \epsilon \Psi_- \phi_-, \qquad\qquad 0 = \phi_- + \epsilon \Psi_+ \phi_+, \tag{5.44}$$

hence

$$\psi_+ = \phi_+ - \epsilon^2 \Psi_- \Psi_+ \phi_+ \equiv (\operatorname{Id}_{\Gamma(V_+)} - \epsilon^2 \Psi_- \Psi_+) \phi_+. \tag{5.45}$$

Analogously one finds that for all $\psi_- \in \Gamma(V_-)$ there are unique $\chi_\pm \in \Gamma(V_\pm)$ such that

$$0 = \chi_+ + \epsilon \Psi_- \chi_-, \qquad\qquad \psi_- = \chi_- + \epsilon \Psi_+ \chi_+, \tag{5.46}$$

and therefore the equality

$$\psi_- = \chi_- - \epsilon^2 \Psi_+ \Psi_- \chi_- \equiv (\operatorname{Id}_{\Gamma(V_-)} - \epsilon^2 \Psi_+ \Psi_-) \chi_- \tag{5.47}$$

is satisfied. Note that ϕ_+ and χ_- are determined uniquely by ψ_+ and ψ_- respectively, and therefore (5.45) and (5.47) can be equivalently rewritten as

$$\phi_+ = (\mathrm{Id}_{\Gamma(V_+)} - \epsilon^2\Psi_-\Psi_+)^{-1}\psi_+, \qquad \chi_- = (\mathrm{Id}_{\Gamma(V_-)} - \epsilon^2\Psi_+\Psi_-)^{-1}\psi_-. \tag{5.48}$$

Take an arbitrary $p \in \operatorname{supp}\Psi_+$. As $(V_{\pm p}, \|.\|^{\pm}_p)$ are finite-dimensional normed vector spaces, they are even Banach spaces, so the following implication holds:

$$\left\|\epsilon^2\Psi_{\mp p}\Psi_{\pm p}\right\| < 1, \qquad \Rightarrow \qquad (\mathrm{Id}_{V_{\pm p}} - \epsilon^2\Psi_{\mp p}\Psi_{\pm p})^{-1} = \mathrm{Id}_{V_{\pm p}} + \sum_{k=1}^{+\infty}(\epsilon^2\Psi_{\mp p}\Psi_{\pm p})^k, \tag{5.49}$$

see [19, Theorem 7.3-1]. Using the inequality $\|AB\| \leq \|A\|\,\|B\|$ satisfied by all bounded operators on normed vector spaces leads to

$$\epsilon < \frac{1}{\sqrt{\left\|\Psi_{-p}\right\|\left\|\Psi_{+p}\right\|}}, \qquad \Rightarrow \qquad (\mathrm{Id}_{V_{\pm p}} - \epsilon^2\Psi_{\mp p}\Psi_{\pm p})^{-1} = \mathrm{Id}_{V_{\pm p}} + \sum_{k=1}^{+\infty}(\epsilon^2\Psi_{\mp p}\Psi_{\pm p})^k. \tag{5.50}$$

Let us now define the continuous function $h : M \to \mathbb{R}^+$ as

$$h(p) := \frac{1}{\sqrt{\left\|\Psi_{-p}\right\|\left\|\Psi_{+p}\right\| + 1}} \tag{5.51}$$

for all $p \in M$. It follows from the fact that $\operatorname{supp}\Psi_+$ is a compact subset of M that

$$\epsilon_2 := \inf_{p\in M} h(p) > 0. \tag{5.52}$$

Taking $\epsilon := \min\{\epsilon_1, \epsilon_2\}$, where ϵ_1 is defined by (5.42), is apparently sufficient for

$$V'_+ := \bigsqcup_{p\in M}\{v + \epsilon\Psi_{+p}v \,|\, v \in V_{+p}\} \tag{5.53}$$

to be a generalized metric, and moreover, for the following equality to hold

$$(\mathrm{Id}_{\Gamma(V_+)} - \epsilon^2\Psi_-\Psi_+)^{-1} = \mathrm{Id}_{\Gamma(V_+)} + \sum_{k=1}^{+\infty}(\epsilon^2\Psi_-\Psi_+)^k = \mathrm{Id}_{\Gamma(V_+)} + O(\epsilon^2), \tag{5.54}$$

and analogously

$$(\mathrm{Id}_{\Gamma(V_-)} - \epsilon^2\Psi_+\Psi_-)^{-1} = \mathrm{Id}_{\Gamma(V_-)} + O(\epsilon^2). \tag{5.55}$$

For an arbitrary $\psi_+ \in \Gamma(V_+)$ there exist unique $\phi_\pm \in \Gamma(V_\pm)$ such that

$$\begin{aligned}
\tau'\psi_+ &= \tau'(\phi_+ + \epsilon\Psi_+\phi_+ + \phi_- + \epsilon\Psi_-\phi_-) \\
&= \phi_+ + \epsilon\Psi_+\phi_+ - \phi_- - \epsilon\Psi_-\phi_- \\
&= (\mathrm{Id}_{\Gamma(V_+)} - \epsilon^2\Psi_-\Psi_+)^{-1}\psi_+ + \epsilon\Psi_+(\mathrm{Id}_{\Gamma(V_+)} - \epsilon^2\Psi_-\Psi_+)^{-1}\psi_+ + \epsilon\Psi_+(\mathrm{Id}_{\Gamma(V_+)} + \epsilon^2\Psi_-\Psi_+)^{-1}\psi_+ \\
&\quad + \epsilon^2\Psi_-\Psi_+(\mathrm{Id}_{\Gamma(V_+)} - \epsilon^2\Psi_-\Psi_+)^{-1}\psi_+ \\
&= \psi_+ + 2\epsilon\Psi_+\psi_+ + O(\epsilon^2) \\
&= \tau\psi_+ + 2\epsilon\Psi_+\psi_+ + O(\epsilon^2),
\end{aligned} \tag{5.56}$$

where we have used (5.44) and (5.48) in the third step. Analogously, one can derive that for all $\psi_- \in \Gamma(V_-)$ there holds

$$\tau'\psi_- = -\psi_- + 2\epsilon \sharp_E \Psi_+^T \flat_E \psi_- + O(\epsilon^2) = \tau\psi_- + 2\epsilon \sharp_E \Psi_+^T \flat_E \psi_- + O(\epsilon^2). \tag{5.57}$$

Therefore,

$$\tau' = \tau + 2\epsilon \begin{pmatrix} 0 & \sharp_E \Psi_+^T \flat_E \\ \Psi_+ & 0 \end{pmatrix} + O(\epsilon^2), \tag{5.58}$$

where the matrix is considered with respect to the decomposition $E = V_+ \oplus V_-$. □

Proof of the theorem. The proof is similar to the proof of theorem 4.1. Let us start with the variation with respect to a volume form. Assume a volume form Vol, any other volume form is related to Vol through multiplication by a smooth function. Take an arbitrary compactly supported smooth function $f \in C^\infty(M)$, and a real number $\epsilon > 0$ small enough for $\mathrm{Vol}' := (1 + \epsilon f)\,\mathrm{Vol}$ being a volume form on M, the existence of such ϵ is guaranteed by the lemma 5.7. Then

$$S[G, \nabla, \mathrm{Vol}'] = \int_M \mathcal{R}_G^\nabla (1 + \epsilon f)\,\mathrm{Vol} = S[G, \nabla, \mathrm{Vol}] + \epsilon \int_M f \mathcal{R}_G^\nabla\,\mathrm{Vol}. \tag{5.59}$$

Then, it follows from *Fundamental lemma of calculus of variations* that $\delta^{\mathrm{Vol}} S[G, \nabla, \mathrm{Vol}] = 0$ if and only if

$$\mathcal{R}_G^\nabla = 0. \tag{5.60}$$

Let us continue with the variation with respect to a generalized metric. Take a generalized metric V_+. Any other generalized metric is then associated with some $C^\infty(M)$-module morphism $\Psi_+ : \Gamma(V_+) \to \Gamma(V_-)$, see lemma 5.8. Take such a morphism Ψ_+, and moreover, assume that it is compactly supported. Then the lemma 5.8 ensures that there is a real number $\epsilon > 0$ small enough for $V'_+ := \bigsqcup_{p\in M} \{v + \epsilon \Psi_{+p} v \,|\, v \in V_{+p}\}$ to be a generalized metric, and furthermore, for the relation (5.29) to be satisfied. The matrix in the relation represents a $C^\infty(M)$-module endomorphism of $\Gamma(E)$, denote it as Ψ. One easily finds how the G-Ricci scalar is varied under the change of generalized metric $V_+ \mapsto V'_+$,

$$\begin{aligned}
\mathcal{R}_{G'}^\nabla &= \mathrm{Ric}_\nabla(\sharp_{G'} \xi^\mu, \xi_\mu) \\
&= G'(\sharp_{G'} \xi^\mu, \sharp_{G'} \xi^\nu)\,\mathrm{Ric}_\nabla(\xi_\mu, \xi_\nu) \\
&= G'(\tau' \sharp_E \xi^\mu, \tau' \sharp_E \xi^\nu)\,\mathrm{Ric}_\nabla(\xi_\mu, \xi_\nu) \\
&= g_E(\sharp_E \xi^\mu, \tau' \sharp_E \xi^\nu)\,\mathrm{Ric}_\nabla(\xi_\mu, \xi_\nu) \\
&= g_E(\sharp_E \xi^\mu, \tau \sharp_E \xi^\nu)\,\mathrm{Ric}_\nabla(\xi_\mu, \xi_\nu) + 2\epsilon \xi^\mu(\Psi \sharp_E \xi^\nu)\,\mathrm{Ric}_\nabla(\xi_\mu, \xi_\nu) + O(\epsilon^2) \\
&= \mathcal{R}_G^\nabla + 2\epsilon(\xi^{+a}(\sharp_E \Psi_+^T \xi^{-b})\,\mathrm{Ric}_\nabla(\xi_a^+, \xi_b^-) + \xi^{-a}(\Psi_+ \sharp_E \xi^{+b})\,\mathrm{Ric}_\nabla(\xi_a^-, \xi_b^+)) + O(\epsilon^2) \\
&= \mathcal{R}_G^\nabla + 4\epsilon \xi^{-a}(\Psi_+ \sharp_E \xi^{+b})\,\mathrm{Ric}_\nabla(\xi_a^-, \xi_b^+) + O(\epsilon^2),
\end{aligned} \tag{5.61}$$

where $\{\xi_\mu\}_{\mu=1}^{\mathrm{Rank}(E)} \equiv \{\xi_a^+\}_{a=1}^{\mathrm{Rank}(V_+)} \cup \{\xi_a^-\}_{a=1}^{\mathrm{Rank}(V_-)}$ is an arbitrary local frame of E adapted to $E = V_+ \oplus V_-$. During the derivation we have used several identities widely discussed throughout the whole thesis. Therefore,

$$\begin{aligned}
S[G', \nabla, \mathrm{Vol}] &= \int_M \mathcal{R}_{G'}^\nabla\,\mathrm{Vol} \\
&= S[G, \nabla, \mathrm{Vol}] + 4\epsilon \int_M \xi^{-a}(\Psi_+ \sharp_E \xi^{+b})\,\mathrm{Ric}_\nabla(\xi_a^-, \xi_b^+)\,\mathrm{Vol} + O(\epsilon^2),
\end{aligned} \tag{5.62}$$

hence using *Fundamental lemma of calculus of variations* leads to

$$\delta^G S[G,\nabla,\mathrm{Vol}] = 0 \qquad \Leftrightarrow \qquad \nabla \text{ is Ricci compatible with } G. \tag{5.63}$$

It remains to perform the variation of the Courant-Einstein-Hilbert action with respect to a Courant algebroid connection. We will follow the approach used before for the variation of the Einstein-Hilbert action with respect to an affine connection. Take an arbitrary trio (G,∇,Vol) of a generalized metric G on E, a Courant algebroid connection ∇ on E, and a volume form Vol on M. As we have shown in the lemma 5.3, there is $\nabla^0 \in \mathrm{LC}(E,G)$ associated with Vol by the equality (5.10). Let us take the one constructed in the proof, that is the one satisfying $\mathrm{div}_\nabla = \mathrm{div}^g \circ \rho$, where div^g is uniquely determined by Vol. It follows from the lemma 1.56 that Courant algebroid connections ∇ and ∇^0 are related by a tensor field $L \in \Omega^1(E)\otimes\Omega^2(E)$ as

$$\nabla_{\psi_1}\psi_2 = \nabla^0_{\psi_1}\psi_2 + \sharp_E\, L(\psi_1,\psi_2,.), \tag{5.64}$$

for all ψ_1, $\psi_2 \in \Gamma(E)$. If we would like to describe any other Courant algebroid connection than ∇, we have to take yet another tensor field from $\Omega^1(E)\otimes\Omega^2(E)$. Therefore, assume an arbitrary $N \in \Omega^1(E)\otimes\Omega^2(E)$ identically vanishing on ∂M and an arbitrary $\epsilon \in \mathbb{R}^+$, and denote

$$\nabla'_{\psi_1}\psi_2 := \nabla_{\psi_2}\psi_1 + \epsilon\sharp_E\, N(\psi_1,\psi_2,.) \equiv \nabla^0_{\psi_1}\psi_2 + \sharp_E\, L(\psi_1,\psi_2,.) + \epsilon\sharp_E\, N(\psi_1,\psi_2,.), \tag{5.65}$$

for all ψ_1, $\psi_2 \in \Gamma(E)$. Now we need to find out how is the G-Ricci scalar changed under the transformation $\nabla \mapsto \nabla'$. Let us do it step by step. For all ψ_1, ψ_2, ψ_3, $\psi_4 \in \Gamma(E)$ there holds

$$\begin{aligned}
&R_{0\nabla'}(\psi_1,\psi_2,\psi_3,\psi_4)\\
&\quad = g_E(\nabla'_{\psi_3}\nabla'_{\psi_4}\psi_2 - \nabla'_{\psi_4}\nabla'_{\psi_3}\psi_2 - \nabla'_{[\psi_3,\psi_4]_E}\psi_2,\psi_1)\\
&\quad = R_{0\nabla}(\psi_1,\psi_2,\psi_3,\psi_4) + \epsilon g_E\Big(\sharp_E\,\big(\nabla_{\psi_3}N(\psi_4,\psi_2,.) + N(\psi_3,\nabla_{\psi_4}\psi_2,.) - \nabla_{\psi_4}N(\psi_3,\psi_2,.)\\
&\qquad - N(\psi_4,\nabla_{\psi_3}\psi_2,.) - N([\psi_3,\psi_4]_E,\psi_2,.)\big),\psi_1\Big) + O(\epsilon^2)\\
&\quad = R_{0\nabla}(\psi_1,\psi_2,\psi_3,\psi_4) + \epsilon\Big(\rho(\psi_3)N(\psi_4,\psi_2,\psi_1) - N(\psi_4,\psi_2,\nabla_{\psi_3}\psi_1) - N(\psi_4,\nabla_{\psi_3}\psi_2,\psi_1)\\
&\qquad - \rho(\psi_4)N(\psi_3,\psi_2,\psi_1) + N(\psi_3,\psi_2,\nabla_{\psi_4}\psi_1) + N(\psi_3,\nabla_{\psi_4}\psi_2,\psi_1) - N(\nabla^0_{\psi_3}\psi_4,\psi_2,\psi_1)\\
&\qquad + N(\nabla^0_{\psi_4}\psi_3,\psi_2,\psi_1) - g_E(\nabla^0_{\sharp_E\,\xi^\mu}\psi_3,\psi_4)N(\xi_\mu,\psi_2,\psi_1)\Big) + O(\epsilon^2)\\
&\quad = R_{0\nabla}(\psi_1,\psi_2,\psi_3,\psi_4) + \epsilon\Big((\nabla^0_{\psi_3}N)(\psi_4,\psi_2,\psi_1) - (\nabla^0_{\psi_4}N)(\psi_3,\psi_2,\psi_1)\\
&\qquad - N(\psi_4,\psi_2,\sharp_E\,\xi^\mu)L(\psi_3,\psi_1,\xi_\mu) - N(\psi_4,\sharp_E\,\xi^\mu,\psi_1)L(\psi_3,\psi_2,\xi_\mu)\\
&\qquad + N(\psi_3,\psi_2,\sharp_E\,\xi^\mu)L(\psi_4,\psi_1,\xi_\mu) + N(\psi_3,\sharp_E\,\xi^\mu,\psi_1)L(\psi_4,\psi_2,\xi_\mu)\\
&\qquad - N(\sharp_E\,\xi^\mu,\psi_2,\psi_1)g_E(\nabla^0_{\xi_\mu}\psi_3,\psi_4)\Big) + O(\epsilon^2).
\end{aligned} \tag{5.66}$$

We have used the compatibility of ∇ with the Courant metric in the second step and the torsion-freeness of ∇^0 in the third step. There is yet another term in the definition of Riemann tensor, see 1.28, which is changing as

$$
\begin{aligned}
& g_E(\nabla'_{\xi_\mu}\psi_3,\psi_4) g_E(\nabla'_{\sharp_E\,\xi^\mu}\psi_2,\psi_1) \\
&\quad = g_E(\nabla_{\xi_\mu}\psi_3,\psi_4) g_E(\nabla_{\sharp_E\,\xi^\mu}\psi_2,\psi_1) + \epsilon\Big(g_E(\nabla_{\xi_\mu}\psi_3,\psi_4) N(\sharp_E\,\xi^\mu,\psi_2,\psi_1) \\
&\qquad + g_E(\nabla_{\xi_\mu}\psi_2,\psi_1) N(\sharp_E\,\xi^\mu,\psi_3,\psi_4)\Big) + O(\epsilon^2) \\
&\quad = g_E(\nabla_{\xi_\mu}\psi_3,\psi_4) g_E(\nabla_{\sharp_E\,\xi^\mu}\psi_2,\psi_1) + \epsilon\Big(N(\sharp_E\,\xi^\mu,\psi_2,\psi_1) g_E(\nabla^0_{\xi_\mu}\psi_3,\psi_4) \\
&\qquad + N(\sharp_E\,\xi^\mu,\psi_3,\psi_4) g_E(\nabla^0_{\xi_\mu}\psi_2,\psi_1) + N(\sharp_E\,\xi^\mu,\psi_2,\psi_1) L(\xi_\mu,\psi_3,\psi_4) \\
&\qquad + N(\sharp_E\,\xi^\mu,\psi_3,\psi_4) L(\xi_\mu,\psi_2,\psi_1)\Big) + O(\epsilon^2),
\end{aligned} \tag{5.67}
$$

for all $\psi_1,\ \psi_2,\ \psi_3,\ \psi_4 \in \Gamma(E)$. In summary, one obtains the transformation relation for the Riemann tensor in the form

$$
\begin{aligned}
& R_{\nabla'}(\psi_1,\psi_2,\psi_3,\psi_4) \\
&\quad = R_{\nabla}(\psi_1,\psi_2,\psi_3,\psi_4) + \epsilon\frac{1}{2}\Big((\nabla^0_{\psi_3}N)(\psi_4,\psi_2,\psi_1) - (\nabla^0_{\psi_4}N)(\psi_3,\psi_2,\psi_1) + (\nabla^0_{\psi_2}N)(\psi_1,\psi_3,\psi_4) \\
&\qquad - (\nabla^0_{\psi_1}N)(\psi_2,\psi_3,\psi_4) - N(\psi_4,\psi_2,\sharp_E\,\xi^\mu) L(\psi_3,\psi_1,\xi_\mu) - N(\psi_4,\sharp_E\,\xi^\mu,\psi_1) L(\psi_3,\psi_2,\xi_\mu) \\
&\qquad + N(\psi_3,\psi_2,\sharp_E\,\xi^\mu) L(\psi_4,\psi_1,\xi_\mu) + N(\psi_3,\sharp_E\,\xi^\mu,\psi_1) L(\psi_4,\psi_2,\xi_\mu) \\
&\qquad - N(\psi_1,\psi_3,\sharp_E\,\xi^\mu) L(\psi_2,\psi_4,\xi_\mu) - N(\psi_1,\sharp_E\,\xi^\mu,\psi_4) L(\psi_2,\psi_3,\xi_\mu) \\
&\qquad + N(\psi_2,\psi_3,\sharp_E\,\xi^\mu) L(\psi_1,\psi_4,\xi_\mu) + N(\psi_2,\sharp_E\,\xi^\mu,\psi_4) L(\psi_1,\psi_3,\xi_\mu) \\
&\qquad + N(\sharp_E\,\xi^\mu,\psi_2,\psi_1) L(\xi_\mu,\psi_3,\psi_4) + N(\sharp_E\,\xi^\mu,\psi_3,\psi_4) L(\xi_\mu,\psi_2,\psi_1)\Big) + O(\epsilon^2),
\end{aligned} \tag{5.68}
$$

for all $\psi_1,\ \psi_2,\ \psi_3,\ \psi_4 \in \Gamma(E)$. Let us now denote the non-trivial partial traces of tensor fields N and L with respect to the Courant and generalized metric as

$$
\mathcal{N} := N(\sharp_E\,\xi^\mu,\xi_\mu,.), \qquad \mathcal{N}_G := N(\sharp_G\,\xi^\mu,\xi_\mu,.), \tag{5.69}
$$

$$
\mathcal{L} := L(\sharp_E\,\xi^\mu,\xi_\mu,.), \qquad \mathcal{L}_G := L(\sharp_G\,\xi^\mu,\xi_\mu,.). \tag{5.70}
$$

The relation for the variation of the Ricci tensor is given by the contraction of (5.68) in the first and the third input with respect to g_E. For arbitrary $\psi_1,\ \psi_2 \in \Gamma(E)$, one obtains

$$
\begin{aligned}
& \mathrm{Ric}_{\nabla'}(\psi_1,\psi_2) \\
&\quad = \mathrm{Ric}_{\nabla}(\psi_1,\psi_2) \\
&\qquad + \epsilon\frac{1}{2}\Big((\nabla^0_{\xi_\nu}N)(\psi_2,\psi_1,\sharp_E\,\xi^\nu) + (\nabla^0_{\psi_2}\mathcal{N})(\psi_1) + (\nabla^0_{\psi_1}\mathcal{N})(\psi_2) - (\nabla^0_{\xi_\nu}N)(\psi_1,\sharp_E\,\xi^\nu,\psi_2) \\
&\qquad - N(\psi_2,\psi_1,\sharp_E\,\xi^\mu)\mathcal{L}(\xi_\mu) - N(\psi_2,\sharp_E\,\xi^\mu,\sharp_E\,\xi^\nu) L(\xi_\nu,\psi_1,\xi_\mu) \\
&\qquad + N(\sharp_E\,\xi^\nu,\psi_1,\sharp_E\,\xi^\mu) L(\psi_2,\xi_\nu,\xi_\mu) - \mathcal{N}(\sharp_E\,\xi^\mu) L(\psi_2,\psi_1,\xi_\mu) - \mathcal{N}(\sharp_E\,\xi^\mu) L(\psi_1,\psi_2,\xi_\mu) \\
&\qquad - N(\sharp_E\,\xi^\nu,\sharp_E\,\xi^\mu,\psi_2) L(\psi_1,\xi_\nu,\xi_\mu) + N(\psi_1,\sharp_E\,\xi^\nu,\sharp_E\,\xi^\mu) L(\xi_\nu,\psi_2,\xi_\mu) \\
&\qquad + N(\psi_1,\sharp_E\,\xi^\mu,\psi_2)\mathcal{L}(\xi_\mu) + N(\sharp_E\,\xi^\mu,\psi_1,\sharp_E\,\xi^\nu) L(\xi_\mu,\xi_\nu,\psi_2) \\
&\qquad + N(\sharp_E\,\xi^\mu,\sharp_E\,\xi^\nu,\psi_2) L(\xi_\mu,\psi_1,\xi_\nu)\Big) + O(\epsilon^2)
\end{aligned} \tag{5.71}
$$

We have used the compatibility of ∇^0 with the Courant metric, see lemma 1.64. By using the skew-symmetry of tensor fields L and N in the last two inputs and reordering the terms on the right hand side of the above equation, one obtains

$$\begin{aligned}
&\mathrm{Ric}_{\nabla'}(\psi_1,\psi_2)\\
&\quad= \mathrm{Ric}_{\nabla}(\psi_1,\psi_2)\\
&\quad+\epsilon\frac{1}{2}\Big((\nabla^0_{\psi_2}\mathcal{N})(\psi_1)+(\nabla^0_{\psi_1}\mathcal{N})(\psi_2)+(\nabla^0_{\xi_\nu}N)(\psi_2,\psi_1,\sharp_E\,\xi^\nu)-(\nabla^0_{\xi_\nu}N)(\psi_1,\sharp_E\,\xi^\nu,\psi_2)\\
&\quad-N(\psi_2,\psi_1,\sharp_E\,\xi^\mu)\mathcal{L}(\xi_\mu)-N(\psi_1,\psi_2,\sharp_E\,\xi^\mu)\mathcal{L}(\xi_\mu)\\
&\quad-\mathcal{N}(\sharp_E\,\xi^\mu)L(\psi_2,\psi_1,\xi_\mu)-\mathcal{N}(\sharp_E\,\xi^\mu)L(\psi_1,\psi_2,\xi_\mu)\\
&\quad-N(\psi_2,\sharp_E\,\xi^\nu,\sharp_E\,\xi^\mu)L(\xi_\nu,\xi_\mu,\psi_1)-N(\psi_1,\sharp_E\,\xi^\nu,\sharp_E\,\xi^\mu)L(\xi_\nu,\xi_\mu,\psi_2)\\
&\quad-N(\sharp_E\,\xi^\nu,\sharp_E\,\xi^\mu,\psi_2)L(\psi_1,\xi_\nu,\xi_\mu)-N(\sharp_E\,\xi^\nu,\sharp_E\,\xi^\mu,\psi_1)L(\psi_2,\xi_\nu,\xi_\mu)\\
&\quad-N(\sharp_E\,\xi^\mu,\sharp_E\,\xi^\nu,\psi_1)L(\xi_\mu,\xi_\nu,\psi_2)-N(\sharp_E\,\xi^\mu,\sharp_E\,\xi^\nu,\psi_2)L(\xi_\mu,\xi_\nu,\psi_1)\Big)+O(\epsilon^2). \qquad (5.72)
\end{aligned}$$

Finally, the formula for the G-Ricci scalar appears by taking the trace of (5.72) with respect to the generalized metric G, it yields

$$\begin{aligned}
\mathcal{R}_G^{\nabla'} &= \mathcal{R}_G^{\nabla}+\epsilon\Big((\nabla^0_{\xi_\kappa}\mathcal{N})(\sharp_G\,\xi^\kappa)+(\nabla^0_{\xi_\nu}\mathcal{N}_G)(\sharp_E\,\xi^\nu)-\mathcal{N}_G(\sharp_E\,\xi^\mu)\mathcal{L}(\xi_\mu)-\mathcal{N}(\sharp_E\,\xi^\mu)\mathcal{L}_G(\xi_\mu)\\
&\quad-N(\sharp_G\,\xi^\kappa,\sharp_E\,\xi^\nu,\sharp_E\,\xi^\mu)L(\xi_\nu,\xi_\mu,\xi_\kappa)-N(\sharp_E\,\xi^\nu,\sharp_E\,\xi^\mu,\sharp_G\,\xi^\kappa)L(\xi_\kappa,\xi_\nu,\xi_\mu)\\
&\quad-N(\sharp_E\,\xi^\mu,\sharp_E\,\xi^\nu,\sharp_G\,\xi^\kappa)L(\xi_\mu,\xi_\nu,\xi_\kappa)\Big)+O(\epsilon^2)\\
&= \mathcal{R}_G^{\nabla}+\epsilon\Big((\nabla^0_{\xi_\kappa}\mathcal{N})(\tau\sharp_E\,\xi^\kappa)+\mathrm{div}_{\nabla^0}\mathcal{N}_G\\
&\quad-N^{\kappa\nu\mu}(G_{\kappa\sigma}g_E{}^{\sigma\lambda}g_{E\lambda\nu}\mathcal{L}_\mu+g_{E\kappa\nu}\mathcal{L}_{G\mu}+G_{\kappa\sigma}g_E{}^{\sigma\lambda}L_{\nu\mu\lambda}+G_{\mu\sigma}g_E{}^{\sigma\lambda}L_{\lambda\kappa\nu}+G_{\mu\sigma}g_E{}^{\sigma\lambda}L_{\kappa\nu\lambda})\Big)\\
&\quad+O(\epsilon^2)\\
&= \mathcal{R}_G^{\nabla}+\epsilon\Big(\mathrm{div}_{\nabla^0}(\tau^T\mathcal{N})+\mathrm{div}_{\nabla^0}\mathcal{N}_G\\
&\quad-N^{\kappa\nu\mu}(G_{\kappa\nu}\mathcal{L}_\mu+g_{E\kappa\nu}\mathcal{L}_{G\mu}+G_\kappa{}^\lambda L_{\nu\mu\lambda}+G_\mu{}^\lambda L_{\lambda\kappa\nu}+G_\mu{}^\lambda L_{\kappa\nu\lambda})\Big)+O(\epsilon^2). \qquad (5.73)
\end{aligned}$$

In the first step, we have used the compatibility of ∇^0 with the generalized metric G in the sense of lemma 1.64. In the second step, an easy to check identity

$$\sharp_G\,\xi^\nu = G(\xi_\mu,\xi_\kappa)g_E(\sharp_E\,\xi^\kappa,\sharp_E\,\xi^\nu)\sharp_E\,\xi^\mu \qquad (5.74)$$

has been used. For the final equality, the compatibility of ∇^0 with the generalized metric G has been used again, but this time in the sense of (1.115). The divergence terms in (5.73) can be further rewritten, see (1.41), as

$$\begin{aligned}
\mathrm{div}_{\nabla^0}(\tau^T\mathcal{N})+\mathrm{div}_{\nabla^0}\mathcal{N}_G &= \mathrm{div}_{\nabla^0}\sharp_E\,(\tau^T\mathcal{N}+\mathcal{N}_G)\\
&= \mathrm{div}^g\,\rho(\sharp_E\,(\tau^T\mathcal{N}+\mathcal{N}_G))\\
&= \mathrm{div}^g\,\flat_g\,\rho(\sharp_E\,(\tau^T\mathcal{N}+\mathcal{N}_G))\\
&= -\,(1,\delta_g\flat_g\,\rho(\sharp_E\,(\tau^T\mathcal{N}+\mathcal{N}_G)))_g, \qquad (5.75)
\end{aligned}$$

where div^g is uniquely determined by Vol. Hence, and from the lemmas 5.1 and 3.5, it follows

$$\begin{aligned}
\int_M \mathrm{div}_{\nabla^0}(\tau^T\mathcal{N}+\mathcal{N}_G)\,\mathrm{Vol} &= \pm\int_M (1,\delta_g\flat_g\,\rho(\sharp_E\,(\tau^T\mathcal{N}+\mathcal{N}_G)))_g\,\mathrm{Vol}_g\\
&= \mp\int_{\partial M}\star_{g,o}\flat_g\,\rho(\sharp_E\,(\tau^T\mathcal{N}+\mathcal{N}_G))=0, \qquad (5.76)
\end{aligned}$$

the last step is an easy consequence of $N|_{\partial M} = 0$. Therefore, for the difference of Courant-Einstein-Hilbert actions one can write

$$
\begin{aligned}
S[G,&\nabla', \mathrm{Vol}] - S[G, \nabla, \mathrm{Vol}] \\
&= \int_M \mathcal{R}_G^{\nabla'} \mathrm{Vol} - \int_M \mathcal{R}_G^{\nabla} \mathrm{Vol} \\
&= -\epsilon \int_M N^{\kappa\nu\mu} (G_{\kappa\nu}\mathcal{L}_\mu + g_{E\kappa\nu}\mathcal{L}_{G\mu} + G_\kappa{}^\lambda L_{\nu\mu\lambda} + G_\mu{}^\lambda L_{\lambda\kappa\nu} + G_\mu{}^\lambda L_{\kappa\nu\lambda}) \mathrm{Vol} + O(\epsilon^2). \quad (5.77)
\end{aligned}
$$

Fundamental lemma of calculus of variations then says $\delta^\nabla S[G, \nabla, \mathrm{Vol}] = 0$ if and only if

$$
\begin{aligned}
0 = &G(\psi_1, \psi_2)\mathcal{L}(\psi_3) - G(\psi_1, \psi_3)\mathcal{L}(\psi_2) + g_E(\psi_1, \psi_2)\mathcal{L}_G(\psi_3) - g_E(\psi_1, \psi_3)\mathcal{L}_G(\psi_2) \\
&+ L(\psi_2, \psi_3, \tau\psi_1) - L(\psi_3, \psi_2, \tau\psi_1) + L(\tau\psi_3, \psi_1, \psi_2) \\
&- L(\tau\psi_2, \psi_1, \psi_3) + L(\psi_1, \psi_2, \tau\psi_3) - L(\psi_1, \psi_3, \tau\psi_2), \quad (5.78)
\end{aligned}
$$

for all ψ_1, ψ_2, $\psi_3 \in \Gamma(E)$. The skew-symmetrization in the last two inputs appears, since N is skew-symmetric in these two.

The last missing piece is to solve the equation of motion (5.78) for an unknown tensor field $L \in \Omega^1(E) \otimes \Omega^2(E)$. Taking the partial trace in ψ_1 and ψ_2 from it with respect to the Courant metric g_E results for all $\psi \in \Gamma(E)$ into

$$
0 = (p - q)\mathcal{L}(\psi) + (\mathrm{Rank}(E) - 2)\mathcal{L}_G(\psi), \quad (5.79)
$$

while on the other taking the same partial trace but with respect to the generalized metric G leads to

$$
0 = (\mathrm{Rank}(E) - 2)\mathcal{L}(\psi) + (p - q)\mathcal{L}_G(\psi), \quad (5.80)
$$

for all $\psi \in \Gamma(E)$. Solving this pair of equations with respect to unknown 1-forms $\mathcal{L}$, $\mathcal{L}_G \in \Omega^1(E)$ is a trivial task, for example we can compute the determinant of the matrix associated with the set of linear equations

$$
\begin{vmatrix} p - q & \mathrm{Rank}(E) - 2 \\ \mathrm{Rank}(E) - 2 & p - q \end{vmatrix} = (p - q)^2 - (p + q - 2)^2 = -4(pq - p - q + 1). \quad (5.81)
$$

It vanishes if and only if $p = 1$ or $q = 1$, which is exactly the case excluded in the assumption of the theorem. Therefore the set of the equations is satisfied if and only if $\mathcal{L} = \mathcal{L}_G = 0$. Employing this back to the equation of motion (5.78) yields

$$
\begin{aligned}
0 = &L(\psi_2, \psi_3, \tau\psi_1) - L(\psi_3, \psi_2, \tau\psi_1) + L(\tau\psi_3, \psi_1, \psi_2) \\
&- L(\tau\psi_2, \psi_1, \psi_3) + L(\psi_1, \psi_2, \tau\psi_3) - L(\psi_1, \psi_3, \tau\psi_2), \quad (5.82)
\end{aligned}
$$

for all ψ_1, ψ_2, $\psi_3 \in \Gamma(E)$. By choosing arbitrary $\psi_{1\pm}$, $\psi_{2\pm}$, $\psi_{3\pm} \in \Gamma(V_\pm)$ for the (5.82), we obtain

$$
L_A(\psi_{1\pm}, \psi_{2\pm}, \psi_{3\pm}) = 0, \quad (5.83)
$$

whereas if we take a trio of inputs such that they are either in $\Gamma(V_+)$ or in $\Gamma(V_-)$, but not that all of them are in the one of these set, we find out that all "mixed" restrictions vanish identically. These are exactly the properties necessary and sufficient for ∇ to be a Levi-Civita Courant algebroid connection on E with respect to G, see 1.61. We have just shown that if some tensor field $L \in \Omega^1(E) \otimes \Omega^2(E)$ satisfies the equation of motion (5.78), then the Courant algebroid connection ∇ defined by (5.64) is Levi-Civita with respect to G, and furthermore, $\operatorname{div}_\nabla = \operatorname{div}^g \circ \rho$. The latter follows directly from employing $\mathcal{L} = 0$ into (1.142).

To finish the proof, let us show the converse. Assume that ∇ related to ∇^0 by (5.64) is an element of $\mathrm{LC}(E, G)$ and $\operatorname{div}_\nabla = \operatorname{div}^g \circ \rho$. Then one immediately sees that $\mathcal{L}$ vanishes thanks to the formula (1.142). As $\nabla \in \mathrm{LC}(E, G)$, for all $\psi \in \Gamma(E)$ there holds

$$\begin{aligned}\mathcal{L}_G(\psi) &= L(\sharp_G\, \xi^\mu, \xi_\mu, \psi) = L(\sharp_E\, \xi^{+a}, \xi^+_a, \psi_+) - L(\sharp_E\, \xi^{-a}, \xi^-_a, \psi_-) = L(\sharp_E\, \xi^\mu, \xi_\mu, \psi_+ - \psi_-)\\ &= \mathcal{L}(\tau\psi),\end{aligned} \tag{5.84}$$

hence also $\mathcal{L}_G = 0$. Therefore, the equation (5.78) simplifies into (5.82). One can easily check that it is satisfied for a difference tensor field between two Levi-Civita Courant algebroid connections with respect to τ. ☐

Remark 5.9. The approach used for the variation of a generalized metric appeared earlier in a different context in [20].

Remark 5.10. Note that all three equations of motion are invariant with respect to the metric and connection preserving Courant algebroid isomorphisms, see propositions 1.48, 1.50, 1.71 and 1.23.

Let us now state the corollary reformulating the previous theorem for the specific class of Courant algebroids, those associated with the generalized geometry, see Chapter 2.

Corollary 5.11. *Let* $(E \xrightarrow{\pi} M, \rho, [\cdot,\cdot]^H_D, g_E)$ *be a Courant algebroid defined in example 2.1 over an oriented connected smooth manifold* M *with* $\dim(M) \neq 1$. *Then a generalized metric* G *on* E, *a Courant algebroid connection* ∇ *on* E *and a volume form* Vol *on* M *extremalize the Courant-Einstein-Hilbert action*

$$S[G, \nabla, \mathrm{Vol}] := \int_M \mathcal{R}^\nabla_G \,\mathrm{Vol}, \tag{5.85}$$

if and only if the G-Ricci scalar $\mathcal{R}^\nabla_G$ *vanishes identically,* ∇ *is Ricci compatible with* G *and Levi-Civita with respect to* G, *and moreover*

$$\mathrm{d}\,\phi \circ \rho = \frac{1}{2}(\operatorname{div}_\nabla - \operatorname{div}^g \circ \rho), \tag{5.86}$$

where g *is a Riemannian metric on* M *uniquely determined by* G *and* Vol *is without loss of generality parametrized as* $\mathrm{Vol} = \pm\, \mathrm{e}^{-2\phi}\, \mathrm{Vol}_g$.

Proof. Since the signature of the chosen Courant algebroid is $(\dim(M), \dim(M))$, the assumptions of the theorem 5.6 are satisfied. Therefore,

$$\delta^{\mathrm{Vol}} S[G, \nabla, \mathrm{Vol}] = 0 \quad \Leftrightarrow \quad \mathcal{R}^\nabla_G = 0, \tag{5.87}$$

$$\delta^{G} S[G, \nabla, \mathrm{Vol}] = 0 \quad \Leftrightarrow \quad \nabla \text{ is Ricci compatible with } G, \tag{5.88}$$

$$\delta^{\nabla} S[G, \nabla, \mathrm{Vol}] = 0 \quad \Leftrightarrow \quad \nabla \in \mathrm{LC}(E, G) \text{ and at the same time } \operatorname{div}_\nabla = \operatorname{div}^{\tilde{g}} \circ \rho, \tag{5.89}$$

where $\operatorname{div}^{\tilde{g}}$ is uniquely determined by Vol, see (5.21), as

$$\operatorname{div}^{\tilde{g}} = \operatorname{div}^g + (\mathrm{d} \log |h|), \tag{5.90}$$

where $\mathrm{Vol} = \pm \mathrm{Vol}_{\tilde{g}} = h \, \mathrm{Vol}_g$, here g denotes without loss of generality, see lemma 5.1, the Riemannian metric on M associated with the generalized metric G on $\mathbb{T}M$ and h is a smooth function on M uniquely determined by the choice of g as an auxiliary metric. As M is connected, the smooth function h does not change its sign on M. We can thus represent it as either $h = \mathrm{e}^{-2\phi}$ or $h = -\,\mathrm{e}^{-2\phi}$ by defining a smooth function $\phi \in C^\infty(M)$ as $\phi := -\frac{1}{2} \log |h|$. The formula (5.90) then goes into the form

$$\mathrm{div}^{\tilde{g}} = \mathrm{div}^g - 2 \, \mathrm{d}\, \phi. \tag{5.91}$$

As $\mathrm{div}_\nabla = \mathrm{div}^{\tilde{g}} \circ \rho$ and ρ is surjective, the equation above is equivalent to

$$\mathrm{d}\, \phi \circ \rho = \frac{1}{2} (\mathrm{div}^g \circ \rho - \mathrm{div}_\nabla). \tag{5.92}$$

□

This is a very interesting result, because we have just discovered that the more or less artificial-looking set-up constructed in Section 3.2 to rewrite the string effective action and the corresponding equations of motion, can be actually derived from the variational principle in a very natural way. This is a rather strong argument for our belief that Courant algebroids provide a natural mathematical framework for the respective physical theory. The result is even more interesting, since we obtain qualitatively the same equations of motion for an arbitrary Courant algebroid completely unrelated to the generalized geometry.

It is good to realize that a Levi-Civita Courant algebroid connection is generally not determined uniquely by fixing its divergence. Therefore, the Palatini approach provides a certain freedom for the choice of the Courant algebroid connection. However, as the following proposition states, the other equations of motion are invariant with respect to this freedom, consequently the physics is not influenced by our choice.

Proposition 5.12. *Let $(E \overset{\pi}{\to} M, \rho, [.\,,.]_E, g_E)$ be a Courant algebroid equipped with a generalized metric G, and assume that we have two Courant algebroid connections $\nabla,\ \nabla' \in \mathrm{LC}(E, G)$ satisfying $\mathrm{div}_\nabla = \mathrm{div}_{\nabla'}$. Then*

$$\mathcal{R}^\nabla_G = \mathcal{R}^{\nabla'}_G, \tag{5.93}$$

and,

$$\nabla \text{ is Ricci compatible with } G \qquad \Leftrightarrow \qquad \nabla' \text{ is Ricci compatible with } G. \tag{5.94}$$

Proof. First of all, realize that $\nabla, \nabla' \in \mathrm{LC}(E, G)$, hence there is a tensor field $K \in \Omega^1(E) \otimes \Omega^2(E)$ possessing the appropriate properties such that

$$\nabla'_{\psi_1} \psi_2 = \nabla_{\psi_1} \psi_2 + \sharp_E \, K(\psi_1, \psi_2, .). \tag{5.95}$$

Since divergences associated with the respective Courant algebroid connections are identical, it follows from (1.142) that the partial trace of K vanishes, that is $\mathcal{K} = 0$. One immediately sees from the proposition 1.69 that $\mathcal{R}^\nabla_G = \mathcal{R}^{\nabla'}_G$. Let us now have a look at the Ricci compatibility. As $\mathcal{K}$ vanishes identically, the relation between the corresponding Ricci tensors, see proposition 1.65, reduces to

$$\begin{aligned} \mathrm{Ric}_{\nabla'}(\psi_1, \psi_2) = \mathrm{Ric}_\nabla(\psi_1, \psi_2) + \frac{1}{2} \Big(& (\nabla_{\xi_\mu} K)(\psi_1, \psi_2, \sharp_E \, \xi^\mu) + (\nabla_{\xi_\mu} K)(\psi_2, \psi_1, \sharp_E \, \xi^\mu) \\ & - K(\psi_2, \sharp_E \, K(\xi_\mu, \psi_1, .), \sharp_E \, \xi^\mu) - K(\sharp_E \, \xi^\mu, \sharp_E \, K(\psi_1, \xi_\mu, .), \psi_2) \\ & + K(\sharp_E \, K(.\,, \xi_\mu, \psi_2), \psi_1, \sharp_E \, \xi^\mu) \Big), \end{aligned} \tag{5.96}$$

for all ψ_1, $\psi_2 \in \Gamma(E)$. Taking arbitrary $\psi_\pm \in \Gamma(V_\pm)$, one has

$$\begin{aligned}\mathrm{Ric}_{\nabla'}(\psi_+, \psi_-) &= \mathrm{Ric}_{\nabla}(\psi_+, \psi_-) + \frac{1}{2}\Big((\nabla_{\xi_\mu} K)(\psi_+, \psi_-, \sharp_E\, \xi^\mu) + (\nabla_{\xi_\mu} K)(\psi_-, \psi_+, \sharp_E\, \xi^\mu) \\ &\quad - K(\xi_\mu, \psi_+, \xi_\nu) K(\psi_-, \sharp_E\, \xi^\nu, \sharp_E\, \xi^\mu) - K(\psi_+, \xi_\mu, \xi_\nu) K(\sharp_E\, \xi^\mu, \sharp_E\, \xi^\nu, \psi_-) \\ &\quad + K(\xi_\nu, \xi_\mu, \psi_-) K(\sharp_E\, \xi^\nu, \psi_+, \sharp_E\, \xi^\mu)\Big) \\ &= \mathrm{Ric}_{\nabla}(\psi_+, \psi_-). \end{aligned} \tag{5.97}$$

The terms with the covariant derivative vanish thanks to (1.116) and the fact that all "mixed" restrictions of K are identically zero. If we choose an adapted local frame, it is easy to see that also the other terms vanish due to the fact that all of them are "mixed". □

Conclusion

Let us now summarize all the accomplishments and failures of this work. In Chapter 1, we have presented a detailed and self-contained text introducing the subject of Courant algebroid connections with all the proofs included. Particularly noteworthy is the proof of the existence of a Levi-Civita Courant algebroid connection for a general Courant algebroid.

Chapter 2 is focused on the specific example of a Courant algebroid. It is the generalized tangent bundle endowed with the H-twisted Dorfman bracket and some other suitable structure. We have successfully illustrated all the abstract concepts introduced in Chapter **??** on this particular example. In particular, we have included the detailed proof that the set of all generalized metrics on the respective Courant algebroid is possible to parametrize by a pair of Riemannian metric and 2-form on the base manifold. We have also showed that the set of all Levi-Civita Courant algebroid connections with respect to some fixed generalized metric can be fully described by a pair of ordinary tensor fields of the rank 3 on the base manifold, which are in addition skew-symmetric in the last two inputs and their complete skew-symmetrizations vanish identically.

Chapter 3 builds on Chapter 2; specifically, we have reformulated the string effective action and the corresponding equations of motion in terms of Courant algebroid connections on the specific Courant algebroid introduced in Chapter 2. In Chapter 4, we have been concerned with the Palatini approach to the Einstein-Hilbert action from general relativity. The complete and detailed proof of the theorem has been carried out in a pure geometrical way, which is more insightful than the standard way usually presented in physics literature.

The final Chapter 5 has provided even more than it was originally intended. The initial aim has been to devise an analogue of the Palatini approach for the reformulated string effective action. Instead, we have taken a step towards a greater generality and introduced a so called Courant-Einstein-Hilbert action living in the framework of a general Courant algebroid and depending on the three mutually independent inputs; namely, a generalized metric and a Courant algebroid connection on the respective Courant algebroid, and a volume form on the base manifold. This action functional reduces to the reformulated string effective action if we take the specific Courant algebroid from Chapter 2, and moreover, if we a priori restrict ourselves only to Levi-Civita Courant algebroid connections related to a given volume form in some way instead of general Courant algebroid connections.

We have indeed succeeded and proved that a trio of a generalized metric, a Courant algebroid connection and a volume form extremalizes the Courant-Einstein-Hilbert action if and only if

1. The respective G-Ricci scalar vanishes identically.
2. The Courant algebroid connection is Ricci compatible with the generalized metric.
3. The Courant algebroid connection is Levi-Civita with respect to the generalized metric, and furthermore, its divergence is given uniquely by the volume form as a divergence associated with a certain Riemannian metric on the base manifold.

For the specific Courant algebroid from Chapter 2, the first two of these equations of motion reduce to the reformulated equations of motion for the string effective action. The third equation of motion precisely coincides with the a priori assumption on the admissible Courant algebroid connections mentioned in the paragraph above.

Although a lot of questions have been answered in this thesis, there still remain some to be solved. First of all, our Palatini approach is working only if $p \neq 1$ and at the same time $q \neq 1$, where (p, q) denotes the signature of the Courant metric. Therefore, it might be interesting to properly examine the case $p = 1$ or $q = 1$, which is not covered yet. Another interesting thing would be to examine the rank of the affine space of a special class of Levi-Civita Courant algebroid connections that appeared as a result of the Platini approach to the Courant-Einstein-Hilbert action. One could also investigate whether it is possible to obtain the generalized supergravity equations from the formalism presented here.

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