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Non–local approximations of general relativistic delta interactions

Master's thesis

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Abstract: The one–dimensional Dirac operator with point interaction formally given by term $c\mathbb{A}|\delta(x)\rangle\langle\delta(x)|$, where $c\mathbb{A}$ is arbitrary complex 2×2 matrix, is rigorously defined as closed, not necessarily self–adjoint operator. A non–local potential in the form of a projection on a fixed scaled function is used as the approximation of the relativistic point interactions. Moreover, spectral analysis of this newly defined operator is done in the text. Finally, non–relativistic limit of the relativistic point interaction and its non–local approximation is found.

Key words: Dirac operator, local potentials, non–self–adjoint operators, non–local potentials, non–relativistic limit, point interactions

Název práce:

Nelokální aproximace obecných relativistických delta interakcí

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Abstrakt: Jednorozměrný Diracův operátor s bodovou interakcí formálně danou jako $c\mathbb{A}|\delta(x)\rangle\langle\delta(x)|$, kde $c\mathbb{A}$ je libovolná 2×2 komplexní matice, je rigorózně zavedený jako uzavřený, ne nutně samosdružený operátor. Nelokální aproximace ve tvaru projekcí na škálovanou funkci jsou v této práci použity jako aproximace těchto relativistických bodových interakcí. Dále je v textu diskutována otázka spektra operátoru bodové interakce. Nakonec je v práci spočtena nerelativistická limita relativistické bodové interakce a její aproximace.

Klíčová slova: bodové interakce, Diracův operátor, lokální potenciály, nelokální potenciály, nerelativistická limita, nesamosdružené operátory

Contents

List of symbols

$\mathcal{H}$	Hilbert space
$\mathcal{B}(\mathcal{H})$	space of bounded linear operators on $\mathcal{H}$
$\mathcal{L}(\mathcal{H})$	space of densely defined linear operators on $\mathcal{H}$
$\operatorname{Dom} B$	domain of a linear operator B
$\operatorname{Ran} B$	range of a linear operator B
$\operatorname{Ker} B$	kernel of a linear operator B
B^*	adjoint operator to B
$\rho(B)$	resolvent set of an operator B
$\sigma(B)$	spectrum of an operator B
$\sigma_p(B)$	point spectrum of an operator B
$\sigma_{ess}(B)$	essential spectrum of an operator B
$B_\varepsilon \xrightarrow{u} B$	convergence of the bounded operator B_ε to the bounded operator B in the operator norm
$B(x, y)$	integral kernel of an integral operator B
$\|v\rangle\langle v\|$	bra–ket notation for an operator of a projection on element v from $\mathcal{H}$
$\langle\cdot\|\cdot\rangle$	scalar product on $\mathcal{H}$
$\|\| \cdot \|\|_{\mathcal{B}}$	norm of an element of a Banach space $\mathcal{B}$
$\|\| \cdot \|\|_2$	Hilbert–Schmidt norm of an operator
$\langle\cdot\|\cdot\rangle_2$	Hilbert–Schmidt inner product of Hilbert–Schmidt operators
$L^p(U; \mathcal{H})$	Banach space of integrable functions in the pth power on the domain U with values in $\mathcal{H}$
$L^p(U)$	stands for $L^p(U; \mathbb{C})$
L^p	stands for $L^p(\mathbb{R})$
$W^{k,p}(\mathbb{R})$	Sobolev space
$\mathcal{D}(U)$	space of test functions on U
$f * g$	convolution of two functions f and g
$\|\mathbb{B}\|^2 = \|\|\mathbb{B}\|\|_2^2$	Frobenius norm $\sum_{i,j=1}^{n,m} \|\mathbb{B}_{ij}\|^2$ of the matrix $\mathbb{B}$
$\det \mathbb{B}$	determinant of a square matrix $\mathbb{B}$
$\operatorname{tr} \mathbb{B}$	trace of a square matrix $\mathbb{B}$
I	identity matrix
σ_j	Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$\operatorname{Im} z$	imaginary part of a complex number z
$\operatorname{Re} z$	real part of a complex number z
$\|z\|$	absolute value of a complex number z
$\mathbb{C}_\pm$	$= \{z \in \mathbb{C} \mid \operatorname{sgn}(\operatorname{Im} z) = \pm 1\}$
$\mathbb{R}_\pm$	$= \{x \in \mathbb{R} \mid 0 < \pm x\}$
$\mathbb{R}_r$	$= \{(-\infty, -r] \cup [r, +\infty)\}$ for $r \geq 0$
$\hat{n}$	set of integers $\{1, \ldots, n\}$
$\delta(x)$	Dirac delta function
$\operatorname{sgn}(x)$	sign function
$\psi(a+), \psi(a-)$	right–hand and left–hand limit at the point a

1 Introduction

The one–dimensional Dirac operator D_0 acting like

$$D_0\psi(x) = -ic\frac{\mathrm{d}}{\mathrm{d}x} \otimes \sigma_1\psi(x) + mc^2 \otimes \sigma_3\psi(x),$$

$$\psi \in \operatorname{Dom} D_0 = W^{1,2}(\mathbb{R}) \otimes \mathbb{C}^2,\ m \geq 0, c > 0,$$

perturbed at one point is an important exactly solvable model of quantum mechanics. Approximations of this mathematical model were rigorously discussed for the first time by Šeba in [21] where he studied exclusively the electrostatic and Lorentz scalar point interactions. More general definition of the self–adjoint relativistic point interaction D^{Λ} was discussed by Benvegnu and Dabrowski in [3], where D^{Λ} acts like the free Dirac operator D_0 on functions from the Sobolev space $W^{1,2}(\mathbb{R} \setminus \{0\}) \otimes \mathbb{C}^2$ with the transmission condition at the point of interaction

$$\psi(0+) = \Lambda\psi(0-),$$

where

$$\Lambda = \omega\begin{pmatrix} \theta & i\tau \\ -i\kappa & \nu \end{pmatrix}, \tag{1}$$

$$\omega = \mathrm{e}^{i\varphi}, \theta\nu - \tau\kappa = 1, \varphi, \theta, \nu, \tau, \kappa \in \mathbb{R}.$$

Non–self–adjoint models extending quantum mechanics theory have been studied since the beginning of 21st century but there are only few papers discussing non–self–adjoint non–relativistic point interactions, for example [10]. However, as far as I know, the non–self–adjoint case of relativistic point interaction has not been studied yet.

This work is focused on studying not necessary self–adjoint non–local potential in the form of the projection $1/\varepsilon^2\, |v(x/\varepsilon)\rangle\langle v(x/\varepsilon)|$ on a fixed scaled function v in $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ multiplied by a complex matrix $c\mathbb{A} \in \mathbb{C}^{2,2}$

$$D_\varepsilon^{\mathbb{A}} = -ic\frac{\mathrm{d}}{\mathrm{d}x} \otimes \sigma_1 + mc^2 \otimes \sigma_3 + c\frac{1}{\varepsilon^2}|v(x/\varepsilon)\rangle\langle v(x/\varepsilon)| \otimes \mathbb{A}. \tag{2}$$

We already proved [11] that in the self–adjoint case the norm resolvent limit of (2) corresponds to the relativistic point interaction discussed in [3]. We will demonstrate that also for non–self–adjoint matrices $\mathbb{A}$ the norm–resolvent limit exists and we will call the limit the non–self–adjoint relativistic point interaction.

Idea of using non–local potential to study approximations of a relativistic point interaction comes from the paper [21], where two matrices $\mathbb{A}$ are studied explicitly. We will generalize this result to any complex matrix $\mathbb{A}$.

In [21] Šeba also discussed comparison of the formal limit and the operator limit. He showed that by starting with the Dirac operator with local potential $1/\varepsilon\, h(x/\varepsilon)$ the formal limit will not correspond to the proper operator limit. This phenomena is called renormalization of the coupling constant. More general setting of the local potential and its limit was already discussed by Hughes [12], [13] and by Tušek [24]. Šeba also proved that for two special choices of self–adjoint matrix $\mathbb{A}$, using a non–local potential will not lead to the renormalization. We already showed that this property is preserved in the most general self–adjoint case of the non–local potential [11]. We will prove that this property is also preserved in the non–self–adjoint setting. Moreover, this approach extends naturally the definition of relativistic point interactions also to the non–self–adjoint case.

Furthermore, we will study properties of this newly defined operator of general relativistic point interactions. Moreover, an implicit equation for eigenvalues of the operator will be derived. We will discuss remarkable spectral transition in the non–self–adjoint case where for special choice of elements of the matrix $\mathbb{A}$ whole complex plane or half–plane will lie in the point spectrum of the operator. We will also find an implicit equation for eigenvalues and eigenfunctions of non–local approximations of the relativistic point interactions. Then the stability of the spectrum of our model with respect to its non–local approximations will be discussed.

In this thesis, we will also derive the physically interesting non–relativistic limit. The process of non–relativistic limit was thoroughly discussed for example in [9]. Firstly, the non–relativistic limit of the newly defined model of the non–self–adjoint relativistic point interaction will be find and then the result will be compared to the non–relativistic model of a general non–self–adjoint point interaction. The latter operator was already studied by many articles. For the reader's convenience, we will try to summarize key properties of such operator.

Since we studied approximations of the relativistic model, we also decided to take the non–relativistic limit for the approximations. By doing this, we will get the corresponding approximations of the non–relativistic model of point interactions. For the non–local approximations we will prove the norm–resolvent convergence to the non–relativistic point interactions. We infer that one can interchange the order of the non–relativistic limit ($c \to +\infty$) and the limit for approximations ($\varepsilon \to 0$). Symbolically,

$$\text{"} \lim_{c\to+\infty} \lim_{\varepsilon\to 0} (D_\varepsilon^{\mathbb{A}} - mc^2) = \lim_{\varepsilon\to 0} \lim_{c\to+\infty} (D_\varepsilon^{\mathbb{A}} - mc^2).\text{"}$$

2 Relativistic point interactions

In the text, we will identify space $L^2(U) \otimes \mathbb{C}^2$ with the space $L^2(U;\mathbb{C}^2)$. Also, slightly abusing the notation, we will denote

$$\langle f|\psi\rangle = \begin{pmatrix} \langle f|\psi_1\rangle \\ \langle f|\psi_2\rangle \end{pmatrix}, \langle f|\mathbb{B}\rangle = \begin{pmatrix} \langle f|\mathbb{B}_{11}\rangle & \langle f|\mathbb{B}_{12}\rangle \\ \langle f|\mathbb{B}_{21}\rangle & \langle f|\mathbb{B}_{22}\rangle \end{pmatrix},$$

for $f \in L^2(\mathbb{R})$, $\psi \in L^2(\mathbb{R};\mathbb{C}^2)$ and $\mathbb{B} \in L^2(\mathbb{R};\mathbb{C}^{2,2})$. Without loss of generality, we choose the Hilbert–Schmidt norm for a complex matrix $\mathbb{A}$

$$|\mathbb{A}|^2 = \|\mathbb{A}\|_2^2 = \sum_{i=1}^{n} \|\vec{a}_i\|_2^2,$$

where $\vec{a}_i$ are columns of the matrix $\mathbb{A}$. Note that the Hilbert–Schmidt norm of matrices is submultiplicative which can be proved using the Cauchy–Schwarz inequality

$$|\mathbb{AB}|^2 = \sum_{i=1}^{n}\sum_{j=1}^{k}\langle \vec{a}_i|\vec{b}_j\rangle^2 \le \sum_{i=1}^{n}\sum_{j=1}^{k} \|\vec{a}_i\|_2^2\|\vec{b}_j\|_2^2 = |\mathbb{A}|^2|\mathbb{B}|^2.$$

For a future convenience we will define the formal differential expression

$$\mathcal{D} := -ic\frac{\mathrm{d}}{\mathrm{d}x} \otimes \sigma_1 + mc^2 \otimes \sigma_3.$$

Now, let us start with the definition of our non–local potential as a projection on a scaled vector v from $L^2(\mathbb{R};\mathbb{R}) \cap L^1(\mathbb{R};\mathbb{R})$ multiplied by a 2×2 complex matrix. Using the bra–ket notation we can write the Dirac operator with the non–local potential in the following way

$$\begin{aligned} D_\varepsilon^{\mathbb{A}} &= D_0 + cW_\varepsilon \otimes \mathbb{A}, \\ W_\varepsilon &= \frac{1}{\varepsilon^2}|v(x/\varepsilon)\rangle\langle v(x/\varepsilon)| =: |v_\varepsilon\rangle\langle v_\varepsilon|, \end{aligned} \tag{3}$$

where $v_\varepsilon := \varepsilon^{-1}v(\varepsilon^{-1}x)$ and D_0 is the free Dirac operator defined as

$$\begin{aligned} (D_0\psi)(x) &= (\mathcal{D}\psi)(x), \forall x \in \mathbb{R} \\ \mathrm{Dom}(D_0) &= W^{1,2}(\mathbb{R}) \otimes \mathbb{C}^2. \end{aligned} \tag{4}$$

Here $W^{1,2}$ stands for the Sobolev space, m is a non–negative constant, c stands for the speed of light and σ_i are the Pauli matrices. The spectrum of the free Dirac operator contains

$$\sigma(D_0) = \sigma_{ess}(D_0) = \{(-\infty, -mc^2] \cup [mc^2, +\infty)\} =: \mathbb{R}_{mc^2}.$$

Its resolvent is the integral operator with the integral kernel given by

$$R_z(x,y) = \frac{i}{2c}(\mathbb{Z}(z) + \mathrm{sgn}(x-y)\sigma_1)\mathrm{e}^{ik(z)|x-y|}, \tag{5}$$

where

$$\mathbb{Z}(z) = \begin{pmatrix} \zeta(z) & 0 \\ 0 & \zeta(z)^{-1} \end{pmatrix},$$

$$\zeta(z) = \frac{z + mc^2}{ck(z)} \quad \text{and} \quad ck(z) = \sqrt{z^2 - (mc^2)^2}, \quad \mathrm{Im}k(z) \geq 0.$$

We already found out [11] that for a self–adjoint matrix $\mathbb{A} = \mathbb{A}^*$ the Dirac operator with the non–local potential converges in the norm–resolvent sense to the operator of the self–adjoint relativistic point interaction $D^{\mathbb{A}}$ [3] acting like

$$\begin{gathered}(D^{\mathbb{A}}\psi)(x) = (\mathcal{D}\psi)(x), \ \forall x \in \mathbb{R} \setminus \{0\}, \\ \psi \in \mathrm{Dom}(D^{\mathbb{A}}) = \{\varphi \in W^{1,2}(\mathbb{R} \setminus \{0\}) \otimes \mathbb{C}^2 \mid (2i - \sigma_1\mathbb{A})\varphi(0+) = (2i + \sigma_1\mathbb{A})\varphi(0-)\}.\end{gathered} \tag{6}$$

Note that if we take the Dirac operator with any scaled potential V_ε which goes to the delta potential

$$D_0 + cV_\varepsilon \otimes \mathbb{A}, \tag{7}$$

one can find a formal limit of this operator as $\mathcal{D}+c\mathbb{A}\delta(x)$. In the following subsection we will discuss how this formal expression of the limit can be rigorously defined as the Dirac operator with the transmission condition

$$(2i + \sigma_1\mathbb{A})\psi(0-) = (2i - \sigma_1\mathbb{A})\psi(0+).$$

Because of this we can see that the formal limit of (7) corresponds to the operator (6).

This formal limit seems to be a good candidate for the operator limit. However, it is now well known [12, 21, 24] that for the special case of the potential V_ε which is the local potential

$$V_\varepsilon = \frac{1}{\varepsilon} h\left(\frac{x}{\varepsilon}\right),$$

the operator limit does not correspond to the formal limit. This phenomena is known as the renormalization of the coupling constant.

We have already shown in the self–adjoint case [11], as one can see above, that for the non–local potential (3) the renormalization of the coupling constant does not occur. In other words, the formal limit for the non–local potential is the same as the norm–resolvent limit of the operator.

We can extend these results also to the non–self–adjoint case of the matrix $\mathbb{A}$. The convergence of the resolvent of the operator $D_\varepsilon^{\mathbb{A}}$ introduced in (3) to a bounded operator will be shown. We will deduce that the limit is, in fact, the resolvent of an unbounded operator also acting like D_0 away from $x = 0$ with certain boundary condition at the point of interaction and we will call this operator the non–self–adjoint relativistic point interaction.

2.1 General relativistic point interactions

To justify following definition we will rewrite the formal limit of the operator $D_\varepsilon^{\mathbb{A}}$ with any complex matrix $\mathbb{A}$ as the Dirac operator with a transmission condition at the point of interaction similarly as described for the self–adjoint case in the beginning of this section. We can easily see that the formal limit of $D_\varepsilon^{\mathbb{A}}$ can be formally written as $\mathcal{D} + c\mathbb{A}|\delta(x)\rangle\langle\delta(x)|$. Firstly, we need to extend the definition of $|\delta(x)\rangle\langle\delta(x)|$ for not necessarily smooth functions as

$$\langle\delta(x)|\psi(x)\rangle := \frac{\psi(0+) + \psi(0-)}{2}.$$

This is only possible extension of the Dirac delta function which preserves properties of $\delta(x)$, mainly $\delta(x) = \delta(-x)$, cf. [16]. Also note that that the formal expression $|\delta(x)\rangle\langle\delta(x)|$ acts on a function ψ in the exactly same manner as the multiplication operator by $\delta(x)$ since for $f \in \mathcal{D}$ we have

$$(\delta(x)\psi(x), f(x)) = \psi(0)(\delta(x), f(x)) = \langle\psi(x)|\delta(x)\rangle(\delta(x), f(x)).$$

Now we will find maximal domain of the formal expression $\mathcal{D} + c\mathbb{A}|\delta(x)\rangle\langle\delta(x)|$ which, in particular, means that we require

$$\mathcal{D}\psi(x) + c\mathbb{A}\langle\delta(x)|\psi(x)\rangle\delta(x) \in L^2(\mathbb{R}; \mathbb{C}^2).$$

In other words, we need singular parts to be cancelled out, see [11], which yields the following condition

$$-ic\sigma_1(\psi(0+) - \psi(0-)) + c\mathbb{A}\frac{\psi(0+) + \psi(0-)}{2} = 0.$$

This is equivalent to

$$(2i + \sigma_1\mathbb{A})\psi(0-) = (2i - \sigma_1\mathbb{A})\psi(0+).$$

Therefore, our formal limit corresponds to the operator $D^{\mathbb{A}}$ acting as the Dirac operator with the transmission condition at the point of interaction.

$$\begin{gathered}(D^{\mathbb{A}}\psi)(x) = (\mathcal{D}\psi)(x), \ x \in \mathbb{R} \setminus \{0\}, \\ \psi \in \operatorname{Dom} D^{\mathbb{A}} = \{\psi \in W^{1,2}(\mathbb{R} \setminus \{0\}) \otimes \mathbb{C}^2 \mid (2i + \sigma_1\mathbb{A})\psi(0-) = (2i - \sigma_1\mathbb{A})\psi(0+)\}.\end{gathered} \tag{8}$$

Definition 2.1.1. *Let $\mathbb{A}$ be any 2×2 complex matrix. Then we will call the operator $D^{\mathbb{A}}$ given by (8) the operator of the relativistic point interaction.*

Note that, for every matrix $\mathbb{A}$, operator $D^{\mathbb{A}}$ is, in fact, an extension of the symmetric operator

$$\begin{gathered}(D_{\min}\psi)(x) = (\mathcal{D}\psi)(x), x \in \mathbb{R}, \\ \operatorname{Dom} D_{\min} = \{\psi \in W^{1,2}(\mathbb{R}; \mathbb{C}^2)|\psi(0) = 0\}.\end{gathered}$$

Also, $D^*_{\min} = D_{\max}$, where

$$\begin{gathered}(D_{\max}\psi)(x) = (\mathcal{D}\psi)(x), x \in \mathbb{R} \setminus \{0\}, \\ \operatorname{Dom} D_{\max} = \{\psi \in W^{1,2}(\mathbb{R} \setminus \{0\}; \mathbb{C}^2)\}.\end{gathered}$$

Therefore, the operator $D^{\mathbb{A}}$ is really a restriction of the operator $D_{\max}$. The deficiency indices of $D_{\min}$ are $(2, 2)$ [3, 17]. From that one can conclude existence of a four–real–parametric family of self–adjoint extensions of the operator $D_{\min}$ that have been studied before [3, 17].

Note that if $\mathbb{A}$ is self–adjoint and matrices $(2i + \sigma_1\mathbb{A})$ and $(2i - \sigma_1\mathbb{A})$ are regular then Definition 2.1.1 coincides with the definition of the relativistic point interaction introduced in [3] as the self–adjoint extension of the Dirac operator perturbed at the point of interaction. In fact, we can calculate the corresponding transmission condition considered in [3],

$$\psi(0+) = \Lambda\psi(0-),$$

where Λ is of the form (1). One can see that

$$\Lambda = (2i - \sigma_1\mathbb{A})^{-1}(2i + \sigma_1\mathbb{A}).$$

Inverting this relation we will get

$$\mathbb{A} = 2i\sigma_1(\Lambda - I)(\Lambda + I)^{-1}.$$

It is clear that for a matrix Λ such that

$$0 = \det(\Lambda + I) = 1 - \omega^2 + \omega(\theta + \nu),$$

our set of operators $D^{\mathbb{A}}$ does not include the corresponding operator. Vice versa, the family of self–adjoint relativistic point interactions considered in [3] does not include all self–adjoint realizations of the operator $D^{\mathbb{A}}$ either, since, as we will see in the following text, the operator $D^{\mathbb{A}}$ is self–adjoint if and only if the matrix $\mathbb{A}$ is self–adjoint.

Of course, we would like to describe also non–self–adjoint realizations of the operator $D^{\mathbb{A}}$. We can try to find out if $D^{\mathbb{A}}$ is at least a closed operator in the most general case. For this recall the trace theorem.

Theorem 2.1.1 (Trace theorem)**.** *Let U be bounded open subset of $\mathbb{R}^n$ with C^1–boundary and $p \in [1, +\infty)$. Then there exists operator $\operatorname{Tr} \in \mathcal{B}(W^{1,p}(U), L^p(\partial U))$ such that $\forall \psi \in W^{1,p}(U) \cap C(\overline{U})$, $\operatorname{Tr}\psi = \psi|_{\partial U}$.*

Proof. For a general case one can found the proof for example in [8]. For the one dimensional setting we even have

$$\sup_{x\in\mathbb{R}} |f(x)| \leq \|f\|_{W^{1,2}},$$

since $\forall f \in \mathcal{D}(\mathbb{R})$,

$$f(x)^2 = \int_{-\infty}^{x} (f(y)^2)' \,\mathrm{d}y = 2\int_{-\infty}^{x} f(y)f'(y)\,\mathrm{d}y \leq 2\|f\|_{L^2}\|f'\|_{L^2} \leq \|f\|_{L^2}^2 + \|f'\|_{L^2}^2.$$

The statement then follows from the fact that $\overline{\mathcal{D}(\mathbb{R})}^{W^{1,2}} = W^{1,2}(\mathbb{R})$. ◻

Theorem 2.1.2. *Let matrix $\mathbb{A}$ be any complex matrix. Then the operator of relativistic point interaction $D^{\mathbb{A}}$ given by* (8) *is densely defined closed operator.*

Proof. We can see that $D^{\mathbb{A}}$ is densely defined operator in $L^2(\mathbb{R};\mathbb{C}^2)$ because $W^{1,2}(\mathbb{R}\setminus\{0\})$ is a dense subset of $L^2(\mathbb{R})$.

If we decompose a function from $\operatorname{Dom} D^{\mathbb{A}}$ into a sum of functions on a positive and negative half–line of $\mathbb{R}$ respectively, and use the Trace theorem 2.1.1 the domain of $D^{\mathbb{A}}$ can be written as follows

$$\operatorname{Dom} D^{\mathbb{A}} = \{\varphi = \varphi_- \oplus \varphi_+ \in W^{1,2}(\mathbb{R}^-;\mathbb{C}^2) \oplus W^{1,2}(\mathbb{R}^+;\mathbb{C}^2) \mid (2i - \sigma_1\mathbb{A}) \operatorname{Tr}\varphi_+ = (2i + \sigma_1\mathbb{A}) \operatorname{Tr}\varphi_-\}. \tag{9}$$

By the Trace theorem, there exists a bounded linear operator $\operatorname{Tr} \in \mathcal{B}(W^{1,2}(\mathbb{R}^\pm;\mathbb{C}^2),\mathbb{C}^2)$ and a certain constant $C \geq 0$ such that

$$|\operatorname{Tr}\varphi_\pm| \leq C\|\varphi_\pm\|_{W^{1,2}(\mathbb{R}^\pm;\mathbb{C}^2)}. \tag{10}$$

To prove that $D^{\mathbb{A}}$ is closed we choose a convergent sequence from $\operatorname{Dom} D^{\mathbb{A}}$ with a convergent sequence of its images.

$$\begin{aligned} \varphi_n &= \varphi_{n,-} \oplus \varphi_{n,+} \in \operatorname{Dom} D^{\mathbb{A}} \\ \varphi_n &\to \varphi = \varphi_- \oplus \varphi_+ \in L^2(\mathbb{R};\mathbb{C}^2) \\ D^{\mathbb{A}}\varphi_n &\to \psi = \psi_- \oplus \psi_+ \in L^2(\mathbb{R};\mathbb{C}^2) \end{aligned} \tag{11}$$

Now we need to prove that $\varphi \in \operatorname{Dom} D^{\mathbb{A}}$ and $\psi = D^{\mathbb{A}}\varphi$. Firstly, note that

$$D^{\mathbb{A}}\varphi_n = (-ic\sigma_1\varphi_{n,-}' + mc^2\sigma_3\varphi_{n,-}) \oplus (-ic\sigma_1\varphi_{n,+}' + mc^2\sigma_3\varphi_{n,+}).$$

Then (11) gives us convergence of functions $\varphi_{n,\pm}$ in $W^{1,2}(\mathbb{R}^{\pm};\mathbb{C}^2)$. Spaces $W^{1,2}(\mathbb{R}^{\pm};\mathbb{C}^2)$ are complete spaces, that implies

$$\varphi_{\pm} \in W^{1,2}(\mathbb{R}^{\pm};\mathbb{C}^2) \text{ and}$$

$$-ic\sigma_1\varphi'_{\pm} + mc^2\sigma_3\varphi_{\pm} = \psi_{\pm}.$$

Now we need to determine if $\varphi_{\pm}$ fulfils the transmission condition of $\operatorname{Dom} D^{\mathbb{A}}$ (9). Since functions $\varphi_{\pm}$ converge in $W^{1,2}(\mathbb{R}^{\pm};\mathbb{C}^2)$, (10) gives us convergence of their traces in $\mathbb{C}^2$ space. Because $\varphi_n \in \operatorname{Dom} D^{\mathbb{A}}$ we get

$$\forall n \in \mathbb{N}, (2i - \sigma_1\mathbb{A}) \operatorname{Tr}\varphi_{n,+} = (2i + \sigma_1\mathbb{A}) \operatorname{Tr}\varphi_{n,-}.$$

Letting $n \to +\infty$ we obtain

$$(2i - \sigma_1\mathbb{A}) \operatorname{Tr}\varphi_{+} = (2i + \sigma_1\mathbb{A}) \operatorname{Tr}\varphi_{-}.$$

We conclude that $\varphi \in \operatorname{Dom} D^{\mathbb{A}}$ and $D^{\mathbb{A}}\varphi = \psi$. This means that the operator $D^{\mathbb{A}}$ is a densely defined closed operator. ◻

Another natural question would be the uniqueness of the definition of the operator $D^{\mathbb{A}}$. More precisely, if different choices of $\mathbb{A}$ yield different operators $D^{\mathbb{A}}$.

Proposition 2.1.1. *$D^{\mathbb{A}} = D^{\mathbb{B}}$ if and only if $\mathbb{A} = \mathbb{B}$.*

Proof. An implication from right to left is easy to see. We will prove the inverse implication. Let us have $D^{\mathbb{A}} = D^{\mathbb{B}}$ which means

$$\operatorname{Dom} D^{\mathbb{A}} = \operatorname{Dom} D^{\mathbb{B}}.$$

From that we deduce $\forall\psi \in \operatorname{Dom} D^{\mathbb{A}} = \operatorname{Dom} D^{\mathbb{B}}$ both of the following conditions hold.

$$\begin{aligned}(2i\sigma_1 - \mathbb{A})\psi(0+) &= (2i\sigma_1 + \mathbb{A})\psi(0-),\\ (2i\sigma_1 - \mathbb{B})\psi(0+) &= (2i\sigma_1 + \mathbb{B})\psi(0-).\end{aligned}$$

If we subtract these conditions, we get

$$(\mathbb{A} - \mathbb{B})(\psi(0+) + \psi(0-)) = 0.$$

If $(\psi(0+) + \psi(0-))$ can be any vector $\vec{c} \in \mathbb{C}^2$, we prove the proposition. Let us then start with arbitrary vector $\vec{c}$. For such vector we can find a function $\psi \in W^{1,2}(\mathbb{R} \setminus \{0\})$ such that

$$\begin{aligned}\psi(0+) &= \frac{1}{4}(2 - i\sigma_1\mathbb{A})\vec{c},\\ \psi(0-) &= \frac{1}{4}(2 + i\sigma_1\mathbb{A})\vec{c}.\end{aligned}$$

One can easily see that the function defined in this way meets the transmission condition of the operator $D^{\mathbb{A}}$, which means $\psi \in \operatorname{Dom} D^{\mathbb{A}} = \operatorname{Dom} D^{\mathbb{B}}$. Also

$$\psi(0+) + \psi(0-) = \vec{c}.$$

Since vector $\vec{c}$ was arbitrary, we have

$$\forall\vec{c} \in \mathbb{C}^2,\ (\mathbb{A} - \mathbb{B})\vec{c} = 0.$$

This implies $\mathbb{A} = \mathbb{B}$. ◻

Theorem 2.1.3. *Let $\mathbb{A}$ be a complex matrix such that $\mathbb{A} = \pm 2i\sigma_1$ or none of the matrices $(2i - \sigma_1\mathbb{A})$ and $(2i + \sigma_1\mathbb{A})$ is invertible. Then operator $D^{\mathbb{A}}$ decouples into a direct sum of operators $D^{\mathbb{A}} = D^{\mathbb{A}}_+ \oplus D^{\mathbb{A}}_-$ on $L^2(\mathbb{R}^+; \mathbb{C}^2) \oplus L^2(\mathbb{R}^-; \mathbb{C}^2)$, where*

$$(D^{\mathbb{A}}_{\pm}\psi)(x) = (\mathcal{D}\psi)(x), x \in \mathbb{R}^{\pm},$$

$$\operatorname{Dom} D^{\mathbb{A}}_{\pm} = \{\psi \in W^{1,2}(\mathbb{R}^{\pm}; \mathbb{C}^2) | (2i \mp \sigma_1\mathbb{A})\psi(0\pm) = 0\}.$$

Proof. If $\mathbb{A} = 2i\sigma_1$ then the transmission condition yields $4i\sigma_1\psi(0-) = 0$, equivalently $\psi(0-) = 0$, and no restriction on $\psi(0+)$. For $\mathbb{A} = -2i\sigma_1$, the roles of $\psi(0-)$ and $\psi(0+)$ interchange. Both cases gave us the stated result.

If $(2i - \sigma_1\mathbb{A})$ or $(2i + \sigma_1\mathbb{A})$ is invertible then the transmission condition can be rewritten as

$$\psi(0+) = \Lambda\psi(0-) \quad \text{or} \quad \psi(0-) = \tilde{\Lambda}\psi(0+).$$

Ranges of matrices Λ and $\tilde{\Lambda}$ are at least one–dimensional. We conclude that in these cases the operator $D^{\mathbb{A}}$ does not decouple.

If none of the matrices $(2i - \sigma_1\mathbb{A})$ and $(2i + \sigma_1\mathbb{A})$ are invertible then writing

$$\mathbb{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

we get

$$0 = \det(2i \pm \sigma_1\mathbb{A}) = -4 - \det\mathbb{A} \pm 2i(\beta + \gamma),$$

which yields

$$\beta + \gamma = 0, \quad \det\mathbb{A} = -4.$$

The matrix satisfying these conditions is of the form

$\alpha \neq 0$:

$$\mathbb{A} = \begin{pmatrix} \alpha & \beta \\ -\beta & -\frac{4+\beta^2}{\alpha} \end{pmatrix}.$$

$\delta \neq 0$:

$$\mathbb{A} = \begin{pmatrix} -\frac{4+\beta^2}{\delta} & \beta \\ -\beta & \delta \end{pmatrix}.$$

$\alpha = \delta = 0$:

$$\mathbb{A} = \begin{pmatrix} 0 & \pm 2i \\ \mp 2i & 0 \end{pmatrix}.$$

In all the cases described matrix $\mathbb{A}$ is satisfying

$$(\sigma_1\mathbb{A})^2 = -4I. \tag{12}$$

Now, multiplying the transmission condition

$$(2i - \sigma_1\mathbb{A})\psi(0+) = (2i + \sigma_1\mathbb{A})\psi(0-)$$

by $(2i \pm \sigma_1\mathbb{A})$ respectively and considering (12) we obtain

$$(-8 \pm 4i\sigma_1\mathbb{A})\psi(\mp) = 0,$$

which is equivalent to

$$(2i\sigma_1 \pm \mathbb{A})\psi(\mp) = 0.$$

On other hand, functions satisfying this condition clearly obey the transmission condition. Finally, we see that the operator decouples into the searched form. □

Theorem 2.1.4. *Let $\mathbb{A}$ be any complex matrix from $\mathbb{C}^{2,2}$ then*

$$(D^{\mathbb{A}})^* = D^{\mathbb{A}^*}.$$

Proof. For an arbitrary complex matrix $\mathbb{A}$ the following holds

$$D_{\min} \subset D^{\mathbb{A}} \subset D_{\max} = D^*_{\min},$$

which implies

$$\overline{D_{\min}} \subset (D^{\mathbb{A}})^* \subset D_{\max}.$$

From this we conclude that $(D^{\mathbb{A}})^*$ is a restriction of an operator $D_{\max}$. Using integration by parts one can deduce that $\forall \varphi \in \operatorname{Dom} D^{\mathbb{A}}$, $\forall \psi \in \operatorname{Dom} D_{\max}$

$$\langle D^{\mathbb{A}}\varphi|\psi\rangle = \int_{\mathbb{R}} \langle -ic\sigma_1\varphi'|\psi\rangle_{\mathbb{C}^2}(x) + \langle mc^2\sigma_3\varphi|\psi\rangle_{\mathbb{C}^2}(x)\,\mathrm{d}x = \int_{\mathbb{R}} \langle \varphi'|ic\sigma_1\psi\rangle_{\mathbb{C}^2}(x) + \langle \varphi|mc^2\sigma_3\psi\rangle_{\mathbb{C}^2}(x)\,\mathrm{d}x =$$

$$= [\langle \varphi|ic\sigma_1\psi\rangle_{\mathbb{C}^2}(x)]_0^{+\infty} + [\langle \varphi|ic\sigma_1\psi\rangle_{\mathbb{C}^2}(x)]_{-\infty}^0 + \int_{\mathbb{R}} \langle \varphi| - ic\sigma_1\psi'\rangle_{\mathbb{C}^2}(x) + \langle \varphi|mc^2\sigma_3\psi\rangle(x)_{\mathbb{C}^2}\,\mathrm{d}x =$$

$$= \langle \varphi(0-)|ic\sigma_1\psi(0-)\rangle_{\mathbb{C}^2} - \langle \varphi(0+)|ic\sigma_1\psi(0+)\rangle_{\mathbb{C}^2} + \langle \varphi|D_{\max}\psi\rangle.$$

This yields, $\psi \in \operatorname{Dom}(D^{\mathbb{A}})^*$ if and only if

$$\forall \varphi \in \operatorname{Dom} D^{\mathbb{A}}, \langle \varphi(0+)|\sigma_1\psi(0+)\rangle_{\mathbb{C}^2} - \langle \varphi(0-)|\sigma_1\psi(0-)\rangle_{\mathbb{C}^2} = 0. \tag{13}$$

We will now distinguish three cases.

1. $(2i - \sigma_1\mathbb{A})$ is invertible. Then transmission condition for the operator $D^{\mathbb{A}}$ can be rewritten as

   $$\forall \varphi \in \operatorname{Dom} D^{\mathbb{A}},\ \varphi(0+) = \Lambda\varphi(0-),$$

   where $\Lambda = (2i - \sigma_1\mathbb{A})^{-1}(2i + \sigma_1\mathbb{A})$. Then (13) will give us

   $$\langle \varphi(0-)|\Lambda^*\sigma_1\psi(0+) - \sigma_1\psi(0-)\rangle_{\mathbb{C}^2} = 0,$$

   from which we conclude

   $$\sigma_1\psi(0-) = \Lambda^*\sigma_1\psi(0+).$$

   Now, we will equivalently rewrite the obtained condition.

   $$\sigma_1\psi(0-) = (-2i + \mathbb{A}^*\sigma_1)(-2i - \mathbb{A}^*\sigma_1)^{-1}\sigma_1\psi(0+),$$
   $$\sigma_1\psi(0-) = 4i(2i + \mathbb{A}^*\sigma_1)^{-1}\sigma_1\psi(0+) - \sigma_1\psi(0+),$$
   $$\psi(0-) = 4i(2i + \sigma_1\mathbb{A}^*)^{-1}\psi(0+) - \psi(0+),$$
   $$(2i + \sigma_1\mathbb{A}^*)\psi(0-) = 4i\psi(0+) - (2i + \sigma_1\mathbb{A}^*)\psi(0+),$$
   $$(2i + \sigma_1\mathbb{A}^*)\psi(0-) = (2i - \sigma_1\mathbb{A}^*)\psi(0+).$$

   Finally, one can see that $(D^{\mathbb{A}})^* = D^{\mathbb{A}^*}$.

2. $(2i + \sigma_1\mathbb{A})$ is invertible. Then transmission condition for the operator $D^{\mathbb{A}}$ can be rewritten as

$$\forall\varphi \in \operatorname{Dom} D^{\mathbb{A}},\ \tilde{\Lambda}\varphi(0+) = \varphi(0-),$$

where $\tilde{\Lambda} = (2i + \sigma_1\mathbb{A})^{-1}(2i - \sigma_1\mathbb{A})$. After a similar calculation as in the case 1., we will get the same result $(D^{\mathbb{A}})^* = D^{\mathbb{A}^*}$.

3. None of the operators $(2i \pm \sigma_1\mathbb{A})$ is invertible. This case is described in Theorem 2.1.3 with the transmission condition

$$\varphi(0\pm) \in \operatorname{Ker}(2i \mp \sigma_1\mathbb{A}) = \operatorname{Ker}(2i\sigma_1 \mp \mathbb{A}). \tag{14}$$

The operator $D^{\mathbb{A}}$ decouples and (13) gives us

$$\langle\varphi(0\pm)|\sigma_1\psi(0\pm)\rangle_{\mathbb{C}^2} = 0.$$

Using (14) we will get the condition on ψ being in $\operatorname{Dom}(D^{\mathbb{A}})^*$,

$$\sigma_1\psi(0\pm) \in (\operatorname{Ker}(2i\sigma_1 \mp \mathbb{A}))^{\perp} = \operatorname{Ran}(-2i\sigma_1 \mp \mathbb{A}^*),$$
$$\psi(0\pm) \in \operatorname{Ran}(-2i \mp \sigma_1\mathbb{A}^*). \tag{15}$$

Recall that in the decoupled case, the matrix $\mathbb{A}$ satisfies (12),i.e.,

$$\mathbb{A}\sigma_1\mathbb{A} = -4\sigma_1.$$

Adjoining both sides of the equation one will get

$$\mathbb{A}^*\sigma_1\mathbb{A}^* = -4\sigma_1,$$
$$(\sigma_1\mathbb{A}^*)^2 = -4I.$$

Using these identities, we will firstly prove $\operatorname{Ran}(-2i \mp \sigma_1\mathbb{A}^*) \subseteq \operatorname{Ker}(2i \mp \sigma_1\mathbb{A}^*)$. The condition $\psi(0\pm) \in \operatorname{Ran}(-2i \mp \sigma_1\mathbb{A}^*)$ yields

$$\exists\vec{C}_\pm \in \mathbb{C}^2, \psi(0\pm) = (-2i \mp \sigma_1\mathbb{A}^*)\vec{C}_\pm,$$
$$(-2i \pm \sigma_1\mathbb{A}^*)\psi(0\pm) = (-2i \pm \sigma_1\mathbb{A}^*)(-2i \mp \sigma_1\mathbb{A}^*)\vec{C}_\pm,$$
$$(2i \mp \sigma_1\mathbb{A}^*)\psi(0\pm) = 0.$$

Now, we need to prove that $\operatorname{Ran}(-2i \mp \sigma_1\mathbb{A}^*) \supseteq \operatorname{Ker}(2i \mp \sigma_1\mathbb{A}^*)$. For $\psi(0\pm) \in \operatorname{Ker}(2i \mp \sigma_1\mathbb{A}^*)$ we have

$$(2i \mp \sigma_1\mathbb{A}^*)\psi(0\pm) = 0,$$
$$(-2i \mp \sigma_1\mathbb{A}^*)\psi(0\pm) = -4i\psi(0\pm),$$
$$\psi(0\pm) = (-2i \mp \sigma_1\mathbb{A}^*)\frac{i}{4}\psi(0\pm).$$

□

From Theorem 2.1.4, we finally get the promised result.

Corollary 2.1.1. *The operator $D^{\mathbb{A}}$ is self–adjoint if and only if $\mathbb{A} = \mathbb{A}^*$. If $D^{\mathbb{A}}$ is normal then the operator $D^{\mathbb{A}}$ is self–adjoint.*

Proof. First part of the corollary comes from the previous Theorem 2.1.4 and from Proposition 2.1.1. The second part of the corollary follows from the fact that the operators $D^{\mathbb{A}}$ and $(D^{\mathbb{A}})^*$ are both restrictions of the operator $D_{\max}$, which means that they act like the free Dirac operator, and from the relation $\operatorname{Dom} D^{\mathbb{A}} = \operatorname{Dom}(D^{\mathbb{A}})^*$, that holds true for every normal operator. □

2.2 Non–local approximations of relativistic point interactions

Let us firstly state one of the main results of this thesis which is the existence of the norm–resolvent limit of the Dirac operator with scaled non–local potential multiplied by any complex matrix $\mathbb{A}$. We will denote elements of the matrix $\mathbb{A}$ as

$$\mathbb{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{16}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. Also without loss of generality we will assume that function v in (3) meets

$$\int_{\mathbb{R}} v(x) \, \mathrm{d}x = 1.$$

Theorem 2.2.1. *Let the matrix $\mathbb{A}$ in the definition of the Dirac operator with the non–local potential (3) be any complex matrix and $z \in \mathbb{C} \setminus \mathbb{R}_{mc^2}$ such that the matrix*

$$(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))$$

is invertible. Then the resolvent of the non–local potential converges in the operator norm to the bounded integral operator

$$R_z^{\mathbb{A}}(x,y) = R_z(x,y) - cR_z(x,0)(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}R_z(0,y), \tag{17}$$

as $\varepsilon \to 0$.

First, we recall the Minkowski integral inequality which plays the main role in the proof of the theorem above.

Proposition 2.2.1 (Minkowski integral inequality). *Let (U_1, μ_1) and (U_2, μ_2) be σ–finite measure spaces and $g : U_1 \times U_2 \to \mathbb{R}$ is a measurable, non–negative function. Let $p \geq 1$ then*

$$\left(\int_{U_2} \left| \int_{U_1} g(x,y) \, \mathrm{d}\mu_1(x) \right|^p \mathrm{d}\mu_2(y) \right)^{\frac{1}{p}} \leq \int_{U_1} \left(\int_{U_2} |g(x,y)|^p \, \mathrm{d}\mu_2(y) \right)^{\frac{1}{p}} \mathrm{d}\mu_1(x). \tag{18}$$

Proof. Denote $F(y) = \int_{U_1} g(x,y) \, \mathrm{d}\mu_1(x)$ then using the Fubini theorem and the Hölder inequality we obtain

$$\|F\|^p_{L^p(U_2, \mathrm{d}\mu_2)} = \int_{U_2} \left| \int_{U_1} g(x,y) \, \mathrm{d}\mu_1(x) \right| |F^{p-1}(y)| \, \mathrm{d}\mu_2(y) \leq \int_{U_1} \int_{U_2} |g(x,y)||F^{p-1}(y)| \, \mathrm{d}\mu_2(y) \, \mathrm{d}\mu_1(x) \leq$$

$$\leq \int_{U_1} \left(\int_{U_2} |g(x,y)|^p \, \mathrm{d}\mu_2(y) \right)^{\frac{1}{p}} \mathrm{d}\mu_1(x) \|F\|^{p-1}_{L^p(U_2, \mathrm{d}\mu_2)}.$$

If $\|F\|_{L^p(U_2, \mathrm{d}\mu_2)} < +\infty$ we get the wanted inequality. If $\|F\|_{L^p(U_2, \mathrm{d}\mu_2)} = +\infty$ we can choose monotone sequences $V_n^1 \subset U_1, V_k^2 \subset U_2$ such that $\forall k, n \in \mathbb{N}, \mu_1(V_n^1), \mu_2(V_k^2) < +\infty$ and $V_n^1 \to U_1$, $V_k^2 \to U_2$. Then for every pair of V_n^1 and V_k^2 inequality (18) holds and by letting $k, n \to +\infty$ we obtain (18). □

Furthermore we will need following lemmas.

Lemma 2.2.1. *The inverse of the matrix* $(I + c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle)$ *exists for sufficiently small* ε *and* z *such that the inverse of* $(I + i/2\ \mathbb{A}\mathbb{Z}(z))$ *exists. Also the following holds*

$$(I + c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle)^{-1} \xrightarrow{u} (I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}.$$

Proof. Let us have the matrix $\mathbb{A}$ of the form (16). Then from the stability of invertibility, see [[15], Theorem IV 1.16], to prove the lemma it is sufficient to show that

$$c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle \xrightarrow{u} \frac{i}{2}\mathbb{A}\mathbb{Z}(z).$$

Firstly, let us calculate exact form of the matrix $c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle$,

$$\mathbb{A}R_z v_\varepsilon = \frac{i}{2c}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}\zeta(z)\int_{\mathbb{R}} \mathrm{e}^{ik(z)|x-y|}v_\varepsilon(y)\,\mathrm{d}y & \int_{\mathbb{R}} \operatorname{sgn}(x-y)\mathrm{e}^{ik(z)|x-y|}v_\varepsilon(y)\,\mathrm{d}y\\ \int_{\mathbb{R}} \operatorname{sgn}(x-y)\mathrm{e}^{ik(z)|x-y|}v_\varepsilon(y)\,\mathrm{d}y & \zeta^{-1}(z)\int_{\mathbb{R}} \mathrm{e}^{ik(z)|x-y|}v_\varepsilon(y)\,\mathrm{d}y\end{pmatrix} =$$
$$= \begin{pmatrix}\beta S + \alpha\zeta(z)E & \alpha S + \beta\zeta^{-1}(z)E\\ \delta S + \gamma\zeta(z)E & \gamma S + \delta\zeta^{-1}(z)E\end{pmatrix}, \text{ where}$$

$$E := \frac{i}{2c}\int_{\mathbb{R}} \mathrm{e}^{ik(z)|x-y|}v_\varepsilon(y)\,\mathrm{d}y,$$
$$S := \frac{i}{2c}\int_{\mathbb{R}} \operatorname{sgn}(x-y)v_\varepsilon(y)\mathrm{e}^{ik(z)|x-y|}\,\mathrm{d}y.$$

This yields

$$c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle = \begin{pmatrix}\beta S_\varepsilon + \alpha\zeta(z)E_\varepsilon & \alpha S_\varepsilon + \beta\zeta^{-1}(z)E_\varepsilon\\ \delta S_\varepsilon + \gamma\zeta(z)E_\varepsilon & \gamma S_\varepsilon + \delta\zeta^{-1}(z)E_\varepsilon\end{pmatrix},$$

$$E_\varepsilon := \frac{i}{2}\int_{\mathbb{R}^2} v_\varepsilon(x)\mathrm{e}^{ik(z)|x-y|}v_\varepsilon(y)\,\mathrm{d}y\,\mathrm{d}x, \tag{19}$$

$$S_\varepsilon := \frac{i}{2}\int_{\mathbb{R}^2} \operatorname{sgn}(x-y)v_\varepsilon(x)\mathrm{e}^{ik(z)|x-y|}v_\varepsilon(y)\,\mathrm{d}y\,\mathrm{d}x.$$

We can see that the integral S_ε is equal to zero because its integrand is an antisymmetric function. From which we get

$$c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle = E_\varepsilon\mathbb{A}\mathbb{Z}(z). \tag{20}$$

Now we just need to prove that $|c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle - \frac{i}{2}\mathbb{A}\mathbb{Z}(z)|$ goes to zero.

$$\left|c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle - \frac{i}{2}\mathbb{A}\mathbb{Z}(z)\right|^2 = \|(E_\varepsilon - \frac{i}{2})\mathbb{A}\mathbb{Z}(z)\|_2^2 \le \|\mathbb{A}\|_2^2(|\zeta(z)|^2 + |\zeta(z)^{-1}|^2)\left|E_\varepsilon - \frac{i}{2}\right|^2, \tag{21}$$

where E_ε converges to $\frac{i}{2}$ by the dominated convergence theorem. Hence the limit of the right–hand side of (21) is zero. □

Lemma 2.2.2. *Let* $\mathbb{B}(x,y)$ *be uniformly bounded matrix valued function and* $w \in \mathbb{C}$ *be such that* $\operatorname{Im} w > 0$. *Then*

$$\sup_{y\in\mathbb{R}} \int_{\mathbb{R}} |\mathbb{B}(x,y)\mathrm{e}^{iw|x-y|}|^2\,\mathrm{d}x < +\infty.$$

Proof. Firstly, we will find an upper bound for the $\mathbb{B}(x,y)\mathrm{e}^{iw|x-y|}$.

$$|\mathbb{B}(x,y)\mathrm{e}^{iw|x-y|}| = \|\mathbb{B}(x,y)\mathrm{e}^{iw|x-y|}\|_2 \leq \|\mathbb{B}(x,y)\|_2|\mathrm{e}^{iw|x-y|}| \leq$$
$$\leq C\mathrm{e}^{-\mathrm{Im}w|x-y|},$$

where $\forall x,y \in \mathbb{R}, \|\mathbb{B}(x,y)\| < C \in \mathbb{R}$. This implies

$$\int_{\mathbb{R}} |\mathbb{B}(x,y)\mathrm{e}^{iw|x-y|}|^2 \,\mathrm{d}x \leq C^2 \int_{\mathbb{R}} \mathrm{e}^{-2\mathrm{Im}w|x-y|} \,\mathrm{d}x =$$

using substitution $x - y = t$ we get

$$= C^2 \int_{\mathbb{R}} \mathrm{e}^{-2\mathrm{Im}w|t|} \,\mathrm{d}t < +\infty.$$

We can see that the final estimate does not depend on $y \in \mathbb{R}$ which proves the lemma. □

Lemma 2.2.3. *Let* $\mathbb{B}(x,y) \in \mathbb{C}^{2,2}$ *be uniformly bounded matrix valued function, continuous in* y *for almost every* $x \in \mathbb{R}$ *and let us have* $w \in \mathbb{C}$ *such that* $\mathrm{Im}w > 0$. *Then*

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\mathbb{B}(x,\varepsilon s)\mathrm{e}^{iw|x-\varepsilon s|} - \mathbb{B}(x,0)\mathrm{e}^{iw|x|}||v(s)| \,\mathrm{d}s \right)^2 \mathrm{d}x = 0. \tag{22}$$

Proof. Using the Minkowski integral inequality from Proposition 2.2.1 we get

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\mathbb{B}(x,\varepsilon s)\mathrm{e}^{iw|x-\varepsilon s|} - \mathbb{B}(x,0)\mathrm{e}^{iw|x|}||v(s)| \,\mathrm{d}s \right)^2 \mathrm{d}x \leq$$
$$\leq \left(\int_{\mathbb{R}} |v(s)| \left(\underbrace{\int_{\mathbb{R}} |\mathbb{B}(x,\varepsilon s)\mathrm{e}^{iw|x-\varepsilon s|} - \mathbb{B}(x,0)\mathrm{e}^{iw|x|}|^2 \,\mathrm{d}x}_{f_\varepsilon(s)} \right)^{\frac{1}{2}} \mathrm{d}s \right)^2.$$

Using Lemma 2.2.2 one can see that $f_\varepsilon(s)$ is uniformly bounded by some constant $C \geq 0$.

$$\forall \varepsilon > 0, \forall s \in \mathbb{R}, |f_\varepsilon(s)| < C \in \mathbb{R}.$$

Because of that we can drag the limit in (22) into the outer integral.

Next, we will deal with the inner integral. Using Young´s inequality we get the following estimate

$$|\mathbb{B}(x,\varepsilon s)\mathrm{e}^{iw|x-\varepsilon s|} - \mathbb{B}(x,0)\mathrm{e}^{iw|x|}|^2 \leq 2|\mathbb{B}(x,\varepsilon s)\mathrm{e}^{iw|x-\varepsilon s|}|^2 + 2|\mathbb{B}(x,0)\mathrm{e}^{iw|x|}|^2.$$

Then for a fixed $s \in \mathbb{R}$, a fixed constant δ and all sufficiently small ε such that $|\varepsilon s| < \delta$ we get

$$|\mathbb{B}(x,\varepsilon s)\mathrm{e}^{iw|x-\varepsilon s|}|^2 \leq C^2\mathrm{e}^{-2\mathrm{Im}w|x-\varepsilon s|} \leq C^2 m(x),$$

where $m(x)$ is the dominating integrable function defined as

$$\forall x \in \mathbb{R},\ m(x) = \mathrm{e}^{-2\mathrm{Im}w(|x|-\delta)}.$$

We can see that

$$2|\mathbb{B}(x,\varepsilon s)\mathrm{e}^{iw|x-\varepsilon s|}|^2 + 2|\mathbb{B}(x,0)\mathrm{e}^{iw|x|}|^2 \leq 2C^2 m(x) + 2|\mathbb{B}(x,0)\mathrm{e}^{iw|x|}|^2 \in L^1(\mathbb{R}).$$

Then by using the Lebesgue dominated convergence theorem two times we obtain the desired result. □

Now we can prove Theorem 2.2.1.

Proof. We will mimic the proof of the limit of the resolvent of the non–local potential from [11] but now using any complex matrix $\mathbb{A}$.

From the resolvent formula and the form of the $D_\varepsilon^{\mathbb{A}}$ we get

$$R_{z,\varepsilon}^{\mathbb{A}} = (D_\varepsilon^{\mathbb{A}} - z)^{-1} = R_z(I + c(|v_\varepsilon\rangle\langle v_\varepsilon| \otimes \mathbb{A})R_z)^{-1},$$

where R_z is the resolvent of the free Dirac operator.

We will find the inverse of the operator $(I + c(|v_\varepsilon\rangle\langle v_\varepsilon| \otimes \mathbb{A})R_z)$.

$$\psi + c\,\underbrace{\langle v_\varepsilon|\mathbb{A}R_z\psi\rangle}_{\vec{k}}\,v_\varepsilon = g \Rightarrow \vec{k} + c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon \otimes \vec{k}\rangle = \langle v_\varepsilon|\mathbb{A}R_z g\rangle,$$

$$\vec{k} + c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle\vec{k} = \langle v_\varepsilon|\mathbb{A}R_z g\rangle,$$

$$\vec{k} = (I + c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle)^{-1}\langle v_\varepsilon|\mathbb{A}R_z g\rangle.$$

Existence of the inverse of the matrix $(I + c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle)$ was already proved in the Lemma 2.2.1. If we substitute for the vector $\vec{k}$ we get

$$\psi = g - c(I + c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle)^{-1}\langle v_\varepsilon|\mathbb{A}R_z g\rangle v_\varepsilon.$$

This yields

$$(I + c(|v_\varepsilon\rangle\langle v_\varepsilon| \otimes \mathbb{A})R_z)^{-1} = I - c(I + c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle)^{-1}(|v_\varepsilon\rangle\langle v_\varepsilon| \otimes \mathbb{A})R_z.$$

Which means that we get the resolvent of the operator $D_\varepsilon^{\mathbb{A}}$ in the following form

$$R_{z,\varepsilon}^{\mathbb{A}} = R_z - cR_z(I + c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle)^{-1}(|v_\varepsilon\rangle\langle v_\varepsilon|\mathbb{A})R_z.$$

The Lemma 2.2.1 implies that the uniform limit of the matrix $(I + c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle)^{-1}$ is

$$(I + c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle)^{-1} \overset{\varepsilon\to 0+}{\to} (I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}. \tag{23}$$

If we denote matrices above as

$$M_\varepsilon = (I + c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle)^{-1},$$

$$M = (I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1},$$

we can rewrite kernels of the resolvent of the non–local potential and its pointwise limit as follows

$$R_{z,\varepsilon}^{\mathbb{A}}(x,y) = R_z(x,y) - \int_{\mathbb{R}^2} cR_z(x,\varepsilon s)v(s)M_\varepsilon\mathbb{A}v(t)R_z(\varepsilon t,y)\,\mathrm{d}s\,\mathrm{d}t, \tag{24}$$

$$R_z^{\mathbb{A}}(x,y) = R_z(x,y) - \int_{\mathbb{R}^2} cR_z(x,0)v(s)M\mathbb{A}v(t)R_z(0,y)\,\mathrm{d}s\,\mathrm{d}t.$$

None of these two resolvents is a Hilbert–Schmidt operator, but their difference is a Hilbert–Schmidt operator. Therefore, we will study the convergence of the operator $R_\varepsilon^{\mathbb{A}} - R^{\mathbb{A}}$ to the zero operator in the Hilbert–Schmidt norm which will imply the convergence of resolvents in the operator norm. Because of that, we will try to find the estimate for the Hilbert–Schmidt norm of the difference of the operator

$R_\varepsilon^{\mathbb{A}}$ and $R^{\mathbb{A}}$. We will use the inequality $(a+b)^2 \leq 2a^2 + 2b^2$ several times in the following series of inequalities.

$$\int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} R_z(x, \varepsilon s) v(s) M_\varepsilon \mathbb{A} v(t) R_z(\varepsilon t, y) - R_z(x, 0) v(s) M \mathbb{A} v(t) R_z(0, y) \, \mathrm{d}s \, \mathrm{d}t \right|^2 \mathrm{d}x \, \mathrm{d}y =$$

$$= \int_{\mathbb{R}^2} \Big| \int_{\mathbb{R}^2} (R_z(x, \varepsilon s) - R_z(x, 0)) v(s) M_\varepsilon \mathbb{A} v(t) R_z(\varepsilon t, y) +$$

$$+ R_z(x, 0) v(s) (M_\varepsilon \mathbb{A} v(t) R_z(\varepsilon t, y) - M \mathbb{A} v(t) R_z(0, y)) \, \mathrm{d}s \, \mathrm{d}t \Big|^2 \mathrm{d}x \, \mathrm{d}y \leq$$

$$\leq \int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^2} |(R_z(x, \varepsilon s) - R_z(x, 0)) v(s) M_\varepsilon \mathbb{A} v(t) R_z(\varepsilon t, y)| +$$

$$+ |R_z(x, 0) v(s) (M_\varepsilon \mathbb{A} v(t) R_z(\varepsilon t, y) - M \mathbb{A} v(t) R_z(0, y))| \, \mathrm{d}s \, \mathrm{d}t \Big)^2 \mathrm{d}x \, \mathrm{d}y \leq$$

$$\leq 2 \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |(R_z(x, \varepsilon s) - R_z(x, 0)) v(s) M_\varepsilon \mathbb{A} v(t) R_z(\varepsilon t, y)| \, \mathrm{d}s \, \mathrm{d}t \right)^2 \mathrm{d}x \, \mathrm{d}y +$$

$$+2 \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |R_z(x, 0) v(s) (M_\varepsilon \mathbb{A} v(t) R_z(\varepsilon t, y) - M \mathbb{A} v(t) R_z(0, y))| \, \mathrm{d}s \, \mathrm{d}t \right)^2 \mathrm{d}x \, \mathrm{d}y \leq$$

$$\leq 2 \underbrace{\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |(R_z(x, \varepsilon s) - R_z(x, 0)) v(s) M_\varepsilon \mathbb{A} v(t) R_z(\varepsilon t, y)| \, \mathrm{d}s \, \mathrm{d}t \right)^2 \mathrm{d}x \, \mathrm{d}y}_{a)} +$$

$$+4 \underbrace{\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |R_z(x, 0) v(s) (M_\varepsilon - M) \mathbb{A} v(t) R_z(\varepsilon t, y)| \, \mathrm{d}s \, \mathrm{d}t \right)^2 \mathrm{d}x \, \mathrm{d}y}_{b)} +$$

$$+4 \underbrace{\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |R_z(x, 0) v(s) M \mathbb{A} v(t) (R_z(\varepsilon t, y) - R_z(0, y))| \, \mathrm{d}s \, \mathrm{d}t \right)^2 \mathrm{d}x \, \mathrm{d}y}_{c)} .$$

We will estimate each of the terms a),b) and c) separately.

a)

$$\text{a)} \leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |R_z(x, \varepsilon s) - R_z(x, 0)| |v(s)| \, \mathrm{d}s \right)^2 \mathrm{d}x \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |M_\varepsilon \mathbb{A}| |v(t)| |R_z(\varepsilon t, y)| \, \mathrm{d}t \right)^2 \mathrm{d}y.$$

The term

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |R_z(x, \varepsilon s) - R_z(x, 0)| |v(s)| \, \mathrm{d}s \right)^2 \mathrm{d}x$$

goes to zero by Lemma 2.2.3. Since matrix M_ε converges to the matrix M by Lemma 2.2.1 it is uniformly bounded by some constant $C \geq 0$. Then by using this observation, Lemma 2.2.2 and the Minkowski integral inequality (Proposition 2.2.1), we get the following

$$\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|M_\varepsilon\mathbb{A}||v(t)||R_z(\varepsilon t,y)|\,\mathrm{d}t\right)^2\mathrm{d}y\le C\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|v(t)||R_z(\varepsilon t,y)|\,\mathrm{d}t\right)^2\mathrm{d}y\le$$
$$\le C\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|v(t)|^2|R_z(\varepsilon t,y)|^2\,\mathrm{d}y\right)^{\frac{1}{2}}\mathrm{d}t\right)^2=C\left(\int_{\mathbb{R}}|v(t)|\left(\int_{\mathbb{R}}|R_z(\varepsilon t,y)|^2\,\mathrm{d}y\right)^{\frac{1}{2}}\mathrm{d}t\right)^2\le\tilde{C}.$$

This yields a) $\to 0$ as $\varepsilon\to 0$.

b) Using Lemmas 2.2.1 and 2.2.2 together with the Minkowski integral inequality we get

$$\text{b)}\le\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}|R_z(x,0)v(s)(M_\varepsilon-M)\mathbb{A}v(t)R_z(\varepsilon t,y)|\,\mathrm{d}s\,\mathrm{d}t\right)^2\mathrm{d}x\,\mathrm{d}y\le$$
$$\le\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|R_z(x,0)||v(s)|\,\mathrm{d}s\right)^2\mathrm{d}x|M_\varepsilon-M||\mathbb{A}|\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|R_z(\varepsilon t,y)||v(t)|\,\mathrm{d}t\right)^2\mathrm{d}y\le$$
$$\le\left(\int_{\mathbb{R}}|v(s)|\,\mathrm{d}s\right)^2\int_{\mathbb{R}}|R_z(x,0)|^2\,\mathrm{d}x|M_\varepsilon-M||\mathbb{A}|\left(\int_{\mathbb{R}}|v(t)|\left(\int_{\mathbb{R}}|R_z(\varepsilon t,y)|^2\,\mathrm{d}y\right)^{\frac{1}{2}}\mathrm{d}t\right)^2\le$$
$$\le\underbrace{\left(\int_{\mathbb{R}}|v(s)|\,\mathrm{d}s\right)^2\int_{\mathbb{R}}|R_z(x,0)|^2\,\mathrm{d}x}_{<+\infty}\underbrace{|M_\varepsilon-M|}_{\to 0}|\mathbb{A}|\underbrace{\left(\int_{\mathbb{R}}|v(t)|\,\mathrm{d}t\right)^2\sup_{\iota\in\mathbb{R}}\int_{\mathbb{R}}|R_z(\iota,y)|^2\,\mathrm{d}y}_{<+\infty}\to 0.$$

c) Similarly to a).

This means that we get the convergence in the Hilbert–Schmidt norm which implies $R^{\mathbb{A}}_{z,\varepsilon}\xrightarrow{u}R^{\mathbb{A}}_z$. □

Thus we get the limit in the operator norm for the resolvent of the Dirac operator with non–local potential but we do not know if it is a resolvent of some operator. In the self–adjoint case we had a candidate in the form of a self–adjoint relativistic point interaction [3] which was also proved to be the norm–resolvent limit of the operator. As we already mentioned the pointwise limit of the self–adjoint operator $D^{\mathbb{A}}_\varepsilon$ with a hermitian matrix $\mathbb{A}$ coincides with its norm resolvent limit. We will prove the same for every choice of $\mathbb{A}$.

Theorem 2.2.2. *Let $\mathbb{A}$ be any complex matrix and $z\in\mathbb{C}\setminus\mathbb{R}_{mc^2}$ be such that*

$$(I+\frac{i}{2}\mathbb{A}\mathbb{Z}(z))$$

is a regular matrix. Then the operator

$$R^{\mathbb{A}}_z=R_z-cR_z(x,0)(I+\frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}R_z(0,y)\tag{25}$$

is the resolvent of the operator $D^{\mathbb{A}}$ given in (8).

Proof. Since the operator $R^{\mathbb{A}}_z$ can be written as

$$R^{\mathbb{A}}_z=R_z-\mathcal{K},$$

where R_z is the resolvent of the free Dirac operator and $\mathcal{K}$ is the Hilbert–Schmidt operator, we get that the operator $R^{\mathbb{A}}_z$ is a bounded operator. Then it is sufficient to check following two statements.

1. $\operatorname{Ran} R_z^{\mathbb{A}} \subset \operatorname{Dom} D^{\mathbb{A}}$ and $\forall \psi \in \operatorname{Dom} R_z^{\mathbb{A}}, (D^{\mathbb{A}} - z)R_z^{\mathbb{A}}\psi = \psi$, 2. $\forall \psi \in \operatorname{Dom} D^{\mathbb{A}}, R_z^{\mathbb{A}}(D^{\mathbb{A}} - z)\psi = \psi$.

1. Firstly, let $\varphi \in \operatorname{Ran} R_z^{\mathbb{A}}$. We will check if $\varphi \in \operatorname{Dom} D^{\mathbb{A}}$. Since $\varphi \in \operatorname{Ran} R_z^{\mathbb{A}}$, there exists $\psi \in \operatorname{Dom} R_z^{\mathbb{A}}$ such that

$$\varphi(x) = R_z^{\mathbb{A}}\psi(x),$$

which in particular means

$$\varphi(x) = \int_{\mathbb{R}} R_z(x,y)\psi(y)\,\mathrm{d}y - cR_z(x,0)(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\int_{\mathbb{R}} R_z(0,y)\psi(y)\,\mathrm{d}y.$$

This yields

$$\begin{aligned}\varphi(0+) &= \int_{\mathbb{R}} R_z(0,y)\psi(y)\,\mathrm{d}y - \frac{i}{2}(\mathbb{Z}(z)+\sigma_1)(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\int_{\mathbb{R}} R_z(0,y)\psi(y)\,\mathrm{d}y,\\ \varphi(0-) &= \int_{\mathbb{R}} R_z(0,y)\psi(y)\,\mathrm{d}y - \frac{i}{2}(\mathbb{Z}(z)-\sigma_1)(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\int_{\mathbb{R}} R_z(0,y)\psi(y)\,\mathrm{d}y.\end{aligned}$$

We can see that the transmission condition

$$(2i - \sigma_1\mathbb{A})\varphi(0+) = (2i + \sigma_1\mathbb{A})\varphi(0-)$$

holds if and only if

$$(2i - \sigma_1\mathbb{A})(I - \frac{i}{2}(\mathbb{Z}(z)+\sigma_1)(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}) = (2i + \sigma_1\mathbb{A})(I - \frac{i}{2}(\mathbb{Z}(z)-\sigma_1)(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}),$$
$$2i - \sigma_1\mathbb{A} + (2i-\sigma_1\mathbb{A})(\mathbb{Z}(z)+\sigma_1)(2i - \mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A} = 2i + \sigma_1\mathbb{A} + (2i+\sigma_1\mathbb{A})(\mathbb{Z}(z)-\sigma_1)(2i - \mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A},$$

$$\begin{aligned}&-\sigma_1\mathbb{A} + 2i\sigma_1(2i - \mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A} - \sigma_1\mathbb{A}\mathbb{Z}(z)(2i - \mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A} =\\ &\qquad = \sigma_1\mathbb{A} - 2i\sigma_1(2i - \mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A} + \sigma_1\mathbb{A}\mathbb{Z}(z)(2i - \mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A},\end{aligned}$$

$$\begin{aligned}-\sigma_1\mathbb{A} + \sigma_1(2i - \mathbb{A}\mathbb{Z}(z))(2i - \mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A} &= \sigma_1\mathbb{A} - \sigma_1(2i - \mathbb{A}\mathbb{Z}(z))(2i - \mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A},\\ -\sigma_1\mathbb{A} + \sigma_1\mathbb{A} &= \sigma_1\mathbb{A} - \sigma_1\mathbb{A}.\end{aligned}$$

We conclude that for any complex matrix $\mathbb{A}$ and $z \in \mathbb{C} \setminus \mathbb{R}_{mc^2}$ such that $(I + i/2\ \mathbb{A}\mathbb{Z}(z))^{-1}$ exists every $\varphi \in \operatorname{Ran} R_z^{\mathbb{A}}$ fulfils the transmission condition of the Hamiltonian of the relativistic point interaction.

Since $R_z(x,y)$ is the resolvent of the Dirac operator and $R_z(x,0)$ standing alone is in $W^{1,2}(\mathbb{R}\setminus\{0\};\mathbb{C}^{2,2})$ we finally get

$$\operatorname{Ran} R_z^{\mathbb{A}} \subset \operatorname{Dom} D^{\mathbb{A}}.$$

Now let us check that $R^{\mathbb{A}}$ is the right inverse of $(D^{\mathbb{A}} - z)$.

$$\begin{aligned}(D^{\mathbb{A}} - z)R_z^{\mathbb{A}}\psi(x) = \Big(-ic\frac{\mathrm{d}}{\mathrm{d}x}\sigma_1 + mc^2\sigma_3 - z\Big)\Big(&\int_{\mathbb{R}} \frac{i}{2c}(\mathbb{Z}(z) + \operatorname{sgn}(x-y)\sigma_1)\mathrm{e}^{ik(z)|x-y|}\psi(y)\,\mathrm{d}y +\\ &+ \frac{1}{4c}(\mathbb{Z}(z) + \sigma_1\operatorname{sgn}(x))\mathrm{e}^{ik(z)|x|}(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\int_{\mathbb{R}}(\mathbb{Z}(z) + \operatorname{sgn}(-y)\sigma_1)\mathrm{e}^{ik(z)|y|}\psi(y)\,\mathrm{d}y\Big).\end{aligned}$$

For $x > 0$, we get the following

$(D^{\mathbb{A}} - z)R_z^{\mathbb{A}}\psi(x) =$

$$= -i\frac{\mathrm{d}}{\mathrm{d}x}\sigma_1 \int_{\mathbb{R}} \frac{i}{2}(\mathbb{Z}(z) + \operatorname{sgn}(x-y)\sigma_1)\mathrm{e}^{ik(z)|x-y|}\psi(y)\,\mathrm{d}y - \tag{26}$$

$$- i\frac{\mathrm{d}}{\mathrm{d}x}\sigma_1\frac{1}{4}(\mathbb{Z}(z) + \sigma_1)\mathrm{e}^{ik(z)|x|}(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\int_{\mathbb{R}}(\mathbb{Z}(z) + \operatorname{sgn}(-y)\sigma_1)\mathrm{e}^{ik(z)|y|}\psi(y)\,\mathrm{d}y - \tag{27}$$

$$- \frac{1}{c}\begin{pmatrix} z - mc^2 & 0 \\ 0 & z + mc^2 \end{pmatrix}\int_{\mathbb{R}} \frac{i}{2}(\mathbb{Z}(z) + \operatorname{sgn}(x-y)\sigma_1)\mathrm{e}^{ik(z)|x-y|}\psi(y)\,\mathrm{d}y - \tag{28}$$

$$- \frac{1}{c}\begin{pmatrix} z - mc^2 & 0 \\ 0 & z + mc^2 \end{pmatrix}\frac{1}{4}(\mathbb{Z}(z) + \sigma_1)\mathrm{e}^{ik(z)|x|}(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\int_{\mathbb{R}}(\mathbb{Z}(z) + \operatorname{sgn}(-y)\sigma_1)\mathrm{e}^{ik(z)|y|}\psi(y)\,\mathrm{d}y. \tag{29}$$

Since

$$\zeta(z)k(z) = \frac{z + mc^2}{c} \text{ and } \zeta(z)^{-1}k(z) = \frac{z - mc^2}{c},$$

we get the following

$$\begin{aligned}
(26) = \frac{1}{2}\sigma_1\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{+\infty}(\mathbb{Z}(z) - \sigma_1)\mathrm{e}^{ik(z)|x-y|}\psi(y)\,\mathrm{d}y + \\
+ \frac{1}{2}\sigma_1\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{x}(\mathbb{Z}(z) + \sigma_1)\mathrm{e}^{ik(z)|x-y|}\psi(y)\,\mathrm{d}y = \\
= -\frac{1}{2}\sigma_1(\mathbb{Z}(z) - \sigma_1)\psi(x) + \frac{1}{2}\sigma_1\int_x^{+\infty}(\mathbb{Z}(z) - \sigma_1)(-ik(z))\mathrm{e}^{ik(z)|x-y|}\psi(y)\,\mathrm{d}y + \\
+ \frac{1}{2}\sigma_1(\mathbb{Z}(z) + \sigma_1)\psi(x) + \frac{1}{2}\sigma_1\int_{-\infty}^{x}(\mathbb{Z}(z) + \sigma_1)(ik(z))\mathrm{e}^{ik(z)|x-y|}\psi(y)\,\mathrm{d}y = \\
= \psi(x) + \frac{1}{c}\begin{pmatrix} z - mc^2 & 0 \\ 0 & z + mc^2 \end{pmatrix}\int_{\mathbb{R}}\frac{i}{2}(\mathbb{Z}(z) + \operatorname{sgn}(x-y)\sigma_1)\mathrm{e}^{ik(z)|x-y|}\psi(y)\,\mathrm{d}y.
\end{aligned}$$

$$(27) = \frac{1}{c}\begin{pmatrix} z - mc^2 & 0 \\ 0 & z + mc^2 \end{pmatrix}\frac{1}{4}(\mathbb{Z}(z) + \sigma_1)\mathrm{e}^{ik(z)|x|}(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\int_{\mathbb{R}}(\mathbb{Z}(z) + \operatorname{sgn}(-y)\sigma_1)\mathrm{e}^{ik(z)|y|}\psi(y)\,\mathrm{d}y.$$

We finally get

$$(D^{\mathbb{A}} - z)R_z^{\mathbb{A}}\psi = (26) + (27) + (28) + (29) = \psi(x),$$

and similarly for $x < 0$ we get the same result.

2. Now we need to check if $R_z^{\mathbb{A}}$ is also a left inverse of $(D^{\mathbb{A}} - z)$.

$$R_z^{\mathbb{A}}(D^{\mathbb{A}} - z)\psi(x) = \int_{\mathbb{R}}\frac{i}{2c}(\mathbb{Z}(z) + \operatorname{sgn}(x-y)\sigma_1)\mathrm{e}^{ik(z)|x-y|}(-ic\frac{\mathrm{d}}{\mathrm{d}y}\sigma_1 + mc^2\sigma_3 - z)\psi(y)\,\mathrm{d}y +$$

$$+\frac{1}{4c}(\mathbb{Z}(z)+\operatorname{sgn}(x)\sigma_1)\mathrm{e}^{ik(z)|x|}(I+\frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\int_{\mathbb{R}}(\mathbb{Z}(z)+\operatorname{sgn}(-y)\sigma_1)\mathrm{e}^{ik(z)|y|}(-ic\frac{\mathrm{d}}{\mathrm{d}y}\sigma_1+mc^2\sigma_3-z)\psi(y)\,\mathrm{d}y =$$

$$= \int_{\mathbb{R}} \frac{1}{2}(\mathbb{Z}(z) + \operatorname{sgn}(x-y)\sigma_1)\mathrm{e}^{ik(z)|x-y|}\frac{\mathrm{d}}{\mathrm{d}y}\sigma_1\psi(y)\,\mathrm{d}y- \tag{30}$$

$$-\frac{i}{4}(\mathbb{Z}(z) + \operatorname{sgn}(x)\sigma_1)\mathrm{e}^{ik(z)|x|}(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\int_{\mathbb{R}}(\mathbb{Z}(z) + \operatorname{sgn}(-y)\sigma_1)\mathrm{e}^{ik(z)|y|}\frac{\mathrm{d}}{\mathrm{d}y}\sigma_1\psi(y)\,\mathrm{d}y- \tag{31}$$

$$-\frac{1}{c}\int_{\mathbb{R}} \frac{i}{2}(\mathbb{Z}(z) + \operatorname{sgn}(x-y)\sigma_1)\mathrm{e}^{ik(z)|x-y|}\begin{pmatrix} z-mc^2 & 0 \\ 0 & z+mc^2\end{pmatrix}\psi(y)\,\mathrm{d}y- \tag{32}$$

$$-\frac{1}{4c}(\mathbb{Z}(z) + \operatorname{sgn}(x)\sigma_1)\mathrm{e}^{ik(z)|x|}(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\int_{\mathbb{R}}(\mathbb{Z}(z) + \operatorname{sgn}(-y)\sigma_1)\mathrm{e}^{ik(z)|y|}\begin{pmatrix} z-mc^2 & 0 \\ 0 & z+mc^2\end{pmatrix}\psi(y)\,\mathrm{d}y. \tag{33}$$

Firstly, we consider $x > 0$. Using integration by parts we get the following

$$\begin{aligned}
(30) = &\int_x^{+\infty} \frac{1}{2}(\mathbb{Z}(z) - \sigma_1)\mathrm{e}^{i\zeta(z)z|x-y|}\frac{\mathrm{d}}{\mathrm{d}y}\sigma_1\psi(y)\,\mathrm{d}y + \int_0^x \frac{1}{2}(\mathbb{Z}(z) + \sigma_1)\mathrm{e}^{ik(z)|x-y|}\frac{\mathrm{d}}{\mathrm{d}y}\sigma_1\psi(y)\,\mathrm{d}y+\\
&+\int_{-\infty}^0 \frac{1}{2}(\mathbb{Z}(z) + \sigma_1)\mathrm{e}^{ik(z)|x-y|}\frac{\mathrm{d}}{\mathrm{d}y}\sigma_1\psi(y)\,\mathrm{d}y =\\
&= \left[\frac{1}{2}(\mathbb{Z}(z) - \sigma_1)\mathrm{e}^{ik(z)|x-y|})\sigma_1\psi(y)\right]_{y\to x}^{y\to+\infty} + \left[\frac{1}{2}(\mathbb{Z}(z) + \sigma_1)\mathrm{e}^{ik(z)|x-y|}\sigma_1\psi(y)\right]_{y\to 0+}^{y\to x} +\\
&+\left[\frac{1}{2}(\mathbb{Z}(z) + \sigma_1)\mathrm{e}^{ik(z)|x-y|}\sigma_1\psi(y)\right]_{y\to-\infty}^{y\to 0-} - \int_x^{+\infty}\frac{1}{2}(\mathbb{Z}(z) - \sigma_1)(ik(z))\mathrm{e}^{ik(z)|x-y|}\sigma_1\psi(y)\,\mathrm{d}y-\\
&-\int_0^x \frac{1}{2}(\mathbb{Z}(z) + \sigma_1)(ik(z))\mathrm{e}^{ik(z)|x-y|}\sigma_1\psi(y)\,\mathrm{d}y - \int_{-\infty}^x \frac{1}{2}(\mathbb{Z}(z) + \sigma_1)(-ik(z))\mathrm{e}^{ik(z)|x-y|}\sigma_1\psi(y)\,\mathrm{d}y =\\
&= \psi(x) + \frac{1}{c}\int_{\mathbb{R}}\frac{i}{2}(\mathbb{Z}(z) + \operatorname{sgn}(x-y)\sigma_1)\mathrm{e}^{ik(z)|x-y|}\begin{pmatrix} z-mc^2 & 0 \\ 0 & z+mc^2\end{pmatrix}\psi(y)\,\mathrm{d}y+\\
&+\frac{1}{2}(\mathbb{Z}(z) + \sigma_1)\mathrm{e}^{ik(z)|x|}\sigma_1(\psi(0-) - \psi(0+)).
\end{aligned}$$

Similarly for $x < 0$ we get

$$\begin{aligned}
(30) = \psi(x) + \frac{1}{c}\int_{\mathbb{R}}\frac{i}{2}(\mathbb{Z}(z) + \operatorname{sgn}(x-y)\sigma_1)\mathrm{e}^{ik(z)|x-y|}&\begin{pmatrix} z-mc^2 & 0 \\ 0 & z+mc^2\end{pmatrix}\psi(y)\,\mathrm{d}y+\\
&+\frac{1}{2}(\mathbb{Z}(z) - \sigma_1)\mathrm{e}^{ik(z)|x|}\sigma_1(\psi(0-) - \psi(0+)).
\end{aligned}$$

In other words for $\forall x \in \mathbb{R}\setminus\{0\}$

$$\begin{aligned}
(30) = \psi(x) + \frac{1}{c}\int_{\mathbb{R}}\frac{i}{2}(\mathbb{Z}(z) + \operatorname{sgn}(x-y)\sigma_1)\mathrm{e}^{ik(z)|x-y|}&\begin{pmatrix} z-mc^2 & 0 \\ 0 & z+mc^2\end{pmatrix}\psi(y)\,\mathrm{d}y+\\
&+\frac{1}{2}(\mathbb{Z}(z) + \operatorname{sgn}(x)\sigma_1)\mathrm{e}^{ik(z)|x|}\sigma_1(\psi(0-) - \psi(0+)).
\end{aligned}$$

$$
\begin{aligned}
(31) = &-\frac{i}{4}(\mathbb{Z}(z) + \mathrm{sgn}(x)\sigma_1)\mathrm{e}^{ik(z)|x|}(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\left[(\mathbb{Z}(z) - \sigma_1)\mathrm{e}^{ik(z)|y|}\sigma_1\psi(y)\right]_0^{+\infty} - \\
&- \frac{i}{4}(\mathbb{Z}(z) + \mathrm{sgn}(x)\sigma_1)\mathrm{e}^{ik(z)|x|}(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\left[(\mathbb{Z}(z) + \sigma_1)\mathrm{e}^{ik(z)|y|}\sigma_1\psi(y)\right]_{-\infty}^0 + \\
&+ \frac{i}{4}(\mathbb{Z}(z) + \mathrm{sgn}(x)\sigma_1)\mathrm{e}^{ik(z)|x|}(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\int_0^{+\infty}(\mathbb{Z}(z) - \sigma_1)(ik(z))\mathrm{e}^{ik(z)|y|}\sigma_1\psi(y)\,\mathrm{d}y + \\
&+ \frac{i}{4}(\mathbb{Z}(z) + \mathrm{sgn}(x)\sigma_1)\mathrm{e}^{ik(z)|x|}(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\int_{-\infty}^{0}(\mathbb{Z}(z) + \sigma_1)(-ik(z))\mathrm{e}^{ik(z)|y|}\sigma_1\psi(y)\,\mathrm{d}y = \\
= &-\frac{i}{4}(\mathbb{Z}(z) + \mathrm{sgn}(x)\sigma_1)\mathrm{e}^{ik(z)|x|}(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\,((\mathbb{Z}(z) + \sigma_1)\sigma_1\psi(0-) - (\mathbb{Z}(z) - \sigma_1)\sigma_1\psi(0+)) + \\
&+ \frac{1}{4c}(\mathbb{Z}(z) + \mathrm{sgn}(x)\sigma_1)\mathrm{e}^{ik(z)|x|}(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\int_{\mathbb{R}}(\mathbb{Z}(z) + \mathrm{sgn}(-y)\sigma_1)\mathrm{e}^{ik(z)|y|}\begin{pmatrix} z - mc^2 & 0 \\ 0 & z + mc^2 \end{pmatrix}\psi(y)\,\mathrm{d}y.
\end{aligned}
$$

The results above imply

$$
\begin{aligned}
&(30) + (31) + (32) + (33) = \\
&= \psi(x) - \frac{i}{4}(\mathbb{Z}(z) + \mathrm{sgn}(x)\sigma_1)\mathrm{e}^{ik(z)|x|}2i\sigma_1(\psi(0-) - \psi(0+)) - \\
&- \frac{i}{4}(\mathbb{Z}(z) + \mathrm{sgn}(x)\sigma_1)\mathrm{e}^{ik(z)|x|}(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}\Big((\mathbb{Z}(z) + \sigma_1)\sigma_1\psi(0-) - (\mathbb{Z}(z) - \sigma_1)\sigma_1\psi(0+)\Big) = \\
&= \psi(x) - \frac{i}{4}(\mathbb{Z}(z) + \mathrm{sgn}(x)\sigma_1)\mathrm{e}^{ik(z)|x|}\underbrace{\Big(2i + (I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}(\mathbb{Z}(z) + \sigma_1)\Big)\sigma_1}_{\mathbb{B}_-}\psi(0-) - \\
&- \frac{i}{4}(\mathbb{Z}(z) + \mathrm{sgn}(x)\sigma_1)\mathrm{e}^{ik(z)|x|}\underbrace{\Big(2i - (I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}(\mathbb{Z}(z) - \sigma_1)\Big)\sigma_1}_{\mathbb{B}_+}\psi(0+).
\end{aligned}
$$

Matrix $\mathbb{B}_-$ next to $\psi(0-)$ is

$$
\begin{aligned}
\mathbb{B}_- = (2i + (I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}(\mathbb{Z}(z) + \sigma_1))\sigma_1 = (2i - 2i(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}(\frac{i}{2}\mathbb{A}\mathbb{Z}(z) + I - I + \frac{i}{2}\mathbb{A}\sigma_1))\sigma_1 = \\
= (I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}(2i\sigma_1 + \mathbb{A}).
\end{aligned}
$$

Similarly matrix $\mathbb{B}_+$ next to $\psi(0+)$ is

$$(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}(2i\sigma_1 - \mathbb{A}).$$

If we take the transmission condition into account

$$(2i - \sigma_1\mathbb{A})\psi(0+) = (2i + \sigma_1\mathbb{A})\psi(0-),$$

we finally get that

$$\forall x \in \mathbb{R} \setminus \{0\},\ R_z^{\mathbb{A}}(D^{\mathbb{A}} - z)\psi(x) = \psi(x).$$

□

3 Spectral analysis

3.1 Spectrum of general relativistic point interactions

In this section we will study the spectrum of the operator $D^{\mathbb{A}}$. Since $D^{\mathbb{A}}$ is not necessarily self–adjoint it may happen that some points of its spectrum lie outside real numbers. We will find out if conditions under which we do not get the resolvent $R_z^{\mathbb{A}}$ from Theorem 2.2.2 are superfluous due to our procedure of finding the resolvent, or if they coincide with conditions for the spectral points.

Theorem 3.1.1. *Let $\mathbb{A}$ be any complex matrix. Then*

$$\sigma(D^{\mathbb{A}}) \setminus \mathbb{R}_{mc^2} = \sigma_p(D^{\mathbb{A}})$$

and $z \in \mathbb{C} \setminus \mathbb{R}_{mc^2}$ is in the spectrum of the operator $D^{\mathbb{A}}$ (8) *if and only if z satisfies the following equation*

$$4 + 2i\,\mathrm{tr}(\mathbb{A}\mathbb{Z}(z)) - \det\mathbb{A} = \det(2I + i\mathbb{A}\mathbb{Z}(z)) = 0. \tag{34}$$

The eigenvalue z has geometric multiplicity equal to $\dim(\mathrm{Ker}(2I + i\mathbb{A}\mathbb{Z}(z))$ *and the corresponding eigenfunction can be found in the following form*

$$\psi(x) = \begin{pmatrix} C_1 \mathrm{e}^{ik(z)|x|} \\ \tilde{C}_1 \zeta(z)^{-1}\mathrm{sgn}(x)\mathrm{e}^{ik(z)|x|} \end{pmatrix}, x \in \mathbb{R} \setminus \{0\},$$

where $(-iC_1 - i\tilde{C}_1, -i\zeta(z)C_1 + i\zeta(z)\tilde{C}_1) \in \mathrm{Ker}(2I + i\mathbb{A}\mathbb{Z}(z)) \setminus \{\vec{0}\}$.

Proof. Let us take $z \in \mathbb{C} \setminus \mathbb{R}_{mc^2}$ then the eigenvalue equation is

$$\begin{aligned} -ic\sigma_1 \frac{\mathrm{d}}{\mathrm{d}x}\psi + mc^2\sigma_3\psi &= z\psi, \psi \in \mathrm{Dom}\, D^{\mathbb{A}}, \\ \frac{\mathrm{d}}{\mathrm{d}x}\psi &= \frac{i}{c}\begin{pmatrix} 0 & z + mc^2 \\ z - mc^2 & 0 \end{pmatrix}\psi. \end{aligned} \tag{35}$$

Since the matrix on the right hand side of the equation is constant, it is easy to get its antiderivative. Using

$$\begin{pmatrix} 0 & z + mc^2 \\ z - mc^2 & 0 \end{pmatrix}^2 = (z^2 - (mc^2)^2)I,$$

we can compute the exponential of the antiderivative.

$$\begin{aligned} \exp\left(\frac{i}{c}\begin{pmatrix} 0 & z + mc^2 \\ z - mc^2 & 0 \end{pmatrix} x\right) &= \sum_{n=0}^{+\infty} \frac{i^n x^n}{c^n n!}\begin{pmatrix} 0 & z + mc^2 \\ z - mc^2 & 0 \end{pmatrix}^n = \\ = \left(\sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n}}{(2n)!c^{2n}}(z^2 - (mc^2)^2)^n\right) I &+ i\left(\sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!c^{2n+1}}(z^2 - (mc^2)^2)^n\right)\begin{pmatrix} 0 & z + mc^2 \\ z - mc^2 & 0 \end{pmatrix} = \\ &= \cos(k(z)x)I + i\sin(k(z)x)\begin{pmatrix} 0 & \zeta(z) \\ \zeta(z)^{-1} & 0 \end{pmatrix}. \end{aligned}$$

This yields that the general solution to the equation (35) can be written in the following form,

$$\psi(x) = \begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} = \begin{pmatrix}\cos(k(z)x) & i\zeta(z)\sin(k(z)x)\\ i\zeta(z)^{-1}\sin(k(z)x) & \cos(k(z)x)\end{pmatrix}\begin{pmatrix}C_1\\ C_2\end{pmatrix}.$$

Now we need to determine constants $C_1, C_2 \in \mathbb{C}$. We will find ψ on the intervals $(-\infty, 0)$ and $(0, +\infty)$ separately and then we will merge them via the transmission condition. Let us write $k(z) = \eta + i\gamma, \gamma > 0$ then

$$\psi_1 = (C_1\cosh(\gamma x) - C_2\zeta(z)\sinh(\gamma x))\cos(\eta x) + i(C_2\zeta(z)\cosh(\gamma x) - C_1\sinh(\gamma x))\sin(\eta x),$$

$$\psi_2 = (C_2\cosh(\gamma x) - C_1\zeta(z)^{-1}\sinh(\gamma x))\cos(\eta x) + i(C_1\zeta(z)^{-1}\cosh(\gamma x) - C_2\sinh(\gamma x))\sin(\eta x).$$

Therefore, to get a square-integrable solution we need $C_1 = \zeta(z)C_2$ on $(0, +\infty)$ and $C_1 = -\zeta(z)C_2$ on $(-\infty, 0)$. We conclude that

$$\psi_1(x) = \begin{cases} C_1 \mathrm{e}^{ik(z)x}, & x \in (0, +\infty),\\ \tilde{C}_1 \mathrm{e}^{-ik(z)x}, & x \in (-\infty, 0),\end{cases}$$

and

$$\psi_2(x) = \begin{cases} C_1\zeta(z)^{-1}\mathrm{e}^{ik(z)x}, & x \in (0, +\infty),\\ -\tilde{C}_1\zeta(z)^{-1}\mathrm{e}^{-ik(z)x}, & x \in (-\infty, 0).\end{cases}$$

Recall that the transmission condition for $\psi \in \operatorname{Dom} D^{\mathbb{A}}$ reads as

$$(2i\sigma_1 - \mathbb{A})\psi(0+) = (2i\sigma_1 + \mathbb{A})\psi(0-),$$

where $\mathbb{A}$ is of the form (16). This implies

$$\begin{aligned}(2i - \alpha\zeta(z) - \beta)C_1 &= (-2i + \alpha\zeta(z) - \beta)\tilde{C}_1,\\ (2i - \gamma - \delta\zeta(z)^{-1})C_1 &= (2i + \gamma - \delta\zeta(z)^{-1})\tilde{C}_1,\end{aligned} \tag{36}$$

which can be rearranged as

$$(2i - \alpha\zeta(z))(C_1 + \tilde{C}_1) - \beta(C_1 - \tilde{C}_1) = 0,$$

$$-\gamma(C_1 + \tilde{C}_1) + (2i - \delta\zeta(z)^{-1})(C_1 - \tilde{C}_1) = 0.$$

Equivalently, we get

$$\underbrace{\begin{pmatrix}2 + i\alpha\zeta(z) & i\beta\zeta(z)^{-1}\\ i\gamma\zeta(z) & 2 + i\delta\zeta(z)^{-1}\end{pmatrix}}_{2I + i\mathbb{A}\mathbb{Z}(z)}\begin{pmatrix}-i(C_1 + \tilde{C}_1)\\ -i\zeta(z)(C_1 - \tilde{C}_1)\end{pmatrix} = \vec{0}. \tag{37}$$

To find an implicit relation for eigenvalues, one must find condition under which a non–trivial solution of (36) exists. Existence of such solution $(C_1, \tilde{C}_1)$ is guaranteed if and only if

$$0 = \frac{1}{2}\begin{vmatrix}2i - \alpha\zeta(z) - \beta & 2i - \alpha\zeta(z) + \beta\\ 2i - \delta\zeta(z)^{-1} - \gamma & -2i + \delta\zeta(z)^{-1} - \gamma\end{vmatrix} = \det(2I + i\mathbb{A}\mathbb{Z}(z)) = 4 + 2i\operatorname{tr}(\mathbb{A}\mathbb{Z}(z)) - \det\mathbb{A}. \tag{38}$$

Recall that

$$z \in \sigma(D^{\mathbb{A}}) \text{ if and only if } R_z^{\mathbb{A}} = (D^{\mathbb{A}} - z)^{-1} \notin \mathcal{B}(\mathcal{H}).$$

Therefore, we can check under which condition, Theorem 2.2.2 will not give us $R_z^{\mathbb{A}}$. By Theorem 2.2.2 only problem may occur if the matrix $(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))$ is not a regular matrix. This is true if and only if

$$0 = \det(2I + i\mathbb{A}\mathbb{Z}(z)) = 4 + 2i\,\mathrm{tr}(\mathbb{A}\mathbb{Z}(z)) - \det\mathbb{A}.$$

We can see that the condition under which $z \in \mathbb{C} \setminus \mathbb{R}_{mc^2}$ is in the point spectrum of the operator $D^{\mathbb{A}}$ (38) is the same as the condition (34). This proves the theorem. □

Theorems 3.1.1 and 2.2.2 give us almost a full picture of the spectral problem. The operator $D^{\mathbb{A}}$ has only point spectrum in $\mathbb{C} \setminus \mathbb{R}_{mc^2}$ which can be found by examining (34). If we start with $\mathbb{A}$ such that the condition (34) is not fulfilled for any $z \in \mathbb{C} \setminus \mathbb{R}$, we receive the operator $D^{\mathbb{A}}$ with purely real spectrum.

We can deal with the remaining set $\mathbb{R}_{mc^2}$ by using [Theorem XIII.14 [18]]. We know that $R_z^{\mathbb{A}} - R_z$ is Hilbert–Schmidt and, thus, it is a compact operator and except cases, in which the inverse of the matrix $(I + i/2\mathbb{A}\mathbb{Z}(z))$ does not exists for whole complex plane or half–plane, we will meet the assumptions of the theorem. Since $\mathbb{R}_{mc^2}$ is equal to essential spectrum of the free Dirac operator the referenced theorem implies that this set is also equal to an essential spectrum of the operator $D^{\mathbb{A}}$. In the next subsection, we will look closer at critical cases, when whole complex plane or half–plane will lie in the point spectrum of the operator $D^{\mathbb{A}}$.

3.2 Spectral transitions

If we choose $m \neq 0$ and the matrix $\mathbb{A}$ such that it has a non–zero diagonal, the condition (34) is at most quadratic in $\zeta(z)$ and this gives us a finite number of points in the spectrum in $\mathbb{C}\setminus\mathbb{R}_{mc^2}$. Nevertheless, we can see remarkable spectral transitions of our model if elements on the diagonal of the matrix $\mathbb{A}$ are equal to zero. Then (34) reduces to

$$0 = 4 - \det\mathbb{A}.$$

This yields that if matrix $\mathbb{A}$ has zeros on the diagonal and $\det\mathbb{A} \neq 4$ then by Theorem 3.1.1 we have no spectrum outside $\mathbb{R}_{mc^2}$. However, if $\mathbb{A}$ has zeros on the diagonal and its determinant is equal to 4, the condition (34) holds for every $z \in \mathbb{C} \setminus \mathbb{R}_{mc^2}$. We can demonstrate this spectral transition by considering following matrix

$$\mathbb{A} = \begin{pmatrix} \alpha & 2\beta \\ -\frac{2}{\beta} & 0 \end{pmatrix}. \tag{39}$$

We can see that for such matrix

$$\sigma_p(D^{\mathbb{A}}) = \begin{cases} \mathbb{C} \setminus \mathbb{R}_{mc^2} & \text{if } \alpha = 0, \\ \emptyset & \text{if } \alpha \neq 0. \end{cases}$$

Also choosing $m = 0$, we can observe another interesting spectral transitions. In this case, matrix $\mathbb{Z}(z)$ takes a following form

$$\mathbb{Z}(z) = \mathrm{sgn}(\mathrm{Im}(z))I.$$

Then (34) reads as

$$4 + \mathrm{sgn}(\mathrm{Im}(z))2i\,\mathrm{tr}(\mathbb{A}) - \det(\mathbb{A}) = 0. \tag{40}$$

We can further investigate (40) and find an exact expression for the spectrum of $D^{\mathbb{A}}$. Firstly, discuss the case when $\mathrm{tr}\,\mathbb{A} = 0$. Then the condition is simplified to the following form

$$4 - \det \mathbb{A} = 0.$$

This means that for the matrices $\mathbb{A}$ such that $\operatorname{tr} \mathbb{A} = 0$ and $\det \mathbb{A} = 4$, we have $\sigma(D^{\mathbb{A}}) = \mathbb{C}$.

Let us now discuss the case when $\operatorname{tr} \mathbb{A} \neq 0$. Then (40) can be divided by $\operatorname{tr} \mathbb{A}$ to get

$$\operatorname{sgn}(\operatorname{Im}(z)) = \frac{\det \mathbb{A} - 4}{2i \operatorname{tr} \mathbb{A}}.$$

This yields that if the following condition holds

$$\frac{\det \mathbb{A} - 4}{2i \operatorname{tr} \mathbb{A}} = \pm 1,$$

then the whole upper respectively lower complex half–plane will be in the spectrum of the operator and the other one will not. We can again demonstrate these spectral transitions by considering the following matrix

$$\mathbb{A} = \begin{pmatrix} i\alpha & 2 + \beta \\ -2 & 0 \end{pmatrix}.$$

One can see that for such matrix in the case $m = 0$, we infer

$$\sigma_p(D^{\mathbb{A}}) = \begin{cases} \mathbb{C} \setminus \mathbb{R} & \text{if } \alpha = \beta = 0, \\ \emptyset & \text{if } \alpha = 0 \wedge \beta \neq 0 \text{ or } 0 \neq \alpha \neq \pm\beta, \\ \mathbb{C}_{\pm} & \text{if } \beta = \mp\alpha \neq 0. \end{cases}$$

We will now summarize our findings.

1. $m = 0$. We have an implicit relation for eigenvalues in the form (40).
   (a) $\operatorname{tr} \mathbb{A} = 0 \wedge \det \mathbb{A} = 4 \Rightarrow \sigma_p(D^{\mathbb{A}}) = \mathbb{C} \setminus \mathbb{R}$.
   (b) $\operatorname{tr} \mathbb{A} = 0 \wedge \det \mathbb{A} \neq 4 \Rightarrow \sigma_p(D^{\mathbb{A}}) = \emptyset$.
   (c) $\operatorname{tr} \mathbb{A} \neq 0 \wedge \det \mathbb{A} - 4 = \pm 2i \operatorname{tr} \mathbb{A} \Rightarrow \sigma_p(D^{\mathbb{A}}) = \mathbb{C}_{\pm}$.
   (d) $\operatorname{tr} \mathbb{A} \neq 0 \wedge \det \mathbb{A} - 4 \neq \pm 2i \operatorname{tr} \mathbb{A} \Rightarrow \sigma_p(D^{\mathbb{A}}) = \emptyset$.
2. $m > 0$. We have implicit relation for eigenvalues in the form (34), which can be equivalently rewritten as
$$\alpha\zeta(z)^2 + \frac{i}{2}(\det \mathbb{A} - 4)\zeta(z) + \delta = 0. \tag{41}$$
   (a) $\alpha = \delta = 0 \wedge \det \mathbb{A} = 4 \Rightarrow \sigma_p(D^{\mathbb{A}}) = \mathbb{C} \setminus \mathbb{R}_{mc^2}$.
   (b) $\alpha = \delta = 0 \wedge \det \mathbb{A} \neq 4 \Rightarrow \sigma_p(D^{\mathbb{A}}) = \emptyset$.
   (c) $\alpha = 0 \wedge \delta \neq 0 \wedge \det \mathbb{A} = 4 \Rightarrow \sigma_p(D^{\mathbb{A}}) = \emptyset$.
   (d) $\alpha = 0 \wedge \delta \neq 0 \wedge \det \mathbb{A} \neq 4$ yields
$$\zeta(z) = \frac{2i\delta}{\det \mathbb{A} - 4}.$$
   We will solve the equation generally
$$\zeta(z) = \zeta \in \mathbb{C},$$

$$\frac{z + mc^2}{\sqrt{z^2 - (mc^2)^2}} = \zeta.$$

Squaring the equation, one will get

$$\frac{z^2 + 2mc^2 z + (mc^2)^2}{z^2 - (mc^2)^2} = \zeta^2,$$

$$(1 - \zeta^2)z^2 + 2mc^2 z + (1 + \zeta^2)(mc^2)^2 = 0.$$

If $\zeta^2 \neq 1$, the solutions to this quadratic equation are

$$z_1 = mc^2 \frac{\zeta^2 + 1}{\zeta^2 - 1}, z_2 = -mc^2.$$

We can immediately exclude the solution z_2, since it does not solve original equation and, in fact, lies in the essential spectrum of the operator $D^{\mathbb{A}}$. If $\zeta \in \mathbb{R}$ then z_1 also lies in the essential spectrum of the operator and we have no solution. If $\zeta \in \mathbb{C} \setminus \mathbb{R}$ we have exactly one solution z_1 in $\mathbb{C} \setminus \mathbb{R}_{mc^2}$.

In the case $\zeta^2 = 1$, we have linear equation and its solution is $z = -mc^2$ and thus we have no solution in $\mathbb{C} \setminus \mathbb{R}_{mc^2}$.

(e) $\alpha \neq 0$ yields quadratic equation for $\zeta(z)$ in the form (41) which yields two solutions. Depending on the character of these solutions using calculation in 2.(d), we have maximum two numbers z as eigenvalues.

We will formulate our findings in the following theorem.

Theorem 3.2.1. *We have*

$$\sigma(D^{\mathbb{A}}) = \sigma_p(D^{\mathbb{A}}) \cup \mathbb{R}_{mc^2}$$

and no points from $\sigma_p(D^{\mathbb{A}})$ are in $\mathbb{R}_{mc^2}$.

We have critical cases

- $m = 0 \wedge \operatorname{tr} \mathbb{A} = 0 \wedge \det \mathbb{A} = 4 \Rightarrow \sigma_p(D^{\mathbb{A}}) = \mathbb{C} \setminus \mathbb{R}$.
- $m = 0 \wedge \operatorname{tr} \mathbb{A} \neq 0 \wedge 4 - \det \mathbb{A} = \mp 2i \operatorname{tr} \mathbb{A} \Rightarrow \sigma_p(D^{\mathbb{A}}) = \mathbb{C}_{\pm}$.
- $m \neq 0 \wedge \det \mathbb{A} = 4 \wedge \alpha = \delta = 0 \Rightarrow \sigma_p(D^{\mathbb{A}}) = \mathbb{C} \setminus \mathbb{R}_{mc^2}$.

In all other cases we have at most two eigenvalues of $D^{\mathbb{A}}$.

3.3 Pseudospectrum of the relativistic point interaction

Now we would like to explain why these wild spectral transitions, mentioned in section 3.2, appeared. We will now consider only $m \neq 0$. It is clear that the spectrum of the point interaction from Definition 2.1.1 will not become denser while approaching critical transmission condition because if we consider for example matrix $\mathbb{A}_\alpha$ of the form (39) then for arbitrarily small $\alpha > 0$ the condition (34) does not hold for any $z \in \mathbb{C} \setminus \mathbb{R}_{mc^2}$. That implies that $D^{\mathbb{A}_\alpha}, \alpha > 0$ does not have any new points in its spectrum. On the other hand, if $\alpha = 0$ then this condition holds for every z and this yields that the spectrum of $D^{\mathbb{A}_0}$ is a whole complex plane.

We will explain this remarkable spectral transitions with the pseudospectrum of the operator. We will show that for arbitrarily small ε by taking τ to zero the whole complex plane will eventually fall into the ε–pseudospectrum of the operator $D^{\mathbb{A}_\tau}$ with $\mathbb{A}_\tau = \mathbb{A}_0 + \tau\mathbb{B}$, where the matrix $\mathbb{A}_0$ is the critical matrix

$$\mathbb{A}_0 = \begin{pmatrix} 0 & 2\beta \\ \frac{-2}{\beta} & 0 \end{pmatrix}$$

and $\mathbb{B}$ is any fixed non–zero complex matrix. Note that for the fixed matrix $\mathbb{B}$ matrix $\mathbb{A} + \tau\mathbb{B}$ cannot be of the critical form.

Definition 3.3.1 (ε–pseudospectrum)**.** *Let A be a linear operator and $R_A(z)$ is its resolvent at $z \in \mathbb{C}$. Then we will call the set*

$$\sigma_\varepsilon(A) = \{z \in \mathbb{C} \mid \|R_A(z)\| > \varepsilon^{-1}\}$$

ε–pseudospectrum of the operator A. Here we use a convention that $\|R_A(z)\| = +\infty$ if $z \in \sigma(A)$.

Our main goal is to prove that the norm of the resolvent of the operator $D^{\mathbb{A}_\tau}$ will go to infinity as τ tends to zero. This will prove that for any ε and any $z \in \mathbb{C} \setminus \mathbb{R}_{mc^2}$ this number z will eventually fall into ε–pseudospectrum.

Theorem 3.3.1. *For any $\varepsilon > 0$ and any number $z \in \mathbb{C} \setminus \mathbb{R}_{mc^2}$ there exists $\tau_0 > 0$ such that for every $0 < \tau < \tau_0$, $z \in \sigma_\varepsilon(D^{\mathbb{A}_\tau})$.*

Proof. Let us denote

$$\mathcal{K} = cR_z(x,0)(I + \frac{i}{2}\mathbb{A}_\tau\mathbb{Z}(z))^{-1}\mathbb{A}_\tau R_z(0,y).$$

Then from a formula for the resolvent $R_z^{\mathbb{A}_\tau}$ of the operator $D^{\mathbb{A}_\tau}$ from Theorem 2.2.2 we get

$$R_z^{\mathbb{A}_\tau} = R_z - \mathcal{K}.$$

R_z is a bounded integral operator and because it is a resolvent of the free Dirac operator we explicitly know the norm of this operator

$$\|R_z\| = \frac{1}{\operatorname{dist}(z, \sigma(D_0))}.$$

On the other hand, we will show that the norm of the operator $\mathcal{K}$ will go to infinity because the matrix $(I + \frac{i}{2}\mathbb{A}_\tau\mathbb{Z}(z))$ is approaching a singular matrix as τ tends to zero.

$$\mathcal{K} = cR_z(x,0)(I + \frac{i}{2}\mathbb{A}_\tau\mathbb{Z}(z))^{-1}\mathbb{A}_\tau R_z(0,y) = c\frac{1}{\det(I + \frac{i}{2}\mathbb{A}_\tau\mathbb{Z}(z))}R_z(x,0)\mathbb{M}_\tau\mathbb{A}_\tau R_z(0,y),$$

where $\mathbb{M}_\tau = \det(I + \frac{i}{2}\mathbb{A}_\tau\mathbb{Z}(z))(I + \frac{i}{2}\mathbb{A}_\tau\mathbb{Z}(z))^{-1} \overset{\tau\to 0}{\to} \begin{pmatrix} 1 & -i\beta\zeta(z)^{-1} \\ \frac{i\zeta(z)}{\beta} & 1 \end{pmatrix}$.

Then we can finally write the norm of the $R_z^{\mathbb{A}_\tau}$ as

$$\|R_z^{\mathbb{A}_\tau}\| = \frac{1}{|\det(I + \frac{i}{2}\mathbb{A}_\tau\mathbb{Z}(z))|}\|\det(I + \frac{i}{2}\mathbb{A}_\tau\mathbb{Z}(z))R_z - cR_z(x,0)\mathbb{M}_\tau\mathbb{A}_\tau R_z(0,y)\| \geq$$

$$\geq \frac{1}{|\det(I + \frac{i}{2}\mathbb{A}_\tau\mathbb{Z}(z))|}\|cR_z(x,0)\mathbb{M}_\tau\mathbb{A}_\tau R_z(0,y)\| - \|R_z\|.$$

Because $\det(I + \frac{i}{2}\mathbb{A}_\tau\mathbb{Z}(z)) \overset{\tau\to 0}{\to} 0$ we conclude that

$$\|R_z^{\mathbb{A}_\tau}\| \overset{\tau\to 0}{\to} +\infty,$$

which proves the theorem. ◻

3.4 Eigenvalues and eigenfunctions of the Dirac operator with the non–local potential

Question of a stability of the spectrum of the perturbation for self–adjoint operators is discussed for example in [Theorems VIII.23,VIII.24 [18]]. In a self–adjoint case if we have the norm–resolvent convergence at our disposal, the spectrum of the limiting operator cannot expand nor contract. Even though a sudden contraction cannot happen also in a non–self–adjoint case, the same is not true for a sudden expansion as discussed in a paragraph after [Theorems VIII.23,VIII.24, [18]] and in [Section IV., §3, 2nd subsection, [15]].

For this reason, we will discuss the spectral problem of the approximations and see if we can prove a stronger statement. One can try to find an implicit relation for eigenvalues of the operator D_0 with the non–local potential (3) similar to the condition for the spectrum of the limiting operator (34) and see how this condition behaves in the limit.

Recall the definition of the non–local approximation $D_\varepsilon^{\mathbb{A}}$ from (3)

$$D_\varepsilon^{\mathbb{A}} = D_0 + |v_\varepsilon\rangle\langle v_\varepsilon| \otimes c\mathbb{A}.$$

Theorem 3.4.1. *A point $z \in \mathbb{C} \setminus \mathbb{R}_{mc^2}$ is an eigenvalue of $D_\varepsilon^{\mathbb{A}}$ if and only if*

$$\det(I + c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle) = 0.$$

In the positive case, the corresponding eigenfunctions is of the form

$$\psi = cR_z\vec{x}v_\varepsilon,$$

where $\vec{x} \in \operatorname{Ker}(I+c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle)$. The geometric multiplicity of such point equals $\dim(\operatorname{Ker}(I+c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle))$.

Proof. Starting with the eigenequation we obtain

$$D_\varepsilon^{\mathbb{A}}\psi = z\psi,\ \psi \in \operatorname{Dom} D_\varepsilon^{\mathbb{A}} = \operatorname{Dom} D_0 = W^{1,2}(\mathbb{R};\mathbb{C}^2),$$

$$-ic\frac{\mathrm{d}}{\mathrm{d}x}\sigma_1\psi + mc^2\sigma_3\psi + c\mathbb{A}\langle v_\varepsilon|\psi\rangle v_\varepsilon = z\psi,$$

$$(D_0 - z)\psi = -c\mathbb{A}\langle v_\varepsilon|\psi\rangle v_\varepsilon \in L^2(\mathbb{R};\mathbb{C}^2). \tag{42}$$

For $z \in \mathbb{C} \setminus \mathbb{R}_{mc^2}$ the equation (42) is equivalent to the following

$$\exists\psi \in \operatorname{Dom}(D_0) \setminus \{0\}, \psi = -cR_z\mathbb{A}\langle v_\varepsilon|\psi\rangle v_\varepsilon. \tag{43}$$

This implies that for a certain vector $\vec{a} \in \mathbb{C}^2$ a function ψ is in the following form

$$\psi = cR_z\mathbb{A}\vec{a}v_\varepsilon. \tag{44}$$

If we substitute (44) into the equation (43) we get another equivalent expression of (42)

$$(\exists\vec{a} \in \mathbb{C}^2)(\mathbb{A}\vec{a} \neq 0 \wedge cR_z\mathbb{A}\vec{a}v_\varepsilon = -c^2R_z\mathbb{A}\langle v_\varepsilon|R_z\mathbb{A}\vec{a}v_\varepsilon\rangle v_\varepsilon), \tag{45}$$

which we can rewrite as

$$(\exists\vec{a} \in \mathbb{C}^2)(\mathbb{A}\vec{a} \neq 0 \wedge \mathbb{A}\vec{a} = -c\mathbb{A}\langle v_\varepsilon|R_z v_\varepsilon\rangle\mathbb{A}\vec{a}). \tag{46}$$

Finally, we get that (46) holds if and only if

$$(\exists\vec{a} \in \mathbb{C}^2)(\mathbb{A}\vec{a} \neq 0 \wedge (I + c\mathbb{A}\langle v_\varepsilon|R_z v_\varepsilon\rangle)\mathbb{A}\vec{a} = 0), \tag{47}$$

which is equivalent to

$$\det(I + c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle) = 0. \tag{48}$$

Firstly, we will prove the equivalence between (47) and (48) for a regular matrix $\mathbb{A}$. In this case (45) can be rewritten to the following form

$$(\exists \vec{d} \in \mathbb{C}^2, \vec{d} \neq 0)(\vec{d} = -c\langle v_\varepsilon|R_z\mathbb{A}v_\varepsilon\rangle\vec{d}).$$

That is true if and only if

$$\det(I + c\langle v_\varepsilon|R_z\mathbb{A}v_\varepsilon\rangle) = 0.$$

Due to the parity in integrals and the form of R_z, this is the exactly the same condition as (48).

For a singular matrix $\mathbb{A}$ a reverse implication from (48) to (47) still remains unproven. Let us start with $\det(I + c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle) = 0$. This implies

$$(\exists \vec{x} \in \mathbb{C}^2, \vec{x} \neq 0)((I + c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle)\vec{x} = 0).$$

We need to prove that there exists some vector $\vec{d} \in \mathbb{C}^2$ such that $\mathbb{A}\vec{d} = \vec{x}$.

$$\begin{aligned}
(I + c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle)\vec{x} &= 0,\\
\vec{x} + c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle\vec{x} &= 0,\\
\vec{x} + c\mathbb{A}\langle v_\varepsilon|R_z v_\varepsilon\rangle\vec{x} &= 0,\\
\vec{x} = \mathbb{A}\,&\underbrace{(-c\langle v_\varepsilon|R_z v_\varepsilon\rangle\vec{x})}_{:=\vec{d}},
\end{aligned}$$

which means that (47) and (48) are indeed equivalent for any matrix $\mathbb{A}$. Also from the previous calculation it is clear that for every $\vec{x} \in \operatorname{Ker}(I + c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle)$ there exists vector $\vec{d}$ such that

$$\vec{x} = \mathbb{A}\vec{d},$$

which, along with (44), gives us the sought form of the eigenfunction of the operator. □

Using similar calculations as in Lemma 2.2.1 we can show that the matrix $c\langle v_\varepsilon|R_z\mathbb{A}v_\varepsilon\rangle$ converges to the matrix $i/2\ \mathbb{Z}(z)\mathbb{A}$. From continuity of the determinant we have the limit of the condition (48)

$$\det(I + c\langle v_\varepsilon|R_z\mathbb{A}v_\varepsilon\rangle) \overset{\varepsilon\to 0}{\to} \det(I + \frac{i}{2}\mathbb{Z}(z)\mathbb{A}),$$

which is exactly the same result as we got in the previous section for the condition for the point spectrum of $D^{\mathbb{A}}$. This result gives us at most asymptotic behaviour of eigenvalues of the approximations nearby the eigenvalues of the limiting operator.

In the following text, we will prove even stronger spectral relation between the approximation and the point interactions then ones we get from the general results. Particularly, we will show that the spectrum of the limiting operator cannot expand also in the non–self–adjoint setting except the critical cases discussed in Section 3.2. Let us denote, for $z \in \mathbb{C} \setminus \mathbb{R}_{mc^2}$, functions

$$\begin{aligned}
\Gamma(z) &:= \det\left(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z)\right),\\
\Gamma_\varepsilon(z) &:= \det(I + c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle).
\end{aligned}$$

Roots of these functions characterize the eigenvalues of the operators $D^{\mathbb{A}}$ and $D^{\mathbb{A}}_\varepsilon$, respectively.

Lemma 3.4.1. *Functions $\Gamma(z)$ and $\Gamma_\varepsilon(z)$ are analytic in $z \in \mathbb{C} \setminus \mathbb{R}_{mc^2}$.*

Proof. Because both functions are smooth compositions of $k(z)$,$\zeta(z)$ and $\zeta(z)^{-1}$, it is sufficient to prove that the latter functions are analytic.

Since function $w \to \sqrt{w}$ is analytic in $\mathbb{C} \setminus [0, +\infty)$, we get that the function

$$ck(z) = \sqrt{z^2 - (mc^2)^2}$$

is also analytic in $\mathbb{C} \setminus \mathbb{R}_{mc^2}$. Finally, we have

$$\zeta(z) = \frac{z + mc^2}{ck(z)},$$

which implies analyticity of $\zeta(z)$ and $\zeta(z)^{-1}$ in $\mathbb{C} \setminus \mathbb{R}_{mc^2}$. □

Theorem 3.4.2. *Operator $D_\varepsilon^{\mathbb{A}}$ has at most countable many eigenvalues in $\mathbb{C} \setminus \mathbb{R}_{mc^2}$. The set of all eigenvalues can have accumulation points only at infinity or in $\mathbb{R}_{mc^2}$. In addition, for $z \in \sigma_p(D_\varepsilon^{\mathbb{A}})$,*

$$|\mathrm{Im} z| \leq \frac{c}{\varepsilon}\|\mathbb{A}\|\|v\|_{L^2}^2.$$

Proof. We can see that

$$\Gamma_\varepsilon(z) = \det(I + c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle) = \det(I + E_\varepsilon \mathbb{A}\mathbb{Z}(z)),$$

where $E_\varepsilon = \frac{i}{2}\int_{\mathbb{R}^2} v_\varepsilon(x)\mathrm{e}^{ik(z)|x-y|}v_\varepsilon(y)\,\mathrm{d}y\,\mathrm{d}x$. If $\Gamma_\varepsilon(z)$ was constant then it would be necessarily equal to 1. We can see that from the limit

$$\lim_{n\to+\infty} \Gamma_\varepsilon(\pm in) = 1.$$

But then there is no point spectrum of the operator $D_\varepsilon^{\mathbb{A}}$.

If the function $\Gamma_\varepsilon(z)$ is not constant then from Lemma 3.4.1 and from the identity theorem for analytic functions it immediately follows that there are at most countable many roots of $\Gamma_\varepsilon(z)$.

Last bit of the theorem comes from the article [7] on non–symmetric perturbations. Since our perturbation of the operator D_0 is

$$cW_\varepsilon \otimes \mathbb{A} = c|v_\varepsilon\rangle\langle v_\varepsilon| \otimes \mathbb{A},$$

which is tensor product of the rank one Hilbert–Schmidt operator $|v_\varepsilon\rangle\langle v_\varepsilon|$ and $\mathbb{A}$ we get the norm of the perturbation

$$\|cW_\varepsilon \otimes \mathbb{A}\| = c\|\mathbb{A}\|\|W_\varepsilon\|_2 = \frac{c}{\varepsilon}\|\mathbb{A}\|\|v\|_{L^2}^2.$$

The assertion then follows from the [Theorem 2.1, [7]]. □

Theorem 3.4.3. *Let $z_0 \in \sigma_p(D^{\mathbb{A}})$ and B be a ball such that z_0 is only eigenvalue of $D^{\mathbb{A}}$ in $\overline{B}$ and $\overline{B} \cap \mathbb{R}_{mc^2} = \emptyset$. Then for $\varepsilon > 0$ sufficiently small, there exists at least one eigenvalue of the operator $D_\varepsilon^{\mathbb{A}}$ inside the ball B.*

Proof. We will begin with the estimation of the difference of $\Gamma(z)$ and $\Gamma_\varepsilon(z)$.

$$|\Gamma(z) - \Gamma_\varepsilon(z)| = \left|\left(\frac{i}{2} - E_\varepsilon\right)\mathrm{tr}\,\mathbb{A}\mathbb{Z}(z) + \left(-\frac{1}{4} - E_\varepsilon^2\right)\det\mathbb{A}\right| \leq \frac{1}{2}|i - 2E_\varepsilon|\left(\frac{1}{2}(1 + 2|E_\varepsilon|)\det\mathbb{A} + |\,\mathrm{tr}\,\mathbb{A}\mathbb{Z}(z)|\right). \tag{49}$$

Since ∂B is a compact set, $k(\partial B)$ and $\operatorname{tr} \mathbb{AZ}(\partial B)$ are also compact by continuity. This implies

$$\sup_{z\in\partial B} |\operatorname{tr} \mathbb{AZ}(z)| < +\infty.$$

If we take into account continuity of $E_\varepsilon(z)$ introduced in (19) together with the fact that $E_\varepsilon \xrightarrow{\varepsilon\to 0} i/2$ we infer that

$$\lim_{\varepsilon\to 0} \sup_{z\in\partial B} |i - 2E_\varepsilon| = 0.$$

Putting this in top of (49) we arrive at

$$\lim_{\varepsilon\to 0} \sup_{z\in\partial B} |\Gamma(z) - \Gamma_\varepsilon(z)| = 0. \tag{50}$$

Since $\forall z \in \partial B, \Gamma(z) \neq 0$ where ∂B is a compact set we have

$$\inf_{\partial B} |\Gamma(z)| > 0.$$

If we use this result and (50) we conclude that

$$(\exists\delta > 0)(\forall\varepsilon < \delta)(\forall z \in \partial B)(|\Gamma(z) - \Gamma_\varepsilon(z)| < |\Gamma(z)|).$$

From the Rouché theorem it follows that there are same number of roots of both functions $\Gamma(z)$ and $\Gamma_\varepsilon(z)$ in the ball B. □

4 Non–relativistic limit

In this section we will consider $m > 0$ and a function $v \in L^2(\mathbb{R};\mathbb{R}) \cap L^1(\mathbb{R};\mathbb{R})$ such that its derivative v' also lies in $L^2(\mathbb{R};\mathbb{R}) \cap L^1(\mathbb{R};\mathbb{R})$. Let us also denote

$$\mu(z) = \sqrt{2mz}, \quad \mathrm{Im}\mu(z) \geq 0.$$

and

$$\tilde{\mathbb{Z}}(z) = \begin{pmatrix} \mu(z)^{-1} & 0 \\ 0 & \mu(z) \end{pmatrix}.$$

Let us also recall the free Schrödinger operator defined as

$$H_0\psi(x) = -\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x), \forall x \in \mathbb{R},$$

$$\mathrm{Dom}\, H_0 = W^{2,2}(\mathbb{R}).$$

Spectrum of this operator contains

$$\sigma(H_0) = \sigma_{ess}(H_0) = [0, +\infty).$$

Its resolvent is the integral operator K_z with an integral kernel

$$K_z(x, y) = \frac{im}{\mu(z)}\mathrm{e}^{i\mu(z)|x-y|}. \tag{51}$$

4.1 Point interactions

A well known result for the Dirac operator tells us that by performing the so called non–relativistic limit, i.e., firstly subtracting rest energy mc^2 from the operator and then sending the speed of light c to $+\infty$, of the Dirac operator one will end up with the Schrödinger operator [Corollary 6.2. [23]]. For this reason, to study the non–relativistic limit of our model of general relativistic point interactions, it is generally a good idea to start with the definition of non–relativistic point interactions. There are dozens of articles dealing with the non–relativistic model of point interactions. If we are interested also in the non–self–adjoint case, the list of the related papers is considerably shorter [2, 10, 19, 20]. Also for a more general view at the problem, the article from Hussein, Krejčiřík and Siegl [14] on non–self–adjoint graphs is recommended.

We will define general non–relativistic point interactions in a similar way as in [10] or [2]. Let us start with the formal expression for the Schrödinger operator with the point interaction

$$H^{\mathbb{A}} = -\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\alpha}{2m}|\delta(x)\rangle\langle\delta(x)| + \frac{i\beta}{2m}|\delta(x)\rangle\langle\delta'(x)| - \frac{i\gamma}{2m}|\delta'(x)\rangle\langle\delta(x)| + \frac{\delta}{2m}|\delta'(x)\rangle\langle\delta'(x)|. \tag{52}$$

Now similarly as in the Dirac case the terms $\langle\delta(x)|\psi\rangle$ and $\langle\delta'(x)|\psi\rangle$ do not make sense since we assume that ψ and ψ' are discontinuous at the point of interaction. Because of this we must extend the definition of the distributions as follows

$$\langle\delta(x)|\psi\rangle = \frac{\psi(0+) + \psi(0-)}{2},$$

$$\langle\delta'(x)|\psi\rangle = -\frac{\psi'(0+) + \psi'(0-)}{2}.$$

Now, to get a well defined operator, the singular parts of $H^{\mathbb{A}}\psi$ must cancel out for every $\psi \in \operatorname{Dom} H^{\mathbb{A}}$. From this transmission conditions will follow. Firstly, let us take $f \in \mathcal{D}(\mathbb{R})$ and consider the following

$$\forall\psi \operatorname{Dom} H^{\mathbb{A}}, \langle\psi''(x)|f(x)\rangle = \langle\psi(x)|f''(x)\rangle =$$
$$= \langle(\psi(0+) - \psi(0-)\delta'(x)|f(x)\rangle + \langle(\psi'(0+) - \psi'(0-))\delta(x)|f(x)\rangle + \langle\{\psi''(x)\}|f\rangle.$$

Therefore, the singular parts of $H^{\mathbb{A}}$ will cancel out if and only if

$$-(\psi(0+) - \psi(0-))\delta'(x) - (\psi'(0+) - \psi'(0-))\delta(x)+$$
$$+\alpha\frac{\psi(0+) + \psi(0-)}{2}\delta(x) - i\beta\frac{\psi'(0+) + \psi'(0-)}{2}\delta(x) - i\gamma\frac{\psi(0+) + \psi(0-)}{2}\delta'(x) - \delta\frac{\psi'(0+) + \psi'(0-)}{2}\delta'(x) = 0.$$

From this condition we will derive the following definition of the general point interaction.

Definition 4.1.1 (General non–relativistic point interactions)**.** *Let $\mathbb{A}$ be any 2×2 complex matrix. Then we will call the operator $H^{\mathbb{A}}$, define as follows, the operator of the point interaction.*

$$H^{\mathbb{A}}\psi(x) = -\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x),\ \ \forall x \in \mathbb{R} \setminus \{0\},$$

$$\operatorname{Dom} H^{\mathbb{A}} = \{\varphi \in W^{2,2}(\mathbb{R} \setminus \{0\}) \mid \Gamma_1\varphi = \mathbb{V}\mathbb{A}\mathbb{V}^*\Gamma_0\varphi\},$$

where

$$\Gamma_0\varphi = \frac{1}{2}\begin{pmatrix} \varphi(0+) + \varphi(0-) \\ -\varphi'(0+) - \varphi'(0-) \end{pmatrix},\ \Gamma_1\varphi = \begin{pmatrix} \varphi'(0+) - \varphi'(0-) \\ \varphi(0+) - \varphi(0-) \end{pmatrix},$$

and

$$\mathbb{V} = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}.$$

For further investigation of the operator $H^{\mathbb{A}}$, it is convenient to look at its eigenequation. The following theorem was already proven in [10]. We rewrite the theorem and its short proof in our notation.

Theorem 4.1.1. *We have $\sigma_p(H^{\mathbb{A}}) \cap [0, +\infty) = \emptyset$ and $z \in \mathbb{C} \setminus [0, +\infty)$ is an eigenvalue of $H^{\mathbb{A}}$ if and only if*

$$\det(2I + i\mathbb{A}\tilde{\mathbb{Z}}(z)) = 4 - \det\mathbb{A} + 2i\left(\frac{\alpha}{\mu(z)} + \mu(z)\delta\right) = 0.$$

In the positive case, the corresponding eigenvalue z has geometric multiplicity equal to

$$\dim\ker(2I + i\mathbb{A}\tilde{\mathbb{Z}}(z)).$$

If $(C_1, C_2) \in \ker(2I + i\mathbb{A}\tilde{\mathbb{Z}}(z)) \setminus \{0\}$ then the associated eigenfunctions are of the form

$$\psi(x) = \begin{cases} C_1\mathrm{e}^{i\mu(z)x}, & x > 0, \\ C_2\mathrm{e}^{-i\mu(z)x}, & x < 0. \end{cases}$$

Proof. Let us start with the eigenequation, $\forall z \in \mathbb{C} \setminus [0, +\infty), x \neq 0$

$$-\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) = z\psi(x),$$
$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) = -2mz\psi(x).$$

One must consider that the corresponding eigenfunction will have discontinuity at the point of interaction. For that reason we will solve the differential equation on $(0,+\infty)$ and $(-\infty,0)$ separately and then we will merge these two solution via the transmission condition from Definition 4.1.1. If we want our solutions to lie in L^2 space, it needs to be in the following form

$$\psi(x) = C_1 \mathrm{e}^{i\mu(z)x},\ x > 0,$$
$$\psi(x) = C_2 \mathrm{e}^{-i\mu(z)x},\ x < 0.$$

From that, we obtain

$$\psi(0+) = C_1,$$
$$\psi(0-) = C_2,$$
$$\psi'(0+) = i\mu(z)C_1,$$
$$\psi'(0-) = -i\mu(z)C_2.$$

Substituting these results into the transmission condition from Definition 4.1.1 we will get

$$\begin{pmatrix} i\mu(z)(C_1+C_2) \\ C_1 - C_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \alpha(C_1+C_2) + \mu(z)\beta(C_1-C_2) \\ -i\gamma(C_1+C_2) - i\mu(z)\delta(C_1-C_2) \end{pmatrix}. \tag{53}$$

Rearranging the equation (53), one infers that

$$\begin{pmatrix} \alpha + \mu(z)\beta - 2i\mu(z) & \alpha - \mu(z)\beta - 2i\mu(z) \\ -i\gamma - i\mu(z)\delta - 2 & -i\gamma + i\mu(z)\delta + 2 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \vec{0}.$$

From that we conclude that the non–trivial solution to the eigenequation will arise if and only if

$$0 = \begin{vmatrix} \alpha + \mu(z)\beta - 2i\mu(z) & \alpha - \mu(z)\beta - 2i\mu(z) \\ -i\gamma - i\mu(z)\delta - 2 & -i\gamma + i\mu(z)\delta + 2 \end{vmatrix} = 4\alpha + 4\mu(z)^2\delta + 2i\mu(z)\det\mathbb{A} - 8i\mu(z)$$

and by dividing the equation by $-2i\mu(z)$ we will equivalently get

$$4 - \det\mathbb{A} + 2i\left(\frac{\alpha}{\mu(z)} + \mu(z)\delta\right) = 0.$$

$\square$

4.2 Non–relativistic limit of relativistic point interactions

We already found the resolvent of the operator $D^{\mathbb{A}}$ in Section 2. To find the non–relativistic limit we will mimic the procedure which can be found in [3]. Firstly, we will subtract rest energy mc^2 from the operator $D^{\mathbb{A}}$. Note that

$$\sigma(D^{\mathbb{A}} - mc^2) = \{(-\infty, -2mc^2] \cup [0,+\infty)\}.$$

Secondly, the elements of the matrix $\mathbb{A}$ must be rescaled in a following way

$$\mathbb{A} \mapsto \mathbb{A}_c = \begin{pmatrix} \frac{1}{2mc}\alpha & \beta \\ \gamma & 2mc\delta \end{pmatrix}. \tag{54}$$

Subsequently, by performing the limit $c \to +\infty$ for the resolvent in the operator norm we will finally get the corresponding non–relativistic limit. One should note that by rescaling the matrix $\mathbb{A}$ in the presented way we did not change determinant of the matrix.

Theorem 4.2.1. *Let $z \in \mathbb{C} \setminus [0,+\infty)$ and $\mathbb{A}$ obeys* (54) *together with*

$$4 - \det \mathbb{A} + 2i\frac{1}{\mu(z)}\alpha + 2i\mu(z)\delta \neq 0.$$

Then the resolvent $R^{\mathbb{A}_c}_{z+mc^2}$ of the operator $D^{\mathbb{A}_c}$ converge in the operator norm, as $c \to +\infty$, to the bounded integral operator $K^{\mathbb{A}}_z$ multiplied by the projection on the upper component of the Dirac wavefunction

$$R^{\mathbb{A}_c}_{z+mc^2} \xrightarrow[c\to+\infty]{u} K^{\mathbb{A}}_z \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$K^{\mathbb{A}}_z(x,y) := \frac{im}{\mu(z)}e^{i\mu(z)|x-y|} + \frac{1}{4 - \det\mathbb{A} + 2i\frac{1}{\mu(z)}\alpha + 2i\mu(z)\delta}\cdot$$
$$\cdot\left(\frac{2m}{\mu(z)^2}\alpha + \frac{im}{\mu(z)}\det\mathbb{A} + \operatorname{sgn}(x)\frac{2m}{\mu(z)}\gamma - \operatorname{sgn}(y)\frac{2m}{\mu(z)}\beta - \operatorname{sgn}(x)\operatorname{sgn}(y)(2m\delta + \frac{im}{\mu(z)}\det\mathbb{A})\right)e^{i\mu(z)(|x|+|y|)}. \tag{55}$$

Proof. Let us recall the resolvent formula of the operator $D^{\mathbb{A}}$.

$$R^{\mathbb{A}_c}_z(x,y) = (D^{\mathbb{A}_c} - z)^{-1}(x,y) =$$
$$= R_z(x,y) - cR_z(x,0)\left(I + \frac{i}{2}\mathbb{A}_c\mathbb{Z}(z)\right)^{-1}\mathbb{A}_cR_z(0,y) = R_z(x,y) - \mathcal{K}(x,y)$$

Then by subtracting mc^2 from the Hamiltonian of the relativistic point interaction we get

$$(D^{\mathbb{A}_c} - z - mc^2)^{-1} = (D^{\mathbb{A}_c} - \tilde{z})^{-1} =$$
$$= \frac{i}{2c}(\mathbb{Z}(\tilde{z}) + \operatorname{sgn}(x-y)\sigma_1)e^{ik(\tilde{z})|x-y|} + \frac{1}{4c}(\mathbb{Z}(\tilde{z}) + \operatorname{sgn}(x)\sigma_1)\left(I + \frac{i}{2}\mathbb{A}_c\mathbb{Z}(\tilde{z})\right)^{-1}\mathbb{A}_c(\mathbb{Z}(\tilde{z}) - \operatorname{sgn}(y)\sigma_1)e^{ik(\tilde{z})(|x|+|y|)}, \tag{56}$$

where

$$\begin{aligned} \tilde{z} &= z + mc^2, \\ \zeta(\tilde{z}) &= c\sqrt{\frac{1}{c^2} + \frac{2m}{z}}, \\ k(\tilde{z}) &= \sqrt{\left(\frac{z}{c}\right)^2 + 2mz}. \end{aligned} \tag{57}$$

We have

$$\frac{4}{\tilde{C}}\mathbb{N} := \left(I + \frac{i}{2}\mathbb{A}_c\mathbb{Z}(\tilde{z})\right)^{-1}\mathbb{A}_c = \frac{1}{1 + i\frac{\zeta(\tilde{z})}{4mc}\alpha + imc\zeta(\tilde{z})^{-1}\delta - \frac{1}{4}\det\mathbb{A}}\begin{pmatrix} \frac{1}{2mc}\alpha + \frac{i}{2}\zeta(\tilde{z})^{-1}\det\mathbb{A} & \beta \\ \gamma & 2mc\delta + \frac{i}{2}\zeta(\tilde{z})\det\mathbb{A} \end{pmatrix},$$

where

$$\tilde{C} = 4 - \det\mathbb{A} + i\frac{\zeta(\tilde{z})}{mc}\alpha + 4imc\zeta(\tilde{z})^{-1}\delta,$$

which yields

$$\mathbb{Z}(\tilde{z})\mathbb{N}\mathbb{Z}(\tilde{z}) = \begin{pmatrix} \frac{\zeta(\tilde{z})^2}{2mc}\alpha + \frac{i}{2}\zeta(\tilde{z})\det\mathbb{A} & \beta \\ \gamma & 2mc\zeta(\tilde{z})^{-2}\delta + \frac{i}{2}\zeta(\tilde{z})^{-1}\det\mathbb{A} \end{pmatrix}, \tag{58}$$

$$\mathbb{Z}(\tilde{z})\mathbb{N}\sigma_1 = \begin{pmatrix} \zeta(\tilde{z})\beta & \frac{\zeta(\tilde{z})}{2mc}\alpha + \frac{i}{2}\det\mathbb{A} \\ 2mc\zeta(\tilde{z})^{-1}\delta + \frac{i}{2}\det\mathbb{A} & \zeta(\tilde{z})^{-1}\gamma \end{pmatrix}, \tag{59}$$

$$\sigma_1\mathbb{N}\mathbb{Z}(\tilde{z}) = \begin{pmatrix} \zeta(\tilde{z})\gamma & 2mc\zeta(\tilde{z})^{-1}\delta + \frac{i}{2}\det\mathbb{A} \\ \frac{\zeta(\tilde{z})}{2mc}\alpha + \frac{i}{2}\det\mathbb{A} & \zeta(\tilde{z})^{-1}\beta \end{pmatrix}, \tag{60}$$

$$\sigma_1\mathbb{N}\sigma_1 = \begin{pmatrix} 2mc\delta + \frac{i}{2}\zeta(\tilde{z})\det\mathbb{A} & \gamma \\ \beta & \frac{1}{2mc}\alpha + \frac{i}{2}\zeta(\tilde{z})^{-1}\det\mathbb{A} \end{pmatrix}. \tag{61}$$

Using the explicit forms of matrices (58),(59),(60), and (61) one can easily take point–wise limit of (56) as $c \to +\infty$.

$$R_{\tilde{z}}^{\mathbb{A}_c} \xrightarrow{c\to+\infty} K_z^{\mathbb{A}} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (K_z - \mathcal{L}) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \left(\frac{im}{\mu(z)}\mathrm{e}^{i\mu(z)|x-y|} + U\mathrm{e}^{i\mu(z)(|x|+|y|)}\right) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{62}$$

where

$$\mathcal{L}(x,y) = -U\mathrm{e}^{i\mu(z)(|x|+|y|)},$$

$$U = \frac{1}{C}\left(\frac{\alpha}{z} + \frac{im}{\mu(z)}\det\mathbb{A} + \operatorname{sgn}(x)\frac{2m}{\mu(z)}\gamma - \operatorname{sgn}(y)\frac{2m}{\mu(z)}\beta - \operatorname{sgn}(x)\operatorname{sgn}(y)(2m\delta + \frac{im}{\mu(z)}\det\mathbb{A})\right), \tag{63}$$

$$C = 4 - \det\mathbb{A} + 2i\frac{1}{\mu(z)}\alpha + 2i\mu(z)\delta.$$

From this we have candidate on the operator limit in the form of (62). Since the first part of $R_{\tilde{z}}^{\mathbb{A}_c}$ is the resolvent $R_{\tilde{z}}$ of the free Dirac operator and the first part of the operator $K_z^{\mathbb{A}}$ is actually the resolvent K_z of the free Schrödinger operator, we know that these two parts will converge to each other in the operator norm [Corollary 6.2. [23]]. Because of that we just need to prove that

$$\mathcal{K} \xrightarrow[c\to+\infty]{u} \mathcal{L} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since both of these two operators $\mathcal{K}$ and $\mathcal{L}$ are Hilbert–Schmidt we will prove the convergence in the Hilbert–Schmidt norm which will consequently imply the convergence in the operator norm.

Let us denote matrix structure of the second part of $R_{\tilde{z}}^{\mathbb{A}_c}$ as

$$\mathbb{U}_c = \frac{1}{c\tilde{C}}(\mathbb{Z}(\tilde{z})\mathbb{N}\mathbb{Z}(\tilde{z}) + \operatorname{sgn}(x)\sigma_1\mathbb{N}\mathbb{Z}(\tilde{z}) - \operatorname{sgn}(y)\mathbb{Z}(\tilde{z})\mathbb{N}\sigma_1 - \operatorname{sgn}(x)\operatorname{sgn}(y)\sigma_1\mathbb{N}\sigma_1), \tag{64}$$

One can see that $\mathbb{U}_c$ converges, as $c \to +\infty$, to

$$\mathbb{U} := U\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

in any matrix norm and because a function $f(w) = \mathrm{e}^{i\sqrt{w}(|x|+|y|)}$, for fixed x and y, is continuos $\forall w \in \mathbb{C} \setminus \mathbb{R}$ we finally get

$$|\mathbb{U}_c\mathrm{e}^{ik(\tilde{z})(|x|+|y|)} - \mathbb{U}\mathrm{e}^{i\mu(z)(|x|+|y|)}| \le |\mathbb{U}_c - \mathbb{U}||\mathrm{e}^{i\mu(z)(|x|+|y|)}| + |\mathbb{U}_c||\mathrm{e}^{ik(\tilde{z})(|x|+|y|)} - \mathrm{e}^{i\mu(z)(|x|+|y|)}| \xrightarrow{c\to+\infty} 0.$$

To finish the prove, one must find the integrable majorant of the integrand $|\mathbb{U}_c e^{ik(\tilde{z})(|x|+|y|)} - \mathbb{U} e^{i\mu(z)(|x|+|y|)}|$ such that it will not depend on c.

$$|\mathbb{U}_c e^{ik(\tilde{z})(|x|+|y|)} - \mathbb{U} e^{i\mu(z)(|x|+|y|)}| \leq |\mathbb{U}_c||e^{i(k(\tilde{z})-\mu(z))(|x|+|y|)}||e^{i\mu(z)(|x|+|y|)}| + |\mathbb{U}||e^{i\mu(z)(|x|+|y|)}|. \tag{65}$$

Now we need to estimate $|\mathbb{U}_c|$, $|\mathbb{U}|$ and $|e^{i(k(\tilde{z})-\mu(z))(|x|+|y|)}|$. Keep in mind that the term $|e^{i\mu(z)(|x|+|y|)}|$ is already square integrable. Since $|\mathbb{U}|$ is uniformly bounded and $\mathbb{U}_c \to \mathbb{U}$ we also have

$$|\mathbb{U}_c| \leq C_1.$$

Now from a continuity of $k(\tilde{z})$ at $\tilde{z}$ it follows

$$(\exists c_1 \geq 0)(\forall c \geq c_1)(|\mathrm{Im}(k(\tilde{z}) - \mu(z))(|x| + |y|)| \leq \frac{\mathrm{Im}(\mu(z))(|x| + |y|)}{2}).$$

Putting this together with (65), we infer that there exists $c_1 \geq 0$ such that $\forall c \geq c_1$,

$$\begin{aligned} |\mathbb{U}_c e^{ik(\tilde{z})(|x|+|y|)} - \mathbb{U} e^{i\mu(z)(|x|+|y|)}| &\leq C_1 e^{-\mathrm{Im}(k(\tilde{z})-\mu(z))(|x|+|y|)} e^{-\mathrm{Im}(\mu(z))(|x|+|y|)} + C_2 e^{-\mathrm{Im}(\mu(z))(|x|+|y|)} \leq \\ &\leq C_1 e^{-\frac{\mathrm{Im}(\mu(z))}{2}(|x|+|y|)} + C_2 e^{-\mathrm{Im}(\mu(z))2(|x|+|y|)}. \end{aligned} \tag{66}$$

This concludes the proof. □

It was already proven in [19] that the operator $K_z^{\mathbb{A}}$, which we got as the non–relativistic limit of the resolvent of the Hamiltonian of a general relativistic point interaction, is indeed the resolvent of $H^{\mathbb{A}}$ from Definition 4.1.1. The proof in the article is rather formal and also misses details of a calculation of $(H^{\mathbb{A}} - z)^{-1}$, which is long but straightforward. For that reason, we decided to present a full proof here.

Theorem 4.2.2. *Let $\mathbb{A}$ be any 2×2 complex matrix and $z \in \mathbb{C} \setminus [0, +\infty)$ such that*

$$4 - \det \mathbb{A} + 2i\frac{1}{\mu(z)}\alpha + 2i\mu(z)\delta \neq 0.$$

Then the operator $K_z^{\mathbb{A}}$ defined in (55) *is the resolvent of the operator of the general point interaction introduced in Definition 4.1.1.*

Proof. Firstly, we can prove that the operator $K_z^{\mathbb{A}}$ is the left inverse of the operator $(H^{\mathbb{A}} - z)$ explicitly by calculating $K_z^{\mathbb{A}}(H^{\mathbb{A}} - z)\psi$ for every $\psi \in \mathrm{Dom}\, H^{\mathbb{A}}$.

Let us now denote few constants to simplify the calculation.

$$C := 4 - \det \mathbb{A} + 2i\frac{1}{\mu(z)}\alpha + 2i\mu(z)\delta,$$

$$\tilde{\alpha} := \frac{1}{C}\left(\frac{\alpha}{z} + \frac{im}{\mu(z)}\det \mathbb{A}\right),$$

$$\tilde{\beta} := \frac{1}{C}\frac{2m\beta}{\mu(z)},$$

$$\tilde{\gamma} := \frac{1}{C}\frac{2m\gamma}{\mu(z)} \text{ and}$$

$$\tilde{\delta} := \frac{1}{C}\left(2m\delta + \frac{im}{\mu(z)}\det \mathbb{A}\right).$$

Let us consider $x > 0$. Then by using integration by parts we get, for the first part of (62), following expression

$$\int_{\mathbb{R}} \frac{im}{\mu(z)} \mathrm{e}^{i\mu(z)|x-y|} \left(-\frac{1}{2m}\psi''(y)\right) \mathrm{d}y =$$

$$= \frac{-i}{2\mu(z)} \left(\int_x^{+\infty} \mathrm{e}^{i\mu(z)|x-y|}\psi''(y)\,\mathrm{d}y + \int_0^x \mathrm{e}^{i\mu(z)|x-y|}\psi''(y)\,\mathrm{d}y + \int_{-\infty}^0 \mathrm{e}^{i\mu(z)|x-y|}\psi''(y)\,\mathrm{d}y \right) =$$

$$= \frac{-i}{2}\Bigg(\left[\tfrac{1}{\mu(z)}\mathrm{e}^{i\mu(z)|x-y|}\psi'(y)\right]_{y\to x}^{y\to+\infty} + \left[\tfrac{1}{\mu(z)}\mathrm{e}^{i\mu(z)|x-y|}\psi'(y)\right]_{y\to 0}^{y\to x} + \left[\tfrac{1}{\mu(z)}\mathrm{e}^{i\mu(z)|x-y|}\psi'(y)\right]_{y\to-\infty}^{y\to 0} -$$

$$- \int_x^{+\infty} i\mathrm{e}^{i\mu(z)|x-y|}\psi'(y)\,\mathrm{d}y + \int_0^x i\mathrm{e}^{i\mu(z)|x-y|}\psi'(y)\,\mathrm{d}y + \int_{-\infty}^0 i\mathrm{e}^{i\mu(z)|x-y|}\psi'(y)\,\mathrm{d}y \Bigg) =$$

$$= \frac{-i}{2}\Bigg(\tfrac{1}{\mu(z)}\mathrm{e}^{i\mu(z)|x|}(\psi'(0-)-\psi'(0+)) - [i\mathrm{e}^{i\mu(z)|x-y|}\psi(y)]_{y\to x}^{y\to+\infty} + [i\mathrm{e}^{i\mu(z)|x-y|}\psi(y)]_{y\to 0}^{y\to x} + [i\mathrm{e}^{i\mu(z)|x-y|}\psi(y)]_{y\to-\infty}^{y\to 0} -$$

$$- \int_{\mathbb{R}} \mu(z)\mathrm{e}^{i\mu(z)|x-y|}\psi(y)\,\mathrm{d}y \Bigg) =$$

$$= \psi(x) + \frac{i}{2\mu(z)}\mathrm{e}^{i\mu(z)|x|}(\psi'(0+)-\psi'(0-)) - \frac{1}{2}\mathrm{e}^{i\mu(z)|x|}(\psi(0+)-\psi(0-)) + \frac{i\mu(z)}{2}\int_{\mathbb{R}} \mathrm{e}^{i\mu(z)|x-y|}\psi(y)\,\mathrm{d}y.$$

Similarly for $x < 0$ we get

$$\int_{\mathbb{R}} \frac{im}{\mu(z)} \mathrm{e}^{i\mu(z)|x-y|} \left(-\frac{1}{2m}\psi''(y)\right) \mathrm{d}y =$$

$$= \psi(x) + \frac{i}{2\mu(z)}\mathrm{e}^{i\mu(z)|x|}(\psi'(0+)-\psi'(0-)) + \frac{1}{2}\mathrm{e}^{i\mu(z)|x|}(\psi(0+)-\psi(0-)) + \frac{i\mu(z)}{2}\int_{\mathbb{R}} \mathrm{e}^{i\mu(z)|x-y|}\psi(y)\,\mathrm{d}y.$$

In other words we get $\forall x \in \mathbb{R}\setminus\{0\}$

$$\frac{im}{\mu(z)}\mathrm{e}^{i\mu(z)|x-y|}(H^{\mathbb{A}}-z)\psi(x) = \psi(x) + \frac{i}{2\mu(z)}\mathrm{e}^{i\mu(z)|x|}(\psi'(0+)-\psi'(0-)) - \operatorname{sgn}(x)\frac{1}{2}\mathrm{e}^{i\mu(z)|x|}(\psi(0+)-\psi(0-)). \quad (67)$$

Now, we need to look at the second part of the operator $K_z^{\mathbb{A}}$ (55). Let us denote

$$\mathcal{L}(x,y) := (\tilde{\alpha} + \tilde{\gamma}\operatorname{sgn}(x) - \tilde{\beta}\operatorname{sgn}(y) - \tilde{\delta}\operatorname{sgn}(x)\operatorname{sgn}(y))\mathrm{e}^{i\mu(z)(|x|+|y|)}.$$

Then we get

$$\int_{\mathbb{R}} \mathcal{L}(x,y)\left(-\frac{1}{2m}\psi''(y)\right)\mathrm{d}y = -\frac{1}{2m}\left(\int_0^{+\infty} \mathcal{L}(x,y)\psi''(y)\,\mathrm{d}y + \int_{-\infty}^0 \mathcal{L}(x,y)\psi''(y)\,\mathrm{d}y\right) =$$

$$= -\frac{1}{2m}\Bigg([(\tilde{\alpha}-\tilde{\beta}+(\tilde{\gamma}-\tilde{\delta})\operatorname{sgn}(x))\mathrm{e}^{i\mu(z)(|x|+|y|)}\psi'(y)]_{y\to 0}^{y\to+\infty} + [(\tilde{\alpha}+\tilde{\beta}+(\tilde{\gamma}+\tilde{\delta})\operatorname{sgn}(x))\mathrm{e}^{i\mu(z)(|x|+|y|)}\psi'(y)]_{y\to-\infty}^{y\to 0} -$$

$$- \int_0^{+\infty} i\mu(z)(\tilde{\alpha}-\tilde{\beta}+(\tilde{\gamma}-\tilde{\delta})\operatorname{sgn}(x))\mathrm{e}^{i\mu(z)(|x|+|y|)}\psi'(y)\,\mathrm{d}y + \int_{-\infty}^0 i\mu(z)(\tilde{\alpha}+\tilde{\beta}+(\tilde{\gamma}+\tilde{\delta})\operatorname{sgn}(x))\mathrm{e}^{i\mu(z)(|x|+|y|)}\psi'(y)\,\mathrm{d}y \Bigg) =$$

$$= -\frac{1}{2m}\Bigg(-(\tilde{\alpha}-\tilde{\beta}+(\tilde{\gamma}-\tilde{\delta})\operatorname{sgn}(x))\mathrm{e}^{i\mu(z)|x|}\psi'(0+) + (\tilde{\alpha}+\tilde{\beta}+(\tilde{\gamma}+\tilde{\delta}\operatorname{sgn}(x))\mathrm{e}^{i\mu(z)|x|}\psi'(0-) +$$

$$+ i\mu(z)(\tilde{\alpha}-\tilde{\beta}+(\tilde{\gamma}-\tilde{\delta})\operatorname{sgn}(x))\mathrm{e}^{i\mu(z)|x|}\psi(0+) + i\mu(z)(\tilde{\alpha}+\tilde{\beta}+(\tilde{\gamma}+\tilde{\delta})\operatorname{sgn}(x))\mathrm{e}^{i\mu(z)|x|}\psi(0-) -$$

$$- \int_{\mathbb{R}} \mu(z)^2(\tilde{\alpha}-\tilde{\beta}\operatorname{sgn}(y)+\tilde{\gamma}\operatorname{sgn}(x)-\tilde{\delta}\operatorname{sgn}(x)\operatorname{sgn}(y))\mathrm{e}^{i\mu(z)(|x|+|y|)}\psi(y)\,\mathrm{d}y \Bigg)$$

Hence, by using the identity $\mu(z)^2 = 2mz$, we obtain

$$\int_{\mathbb{R}} \mathcal{L}(x,y)(H^{\mathbb{A}} - z)\psi(y)\,\mathrm{d}y =$$
$$= -\frac{1}{2m}\Big(-(\tilde{\alpha} - \tilde{\beta} + (\tilde{\gamma} - \tilde{\delta})\mathrm{sgn}(x))\mathrm{e}^{i\mu(z)|x|}\psi'(0+) + (\tilde{\alpha} + \tilde{\beta} + (\tilde{\gamma} + \tilde{\delta}\mathrm{sgn}(x))\mathrm{e}^{i\mu(z)|x|}\psi'(0-)+$$
$$+ i\mu(z)(\tilde{\alpha} - \tilde{\beta} + (\tilde{\gamma} - \tilde{\delta})\mathrm{sgn}(x))\mathrm{e}^{i\mu(z)|x|}\psi(0+) + i\mu(z)(\tilde{\alpha} + \tilde{\beta} + (\tilde{\gamma} + \tilde{\delta})\mathrm{sgn}(x))\mathrm{e}^{i\mu(z)|x|}\psi(0-)\Big). \quad (68)$$

Now, in view of (67) and (68), $K_z^{\mathbb{A}}(H^{\mathbb{A}} - z) = Id|_{\mathrm{Dom}\,H^{\mathbb{A}}}$ if and only if

$$\frac{1}{2m}(\tilde{\alpha} + \mathrm{sgn}(x)\tilde{\gamma} - \tilde{\beta} - \mathrm{sgn}(x)\tilde{\delta})\psi'(0+) - \frac{1}{2m}(\tilde{\alpha} + \mathrm{sgn}(x)\tilde{\gamma} + \tilde{\beta} + \mathrm{sgn}(x)\tilde{\delta})\psi'(0-)-$$
$$- \frac{i\mu(z)}{2m}(\tilde{\alpha} + \mathrm{sgn}(x)\tilde{\gamma} - \tilde{\beta} - \mathrm{sgn}(x)\tilde{\delta})\psi(0+) - \frac{i\mu(z)}{2m}(\tilde{\alpha} + \mathrm{sgn}(x)\tilde{\gamma} + \tilde{\beta} + \mathrm{sgn}(x)\tilde{\delta})\psi(0-)+$$
$$+ \frac{i}{2\mu(z)}(\psi'(0+) - \psi'(0-)) - \mathrm{sgn}(x)\frac{1}{2}(\psi(0+) - \psi(0-)) = 0, \quad (69)$$

which can be rewritten, by substituting for $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ and $\tilde{\delta}$, as

$$\left(\frac{2\alpha}{\mu(z)^2} + \frac{i}{\mu(z)}\det\mathbb{A} - \frac{2\beta}{\mu(z)} + \frac{2\gamma}{\mu(z)}\mathrm{sgn}(x) - (2\delta + \frac{i}{\mu(z)}\det\mathbb{A})\mathrm{sgn}(x)\right)\psi'(0+)-$$
$$-\left(\frac{2\alpha}{\mu(z)^2} + \frac{i}{\mu(z)}\det\mathbb{A} + \frac{2\beta}{\mu(z)} + \frac{2\gamma}{\mu(z)}\mathrm{sgn}(x) + (2\delta + \frac{i}{\mu(z)}\det\mathbb{A})\mathrm{sgn}(x)\right)\psi'(0-)-$$
$$- i\mu(z)\left(\frac{2\alpha}{\mu(z)^2} + \frac{i}{\mu(z)}\det\mathbb{A} - \frac{2\beta}{\mu(z)} + \frac{2\gamma}{\mu(z)}\mathrm{sgn}(x) - (2\delta + \frac{i}{\mu(z)}\det\mathbb{A})\mathrm{sgn}(x)\right)\psi(0+)-$$
$$- i\mu(z)\left(\frac{2\alpha}{\mu(z)^2} + \frac{i}{\mu(z)}\det\mathbb{A} + \frac{2\beta}{\mu(z)} + \frac{2\gamma}{\mu(z)}\mathrm{sgn}(x) + (2\delta + \frac{i}{\mu(z)}\det\mathbb{A})\mathrm{sgn}(x)\right)\psi(0-)+$$
$$+ \frac{i}{\mu(z)}\left(4 - \det\mathbb{A} + 2i(\frac{\alpha}{\mu(z)} + \mu(z)\delta)\right)(\psi'(0+) - \psi'(0-))-$$
$$- \mathrm{sgn}(x)\left(4 - \det\mathbb{A} + 2i(\frac{\alpha}{\mu(z)} + \mu(z)\delta)\right)(\psi(0+) - \psi(0-)) = 0. \quad (70)$$

From this we can see that several terms in (70) will cancel out and for the remaining terms we get the following

$$\left(-\frac{2\beta}{\mu(z)} + \frac{2\gamma}{\mu(z)}\mathrm{sgn}(x) - (2\delta + \frac{i}{\mu(z)}\det\mathbb{A})\mathrm{sgn}(x)\right)\psi'(0+)-$$
$$-\left(\frac{2\beta}{\mu(z)} + \frac{2\gamma}{\mu(z)}\mathrm{sgn}(x) + (2\delta + \frac{i}{\mu(z)}\det\mathbb{A})\mathrm{sgn}(x)\right)\psi'(0-)-$$
$$-\left(\frac{2i\alpha}{\mu(z)} - \det\mathbb{A} - 2i\beta + 2i\gamma\mathrm{sgn}(x)\right)\psi(0+)-$$
$$-\left(\frac{2i\alpha}{\mu(z)} - \det\mathbb{A} + 2i\beta + 2i\gamma\mathrm{sgn}(x)\right)\psi(0-)+$$
$$+\left(\frac{4i}{\mu(z)} - 2\delta\right)(\psi'(0+) - \psi'(0-)) - \mathrm{sgn}(x)\left(4 + \frac{2i}{\mu(z)}\alpha\right)(\psi(0+) - \psi(0-)) = 0 \quad (71)$$

We will look at the parts of (71) with $\mu(z)^{-1}$ and the rest separately.

1.

$$(-2\beta + 2\gamma\operatorname{sgn}(x) - i\det\mathbb{A}\operatorname{sgn}(x))\psi'(0+) - (2\beta + 2\gamma\operatorname{sgn}(x) + i\det\mathbb{A}\operatorname{sgn}(x))\psi'(0-)-$$
$$- 2i\alpha(\psi(0+) + \psi(0-)) + 4i(\psi'(0+) - \psi'(0-)) - 2i\alpha\operatorname{sgn}(x)(\psi(0+) - \psi(0-)) = 0,$$

which we can rewrite as

$$- (2\beta + i\det\mathbb{A}\operatorname{sgn}(x))(\psi'(0+) + \psi'(0-)) + (4i + 2\gamma\operatorname{sgn}(x))(\psi'(0+) - \psi'(0-))-$$
$$- 2i\alpha(\psi(0+) + \psi(0-)) - 2i\alpha\operatorname{sgn}(x)(\psi(0+) - \psi(0-)) = 0. \quad (72)$$

2.

$$- 2\delta\operatorname{sgn}(x)(\psi'(0+) + \psi'(0-)) + (\det\mathbb{A} + 2i\beta - 2i\gamma\operatorname{sgn}(x))\psi(0+)+$$
$$+ (\det\mathbb{A} - 2i\beta - 2i\gamma\operatorname{sgn}(x))\psi(0-) - 2\delta(\psi'(0+) - \psi'(0-)) - \operatorname{sgn}(x)4(\psi(0+) - \psi(0-)) = 0,$$

which is equivalent to

$$- 2\delta\operatorname{sgn}(x)(\psi'(0+) + \psi'(0-)) - 2\delta(\psi'(0+) - \psi'(0-))+$$
$$+ (\det\mathbb{A} - 2i\gamma\operatorname{sgn}(x))(\psi(0+) + \psi(0-)) + (2i\beta - 4\operatorname{sgn}(x))(\psi(0+) - \psi(0-)) = 0. \quad (73)$$

Now from (72) and (73) we get the following four conditions

$$-2\beta\psi'_+ + 4i\psi'_- - 2i\alpha\psi_+ = 0, \quad (74)$$
$$-i\det\mathbb{A}\psi'_+ + 2\gamma\psi'_- - 2i\alpha\psi_- = 0, \quad (75)$$
$$-2\delta\psi'_- + \det\mathbb{A}\psi_+ + 2i\beta\psi_- = 0, \quad (76)$$
$$-2\delta\psi'_+ - 2i\gamma\psi_+ - 4\psi_- = 0, \quad (77)$$

where

$$f_+ := f(0+) + f(0-),\ f_- := f(0+) - f(0-).$$

One can check that the four conditions (74), (75), (76) and (77) are not linearly independent and, in fact, it is sufficient for the four conditions to hold that the following two conditions are satisfied

$$\psi'_- = -\frac{i\beta}{2}\psi'_+ + \frac{\alpha}{2}\psi_+,$$
$$\psi_- = -\frac{\delta}{2}\psi'_+ - \frac{i\gamma}{2}\psi_+.$$

These two conditions are our transmission conditions for the operator $H^{\mathbb{A}}$ of general point interactions. This yields

$$K_z^{\mathbb{A}}(H^{\mathbb{A}} - z)\psi(x) = \psi(x).$$

Now, we need to prove that $K_z^{\mathbb{A}}$ is also right inverse. Above all, we need to prove that $K_z^{\mathbb{A}}\psi \in \operatorname{Dom} H^{\mathbb{A}}$. Putting

$$\varphi(x) := K_z\psi(x) =$$
$$= \int_{\mathbb{R}}\left(\frac{im}{\mu(z)}e^{i\mu(z)|x-y|} + (\tilde{\alpha} + \tilde{\gamma}\operatorname{sgn}(x) - \tilde{\beta}\operatorname{sgn}(y) - \tilde{\delta}\operatorname{sgn}(x)\operatorname{sgn}(y))e^{i\mu(z)(|x|+|y|)}\right)\psi(y)\,dy,$$

we see that

$$\varphi(0+) = \int_{\mathbb{R}} \left(\frac{im}{\mu(z)} \mathrm{e}^{i\mu(z)|y|} + (\tilde{\alpha} + \tilde{\gamma} - \tilde{\beta}\mathrm{sgn}(y) - \tilde{\delta}\mathrm{sgn}(y))\mathrm{e}^{i\mu(z)|y|} \right) \psi(y)\,\mathrm{d}y \tag{78}$$

and

$$\varphi(0-) = \int_{\mathbb{R}} \left(\frac{im}{\mu(z)} \mathrm{e}^{i\mu(z)|y|} + (\tilde{\alpha} - \tilde{\gamma} - \tilde{\beta}\mathrm{sgn}(y) + \tilde{\delta}\mathrm{sgn}(y))\mathrm{e}^{i\mu(z)|y|} \right) \psi(y)\,\mathrm{d}y. \tag{79}$$

Next we gave

$$\frac{\mathrm{d}}{\mathrm{d}x}\varphi(x) = \int_{\mathbb{R}} \left(-m\mathrm{sgn}(x-y)\mathrm{e}^{i\mu(z)|x-y|} + (\tilde{\alpha} + \tilde{\gamma} - \tilde{\beta}\mathrm{sgn}(y) - \tilde{\delta}\mathrm{sgn}(y)) i\mu(z)\mathrm{e}^{i\mu(z)(|x|+|y|)} \right) \psi(y)\,\mathrm{d}y,$$

$$\varphi'(0+) = \int_{\mathbb{R}} \left(m\mathrm{sgn}(y)\mathrm{e}^{i\mu(z)|y|} + (\tilde{\alpha} + \tilde{\gamma} - \tilde{\beta}\mathrm{sgn}(y) - \tilde{\delta}\mathrm{sgn}(y)) i\mu(z)\mathrm{e}^{i\mu(z)|y|} \right) \psi(y)\,\mathrm{d}y. \tag{80}$$

for $x > 0$, and

$$\frac{\mathrm{d}}{\mathrm{d}x}\varphi(x) = \int_{\mathbb{R}} \left(-m\mathrm{sgn}(x-y)\mathrm{e}^{i\mu(z)|x-y|} - (\tilde{\alpha} - \tilde{\gamma} - \tilde{\beta}\mathrm{sgn}(y) + \tilde{\delta}\mathrm{sgn}(y)) i\mu(z)\mathrm{e}^{i\mu(z)(|x|+|y|)} \right) \psi(y)\,\mathrm{d}y,$$

$$\varphi'(0-) = \int_{\mathbb{R}} \left(m\mathrm{sgn}(y)\mathrm{e}^{i\mu(z)|y|} - (\tilde{\alpha} - \tilde{\gamma} - \tilde{\beta}\mathrm{sgn}(y) + \tilde{\delta}\mathrm{sgn}(y)) i\mu(z)\mathrm{e}^{i\mu(z)|y|} \right) \psi(y)\,\mathrm{d}y. \tag{81}$$

for $x < 0$, respectively

From (78),(79),(80), and (81) we get the following

$$\begin{aligned}
\varphi_+ &= \int_{\mathbb{R}} \left(2\frac{im}{\mu(z)} + 2(\tilde{\alpha} - \tilde{\beta}\mathrm{sgn}(y)) \right) \mathrm{e}^{i\mu(z)|y|}\psi(y)\,\mathrm{d}y, \\
\varphi_- &= \int_{\mathbb{R}} 2(\tilde{\gamma} - \tilde{\delta}\mathrm{sgn}(y))\mathrm{e}^{i\mu(z)|y|}\psi(y)\,\mathrm{d}y, \\
\varphi'_+ &= \int_{\mathbb{R}} (2\mathrm{sgn}(y)m + (2\tilde{\gamma} - 2\tilde{\delta}\mathrm{sgn}(y)) i\mu(z))\mathrm{e}^{i\mu(z)|y|}\psi(y)\,\mathrm{d}y, \\
\varphi'_- &= \int_{\mathbb{R}} 2(\tilde{\alpha} - \tilde{\beta}\mathrm{sgn}(y)) i\mu(z)\mathrm{e}^{i\mu(z)|y|}\psi(y)\,\mathrm{d}y.
\end{aligned}$$

Now, we want to check if the transmission condition holds,i.e., whether

$$\begin{aligned}
\int_{\mathbb{R}} 2(\tilde{\alpha} - \tilde{\beta}\mathrm{sgn}(y)) i\mu(z)\mathrm{e}^{i\mu(z)|y|}\psi(y)\,\mathrm{d}y = -ib \int_{\mathbb{R}} (m\mathrm{sgn}(y) + i\mu(z)(\tilde{\gamma} - \tilde{\delta}\mathrm{sgn}(y)))\mathrm{e}^{i\mu(z)|y|}\psi(y)\,\mathrm{d}y + \\
+a \int_{\mathbb{R}} \left(\frac{im}{\mu(z)} + (\tilde{\alpha} - \tilde{\delta}\mathrm{sgn}(y)) \right) \mathrm{e}^{i\mu(z)|y|}\psi(y)\,\mathrm{d}y,
\end{aligned}$$

which is true if and only if

$$2(\tilde{\alpha} - \tilde{\beta}\mathrm{sgn}(y)) i\mu(z) = -ib(m\mathrm{sgn}(y) + i\mu(z)(\tilde{\gamma} - \tilde{\delta}\mathrm{sgn}(y))) + a\left(\frac{im}{\mu(z)} + (\tilde{\alpha} - \tilde{\beta}\mathrm{sgn}(y)) \right).$$

Substituting back for $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$, and $\tilde{\delta}$, we obtain

$$\frac{2}{C}\left(\frac{2m}{\mu(z)^2}a + i\frac{m}{\mu(z)}\det\mathbb{A} - \frac{2mb}{\mu(z)}\mathrm{sgn}(y)\right)i\mu(z) = -ib\left(m\mathrm{sgn}(y) + \frac{i\mu(z)}{C}\left(\frac{2mc}{\mu(z)} - \left(2md + \frac{im}{\mu(z)}\det\mathbb{A}\right)\mathrm{sgn}(y)\right)\right) + \\ +a\left(\frac{im}{\mu(z)} + \frac{1}{C}\left(\frac{2m}{\mu(z)^2}a + \frac{im}{\mu(z)}\det\mathbb{A} - \frac{2mb}{\mu(z)}\mathrm{sgn}(y)\right)\right)$$

Now, we will try to rearrange the equation above to proof its validity. Using the definition of the constant C, we get

$$\frac{4ima}{\mu(z)} - 2m\det\mathbb{A} - 4imb\mathrm{sgn}(y) = -ib\mathrm{sgn}(y)C + b\mu(z)\left(\frac{2mc}{\mu(z)} - \left(2md + im\frac{\det\mathbb{A}}{\mu(z)}\right)\mathrm{sgn}(y)\right) + \\ +a\frac{im}{\mu(z)}C + \frac{2m}{\mu(z)^2}a^2 + im\frac{a\det\mathbb{A}}{\mu(z)} - \frac{2mab}{\mu(z)}\mathrm{sgn}(y),$$

$$\begin{aligned}\frac{4ima}{\mu(z)} - 2m\det\mathbb{A} - 4imb\mathrm{sgn}(y) = &-4imb\mathrm{sgn}(y) + imb\det\mathbb{A}\mathrm{sgn}(y) + 2bm\left(\frac{a}{\mu(z)} + \mu(z)d\right)\mathrm{sgn}(y) + \\ &+2mbc - 2mdb\mu(z)\mathrm{sgn}(y) - imb\det\mathbb{A}\mathrm{sgn}(y) + \\ &+4a\frac{im}{\mu(z)} - a\frac{im}{\mu(z)}\det\mathbb{A} - 2a\frac{m}{\mu(z)}\left(\frac{a}{\mu(z)} + \mu(z)d\right) + \frac{2ma^2}{\mu(z)^2} + \\ &+im\frac{a}{\mu(z)}\det\mathbb{A} - \frac{2mab}{\mu(z)}\mathrm{sgn}(y).\end{aligned} \tag{82}$$

Hence, we see that all terms in (82) will cancel out, and so the transmission condition holds. We can prove in a similar way that the second transmission condition holds. Since $K_z(x, y)$ is the resolvent of the free Schrödinger operator and $K_z(x, 0)$ is in $W^{2,2}(\mathbb{R} \setminus \{0\})$ we arrive at

$$\mathrm{Ran}\, K_z^{\mathbb{A}} \subset \mathrm{Dom}\, H^{\mathbb{A}}.$$

It remains to show that

$$(H^{\mathbb{A}} - z)K_z^{\mathbb{A}}\psi = \psi. \tag{83}$$

$$\left(-\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - z\right)\int_{\mathbb{R}}\left(\frac{im}{\mu(z)}\mathrm{e}^{i\mu(z)|x-y|} + (\tilde{\alpha} - \tilde{\beta}\mathrm{sgn}(y) + \tilde{\gamma}\mathrm{sgn}(x) - \tilde{\delta}\mathrm{sgn}(x)\mathrm{sgn}(y))\mathrm{e}^{i\mu(z)(|x|+|y|)}\right)\psi(y)\,\mathrm{d}y =$$

$$= -\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_{\mathbb{R}}\frac{im}{\mu(z)}\mathrm{e}^{i\mu(z)|x-y|}\psi(y)\,\mathrm{d}y- \tag{84}$$

$$-\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{e}^{i\mu(z)|x|}\int_{\mathbb{R}}(\tilde{\alpha} - \tilde{\beta}\mathrm{sgn}(y) + \mathrm{sgn}(x) - \tilde{\delta}\mathrm{sgn}(x)\mathrm{sgn}(y))\mathrm{e}^{i\mu(z)|y|}\psi(y)\,\mathrm{d}y- \tag{85}$$

$$-\frac{izm}{\mu(z)}\int_{\mathbb{R}}\mathrm{e}^{i\mu(z)|x-y|}\psi(y)\,\mathrm{d}y- \tag{86}$$

$$-z\mathrm{e}^{i\mu(z)|x|}\int_{\mathbb{R}}(\tilde{\alpha} - \tilde{\beta}\mathrm{sgn}(y) + \tilde{\gamma}\mathrm{sgn}(x) - \tilde{\delta}\mathrm{sgn}(x)\mathrm{sgn}(y))\mathrm{e}^{i\mu(z)|y|}\psi(y)\,\mathrm{d}y. \tag{87}$$

Now we will consider only $x > 0$ for $x < 0$ the approach is similar.

$$(84) = -\frac{i}{2\mu(z)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_x^{+\infty} \mathrm{e}^{i\mu(z)|x-y|}\psi(y)\,\mathrm{d}y - \frac{i}{2\mu(z)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_{-\infty}^{x} \mathrm{e}^{i\mu(z)|x-y|}\psi(y)\,\mathrm{d}y =$$
$$= -\frac{i}{2\mu(z)}\frac{\mathrm{d}}{\mathrm{d}x}\left(-\psi(x) + \psi(x) + \int_x^{+\infty} -\mu(z)\mathrm{e}^{i\mu(z)|x-y|}\psi(y)\,\mathrm{d}y + \int_{-\infty}^{x} i\mu(z)\mathrm{e}^{i\mu(z)|x-y|}\psi(y)\,\mathrm{d}y\right) =$$
$$= -\frac{i}{2\mu(z)}\left(2i\mu(z)\psi(x) + \int_x^{+\infty} (-i\mu(z))^2\mathrm{e}^{i\mu(z)|x-y|}\psi(y)\,\mathrm{d}y + \int_{-\infty}^{x} (i\mu(z))^2\mathrm{e}^{i\mu(z)|x-y|}\psi(y)\,\mathrm{d}y\right) =$$
$$= \psi(x) + \frac{i\mu(z)}{2}\int_{\mathbb{R}} \mathrm{e}^{i\mu(z)|x-y|}\psi(y)\,\mathrm{d}y$$

and
$$(85) = -\frac{(i\mu(z))^2}{2m}\mathrm{e}^{i\mu(z)|x|}\int_{\mathbb{R}} (\tilde{\alpha} - \tilde{\beta}\mathrm{sgn}(y) + \tilde{\gamma} - \tilde{\delta}\mathrm{sgn}(y))\mathrm{e}^{i\mu(z)|y|}\psi(y)\,\mathrm{d}y.$$

Since
$$\frac{\mu(z)}{2} = \frac{zm}{\mu(z)},$$
(84),(85),(86), and (87) imply (83). □

4.3 Non–relativistic limit of non–local approximations

Similarly as in the case of the relativistic point interactions, we can try to take the non–relativistic limit for its non–local approximations we introduced in Section 2.2. Doing this we will be able to get non–local approximation of general non–relativistic point interactions. Let us start with the operator

$$D_\varepsilon^{\mathbb{A}} = D_0 + c|v_\varepsilon\rangle\langle v_\varepsilon| \otimes \mathbb{A}.$$

We already found the resolvent of this operator in the form of the integral operator $R_{z,\varepsilon}^{\mathbb{A}}$ with the integral kernel given in (24)

$$R_{z,\varepsilon}^{\mathbb{A}} = R_z - cR_z(I + c\langle v_\varepsilon|\mathbb{A}R_z v_\varepsilon\rangle)^{-1}|v_\varepsilon\rangle\langle v_\varepsilon|\mathbb{A}R_z.$$

To take the non–relativistic limit for the operator $D_\varepsilon^{\mathbb{A}}$ we need to rescale matrix $\mathbb{A}$ in the same way as we did in the previous Section 4.2 and we need to take the resolvent at the point $\tilde{z} = z + mc^2$ and then send c to infinity.

$$\mathbb{A} \mapsto \mathbb{A}_c = \begin{pmatrix} \frac{1}{2mc}\alpha & \beta \\ \gamma & 2mc\delta \end{pmatrix}.$$

Then we get

$$c\langle v_\varepsilon|\mathbb{A}_c R_{\tilde{z}} v_\varepsilon\rangle = \begin{pmatrix} \frac{\alpha\zeta(\tilde{z})E_\varepsilon}{2mc} & \beta\zeta(\tilde{z})^{-1}E_\varepsilon \\ \gamma\zeta(\tilde{z})E_\varepsilon & \delta 2mc\zeta(\tilde{z})^{-1}E_\varepsilon \end{pmatrix}.$$

From this we conclude

$$(I + c\langle v_\varepsilon|\mathbb{A}_c R_{\tilde{z}} v_\varepsilon\rangle)^{-1} = \frac{1}{1 + E_\varepsilon\left(\alpha\frac{\zeta(\tilde{z})}{2mc} + \delta\frac{2mc}{\zeta(\tilde{z})}\right) + E_\varepsilon^2 \det\mathbb{A}}\begin{pmatrix} 1 + \delta 2mc\zeta(\tilde{z})^{-1}E_\varepsilon & -\beta\zeta(\tilde{z})^{-1}E_\varepsilon \\ -\gamma\zeta(\tilde{z})E_\varepsilon & 1 + \alpha\frac{\zeta(\tilde{z})E_\varepsilon}{2mc} \end{pmatrix}. \quad (88)$$

Finally, (88) yields

$$(I + c\langle v_\varepsilon | \mathbb{A}_c R_{\tilde{z}} v_\varepsilon\rangle)^{-1}\mathbb{A}_c = \frac{1}{1 + E_\varepsilon\left(\alpha\frac{\zeta(\tilde{z})}{2mc} + \delta\frac{2mc}{\zeta(\tilde{z})}\right) + E_\varepsilon^2 \det\mathbb{A}}\mathbb{M},$$

where

$$\mathbb{M} := \begin{pmatrix} \frac{\alpha}{2mc} + \zeta(\tilde{z})^{-1}E_\varepsilon \det\mathbb{A} & \beta \\ \gamma & \delta 2mc + \zeta(\tilde{z})E_\varepsilon \det\mathbb{A} \end{pmatrix}.$$

We also denote

$$C_\varepsilon := 1 + E_\varepsilon\left(\alpha\frac{\zeta(\tilde{z})}{2mc} + \delta\frac{2mc}{\zeta(\tilde{z})}\right) + E_\varepsilon^2 \det\mathbb{A}.$$

Since the matrix part of the resolvent consists of matrices σ_1 and $\mathbb{Z}(\tilde{z})$, we will look at all combinations of matrices σ_1, $\mathbb{Z}(\tilde{z})$ and $\mathbb{M}$ separately.

$$\mathbb{Z}(\tilde{z})\mathbb{M}\mathbb{Z}(\tilde{z}) = \begin{pmatrix} \alpha\frac{\zeta(\tilde{z})^2}{2mc} + E_\varepsilon\zeta(\tilde{z})\det\mathbb{A} & \beta \\ \gamma & \delta\frac{2mc}{\zeta(\tilde{z})^2} + E_\varepsilon\frac{1}{\zeta(\tilde{z})}\det\mathbb{A} \end{pmatrix}$$

$$\mathbb{Z}(\tilde{z})\mathbb{M}\sigma_1 = \begin{pmatrix} \beta\zeta(\tilde{z}) & \alpha\frac{\zeta(\tilde{z})}{2mc} + E_\varepsilon\det\mathbb{A} \\ \delta\frac{2mc}{\zeta(\tilde{z})} + E_\varepsilon\det\mathbb{A} & \frac{\gamma}{\zeta(\tilde{z})} \end{pmatrix}$$

$$\sigma_1\mathbb{M}\mathbb{Z}(\tilde{z}) = \begin{pmatrix} \gamma\zeta(\tilde{z}) & \delta\frac{2mc}{\zeta(\tilde{z})} + E_\varepsilon\det\mathbb{A} \\ \alpha\frac{\zeta(\tilde{z})}{2mc} + E_\varepsilon\det\mathbb{A} & \frac{\beta}{\zeta(\tilde{z})} \end{pmatrix}$$

$$\sigma_1\mathbb{M}\sigma_1 = \begin{pmatrix} \delta 2mc + E_\varepsilon\zeta(\tilde{z})\det\mathbb{A} & \gamma \\ \beta & \frac{\alpha}{2mc} + E_\varepsilon\frac{1}{\zeta(\tilde{z})}\det\mathbb{A} \end{pmatrix}$$

We will find the point–wise limit as $c \to +\infty$ and for this limit we will find corresponding operator and then proof norm–resolvent limit similarly as in the previous case $D^{\mathbb{A}} \xrightarrow{c\to+\infty} H^{\mathbb{A}}$.

$$E_\varepsilon = \frac{i}{2}\int_{\mathbb{R}^2} v_\varepsilon(x)\mathrm{e}^{ik(\tilde{z})|x-y|}v_\varepsilon(y)\,\mathrm{d}x\,\mathrm{d}y \xrightarrow{c\to+\infty} \tilde{E}_\varepsilon := \frac{i}{2}\int_{\mathbb{R}^2} v_\varepsilon(x)\mathrm{e}^{i\mu(z)|x-y|}v_\varepsilon(y)\,\mathrm{d}x\,\mathrm{d}y. \tag{89}$$

Similarly to (62) we can easily see that the point– wise limit of the integral kernel of the operator $R^{\mathbb{A}}_{z,\varepsilon}$ is

$$R^{\mathbb{A}_c}_{z,\varepsilon} \xrightarrow{c\to+\infty} K^{\mathbb{A}}_{z,\varepsilon} := \left(\frac{im}{\mu(z)}\mathrm{e}^{i\mu(z)|x-y|} - \mathcal{L}_\varepsilon\right) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{90}$$

where

$$\begin{aligned}\mathcal{L}_\varepsilon := -\frac{1}{4\tilde{C}_\varepsilon}\int_{\mathbb{R}^2} v(s)\mathrm{e}^{i\mu(z)|x-\varepsilon s|}\Big(\Big(\tfrac{\alpha}{z} + 2\tilde{E}_\varepsilon\tfrac{m}{\mu(z)}\det\mathbb{A}\Big) + \operatorname{sgn}(\varepsilon t - y)\tfrac{2m}{\mu(z)}\beta + \\ + \operatorname{sgn}(x-\varepsilon s)\frac{2m}{\mu(z)}\gamma + \operatorname{sgn}(x-\varepsilon s)\operatorname{sgn}(\varepsilon t - y)\left(2m\delta + 2\tilde{E}_\varepsilon\frac{m}{\mu(z)}\det\mathbb{A}\right)\Big)v(t)\mathrm{e}^{i\mu(z)|\varepsilon t-y|}\,\mathrm{d}t\,\mathrm{d}s,\end{aligned}$$

with

$$\tilde{C}_\varepsilon := 1 + \tilde{E}_\varepsilon\left(\frac{\alpha}{\mu(z)} + \delta\mu(z)\right) + \tilde{E}_\varepsilon^2\det\mathbb{A}.$$

Theorem 4.3.1. *Let $\mathbb{A}$ be any 2×2 complex matrix and $z \in \mathbb{C} \setminus [0, +\infty)$ be such that*

$$\tilde{C}_\varepsilon \neq 0.$$

Then the resolvent $R^{\mathbb{A}_c}_{\tilde{z},\varepsilon}$ of the operator $D^{\mathbb{A}_c}_{\varepsilon}$ converges in the operator norm, as $c \to +\infty$, to the bounded integral operator $K^{\mathbb{A}}_{z,\varepsilon}$ multiplied by the projection to the upper component of the Dirac wavefunction.

Proof. One can see that the first part of the operator $R^{\mathbb{A}_c}_{\tilde{z},\varepsilon}$ is the resolvent of the free Dirac operator in $z + mc^2$ and the first part of the operator $K^{\mathbb{A}}_{z,\varepsilon}$ is the resolvent of the free Schrödinger operator in z. For that reason we just need to prove that the operator

$$\mathcal{K}_\varepsilon = cR_{\tilde{z}}(I + c\langle v_\varepsilon|\mathbb{A}R_{\tilde{z}}v_\varepsilon\rangle)^{-1}|v_\varepsilon\rangle\langle v_\varepsilon|\mathbb{A}R_{\tilde{z}}$$

will converge in the operator norm to the operator

$$\mathcal{L}_\varepsilon \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since both of these operators are Hilbert–Schmidt, we will prove their convergence in the Hilbert–Schmidt norm.

Let us firstly denote

$$\mathbb{U}^\varepsilon_c = \frac{1}{4cC_\varepsilon}(\mathbb{Z}(\tilde{z})\mathbb{M}\mathbb{Z}(\tilde{z}) + \operatorname{sgn}(\varepsilon t - y)\mathbb{Z}(\tilde{z})\mathbb{M}\sigma_1 + \operatorname{sgn}(x - \varepsilon s)\sigma_1\mathbb{M}\mathbb{Z}(\tilde{z}) + \operatorname{sgn}(x - \varepsilon s)\operatorname{sgn}(\varepsilon t - y)\sigma_1\mathbb{M}\sigma_1),$$

$$\begin{aligned} U^\varepsilon = \frac{1}{4\tilde{C}_\varepsilon}\Big(\Big(\tfrac{\alpha}{z} + 2\tilde{E}_\varepsilon \tfrac{m}{\mu(z)} \det\mathbb{A}\Big) + \operatorname{sgn}(\varepsilon t - y)\tfrac{2m}{\mu(z)}\beta + \\ + \operatorname{sgn}(x - \varepsilon s)\frac{2m}{\mu(z)}\gamma + \operatorname{sgn}(x - \varepsilon s)\operatorname{sgn}(\varepsilon t - y)\left(2m\delta + 2\tilde{E}_\varepsilon\frac{m}{\mu(z)}\det\mathbb{A}\right)\Big), \end{aligned} \tag{91}$$

$$\mathbb{U}^\varepsilon = U^\varepsilon \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now

$$\begin{aligned} |\mathcal{K}_\varepsilon - \mathcal{L}_\varepsilon| = \left|\int_{\mathbb{R}^2} \mathrm{e}^{ik(\tilde{z})|x-\varepsilon s|}v(s)\mathbb{U}^\varepsilon_c v(t)\mathrm{e}^{ik(\tilde{z})|\varepsilon t-y|} - \mathrm{e}^{i\mu(z)|x-\varepsilon s|}v(s)\mathbb{U}^\varepsilon v(t)\mathrm{e}^{i\mu(z)|\varepsilon t-y|}\,\mathrm{d}t\,\mathrm{d}s\right| \leq \\ \leq \int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{Im}k(\tilde{z})|x-\varepsilon s|}|v(s)||\mathbb{U}^\varepsilon_c||v(t)|\mathrm{e}^{-\mathrm{Im}k(\tilde{z})|\varepsilon t-y|} + \mathrm{e}^{-\mathrm{Im}\mu(z)|x-\varepsilon s|}|v(s)||\mathbb{U}^\varepsilon||v(t)|\mathrm{e}^{-\mathrm{Im}\mu(z)|\varepsilon t-y|}\,\mathrm{d}t\,\mathrm{d}s \leq \end{aligned} \tag{92}$$

Since the matrix $\mathbb{U}^\varepsilon_c$ converges to the matrix $\mathbb{U}^\varepsilon$ and is uniformly bounded in x and y, we can find an upper bound $C_1 \in \mathbb{R}$ for its norm. Similarly, we will find $|\mathbb{U}^\varepsilon| \leq C_2 \in \mathbb{R}$. This yields the following

$$(92) \leq C_1 \int_{\mathbb{R}^2} |v(s)||v(t)|\mathrm{e}^{-\mathrm{Im}k(\tilde{z})(|x-\varepsilon s|+|\varepsilon t-y|)}\,\mathrm{d}s\,\mathrm{d}t + C_2 \int_{\mathbb{R}^2} |v(s)||v(t)|\mathrm{e}^{-\mathrm{Im}\mu(z)(|x-\varepsilon s|+|\varepsilon t-y|)}\,\mathrm{d}s\,\mathrm{d}t. \tag{93}$$

To find integrable majorant that is not dependent on c, it remains to estimate

$$\mathrm{e}^{-\mathrm{Im}k(\tilde{z})(|x-\varepsilon s|+|\varepsilon t-y|)}$$

from above. From continuity of $k(\tilde{z})$ we see that

$$(\exists c_1 \geq 0)(\forall c \geq c_1)(|\mathrm{Im}(k(\tilde{z}) - \mu(z))(|x - \varepsilon s| + |\varepsilon t - y|)| \leq \frac{\mathrm{Im}(\mu(z))(|x - \varepsilon s| + |\varepsilon t - y|)}{2}.$$

Hence, we get for $c \geq c_1$

$$\mathrm{e}^{-\mathrm{Im}k(\tilde{z})(|x-\varepsilon s|+|\varepsilon t-y|)} = \mathrm{e}^{-(\mathrm{Im}k(\tilde{z})-\mathrm{Im}\mu(z))(|x-\varepsilon s|+|\varepsilon t-y|)}\mathrm{e}^{-\mathrm{Im}\mu(z)(|x-\varepsilon s|+|\varepsilon t-y|)} \leq \mathrm{e}^{-\frac{\mathrm{Im}\mu(z)}{2}(|x-\varepsilon s|+|\varepsilon t-y|)}.$$

Consequently, we get the square integrable majorant as

$$|\mathcal{K}_\varepsilon - \mathcal{L}_\varepsilon| \leq C_1 \int_{\mathbb{R}^2} |v(s)||v(t)|\mathrm{e}^{-\frac{\mu(z)}{2}(|x-\varepsilon s|+|\varepsilon t-y|)}\,\mathrm{d}s\,\mathrm{d}t + C_2 \int_{\mathbb{R}^2} |v(s)||v(t)|\mathrm{e}^{-\mathrm{Im}\mu(z)(|x-\varepsilon s|+|\varepsilon t-y|)}\,\mathrm{d}s\,\mathrm{d}t. \tag{94}$$

We can prove that the right–hand side of (94) is indeed a square integrable function in variables x and y using the Minkowski integral inequality, because, for $w > 0$ we get

$$\left(\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}^2} |v(s)||v(t)|\mathrm{e}^{-w(|x-\varepsilon s|+|\varepsilon t-y|)}\,\mathrm{d}s\,\mathrm{d}t\right|^2 \mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \leq \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2} |v(s)|^2|v(t)|^2\mathrm{e}^{-2w(|x-\varepsilon s|+|\varepsilon t-y|)}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \mathrm{d}s\,\mathrm{d}t =$$

$$= \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2} \mathrm{e}^{-2w(|x-\varepsilon s|+|\varepsilon t-y|)}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} |v(s)||v(t)|\,\mathrm{d}s\,\mathrm{d}t = \left|\begin{matrix} x = x - \varepsilon s \\ y = y - \varepsilon t \end{matrix}\right|$$

$$= \left(\int_{\mathbb{R}^2} \mathrm{e}^{-2w(|x|+|y|)}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \int_{\mathbb{R}^2} |v(s)||v(t)|\,\mathrm{d}s\,\mathrm{d}t < +\infty.$$

The proof then follows from the dominated convergence theorem. □

Firstly, we will rewrite operator $K_{z,\varepsilon}^{\mathbb{A}}$ to the more convenient form, using the following

$$\begin{aligned}\int_{\mathbb{R}} v(s)\mathrm{e}^{i\mu(z)|x-\varepsilon s|}\mathrm{sgn}(x-\varepsilon s)\,\mathrm{d}s &= \int_{\mathbb{R}} v_\varepsilon(s)\mathrm{e}^{i\mu(z)|x-s|}\mathrm{sgn}(x-s)\,\mathrm{d}s = \\ &= -\int_x^{+\infty} v_\varepsilon(s)\mathrm{e}^{i\mu(z)|x-s|}\,\mathrm{d}s + \int_{-\infty}^{x} v_\varepsilon(s)\mathrm{e}^{i\mu(z)|x-s|}\,\mathrm{d}s = \\ &= \int_x^{+\infty} \frac{1}{\varepsilon^2}v'(x/\varepsilon)\frac{\mathrm{e}^{i\mu(z)|x-s|}}{i\mu(z)}\,\mathrm{d}s + \int_{-\infty}^{x} \frac{1}{\varepsilon^2}v'(x/\varepsilon)\frac{\mathrm{e}^{i\mu(z)|x-s|}}{i\mu(z)}\,\mathrm{d}s = \\ &= \frac{1}{i\mu(z)}\int_{\mathbb{R}}(v_\varepsilon(s))'\mathrm{e}^{i\mu(z)|x-s|}\,\mathrm{d}s.\end{aligned} \tag{95}$$

Incorporated (95) and the resolvent of the free Schrödinger operator K_z, one can see that $K_{z,\varepsilon}^{\mathbb{A}}$ can be rewritten as follows

$$K_{z,\varepsilon}^{\mathbb{A}} = K_z - \alpha_\varepsilon K_z|v_\varepsilon\rangle\langle v_\varepsilon|K_z + \beta_\varepsilon K_z|(v_\varepsilon)'\rangle\langle v_\varepsilon|K_z - \gamma_\varepsilon K_z|v_\varepsilon\rangle\langle (v_\varepsilon)'|K_z - \delta_\varepsilon K_z|(v_\varepsilon)'\rangle\langle (v_\varepsilon)'|K_z, \tag{96}$$

where

$$\alpha_\varepsilon = \frac{1}{2m\tilde{C}_\varepsilon}(\alpha + \tilde{E}_\varepsilon\mu(z)\det\mathbb{A}),$$

$$\beta_\varepsilon = \frac{1}{2im\tilde{C}_\varepsilon}\beta,$$

$$\gamma_\varepsilon = \frac{1}{2im\tilde{C}_\varepsilon}\gamma \text{ and}$$

$$\delta_\varepsilon = \frac{1}{2m\tilde{C}_\varepsilon}(\delta + \tilde{E}_\varepsilon \mu(z)^{-1} \det \mathbb{A}).$$

We assume that the operator $K^{\mathbb{A}}_{z,\varepsilon}$ is the resolvent of a Schrödinger operator perturbed by some non–local potential. To find this operator we will invert $K^{\mathbb{A}}_{z,\varepsilon}$. Firstly, we need to prove following identities.

Lemma 4.3.1. *Let $v \in L^1(\mathbb{R};\mathbb{R}) \cap L^2(\mathbb{R};\mathbb{R})$ be such that $v' \in L^1(\mathbb{R};\mathbb{R}) \cap L^2(\mathbb{R};\mathbb{R})$, K_z be the resolvent of the free Schrödinger operator defined in* (51)*, and $\tilde{E}_\varepsilon$ be in the form of* (89)*. The following identities holds*

$$\langle (v_\varepsilon)' | K_z (v_\varepsilon)' \rangle = 2m\|v_\varepsilon\|^2_{L^2} + 2mz\langle v_\varepsilon | K_z v_\varepsilon \rangle,$$

$$\langle (v_\varepsilon)' | K_z v_\varepsilon \rangle = \langle v_\varepsilon | K_z (v_\varepsilon)' \rangle = 0,$$

$$\tilde{E}_\varepsilon = \frac{\mu(z)}{2m}\langle v_\varepsilon | K_z v_\varepsilon \rangle.$$

Proof. Firstly, we will prove the first identity.

$$\begin{aligned}
\langle (v_\varepsilon)' | K_z (v_\varepsilon)' \rangle &= \frac{im}{\mu(z)} \int_{\mathbb{R}^2} (v_\varepsilon)'(x) \mathrm{e}^{i\mu(z)|x-y|} (v_\varepsilon)'(y) \,\mathrm{d}y \,\mathrm{d}x = \\
= \frac{im}{\mu(z)} \int_{\mathbb{R}} (v_\varepsilon)'(x) &\left([\mathrm{e}^{i\mu(z)(y-x)} v_\varepsilon(y)]^{+\infty}_x + [\mathrm{e}^{i\mu(z)(x-y)} v_\varepsilon(y)]^x_{-\infty} - i\mu(z) \int_{\mathbb{R}} \mathrm{sgn}(x-y) \mathrm{e}^{i\mu(z)|x-y|} v_\varepsilon(y) \,\mathrm{d}y \right) \mathrm{d}x = \\
&= -m \int_{\mathbb{R}^2} (v_\varepsilon)'(x) \mathrm{sgn}(x-y) \mathrm{e}^{i\mu(z)|x-y|} v_\varepsilon(y) \,\mathrm{d}y \,\mathrm{d} = \\
= -m \int_{\mathbb{R}} v_\varepsilon(y) &\left([\mathrm{e}^{i\mu(z)(x-y)} v_\varepsilon(x)]^{+\infty}_y - [\mathrm{e}^{i\mu(z)(y-x)} v_\varepsilon(x)]^y_{-\infty} - i\mu(z) \int_{\mathbb{R}} \mathrm{e}^{i\mu(z)|x-y|} v_\varepsilon(x) \,\mathrm{d}x \right) \mathrm{d}y = \\
&= 2m\|v_\varepsilon\|^2_{L^2} + 2mz\langle v_\varepsilon | K_z v_\varepsilon \rangle.
\end{aligned}$$

Now, we will show that the second identity holds.

$$\begin{aligned}
\langle v_\varepsilon | K_z (v_\varepsilon)' \rangle &= \frac{im}{\mu(z)} \int_{\mathbb{R}^2} v_\varepsilon(x) \mathrm{e}^{i\mu(z)|x-y|} (v_\varepsilon)'(y) \,\mathrm{d}y \,\mathrm{d}x = \\
= \frac{im}{\mu(z)} \int_{\mathbb{R}} v_\varepsilon(x) &\left([\mathrm{e}^{i\mu(z)(y-x)} v_\varepsilon(y)]^{+\infty}_x + [\mathrm{e}^{i\mu(z)(x-y)} v_\varepsilon(y)]^x_{-\infty} - i\mu(z) \int_{\mathbb{R}} \mathrm{sgn}(x-y) \mathrm{e}^{i\mu(z)|x-y|} v_\varepsilon(y) \,\mathrm{d}y \right) \mathrm{d}x = \\
&= m \int_{\mathbb{R}^2} v_\varepsilon(x) \mathrm{sgn}(x-y) \mathrm{e}^{i\mu(z)|x-y|} v_\varepsilon(y) \,\mathrm{d}x \,\mathrm{d}y.
\end{aligned}$$

Since the integrand of the obtained integral is an antisymmetric function the integral is identically zero. Similarly for $\langle (v_\varepsilon)' | K_z v_\varepsilon \rangle$.

Third identity is easy to see from definitions. □

Theorem 4.3.2. *Let us take $\mathbb{A} \in \mathbb{C}^{2,2}$ and $z \in \mathbb{C} \setminus [0, +\infty)$ such that $\tilde{C}_\varepsilon \neq 0$. Then $K^{\mathbb{A}}_{z,\varepsilon}$ is the resolvent of the operator*

$$H^{\mathbb{A}}_\varepsilon = H_0 + \frac{1}{2m(1 - \delta\|v_\varepsilon\|^2_{L^2})} \langle \mathbb{W}_\varepsilon | \hat{\mathbb{A}} \rangle_2 + \frac{1}{2m} \frac{\beta\gamma}{1 - \delta\|v_\varepsilon\|^2_{L^2}} |v_\varepsilon\rangle\langle v_\varepsilon|,$$

where

$$\hat{\mathbb{A}} = \begin{pmatrix} \alpha(1 - \delta\|v_\varepsilon\|^2_{L^2}) & i\beta \\ -i\gamma & \delta \end{pmatrix} \textit{ and}$$

$$\mathbb{W}_\varepsilon = \begin{pmatrix} |v_\varepsilon\rangle\langle v_\varepsilon| & |v_\varepsilon\rangle\langle (v_\varepsilon)'| \\ |(v_\varepsilon)'\rangle\langle v_\varepsilon| & |(v_\varepsilon)'\rangle\langle (v_\varepsilon)'| \end{pmatrix}.$$

Proof. Let us start with the operator

$$H_\varepsilon^{\mathbb{A}} = H_0 + W_\varepsilon,$$

where

$$W_\varepsilon = \frac{a}{2m}|v_\varepsilon\rangle\langle v_\varepsilon| - \frac{b}{2im}|v_\varepsilon\rangle\langle (v_\varepsilon)'| + \frac{c}{2im}|(v_\varepsilon)'\rangle\langle v_\varepsilon| + \frac{d}{2m}|(v_\varepsilon)'\rangle\langle (v_\varepsilon)'|.$$

We will find the resolvent for this operator and we will compare the result with the limit from Theorem 4.3.1 and from that we will derive coefficient a,b,c and d.

$$(H_\varepsilon^{\mathbb{A}} - z)^{-1} = (H_0 + W_\varepsilon - z)^{-1} = K_z(I + W_\varepsilon K_z)^{-1}.$$

Now, we just need to invert $(I + W_\varepsilon K_z)$.

$$\psi + \frac{a}{2m}\langle v_\varepsilon|K_z\psi\rangle v_\varepsilon - \frac{b}{2im}\langle (v_\varepsilon)'|K_z\psi\rangle v_\varepsilon + \frac{c}{2im}\langle v_\varepsilon|K_z\psi\rangle (v_\varepsilon)' + \frac{d}{2m}\langle (v_\varepsilon)'|K_z\psi\rangle (v_\varepsilon)' = g. \tag{97}$$

Acting on the equation (97) by $\langle v_\varepsilon|\cdot\rangle K_z$ and by $\langle (v_\varepsilon)'|\cdot\rangle K_z$, and using Lemma 4.3.1, we will arrive at the following system of linear equations

$$\begin{pmatrix} 1 + \frac{a}{2m}\langle v_\varepsilon|K_z v_\varepsilon\rangle & -\frac{b}{2im}\langle v_\varepsilon|K_z v_\varepsilon\rangle \\ \frac{c}{2im}\langle (v_\varepsilon)'|K_z(v_\varepsilon)'\rangle & 1 + \frac{d}{2m}\langle (v_\varepsilon)'|K_z(v_\varepsilon)'\rangle \end{pmatrix} \begin{pmatrix} \langle v_\varepsilon|K_z\psi\rangle \\ \langle (v_\varepsilon)'|K_z\psi\rangle \end{pmatrix} = \begin{pmatrix} \langle v_\varepsilon|K_z g\rangle \\ \langle (v_\varepsilon)'|K_z g\rangle \end{pmatrix}.$$

Hence, we have

$$\begin{pmatrix} \langle v_\varepsilon|K_z\psi\rangle \\ \langle (v_\varepsilon)'|K_z\psi\rangle \end{pmatrix} = \frac{1}{\sigma}\begin{pmatrix} 1 + \frac{d}{2m}\langle (v_\varepsilon)'|K_z(v_\varepsilon)'\rangle & \frac{b}{2im}\langle v_\varepsilon|K_z v_\varepsilon\rangle \\ -\frac{c}{2im}\langle (v_\varepsilon)'|K_z(v_\varepsilon)'\rangle & 1 + \frac{a}{2m}\langle v_\varepsilon|K_z v_\varepsilon\rangle \end{pmatrix} \begin{pmatrix} \langle v_\varepsilon|K_z g\rangle \\ \langle (v_\varepsilon)'|K_z g\rangle \end{pmatrix},$$

where

$$\sigma = 1 + \frac{a}{2m}\langle v_\varepsilon|K_z v_\varepsilon\rangle + \frac{d}{2m}\langle (v_\varepsilon)'|K_z(v_\varepsilon)'\rangle + \frac{ad-bc}{4m^2}\langle v_\varepsilon|K_z v_\varepsilon\rangle\langle (v_\varepsilon)'|K_z(v_\varepsilon)'\rangle.$$

We can put the solution back to the equation (97) and get the resolvent of $H_\varepsilon^{\mathbb{A}}$.

$$\begin{aligned}\psi = g &- \frac{a}{2m\sigma}\left(1 + \frac{d}{2m}\langle (v_\varepsilon)'|K_z(v_\varepsilon)'\rangle\right)\langle v_\varepsilon|K_z g\rangle v_\varepsilon - \frac{ab}{4im^2\sigma}\langle v_\varepsilon|K_z v_\varepsilon\rangle\langle (v_\varepsilon)'|K_z g\rangle v_\varepsilon - \\ &+ \frac{bc}{4m^2\sigma}\langle (v_\varepsilon)'|K_z(v_\varepsilon)'\rangle\langle v_\varepsilon|K_z g\rangle v_\varepsilon + \frac{b}{2im\sigma}\left(1 + \frac{a}{2m}\langle v_\varepsilon|K_z v_\varepsilon\rangle\right)\langle (v_\varepsilon)'|K_z g\rangle v_\varepsilon - \\ &- \frac{c}{2im\sigma}\left(1 + \frac{d}{2m}\langle (v_\varepsilon)'|K_z(v_\varepsilon)'\rangle\right)\langle v_\varepsilon|K_z g\rangle (v_\varepsilon)' + \frac{bc}{4m^2\sigma}\langle v_\varepsilon|K_z v_\varepsilon\rangle\langle (v_\varepsilon)'|K_z g\rangle (v_\varepsilon)' + \\ &+ \frac{cd}{4im^2\sigma}\langle (v_\varepsilon)'|K_z(v_\varepsilon)'\rangle\langle v_\varepsilon|K_z g\rangle (v_\varepsilon)' - \frac{d}{2m\sigma}\left(1 + \frac{a}{2m}\langle v_\varepsilon|K_z v_\varepsilon\rangle\right)\langle (v_\varepsilon)'|K_z g\rangle (v_\varepsilon)',\end{aligned}$$

which can be simplified as

$$\begin{aligned}\psi = g &- \frac{1}{\sigma}\left(\frac{a}{2m} + \frac{ad-bc}{4m^2}\langle (v_\varepsilon)'|K_z(v_\varepsilon)'\rangle\right)\langle v_\varepsilon|K_z g\rangle v_\varepsilon + \frac{b}{2im\sigma}\langle (v_\varepsilon)'|K_z g\rangle v_\varepsilon - \\ &- \frac{c}{2im\sigma}\langle v_\varepsilon|K_z g\rangle (v_\varepsilon)' - \frac{1}{\sigma}\left(\frac{d}{2m} + \frac{ad-bc}{4m^2}\langle v_\varepsilon|K_z v_\varepsilon\rangle\right)\langle (v_\varepsilon)'|K_z g\rangle (v_\varepsilon)'.\end{aligned}$$

Using Lemma 4.3.1 and a previous calculation, we infer that the resolvent of $H_\varepsilon^{\mathbb{A}}$ obeys

$$
\begin{aligned}
(H_\varepsilon^{\mathbb{A}} - z)^{-1} = K_z - \frac{1}{2m\sigma}\left(a + (ad-bc)\|v_\varepsilon\|_{L^2}^2 + \mu(z)(ad-bc)\tilde{E}_\varepsilon\right) K_z|v_\varepsilon\rangle\langle v_\varepsilon|K_z + \frac{b}{2im\sigma}K_z|v_\varepsilon\rangle\langle (v_\varepsilon)'|K_z - \\
- \frac{c}{2im\sigma}K_z|(v_\varepsilon)'\rangle\langle v_\varepsilon|K_z - \frac{1}{2m\sigma}\left(d + \frac{ad-bc}{\mu(z)}\tilde{E}_\varepsilon\right) \text{ and}
\end{aligned} \tag{98}
$$

$$
\sigma = 1 + d\|v_\varepsilon\|_{L^2}^2 + \left(\frac{a + (ad-bc)\|v_\varepsilon\|_{L^2}^2}{\mu(z)} + \mu(z)d\right)\tilde{E}_\varepsilon + (ad-bc)\tilde{E}_\varepsilon^2.
$$

Now if we just compare the resolvent $(H_\varepsilon^{\mathbb{A}} - z)^{-1}$ (98) with the limit (96), we are getting the following relations between coefficients.

$$
\alpha = \frac{a + (ad-bc)\|v_\varepsilon\|_{L^2}^2}{1 + d\|v_\varepsilon\|_{L^2}^2},
$$

$$
\beta = \frac{b}{1 + d\|v_\varepsilon\|_{L^2}^2},
$$

$$
\gamma = \frac{c}{1 + d\|v_\varepsilon\|_{L^2}^2},
$$

$$
\delta = \frac{d}{1 + d\|v_\varepsilon\|_{L^2}^2}.
$$

Inverting identities above, one will conclude that the non–relativistic limit $K_{z,\varepsilon}^{\mathbb{A}}$ is actually the resolvent of the Schrödinger operator with non–local potential $H_\varepsilon^{\mathbb{A}}$ with coefficients

$$
d = \frac{\delta}{1 - \delta\|v_\varepsilon\|_{L^2}^2},
$$

$$
c = \frac{\gamma}{1 - \delta\|v_\varepsilon\|_{L^2}^2},
$$

$$
b = \frac{\beta}{1 - \delta\|v_\varepsilon\|_{L^2}^2} \text{ and}
$$

$$
a = \alpha + \frac{\beta\gamma\|v_\varepsilon\|_{L^2}^2}{1 - \delta\|v_\varepsilon\|_{L^2}^2}.
$$

□

Let us look more closely at the operator from Theorem 4.3.2.

If $\beta = \gamma = \delta = 0$, then one will get the operator

$$
H_0 + \frac{\alpha}{2m}|v_\varepsilon\rangle\langle v_\varepsilon|.
$$

It is well known fact [1] that this operator converges in the norm resolvent sense to the corresponding Schrödinger point interaction $H^{\mathbb{A}}$. Now let us discuss a more interesting situation.

If $\alpha = \beta = \gamma = 0$ and $\delta = 1$, then for such combination one will arrive at the following operator

$$
H_\varepsilon^{\mathbb{A}} = H_0 + \frac{1}{2m(1 - \|v_\varepsilon\|_{L^2}^2)}|(v_\varepsilon)'\rangle\langle (v_\varepsilon)'|.
$$

We can compare this result with the article [20], where the non–local approximation of δ'–interaction was found. In the spirit of the article we will define

$$u(x) := v'(x).$$

Then one can easily deduce that

$$-\frac{1}{2}\int_{\mathbb{R}^2} u(x)|x-y|u(y)\,\mathrm{d}x\,\mathrm{d}y = -\|v\|_{L^2}^2 \text{ and}$$

$$\frac{\|v\|_{L^2}^2}{\varepsilon} = \|v_\varepsilon\|_{L^2}^2.$$

Also since, $\int_{\mathbb{R}} v(x)\,\mathrm{d}x = 1$ then

$$1 = \int_{\mathbb{R}} v(x)\,\mathrm{d}x = \int_{\mathbb{R}} v(x+t)\,\mathrm{d}x$$

and differentiating with respect to t yields

$$0 = \int_{\mathbb{R}} v'(x+t)\,\mathrm{d}x = \int_{\mathbb{R}} u(x)\,\mathrm{d}x.$$

We can easily see that the operator $H_\varepsilon^{\mathbb{A}}$ is exactly the same non–local approximation H_ε of δ'–interaction Šeba discussed in the article [20] where norm–resolvent limit was deduce for this type of approximation.

We will now consider the most general situation of the non–local approximation $H_\varepsilon^{\mathbb{A}}$ from Theorem 4.3.2. For this operator, a convergence to the operator $H^{\mathbb{A}}$ in the norm–resolvent sense will be proved.

Theorem 4.3.3. *Let matrix $\mathbb{A}$ be any 2×2 complex matrix and $z \in \mathbb{C} \setminus [0, +\infty)$ such that*

$$\det(2I + i\mathbb{A}\tilde{\mathbb{Z}}(z)) \neq 0.$$

Then the operator $(H_\varepsilon^{\mathbb{A}} - z)^{-1}$ converges in the operator norm to the operator $(H^{\mathbb{A}} - z)^{-1}$ as ε tends to 0.

Proof. In Theorem 4.3.2 and Theorem 4.2.2 we proved that

$$(H_\varepsilon^{\mathbb{A}} - z)^{-1} = K_{z,\varepsilon}^{\mathbb{A}},$$
$$(H^{\mathbb{A}} - z)^{-1} = K_z^{\mathbb{A}}.$$

None of these two operators is a Hilbert–Schmidt operator but in similarly as in the relativistic case, one can see that the difference of these two operators is Hilbert–Schmidt, and so, to prove the theorem it is sufficient to show that

$$K_{z,\varepsilon}^{\mathbb{A}} - K_z^{\mathbb{A}} \xrightarrow[\varepsilon\to 0]{HS} 0.$$

In the same way as in Theorem 4.3.2, let us denote U^ε as in (91) and also U as in (63). It can be easily seen that $U^\varepsilon \xrightarrow{\varepsilon\to 0} U$.

Now we will deal with the Hilbert–Schmidt norm of $K_{z,\varepsilon}^{\mathbb{A}} - K_z^{\mathbb{A}}$ in a similar way as in Theorem 2.2.1. Using Young's inequality we obtain

$$\|K^{\mathbb{A}}_{z,\varepsilon} - K^{\mathbb{A}}_{z}\|_2^2 = \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} \mathrm{e}^{i\mu(z)|x-\varepsilon y|} v(s) U^{\varepsilon}(s,t) \mathrm{e}^{i\mu(z)|\varepsilon t - y|} v(t) - \mathrm{e}^{i\mu(z)|x|} v(s) U(s,t) \mathrm{e}^{i\mu(z)|y|} v(t)\, \mathrm{d}t\, \mathrm{d}s \right|^2 \mathrm{d}x\, \mathrm{d}y =$$

$$= \int_{\mathbb{R}^2} \Big| \int_{\mathbb{R}^2} (\mathrm{e}^{i\mu(z)|x-\varepsilon s|} - \mathrm{e}^{i\mu(z)|x|}) v(s) U^{\varepsilon}(s,t) \mathrm{e}^{i\mu(z)|\varepsilon t - y|} v(t) + \mathrm{e}^{i\mu(z)|x|} v(s) (U^{\varepsilon}(s,t) - U(s,t)) \mathrm{e}^{i\mu(z)|\varepsilon t - y|} v(t) +$$

$$+ \mathrm{e}^{i\mu(z)|x|} v(s) U(s,t) (\mathrm{e}^{i\mu(z)|\varepsilon t - y|} - \mathrm{e}^{i\mu(z)|y|}) v(t)\, \mathrm{d}t\, \mathrm{d}s \Big|^2 \mathrm{d}x\, \mathrm{d}y \leq$$

$$\leq 2 \underbrace{\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |(\mathrm{e}^{i\mu(z)|x-\varepsilon s|} - \mathrm{e}^{i\mu(z)|x|}) v(s) U^{\varepsilon}(s,t) \mathrm{e}^{i\mu(z)|\varepsilon t - y|} v(t)|\, \mathrm{d}s\, \mathrm{d}t \right)^2 \mathrm{d}x\, \mathrm{d}y}_{a)} +$$

$$+ 4 \underbrace{\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |\mathrm{e}^{i\mu(z)|x|} v(s) (U^{\varepsilon}(s,t) - U(s,t)) \mathrm{e}^{i\mu(z)|\varepsilon t - y|} v(t)|\, \mathrm{d}t\, \mathrm{d}s \right)^2 \mathrm{d}x\, \mathrm{d}y}_{b)} +$$

$$+ 4 \underbrace{\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |\mathrm{e}^{i\mu(z)|x|} v(s) U(s,t) (\mathrm{e}^{i\mu(z)|\varepsilon t - y|} - \mathrm{e}^{i\mu(z)|y|}) v(t)|\, \mathrm{d}t\, \mathrm{d}s \right)^2 \mathrm{d}x\, \mathrm{d}y}_{c)} .$$

Now, we will prove that every term a),b) and c) goes to zero.

a)

$$a) \leq \underbrace{\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\mathrm{e}^{i\mu(z)|x-\varepsilon s|} - \mathrm{e}^{i\mu(z)|x|}||v(s)|\, \mathrm{d}s \right)^2 \mathrm{d}x}_{\text{By Lemma 2.2.3}\to 0} \underbrace{\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |U^{\varepsilon}(s,t)||v(t)||\mathrm{e}^{i\mu(z)|\varepsilon t - y|}|\, \mathrm{d}t \right)^2 \mathrm{d}y}_{\tilde{K}_{\varepsilon}} .$$

U^{ε} is converging to U and thus it can be uniformly bounded by some constant C_0. Then using the Minkowski integral inequality from Proposition 2.2.1, we infer that

$$\tilde{K}_{\varepsilon} \leq C_0 \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |v(t)||\mathrm{e}^{i\mu(z)|\varepsilon t - y|}\, \mathrm{d}t \right)^2 \mathrm{d}y \leq C_0 \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |v(t)|^2 |\mathrm{e}^{i\mu(z)|\varepsilon t - y|}|^2\, \mathrm{d}y \right)^{\frac{1}{2}} \mathrm{d}t \right)^2 \leq$$

$$\leq C_0 \left(\int_{\mathbb{R}} |v(t)| \underbrace{\left(\int_{\mathbb{R}} |\mathrm{e}^{i\mu(z)|\varepsilon t - y|}|^2\, \mathrm{d}y \right)^{\frac{1}{2}}}_{\text{by Lemma 2.2.2} < +\infty} \mathrm{d}t \right)^2 < +\infty.$$

b) Using the Minkowski integral inequality and Lemma 2.2.2 we deduce that

$$
\begin{aligned}
b) \leq & \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \left| \mathrm{e}^{i\mu(z)|x|} v(s) (U^{\varepsilon}(s,t) - U(s,t)) \mathrm{e}^{i\mu(z)|\varepsilon t - y|} v(t) \right| \,\mathrm{d}s\,\mathrm{d}t \right)^2 \mathrm{d}x\,\mathrm{d}y \leq \\
\leq & \left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \mathrm{e}^{-2\mathrm{Im}\mu(z)|x|} |v(s)|^2 |U^{\varepsilon}(s,t) - U(s,t)|^2 \mathrm{e}^{-2\mathrm{Im}\mu(z)|\varepsilon t - y|} |v(t)|^2 \,\mathrm{d}x\,\mathrm{d}y \right)^{\frac{1}{2}} \mathrm{d}t\,\mathrm{d}s \right)^2 = \\
= & \left(\int_{\mathbb{R}^2} |v(t)||v(s)||U^{\varepsilon}(s,t) - U(s,t)| \left(\underbrace{\int_{\mathbb{R}^2} \mathrm{e}^{-2\mathrm{Im}\mu(z)|x|} \mathrm{e}^{-2\mathrm{Im}\mu(z)|\varepsilon t - y|} \,\mathrm{d}x\,\mathrm{d}y}_{<C_1<+\infty} \right)^{\frac{1}{2}} \mathrm{d}t\,\mathrm{d}s \right)^2 \leq \\
& \leq C_1 \left(\int_{\mathbb{R}^2} |v(t)||v(s)||U^{\varepsilon}(s,t) - U(s,t)| \,\mathrm{d}t\,\mathrm{d}s \right)^2 .
\end{aligned}
$$

Since

$$
\begin{aligned}
& |U^{\varepsilon}(s,t) - U(s,t)| \leq |U^{\varepsilon}(s,t)| + |U(s,t)| \leq \\
& \leq 2\left|\frac{\alpha}{z}\right| + \left|2\tilde{E}_{\varepsilon}\frac{m}{\mu(z)}\det\mathbb{A}\right| + \left|\frac{im}{\mu(z)}\det\mathbb{A}\right| + 2\left|\frac{2m}{\mu(z)}\gamma\right| + 2\left|\frac{2m}{\mu(z)}\beta\right| + 2\,|2m\delta| + \left|2\tilde{E}_{\varepsilon}\frac{m}{\mu(z)}\det\mathbb{A}\right| + \left|\frac{im}{\mu(z)}\det\mathbb{A}\right|,
\end{aligned}
$$

using the Lebesgue dominated convergence theorem yields $b) \to 0$.

c) similarly to a).

We conclude that $\|K^{\mathbb{A}}_{z,\varepsilon} - K^{\mathbb{A}}_{z}\|_2 \to 0$ which proves the theorem. □

Theorems 2.2.1, 4.2.1, 4.3.1 and 4.3.3 prove that the following diagram commutes.

$$
\begin{tikzcd}
(D^{\mathbb{A}_c}_{\varepsilon} - mc^2) \arrow[r, "\varepsilon\to 0", "u"'] \arrow[d, "c\to+\infty"', "u"] & (D^{\mathbb{A}_c} - mc^2) \arrow[d, "u"', "c\to+\infty"] \\
H^{\mathbb{A}}_{\varepsilon} \arrow[r, "u", "\varepsilon\to 0"'] & H^{\mathbb{A}}
\end{tikzcd}
$$

Interestingly enough, we observe that for the one branch of the diagram, more precisely for

$$
(D^{\mathbb{A}_c}_{\varepsilon} - mc^2) \to (D^{\mathbb{A}_c} - mc^2) \to H^{\mathbb{A}},
$$

we see no renormalization. On the other hand, in the latter branch of the diagram the renormalization does happen.

This problem with our non–local potential as an approximation of the non–relativistic point interactions was already addressed in the literature [2, 20]. Possible solution that would most probably work without renormalization of coupling constants would be to choose starting potential W_{ε} in (3) as

$$
W_{\varepsilon} = |v_{\varepsilon}\rangle\langle u_{\varepsilon}|,
$$

where functions v and u have disjoint supports, which was considered in [2] and the norm resolvent limit $H^{\mathbb{A}}_{\varepsilon} \to H^{\mathbb{A}}$ without the renormalization was proved.

5 Conclusion

In this work we proved that the free Dirac operator D_0 with not necessarily self–adjoint non–local potential
$$D_\varepsilon^{\mathbb{A}} = D_0 + \frac{1}{\varepsilon^2}|v(x/\varepsilon)\rangle\langle v(x/\varepsilon)| \otimes c\mathbb{A}$$
converges in the norm–resolvent sense to some unbounded operator acting like the Dirac operator with certain transmission condition, which describes the character of the interaction. In the self–adjoint case the limit corresponds to the relativistic point interaction described in [3].

Šeba conjectured and proved for two special cases of the self–adjoint matrix $\mathbb{A}$ in [21] that the norm–resolvent limit of $D_\varepsilon^{\mathbb{A}}$ is the same as its formal limit. We concluded in this manuscript that this so called renormalization of coupling constant does not occur for the most general case of the non–local potential used by Šeba. This property led us to the natural generalization of the definition of the relativistic point interaction also to the non–self–adjoint setting as the limit of $D_\varepsilon^{\mathbb{A}}$. More precisely, we rigorously introduced for all complex matrices $\mathbb{A} \in \mathbb{C}^{2,2}$ the formal limit
$$D_\varepsilon^{\mathbb{A}} \xrightarrow{\varepsilon \to 0} D_0 + |\delta(x)\rangle\langle\delta(x)| \otimes c\mathbb{A}$$
as the well–defined, closed operator
$$\begin{gathered}(D^{\mathbb{A}}\psi)(x) = (D_0\psi)(x), \ \forall x \in \mathbb{R} \setminus \{0\} \text{ on domain} \\ \psi \in \operatorname{Dom} D^{\mathbb{A}} = \{\psi \in W^{1,2}(\mathbb{R}\backslash\{0\}) \otimes \mathbb{C}^2 \mid (2i + \sigma_1\mathbb{A})\psi(0-) = (2i - \sigma_1\mathbb{A})\psi(0+)\}.\end{gathered}$$

Recall that in the case of a local potential with the Dirac operator coupling constants do renormalize. From that we can deduce that the character of the relativistic point interactions is rather non–local.

Furthermore, we discussed the spectrum of $D^{\mathbb{A}}$ and found that $z \in \mathbb{C} \setminus \mathbb{R}_{mc^2}$ is in the spectrum of the operator $D^{\mathbb{A}}$ if and only if the following equation holds
$$0 = 4 + 2i\,\mathrm{tr}(\mathbb{A}\mathbb{Z}(z)) - \det\mathbb{A}.$$
We also observed wild spectral transitions described in Section 3.2. We showed that for special choices of matrix $\mathbb{A}$ we have the whole complex plane or half–plane as the point spectrum of $D^{\mathbb{A}}$. On the other hand, we proved, apart of these critical cases of $\mathbb{A}$, that the operator $D^{\mathbb{A}}$ has at most two eigenvalues. From that, we concluded that the general relativistic point interaction behaves similarly as its self–adjoint realization.

We also found the implicit formula for the eigenvalues of the non–local approximation. The stability of the spectrum was discussed in the thesis.

Ultimately, we found the non–relativistic norm–resolvent limit of the operator of not necessary self–adjoint relativistic point interactions. Doing this we obtained the corresponding model of non–relativistic point interactions. We also took the non–relativistic limit of the non–local approximation $D_\varepsilon^{\mathbb{A}}$. Consequently, we arrive at the non–local approximation of non–relativistic point interactions. Finally, we proved interchangeability of the non–relativistic norm resolvent limit and norm resolvent limit of approximations.

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