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F3

Faculty of Electrical Engineering
Department of Cybernetics

Reichenbach's Common Cause Principle

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BACHELOR‘S THESIS ASSIGNMENT

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II. Bachelor’s thesis details

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Reichenbach's Common Cause Principle

Bachelor’s thesis title in Czech:

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Guidelines:

Reichenbach's Common Cause Principle is a philosophical principle with a clear mathematical formulation, studied by mathematicians and physicists. There are many problems and open questions concerning the existence of the common cause or the possibility of embedding a system into a larger one in which the common cause of any couple of correlated events exists.
1. Demonstrate the principle on illustrative examples.
2. Make critical research of the existing works on the mathematical aspects of the principle (for both classical and quantum probability models) and suggest improvements or extensions of some of them.

Bibliography / sources:

[1] Gábor Hofer-Szabó, Miklós Rédei, and László E. Szabó. “Common Cause Completability of Classical and Quantum Probability Spaces”. International Journal of Theoretical Physics 39.3 (Mar. 2000), pp. 913–919. DOI: 10.1023/A:1003643300514.
[2] Yuichiro Kitajima. “Reichenbach’s Common Cause in an Atomless and Complete Orthomodular Lattice”. International Journal of Theoretical Physics 47.2 (Feb. 2008), pp. 511–519. DOI: 10. 1007/s10773-007-9475-2.
[3] Gábor Hofer-Szabó, Miklós Rédei, and László E. Szabó. The Principle of the Common Cause. Cambridge University Press, 2013. DOI: 10.1017/CBO9781139094344.

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Declaration

I declare that the presented work was developed independently and that I have listed all sources of information used within it in accordance with the methodical instructions for observing the ethical principles in the preparation of university theses.

Prague, 26. May 2023

Abstract

This thesis deals with Reichenbach's principle of common cause. This principle was published in 1956 and its author is Hans Reichenbach. This principle slightly interferes with the philosophy of science. In particular, it tries to explain some macro statistical asymmetries that arise from the second law of thermodynamics. This principle has already been discussed in depth in a variety of publications. In this thesis, we provide and amend some proofs that we did not find. We also add, modify and correct some lemmas and conclusions from the already published literature and answer some open questions.

Keywords: Orthomodular lattice, Hans Reichenbach, common cause, Reichenbach's common cause principle

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Abstrakt

Tato práce se zabývá Reichenbachovým principem společné příčiny. Tento princip byl publikován v roce 1956 a jeho autorem je Hans Reichenbach. Tento princip mírně zasahuje do filozofie vědy. Zejména se snaží vysvětlit některé makrostatistické asymetrie, které vyplývají z druhého zákona termodynamického. Tento princip byl již do hloubky rozebrán v řadě publikací. V této práci poskytujeme a doplňujeme některé důkazy, které jsme nenalezli. Dále doplňujeme, upravujeme a opravujeme některá lemmata a závěry z již publikované literatury a odpovídáme na některé otevřené otázky.

Klíčová slova: Ortomodulární svaz, Hans Reichenbach, sploleční příčina, Reichenbachův princip společné příčiny

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Chapter 1

Introduction

1.1 History

This thesis deals with *Reichenbach's common cause principle* introduced by Hans Reichenbach in his book [15]. [1]

Hans Reichenbach was a philosopher of science who looked deeper into *correlation* and its relation with *causation*. Reichenbach claimed that if there is a *correlation* between two *events* and there is no direct link between the *events*, then there exists a third *event* which is called the *common cause* of the *correlation*. In his own words [15]:

> If an improbable coincidence has occurred, there must exist a common cause.

He studied it in the context of the *second law of thermodynamics*, which states that *entropy* of a closed system may only increase over time. This fact implies some *macrostatistical asymmetries* which are still not fully understood.

However, the *common cause principle* has also been subject to criticism and debate. One of the main criticisms is that the principle assumes that there are no hidden causal factors that could explain the *correlation* between two events, which is often different in real-world scenarios. Additionally, the principle does not provide a way to identify the *common cause*, and it is often difficult to distinguish between spurious *correlations* and genuine causal relationships.

Despite these criticisms, *Reichenbach's common cause principle* remains an important concept in the philosophy of science and has contributed to our

[1] respectively by his wife Maria Reichenbach, who published the book since Hans Reichenbach died in 1953

understanding of *causation* and *correlation*. We will illustrate an outline of the *common cause* in the following example:

Example 1.1 (Common cause)**.** Here we illustrate *events* A and B having *event* C as their *common cause*.

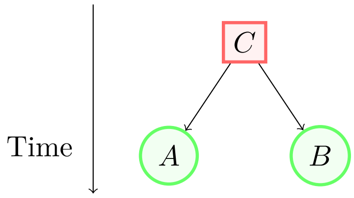

Figure 1.1: Illustration of the common cause

We will analyze the following statement[2]:

> People who eat caviar live longer. The common cause is wealth.

In this case, the event A is eating caviar. The event B is longer life, and the event C is wealth.

Reichenbach's common cause principle was studied in the context of the *classical probability theory* as well as in the context of *non-classical probability theory*. This thesis mostly follows up on [10], [7] and [8].

1.2 Introductory definitions

First of all, we will define some basic mathematical concepts used in this text.

Definition 1.2 (Partially ordered set)**.** Let S be a *set* and let $\leq$ be a *binary relation* on S which a, b, c S satisfies:

1. *Reflexivity*: $a \leq a$
2. *Antisymmetry*: If $a \leq b$ and $b \leq a$ then $a = b$
3. *Transitivity*: If $a \leq b$ and $b \leq c$ then $a \leq c$

The pair $(S, \leq)$ is then called a *partially ordered set*.

Partially ordered sets are also called *posets*. *Posets* formalize and generalize the concept of ordering and arrangement of the elements of a *set*. Note that

[2] Taken from Ján Markoš: Sila rozumu v bláznivej dobe. N Press, 2019.

the *order* is only *partial*. This, in contrast to *total order*, means that not every pair of elements is *comparable*.

Definition 1.3 (Lattice)**.** A *partially ordered set* $(L, \leq)$ is called a *lattice* if and only if any pair of elements $a, b \in L$ has a unique:

1. *Infimum* inf(a,b) denoted as $a \wedge b$. The *infimum* is a *greatest lower bound* of a and b. In other words, $a \wedge b$ is the largest element in L such that $a \wedge b \leq a$ and $a \wedge b \leq b$.
2. *Supremum* sup(a,b) denoted as $a \vee b$. The *supremum* is a *least upper bound* of a and b. That is, $a \vee b$ is the smallest element in L such that $a \leq a \vee b$ and $b \leq a \vee b$.

In a *lattice*, the *infimum* and *supremum* are unique for any pair of *elements* $a, b \in L$.

Definition 1.4 (Orthomodular lattice)**.** A *lattice* L is called an *orthomodular lattice* when $0, 1 \in L$ where 0 is the *least element* and 1 is the *greatest element* and there is a given *mapping* $\perp : L \to L$, called an *orthocomplementation*, with the following properties for all $a, b \in L$

1. $a \vee a^{\perp} = 1$
2. $a \leq b \implies a^{\perp} \geq b^{\perp}$
3. $(a^{\perp})^{\perp} = a$
4. $a \leq b \implies b = a \vee (b \wedge a^{\perp})$

When $a \leq b^{\perp}$ we say that a and b are *orthogonal*.

The *symmetric difference* of two *sets* X and Y is defined as the *set* of *elements* that belong to either X or Y, but not to both. Mathematically, the symmetric difference can be defined as follows:

Definition 1.5 (Symmetric difference in a set)**.** Let X and Y be sets. Then we define the operation of *symmetric difference* as follows:

$$\Delta(X, Y) = (X \setminus Y) \cup (Y \setminus X)$$

The *symmetric difference* operation is *commutative*, meaning that $\Delta(X, Y) = \Delta(Y, X)$. It is also *associative*, so:

$$\Delta(\Delta(X, Y), Z) = \Delta(X, \Delta(Y, Z))$$

An alternative way to define the *symmetric difference* using *set* operations is:

$$\Delta(X, Y) = (X \cup Y) \setminus (X \cap Y)$$

This definition highlights that the *symmetric difference* is the *set* of *elements* in the union of X and Y, excluding the elements in their intersection.

In an *orthomodular lattice*, the *symmetric difference* operation can be defined using the *join* operation, *meet* operation and the *orthocomplementation* operation:

Definition 1.6 (Symmetric difference in an orthomodular lattice)**.** Let L be an *orthomodular lattice*. Given elements $a, b \quad L$, the *symmetric difference operation*, denoted as Δ, can be defined as follows:

$$\Delta(a, b) = (a \quad b \;) \quad (a \qquad b)$$

Definition 1.7 (Interval in an orthomodular lattice)**.** Let L be an orthomodular lattice and $x, y \quad L$ such that $x \leq y$. We define an interval, which consists of all elements $z \quad L$ such that $x \leq z \leq y$. We denote it by $[x, y]$.

Definition 1.8 (Atom)**.** Let L be an *orthomodular lattice*. An *atom* $a \quad L$ is a non-zero element of L such that there is no element b satisfying $0 < b < a$.

An *atom* is a *minimal element* immediately above the *zero element*, with no other elements between them. *Atoms* represent the smallest non-trivial elements in an *orthomodular lattice*.

Example 1.9 (Lattice MO2)**.** Let L be an *orthomorudular lattice* such that $L = \{1, 0, a, b, a \;\;, b \;\;\}$. Such a *lattice* is called the *lattice* $MO2$. There are several ways to visualize an *orthomodular lattice*. In this thesis, we will use *Hasse diagrams* and *Greechie diagrams*.

1. *Hasse diagram*
   The most straightforward way to visualize a *partially ordered set* is using a *Hasse diagram*. The diagram consists of nodes, which represent the elements of the *partially ordered set*, and edges, which represent the *partial order relation*. The edges are drawn only between nodes that have direct predecessor-successor relation. The omitted connections follow from the *transitivity*.

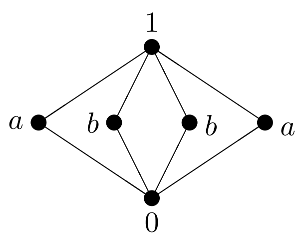

Figure 1.2: Lattice MO2 displayed using Hasse diagram

2. *Greechie diagram*
   A Greechie diagram is a graphical representation of an *orthomodular lattice*, where nodes represent *atoms* of the *orthomodular lattice* and hyperedges connect maximal *sets* of *mutally orthogonal* elements.

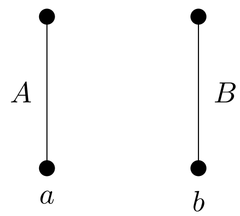

Figure 1.3: Lattice MO2 displayed using Greechie diagram

Definition 1.10 (σ-orthomodular lattice)**.** Let L be an *orthomodular lattice*. We call it a σ-*orthomodular lattice* when it is closed under countable *meets* and *joins*, meaning that for any countable subset $\{x_i\}_{i \in \mathbb{N}} \subseteq L$, the *meet* $\bigwedge_{i \in \mathbb{N}} x_i$ and the *join* $\bigvee_{i \in \mathbb{N}} x_i$ exists.

Definition 1.11 (Probability measure)**.** Let L be a σ-*orthomodular lattice*. A mapping $\mu: L \to [0,1]$ is called a *probability measure* on L when μ satisfies the following conditions:

1. $\mu(1) = 1$
2. $\mu(\bigvee_{n \in \mathbb{N}} a_n) = \sum_{n \in \mathbb{N}} \mu(a_n)$ whenever all $a_n \in L$ and $a_i \perp a_j = 0$ for $i \neq j$

Chapter 2

Reichenbach's common cause principle

Definition 2.1 (Classical probability space)**.** A *classical probability space* is a structure $(\Omega, \mathsf{A}, \mu)$, where Ω denotes a *non-empty set*, A is a *σ-algebra* of subsets of Ω and μ is a *probability measure* on A.

The complement of an event $a \quad \mathsf{A}$ is denoted as a .

Definition 2.2 (Random variable)**.** Let $(\Omega, \mathsf{A}, \mu)$ be a *classical probability space*. A *random variable* X is a mapping $X \colon \Omega \to \mathsf{R}$ measurable with respect to σ-algebra A.

Definition 2.3 (σ-homomorphism)**.** Let $(\Omega_1, \mathsf{A}_1, \mu_1)$ and $(\Omega_2, \mathsf{A}_2, \mu_2)$ be two *classical probability spaces*. A mapping $h \colon \Omega_1 \to \Omega_2$ is called a *σ-homomorphism* if and only if the following conditions are satisfied:

1. $h(a\ ) = h(a)$
2. $h(\bigvee_{i \ \mathsf{N}} a_i) = \bigvee_{i \ \mathsf{N}} h(a_i)$

In contrast to the *random variable* definition, we define an *observable*.

Definition 2.4 (Observable)**.** Let $(\Omega, \mathsf{A}, \mu)$ be a *classical probability space*. Then we say that *observable* ϕ is a *σ-homomorphism* $\phi \colon \mathsf{B}(\mathsf{R}) \to \mathsf{A}$, where $\mathsf{B}(\mathsf{R})$ denotes the *Borel σ-algebra*.

One advantage of the *observable definition* is the fact that the structure (A, μ) is sufficient as a *classical probability space*. An observable can be defined without the set of elementary events.

A random variable X can be identified with an observable ϕ:

$$\phi(A) = X^{-1}(A) = \{\omega \quad \Omega \mid X(\omega) \quad A\}$$

Definition 2.5 (Dependence of events)**.** Let L be a σ-orthomodular lattice and suppose to have events $a, b \quad L$. These events are called:

1. *Independent* if and only if $\mu(a \quad b) = \mu(a)\mu(b)$
2. *Positively correlated* if and only if $\mu(a \quad b) > \mu(a)\mu(b)$
3. *Negatively correlated* if and only if $\mu(a \quad b) < \mu(a)\mu(b)$

Definition 2.6 (Common cause in classical probability theory)**.** Let (Ω, A, P) be a *classical probability space* and let $a, b \quad A$ be events which are *positively correlated*. Then we call $c \quad A$ a *common cause* of a and b if the following conditions hold:

$$P(a \cap b|c) = P(a|c)P(b|c) \tag{2.1}$$

$$P(a \cap b|c\ ) = P(a|c\ )P(b|c\ ) \tag{2.2}$$

$$P(a|c) > P(a|c\ ) \tag{2.3}$$

$$P(b|c) > P(b|c\ ) \tag{2.4}$$

where we require that $0 < P(c) < 1$ and $P(x|y) = \frac{P(x \cap y)}{P(y)}$ denotes the *conditional probability* of x given y.

2.1 The independence of Reichenbach common cause conditions

It is known[1] that the conditions from definition 2.6 are independent, but we did not find a proof, so we prove it here. Moreover, we add the assumption of positive correlation, which need not follow from the reduced set of conditions.

Example 2.7. Let A be the *Boolean algebra* with 8 *atoms* of the form $a \quad b \quad c\ $, where $a \quad \{a, a\ \}, b \quad \{b, b\ \}, c \quad \{c, c\ \}$. We define *states* s_0, s_1, s_2, s_3, s_4 by their values at the *atoms* given in table 2.1 Then a, b are positively correlated in all of these states.

0. In state s_0, each of the conditions from definition 2.6 is satisfied.
1. In state s_1, conditions (2.2), (2.3), (2.4) are satisfied and (2.1) is violated.
2. In state s_2, conditions (2.1), (2.3), (2.4) are satisfied and (2.2) is violated.

[1] [8], Definition 2.4

3. In state s_3, conditions (2.1), (2.2), (2.4) are satisfied and (2.3) is violated.
4. In state s_4, conditions (2.1), (2.2), (2.3) are satisfied and (2.4) is violated.

	$a \quad b \quad c$	$a \quad b \quad c$	$a \quad b \quad c$	$a \quad b \quad c$	$a \quad b \quad c$	$a \quad b \quad c$	$a \quad b \quad c$	$a \quad b \quad c$
s_0	$2/9$	$1/9$	$1/9$	$1/18$	$1/8$	$1/8$	$1/8$	$1/8$
s_1	$2/9$	$1/9$	$1/9 + \epsilon$	$1/18 - \epsilon$	$1/8$	$1/8$	$1/8$	$1/8$
s_2	$2/9$	$1/9$	$1/9$	$1/18$	$1/8$	$1/8$	$1/8 + \epsilon$	$1/8 - \epsilon$
s_3	$1/9$	$1/18$	$2/9$	$1/9$	$1/8$	$1/8$	$1/8$	$1/8$
s_4	$1/9$	$2/9$	$1/18$	$1/9$	$1/8$	$1/8$	$1/8$	$1/8$

Table 2.1: Values of states s_0, s_1, s_2, s_3 and s_4 on atoms, where $\epsilon > 0$ is sufficiently small

	(2.1) LHS	(2.1) RHS	(2.2) LHS	(2.2) RHS	(2.3) LHS	(2.3) RHS	(2.4) LHS	(2.4) RHS
s_0	$4/9$	$4/9$	$1/4$	$1/4$	$2/3$	$1/2$	$2/3$	$1/2$
s_1	$4/9$	$4/9 + 4\epsilon/3$	$1/4$	$1/4$	$2/3$	$1/2$	$2/3 + 2\epsilon$	$1/2$
s_2	$4/9$	$4/9$	$1/4$	$1/4 + \epsilon$	$2/3$	$1/2$	$2/3$	$1/2 + 2\epsilon$
s_3	$2/9$	$2/9$	$1/4$	$1/4$	$1/3$	$1/2$	$2/3$	$1/2$
s_4	$2/9$	$2/9$	$1/4$	$1/4$	$2/3$	$1/2$	$1/3$	$1/2$

Table 2.2: Specific values computed using equations from definition 2.6 and values from table 2.1

2.2 Illustration and examples

Definition 2.8 (Types of the common cause)**.** Let (Ω, A, μ) be a *probablility space*. Let $a, b, c \quad A$ such that a, b are positively correlated and c is a common cause of the correlation. Following Rédei, we distinguish the following types of a *common cause*:

1. *Deterministic*
   A common cause c is *called deterministic* if:

$$\mu(a|c) = \mu(b|c) = 1$$
$$\mu(a|c\ ) = \mu(b|c\ ) = 0$$

2. *Genuinely probabilistic*
   If c is a common cause such that c * a and c * b, then c is called a *genuinely probabilistic common cause.*

3. *Proper*
   A *common cause* c of the correlation between $a, b \quad A$ is called *proper* if:

$$\mu(\Delta(b, c)) = 0$$
$$\mu(\Delta(a, c)) = 0$$

4. *Improper*
   The *common cause* c is called *improper* when it is not *proper*.

Chapter 3

Reichenbach's common cause principle in non-classical probability theory

In this chapter, we will examine the *common cause* in *non-classical probability theory* and its existence. We will extend the definition of the *common cause* in *classical probability space* to the *non-classical probability space*.

First of all, we define the *non-classical probability space*:

Definition 3.1 (Non-classical probability space)**.** A *non-classical probability space* is a pair (L, μ), where L is a *σ-orthomodular lattice* of events and μ is a *probability measure* on L.

Definition 3.2 (Covariance)**.** Let (L, μ) be a *non-classical probability space* and let $a, b \in L$. Then we define a *covariance* of a and b as:

$$\mathrm{Cov}(a, b) = \mu(a \wedge b) - \mu(a)\mu(b)$$

Definition 3.3 (Atomless orthomodular lattice)**.** An *orthomodular lattice* is called *atomless* when it has no *atoms*.

Definition 3.4 (Commutator of elements on an orthomodular lattice)**.** Let L be an *orthomodular lattice* and $a, b \in L$. We define mapping $\mathbf{C}\colon L \times L \to L$ as $\mathbf{C}(a, b) = (a \wedge b) \vee (a \wedge b^{\perp}) \vee (a^{\perp} \wedge b) \vee (a^{\perp} \wedge b^{\perp})$.

Furthermore, we define the *relation of commutation* C. We say that $a \,\mathsf{C}\, b$ if and only if $\mathbf{C}(a, b) = 1$. This is equivalent to $a = (a \wedge b) \vee (a \wedge b^{\perp})$. It is easy to see that the *relation of commutation* is *symmetric*.

Note that if $a, b \in B$, where B denotes the *Boolean algebra*, the condition $\mathbf{C}(a, b) = 1$ is always satisfied.

Now we can rewrite conditions from definition 2.6 using the lattice operations instead of the set operations. The conditional probabilities can also be rewritten:

Definition 3.5 (Common cause in non-classical probability theory)**.** Let L be an *orthomodular lattice*, let μ be a *probability measure* on L and let $a, b, c \quad L$ such that $c\,\mathrm{C}\,a$, $c\,\mathrm{C}\,b$ and $0 < \mu(c) < 1$. Then we say that c is the *common cause* of a and b when the following conditions are met:

$$\frac{\mu(a \quad b \quad c)}{\mu(c)} = \frac{\mu(a \quad c)}{\mu(c)} \frac{\mu(b \quad c)}{\mu(c)} \tag{3.1}$$

$$\frac{\mu(a \quad b \quad c\ \ )}{\mu(c\ \ )} = \frac{\mu(a \quad c\ \ )}{\mu(c\ \ )} \frac{\mu(b \quad c\ \ )}{\mu(c\ \ )} \tag{3.2}$$

$$\frac{\mu(a \quad c)}{\mu(c)} > \frac{\mu(a \quad c\ \ )}{\mu(c\ \ )} \tag{3.3}$$

$$\frac{\mu(b \quad c)}{\mu(c)} > \frac{\mu(b \quad c\ \ )}{\mu(c\ \ )} \tag{3.4}$$

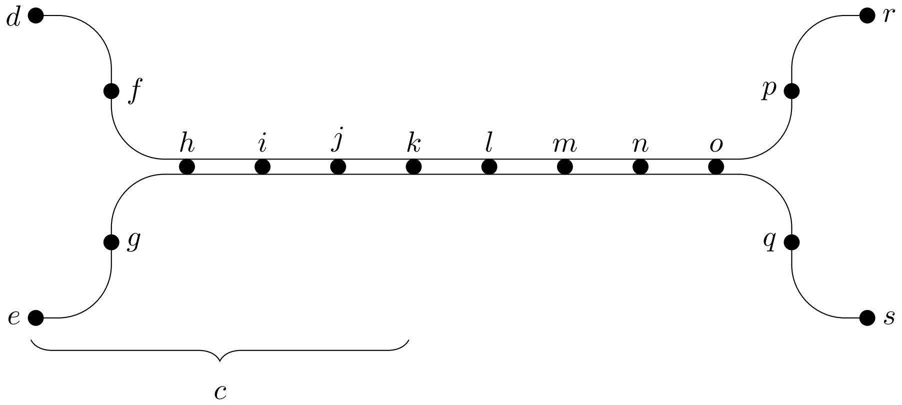

Figure 3.1: Three elements a, b and c, such that $a\,\mathrm{C}\,c$ and $b\,\mathrm{C}\,c$, generate a sub-orthomodular lattice, which in the most general case is the one we display. There are no other variables involved. All conditions of the common cause can be tested on this orthomodular lattice.

We can express a, b and c in the following way:

$$\begin{aligned} a &= g \quad h \quad i \quad n \quad o \quad p \\ b &= f \quad i \quad j \quad m \quad n \quad q \\ c &= d \quad f \quad h \quad i \quad j \quad k \\ &= e \quad g \quad h \quad i \quad j \quad k \end{aligned}$$

3.1 Existence of common cause in an orthomodular lattice

In this chapter, we will examine the common cause in *non-classical probability theory*. We will provide a proof of the existence of the *common cause* in a *σ-orthomodular lattice*.

Then, we will discuss some conclusions taken from [10]. We will reformulate, amend and prove some lemmas which we found incomplete. Those lemmas were used to prove the existence of the *common cause* in an *atomless* and *complete orthomodular lattice*. We will show a proof based on more general assumptions. We will also answer the question from the conclusion of [10].

3.1.1 Enhanced proof of the existence of a common cause

Proposition 3.6. *In classical probability,*

$$Cov(a, b) = -Cov(a\ \ , b) = -Cov(a, b\ \ ) = Cov(a\ \ , b\ \ ) \tag{3.5}$$

Note that in quantum probability, equalities (3.5) need not hold, even the signs might be different.

Proof. We will prove that:

$$\mathrm{Cov}(a, b) = -\mathrm{Cov}(a\ \ , b)$$

First of all, we realize:

$$\mathrm{Cov}(a, b) = \mu(a\quad b) - \mu(a)\mu(b)$$

Because we are in classical probability, we can use the fact that $a\,\mathsf{C}\,b$, which means $\mu(b) - \mu(a\qquad b) = \mu(a\quad b)$:

$$\begin{aligned}
\mu(b) - \mu(a\qquad b) - \mu(a)\mu(b) &= -(\mu(a\qquad b) - \mu(b) + \mu(a)\mu(b)) \\
&= -(\mu(a\qquad b) - \mu(b)(1 - \mu(a))) \\
&= -(\mu(a\qquad b) - \mu(a\ \ )\mu(b)) \\
&= -\mathrm{Cov}(a\ \ , b)
\end{aligned}$$

We can use the commutativity to prove the other two equalities, too. □

Definition 3.7 (Logically independent events [8, Definition 2.1])**.** Events $a, b \in \mathsf{L}$ are called *logically independent* when the following conditions are satisfied:

1. $a \wedge b = 0$
2. $a \wedge b^{\perp} = 0$
3. $a^{\perp} \wedge b = 0$
4. $a^{\perp} \wedge b^{\perp} = 0$

Definition 3.8 (Faithful measure)**.** Let μ be a *probability measure* on a σ-*orthomodular lattice* L. We call μ a *faithful measure* when $\mu(x) = 0$ for any non-zero element of L.

Definition 3.9 (Darboux property[1])**.** We say that a *probability measure* μ defined on a σ-*orthomodular lattice* L has the *Darboux property* when $\forall x, y \in \mathsf{L}$ such that $x \leq y$ it holds:

$$\forall r \in [\mu(x), \mu(y)] \; \exists z \in [x, y] \colon \mu(z) = r$$

In other words, a function with the *Darboux property* assumes every intermediate value between any two points in its domain. This means that the function cannot have any discontinuities that prevent it from taking on intermediate values between any two of its points.

Theorem 3.10 (Existence of a common cause)**.** *Let (L, μ) be a non-classical probability space. Let $a, b \in \mathsf{L}$ such that $a \mathsf{C} c$, $b \mathsf{C} c$ and $\mathrm{Cov}(a, b) > 0$ i.e. positively correlated. Let the measure μ be faithful and satisfy the Darboux property. Then there exists a common cause $c \in \mathsf{L}$.*

Proof. We will look for a *common cause* $c \leq a \wedge b$ of a and b, provided $\mu(a \wedge b) - \mu(a)\mu(b) > 0$. We will show that conditions (3.5) hold under assumptions mentioned above:

1. First of all, we multiply both sides of equation (3.1) by $\mu(c)^2$:

$$\mu(c)\mu(a \wedge b \wedge c) = \mu(a \wedge c)\mu(b \wedge c)$$

Then we separate the term $\mu(c)$:

$$\mu(c) = \frac{\mu(a \wedge c)\mu(b \wedge c)}{\mu(a \wedge b \wedge c)}$$

This equation holds because we know that $\mu(a \wedge b \wedge c) = 0$.

[1] Also called *denseness property* according to [8]

2. We can rewrite equation (3.2) as follows:

$$\frac{\mu(a \quad b) - \mu(c)}{1 - \mu(c)} = \frac{(\mu(a) - \mu(c))(\mu(b) - \mu(c))}{(1 - \mu(c))^2}$$

$$\mu(a \quad b) - \mu(c) = \frac{(\mu(a) - \mu(c))(\mu(b) - \mu(c))}{1 - \mu(c)} \tag{3.6}$$

We know that $\mu(a \quad b) > \mu(c) > 0$ so $\mu(c) \quad [0, \mu(a \quad b)]$. We can consider the left-hand side of equality (3.6) as a function:

$$F(\mu(c)) = \mu(a \quad b) - \mu(c)$$

and the right-hand side of equality (3.6) as a function:

$$G(\mu(c)) = \frac{(\mu(a) - \mu(c))(\mu(b) - \mu(c))}{1 - \mu(c)}$$

Those functions are continuous on $[0, 1]$ but we are interested only in the interval $[0, \mu(a \quad b)]$:

$$F(0) = \mu(a \quad b) > G(0) = \mu(a)\mu(b)$$

$$F(\mu(a \quad b)) = 0 < G(\mu(a \quad b))$$

By our assumption, the measure μ has the *Darboux property*, so we can say, that there exists $\mu(c) \quad (0, \mu(a \quad b))$ such that $F(\mu(c)) = G(\mu(c))$.

3. After multiplying both sides by denominators of (3.3), we obtain an inequality:

$$\mu(a \quad c) - \mu(c)\mu(a \quad c) > \mu(c)\mu(a \quad c \quad)$$

By rearranging and using the commutation of a and c we get:

$$\begin{aligned} \mu(a \quad c) &> \mu(c)\mu(a) > \mu(a \quad c)\mu(a) \\ \mu(a \quad c) &> \mu(a \quad c)\mu(a) \\ 1 &> \mu(a) \end{aligned}$$

Every operation done on the inequality is equivalent. The proof is done.

4. Condition (3.4) is in the same form as condition (3.3). We use commutation of b and c instead of commutation of a and c and the rest of the proof is the same as above.

□

Note that since $a \, C \, c$ and $b \, C \, c$, we can use the result of proposition 3.6

$$\mathrm{Cov}(a, b) = \mathrm{Cov}(a \quad, b \quad)$$

So when a and b are *positively correlated*, then a and b must be *positively correlated*. Additionally, we notice using a and b instead of a and b in conditions (3.5) results in the same set of conditions. Therefore, the proof of the *common cause* existence would be the same as above. In other words, if we do not insist on $c \leq a \quad b$, another common cause $d \leq a \quad b$ could be found assuming the *measure* μ satisfies the *Darboux property* on the interval $[0, a \quad b\ ]$.

As mentioned in [8, pp.72-73], what is really needed in [10] is the *Darboux property* of μ. Moreover, we need it only for the restriction of μ to the interval $[0, a \quad b]$ and even less, we need that for each $r \quad [0, \mu(a \quad b)]$, there is a $c \leq a \quad b$ such that $\mu(c) = r$.

Thus we may formulate our result as follows:

Corollary 3.11. *For the existence of a* common cause $c \quad L$, *the* measure μ *from theorem 3.10 has to be faithful and satisfy the* Darboux property *on the interval* $[0, a \quad b]$.

3.1.2 Comparison to previous results

In [10] there is examined the existence of a *common cause* of two *positively correlated* elements in an *atomless* and *complete orthomodular lattice*. The key tool in [10] to prove the *existence* of a *non-trivial common cause* are [10, Lemma 3.4] and [10, Lemma 3.8].

We will look closer at those lemmas, we will find simpler proofs and for [10, Lemma 3.4], we will amend a missing assumption.

The [10, Lemma 3.4] is formulated as follows:

Lemma 3.12 ([10, Lemma 3.4])**.** *Let μ be a completely additive, faithful probability measure on a σ-orthomodular lattice L and let a and b be elements in L such that $\mu(a \quad b) > \mu(a)\mu(b)$. Then $1 - \mu(a) - \mu(b) + \mu(a \quad b) > 0$ and the following facts hold:*

1. *If $\mu(a) > \mu(a \quad b)$, then*
$$\frac{\mu(a \quad b) - \mu(a)\mu(b)}{1 - \mu(a) - \mu(b) + \mu(a \quad b)} < \mu(a \quad b) \tag{3.7}$$

2. *If $\mu(a) = \mu(a \quad b)$, then*
$$\frac{\mu(a \quad b) - \mu(a)\mu(b)}{1 - \mu(a) - \mu(b) + \mu(a \quad b)} = \mu(a \quad b). \tag{3.8}$$

Example 3.13. Assumption $\mu(a) > \mu(a \quad b)$ is too weak. Inequality (3.7) is for example violated when we say that $\mu(b) = \mu(a \quad b)$, so:

$$\frac{\mu(b) - \mu(a)\mu(b)}{1 - \mu(a) - \mu(b) + \mu(b)} = \mu(b)$$

□

Let's rephrase [10, Lemma 3.4]:

Lemma 3.14. *Let μ be a completely additive probability measure on a complete orthomodular lattice* **L** *and let $a, b \quad \mathbf{L}$ such that $\mu(a \quad b) > \mu(a)\mu(b)$. Then the following facts hold:*

$$1 - \mu(a) - \mu(b) + \mu(a \quad b) > 0 \tag{3.9}$$

If $\mu(a) > \mu(a \quad b)$ and $\mu(b) > \mu(a \quad b)$, then:

$$\frac{\mu(a \quad b) - \mu(a)\mu(b)}{1 - \mu(a) - \mu(b) + \mu(a \quad b)} < \mu(a \quad b) \tag{3.10}$$

If $\mu(a) = \mu(a \quad b)$ or $\mu(b) = \mu(a \quad b)$, then:

$$\frac{\mu(a \quad b) - \mu(a)\mu(b)}{1 - \mu(a) - \mu(b) + \mu(a \quad b)} = \mu(a \quad b) \tag{3.11}$$

Proof. Let's first prove (3.9):
We assume that $\mu(a \quad b) > \mu(a)\mu(b)$, which results in $0 < \mu(a) < 1$ and $0 < \mu(b) < 1$. It can then be rewritten as follows:

$$1-\mu(a)-\mu(b)+\mu(a \quad b) > 1-\mu(a)-\mu(b)+\mu(a)\mu(b) = (1-\mu(a))(1-\mu(b)) > 0$$

Now let's rewrite assumptions $\mu(a) > \mu(a \quad b)$ and $\mu(b) > \mu(a \quad b)$ as follows:

$$\mu(a) = \mu(a \quad b) + \varepsilon_a \tag{3.12}$$

$$\mu(b) = \mu(a \quad b) + \varepsilon_b \tag{3.13}$$

where ε_a and ε_b are greater than zero.
Note that multiplying (3.12) with $\mu(b)$ results in:

$$\mu(a)\mu(b) = \mu(b)\mu(a \quad b) + \mu(b)\varepsilon_a \tag{3.14}$$

Let's start with inequality:

$$\mu(b) > \mu(a \quad b)$$

then we multiply both sides by ε_a and we add $\mu(b)\mu(a \quad b)$ to both sides getting:

$$\mu(b)\mu(a \quad b) + \varepsilon_a\mu(b) > \mu(b)\mu(a \quad b) + \varepsilon_a\mu(a \quad b)$$

which can be rewritten as:

$$\mu(b)\mu(a \quad b) + \varepsilon_a\mu(b) > \mu(a \quad b)(\mu(b) + \mu(a) - \mu(a) + \varepsilon_a)$$

We use (3.14) on the left-hand side and (3.12) on the right-hand side resulting in:

$$\mu(a)\mu(b) > \mu(a \wedge b)(\mu(a) + \mu(b) - \mu(a \wedge b))$$

We multiply both sides of the inequality by -1 and we add $\mu(a \wedge b)$ to both sides:

$$\mu(a \wedge b) - \mu(a)\mu(b) < \mu(a \wedge b) - \mu(a \wedge b)(\mu(a) + \mu(b) - \mu(a \wedge b))$$

$$\mu(a \wedge b) - \mu(a)\mu(b) < \mu(a \wedge b)(1 - \mu(a) - \mu(b) + \mu(a \wedge b))$$

which results in

$$\frac{\mu(a \wedge b) - \mu(a)\mu(b)}{1 - \mu(a) - \mu(b) + \mu(a \wedge b)} < \mu(a \wedge b)$$

□

The following two lemmas are used in [10] to prove the existence of a *common cause* on an *atomless complete orthomodular lattice*. However, the lemmas show, that a *faithful meaure* μ on an *atomless complete orthomodular lattice* L has the *Darboux property*, which, together with the *faithfulness* of μ, is the only required assumption.

Therefore, we omit proofs of 3.15 and 3.16. We have already shown that the *Darboux property* is the sufficient assumption for the existence of a *common cause* on an *orthomodular lattice*. Moreover, proofs of those lemmas in [10] are rather complex.

Lemma 3.15 ([10, Lemma 3.6]). *Let μ be a completely additive probability measure on an atomless and complete orthomodular lattice L and let r be an element in L such that $\mu(r) = 0$. For any real number α such that $0 < \alpha < \mu(r)$ there exists $c \in \mathsf{L}$ such that $c < r$ and $\mu(c) = \alpha$.*

Lemma 3.16 ([10, Lemma 3.7]). *Let μ be a faithful completely additive probability measure on an atomless and complete orthomodular lattice L, let r be a nonzero element in L and let α be a real number such that $0 < \alpha < \mu(r)$. Then the set $\{c \in \mathsf{L} | c < r,\ \mu(c) = \alpha\}$ is an uncountably infinite set.*

The following lemma is mentioned and proved in [15]. It is also stated in [8] as a fact. Here we present a proof:

Lemma 3.17 ([10, Lemma 3.8]). *Let μ be a* probability measure *on a σ-*orthomodular lattice L *and let $a, b, c \in \mathsf{L}$. If c is a* common cause *of a and b, then*

$$\mu(a \wedge b) > \mu(a)\mu(b)$$

Proof. First of all, we will look closer at inequality (3.3):

$$\frac{\mu(a \quad c)}{\mu(c)} > \frac{\mu(a \quad c\ )}{\mu(c\ )}$$

$$\mu(a \quad c) - \mu(c)\mu(a \quad c) > \mu(c)\mu(a \quad c\ )$$

$$\mu(a \quad c) > \mu(c)(\mu(a \quad c) + \mu(a \quad c\ ))$$

$$\mu(a \quad c) > \mu(a)\mu(c)$$

We used the fact that $a \, \mathsf{C} \, c$ in the last step. The same transformations could be applied to (3.4):

$$\mu(b \quad c) > \mu(b)\mu(c)$$

Every nontrivial convex combination of $\frac{\mu(a \ c)}{\mu(c)}$ and $\frac{\mu(a \ c^{\perp})}{\mu(c^{\perp})}$ must be between them. We will be interested in the following convex combination:

$$\frac{\mu(a \quad c)}{\mu(c)} > \mu(c)\frac{\mu(a \quad c)}{\mu(c)} + \mu(c\ )\frac{\mu(a \quad c\ )}{\mu(c\ )} > \frac{\mu(a \quad c\ )}{\mu(c\ )}$$

because

$$\mu(c)\frac{\mu(a \quad c)}{\mu(c)} + \mu(c\ )\frac{\mu(a \quad c\ )}{\mu(c\ )} = \mu(a \quad c) + \mu(a \quad c\ ) = \mu(a)$$

Using this fact we obtain:

$$\mu(a)\mu(c\ ) > \mu(a \quad c\ ) \tag{3.15}$$

Similarly, we could have obtained:

$$\mu(b)\mu(c\ ) > \mu(b \quad c\ ) \tag{3.16}$$

Now we take equation (3.1) and we multiply its both sides by the right-hand side denominator:

$$\mu(c)\mu(a \quad b \quad c) = \mu(a \quad c)\mu(b \quad c)$$

Then we use inequalities $\mu(a \quad c) > \mu(a)\mu(c)$ and $\mu(b \quad c) > \mu(b)\mu(c)$:

$$\mu(c)\mu(a \quad b \quad c) = \mu(a \quad c)\mu(b \quad c) > \mu(a)\mu(b)\mu(c)^2$$

resulting in:

$$\mu(a \quad b \quad c) > \mu(a)\mu(b)\mu(c) \tag{3.17}$$

From equation (3.2) we can obtain:

$$\mu(a \quad b \quad c\ ) = \frac{\mu(a \quad c\ )\mu(b \quad c\ )}{\mu(c\ )} \tag{3.18}$$

Now we realize:

$$\mu(a \quad b \quad c) + \mu(a \quad b \quad c\ ) = \mu(a \quad b)$$

Adding equalities (3.17),(3.18) and using inequalities (3.15),(3.16) results in:

$$\mu(a\quad b) > \mu(a)\mu(b)\mu(c) + \frac{\mu(a\quad c\ \ )\mu(b\quad c\ \ )}{\mu(c\ \ )} > \mu(a)\mu(b)\mu(c) + \frac{\mu(a)\mu(b)\mu(c\ \ )^2}{\mu(c\ \ )}$$

which can be further simplified:

$$\mu(a\quad b) > \mu(a)\mu(b)(\mu(c) + \mu(c\ \ ))$$
$$\mu(a\quad b) > \mu(a)\mu(b)$$

□

In the conclusion of [10], Kitajima says without proof:

Proposition 3.18. *If* $a \, C \, b$ *and* $\mu(a\quad b) - \mu(a)\mu(b) > 0$, *then:*

$$1 - \frac{\mu(a)\mu(b)}{\mu(a\quad b)} < \mu(a\qquad b\ \ )$$

We will reformulate the proposition and show proof:

Lemma 3.19 (Reformulation of the conclusion of [10])**.** *Let* L *be a* σ-orthomodular lattice. *Let* μ *be a* probability measure *on* L, *let* $a, b\quad L$ *such that* $a \, C \, b$ *and* $\mu(a\quad b) = 0$. *Then:*

1.

$$1 - \frac{\mu(a)\mu(b)}{\mu(a\quad b)} \leq \mu(a\qquad b\ \ ) \tag{3.19}$$

2. *Additionally, when the measure* μ *is faithful, then:*

$$1 - \frac{\mu(a)\mu(b)}{\mu(a\quad b)} < \mu(a\qquad b\ \ ) \tag{3.20}$$

Proof. 1. First of all, we multiply both sides of inequality (3.19) by $\mu(a\quad b)$:

$$\mu(a\quad b) - \mu(a)\mu(b) \leq \mu(a\qquad b\ \ )\mu(a\quad b)$$

Then we subtract the left-hand side of the inequality so we get:

$$0 \leq \mu(a\qquad b\ \ )\mu(a\quad b) + \mu(a)\mu(b) - \mu(a\quad b)$$

Now we rearrange the terms in the inequality:

$$0 \leq \mu(a)\mu(b) - \mu(a\quad b)(1 - \mu(a\qquad b\ \ ))$$

Now we realize that:

$$1 - \mu(a\qquad b\ \ ) = \mu((a\qquad b\ \ )\ \ ) = \mu(a\quad b) = \mu(a) + \mu(b) - \mu(a\quad b)$$

And we use it in the inequality we want to prove:

$$0 \leq \mu(a)\mu(b) - \mu(a)\mu(a \quad b) - \mu(b)\mu(a \quad b) + \mu(a \quad b)\mu(a \quad b)$$

This allows us to use the formula $xy - xc - yc + c^2 = (x - c)(y - c)$ for $x = \mu(a)$, $y = \mu(b)$, $c = \mu(a \quad b)$:

$$0 \leq (\mu(a) - \mu(a \quad b))(\mu(b) - \mu(a \quad b))$$

Now using commutation of a and b we get:

$$0 \leq \mu(a \qquad b)\mu(b \qquad a)$$

Every operation that we have applied to the inequality is equivalent.

2. The measure μ is faithful, so we know that $\mu(a \qquad b) = 0$ and $\mu(a \quad b \quad) = 0$, which means:

$$0 < \mu(a \qquad b)\mu(a \quad b \quad)$$

The proof is done. □

The following question is posed in [10]:

Question 3.1.1. Does 3.19 hold when a does not commute with b?

Answer 3.1.1. The answer to this question is negative. We can find a counterexample:

Let us consider the product $MO2 \times \mathsf{B}_2$, where B_2 denotes a two-element *Boolean algebra*. The Greechie diagram of such a structure is displayed in figure 3.2.

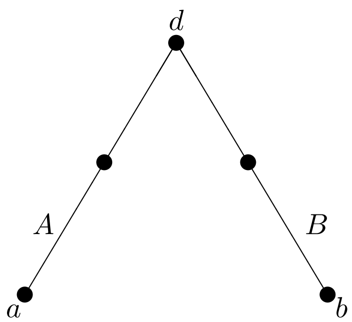

Figure 3.2: Lattice $MO2 \times \mathsf{B}_2$

Then we choose measure:

$$\mu(a \quad b) = \frac{1}{2} = \mu(b \qquad (a \quad b \quad) \qquad (a \qquad b \quad) \quad)$$

$$\mu(a) = \mu(b) = \frac{1}{2}$$

Then:

$$1 - \frac{\mu(a)\mu(b)}{\mu(a \quad b)} = 1 - \frac{1/4}{1/2} = \frac{1}{2}$$

But when we look at the right-hand side of the inequality from lemma 3.19:

$$\mu(a \qquad b \quad) = 0$$

So the inequality does not hold in this case. The counterexample is done. □

Chapter 4

Conclusions

We have shown in theorem 3.10 that if (L, μ) is a *non-classical probability space*, where L is an *atomless orthomodular lattice*, and a, b L are such that $a \, C \, c$ and $b \, C \, c$, then there exists a *common cause* c L. The condition that the measure μ is *faithful* and satisfies the *Darboux property* is crucial for the proof.

We have also provided a counterexample to the claim in [10] that the inequality in Lemma 3.19 holds when a and b do not commute with c. This demonstrates that the *commutation* is an essential requirement for the existence of a *common cause* in a *non-classical probability space*.

In summary, we have reformulated, amended, and proved some lemmas related to the existence of a *common cause* in the *non-classical probability theory*. We showed that under certain conditions, a *common cause* does indeed exist.

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