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DIRAC OSCILLATOR IN DYNAMICAL NONCOMMUTATIVE SPACE

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Abstract. In this paper, we address the energy eigenvalues of two-dimensional Dirac oscillator perturbed by a dynamical noncommutative space. We derived the relativistic Hamiltonian of Dirac oscillator in the dynamical noncommutative space, in which the space-space Heisenberg-like commutation relations and noncommutative parameter are position-dependent. Then, we used this Hamiltonian to calculate the first-order correction to the eigenvalues and eigenvectors, based on the language of creation and annihilation operators and using the perturbation theory. It is shown that the energy shift depends on the dynamical noncommutative parameter τ. Knowing that, with a set of two-dimensional Bopp-shift transformation, we mapped the noncommutative problem to the standard commutative one.

Keywords: Dynamical noncommutative space, τ-space, noncommutative Dirac oscillator, perturbation theory.

1. Introduction

In the last few decades, physicists and mathematicians have developed a mathematical theory called noncommutative geometry, which has quickly become a topic of great interest and has been finding applications in many areas of modern physics, such as high energy [1], cosmology [2, 3], gravity [4], quantum physics [5–7] and field theory [8, 9]. Substantially, the study on noncommutative (NC) spaces is very important for understanding phenomena at a tiny scale of physical theories. The idea behind the extension of noncommutativity to the coordinates was first suggested by Heisenberg in 1930 as a solution to remove the infinite quantities of field theories. The NC space-time structures were first mentioned by Snyder in 1947 [10, 11], in which he introduced noncommutativity in the hope of regularizing the divergencies that plagued quantum field theory.

Motivated by the attempts to understand the string theory, the quantum gravitation and black holes through NC spaces and by seeking to highlight more phenomenological implications, we consider the Dirac oscillator (DO) within a two-dimensional dynamical noncommutative (DNC) space (also known as a position-dependent NC space).

Unlike the simplest possible type of NC spaces, in which the NC parameter is constant, here we talk about a different type of NC spaces, where the deformation parameter will no longer be constant. However, there are many other possibilities that cannot be excluded. In fact, in the first paper by Snyder himself [10], the noncommutativity parameter was taken to depend on the coordinates and the momenta. Considerable different possibilities have been explored since then, especially in the Lie-algebraic approaches [12], κ-Poincaré noncommutativity [13], other fuzzy spaces [14]. Besides, more recently in position-dependent approach [15–17], the authors considered $\Theta_{\mu\nu}$ to be a function of the position coordinates, i.e., $\Theta \to \Theta(X, Y)$.

The relativistic DO has a great potential both for the theoretical and practical applications. The potential term is introduced linearly, by substitution $\overrightarrow{p} \to \overrightarrow{p} - im\beta\omega\overrightarrow{r}$ in free Dirac Hamiltonian, this was considered for the first time by Ito et al. [18], with $\overrightarrow{r}$ being the position vector and m, β, $\omega > 0$ being the rest mass of the particle, Dirac matrix and constant oscillator frequency, respectively. It was named Dirac oscillator by Moshinsky and Szczepaniak [19] because it is a relativistic generalization of the non-relativistic harmonic oscillator and, exactly in a non-relativistic limit, it reduces to a standard harmonic oscillator with a strong spin-orbit coupling term.

Physically, DO has attracted a lot of attention because of its considerable physical applications, it is widely studied and illustrated. It can be shown that it is a physical system, which can be interpreted as an interaction of the anomalous magnetic moment with a linear electric field [20]. In addition, it can be associated with the electromagnetic potential [21]. As an exactly solvable model, DO in the background of a perpendicular uniform magnetic field has been widely studied. However, we mention, for instance, the following: In ref. [19], the spectra of (3+1)-dimensional DO are solved and the non-relativistic limit is discussed, as well, in ref. [22], the symmetrical properties of the DO are studied. The operators of shift for symmetries are constructed explicitly [23]. Interestingly, the DO may offer a new approach to study quantum optics, where it was found that there is an exact map from (2+1)-dimensional DO to Jaynes-Cummings (JC) model [24], which describes the atomic transitions in a two level system. Subsequently, it was found that this model can be mapped either to JC or anti-JC models, depending on the magnitude of the magnetic field [25].

Basically, DO became more and more important since the experimental observations. For instance, we mention that Franco-Villafañe et al. [26] came with the proposal of a first-experimental microwave realisation of the one-dimensional DO. The experiment depends on a relation of the DO to a corresponding tight-binding system. The experimental results obtained show that the spectrum of the one-dimensional DO is in good agreement with that of the theory. Quimbay et al. [27, 28] show that the DO may describe a naturally occurring physical system. More precisely, that case of a two-dimensional DO can be used to describe the dynamics of the charge carriers in graphene, and hence its electronic properties [29].

This paper is organized as follows. In section 2, the DNC geometry is briefly reviewed. In section 3, the two-dimensional DNC DO is investigated, where in sub-section 3.2, the energy spectrum in NC space is obtained. In sub-section 3.3, based on the perturbation theory and Fock basis, the energy spectrum including the dynamical noncommutativity effect is obtained, therefore, we summarize the results and discussions. Section 4, is then devoted to the conclusions.

2. Review of dynamical noncommutativity

Let us present the essential formulas of the DNC space algebra we need in this study. As is known, at the tiny scale (string scale), the position coordinates do not commute with each other, thus the canonical variables satisfy the following deformed Heisenberg commutation relation

$$\left[x_\mu^{nc}, x_\nu^{nc}\right] = i\Theta_{\mu\nu}, \tag{1}$$

with $\Theta_{\mu\nu}$ being an anti-symmetric tensor. In simplest way, the deformation parameter is considered a real constant. But, in general, $\Theta_{\mu\nu}$ can be a function of coordinates. Fring et al. [16] made a generalization of an NC space to a position-dependent space by introducing a set of new variables X, Y, P_x, P_y and converting the constant Θ into a function of coordinates $\theta(X,Y) = \Theta\left(1+\tau Y^2\right)$. As another example of $\theta(X,Y)$, we also mention that Gomes et al. [17] chose in their study $\theta(X,Y) = \Theta/[1+\Theta\alpha\left(1+Y^2\right)]$.

However, as a deformation of this NC parameter form will almost inevitably lead to non-Hermitian coordinates, it was pointed out [30] that these types of structures are related directly to non-Hermitian Hamiltonian systems. Later, it is explained how this problem was solved.

In the new type of the two-dimensional NC space, which is known as the DNC space or $\tau-$space, the commutation relations are [16]

$$\begin{array}{rclrcl} [X,Y] = & i\Theta\left(1+\tau Y^2\right), & [Y,P_y] = & i\hbar\left(1+\tau Y^2\right), \\ [X,P_x] = & i\hbar\left(1+\tau Y^2\right), & [Y,P_x] = & 0, \\ [X,P_y] = & 2i\tau Y\left(\Theta P_y + \hbar X\right), & [P_x,P_y] = & 0. \end{array} \tag{2}$$

It is interesting to note that $\sqrt{\Theta}$ and $\sqrt{\tau}$ have dimensions of L and L^{-1}, respectively.

In the limit $\tau \to 0$, we obtain the following non-dynamical NC commutation relations

$$\begin{array}{rlrl} [x^{nc}, y^{nc}] = & i\Theta, & \left[y^{nc}, p_y^{nc}\right] = & i\hbar, \\ [x^{nc}, p_x^{nc}] = & i\hbar, & \left[y^{nc}, p_x^{nc}\right] = & 0, \\ \left[x^{nc}, p_y^{nc}\right] = & 0, & \left[p_x^{nc}, p_y^{nc}\right] = & 0. \end{array} \tag{3}$$

The coordinate X and the momentum P_y are not Hermitian, which makes the Hamiltonian that includes these variables non-Hermitian. We represent algebra (2) in terms of the standard Hermitian NC variable operators $x^{nc}, y^{nc}, p_x^{nc}, p_y^{nc}$ as

$$\begin{aligned} X &= \left(1+\tau\left(y^{nc}\right)^2\right)x^{nc}, \quad Y = y^{nc}, \\ P_y &= \left(1+\tau\left(y^{nc}\right)^2\right)p_y^{nc}, \quad P_x = p_x^{nc}. \end{aligned} \tag{4}$$

From this representation, we can see that some of the operators involved above are no longer Hermitian. However, to convert the non-Hermitian variables into a Hermitian one, we use a similarity transformation as a Dyson map $\eta O \eta^{-1} = o = O^\dagger$ with $\eta = (1+\tau Y^2)^{-\frac{1}{2}}$, as stated in [16]. Therefore, we express the new Hermitian variables x, y, p_x and p_y in terms of NC variables as follows

$$\begin{aligned} x=\eta X\eta^{-1} &= (1+\tau Y^2)^{-\frac{1}{2}}X(1+\tau Y^2)^{\frac{1}{2}} &&= (1+\tau\left(y^{nc}\right)^2)^{\frac{1}{2}}x^{nc}(1+\tau\left(y^{nc}\right)^2)^{\frac{1}{2}},\\ y=\eta Y\eta^{-1} &= (1+\tau\left(y^{nc}\right)^2)^{-\frac{1}{2}}y^{nc}(1+\tau\left(y^{nc}\right)^2)^{\frac{1}{2}} &&= y^{nc},\\ p_x=\eta P_x\eta^{-1} &= (1+\tau\left(y^{nc}\right)^2)^{-\frac{1}{2}}p_x^{nc}(1+\tau\left(y^{nc}\right)^2)^{\frac{1}{2}} &&= p_x^{nc},\\ p_y=\eta P_y\eta^{-1} &= (1+\tau\left(y^{nc}\right)^2)^{-\frac{1}{2}}P_y(1+\tau\left(y^{nc}\right)^2)^{\frac{1}{2}} &&= (1+\tau\left(y^{nc}\right)^2)^{\frac{1}{2}}p_y^{nc}(1+\tau\left(y^{nc}\right)^2)^{\frac{1}{2}}. \end{aligned} \tag{5}$$

These new Hermitian DNC variables satisfy the following commutation relations

$$\begin{aligned} [x,y] &= i\Theta\left(1+\tau y^2\right), & [y,p_y] &= i\hbar\left(1+\tau y^2\right),\\ [x,p_x] &= i\hbar\left(1+\tau y^2\right), & [y,p_x] &= 0,\\ [x,p_y] &= 2i\tau y\left(\Theta p_y+\hbar x\right), & [p_x,p_y] &= 0. \end{aligned} \tag{6}$$

Now, using Bopp-shift transformation [31], one can express the NC variables in terms of the standard commutative variables [5]

$$\begin{aligned} x^{nc} &= x^s - \tfrac{\Theta}{2\hbar}p_y^s, & p_x^{nc} &= p_x^s,\\ y^{nc} &= y^s + \tfrac{\Theta}{2\hbar}p_y^s, & p_y^{nc} &= p_y^s, \end{aligned} \tag{7}$$

where the index s refers to the standard commutative space. The interesting point is that in the DNC space, there is a minimum length for X in a simultaneous X, Y measurement [16]:

$$\triangle X_{min} = \Theta\sqrt{\tau}\sqrt{1+\tau\left\langle Y\right\rangle_\rho^2}, \tag{8}$$

as well, in a simultaneous Y, P_y measurement, we find a minimal momentum as

$$\triangle\left(P_y\right)_{min} = \hbar\sqrt{\tau}\sqrt{1+\tau\left\langle Y\right\rangle_\rho^2}. \tag{9}$$

The motivation and interesting physical consequence for position-dependent noncommutativity is that objects in two-dimensional spaces are string-like [16]. However, investigating the DO in DNC geometry gives rise to some phenomenological consequences that may be very important and useful.

3. Two-dimensional Dirac oscillator in dynamical noncommutative space

3.1. Extension to dynamical noncommutative space

The dynamics of the DO in the presence of a uniform external magnetic field is governed by the following Hamiltonian

$$H_D = c\overrightarrow{\alpha}.\left(\overrightarrow{p}^s - \frac{e}{c}\overrightarrow{A}^s - imc\omega\beta\overrightarrow{r}^s\right) + \beta mc^2, \tag{10}$$

where $\overrightarrow{A}^s\left(A_x^s, A_y^s, A_z^s\right)$ is the vector potential produced by the external magnetic field and e is the charge of the DO (the electron charge). The $\vec{\alpha}$ matrices, in two dimensions, are represented by the following Pauli matrices

$$\alpha_1=\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix}, \alpha_2=\sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix}, \beta=\sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \tag{11}$$

which satisfy the following relations

$$\begin{aligned} &\alpha_i^2=\beta^2=1,\\ &\alpha_i\alpha_j+\alpha_j\alpha_i=0, \qquad i=1,2,3\\ &\alpha_i\beta+\beta\alpha_i=0. \end{aligned} \tag{12}$$

In two dimensions, equation (10) becomes

$$H_D = c\left(\alpha_1 p_x^s + \alpha_2 p_y^s\right) - e\left(\alpha_1 A_x^s + \alpha_2 A_y^s\right) - imc\omega\left(\alpha_1\beta x^s + \alpha_2\beta y^s\right) + \beta mc^2. \tag{13}$$

Let us consider $\overrightarrow{B}$ to be along the z axis, thus the vector potential $\vec{A}^s$ is given in the Landau gauge by

$$\overrightarrow{A} = \frac{B}{2}\left(-y^s, x^s, 0\right), \tag{14}$$

therefore, we have

$$H_D\left(x_i^s, p_i^s\right) = c\left(\alpha_1 p_x^s + \alpha_2 p_y^s\right) + e\frac{B}{2}\left(\alpha_1 y^s - \alpha_2 x^s\right) - imc\omega\left(\alpha_1\beta x^s + \alpha_2\beta y^s\right) + \beta mc^2. \tag{15}$$

The above Hamiltonian in DNC space turns to

$$H_D\left(x_i, p_i\right) = c\left(\alpha_1 p_x + \alpha_2 p_y\right) + e\frac{B}{2}\left(\alpha_1 y - \alpha_2 x\right) - imc\omega\left(\alpha_1\beta x + \alpha_2\beta y\right) + \beta mc^2. \tag{16}$$

Now, using equation (5), we express the Hamiltonian above in terms of NC variables

$$\begin{aligned} H_D\left(x_i^{nc}, p_i^{nc}\right) = \beta mc^2 + c[\alpha_1 p_x^{nc} + \alpha_2\left(1+\tau\left(y^{nc}\right)^2\right)^{\frac{1}{2}} p_y^{nc}\left(1+\tau\left(y^{nc}\right)^2\right)^{\frac{1}{2}}] \\ + e\frac{B}{2}[\alpha_1 y^{nc} - \alpha_2\left(1+\tau\left(y^{nc}\right)^2\right)^{\frac{1}{2}} x^{nc}\left(1+\tau\left(y^{nc}\right)^2\right)^{\frac{1}{2}}] \\ - imc\omega[\alpha_1\beta\left(1+\tau\left(y^{nc}\right)^2\right)^{\frac{1}{2}} x^{nc}\left(1+\tau\left(y^{nc}\right)^2\right)^{\frac{1}{2}} + \alpha_2\beta y^{nc}]. \end{aligned} \tag{17}$$

Since τ is very small, the parentheses can be expanded to the first-order through

$$\left(1+\tau\left(y^{nc}\right)^2\right)^{\frac{1}{2}} = 1 + \frac{1}{2}\tau\left(y^{nc}\right)^2, \tag{18}$$

so that equation (17) turns to

$$\begin{aligned} H_D\left(x_i^{nc}, p_i^{nc}\right) = c\left[\alpha_1 p_x^{nc} + \alpha_2\left\{p_y^{nc} + \frac{1}{2}\tau\left(y^{nc}\right)^2 p_y^{nc} + \frac{1}{2}\tau p_y^{nc}\left(y^{nc}\right)^2\right\}\right] \\ + \beta mc^2 + e\frac{B}{2}\left[\alpha_1 y^{nc} - \alpha_2\left\{x^{nc} + \frac{1}{2}\tau\left(y^{nc}\right)^2 x^{nc} + \frac{1}{2}\tau x^{nc}\left(y^{nc}\right)^2\right\}\right] \\ - imc\omega\left[\alpha_2\beta y^{nc} + \alpha_1\beta\left\{x^{nc} + \frac{1}{2}\tau\left(y^{nc}\right)^2 x^{nc} + \frac{1}{2}\tau x^{nc}\left(y^{nc}\right)^2\right\}\right]. \end{aligned} \tag{19}$$

Using the Bopp-shift transformation (7), Hamiltonian (19) can be expressed in terms of the standard commutative variables

$$\begin{aligned} H_D\left(x_i^s, p_i^s\right) = c\alpha_1 p_x^s + \beta mc^2 + c\alpha_2 p_y^s + c\alpha_2\left\{\frac{1}{2}\tau\left(y^s + \frac{\Theta}{2\hbar}p_x^s\right)^2 p_y^s + \frac{1}{2}\tau p_y^s\left(y^s + \frac{\Theta}{2\hbar}p_x^s\right)^2\right\} \\ + \frac{eB}{2}\left[\alpha_1\left(y^s + \frac{\Theta}{2\hbar}p_x^s\right) - \alpha_2\left\{\frac{\tau}{2}\left(y^s + \frac{\Theta}{2\hbar}p_x^s\right)^2\left(x^s - \frac{\Theta}{2\hbar}p_y^s\right) + x^s - \frac{\Theta}{2\hbar}p_y^s + \frac{1}{2}\tau\left(x^s - \frac{\Theta}{2\hbar}p_y^s\right)\left(y^s + \frac{\Theta}{2\hbar}p_x^s\right)^2\right\}\right] \\ - imc\omega\left[\alpha_2\beta\left(y^s + \frac{\Theta}{2\hbar}p_x^s\right) + \alpha_1\beta\left\{x^s - \frac{\Theta}{2\hbar}p_y^s + \frac{\tau}{2}\left(y^s + \frac{\Theta}{2\hbar}p_x^s\right)^2\left(x^s - \frac{\Theta}{2\hbar}p_y^s\right) + \frac{\tau}{2}\left(x^s - \frac{\Theta}{2\hbar}p_y^s\right)\left(y^s + \frac{\Theta}{2\hbar}p_x^s\right)^2\right\}\right]. \end{aligned} \tag{20}$$

Therefore, to the first-order in Θ and τ, we have (noting that terms containing $\Theta\tau$ are neglected too)

$$H_D\left(x_i^s,p_i^s\right)=c\left[\alpha_1 p_x^s+\alpha_2\left\{p_y^s+\frac{\tau}{2}\left(y^s\right)^2 p_y^s+\frac{1}{2}\tau p_y^s\left(y^s\right)^2\right\}\right]$$
$$+\beta mc^2+e\frac{B}{2}\left[\alpha_1\left(y^s+\frac{\Theta}{2\hbar}p_x^s\right)-\alpha_2\left\{x^s-\frac{\Theta}{2\hbar}p_y^s+\tau x^s\left(y^s\right)^2\right\}\right]$$
$$-imc\omega\left[\alpha_2\beta\left(y^s+\frac{\Theta}{2\hbar}p_x^s\right)+\alpha_1\beta\left\{x^s-\frac{\Theta}{2\hbar}p_y^s+\tau x^s\left(y^s\right)^2\right\}\right], \quad (21)$$

which can be written as

$$H_D=H_0+H_\Theta+H_\tau, \quad (22)$$

with

$$H_0=c\alpha_1 p_x^s+c\alpha_2 p_y^s+\frac{eB}{2}\left(\alpha_1 y^s-\alpha_2 x^s\right)-imc\omega\left(\alpha_1\beta x^s+\alpha_2\beta y^s\right)+\beta mc^2, \quad (23)$$

$$H_\Theta=\frac{\Theta}{2\hbar}\left[\frac{eB}{2}\left(\alpha_1 p_x^s+\alpha_2 p_y^s\right)-imc\omega\left(\alpha_2\beta p_x^s-\alpha_1\beta p_y^s\right)\right], \quad (24)$$

$$H_\tau=\frac{1}{2}\tau\left[c\alpha_2\left(y^s\right)^2 p_y^s+c\alpha_2 p_y^s\left(y^s\right)^2-\left(eB\alpha_2 x^s\left(y^s\right)^2-i2mc\omega\alpha_1\beta x^s\left(y^s\right)^2\right]\right.$$
$$=\frac{\tau}{2}\left[\alpha_2\mathcal{V}_1+\alpha_2\mathcal{V}_2-\left(eB\alpha_2+i2mc\omega\alpha_1\beta\right)\mathcal{V}_3\right], \quad (25)$$

where

$$\mathcal{V}_1=\left(y^s\right)^2 p_y^s,\ \mathcal{V}_2=p_y^s\left(y^s\right)^2,\ \text{and } \mathcal{V}_3=x^s\left(y^s\right)^2. \quad (26)$$

Knowing that H_τ is the perturbation Hamiltonian in which it reflects the effects of dynamical noncommutativity of space on the DO Hamiltonian. We can also treat the term proportional to Θ, given in equation (24) as a perturbation term. But here, and in a different way, we will accurately calculate the energy of the deformed system H_0+H_Θ and employ it to test the effect of the DNC space on the DO. Thus, we consider the following unperturbed system

$$H_{UNP}=H_0+H_\Theta. \quad (27)$$

While the noncommutativity parameter τ is non-zero and very small, one can use the perturbation theory to find the spectrum of the systems in question.

The two-dimensional DO equation in the DNC space is written as follows

$$H_D\left|\psi_D\right\rangle=\left(H_{UNP}+H_\tau\right)\left|\psi_D\right\rangle=E_{\Theta,\tau}\left|\psi_D\right\rangle, \quad (28)$$

with

$$\left|\psi_D\right\rangle=\left(\left|\psi_1\right\rangle,\left|\psi_2\right\rangle\right)^T, \quad (29)$$

being the wave function of the system in question.

3.2. Unperturbed eigenvalues and eigenvectors

We introduce the following complex coordinates

$$z^s = x^s + iy^s,\ \overline{z}^s = x^s - iy^s, \tag{30}$$

$$\begin{array}{l} p_z^s = -i\hbar\frac{d}{dz^s} = \frac{1}{2}\left(p_x^s - ip_y^s\right), \\ p_{\overline{z}}^s = -i\hbar\frac{d}{d\overline{z}^s} = \frac{1}{2}\left(p_x^s + ip_y^s\right), \end{array} \tag{31}$$

where

$$[z^s, p_z^s] = [\overline{z}^s, p_{\overline{z}}^s] = i\hbar, \qquad [z^s, p_{\overline{z}}^s] = [\overline{z}^s, p_z^s] = 0. \tag{32}$$

Using equation (11), our unperturbed system (24), in the complex formalism, merely becomes

$$H_{UNP} = \begin{bmatrix} mc^2 & 2c\Omega p_z^s + imc\overline{z}^s\tilde{\omega} \\ 2c\Omega p_{\overline{z}}^s - imc\omega z^s\tilde{\omega} & -mc^2 \end{bmatrix}, \tag{33}$$

where

$$\Omega = 1 + m\frac{\Theta\tilde{\omega}}{2\hbar} \text{ with } \tilde{\omega} = \omega - \frac{\omega_c}{2}, \tag{34}$$

knowing that $\omega_c = \frac{|e|B}{mc}$ is the cyclotron frequency.
Now, let us introduce the following creation and annihilation operators

$$a = i\left(\frac{\Omega}{\sqrt{m\varpi\hbar}}p_{\overline{z}}^s - \frac{i}{2\Omega}\sqrt{\frac{m\varpi}{\hbar}}z^s\right), \tag{35}$$

$$a^\dagger = -i\left(\frac{\Omega}{\sqrt{m\varpi\hbar}}p_z^s + \frac{i}{2\Omega}\sqrt{\frac{m\varpi}{\hbar}}\overline{z}^s\right), \tag{36}$$

that satisfy the following commutations relations

$$\left[a, a^\dagger\right] = 1,\ [a, a] = \left[a^\dagger, a^\dagger\right] = 0. \tag{37}$$

Thus, in terms of the creation and annihilation operators, the Hamiltonian (33) takes the following form

$$H_\Theta = \begin{bmatrix} mc^2 & i2c\sqrt{m\varpi\hbar}a^\dagger \\ -i2c\sqrt{m\varpi\hbar}a & -mc^2 \end{bmatrix} = \begin{bmatrix} mc^2 & i\overline{g}a^\dagger \\ -i\overline{g}a & -mc^2 \end{bmatrix}, \tag{38}$$

with $\overline{g} = 2c\sqrt{m\hbar\varpi}$ being a parameter that describes the coupling between different states in NC space, and $\varpi = \Omega\tilde{\omega}$ being the correction of the frequency $\tilde{\omega}$ of the commutative space. In addition, the parameter $g = 2c\sqrt{m\tilde{\omega}\hbar}$ describes the coupling between different states in the commutative space.

Now, we solve the following equation

$$H_{UNP}\left|\overline{\psi}_D\right\rangle = E_\Theta\left|\overline{\psi}_D\right\rangle, \tag{39}$$

where E_Θ, $\left|\overline{\psi}_D\right\rangle$ are the eigenenergy and wave function of the Dirac equation above, respectively.
By inserting equation (29) in (39), we obtain the following system of equations

$$\begin{pmatrix} mc^2 & i\overline{g}a^\dagger \\ -i\overline{g}a & -mc^2 \end{pmatrix}\begin{pmatrix} |\overline{\psi}_1> \\ |\overline{\psi}_2> \end{pmatrix} = E_\Theta\begin{pmatrix} |\overline{\psi}_1> \\ |\overline{\psi}_2> \end{pmatrix}, \tag{40}$$

where

$$\left(mc^2 - E_\Theta\right) \mid \overline{\psi}_1 > +i\overline{g}a^\dagger \mid \overline{\psi}_2 >= 0, \tag{41}$$

$$-i\overline{g}a \mid \overline{\psi}_1 > - \left(mc^2 + E_\Theta\right) \mid \overline{\psi}_2 >= 0. \tag{42}$$

From the equations (41) and (42), we have

$$\mid \overline{\psi}_2 >= \frac{-i\overline{g}a}{E_\Theta + mc^2} \mid \overline{\psi}_1 >, \tag{43}$$

subsequently

$$\left[\overline{g}^2 a^\dagger a + m^2c^4 - (E_\Theta)^2\right] \mid \overline{\psi}_1 >= 0. \tag{44}$$

On the basis of the second quantization, of which $\mid \overline{\psi}_1 >\equiv\mid n >$, we have

$$\left[\overline{g}^2 n + m^2c^4 - (E_\Theta)^2\right] \mid n >= 0,\ a^\dagger a \mid n >= n \mid n > . \tag{45}$$

Thus, the energy spectrum is given by

$$E^{\pm}_{\Theta,n} = \pm\sqrt{m^2c^4 + \overline{g}^2 n}, \tag{46}$$

which can be rewritten as

$$E^{\pm}_{\Theta,n} = \pm mc^2\sqrt{1 + \frac{4\hbar\tilde{\omega}}{mc^2}\left(1 + m\frac{\Theta\tilde{\omega}}{2\hbar}\right)n},\ n = 0, 1, 2, ... \tag{47}$$

Furthermore, we have the reduced energy spectrum

$$\frac{E^{\pm}_n}{E_0} = \pm\sqrt{1 + 4w\left(1 + \frac{1}{2}qw\right)n}, \tag{48}$$

where the non-relativistic limit feature is reduced in $w = \frac{\hbar\tilde{\omega}}{mc^2}$, which is a parameter that controls the non-relativistic limit within the NC space (as well as in commutative case, if $\Theta = 0$), and $E_0 = mc^2$ is a background energy, which corresponds to $n = 0$. And $q = \frac{\Theta}{\Theta_0}$ with $\Theta_0 = \left(\frac{\hbar}{mc}\right)^2$of the dimension $\left[\frac{\hbar}{mc}\right]^2 = L^2 \equiv m^2$.

The corresponding wave function is written as a function of the basis $\mid n >= \frac{\left(a^\dagger\right)^n}{\sqrt{n!}} \mid 0 >$, and it is given by the following formula

$$\mid \overline{\psi}^{\pm}_n >= c^{\pm}_n \mid n; \frac{1}{2} > +id^{\pm}_n \mid n-1; -\frac{1}{2} >, \tag{49}$$

where the coefficients $c^{\pm}_n$ and $d^{\pm}_n$ are determined from the normalization condition. We thus obtain [24]

$$c^{\pm}_n = \sqrt{\frac{E^{+}_n \pm mc^2}{2E^{+}_n}},\ d^{\pm}_n = \mp\sqrt{\frac{E^{+}_n \mp mc^2}{2E^{+}_n}}. \tag{50}$$

In the limit $\Theta \to 0$, the NC energy spectrum becomes commutative one, i.e., equation (47) turns to equation (10) of ref. [32], which confirms that we are in good agreement. As well, in ref. [33] Boumali et al. made a study of a DO in an NC phase-space, where, if $\overline{\Theta} \to 0$, the energy eigenvalues (eq:50) will be similar as ours in equation (47).

We plot the reduced energy spectrum in terms of quantum number n, for the cases $w = 1$, $q = 1$; $w = 1$, $q = 2$ and the commutative case with $w = 1$.

The $\frac{E_n^\pm}{E_0}$, as a function of quantum number n of equation (48) in both commutative ($\Theta = q = 0$) and NC ($q = 1$; $q = 2$) spaces, is illustrated in Fig. 1. Knowing that Fig. 1 discloses that the influence of the NC parameter on the energy spectrum is considerable and significant.

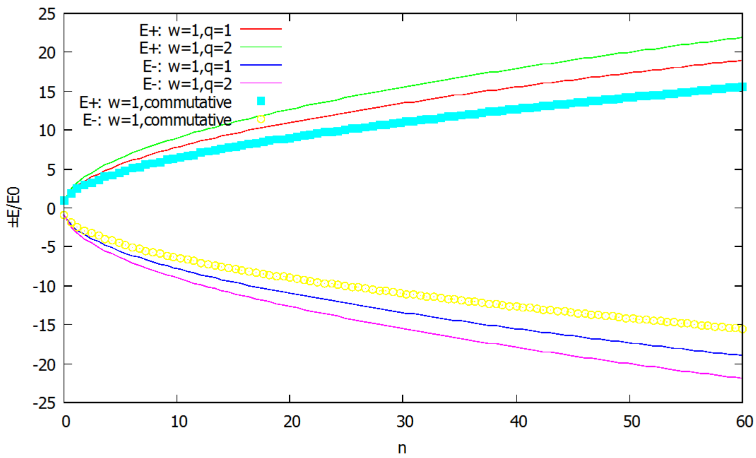

Figure 1. A reduced energy versus quantum number in both cases of NC and commutative spaces.

The following figure shows the coupling parameters g and $\bar{g}$, between different levels for the two cases in the NC space.

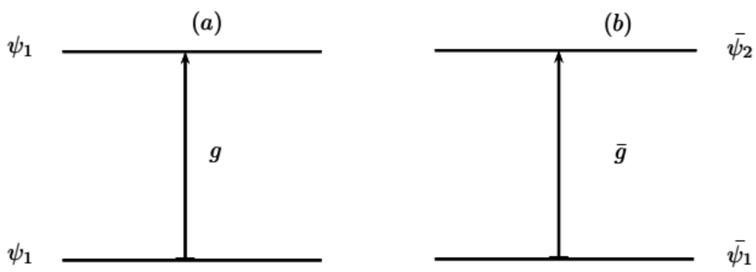

Figure 2. The coupling parameter between different levels: (a) in the case of commutative space, (b) in the case of NC space.

While n are non-negative integers, we explicitly observe that our eigenvalues are non-degenerated (the spectrum has no degeneracy), this case can be explained by the fact that the particle is restricted to moving in two dimensions, and the third dimension does not contribute in the form of energy. Knowing that, it will be an infinite degeneracy when there is a contribution of an element related to the third dimension, such as k_z or p_z.

In more detail and indirectly (in other sense), the energy spectrum is degenerated. As is known, this is related to the Landau problem, and it is known that there is an infinite degeneracy. Nevertheless, we consider the energy spectrum non-degenerated, because we do not rely on the states with different angular momentum, which is not useful here. The reason is that when we use chiral creation and annihilation operators (a_l, $a_l^\dagger$ and a_r, $a_r^\dagger$), we see that the number of particles n_r created by right operators does not appear in the form of energy, we see only number of particles n_l generated by left operators. However, right operators create excitations with a definite angular momentum in one or the other direction; thus, in this sense, we have the degeneracy.

This point is very important to clarify because our calculations in the perturbation theory depend on this point. As many researchers have dealt with this sensitive point and considered that the spectrum has no degeneracy such as [33]. Besides, differently, for instance, energy levels can appear explicitly degenerated, as in a study [34] about the mesoscopic states in a relativistic Landau levels, the authors found that the energy spectrum depends on p_z^2 (check eq. 13 in this cited reference), which is the underlying reason for an infinite degeneracy of all levels.

3.3. Perturbed system

In this sub-section, we aim to determine the correction of first-order energy by using first-order energy shift formulas. To explain the structure of our spectrum, we will use time-independent perturbation theory for small values of the parameter τ. In view that energies are non-degenerated, we use the non-degenerated time-independent perturbation theory

$$\left|\overline{\psi_n}\right\rangle = \left|\overline{\psi}_n^{(0)}\right\rangle + \tau\left|\overline{\psi}_n^{(1)}\right\rangle + \tau^2\left|\overline{\psi}_n^{(2)}\right\rangle + ... \tag{51}$$

$$E_n = E_n^{(0)} + \tau E_n^{(1)} + \tau^2 E_n^{(2)} + ... \tag{52}$$

Here, the (0) superscript denotes the quantities that are associated with the unperturbed system.

The first-order correction to the eigenvalues and eigenvectors in perturbation theory are simply given by

$$E_n^{(1)} = \triangle E_n =< \overline{\psi}_n^{(0)} \mid \frac{1}{\tau}H_\tau \mid \overline{\psi}_n^{(0)} >, \tag{53}$$

$$\left|\overline{\psi}_n^{(1)}\right\rangle = \sum_{k\neq n}\frac{< \overline{\psi}_k^{(0)} \mid \frac{1}{\tau}H_\tau \mid \overline{\psi}_n^{(0)} >}{E_n^{(0)} - E_k^{(0)}}\left|\overline{\psi}_k^{(0)}\right\rangle. \tag{54}$$

Inserting equation (25) into the equation above, we find

$$E_n^{(1)} =< \overline{\psi}_n^{(0)} \mid \frac{1}{2}\{\alpha_2(\mathcal{V}_1 + \mathcal{V}_2) - (eB\alpha_2 + i2mc\omega\alpha_1\beta)\,\mathcal{V}_3\} \mid \overline{\psi}_n^{(0)} >, \tag{55}$$

the operator method can also be used to obtain the energy shift in Fock space. In our scenario, we require adopting the notation of the state as follows

$$\mid \overline{\psi}_n^{(0)} >=\mid n_x, n_y > . \tag{56}$$

The perturbation matrix is given by

$$\mathcal{M} =< n_x, n_y \mid \frac{i}{2}\begin{pmatrix} 0 & -\mathcal{V}_1 - \mathcal{V}_2 + \Upsilon\mathcal{V}_3 \\ \mathcal{V}_1 + \mathcal{V}_2 - \Upsilon\mathcal{V}_3 & 0 \end{pmatrix} \mid n_x^{'}, n_y^{'} >, \tag{57}$$

with $\Upsilon = eB + 2mc\omega$. To calculate the influence of $\mathcal{V}_i$ $(i = 1, .., 3)$ on the element of the Fock basis, we conveniently use the following b_j, $b_j^\dagger$ $(j = x, y)$ operators

$$b_j = \sqrt{\frac{m\omega}{2\hbar}}\left(x_j^s + i\frac{p_j^s}{m\omega}\right) \text{ and } b_j^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x_j^s - i\frac{p_j^s}{m\omega}\right), \tag{58}$$

where

$$\left[b_j, b_j^\dagger\right] = 1, \text{ with } b_j^\dagger b_j = N_j. \tag{59}$$

The above creation and annihilation operators are, in fact, extracted from the one in 3.2, when $(a, a^\dagger) = \left\{i\left(a_x + ia_y\right), -i\left(a_x^\dagger - ia_y^\dagger\right)\right\} |_{\Theta\to 0}\to (b, b^\dagger) = \left\{i\left(b_x + ib_y\right), -i\left(b_x^\dagger - ib_y^\dagger\right)\right\}$ (because in 3.3, we deal only with τ). In fact, we have only one integer, which is n, but with the feature $n = n_x + n_y$. We deliberately use n_x, n_y instead of n because in the perturbed Hamiltonian, we cannot use a complex formalism, thus we divide n into n_x and n_y.

With the help of the following definitions of eigenkets and central properties of creation and annihilation operators [35]

$$\begin{array}{rcl} b_j \mid n_j > & = & \sqrt{n_j} \mid n_j - 1 >, \\ b_j^\dagger \mid n_j > & = & \sqrt{n_j + 1} \mid n_j + 1 >, \\ b_j^2 \mid n_j > & = & \sqrt{n_j\left(n_j - 1\right)} \mid n_j - 2 >, \\ b_j^{\dagger 2} \mid n_j > & = & \sqrt{\left(n_j + 1\right)\left(n_j + 2\right)} \mid n_j + 2 >, \\ b_j^3 \mid n_j > & = & \sqrt{n_j\left(n_j - 1\right)\left(n_j - 2\right)} \mid n_j - 3 >, \\ b_j^{\dagger 3} \mid n_j > & = & \sqrt{\left(n_j + 1\right)\left(n_j + 2\right)\left(n_j + 3\right)} \mid n_j + 3 >, \\ & \vdots & \end{array} \tag{60}$$

with $< n_j^{'} \mid n_j >= \delta_{n_j^{'},n_j}$ and

$$\left[b^{\dagger}b, b^{\dagger}\right] = b^{\dagger}bb^{\dagger} - b^{\dagger 2}b = b^{\dagger} \text{ and } \left[b^{\dagger}b, b\right] = b^{\dagger}b^2 - bb^{\dagger}b = -b, \tag{61}$$

$$b^{\dagger}bb^{\dagger} \mid n >= Nb^{\dagger} \mid n >= (n+1)\, b^{\dagger} \mid n >, \tag{62}$$

$$b^{\dagger}bb \mid n >= Nb \mid n >= (n-1)\, b \mid n >, \tag{63}$$

$$[N, b] = -b \text{ and } bN = Nb + b, \tag{64}$$

$$\left[N, b^{\dagger}\right] = b^{\dagger} \text{ and } b^{\dagger}N = Nb^{\dagger} - b^{\dagger}. \tag{65}$$

Knowing that

$$x_j^s = \sqrt{\frac{\hbar}{2m\omega}} \left(b_j + b_j^{\dagger}\right) \text{ and } p_j^s = i\sqrt{\frac{\hbar m\omega}{2}} \left(b_j^{\dagger} - b_j\right). \tag{66}$$

The contributions of the different parts of the perturbed Hamiltonian are as follows

$$\begin{aligned} < n_x, n_y \mid \mathcal{V}_1 \mid n_x^{'}, n_y^{'} >=& < n_x, n_y \mid (y^s)^2\, p_y^s \mid n_x^{'}, n_y^{'} > \\ =& \frac{-i\hbar}{2}\sqrt{\frac{\hbar}{2m\omega}}\delta_{n_x,n_x^{'}} \left\{\sqrt{n_y^{'}\left(n_y^{'}-1\right)\left(n_y^{'}-2\right)}\delta_{n_y,n_y^{'}-3}\right. \\ & \left. - \sqrt{\left(n_y^{'}+1\right)\left(n_y^{'}+2\right)\left(n_y^{'}+3\right)}\delta_{n_y,n_y^{'}+3} + \left(n_y^{'}-2\right)\sqrt{n_y^{'}}\delta_{n_y,n_y^{'}-1} - \left(n_y^{'}+3\right)\sqrt{n_y^{'}+1}\delta_{n_y,n_y^{'}+1}\right\}. \end{aligned} \tag{67}$$

$$\begin{aligned} < n_x, n_y \mid \mathcal{V}_2 \mid n_x^{'}, n_y^{'} >=& < n_x, n_y \mid p_y^s\, (y^s)^2 \mid n_x^{'}, n_y^{'} > \\ =& \frac{-i\hbar}{2}\sqrt{\frac{\hbar}{2m\omega}}\delta_{n_x,n_x^{'}} \left\{\sqrt{n_y^{'}\left(n_y^{'}-1\right)\left(n_y^{'}-2\right)}\delta_{n_y,n_y^{'}-3}\right. \\ & \left. - \sqrt{\left(n_y^{'}+1\right)\left(n_y^{'}+2\right)\left(n_y^{'}+3\right)}\delta_{n_y,n_y^{'}+3} + \left(n_y^{'}+2\right)\sqrt{n_y^{'}}\delta_{n_y,n_y^{'}-1} + \left(3-n_y^{'}\right)\sqrt{n_y^{'}+1}\delta_{n_y,n_y^{'}+1}\right\}. \end{aligned} \tag{68}$$

$$\begin{aligned} < n_x, n_y \mid \mathcal{V}_3 \mid n_x^{'}, n_y^{'} >=& < n_x, n_y \mid x^s\, (y^s)^2 \mid n_x^{'}, n_y^{'} > \\ =& \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left(\sqrt{n_x^{'}}\delta_{n_x,n_x^{'}-1} + \sqrt{n_x^{'}+1}\delta_{n_x,n_x^{'}+1}\right) \\ & \times \left(\sqrt{n_y^{'}\left(n_y^{'}-1\right)}\delta_{n_y,n_y^{'}-2} + \sqrt{\left(n_y^{'}+1\right)\left(n_y^{'}+2\right)}\delta_{n_y,n_y^{'}+2} + \left(1+2n_y^{'}\right)\delta_{n_y,n_y^{'}}\right). \end{aligned} \tag{69}$$

The relevant perturbation matrix is given by

$$\mathcal{M} = i\begin{pmatrix} 0 & \mathcal{W}_{12} \\ \mathcal{W}_{21} & 0 \end{pmatrix}, \tag{70}$$

with

$$\mathcal{W}_{12} = -\mathcal{W}_{21} =< n_x, n_y \mid \frac{1}{2}\{\Upsilon\mathcal{V}_3 - (\mathcal{V}_1 + \mathcal{V}_2)\} \mid n_x^{'}, n_y^{'} > . \tag{71}$$

The DO Hamiltonian $H_D = H_0 + H_\Theta + H_\tau$ may be represented by a square matrix as follows (we have used the basis given by unperturbed energy eigenkets)

$$H_D \equiv \begin{pmatrix} E^{(0)}_{n,+}(\Theta) & i\tau\mathcal{W}_{12} \\ i\tau\mathcal{W}_{21} & E^{(0)}_{n,-}(\Theta) \end{pmatrix}. \tag{72}$$

The eigenvalues of the problem above are

$$\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \frac{E^{(0)}_- + E^{(0)}_+}{2} \pm \sqrt{\frac{\left(E^{(0)}_- - E^{(0)}_+\right)^2}{4} + \lambda^2 \left|\mathcal{W}_{12}\right|^2}. \tag{73}$$

Here, we set $\lambda = i\tau$. Supposing that $\lambda\left|\mathcal{W}_{12}\right|$ is small as compared to relevant energy scale, so that the difference of the energy eigenvalues of the unperturbed system equals

$$\lambda\left|\mathcal{W}_{12}\right| < \left|E^{(0)}_- - E^{(0)}_+\right|. \tag{74}$$

To obtain the expansion of the energy eigenvalues in the presence of a perturbation, namely (a perturbation expansion always exists for a sufficiently weak perturbation)

$$\begin{aligned} E_1 &= E^{(0)}_- + \tfrac{\lambda^2\left|\mathcal{W}_{12}\right|^2}{E^{(0)}_- - E^{(0)}_+} + \ldots \\ E_2 &= E^{(0)}_+ + \tfrac{\lambda^2\left|\mathcal{W}_{12}\right|^2}{E^{(0)}_+ - E^{(0)}_-} + \ldots \end{aligned} \tag{75}$$

We terminate the calculation by the radius of convergence of series expansion (75), so while λ is a complex variable, λ is increased from zero, branch points are encoutered at [34]

$$\lambda\left|\mathcal{W}_{12}\right| = \frac{\pm i\left(E^{(0)}_- - E^{(0)}_+\right)}{2}, \tag{76}$$

the condition for the convergence of (75) for the $\lambda = 1$ full strength case is

$$\left|\mathcal{W}_{12}\right| = \frac{\left|E^{(0)}_- - E^{(0)}_+\right|}{2}. \tag{77}$$

If this condition is not met, the expansion (75) is meaningless.

It can be checked that all the results of the NC case can be obtained from the DNC case directly by taking the limit of $\tau \to 0$, for instance, equations (51) and (52) give the same values as the eigenvalues and eigenvectors in the NC space, i.e., equations (47) and (49), respectively.

It may be also useful to mention that the DO in deformed spaces (including NC spaces) has been investigated in [36–43].

We can regard equation (75) as the eigenvalues of our system, where we restrict ourselves to the first-order correction to the eigenvalues and eigenvectors, which leads the energy shift for the ground state. Besides, it is easy to obtain the eigensolutions for excited states.

It is interesting to illustrate the DNC effect on DO energy levels. This effect is reduced in the energy shifts obtained, hence we do the following sample, with

$$a_1 = \frac{\Upsilon}{2}\left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}}, a_2 = \frac{i\hbar}{2\omega}\left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}}, a_3 = i\hbar\left(\frac{\hbar}{m\omega}\right)^{\frac{1}{2}}, a_4 = \frac{i\hbar}{2}\left(\frac{\hbar}{m\omega}\right)^{\frac{1}{2}}. \tag{78}$$

In Table 1, all numerical values of the energy are in units of eV. It may be worth to underline that thanks to the Kronecker's delta, the elements of the perturbed Hamiltonian will seldom take many values.

Now, the DNC and non-DNC effects on the energy levels of the DO are illustrated in Fig. 3.

The upper bound on the value of the NC parameter Θ is $\sqrt{\Theta} \leq 2 \times 10^{-20}$ m [44], as well for τ is $\sqrt{\tau} \leq 10^{-17}$ eV [45]. The bound on $\sqrt{\tau}$ is consistent with the accuracy in the energy measurement 10^{-12} eV.

$\left(n_x, n_y, n_x^{'}, n_y^{'}\right)$	$\mathcal{W}_{12}$	E_1	E_2	$\triangle E$
$(1,1,1,1)$	0	$-5, 11.10^5$	$5, 11.10^5$	0
$(1,0,0,0)$	a_1	$-5, 11.10^5 + \frac{a_1^2\tau^2}{10{,}22.10^5}$	$5, 11.10^5 - \frac{a_1^2\tau^2}{10{,}22.10^5}$	$\frac{a_1^2\tau^2}{10{,}22.10^5}$
$(0,0,0,1)$	a_2	$-5, 11.10^5 + \frac{a_2^2\tau^2}{10{,}22.10^5}$	$5, 11.10^5 - \frac{a_2^2\tau^2}{10{,}22.10^5}$	$\frac{a_2^2\tau^2}{10{,}22.10^5}$
$(0,1,0,2)$	a_3	$-5, 11.10^5 + \frac{a_3^2\tau^2}{10{,}22.10^5}$	$5, 11.10^5 - \frac{a_3^2\tau^2}{10{,}22.10^5}$	$\frac{a_3^2\tau^2}{10{,}22.10^5}$
$(0,2,0,1)$	-a_4	$-5, 11.10^5 + \frac{a_4^2\tau^2}{10{,}22.10^5}$	$5, 11.10^5 - \frac{a_4^2\tau^2}{10{,}22.10^5}$	$\frac{a_4^2\tau^2}{10{,}22.10^5}$

Table 1. Energy levels due to DNC space, where we suffice with eigenvalues corrections to the ground state, i.e. $E_{1,2} = \pm mc^2$.

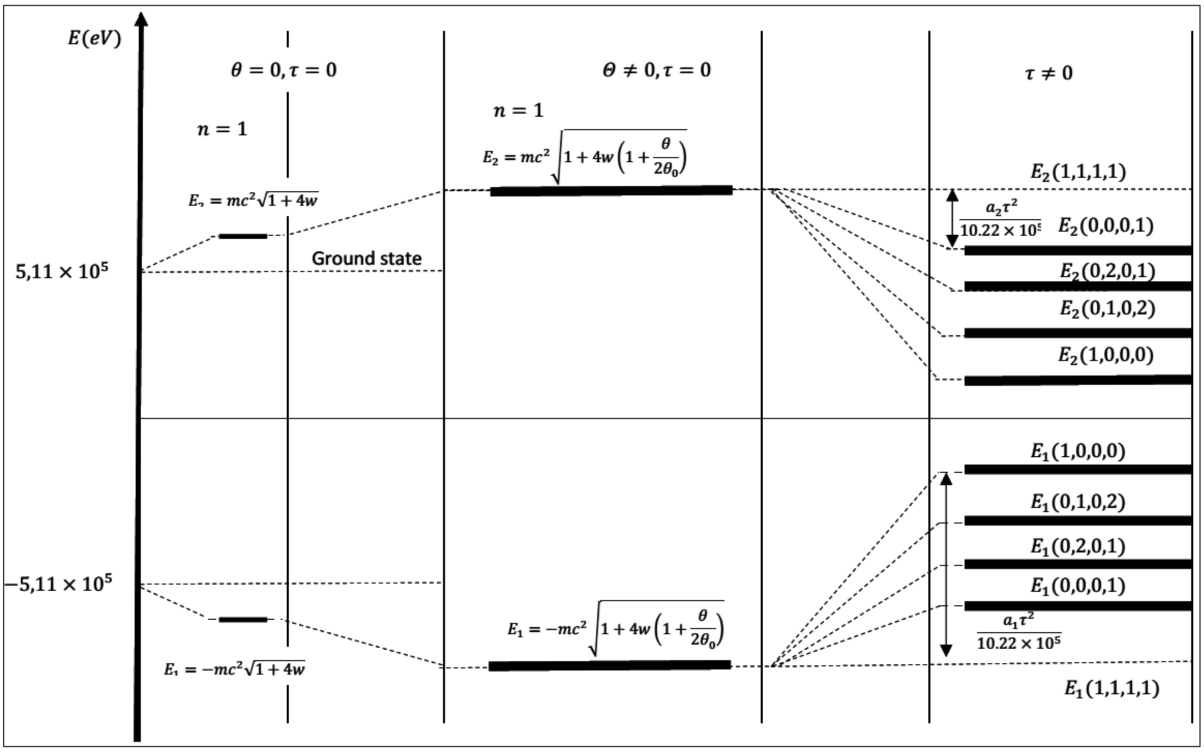

Figure 3. Diagram of splittings for energy levels due to DNC and non-DNC spaces.

It is important to clarify that the presence of τ^2 in the eigenvalues is not due to the action of the second-order correction, but rather due to the Dirac matrices in the perturbed Hamiltonian term.

In Fig. 3, we see the energy levels are splitting as the Θ and τ parameters turn on. Values of the eigenvalues E_1 and E_2 are shown in Table 1. First, in Fig. 3, we show the effect of the NC space when the perturbation parameter is off ($\tau = 0$), where the effect of noncommutativity is very significative as explained in Fig. 1. Thereafter, the effect of noncommutativity is more significative when the DNC perturbation term is present ($\tau \neq 0$), the presence of this term exhibits the energy shifts. The last part of Fig. 3 shows the combined effect of both the DNC and NC parameters, where the effect is very evident.

4. Conclusion

In conclusion, we have investigated the effects of a DNC space on the 2D DO in the presence of an external magnetic field in terms of creation and annihilation operator languages and through properly chosen canonical pairs of coordinates and its corresponding momenta in a complex NC space. However, the dynamical noncommutativity was treated as a perturbation. More precisely, we have solved the DO problem in a two-dimensional NC space to find the exact energy spectrum and wave functions. Therefore, we have employed these obtained results to find the first-order correction to the eigenvalues and eigenvectors. It is worth noting that we addressed the system in the NC space as an unperturbed system instead of considering the fundamental system in a commutative space and noncommutativity as a perturbation. The first-order correction for the ground state of the DO due to the noncommutativity of space is zero for the non-DNC case while it has a nonvanishing value in the DNC case. Knowing that, the result reduces to that of a usual DO in the limits of $\tau \to 0$, $\Theta \to 0$.

As mentioned in section 2, some operators in the DNC space are non-Hermitian. This mixture of DNCS, the non-Hermiticity theory and the string theory can lead to fundamental new insights in these three fields.

Distinctly, there are plenty of interesting problems arising from our investigation, such as the investigation of further possibilities of consistent deformations, and studies of additional models in terms of the DNC variables.

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