Large multipartite subgraphs in H-free graphs

Ping Hu[1], Bernard Lidický[2], Taísa Martins[3], Sergey Norin[4], and Jan Volec[5]

[1] School of Mathematics, Sun Yat-sen University, Guangzhou, 510275, China. E-mail: huping9@mail.sysu.edu.cn.
[2] Iowa State University, Department of Mathematics, Iowa State University, Ames, Iowa, USA. E-mail: lidicky@iastate.edu
[3] Instituto de Mathemática, Universidade Federal Fluminense, Niterói, Brazil. E-mail: tlmartins@id.uff.br.
[4] Department of Mathematics and Statistics, McGill Univ ersity, Montréal, Canada. E-mail: sergey.norin@mcgill.ca.
[5] Department of Mathematics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague, Trojanova 13, 120 00 Prague, Czech Republic. E-mail: jan@ucw.cz.

Abstract. In this work, we discuss a strengthening of a result of Füredi that every n-vertex K_{r+1}-free graph can be made r-partite by removing at most $T(n,r)-e(G)$ edges, where $T(n,r)=\frac{r-1}{2r}n^2$ denotes the number of edges of the n-vertex r-partite Turán graph. As a corollary, we answer a problem of Sudakov and prove that every K_6-free graph can be made bipartite by removing at most $4n^2/25$ edges. The main tool we use is the flag algebra method applied to locally definied vertex-partitions.

Keywords: Max-Cut, Turán graph, Flag Algebras

1 Introduction

Let $G=(V,E)$ be an n-vertex graph and $r \geq 2$ an integer. Denote by $\mathrm{del}_r(G)$ the minimum size of an edge-subset $X \subseteq E$ such that the graph $G-X$ is r-partite. Note that $\mathrm{del}_2(G)$ is the dual problem to Max-Cut, i.e., finding the largest bipartite subgraph in G. For convenience, we also define $\mathrm{del}_1(G) := e(G)$.

Our aim is to obtain upper bounds on $\mathrm{del}_r(G)$ and $\mathrm{del}_2(G)$, respectively, when G is a K_{r+1}-free graph, i.e., a graph with no complete subgraph on $r+1$ vertices. A beautiful stability-type argument of Füredi [6] provides the following upper bound on $\mathrm{del}_r(G)$.

Theorem 1 (Füredi [6]). *Fix an integer $r \geq 2$. If G is an n-vertex K_{r+1}-free graph, then* $\mathrm{del}_r(G) \leq \frac{r-1}{2r} \cdot n^2 - e(G)$.

Note that the number of edges in every K_{r+1}-free graph on n vertices is bounded from above by the number of edges in the Turán graph $T(n,r)$, which is equal to $\frac{r-1}{2r} \cdot n^2$. In other words, the result of Füredi can be stated as follows: if a K_{r+1}-free graph is missing t edges to being extremal, then removing at most t edges from it makes it r-partitie.

When the number of edges of G is very close to the extremal value, Theorem 1 was sharpened in [2,7]. Here we focus on a global improvement, and conjecture that Theorem 1 can be strengthened as follows.

Conjecture 1. Fix an integer $r \geq 2$. If G is an n-vertex K_{r+1}-free graph, then $\mathrm{del}_r(G) \leq 0.8 \left(\frac{r-1}{2r} \cdot n^2 - e(G)\right)$.

If true, Conjecture 1 would be best possible, and we present tight constructions in Section 3. Note that for $r \geq 4$, the conjecture does not have a unique extremal example. To provide an evidence for Conjecture 1, we prove it for $r \in \{2, 3, 4\}$.

Theorem 2. *Fix an integer $r \in \{2, 3, 4\}$. If G is an n-vertex K_{r+1}-free graph, then* $\mathrm{del}_r(G) \leq 0.8 \left(\frac{r-1}{2r} \cdot n^2 - e(G)\right)$.

We also establish the following general improvement on Theorem 1.

Theorem 3. *For every $r \geq 5$ there exists $\varepsilon := \varepsilon(r) > 0$ such that the following holds. If G is an n-vertex K_{r+1}-free graph, then* $\mathrm{del}_r(G) \leq (1-\varepsilon) \left(\frac{r-1}{2r} \cdot n^2 - e(G)\right)$.

The bound on $\varepsilon(r)$ we establish monotonically decreases to 0 as r tends to infinity, while Conjecture 1 claims that $\varepsilon(r) = 0.2$ for every r.

A closely related problem inspired by a well-known problem of Erdős on Max-Cuts in dense triangle-free graphs is the following conjecture of Sudakov [9].

Conjecture 2. Fix $r \geq 3$. For every K_{r+1}-free graph G, it holds that

$$\mathrm{del}_2(G) \leq \begin{cases} \frac{(r-1)^2}{4r^2} \cdot n^2 & r \text{ odd, and} \\ \frac{r-2}{4r} \cdot n^2 & r \text{ even.} \end{cases}$$

Note that the conjectured value corresponds to the value of $\mathrm{del}_2(T(n,r))$. Sudakov [9] proved the conjecture for $r = 3$.

Theorem 4 (Sudakov [9]). *An n-vertex K_4-free graph G can be made bipartite by removing $n^2/9$ edges, i.e., $\mathrm{del}_2(G) \leq n^2/9$. Moreover, if $\mathrm{del}_2(G) = n^2/9$, then G is the Turán graph $T(n,3)$.*

We prove the conjecture for $r = 5$.

Theorem 5. *If G is an n-vertex K_6-free graph, then $\mathrm{del}_2(G) \leq 4n^2/25$. Moreover, if $\mathrm{del}_2(G) = 4n^2/25$, then G is the Turán graph $T(n,5)$.*

As we have already mentioned, Erdős [4] made a conjecture on the size of the largest bipartite subgraph in triangle-free graphs. Specifically, he conjectured that $\mathrm{del}_2(G) \leq n^2/25$ for every triangle-free n-vertex graph G. A result of Erdős, Faudree, Pach, and Spencer [5] states that $\mathrm{del}_2(G) \leq n^2/18$. Using flag algebras in a manner analogous to the one we use here, an improvement on the last bound was recently announced by Balogh, Clemen, and Lidický [3].

Note that for all the theorems in this section, a straightforward application of the regularity lemma yields the corresponding asymptotic results for H-free graphs, where H is a fixed r-colorable graph.

In our work, we extensively use flag algebras, a versatile tool developed by Razborov [8], applied to K_{r+1}-free graph limits. We use as a convention that unlabeled vertices are depicted as black circles, labeled vertices as yellow squares, and edges as blue lines. Dashed lines indicate that both edge and non-edge are admissible. We write $[\![.]\!]$ to denote the so-called unlabeling/averaging operator.

The rest of this extended abstract is organized as follows: In Section 2, we describe an alternative proof of Theorem 1 using flag algebras, which demonstrates the technique we use. In Section 3, we examine the set of possible extremal constructions for Conjecture 1, and give a sketch of the proof of Theorem 2 for the case $r = 2$. We conclude the extended abstract by Section 4, where we briefly discuss the case $r \geq 3$ as well as the ideas for the proof of Theorem 5.

2 Theorem 1 in Flag Algebras

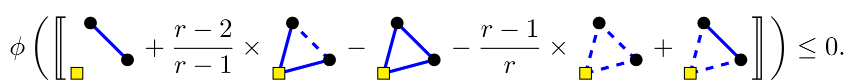
As a warm-up to our flag algebra technique, we present a proof of Theorem 1. Suppose Theorem 1 is false, and let r be the smallest integer for which it fails. Let G be an n-vertex K_{r+1}-free graph G such that $\mathrm{del}_r(G) > \frac{r-1}{2r} \cdot n^2 - e(G)$.

For a vertex $v \in V(G)$, consider an r-partition of $V(G)$ with $A_r := V \setminus N(v)$ being one part, and $(A_1, A_2, \ldots, A_{r-1})$ being an $(r-1)$-partition of $N(v)$ given by Theorem 1 if $r \geq 3$, and $A_1 := N(v)$ in case $r = 2$. Note that if $r = 2$ then $N(v)$ induces no edges in G. It follows that the number of edges inside the parts is at most $e(G[A_r]) + \mathrm{del}_{r-1}(G[N(v)])$, which is as most

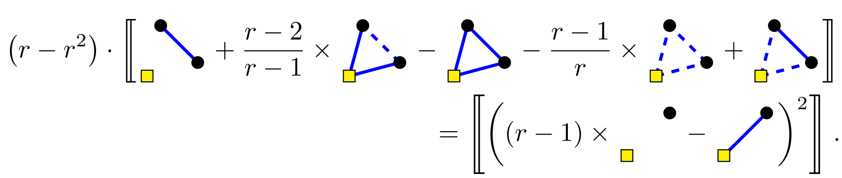
$$e(G[A_r]) + \frac{r-2}{r-1} \cdot \frac{|N(v)|^2}{2} - e(G[N(v)]). \tag{1}$$

On the other hand, this is at least $\mathrm{del}_r(G) > \frac{r-1}{2r} \cdot n^2 - e(G)$. This is in direct contradiction with the following simple flag algebra proposition, which shows that if we choose a vertex v uniformly at random, then the expectation of (1) is at most $\frac{r-1}{2r} \cdot n^2 - e(G)$.

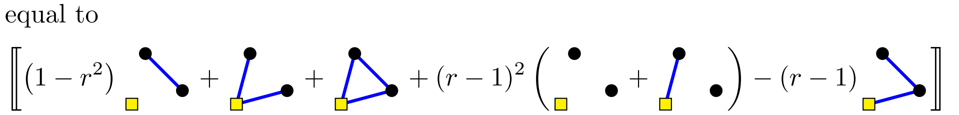
Proposition 1. *Fix $r \geq 2$. If ϕ is a K_{r+1}-free graph limit, then*



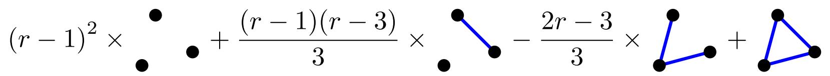
Proof. We will show that the following identity holds for every $r \geq 2$.



Note that the identity immediately proves the statement since the right-hand side is non-negative while $r - r^2 < 0$. Firstly, observe that the left-hand side is equal to

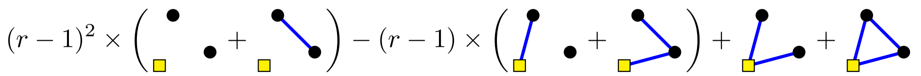


By the definition of $[\![\cdot]\!]$, the previous expression averages to the following:



(2)

On the other hand, the right-hand side of the identity is equal to



which again averages to (2). This finished the proof. □

Proposition 1 and the following lemma yield the statement of Theorem 1.

Lemma 1. *Fix positive integers r, b and ℓ. If G is a K_{r+1}-free graph then its b-blow-up $G[b]$ is K_{r+1}-free and* $\mathrm{del}_\ell(G[b]) = b^2 \cdot \mathrm{del}_\ell(G)$.

An inspection of the just presented proof yields that the bound in Theorem 1 is tight only if G is a Turán graph. Indeed when $G = T(n,r)$, Theorem 1 does not allow to remove any edge. However, this is rather a technical "obstacle" and Conjecture 1 can be seen as a way how to bypass it.

3 Tight constructions for Conjecture 1

Clearly, Conjecture 1 is tight for Turán graphs since the bound $T(n,r) - e(G)$ does not allow deletion of any edges. When $r = 2$, the complete balanced bipartite graph and a balanced blow-up of C_5 attains the bound $0.8(n^2/4 - e(G))$. Therefore, blow-ups of C_5 behave similarly as a complete bipartite graph with respect to Conjecture 1, and this propagates to larger r.

Given $r \geq 2$, a tight construction for Conjecture 1 can be obtained as follows: Let H be a join of a copies of K_1 and b copies of C_5, where $a + 2b = r$. Let G be a blow-up of H, such that all the vertices corresponding to K_1s have the weight $1/r$ and all the vertices corresponding to C_5s have the weight $2/(5r)$.

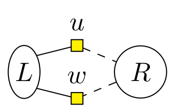
When $r \in \{2, 3, 4\}$, we prove the above description of the tight constructions for Theorem 2 is complete, see also Figure 1.

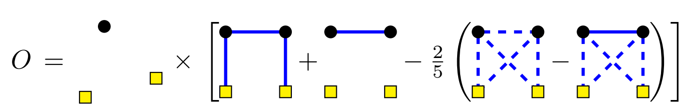
3.1 Proof of Theorem 2 when $r = 2$

Let N be the non-edge type with labels u and w, and let C be the combination of N-flags that expresses the size of the cut (L, R) with $L := N(u) \cup N(v)$ and $R := V \setminus L$. Next, we define

$$O := \overline{K_3^N} \times (C - 0.8(1/2 - d(G))) = \overline{K_3^N} \times \left(C - 0.4(d(\overline{G}) - d(G))\right),$$

which can be expressed using flag algebras as follows:





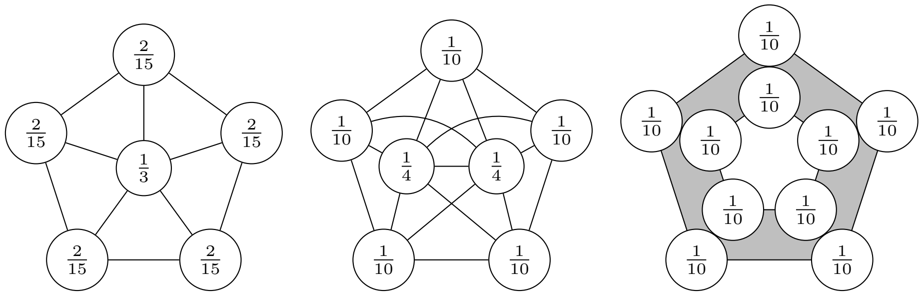
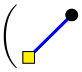
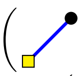
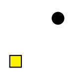
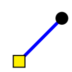
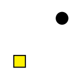
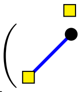


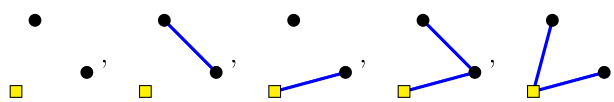
Fig. 1. Non-Turán tight constructions for Theorem 2 when $r = 3$ and $r = 4$.

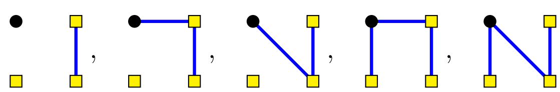
Notice that $\frac{1}{2} - d(G)$ is the density of missing edges to the complete bipartite graph, and $0.8(\frac{1}{2} - d(G))$ is the normalized number of edges we are allowed to delete in Conjecture 1 when $r = 2$. In order to prove Conjecture 1, we need to show that the expression O is non-positive in triangle-free graphs.

Theorem 6. *If ϕ is a K_3-free graph limit, then $\phi([\![O]\!]) \leq 0$. Moreover, if $\phi([\![O]\!]) = 0$, then ϕ^1* () $\in \{0.4, 0.5\}$ *almost surely.*

Proof. First, let $F_1 :=$ ( $-$ ) $\times$ ($6 \times$  $- 4 \times$ ). Observe that if $\phi([\![F_1^2]\!]) = 0$ then ϕ^1 () $\in \{0.4, 0.5\}$ almost surely.

Next, consider the following two vectors X and Y of σ-flags, where σ is the one-vertex type and the co-cherry type, respectively, and the following 7 linear combinations of flags using X and Y:

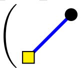
$X =$  , (3)

$Y =$  . (4)

$F_1 = X \cdot (4, 4, -5, -5, 6)$, $\quad F_4 = Y \cdot (0, 1, -1, 1, -1)$, $\quad F_7 = Y \cdot (6, 1, 1, -4, -4)$,
$F_2 = X \cdot (6, -9, 0, 0, -6)$, $\quad F_5 = Y \cdot (0, 1, -1, 2, -2)$, $\quad F_8 = Y \cdot (2, -2, -2, 1, 1)$.
$F_3 = X \cdot (4, 0, -3, -4, 4)$, $\quad F_6 = Y \cdot (0, 2, -2, 1, -1)$,

We express each term as a linear combination of 5-vertex unlabeled flags and establish the following estimate on $[\![O]\!]$ for some non-positive rationals $w_1, w_2, \ldots, w_8$:

$$[\![O]\!] \leq \sum_{i \in \{1,2,\ldots,8\}} w_i \times [\![F_i^2]\!] \,.$$

Hence, $\phi([\![O]\!]) \leq 0$. Moreover, if the equality is attained for some limit ϕ, then $\phi([\![F_i^2]\!]) = 0$ for all $i \in [8]$ by complementary slackness. In particular, we have $\phi^1\left(\right.$  $\left.\right) \in \{0.4, 0.5\}$ for almost every choice of the root. □

Lemma 1 readily translates Theorem 6 to the setting of finite graphs, and a result of Andrásfai, Erdős and Sós [1] yields that the only non-bipartite tight graph in Theorem 2 when $r = 2$ is a balanced blow-up of C_5.

4 Concluding remarks

An analogous approach to Conjecture 1 when $r = 2$ can be applied to the cases $r = 3$ and $r = 4$, although more locally defined partitions and more sum-of-squares are needed. The proof of Theorem 5 is also very similar, and in fact the simplest form we have found consists only of five sum-of-squares, a natural partition tuned to perform optimally on the corresponding Turán graphs, and an application of Theorem 6.

One of the main reasons why the complexity of the proof grows with r is the increasing number of tight constructions, and it is not obvious how to generalize this approach to all r. Nevertheless, bootstraping from Theorem 6, we establish a much more modest improvement described in Theorem 3.

Acknowledgements. The first author is supported in part by National Natural Science Foundation of China grants 11801593 and 11931002. The second author is supported in part by NSF grant DMS-1855653. The fourth author is supported in part by an NSERC Discovery grant.

References

1. B. Andrásfai, P. Erdős, and V. T. Sós. On the connection between chromatic number, maximal clique and minimal degree of a graph. *Discrete Math.*, 8:205–218, 1974.
2. József Balogh, Felix C. Clemen, Mikhail Lavrov, Bernard Lidický, and Florian Pfender. Making K_{r+1}-free graphs r-partite. *Combin. Probab. Comput.*, 2020.
3. József Balogh, Felix C. Clemen, and Bernard Lidický. Max cuts in triangle-free graphs. Manuscript, 2021.
4. P. Erdős. Problems and results in graph theory and combinatorial analysis. In *Proceedings of the Fifth British Combinatorial Conference (Univ. Aberdeen, Aberdeen, 1975)*, pages 169–192. Congressus Numerantium, No. XV, 1976.
5. Paul Erdős, Ralph Faudree, János Pach, and Joel Spencer. How to make a graph bipartite. *J. Combin. Theory Ser. B*, 45(1):86–98, 1988.
6. Zoltán Füredi. A proof of the stability of extremal graphs, Simonovits' stability from Szemerédi's regularity. *J. Combin. Theory Ser. B*, 115:66–71, 2015.
7. Dániel Korándi, Alexander Roberts, and Alex Scott. Exact stability for Turán's theorem, 2020. arXiv:2004.10685.
8. Alexander A. Razborov. Flag algebras. *J. Symbolic Logic*, 72(4):1239–1282, 2007.
9. Benny Sudakov. Making a K_4-free graph bipartite. *Combinatorica*, 27(4):509–518, 2007.