Master Thesis

Czech Technical University in Prague

F3 Faculty of Electrical Engineering
Department of Economics, Management and Humanities

Common Cause Principle in Classical and Non-classical Probability

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Field of study: Electrical Engineering, Power Engineering and Management
Subfield: Management of Power Engineering and Electrotechnics
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Diplomantka bere na vědomí, že je povinna vypracovat diplomovou práci samostatně, bez cizí pomoci, s výjimkou poskytnutých konzultací.
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Pokyny pro vypracování:

The dependence of events is a necessary part of a description of real-world systems. It is formulated in terms of probability, not necessarily classical because some "quantum" phenomena are present in all complex systems, including psychology, sociology, AI, etc. According to Reichenbach [1], each two positively correlated events should have a common cause which makes them conditionally independent. If this is satisfied, we call the event structure common cause closed. A mathematical question is which structures possess this property. It led to a series of results, summarized in the monograph [3]. The thesis should bring progress in at least 3 of the following directions:

1. Summarize current knowledge of common cause closed systems and try to improve some of the proofs.
2. Find minimized examples of common cause closed systems.
3. The common cause has only one description in classical probability. In the more general context of quantum logics, it has several possible generalisations, among which only one was intensively studied in [2], [3], etc. Discuss the possibility of extention of the results to the remaining cases.
4. For systems without common causes, try to find their extensions to common cause complete ones.

The output should be a theoretical analysis, supported by own examples. Discuss consequences for philosophy of science and possibilities of a probabilistic description of real world and for decision making, including managerial applications.

Seznam doporučené literatury:

[1] Hans Reichenbach. The Direction of Time. Dover Publications, 1956.
[2] Yuichiro Kitajima. “Reichenbach’s Common Cause in an Atomless and Complete Orthomodular Lattice”. International Journal of Theoretical Physics 47.2 (Feb. 2008), pp. 511–519. DOI: 10. 1007/s10773-007-9475-2.
[3] Gábor Hofer-Szabó, Miklós Rédei, and László E. Szabó. The Principle of the Common Cause. Cambridge University Press, 2013. DOI: 10.1017/CBO9781139094344.

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DECLARATION

I, the undersigned

Student's surname, given name(s): Burešová Dominika
Personal number: 492146
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declare that I have elaborated the master's thesis entitled

Common Cause Principle in Classical and Non-classical Probability

independently, and have cited all information sources used in accordance with the Methodological Instruction on the Observance of Ethical Principles in the Preparation of University Theses and with the Framework Rules for the Use of Artificial Intelligence at CTU for Academic and Pedagogical Purposes in Bachelor's and Continuing Master's Programmes.

I declare that I did not use any artificial intelligence tools during the preparation and writing of my thesis. I am aware of the consequences if manifestly undeclared use of such tools is determined in the elaboration of any part of my thesis.

In Prague on 16.05.2025

Bc. Dominika Burešová
....
student's signature

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I would like to thank Professor Pták, Professor Navara, and Jan Ševic, BSc., for their valuable advice and support during the writing of this thesis.

Declaration

I declare that I have prepared the final thesis independently and have listed all information sources used in accordance with the Methodological Guidelines on the observance of ethical principles in the preparation of university final theses and the Framework Rules for the use of artificial intelligence at CTU for study and pedagogical purposes in Bc and NM studies.

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Prague, 16. May 2025

Abstract

Reichenbach's common cause principle (RCCP) is a metaphysical claim about the causal structure of the world. In this work, we adopt its mathematical formulation. We find explicit requirements for a state to have a common cause in a quantum logic. We show that the notion of *maximal* correlation in quantum logics differs considerably from the classical case. We answer an open question by providing a counterexample based on a quantum logic given by a free orthomodular lattice. The rest of our work is devoted to the notion of common cause completeness (CCC). We find a smallest possible system that is non-trivially CCC. Then we find a relation between the Darboux property and RCCP. We invent a new technique for obtaining the result that atomless σ-complete quantum logics are CCC. We correct the claim that quantum probability spaces with one atom must be CCC. For compatible events, we give a simplified proof of the existence of a common cause; for non-compatible ones, we present a counterexample. The abundance of quantum logics which are common cause incomplete raises the question whether they can be *embedded* into common cause complete ones. This was formulated as one of the most important open questions in the monograph *"The Principle of the Common Cause"(Cambridge Press)*, and before our work, no such result has been published. We succeeded in constructing such an embedding for quantum logics with finitely many atoms. Thus, we substantially contributed to the principal open question of the theory.

Keywords: Reichenbach's common cause principle, RCCP, Boolean algebra, orthomodular lattice, atomless quantum probability space, common cause completeness

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Abstrakt

Reichenbachův princip společné příčiny (RCCP) je metafyzickým tvrzením o kauzální struktuře světa. V této práci studujeme jeho matematickou formulaci. Nejprve určujeme explicitní podmínky, za nichž má stav společnou příčinu v kvantové logice. Ukazujeme, že pojem *maximální* korelace v kvantových logikách se podstatně liší od klasického (Booleovského) případu. Odpovídáme na otevřenou otázku tak, že předkládáme protipříklad založený na kvantové logice dané volným ortomodulárním svazem. Zbytek práce je věnován pojmu *úplnosti společných příčin* (CCC). Nacházíme nejmenší možný systém, který je netriviálně CCC. Odhalujeme vztah mezi Darbouxovou vlastností a CCC. Podáváme nový důkaz toho, že bezatomické σ-úplné kvantové logiky jsou CCC. Upřesňujeme tvrzení, že kvantové pravděpodobnostní prostory s jedním atomem musí být CCC. Pro kompatibilní jevy podáváme zjednodušený důkaz existence společné příčiny; pro nekompatibilní uvádíme protipříklad. Skutečnost, že existuje mnoho kvantových logik, které nejsou CCC, vyvolává otázku, zda je lze alespoň *vnořit* do logik, jež CCC jsou. Tato otázka byla formulována jako jeden z nejdůležitějších otevřených problémů v monografii *„The Principle of the Common Cause" (Cambridge Press)* a před naší prací neexistoval žádný publikovaný výsledek. Nám se podařilo takové vnoření zkonstruovat pro kvantové logiky s konečným počtem atomů. Tím zásadně přispíváme k řešení významné otevřené otázky.

Klíčová slova: Reichenbachův princip společné příčiny, RCCP, Booleova algebra, ortomodulární svaz, bezatomický kvantový pravděpodobnostní prostor, úplnost společných příčin

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Chapter 1

Introduction

Reichenbach's common cause principle (RCCP), introduced by Hans Reichenbach in his book "The Direction of Time" [37] is an important concept in the philosophy of science. It was discussed as one of possible principles of physics. It has contributed significantly to the understanding of *causation* and *correlation*. The basic idea of RCCP is, roughly, that there is no correlation without causation. The causation is indeed *linked with the direction of time*.

In classical physics, physical laws are time-reversible. In quantum mechanics, time reversibility is represented by a unitary or antiunitary operator, described by Wigner's theorem [45, 47].

In thermodynamics, the direction of time can be seen in one of its key concepts—the concept of entropy. The second law of thermodynamics states that the entropy of an isolated physical system can be constant with respect to time if the processes in the system are reversible. If the processes are not reversible, then the entropy increases with time. This increase in entropy reflects the statistical behavior of physical systems which tend to evolve towards a state of greater entropy. Hans Reichenbach (American-German philosopher, 1891–1953) demonstrated the strong connection between the principle of common cause and the second law of thermodynamics by deriving the principle of common cause from the second law of thermodynamics.

This paper concerns the RCCP in the mathematical sense as introduced in [37] and further studied in [28] and [24]. We address the questions that came into existence by adopting the axioms of RCCP in quantum logics.

We are particularly interested in *common cause complete* spaces, i.e., spaces where each positively correlated pair has a common cause. We substantially simplify the proof that a σ-complete quantum probability space is common cause complete when the state is purely non-atomic. If this is not the case, we still may *embed* the quantum probability space into one that *is* common cause complete. This has been an open question and no essential progress was made during a decade. We give a positive answer for quantum logics with finitely many atoms and some quantum logics with countably many atoms.

Since quantum logics are associated with the events of quantum experiments (see [12, 16, 36, 42] etc.) and since they also found applications in areas of social sciences (see [19, 31, 34, 46] etc.), these results might be applicable in various scientific disciplines.

Chapter 2

Basic notions

Let us recall the basic definitions as we shall use them in the sequel.

Definition 2.1. A **quantum logic** (abbr. **QL**) is a pentuple $(L, \leq, \ , 0, 1)$, where

1. $(L, \leq)$ is a lattice with a least element 0 and a greatest element 1,
2. $\ : L \to L$ is an orthocomplementation operation $(a \leq b = \quad a \geq b \ ,$ $(a \ ) \ = a$, and $a \quad a \ = 0)$,
3. the **orthomodular law** holds, $a \leq b = \quad b = a \quad (b \quad a \ )$.

In algebraic language, a quantum logic is precisely an **orthomodular lattice**. We usually speak simply of a quantum logic L. On the other hand, when we need to distinguish operations and relations in different quantum logics, we do so by indexing: $\leq_L$, ^{L}, $0_L, 1_L$. A quantum logic L is said to be ***σ*-complete** [resp., **complete**] provided for each countable subset [resp., for each subset] $K \quad L$ the supremum $\bigvee K$ exists in L. Elements $a, b \quad L$ are called **orthogonal** (in symbols, $\boldsymbol{a} \perp \boldsymbol{b}$) if $a \leq b \ $.

Let us present an important example of quantum logics that is close to Boolean algebras.

Definition 2.2. Let S be a set and let $\boldsymbol{\Delta}$ be a collection of subsets of S $(\boldsymbol{\Delta} \quad \exp S)$. Let $\leq$ be given by inclusion $(A \leq B \qquad A \quad B)$ and be the set complementation in S. Let $\boldsymbol{\Delta}$ be a lattice with respect to $\leq$. Let us further suppose that $\boldsymbol{\Delta}$ is subject to the following requirements:

1. $S \quad \mathbf{\Delta}$,
2. if $A \quad \mathbf{\Delta}$ then $A \quad \mathbf{\Delta} \quad (A \ = S \setminus A)$,
3. if $A, B \quad \mathbf{\Delta}$ and $A \cap B = \quad$ then $A \quad B \quad \mathbf{\Delta}$.

Then $(\mathbf{\Delta}, \ , \ , \ , S)$ is a quantum logic. We say that $(S, \mathbf{\Delta})$ is a **set-representable quantum logic**.

Definition 2.3. Let L be a quantum logic. Let $s\colon L \to [0,1]$ be a mapping that satisfies the following conditions:

1. $s(1) = 1$,
2. $s(a \quad b) = s(a) + s(b)$ provided $a \quad b$.

Then s is called a **state** on L and the couple (L, s) is called a **quantum probability space**.[1] A state is said to be ***σ*-additive** [resp., **completely additive**] if for each countable subset [resp., for each subset] $K \quad L$ of mutually orthogonal elements we have $s(\bigvee K) = \sum_{x \quad K} s(x)$. If L is σ-complete and s is σ-additive, we call (L, s) a ***σ*-complete quantum probability space**. If a state s satisfies the implication $s(a) = 0 \ = \quad a \ = 0$, then s is said to be a **faithful state**.

Example 2.4.

1. Any Boolean algebra is a QL. The states are finitely additive probability measures.
2. The set $L(\mathsf{R}^n)$ of all linear subspaces of R^n, $n \quad \mathsf{N}$, is a QL when endowed with the inclusion ordering and with the orthocomplementation defined as the orthogonal complement. Each one-dimensional subspace $a \quad L(\mathsf{R}^n)$ determines a *vector state* s_a by $s_a(x) = \cos^2 \angle(a, x)$ for all one-dimensional subspaces $x \quad L(\mathsf{R}^n)$; this formula extends uniquely to all elements of $L(\mathsf{R}^n)$. Notably, the states on $L(\mathsf{R}^n)$, $n \geq 3$, are exactly convex combinations of vector states (see [13, 3]).
3. Let us take $S = \{1, 2, 3, 4\}$ and denote by $\mathbf{\Delta}$ the collection of sets $\ , S$, and all two-element subsets of S. Then $(S, \mathbf{\Delta})$ is a set-representable QL. Quantum logics obtained by a generalization of this example play an important role (see e.g., [9, 11, 29, 31, 43]).

[1] We follow the terminology of [23]. In other sources, mainly [24], it is called a *non-standard probability space* or *general probability space*.

Definition 2.5. Let L be a QL [resp., a σ-complete QL]. A subset K of L, $K \quad L$, is said to be a **sublogic** [resp., a **σ-sublogic**] if

1. $0 \quad K$,
2. $a \quad K = \quad a \quad K$,
3. if $a, b \quad K$, then $a \quad_K b$ (i.e., $a \quad b$ taken in K) equals $a \quad_L b$ (i.e., $a \quad b$ taken in L) [resp., for a countable family $(a_i)_{i \quad \mathbb{N}}$, $\bigvee_i a_i$ taken in K equals $\bigvee_i a_i$ taken in L].

If, moreover, K satisfies the laws of a Boolean algebra, it is called a **Boolean subalgebra** of L. A *maximal* Boolean subalgebra is called a **block**. Elements $a, b \quad L$ are said to be **compatible** (in symbols, $\boldsymbol{a \, \mathcal{C} \, b}$) if there is a Boolean subalgebra of L containing $\{a, b\}$.[2]

In particular, if $a \quad b$ or $a \leq b$, then $a \, \mathsf{C} \, b$. Every quantum logic is a union of its blocks [27].

Let us recall basic properties of states on QLs.

Proposition 2.6. *Let L be a QL. Let s be a state on L. Then we have, for all $a, b \quad L$,*

1. $s(a\ ) = 1 - s(a)$.
2. *If $a \, \mathsf{C} \, b$ then $s(a) = s(a \quad b) + s(a \quad b\ )$ and $s(a \quad b) = s(a) + s(b) - s(a \quad b)$.*
3. *If $a \leq b$ then $s(a) \leq s(b)$.*

The proof is elementary since the elements a, b can be viewed as elements of a Boolean subalgebra of L.

[2] See [1, 27] for other equivalent definitions of compatibility.

Chapter 3

Reichenbach's common cause principle

The principle applies to the events which are correlated in the following sense. This chapter is based upon [6, 7, 8, 5, 39].

Definition 3.1. Let (L, s) be a quantum probability space. A random event a, $a \quad L$, defines a random variable, denoted here by the same symbol; it attains the value 1 if the event occurs, 0 otherwise. The mean of the random variable a at the state s is $s(a)$. For two such random variables determined by events a and b, their **covariance**, $\mathrm{Cov}_s(a, b)$, is defined as follows:

$$\mathrm{Cov}_s(a, b) = s(a \quad b) - s(a) \cdot s(b)\,.$$

Two variables (and also the corresponding events) a, b are called **positively correlated** if $\mathrm{Cov}_s(a, b) > 0$.

Let us formulate the definition of the main notion of this work (see [37, 24]). We use the notation

$$s(x \mid c) = \frac{s(x \quad c)}{s(c)} \tag{3.1}$$

for the expression which is formally ***a conditional probability***. [1]

Definition 3.2 (RCCP). Let (L, s) be a quantum probability space. Let $a, b \quad L$. Then the element c, $c \quad L$, is said to be a **common cause** of a, b

[1] Introducing conditional probabilities in quantum theory is a problem and this is not the only way. However, when x is compatible with c, this is the classical formula with the proper meaning.

provided $a \,\mathsf{C}\, c$, $b \,\mathsf{C}\, c$, $0 < s(c) < 1$, and the following conditions are satisfied:

$$s(a \quad b \mid c) = s(a \mid c)\, s(b \mid c)\,, \tag{RCCP1}$$

$$s(a \quad b \mid c\ ) = s(a \mid c\ )\, s(b \mid c\ )\,, \tag{RCCP2}$$

$$s(a \mid c) > s(a \mid c\ )\,, \tag{RCCP3}$$

$$s(b \mid c) > s(b \mid c\ )\,. \tag{RCCP4}$$

When c is a common cause of a, b in a state s, we say that the quadruple $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \mid \boldsymbol{s})$ **adheres to RCCP**.

We shall often use the following reformulation of RCCP:

Proposition 3.3. *In Definition 3.2, condition* (RCCP3)*, resp.* (RCCP4)*, is equivalent to*

$$\frac{s(a \quad c)}{s(c)} > s(a)\,, \tag{RCCP3'}$$

resp.

$$\frac{s(b \quad c)}{s(c)} > s(b)\,. \tag{RCCP4'}$$

Proof. We may express $s(a) = s(a \quad c) + s(a \quad c\ )$ as a proper convex combination of $s(a \mid c)$ and $s(a \mid c\ )$:

$$\frac{s(a \quad c)}{s(c)} > \overbrace{s(c)\frac{s(a \quad c)}{s(c)} + s(c\ )\frac{s(a \quad c\ )}{s(c\ )}}^{s(a)} > \frac{s(a \quad c\ )}{s(c\ )}\,.$$

The other implication uses the same principle and the proof for b is analogous. □

Condition (RCCP1), resp. (RCCP2), expresses the conditional independence of a, b given c, resp. c . Condition (RCCP3'), resp. (RCCP4'), says that c is positively correlated to a (resp. b),

$$\mathrm{Cov}_s(a, c) > 0 > \mathrm{Cov}_s(a, c\ )\,, \qquad \mathrm{Cov}_s(b, c) > 0 > \mathrm{Cov}_s(b, c\ )\,.$$

RCCP expresses that *a* given pair of events has a common cause. From the philosophical point of view, it is crucial to know whether *every* pair of positively correlated events has a common cause. Let us employ the following definition.

Definition 3.4. [22] A quantum probability space (L, s) is said to be **common cause complete**[2] if for all $a, b \in L$ with $\mathrm{Cov}_s(a, b) > 0$ there exists a $c \in L$ such that $(a, b, c \mid s)$ adheres to RCCP.

If there is no positively correlated pair of events, then RCCP is trivially satisfied. Such QLs are called **trivially common cause complete** [24]. In this paper, we are interested in QLs which are common cause complete, but not trivially (i.e., positively correlated pairs exist).

Remark 3.5. Spurious correlations are occasionally observed; see an interesting collection [44]. Some of these correlations admit an explanation (sharing the same time dependence, thus the time represents a common cause), but many refer to situations in which we cannot imagine any relation. However, these are *sample correlations*, based on *empirical distributions*. These observations can be explained by their fluctuations and the abundance of possible correlations. In contrast, Reichenbach's common cause principle refers to the *theoretical distributions* that are free of these phenomena and explain *true correlations*.

Ramsey theory suggests that some accidental true correlations may occur [10]. Reichenbach's approach omits this.

[2] Also *common cause closed*.

Chapter 4

Properties of RCCP

Let us collect basic properties of RCCP. In the considerations, we substantially economize on the proofs of some of the results (see [24, 28]). The basic observation is that the existence of a common cause implies that the events are positively correlated. This chapter is based upon [6, 7, 8, 5].

Remark 4.1. The fact that RCCP implies $\mathrm{Cov}_s(a, b) > 0$ is principal, but its history is complicated. In [24], it is formulated as Proposition 2.5. For the proof, they refer to a more general Proposition 7.2, then to Lemma 7.3, and its proof is left as an exercise (at least a useful hint is given). We present a detailed proof here.

Proposition 4.2. *Let* (L, s) *be a quantum probability space. Let* $a, b, c \in L$ *satisfy* (RCCP1)–(RCCP4)*. Then* $\mathrm{Cov}_s(a, b) > 0$*.*

Proof. Using the formula for total probability, (RCCP1), and (RCCP2),

$$\begin{aligned} s(a) &= s(a \mid c)\, s(c) + s(a \mid c^\perp)\, s(c^\perp)\,, \\ s(b) &= s(b \mid c)\, s(c) + s(b \mid c^\perp)\, s(c^\perp)\,, \\ s(a \wedge b) &= s(a \wedge b \mid c)\, s(c) + s(a \wedge b \mid c^\perp)\, s(c^\perp) \\ &= s(a \mid c)\, s(b \mid c)\, s(c) + s(a \mid c^\perp)\, s(b \mid c^\perp)\, s(c^\perp)\,. \end{aligned}$$

Substituting these expressions to $\mathrm{Cov}_s(a, b)$ and using the equalities

$$s(c) - (s(c))^2 = s(c)\, s(c^\perp) = s(c^\perp) - (s(c^\perp))^2\,,$$

we obtain

$$
\begin{aligned}
\mathrm{Cov}_s(a,b) &= s(a\quad b) - s(a)\,s(b) \\
&= s(a \mid c)\,s(b \mid c)\,s(c) + s(a \mid c\ )\,s(b \mid c\ )\,s(c\ ) \\
&\quad - s(a \mid c)\,s(b \mid c)\,(s(c))^2 - s(a \mid c\ )\,s(b \mid c\ )\,(s(c\ ))^2 \\
&\quad - s(a \mid c)\,s(b \mid c\ )\,s(c)\,s(c\ ) - s(a \mid c\ )\,s(b \mid c)\,s(c)\,s(c\ ) \\
&= s(c)\,s(c\ )\Big(s(a \mid c)\,s(b \mid c) + s(a \mid c\ )\,s(b \mid c\ ) \\
&\quad - s(a \mid c)\,s(b \mid c\ ) - s(a \mid c\ )\,s(b \mid c)\Big) \\
&= s(c)\,s(c\ )\,\underbrace{(s(a \mid c) - s(a \mid c\ ))}_{>0}\,\underbrace{(s(b \mid c) - s(b \mid c\ ))}_{>0} > 0\,.
\end{aligned}
$$

Notice that we utilized (RCCP3) and (RCCP4) and hence all four properties RCCP were used. □

The following proposition says that for $c \le a\quad b$ all the conditions of RCCP reduce to (RCCP2).

Proposition 4.3. *Let* (L, s) *be a quantum probability space. Let* $a, b, c\quad L$ *satisfy* $\mathrm{Cov}_s(a,b) > 0$, $c \le a\quad b$, *and* $0 < s(c) < 1$. *Then* $(a, b, c \mid s)$ *fulfills* (RCCP1), (RCCP3), *and* (RCCP4).

Proof. The assumption $\mathrm{Cov}_s(a,b) > 0$ gives us $s(a) < 1$, $s(b) < 1$. As $c \le a\quad b$, we have $s(a\quad b \mid c) = s(a \mid c) = s(b \mid c) = \frac{s(c)}{s(c)} = 1$, which implies (RCCP1), (RCCP3), and (RCCP4). □

For $c \le a\quad b$, the value $s(c)$ is determined as the unique solution to (RCCP2):

Theorem 4.4. *Let* (L, s) *be a quantum probability space. Let* $a, b, c\quad L$, $Cov_s(a,b) > 0$, $c \le a\quad b$. *Then* $(a, b, c \mid s)$ *adheres to RCCP if and only if*

$$
s(c) = \frac{s(a\quad b) - s(a)\cdot s(b)}{1 - s(a) - s(b) + s(a\quad b)}\,. \tag{4.1}
$$

The value of the right-hand side belongs to the interval $(0, 1]$.

Proof. By Proposition 4.3, we only have to check (RCCP2). Its equivalent formulations are

$$\frac{s(a \quad b \quad c\ )}{s(c\ )} = \frac{s(a \quad c\ )}{s(c\ )}\,\frac{s(b \quad c\ )}{s(c\ )}\,,$$

$$s(a \quad b \quad c\ ) \cdot s(c\ ) = s(a \quad c\ ) \cdot s(b \quad c\ )\,,$$

$$(s(a \quad b) - s(c)) \cdot (1 - s(c)) = (s(a) - s(c)) \cdot (s(b) - s(c))\,,$$

$$s(c)^2 + s(a \quad b) - s(c) - s(c) \cdot (a \quad b) = s(a) \cdot s(b) - s(a) \cdot s(c) - s(b) \cdot s(c) + s(c)^2\,,$$

$$s(a \quad b) - s(c) - s(c) \cdot (a \quad b) = s(a) \cdot s(b) - s(a) \cdot s(c) - s(b) \cdot s(c)\,,$$

$$s(a \quad b) - s(a) \cdot s(b) = s(c) - s(a) \cdot s(c) - s(b) \cdot s(c) + s(c) \cdot s(a \quad b)\,,$$

$$s(c) = \frac{s(a \quad b) - s(a) \cdot s(b)}{1 - s(a) - s(b) + s(a \quad b)}\,.$$

This expression can be written as

$$s(c) = \frac{\mathrm{Cov}_s(a,b)}{s(a\ ) \cdot s(b\ ) + \mathrm{Cov}_s(a,b)}\,.$$

The inequalities

$$0 < \frac{\mathrm{Cov}_s(a,b)}{s(a\ ) \cdot s(b\ ) + \mathrm{Cov}_s(a,b)} \leq 1$$

follow from the positive correlation of a, b. □

Chapter 5

Comparison of RCCP in classical and quantum systems

The calculations in the standard (Boolean) and the quantum setup differ considerably. Let us point out some striking differences. This chapter is based upon [6, 7, 8, 5] and is mostly a work of the co-authors. We included it for the self-containedness.

Proposition 5.1. *Let* (L, s) *be a quantum probability space and* $a, b \in L$. *If* $a \,\mathsf{C}\, b$ *then* (RCCP1)–(RCCP4) *are satisfied when* a *or* b *is substituted for* c.

Proof. Assume $a \,\mathsf{C}\, b$ and consider the quadruple $(a, b, a \mid s)$. The conditions (RCCP1) and (RCCP2) are always valid—their formulations reduce to

$$(\mathrm{RCCP1}) : \frac{s(a \wedge b)}{s(a)} = 1 \cdot \frac{s(b \wedge a)}{s(a)}\,, \text{ and } (\mathrm{RCCP2}) :\ 0 = 0\,.$$

Let us take up the inequalities (RCCP3) and (RCCP4). We have the implications

$$\begin{aligned} s(a \wedge b) > s(a) \cdot s(b) &\implies s(a \wedge b) > s(b) \cdot (s(a \wedge b^\perp) + s(a \wedge b)) \\ &\implies s(a \wedge b) \cdot (1 - s(b)) > s(b) \cdot s(a \wedge b^\perp) \\ &\implies \frac{s(a \wedge b)}{s(b)} > \frac{s(a \wedge b^\perp)}{s(b^\perp)} \end{aligned}$$

and the final inequality is (RCCP3). Analogously, we obtain (RCCP4). The argument for $(a, b, b \mid s)$ is analogous. □

In Proposition 5.1, a and b are common causes of a, b. In the terminology of [24], they are ***improper common causes***. More generally, we may take for

c any element which differs from a or from b only at an element of measure zero. In [24], a common cause c of a, b is called **proper** if it satisfies the following inequalities

$$s((a \quad c\ ) \quad (a \quad c)) > 0\,,$$
$$s((b \quad c\ ) \quad (b \quad c)) > 0\,.$$

Otherwise, it is called **improper**.

Remark 5.2. In Definition 3.2, the common cause is assumed to be ***compatible*** with a, b. Detailed arguments for this choice are presented in [20, 21, 24]. This induces a new situation in a ***quantum logic***. For non-compatible a, b, the compatibility condition already excludes them from being their common cause and an improper common cause does not exist. Surprisingly, this crucial difference between the meaning of a common cause in classical and quantum logic remained unnoticed in [24] and other sources.

A common cause c of a, b is called **deterministic** [24, Definition 2.6] if both inequalities

$$s(a \mid c) = 1 = s(b \mid c)\,,$$
$$s(a \mid c\ ) = 0 = s(b \mid c\ )\,.$$

are valid.

For $s(y) = 0$, the equality

$$s(x \mid y) = 1 \tag{5.1}$$

has equivalent formulations

$$s(x \quad y) = s(y)\,, \tag{5.2}$$
$$s(y \quad (x \quad y)\ ) = 0\,. \tag{5.3}$$

In Boolean algebras, the latter equations are usually written in one of the following forms:

$$s(x \quad y) = 0\,, \tag{5.4}$$
$$s(x \ \mid y) = 0\,. \tag{5.5}$$

In quantum logics, the situation is not as simple, we only have the following relations:

$$(5.1) \qquad (5.2) \qquad (5.3) \;=\; (5.4) \qquad (5.5)\,.$$

The implication (5.4) $=$ (5.3) need not hold because

$$y = (x \quad y) \quad (x \quad y) \quad (\underline{y \quad (x \quad y) \quad (x \quad y)})\,,$$
$$s(y) = s(x \quad y) + s(x \quad y) + s(y \quad (x \quad y) \quad (x \quad y))\,,$$

where the underlined expression is zero if and only if x, y are compatible. Thus it is possible that

$$s(x \quad y) = s(x \quad y) = 0$$

and $s(y)$ is non-zero; it can even be the case that $s(y) = 1$.

Proposition 5.3. *[24] In a Boolean algebra, if a and b are* maximally correlated*, that is, if any of the following (equivalent) conditions holds:*

$$s(a \,|\, b) = s(b \,|\, a) = 1\,, \tag{5.6}$$
$$s(a \,|\, b\ ) = s(b \,|\, a\ ) = 0\,, \tag{5.7}$$

then the correlation between a and b can only have a deterministic cause.

Proposition 5.3 is presented in [24, Proposition 2.7] without any proof and with the reference to [41]. For (5.6), we provide a proof in the more general context of quantum logics. On the other hand, we shall show that (5.7) is not strong enough to force the quantum version of an analogous result.

Proposition 5.4. *In a quantum logic,* (5.6) *implies that the correlation between a and b can only have a deterministic cause.*

Proof. In view of (5.2),

$$s(a \quad b) = s(a) = s(b)\,. \tag{5.8}$$

This means that

$$s(a \quad (a \quad b)\ ) = 0$$

and the same holds for any argument of s under $a \quad (a \quad b)\ $, in particular,

$$s(a \quad (a \quad b) \quad c) = 0\,. \tag{5.9}$$

As c is compatible with a and b, it distributes over all formulas formed from a and b (see [14, 15] or [25]), hence

$$(a \quad b) \quad c = ((a \quad b) \quad c) \quad (c \quad c) = (a \quad b \quad c) \quad c$$

and (5.9) can be rewritten as follows:

$$\begin{aligned} s(a \quad (a \quad b) \quad c) &= s((a \quad c) \quad ((a \quad b) \quad c)) \\ &= s((a \quad c) \quad ((a \quad b \quad c) \quad c)) \\ &= s((a \quad c) \quad (a \quad b \quad c)\ ) = 0\,. \end{aligned}$$

This means that

$$s(a \quad b \quad c) = s(a \quad c)\,.$$

Similar arguments work for b in place of a, so

$$s(a \quad b \quad c) = s(b \quad c)\,.$$

We obtained an analogue of (5.8). Dividing it by $s(c)$, we find a single value γ satisfying

$$\gamma = \frac{s(a \quad b \quad c)}{s(c)} = \frac{s(a \quad c)}{s(c)} = \frac{s(b \quad c)}{s(c)} = s(a \quad b \mid c) = s(a \mid c) = s(b \mid c)\,.$$

Then (RCCP1) attains the form

$$\gamma = \gamma^2\,.$$

This is satisfied only for $\gamma \quad \{0,1\}$. Similar arguments can be used for (RCCP2). Hence c is a deterministic common cause. □

In Boolean algebras, conditions (5.6), (5.7) are equivalent, but in quantum logics they are not. An analogue of Proposition 5.4 with (5.7) does not hold. We demonstrate it by the Counterexample 5.7 bellow. Prior to that, let us recall the basic stones of the construction.

Definition 5.5. Let $(L_\alpha)_{\alpha \quad I}$ be a collection of QLs. Let us denote by P their cartesian product, $P = \prod_{\alpha \quad I} L_\alpha$, endowed with the coordinatewise operations. Then P is a QL called the **product** of $(L_\alpha)_{\alpha \quad I}$.

Suppose that a mapping $p\colon P \to [0,1]$ is such that there is some $\beta \quad I$ and a state s_β on L_β satisfying $p\big((x_\alpha)_{\alpha \quad I}\big) = s_\beta(x_\beta)$ for all $(x_\alpha)_{\alpha \quad I} \quad P$. Then p is a state on P, called a **preimage state**.

Proposition 5.6. *[32] Let I be a finite set and let $P = \prod_{\alpha \quad I} L_\alpha$ be a product of QLs. Each state on P is a convex combination of preimage states.*[1]

Now we may present our examples.

Example 5.7. (1) The Hasse diagram in Figure 5.1 shows the quantum logic called MO_2. As $x \quad y = 0$ (and similarly for $x\ ,\ y\ $), there is no positively correlated pair of events for any state s and (MO_2, s) is trivially common cause complete.

(2) We take the Boolean algebra $B = \exp\{1,2,3,4\}$ and form the product $F = \mathrm{MO}_2 \times B$. It is the *free orthomodular lattice with two free generators* f, g, where $f = (x, \{1,2\})$, $g = (y, \{2,3\})$ (see [27]).

[1] See [32] for the proof and further analysis.

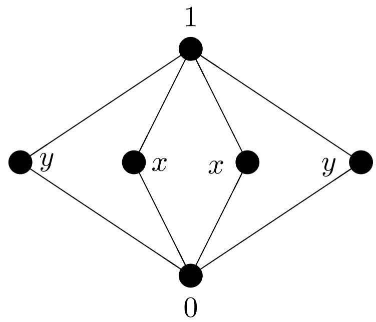

Figure 5.1: Quantum logic MO_2

(3) Let us take the product of two copies of the QL F from (2), $L = F \times F$. We take elements $a, b, c \quad L$, $a = (f, f)$, $b = (g, g)$, $c = (0_F, 1_F)$. There is a unique state s on L such that

$$
\begin{aligned}
s(c) = s(c\ ) &= 1/2\,, & s(a \quad b\ ) &= 0\,,\\
s(a \quad b) &= 0\,, & s(a \quad b\ ) &= 0\,,\\
s(a \quad b \quad c) &= 2/9\,, & s(a \quad b \quad c\ ) &= 1/18\,,\\
s(a \quad (a \quad b) \quad c) &= 1/9\,, & s(a \quad (a \quad b) \quad c\ ) &= 1/9\,,\\
s(b \quad (b \quad a) \quad c) &= 1/9\,, & s(b \quad (b \quad a) \quad c\ ) &= 1/9\,.
\end{aligned}
$$

Checking RCCP, we obtain

$$
\begin{aligned}
s(a \quad c) = s(b \quad c) &= 1/3\,, & s(a \quad c\ ) = s(b \quad c\ ) &= 1/6\,,\\
s(a \mid c) = s(b \mid c) &= 2/3\,, & s(a \mid c\ ) = s(b \mid c\ ) &= 1/3\,,\\
s(a \quad b \mid c) &= 4/9\,, & s(a \quad b \mid c\ ) &= 1/9\,,
\end{aligned}
$$

so $(a, b, c \mid s)$ adheres to RCCP for a non-deterministic common cause c.

The following observations reveal further comparisons of the standard and quantum logics.

If a, b are compatible then

$$
\mathrm{Cov}_s(a, b) = -\,\mathrm{Cov}_s(a, b\ ) = -\,\mathrm{Cov}_s(a\ , b) = \mathrm{Cov}_s(a\ , b\ )\,. \tag{5.10}
$$

If a, b are non-compatible, then (5.10) need not hold. For example, all four covariances can be negative. However, they cannot all be positive. This non-symmetry is caused by the fact that the classical conjunction has six corresponding generalizations in orthomodular lattices (see [1]), the lattice infimum (meet) being the smallest of them. Alternatively, one could, e.g., define

$$
\mathrm{Cov}_s(a, b) = s(a \quad b) - s(a) \cdot s(b)\,,
$$

where

$$a \quad b = (a \quad b) \quad (a \quad b\ ) \quad (a \quad b)$$

is the maximal binary operation in orthomodular lattices which reduces to $a \quad b$ if a, b are compatible. Then all values $\mathrm{Cov}_s(a, b)$, $\mathrm{Cov}_s(a, b\ )$, $\mathrm{Cov}_s(a\ , b)$, $\mathrm{Cov}_s(a\ , b\ )$ may be positive, but they cannot all be negative.

We see that the form of (RCCP1)–(RCCP4) was chosen from several options. Trying to generalize the formula for the Boolean conditional probability, another operation could be chosen instead of the lattice-theoretical infimum . We do not see any privileged choice. The *Sasaki projection* (see [27]), $a \quad (a \quad b\ )$, could be an interesting alternative worth future research.

A common cause may exist in quantum logics even if the covariances $\mathrm{Cov}_s(a, b)$, $\mathrm{Cov}_s(a\ , b\ )$ have opposite signs.

Example 5.8. In Example 5.7,

$$\mathrm{Cov}_s(a, b) = s(a \quad b) - s(a)\, s(b) = \frac{5}{18} - \frac{1}{2}\frac{1}{2} = \frac{1}{36}\,,$$
$$\mathrm{Cov}_s(a\ , b\ ) = s(a \quad b\ ) - s(a\ )\, s(b\ ) = 0 - \frac{1}{2}\frac{1}{2} = -\frac{1}{4}\,.$$

In comparison of the standard and quantum situations, let us consider a result of [28]. The author formulates a Boolean state inequality and asks whether it is true even for general QLs. We correct the inequality, generalize it for compatible elements of a QL, and show that it may fail for non-compatible elements.

Proposition 5.9. *Let (L, s) be a quantum probability space and $a, b \quad L$ such that a C b and $s(a \quad b) = 0$. Then*

$$1 - \frac{s(a) \cdot s(b)}{s(a \quad b)} \leq s(a \quad b\ )\,. \tag{5.11}$$

Proof. We have $0 \leq (s(a) - s(a \quad b)) \cdot (s(b) - s(a \quad b))$.

This means that

$$0 \leq s(a) \cdot s(b) - s(a) \cdot s(a \quad b) - s(b) \cdot s(a \quad b) + s(a \quad b)^2.$$

Hence

$$0 \leq s(a) \cdot s(b) - s(a \quad b) \cdot (s(a) + s(b) - s(a \quad b)).$$

Applying the equality $s(a) + s(b) - s(a \quad b) = 1 - s(a \quad b\ )$, we infer that

$$0 \leq s(a) \cdot s(b) - s(a \quad b) \cdot (1 - s(a \quad b\ )).$$

Hence

$$s(a \quad b) - s(a) \cdot s(b) \leq s(a \quad b) \cdot s(a \quad\; b\ ).$$

Therefore

$$1 - \frac{s(a) \cdot s(b)}{s(a \quad b)} \leq s(a \quad\; b\ ).$$

□

Remark 5.10. In [28], L is assumed to be a Boolean algebra and strict inequality occurs in (5.11). As seen from our proof, strict inequality holds if and only if $s(a) - s(a \quad b) > 0$ and $s(b) - s(a \quad b) > 0$ (in particular, when a and b are non-comparable and s is faithful). Otherwise, it is easy to construct s with $s(a) = s(a \quad b)$ and $s(b) = s(a \quad b)$ and obtain an equality in (5.11).

Paper [28] ends up with an open question whether Proposition 5.9 can be proved without the compatibility assumption. We give a negative answer:

Example 5.11. Let us consider the QL F of Example 5.7 and its elements $f = (x, \{1, 2\})$, $g = (y, \{2, 3\})$. Then $f \quad g = (0, \{2\}) = (0, \quad)$. There is a state s on F that satisfies the equations

$$s(f) = 1/2\,, \qquad s(g) = 1/2\,,$$
$$s(f \quad g) = 1/2\,, \qquad s(f \quad\; g\ ) = 0\,.$$

Then

$$1 - \frac{s(f) \cdot s(g)}{s(f \quad g)} = 1/2 > s(f \quad\; g\ ) = 0\,.$$

Chapter 6

Finding a countable system that is common cause complete

It is desirable to construct as simple CCC Boolean algebra as possible. In this Chapter, we present a countable example, exhibiting thus a simplest example of CCC Boolean algebra. We also formulate a few consequences of this result in quantum logics. This chapter is based upon [4, 5].

Proposition 6.1. *Let B be a Boolean algebra and let s be a finitely additive state. If $a, b \in B$ and $a \setminus b$ is the difference $a \setminus b = a \wedge b'$, then*

$$s(a) \cdot s(b) - s(a \setminus b) \cdot s(b \setminus a) = s(a \wedge b) \cdot s(b \vee a)\,.$$

A consequence: $s(a) \cdot s(b) \geq s(a \wedge b) \cdot s(b \vee a)$.

Proof. By the Stone theorem [38], take the set representation, (D, Δ) of B. So, B is Boolean-isomorphic with Δ. In the isomorphism, let $a \in \Delta$ and $b \in \Delta$. Write $\alpha = s(a \setminus b)$, $\beta = s(b \setminus a)$ and $\gamma = s(a \wedge b)$.

Then the equality $s(a) \cdot s(b) - s(a \setminus b) \cdot s(b \setminus a) = s(a \wedge b) \cdot s(b \vee a)$ rewrites as

$$(\alpha + \gamma) \cdot (\beta + \gamma) - \alpha \cdot \beta = (\alpha + \beta + \gamma) \cdot \gamma.$$

But this equation is obvious. □

Proposition 6.2. *Let B a Boolean algebra and let s be a state on B. Let* $\mathrm{Cov}_s(a, b) > 0$*. Then* $0 < \frac{s(a \wedge b) - s(a) \cdot s(b)}{1 + s(a \wedge b) - s(a) - s(b)} \leq s(a \wedge b)$.

Proof. Since $(1 - s(a)) \cdot (1 - s(b)) > 0$, we see that

$$1 - s(b) - s(a) + s(a) \cdot s(b) > 0$$

and therefore

$$-s(a)\cdot s(b) < 1 - s(b) - s(a).$$

Adding $s(a \quad b)$ to both sides, we obtain

$$s(a \quad b) - s(a)\cdot s(b) < 1 + s(a \quad b) - s(a) - s(b)$$

and therefore

$$\frac{s(a \quad b) - s(a)\cdot s(b)}{1 + s(a \quad b) - s(a) - s(b)} > 0.$$

In order to prove the other inequality, recall first the inequality of Proposition 6.1:

$$s(a)\cdot s(b) \geq s(a \quad b)\cdot s(a \quad b).$$

When we adjust the desired inequality $\frac{s(a \quad b) - s(a)\cdot s(b)}{1 + s(a \quad b) - s(a) - s(b)} \leq s(a \quad b)$, by multiplying with the denominator we obtain

$$s(a \quad b) + s(a \quad b)^2 - s(a \quad b)\cdot s(a) - s(a \quad b)\cdot s(b) \geq s(a \quad b) - s(a)\cdot s(b)\,.$$

Manipulating with this inequality, we obtain

$$s(a \quad b)^2 - s(a \quad b)\cdot s(a) - s(a \quad b)\cdot s(b) + s(a)\cdot s(b) \geq 0\,.$$

This is equivalent to

$$\left(s(a) - s(a \quad b)\right)\cdot\left(s(b) - s(a \quad b)\right) \geq 0\,,$$

which is obviously true. □

Note 6.3. *The inequality $s(a)\cdot s(b) \geq s(a \quad b)\cdot s(b \quad a)$ in Proposition 6.2 was dealt with in the book [24]. ver, the proof presented there was erroneous in the fundamental statement (4.92). We present a correct and new proof.*

Let us note that finite Boolean algebras can be only trivially common cause complete (Definition 3.4). Hence a meaningful case for looking for the CCC begins with a countable cardinality of B.

Proposition 6.4. *There is a countable Boolean algebra B with a finitely additive state s such that (B, s) is non-trivially common cause complete.*

Construction 6.5. Take the interval $(0,1]$ and the collection of all finite disjoint unions of sub-intervals of $(0,1]$ that have rational endpoints and that are left-open and right-closed. This collection forms a Boolean algebra. Indeed, this collection is closed under the formation of unions, intersections, and complements in $(0,1]$. Also, this Boolean algebra is countable since it has the same cardinality as $\bigcup_{i \quad \mathbb{N}} \mathbb{Q}^i$ ($\mathbb{Q}$ denotes the rational numbers of $(0,1]$). So the cardinality of the Boolean algebra in question is a countable union of countable sets and therefore it is countable. Take the state s as the restriction of the Lebesgue measure to B. By the construction, s is common cause complete. Indeed, consider the value $v = \frac{s(a \quad b) - s(a)\cdot s(b)}{1 + s(a \quad b) - s(a) - s(b)} > 0$. So v is a rational number with $v \leq s(a \quad b)$. By the definition of v and s, it is easily seen that there is an element c such that $c \leq a \quad b$ and $s(c) = v$. This completes the proof.

Let us briefly comment on the result and its proof. Firstly, the same method also gives us an uncountable Boolean algebra that is non-trivially common cause complete adding thus to the examples of [24]. Indeed, the classic results of the division ring theory assert that there is a division ring $\tilde{R}$ in the real numbers R such that $\tilde{R}$ is uncountable and dense in R. In analogy with Construction 6.4, one takes for B the collection of disjoint unions of intervals in $\tilde{R} \cap (0, 1]$. It again suffices to use the Lebesgue measure.

Secondly, though we mainly addressed the Boolean line of CCC, if we stepped into quantum logics, we see that the situation is analogous there (including the fact that finite quantum logics cannot be non-trivially CCC). We can easily construct "finitely additive" non-trivially CCC quantum logics, too. We achieve this goal by a construction called the horizontal sum (see 6.6). It may be noted that if we take the technique of the horizontal sum of a given cardinality, we can obtain arbitrarily large CCC examples.

Definition 6.6. [27] Let $(L_\alpha)_{\alpha \;\; I}$ be a collection of QLs and let L be their disjoint union, $L = \bigcup_{\alpha \;\; I} L_\alpha$, in which we identify the 0's and 1's of all L_α. Formally,

$$
\begin{aligned}
0_L &= 0_{L_\alpha}, && (\alpha \quad I, \alpha \text{ arbitrary}) \\
1_L &= 1_{L_\alpha}, && (\alpha \quad I, \alpha \text{ arbitrary}) \\
\leq_L &= \bigcup_{\alpha \;\; I} \leq_{L_\alpha}, && (\leq \ \text{understood as a binary relation}) \\
{}^{L} &= \bigcup_{\alpha \;\; I} {}^{L_\alpha}. && (\ \ \text{understood as a unary operation})
\end{aligned}
$$

Then $(L, \leq_L, {}^{L}, 0_L, 1_L)$ is a quantum logic, called the **horizontal sum** of $(L_\alpha)_{\alpha \;\; I}$.

If card $L_\alpha > 2$ for at least two values of α, $\alpha \quad I$, then the horizontal sum L is not Boolean. E.g., MO_2 (Example 5.7(1)) is the horizontal sum of two four-element Boolean algebras, $\{0, x, x\ , 1\}$ and $\{0, y, y\ , 1\}$.

Chapter 7

Results on the Darboux property in connection with RCCP

In this Chapter, we pursue QLs that meet notable properties with respect to RCCP. This chapter is based upon [6, 7, 8, 5].

We shall need the definition of an interval in a QL.

Definition 7.1. Let L be a QL and let $a, b \in L$, $a \leq b$. The set $[a, b]_L = \{c \mid c \in L,\ a \leq c \leq b\}$ is said to be an **interval** in L.

An interval in a QL becomes also a QL with the inherited structure (see e.g., [27]).

Proposition 7.2. *Let L be a QL, $a, b \in L$, $a \leq b$. For all x, y from the interval $Q = [a, b]_L$, we define*

1. $x \leq_Q y \iff x \leq_L y$,
2. $x^{\perp_Q} = (x^{\perp_L} \wedge_L b) \vee_L a$.

Then $(Q, \leq_Q, {}^{\perp_Q}, a, b)$ is a quantum logic and $x \vee_Q y = x \vee_L y$ for all $x, y \in Q$. If L is σ-complete [resp. complete], so is Q.[1]

[1] We always index intervals in QLs by the respective logic; the operations used inside the brackets (inside the bounds of the interval) are tacitly assumed to be computed in the same logic, without their explicit indexing, unless there is a risk of confusion.

Let us now recall the definition of the Darboux property (also called *denseness* in the context of Boolean algebras, see [17]).

Definition 7.3. Let (L, s) be a quantum probability space. We say that s has the **Darboux property** if for any $x, y \quad L$ with $x \leq y$ and any r, $r \quad [s(x), s(y)]$, there is some z, $z \quad [x, y]_L$, such that $s(z) = r$.

Theorem 7.4. *Let L be a QL. Let $a, b \quad L$ be positively correlated. Let s be a state on L whose restriction to the interval $[0, a \quad b]_L$ has the Darboux property. Then there is a common cause c, $c \quad [0, a \quad b]_L$, such that $(a, b, c \mid s)$ adheres to RCCP.*

Proof. Following Proposition 4.3, we only have to check (RCCP2).

Let us discuss first the case $s(a \quad b) = s(a)$. Then we may set $c = a \quad b$. Since $s(a \quad c\ ) = s(a) - s(c)$, we see that (RCCP2) amounts to $0 = 0$. Analogous arguments work for $s(a \quad b) = s(b)$.

Suppose therefore that $s(a \quad b) < s(a)$ and $s(a \quad b) < s(b)$. Let us consider two real functions, F and G, defined on the interval $[0, s(a \quad b)]$,

$$F(t) = s(a \quad b) - t\,, \qquad G(t) = \frac{(s(a) - t) \cdot (s(b) - t)}{1 - t}\,.$$

Both functions F, G are continuous on $[0, s(a \quad b)]$. Write $H(t) = F(t) - G(t)$. Then $H(0) = \mathrm{Cov}_s(a, b) > 0$ and $H(s(a \quad b)) = -G(s(a \quad b)) < 0$.

By the classical Darboux property for continuous functions, there is an r, $r \quad (0, s(a \quad b))$, such that $H(r) = 0$. Hence $F(r) = G(r)$.

Moreover, by the Darboux property of s there is an element c, $c \quad L$, such that $s(c) = r$. We will check that c is a common cause for a, b. We claim that the construction above guarantees (RCCP2). Indeed, since $F(s(c)) = G(s(c))$, we have

$$s(a \quad b) - s(c) = \frac{(s(a) - s(c)) \cdot (s(b) - s(c))}{1 - s(c)}\,.$$

It gives us the equality

$$\frac{s(a \quad b \quad c\ )}{1 - s(c)} = \frac{(s(a) - s(c)) \cdot (s(b) - s(c))}{(1 - s(c))^2}\,.$$

This verifies (RCCP2). □

Observe that for a, b compatible, we have a dual formulation of Theorem 7.4, using the interval $[a \quad b, 1]_L$ instead of $[0, a \quad b]_L$:

Theorem 7.5. *Let L be a QL. Let $a, b \quad L$ be compatible and positively correlated. Let s be a state on L whose restriction to the interval $[a \quad b, 1]_L$ has the Darboux property. Then there is a common cause c, $c \quad [a \quad b, 1]_L$, such that $(a, b, c \mid s)$ adheres to RCCP.*

Proof. According to (5.10), $\mathrm{Cov}_s(a, b) = \mathrm{Cov}_s(a\ , b\ )$. Thus also $a\ , b\ $ are positively correlated. By Theorem 7.4, they have a common cause $d \leq a \quad b\ $. (This fact is, in a modified version, mentioned in [24, p. 16].) Then a, b have a common cause $c = d\ \geq a \quad b$, distinct from that constructed in the proof of Theorem 7.4. $\square$

For a, b *non-compatible*, an analogue of Theorem 7.5 does not hold.

Example 7.6. For $s(a), s(b) \quad (0, 1)$, the value of $s(a \quad b\ )$ can be arbitrarily small. If we choose $s(a \quad b\ ) = 0$, then $s(a \quad b) = 1$ and $s(c) = 1$ for all $c \quad [a \quad b, 1]_L$. The Darboux property on $[a \quad b, 1]_L$ is trivially satisfied, but c cannot be a common cause of a, b.

Chapter 8

Quantum probability spaces without atoms

A common cause need not exist for all a, b with $\mathrm{Cov}_s(a, b) > 0$. A natural question is whether common causes always exist at least in some quantum probability spaces.

After a series of negative results, a question was posed in [22, 23] whether common cause complete quantum probability spaces exist. A positive result was formulated in [28] (see the following Theorem 8.6). Here we shall provide a new proof of this result.

As shown in the previous Chapter, the Darboux property allows to find the common cause. However, it presents a rather strict condition on the quantum probability space we deal with. Do such quantum probability spaces exist at all? The following notions prove themselves significant in answering the question. This chapter is based upon [6, 7, 8, 5].

Definition 8.1. [24] Let (L, s) be a quantum probability space. Let us call an element $a \in L$ an ***s*-atom** if $s(a) > 0$ and there is no b, $b \in L$, such that $0 < b < a$ and $0 < s(b) < s(a)$. The state s on L is called **purely nonatomic** if there are no s-atoms.

A related notion can be defined without any reference to the state.

Definition 8.2. Let L be a QL. Let us call a minimal non-zero element in L an **atom** in L. Formally, an element a, $a \in L$, $a > 0$, is said to be an atom in L if there is no b, $b \in L$, such that $0 < b < a$. Let us denote by $\boldsymbol{\mathcal{A}t(L)}$ the set of all atoms in L. If L does not have any atoms, we say that L is **atomless**.

Obviously, if a state s on a quantum logic L is faithful, then s is purely nonatomic if and only if L is atomless. For a non-faithful state, these properties may differ.

Example 8.3. Consider the σ-algebra $\mathcal{B}([0,1])$ of all Borel sets on $[0,1]$ with the Lebesgue measure μ; $\mathcal{B}([0,1])$ obviously has atoms that are not μ-atoms, so are the singletons of $[0,1]$. Let us factorize $\mathcal{B}([0,1])$ over the σ-ideal J of the sets of Lebesgue measure zero, $J = \{X \in \mathcal{B}([0,1]) \mid \mu(X) = 0\}$. The factor Boolean algebra $\tilde{B} = \mathcal{B}([0,1])/J$ is complete [17]. The Lebesgue measure corresponds to a *completely additive* measure $s_{\tilde{B}}$ on $\tilde{B}$, $s_{\tilde{B}}(X/J) = \mu(X)$. As known (see [17]), $\tilde{B}$ is atomless and $s_{\tilde{B}}$ is purely nonatomic.

Let us recall a definition which will be needed for the proof of the next theorem.

Definition 8.4. Let us have two QLs M and L. A mapping $f\colon M \to L$ is a **morphism** if it preserves suprema, infima, and orthocomplementation. If, moreover, f is surjective and any countable mutually orthogonal subset $K \subseteq M$ satisfies $f(\bigvee K) = \bigvee_{x \in K} f(x)$, then f is said to be a **σ-epimorphism**. Let us call a morphism f an **isomorphism** if it is one-to-one and f^{-1} is also a morphism. If $f\colon M \to f(M) \subseteq L$ is an isomorphism, then f is called an **embedding**.

The following theorem was inspired by [30].

Theorem 8.5. *Let (L, s) be a σ-complete quantum probability space. If s is purely nonatomic then s has the Darboux property.*

Proof. Suppose that $x, y \in L$, $x \leq y$. Take a maximal Boolean subalgebra (block) B of L that contains both x and y. Then B is σ-complete (see [27]). Using the Loomis–Sikorski Theorem (see [38]), there is a set-representable Boolean σ-algebra, $(S, \boldsymbol{\Delta})$, and a σ-epimorphism (see Definition 8.4) $e\colon (S, \boldsymbol{\Delta}) \to B$. Moreover, there are two sets $X, Y \in \boldsymbol{\Delta}$, such that $X \subseteq Y$, $e(X) = x$, and $e(Y) = y$. Let s be a σ-additive purely nonatomic state on L. Then the restriction of s to B becomes a σ-additive purely nonatomic state on B. Further, $s \circ e$ naturally defines a "preimage" σ-additive purely nonatomic state on QL $(S, \boldsymbol{\Delta})$. By the classical result (see e.g., [17, p. 174] and [30]), for each r, $r \in [(s \circ e)(X), (s \circ e)(Y)]$, there is a Z, $X \subseteq Z \subseteq Y$, such that $(s \circ e)(Z) = s(e(Z)) = r$ and we are done. □

As shown in [35], finitely additive purely nonatomic measures need not have the Darboux property. Another example is the Boolean algebra of all subsets of the real interval $[0,1]$ which are finite unions of intervals with rational bounds. The Lebesgue measure is purely nonatomic, but the Darboux property is violated. Thus, the σ-completeness of the quantum logic and

σ-additivity of the state will be required in the sequel.

If we combine Theorems 7.4 and 8.5, we obtain the following result.

Theorem 8.6. *Let (L, s) be a σ-complete quantum probability space. If s is purely nonatomic then (L, s) is common cause complete.*
A consequence: If L is atomless and s faithful then (L, s) is common cause complete.

Remark 8.7. In [28], the author takes the effort to give a self-contained direct proof of Theorem 8.6. Though there seems to be some serious overlooking in his proof, the result presents a pioneering contribution and the technique can be remedied (see [24]). Our method provides a short proof.

As regards [24], the proof in [28] assumes *logical independence* of a, b, i.e., all expressions $a \quad b$, $a \quad b$, $a \quad b$, $a \quad b$ are supposed to be be non-zero. However, [24, Proposition 6.11] apparently excludes several cases and then solves the case of logical independence, although the case $a \quad b = 0$ was not excluded. Besides, its proof starts with the assumption that a, b are *compatible*. This assumption is not used, so this does not affect the generalized contribution.

Theorem 8.6 may have a potential for further generalization, see [2, 18, 30].

In both [24, 28], the authors consider the projection logic of the von Neumann algebra of the type II_1 (finite von Neumann algebra with a continuous dimension function) as an example of a modular atomless σ-complete QL (see e.g., [26]). However, this may hardly be accessible for the readers that are not well-trained in functional analysis.

We shall present a natural example by making use of the *horizontal sum* (and obtain an example that is *not* modular).

Example 8.8. Let us take the complete Boolean algebra $\tilde{B}$ with a faithful completely additive state $s_{\tilde{B}}$ of Example 8.3 (its properties follow from [17]). We form a horizontal sum of two copies, $\tilde{B}$ and $\tilde{A}$, of this Boolean algebra. We obtain a non-Boolean atomless complete QL, Q, which is not modular, and a faithful completely additive state s_Q on it. It has the Darboux property and Theorem 8.6 applies to (Q, s_Q).

Another tangible example can be found as follows. Take the algebra $\mathcal{B}([0, \infty))$ of Borel subsets of the interval $[0, \infty)$ and define a state s by the formula $s(A) = \int_A \mathrm{e}^{-x}\, \mathrm{d}x$. When we factorize over the null sets of s, we obtain an atomless complete Boolean algebra with a faithful completely additive state. It may be observed that by this factorization technique with the algebra of ***all subsets*** of a set, we have a good chance to obtain a trivial σ-algebra—this is an interesting case when a philosophical principle meets a measure-theoretical phenomenon (the axiom of real-measurable cardinals, see e.g., [40]).

Chapter 9

Quantum probability spaces with exactly one atom

This chapter is based upon [6, 7, 8, 5]. We can extend the previous results as follows:

Theorem 9.1. *Let* (L, s) *be a* σ*-complete quantum probability space. Let there be at most one* s*-atom. Let* a, b L *be compatible and positively correlated. Then there is an element* c*,* c L*, such that* $(a, b, c \mid s)$ *adheres to RCCP.*

Proof. The case of no s-atoms was already proved. Suppose that there is exactly one s-atom, say e L. Then s violates the Darboux property. However, if e / $[0, a \quad b]_L$, the restriction of s to this interval satisfies the Darboux property and Theorem 7.4 applies. Analogously, if e / $[0, a \quad b\ ]_L$, the Darboux property holds for the restriction of s to $[0, a \quad b\ ]_L$. Applying it to complements (to the function $1 - s$), the Darboux property holds also for the restriction of s to $[a \quad b, 1]_L$ and Theorem 7.5 applies. As e cannot be simultaneously in $[0, a \quad b]_L$ and in $[0, a \quad b\ ]_L$, we solved all possible cases. □

An analogous result for Boolean algebras is presented in [24, Proposition 4.17], with a rather long proof. Then it is generalized to quantum probability spaces in [24, Proposition 6.15]: under an additional assumption that the state s is faithful and e-*decomposable* (e being the only atom of L), i.e., s is a convex combination of a purely nonatomic state s_1 and a state s_2 given by

$$s_2(x) = \begin{cases} 1 & \text{if } e \leq x \\ 0 & \text{otherwise} \end{cases}$$

for any x L. This result was presented as a proposition; however, its validity will be jeopardized, thus we formulate it as a conjecture:[1]

Conjecture 9.2. *[24, Proposition 6.15] Let (L, s) be a quantum probability space with a faithful s. If there is a single s-atom e in L and s is e-decomposable, then (L, s) is common cause complete.*

The proof in [24] is again long, even with extensive references to analogous parts of the proof of the Boolean case. In contrast to this, we found a short proof of more general Theorem 9.1 without the use of e-decomposability. The σ-completeness of L and σ-additivity of s are not mentioned in Conjecture 9.2. What is crucial is that *compatibility of a and b is not assumed* in the formulation of [24, Proposition 6.15], but assumed and *used* since the very beginning of its proof. Thus *the non-compatible case of Conjecture 9.2 is not proved* there. It cannot be because it does not hold in general, as we witness by the following counterexample.

Example 9.3. Let us take the quantum probability space (Q, s_Q) from Example 8.8. We take a two-element Boolean algebra, $T = \{0_T, 1_T\}$ with its unique state s_T and construct the product $L = Q \times T$ (Definition 5.5). It has exactly one atom, $c = (0_Q, 1_T)$.

Given any p, q $(0, 1)$, we may choose a_0 $\tilde{A} \setminus \{0, 1\}$, b_0 $\tilde{B} \setminus \{0, 1\}$ such that $s_Q(a_0) = p$, $s_Q(b_0) = q$. The elements a_0, b_0 Q are non-compatible, a_0 $_Q$ $b_0 = 0_Q$. Also elements $a = (a_0, 1_T)$, $b = (b_0, 1_T)$ of L are non-compatible. They satisfy a $_L$ $b = c$.

For each r $[0, 1]$, there is a unique state s on L such that

$$s(x, y) = r\, s_T(y) + (1 - r)\, s_Q(x)$$

(Proposition 5.6). In particular,

$$\begin{aligned} s(c) &= r\,, \\ s(a) &= r + (1 - r)\, p\,, \\ s(b) &= r + (1 - r)\, q\,. \end{aligned}$$

Events a, b are positively correlated if and only if

$$r > (r + (1 - r)\, p) \cdot (r + (1 - r)\, q)\,, \tag{9.1}$$

which can be easily satisfied. Any common cause of a, b must be compatible with both of them. The bounds $0_L, 1_L$ are excluded, so c and c remain the only candidates. In both cases, the conditional independence (RCCP2) is required.

[1] We cite [24, Proposition 6.15] literally, only some terms are replaced by synonyms used in this paper.

To complete the counterexample, we may choose, e.g., $p = q = \frac{1}{4}$, $r = \frac{1}{2}$; then $s(c) = s(c\ ) = \frac{1}{2}$, $s(a) = s(b) = \frac{5}{8}$, (9.1) becomes a valid inequality $\frac{1}{2} > \left(\frac{5}{8}\right)^2$ and (RCCP2) is violated because $s(a\quad b \mid c\ ) = 0$, $s(a \mid c\ ) = s(b \mid c\ ) = \frac{1}{4}$.

As shown in [24, Proposition 4.16], an analogue of Theorem 9.1 does not hold for two and more atoms even in Boolean algebras.

Chapter 10

Quantum probability spaces with exactly one atom—embeddings

This chapter is based upon [6, 7, 8, 5]. As shown in the previous Chapter, [24, Proposition 6.15] (cited as Conjecture 9.2 here) is not valid and quantum probability spaces need not admit common causes if they contain atoms (even a single atom). The next question (formulated in [24]) was whether quantum logics which are *not* common cause complete can be enlarged to (embeddeded in) common cause complete ones. One can imagine that an added part can contain all common causes of positively correlated events *from the original QL*. (A partial positive result for finitely many correlated pairs in a Boolean algebra is [24, Proposition 3.9].) However, the added elements introduce new correlations which need not have common causes. Thus the search for a common cause complete system may require an infinite procedure with a problematic result. We found a way around the question and met the goal: We can embed some QLs into those which are common cause complete. This represents a progress in one of the principal open problems in the monograph [24].

One source of inspiration was [24, Proposition 6.11] (Theorem 8.6). According to it, starting from the original σ-complete quantum probability space (L, s_L), it suffices to find a σ-complete quantum probability space (P, s_P) and an embedding $f\colon L \to P$ such that

- $x \quad L : s_L(f(x)) = s_P(x)\,,$
- s_P is purely nonatomic.

Then (P, s_P) is common cause complete. In our constructions, L will be a subset of P and the embedding f will be the identity; thus the task is to extend the state s_L from L to a state s_P on P. In a weaker version, if s_L is faithful, it suffices to ensure that P is atomless and s_P is faithful.

The first step is to embed a QL K with an atom a into a QL P such that $At(P) = At(K) \setminus \{a\}$. We accomplish this through combining several techniques found in the literature. To make the paper self-contained (and independent of the specifics of different sources), let us describe the whole construction in detail. At the end of this Chapter, we summarize the result and its properties in Theorem 10.5 and its Corollary 10.6.

In Definition 6.6, we introduced a new quantum logic from the collection of QLs that have only the least and the greatest element in common. This is a special case of a more general construction of a *pasting* (see [33]). We shall need another special case where two QLs intersect in two intervals.

Let a be an atom in a QL K. The intervals $[a, 1]_K$, $[0, a\ ]_K$ are isomorphic under the isomorphism $h(y) = y \quad a$ for all $y \geq a$. Its inverse is $h^{-1}(x) = x \quad a$ for all $x \leq a\ $. Therefore, $h^{-1}(h(y)) = (y \quad a\ ) \quad a\ = y$. (These computations are easily verified in a Boolean algebra and this is a sufficient argument because a is compatible with x and y.)

The union $D = [0, a\ ]_K \quad [a, 1]_K$ is the set of all elements of K compatible with a. It is a sublogic of K. The logic K may contain also other elements, not only those that belong to D. We take the product, $K \times T$, (see Definition 5.5) of K with a two-element Boolean algebra, $T = \{0_T, 1_T\}$, and select its sublogic $G = \{(x, 0_T), x \leq a\ \} \quad \{(y, 1_T), y \geq a\} \quad K \times T$. We define the mapping $i\colon D = [0, a\ ]_K \quad [a, 1]_K \to G$ such that $i(x) = (x, 0_T)$, $i(y) = (h(y), 1_T)$ for all $x \leq a\ $, $y \geq a$. It is easy to verify that i is an isomorphism.

Let us take an atomless Boolean σ-algebra $\tilde{B}$ with a faithful σ-additive state $s_{\tilde{B}}$, e.g., that of Example 8.3. We construct the product $M = [0, a\ ]_K \times \tilde{B}$. We define an embedding $f\colon G \to M$ by $f(x, 0_T) = (x, 0_{\tilde{B}})$, $f(x, 1_T) = (x, 1_{\tilde{B}})$. We will use a composed embedding $g = f \circ i\colon D \to M$, $g(x) = (x, 0_{\tilde{B}})$, $g(y) = (h(y), 1_{\tilde{B}})$ for all $x \leq a\ $, $y \geq a$.

The next step is the construction of the *pasting* of the QLs K and M (see [27], generalized in [33]). These logics K and M contain isomorphic sublogics $D = [0, a\ ]_K \quad [a, 1]_K \quad K$ and $g(D) = [(0_K, 0_{\tilde{B}}), (a^{\ K}, 0_{\tilde{B}})]$ $[(0_K, 1_{\tilde{B}}), (a^{\ K}, 1_{\tilde{B}})] \quad M$. In the disjoint union of K and M, let us identify z with $g(z)$ for all $z \quad D$. This means that we have identified

- $z \quad [0, a\,]_K$ with $(z, 0_{\tilde{B}}) \quad M\,$,
- $z \quad [a, 1]_K$ with $(h(z), 1_{\tilde{B}}) = (z \quad a\,, 1_{\tilde{B}}) \quad M\,$.

(In particular, we identify $a \quad [0, a\,]_K$ with $(a\,, 0_{\tilde{B}}) \quad M$ and $a \quad [a, 1]_K$ with $(0_K, 1_{\tilde{B}}) \quad M$.) Let us denote by P the union of K and M subject to this identification. Let us equip P with the ordering and orthocomplementation inherited from K and M. Formally,

$$\leq_P = \leq_K \quad \leq_M\,,$$
$$^P = \;^K \quad ^M\,.$$

When expressed explicitly, we define

- $z \leq_P w$ if and only if at least one of the following alternatives holds:
  - $z, w \quad K$, $z \leq_K w$,
  - $z, w \quad M$, $z \leq_M w$;
- $z = w^{\,P}$ if and only if at least one of the following alternatives holds:
  - $z, w \quad K$, $z = w^{\,K}$,
  - $z, w \quad M$, $z = w^{\,M}$.

When $z, w \quad K \cap M$, the ordering and orthocomplementation inherited from K and M coincide because D and G are isomorphic. Thus, the definitions of these operations are consistent.

Assume that two elements $x, y \quad P$ are orthogonal in P, $x \quad _P\, y$, then by definition $x \leq_P y^{\,P}$, and $x, y^{\,P} \quad K$ or $x, y^{\,P} \quad M$. We have two (non-exclusive) options:

- If $x \quad _K\, y$, then $x, y^{\,P} \quad K$, also $y \quad K$, $y^{\,P} = y^{\,K}$ and $x \leq_K y^{\,K}$, thus $x \quad _K\, y$.
- If $x \quad _M\, y$, then $x, y^{\,P} \quad M$, also $y \quad M$, $y^{\,P} = y^{\,M}$ and $x \leq_M y^{\,M}$, thus $x \quad _M\, y$.

Hence the orthogonality relation on P is the union of orthogonalities on K and M,

$$_P = \;_K \quad _M\,.$$

Let us outline the verification that P endowed with this ordering and orthocomplementation is a QL. The properties of the ordering and orthocomplementation are verified in each of the QLs K and M separately, and they coincide in their intersection. The only step requiring more explanation is how the transitivity of the ordering $\leq_P$ of P follows from the transitivity of $\leq_K$ and $\leq_M$. Suppose that $x, y, z \in P$, $x \leq_P y \leq_P z$. If $x \leq_K y \leq_K z$, then $x \leq_K z$, hence $x \leq_P z$. The same holds for M. If $x \leq_K y \leq_M z$, then $x \in K$, $z \in M$, and $y \in D = K \cap M$. From the special choice of $D = [0, a\,]_K \cup [a, 1]_K$, we have two (non-exclusive) options:

- If $y \in [0, a\,]_K$, then $x \in M$.
- If $y \in [a, 1]_K$, then $z \in K$.

The case $x \leq_M y \leq_K z$ is solved analogously. In any case, one of the QLs K or M contains all three elements x, y, z and guarantees $x \leq_P z$.

Let $x, y \in K$. Each upper bound z of $\{x, y\}$ must be in K. Indeed, if $z \in M \setminus K$, we must have $x \leq_M z$ and $y \leq_M z$, which is possible only for $\{x, y\} \subseteq D = K \cap M$ and, as D is a sublogic of both K and M, $z \geq x \vee_D y = x \vee_K y = x \vee_M y$. In any case, $x \vee_K y = x \vee_P y$. The same argument can be used for suprema of elements of M instead of K, and also dually for infima. The conclusion is that the finite lattice operations of K and M are preserved in their pasting P. As a consequence, we obtain the orthomodular law in P because it acts separately in K and M and consistently in their intersection.

Let us prove that P is a lattice. What remains is to find the supremum of $x \in K \setminus M$ and $y \in M \setminus K$. Then y must be of the form $y = (z, w) \in M$ for some $z \in [0, a\,]_K$ and $0_{\tilde{B}} < w < 1_{\tilde{B}}$. All upper bounds of $\{x, y\}$ must be in D. All upper bounds of y which are in D are greater than or equal to $(z, 1_{\tilde{B}}) \in M$, which was identified with $z \vee_K a \in D$. Thus $x \vee_K z \vee_K a$ is the supremum $x \vee_P y$ and P is a lattice.

We call the quantum logic P the **pasting of K and M**. (We refer to [33, Propositions 4.2 and 4.3] for a detailed analysis.) We consider K a subset of P, the natural embedding of K into P being the identity mapping.

The element $a \in \mathcal{A}t(K)$ is not an atom of P. Indeed, as $a \in K \subseteq P$ was identified with $(0_K, 1_{\tilde{B}}) \in M \subseteq P$, the interval $[0, a]_P = [(0_K, 0_{\tilde{B}}), (0_K, 1_{\tilde{B}})]_P$ is isomorphic to $\tilde{B}$ and contains no atoms. We say that **P originated by the substitution of the atom a in K by $\tilde{B}$**. (We used a particular case of the substitution described in [33].)

Let us now summarize properties of the substitution of an atom which we shall use in the sequel.

Lemma 10.1. Atoms of P are exactly all atoms of K different from a, i.e.,

$$\mathsf{A}t(P) = \mathsf{A}t(K) \setminus \{a\}.$$

Proof. Since P is the disjoint union of K and M with some elements identified, the only possible atoms are the atoms of K or M. As already mentioned, a is not an atom in P. The only atoms of M are atoms of $[0, a\,]_K$, thus atoms of K. It remains to be shown that all other atoms of K are atoms of P. Atoms of K which were not compatible with a were not identified with any elements of M, so in P they are ordered only with respect to the original elements of K, thus they remain atoms in P. An atom x which was compatible with a was identified with element $(x, 0_{\tilde{B}})$. Clearly there is no element smaller than $0_{\tilde{B}}$ in $\tilde{B}$ and since x was an atom in K, there is no element smaller than x in the interval $[0, a\,]_K$, thus x is an atom of P. □

Lemma 10.2. Each block of P is either

- a block of K, or
- the result of a substitution of the atom a in a block of K by $\tilde{B}$.

This determines a natural one-to-one correspondence between the blocks of K and blocks of P.

Proof. In the substitution of a K, we only added the elements that are compatible with all elements compatible with a. Also, we take into account the regularity of QLs—if a is pairwisely compatible with all elements $x_1, x_2, ..., x_n$ [36], then a is compatible with all elements of the QL generated by $\{x_1, x_2, ..., x_n\}$. Thus the elements of $M \setminus K$ were added only to the blocks of K containing a and did not generate any new block. □

Lemma 10.3. If K is σ-complete, then P is σ-complete, too.

Proof. Suprema of orthogonal sequences exist; they are realized in blocks, which are finite powers of $\tilde{B}$. However, the existence of the suprema of *all* sequences require a more complicated argument.

Let $(x_i)_{i\in\mathbb{N}}$ be a sequence in P. Define a sequence $(y_i)_{i\in\mathbb{N}}$ in P by induction:

$$\begin{aligned}
y_1 &= x_1\,,\\
y_2 &= (x_1 \vee x_2) \wedge y_1^{\perp}\,,\\
y_3 &= (x_1 \vee x_2 \vee x_3) \wedge (y_1 \vee y_2)^{\perp}\,,\\
&\dots
\end{aligned}$$

in general

$$y_{j+1} = \Big(\bigvee_{i\le j+1} x_i\Big) \wedge \Big(\bigvee_{i\le j} y_i\Big)^{\perp}$$

for all $j \in \mathbb{N}$. Then

$$\bigvee_{i\le j} x_i = \bigvee_{i\le j} y_i$$

for all $j \in \mathbb{N}$. The sequence $(y_i)_{i\in\mathbb{N}}$ is orthogonal, thus it has a supremum in P, which is also the supremum of $(x_i)_{i\in\mathbb{N}}$,

$$\bigvee_{i\in\mathbb{N}} x_i = \bigvee_{i\in\mathbb{N}} y_i\,.$$

□

So far, we only dealt with the structure of the quantum logic obtained by a substitution of an atom. Now we shall describe how this construction behaves with respect to states.

Lemma 10.4. Let the QL K be σ-complete and let s_K be a σ-additive state on K. Let us take the (completely additive) state $s_{\tilde B}$ on $\tilde B$ of Example 8.3 and define

$$s_P(z) = \begin{cases} s_K(z) & \text{if } z \in K\,,\\ s_K(x) + s_K(a)\cdot s_{\tilde B}(y) & \text{if } z = (x,y) \in M\,. \end{cases} \tag{10.1}$$

Than s_P is a σ-additive state on P. Moreover, s_P is faithful provided s_K is faithful.

Proof. For $z = (x,y) \in D = K \cap M$, both cases in (10.1) apply and give the same result:

- $z \in [0, a\,]_K$ was identified with $g(z) = (z, 0_{\tilde B})$ and $s_P(z) = s_K(z)$,
- $z \in [a, 1]_K$ was identified with $g(z) = (h(z), 1_{\tilde B}) = (z \wedge a^{\perp}, 1_{\tilde B})$ and $s_P(z) = s_K(z \wedge a^{\perp}) + s_K(a) = s_K(z)$.

Thus s_P is correctly defined. Let us check that s_P is a state on P.

We see by our definition that $s_P(1) = s_K(1) = 1$.

Let us verify the σ-additivity of s_P. The ordering and the orthocomplementation in P are inherited from K and M. The orthogonality relation on P is just the union of orthogonalities on K and M. This means that any countable set of mutually orthogonal elements in P must be a subset of one of the logics K and M. The restriction of s_P to K is s_K, the restriction of s_P to M is a state on the product (isomorphic to) $[0, a\ ]_K \times \tilde{B}$, obtained as a convex combination of states (see Proposition 5.6)

$$s_P((x, y)) = s_K(a\ ) \cdot \frac{s_K(x)}{s_K(a\ )} + s_K(a) \cdot s_{\tilde{B}}(y)\,,$$

where $\frac{s_K}{s_K(a^{□})}$ is a state on $[0, a\ ]_K$ provided that $s_K(a\ ) = 0$ (see also Proposition 5.6); otherwise, the first summand is omitted and $s_P((x, y)) = s_{\tilde{B}}(y)$. Both summands are σ-additive. This completes the proof that s_P is a σ-additive state on P.

Preservation of faithfulness is verified easily. □

Let us summarize the preceding construction:

Theorem 10.5. *Let K be a QL and a be an atom of K. Then K is a sublogic of a QL, P, $P \quad K$ such that $At(P) = At(K) \setminus \{a\}$. Each block of P is either a block of K or the result of a substitution of an atom in a block of K by $\tilde{B}$. If K is σ-complete, so is P and each σ-additive state s_K on K can be extended to a σ-additive state s_P on P. If s_K is faithful, s_P can be chosen faithful, too.*

Corollary 10.6. *Let K be a σ-complete QL with a σ-additive state s_K that is faithful. Let K have precisely one atom. Then we can embed K in a QL P with an extension s_P of s_K such that (P, s_P) is common cause complete.*

Proof. We apply Theorem 10.5 and obtain a QL P with a state s_P extending s_K. Moreover, s_P is faithful. The set of atoms becomes empty, $At(P) = At(K) \setminus \{a\} = \ $, thus P is atomless. According to Theorem 8.6, (P, s_P) is common cause complete. □

Chapter 11

Quantum probability spaces with more atoms—embeddings

Preceding works indicated that common cause complete quantum logics are "rare", e.g., atoms are not allowed. In contrast to that, we show that the class of common cause complete quantum logics is quite large; they allow to embed almost arbitrary quantum logics, under mild restrictive assumptions. Philosophically, all common causes can be (in many cases) found in a larger quantum logic, which, moreover, *does not introduce any new correlations without common causes*. This result changes our view on the role of Reichenbach's common cause principle—it is not an "exotic" requirement which would be hardly achievable. Embedding into a larger quantum logic can satisfy it. This chapter is based upon [6, 7, 8, 5].

The construction from Theorem 10.5 and Corollary 10.6 can be repeated, which arrives to the following consequence:

Theorem 11.1. *Let K be a σ-complete QL with a faithful σ-additive state s_K. Let K have only a finite number of atoms. Then we can embed K in a σ-complete QL Q with a state s_Q extending s_K such that (Q, s_Q) is common cause complete.*

Proof. In the case of one atom, the embedding is obtained by Corollary 10.6. In the case of finitely many atoms, we only compose the embeddings of Corollary 10.6 finitely many times to get rid of all atoms. We obtain an embedding into an atomless σ-complete QL Q with the desired faithful state. □

Infinite repetition of the same procedure requires more tools, but it is possible at least under additional conditions. Let us formulate the final embedding result in paper [6, 7, 8, 5]. This result is mostly a work of the co-authors, we included it for the self-containedness.

Definition 11.2. Strictly increasing sequences in a QL are called **chains**. A QL L is said to be **chain-finite** if all chains in L are finite.

Theorem 11.3. *Let K be a countable chain-finite QL and let s_K be a faithful state on K. Then K can be embedded in a complete quantum logic Q in such a manner that the state s_K can be extended to a faithful completely additive state, s_Q, on Q and (Q, s_Q) is common cause complete.*

Proof. We start from the logic $Q_0 = K$ with a state $s_0 = s_K$. We choose an atom, $a_0 \in At(Q_0)$, and substitute it by $\tilde{B}$ as in Corollary 10.6. We obtain a QL Q_1, $Q_1 \supseteq Q_0$, with a state s_1 that extends s_0. Then we choose an atom $a_1 \in At(Q_1)$. It is an atom of K and we repeat the construction, substituting a_1 by $\tilde{B}$ in Q_1. The result is a logic Q_2 with a state s_2, etc. This can be done until we exhaust all atoms of K. We obtain an increasing sequence $(Q_i)_{i \in I}$ of QLs with the corresponding extended states, $(s_i)_{i \in I}$. Since the sequence is infinite, we need an additional step, the *limit* of a sequence of logics (see [33, Theorem 4.10]). In this particular application, it admits a simplified proof which we present here. Besides, we need properties which are not guaranteed in general limits of QLs, but they are satisfied in our particular case.

We have an infinite sequence of logics $(Q_i)_{i \in \mathsf{N}}$ with states $(s_i)_{i \in \mathsf{N}}$ such that $Q_i \subseteq Q_{i+1}$ for all $i \in \mathsf{N}$, the identity mapping being an embedding $Q_i \to Q_{i+1}$, and s_{i+1} extending s_i. Let us take the union

$$Q = \bigcup_{i \in \mathsf{N}} Q_i$$

and embed it with the inherited ordering and orthocomplementation. Formally,

$$\leq_Q = \bigcup_{i \in \mathsf{N}} \leq_{Q_i}, \qquad (\leq \text{ understood as a binary relation})$$

$${}^{\perp_Q} = \bigcup_{i \in \mathsf{N}} {}^{\perp_{Q_i}}. \qquad ({}^{\perp} \text{ understood as a unary operation})$$

Explicitly:

- $z \leq_Q w$ if and only if $\exists i \in \mathsf{N} : z, w \in Q_i,\ z \leq_{Q_i} w$,
- $z = w^{\perp_Q}$ if and only if $\exists i \in \mathsf{N} : z, w \in Q_i,\ z = w^{\perp_{Q_i}}$.

Such a definition of operations on Q is easily seen to be correct. All properties concerning finitely many elements are easily verified in a sufficiently large QL Q_i which contains all of them. Thus

$$Q = \bigcup_{i \quad \mathsf{N}} Q_i$$

with the inherited operations is a quantum logic.

The only property which has not been proved yet is the completeness of Q. One uses the fact that blocks of K were supposed finite. In the course of the construction of Q, each atom a of K was substituted by $\tilde{B}$. Thus the interval $[0, a]_K$ was "inflated" to an interval $[0, a]_Q$ isomorphic to $\tilde{B}$. In finitely many steps, all atoms of the block were substituted, and then the block did not change. Thus the blocks of Q are (exactly) products of finitely many copies of $\tilde{B}$, they are of the form $\tilde{B}^n$ for $n \quad \mathsf{N}$ equal to the number of atoms in the corresponding block of K. The blocks are obviously complete.

Let us construct a state, s_Q, on Q, which extends all $(s_i)_{i \quad \mathsf{N}}$. Making use of Theorem 8.6, s_Q can be correctly defined by setting

$$s_Q(x) = s_i(x) \text{ whenever } x \quad Q_i\,.$$

Again, its properties concerning finitely many elements can be derived from a sufficiently large logic Q_i which contains all of them. Thus s_i is a state. The only problem is the complete additivity. This can be proved by looking at blocks. Each orthogonal sequence belongs to some block, which is of the form $\tilde{B}^n$ for some $n \quad \mathsf{N}$. This block is contained in some logic Q_i for a sufficiently large i; s_Q restricted to Q_i is s_i, which is σ-additive by Theorem 8.6. If there is a faithful σ-additive state, all orthogonal sets must be countable. Hence the QL Q_i is complete and the state s_i is completely additive. The block remains unchanged in all subsequent quantum logics in the sequence. The block is a complete Boolean algebra and contains the supremum of the sequence in Q. Thus also Q is complete and s_Q is completely additive. □

Let us shortly comment on the results above. Considering the construction, it seems that K can be more complex. Not all conditions assumed in Theorem 11.3 are necessary.

Further generalizations are left for future work. What we consider important is that, eleven years after the monograph [24] was published, we succeeded to substantially contribute to one of its principal unsolved questions. As far as we know, no other progress in this direction was made. Moreover, our result (Theorem 11.3) is applicable to quite a large class of quantum logics.

Chapter 12

Conclusion

This work contains the main results of two research papers: in the first, the author of this thesis is the main author [6], and in the second the sole author [4]. Moreover, these results also appear in three conference contributions [7, 8, 5]; in two of them [8, 5], the author of this thesis serves as the presenting author.

We adopt the mathematical formulation of Reichenbach's common cause principle (RCCP). We revisit some known results and simplify their proofs. Then we address a few topical problems and provide their solutions. We find which statements on the deterministic causes can be generalized to quantum logics and which cannot. In particular, the notion of maximal correlation considerably differs in these two cases. We provide an important counterexample based on the free quantum logic.

The rest of this paper is devoted to main results. In the 90's, several negative results were obtained; e.g., that finite quantum logics are necessarily common cause incomplete, which means that they contain correlated events without a common cause. The question was posed whether we can enlarge (embed) such a quantum logic into one that contains the missing causes. This is sometimes possible, see [24, Proposition 3.9], but the newly added elements introduce new correlations without common causes. Thus a much more ambitious goal was formulated in 1999 [22, 23]: Are there quantum logics which are common cause complete, i.e., each positive correlation has a common cause? A positive answer was presented in [28]. We find a smallest possible system that is non-trivially CCC in Chapter 6 and in Chapter 7 we give a much simpler proof of a slightly more general result than the one presented in [28]. Our main tool is the relation of the Darboux property in connection with RCCP. By using deeper results in orthomodular structures, we prove that atomless σ-complete logics with faithful σ-additive states are always common cause complete. This part is mathematically involved and a variety of new examples is presented. Notably, certain projection logics in von Neumann algebras and appropriate horizontal sums are such examples.

We correct the claim that quantum probability spaces with one atom must be common cause complete [24, Proposition 6.15]. For compatible events, we give a simplified proof of the existence of a common cause; for non-compatible ones, we present a counterexample.

The abundance of quantum logics which are common cause incomplete rises the question whether they can be ***embedded*** into common cause complete ones. This was formulated as one of the most important open questions in the monograph [24], and 11 years after its publication, no such result has been published. We succeeded to construct such an embedding for quantum logics which have finitely many atoms. We can even allow a countably infinite set of atoms under an additional condition that the embedded quantum logic is chain-finite. Thus we give a rather general answer to the principal open question. Possibilities of further generalizations and related results should be a topic of future research.

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Appendix A

Generalizations of RCCP

In Boolean algebras with two generators, we start with two distinct elements and apply the three operations (meet, join, complement) in order to obtain all elements of the Boolean algebra; the initial elements, their complements, joins, meets, and joins of meets. In total, that makes 16 distinct elements (it agrees with the free Boolean algebra over 2 generators, of course). These 16 elements can be viewed as all the possible operations with two elements in a Boolean algebra. However, the quantum case is more complicated. If the generators are not compatible, then we end up with 96 elements (again, this is the number of the elements of the free orthomodular lattice over 2 generators) that can once again be viewed as all the possible operations on two elements in a quantum logic. In order to distinguish them, we use their Beran numbers [1]. A diagram of Beran numbers and their associated operations can be seen in figure A.1. These extra operations collapse into their classical counterparts in Boolean algebras if the elements in question are compatible; we can categorize them based on the operations they collapse into and observe that we have 6 'meets', 6 'joins', 6 implications, and so on. This brings about a potentially interesting question in quantum theories. The quantum case admits 6 possible 'meets'—why does the Reichenbach's Common Cause Principle in quantum logics use only one of them? What happens if we define RCCP using a different 'meet'? The Beran numbers of the candidate operations are 2, 18, 34, 50, 66, 82.

Operations 18, 34, 50, and 66 are non-commutative. It may happen that we arrive at the conclusion that A depends on B, but B is independent of A. This sounds strange, but it can have some meaning. E.g., our plans for a trip depend on the weather, but the weather does not depend on our plans. Thus the non-commutativity need not be a reason for rejection of these operations.

None of the six operations represents a real conditional probability in the classical sense because non-compatible events cannot be tested simultaneously. Hence it is meaningful to analyze all these operations.

The classical (lattice) meet (exclusively used in previous works) is associated with operation 2 and is commutative—along with operation 82, which is defined as follows: $a \quad b := (a \quad b) \quad (a \quad b\ ) \quad (a \quad b)$.

In particular, we tried to investigate if the results obtained on the Darboux property in this generalized setup remain true (and for which operations).

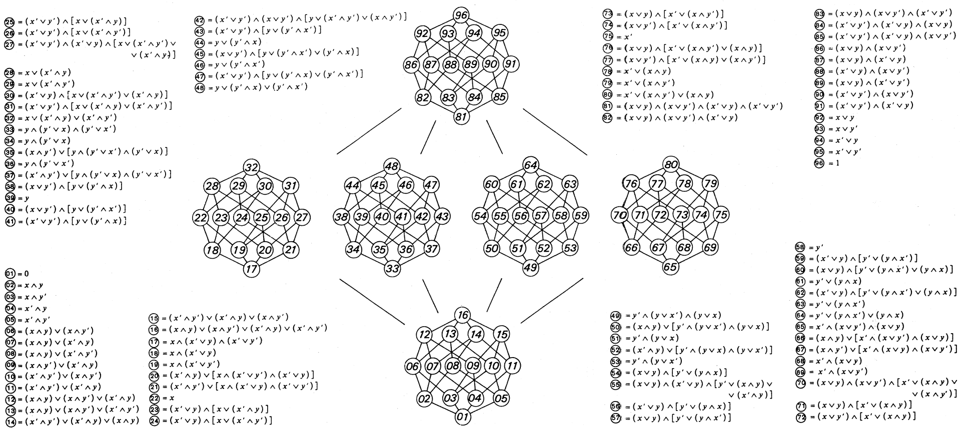

Figure A.1: Beran Numbers

Although we omitted the requirement of positive correlation between the events from the original definition of RCCP, we were only forced to do so because we knew that it emerges as a classical property of RCCP. We lack this knowledge in our exploration, and the philosophical motivations of RCCP require positive correlation, as it would not make sense to speak of a common cause of events that make each other less likely to occur. For these reasons, if we want to reformulate RCCP using the Beran operation 82 (or any other candidate operation), we first need to redefine the concept of positive correlation using the Beran operation 82.

Definition A.1 (Positive correlation using Beran operation 82**).** Let (L, s) be a quantum probability space. We say that two random events a, b, $a, b \quad L$, are positively correlated under the Beran operation 82 if $Cov_s(a, b) > 0$, with Cov_s defined as follows.

$$\mathrm{Cov}_s(a, b) = s((a \quad b) \quad (a \quad b\ ) \quad (a \quad b)) - s(a) \cdot s(b)\,.$$

This condition is weaker than the original one. In fact, the meet is the smallest of all six candidate operations, thus a positive correlation with respect to it (as used in all previous works) was the strongest condition of positive correlation. On the other hand, the operation with Beran number 82 is the greatest among all six. This means that there may be more pairs of events a, b such that the correlation $\mathrm{Cov}_s(a, b)$ is positive, and we need to find more common causes.

With the new definition of positive correlation in hand, we are able to reformulate RCCP.

Definition A.2 (RCCP using Beran operation 82)**.** Let (L, s) be a quantum probability space. Let $a, b \quad L$ such that $Cov_s(a, b) > 0$. Then the element c, $c \quad L$, is said to be a **common cause** of a, b provided $a \mathsf{C} c$, $b \mathsf{C} c$, $0 < s(c) < 1$, and the following conditions are satisfied:

$$s((a \quad b) \quad (a \quad b\ ) \quad (a \quad b) \mid c) = s(a \mid c)\, s(b \mid c)\,, \tag{RCCP1*}$$

$$s((a \quad b) \quad (a \quad b\ ) \quad (a \quad b) \mid c\ ) = s(a \mid c\ )\, s(b \mid c\ )\,, \tag{RCCP2*}$$

$$s(a \mid c) > s(a \mid c\ )\,, \tag{RCCP3*}$$

$$s(b \mid c) > s(b \mid c\ )\,. \tag{RCCP4*}$$

Note A.3. *Note that $s(x \mid c)$, resp. $s(x \mid c\ )$, is still defined as $\frac{s(x \quad c)}{s(c)}$, resp. $\frac{s(x \quad c\ )}{s(c\ )}$; there is no need to redefine it using Beran operation 82 because c is assumed to be compatible with both a and b, meaning that Beran operation 82 between $(a \quad b) \quad (a \quad b\ ) \quad (a \quad b)$ (respectively a, b) and c (respectively $c\ $) reduces to the standard lattice meet.*

We want to prove an analog of Theorem 8.6 under our redefinition of RCCP. It requires Theorems 8.5 and 7.4. Theorem 8.5 is not affected by RCCP and so we need not worry about it. However, Theorem 7.4 relies on RCCP and so we have to verify it for our redefinition. Its proof in the presented form does not work. We do not exclude the possibility of finding a common cause of a, b outside the interval $[0, a \quad b]$ or $[0, a \quad b]$, applying possibly the Darboux property similarly. However, we have found a serious drawback of this approach which denies its motivation.

The formula for conditional probability works well if the conditioning event (c or $c\ $) is compatible with all elements of the quantum logic, or at least with all elements involved in the equations for RCCP. This is the case here and the probability conditioned by c or $c\ $ is a probability. In particular, it satisfies the additivity. If $a \quad b$, then all three elements a, b, and c are compatible and contained in a Boolean subalgebra, which guarantees that $s(a \quad b \mid c) = s(a \mid c) + s(b \mid c)$. Similar arguments ensure additivity for more orthogonal events.

Although the conditional probability satisfies the axioms, it has quite strange properties. Problems arise with (RCCP1*) or (RCCP2*). It may happen that $s((a \quad (a \quad b) \quad (a \quad b)) \quad c)$ or $s((b \quad (a \quad b) \quad (a \quad b)) \quad c)$ is "large", while $s(b \quad c)$ can be an arbitrarily small positive number. Then

$$s(a \quad b \mid c) > s(b \mid c)\,,$$

which is counterintuitive. This is possible because the inequality $a \quad b \leq b$, resp. $(a \quad b) \quad c \leq b \quad c$, can be violated. The same can happen for all five candidate operations different from the lattice meet.

We see that the conditional probability based on any operation different from the meet (Beran number 2) can hardly be defended as a meaningful probability model. Therefore we did not investigate the possibility of extension of the result on RCCP to this context.