# Sharp bounds for decomposing graphs into edges and triangles

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#### Abstract

For a real constant  $\alpha$ , let  $\pi_3^{\alpha}(G)$  be the minimum of twice the number of  $K_2$ 's plus  $\alpha$  times the number of  $K_3$ 's over all edge decompositions of G into copies of  $K_2$  and  $K_3$ , where  $K_r$  denotes the complete graph on r vertices. Let  $\pi_3^{\alpha}(n)$  be the maximum of  $\pi_3^{\alpha}(G)$  over all graphs G with n vertices.

The extremal function  $\pi_3^3(n)$  was first studied by Győri and Tuza [Decompositions of graphs into complete subgraphs of given order, *Studia Sci. Math. Hungar.* 22 (1987), 315–320]. In a recent progress on this problem, Král', Lidický, Martins and Pehova [Decomposing graphs into edges and triangles, *Combin. Prob. Comput.* 28 (2019) 465–472] proved via flag algebras that  $\pi_3^3(n) \leq (1/2+o(1))n^2$ . We extend their result by determining the exact value of  $\pi_3^{\alpha}(n)$  and the set of extremal graphs for all  $\alpha$  and sufficiently large n. In particular, we show for  $\alpha = 3$  that  $K_n$  and the complete bipartite graph  $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$  are the only possible extremal examples for large n.

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## 1 Introduction

In a recent progress on a problem of Győri and Tuza [27], Král', Lidický, Martins and Pehova [19] proved via flag algebras that the edges of any n-vertex graph can be decomposed into copies of  $K_2$  and  $K_3$  whose total number of vertices is at most  $(1/2 + o(1))n^2$ , where  $K_r$  denotes the clique on r vertices. The origins of this problem can be traced back to Erdős, Goodman and Pósa [10] who considered the problem of minimising the total number of cliques in an edge decomposition of an arbitrary n-vertex graph. They showed the following:

**Theorem 1** (Erdős, Goodman, Pósa [10]). The edges of every n-vertex graph can be decomposed into at most  $\lfloor n^2/4 \rfloor$  complete graphs.

The only extremal example for this bound is the (bipartite) Turán graph  $T_2(n) := K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ , where  $K_{a,b}$  denotes the complete bipartite graph with part sizes a and b. Moreover, this result still holds if we restrict the sizes of the cliques used in the decomposition to 2 and 3 (that is, single edges and triangles). In a series of papers published independently by Chung [4], Győri and Kostochka [11], and Kahn [18], they proved that in fact something stronger than Theorem 1 is true, confirming a conjecture by Katona and Tarján:

**Theorem 2** (Chung [4], Győri and Kostochka [11], Kahn [18]). Every n-vertex graph can be edge decomposed into cliques whose total number of vertices is at most  $\lfloor n^2/2 \rfloor$ .

For a given graph G on n vertices, let  $\pi_k(G)$  be the minimum over all decompositions of the edges of G into cliques  $C_1, \ldots, C_\ell$  of size at most k of the sum  $|C_1| + |C_2| + \cdots + |C_\ell|$ , where |G| := |V(G)| denotes the *order* of a graph G. Let  $\pi_k(n)$  be the maximum of  $\pi_k(G)$  over all graphs G with n vertices. With this notation, the conclusion of the above theorem is that  $\min_{k \in \mathbb{N}} \pi_k(n) \leq \lfloor n^2/2 \rfloor$ . In light of Theorem 2, Tuza [27] conjectured that  $\pi_3(n) \leq n^2/2 + o(n^2)$ , and in fact that  $\pi_3(n) \leq n^2/2 + O(1)$ . Győri and Tuza [16] showed that  $\pi_3(n) \leq 9n^2/16$ . This was the best known bound until recently, when using the celebrated flag algebra method of Razborov [24], Král', Lidický, Martins and Pehova [19] proved the asymptotic version of Tuza's conjecture:

**Theorem 3** (Král', Lidický, Martins and Pehova [19]). We have  $\pi_3(n) \leq (1/2 + o(1))n^2$  as  $n \to \infty$ .

In this paper we show, by building upon the proof in [19], that for all large n it holds in fact  $\pi_3(n) \leq n^2/2+1$ . Moreover, if a graph G of order n attains  $\pi_3(n)$  then G is the complete graph  $K_n$  or the Turán graph  $T_2(n)$ .

Which of these two graphs is extremal is a matter of divisibility of n by 6. In the case of the Turán graph, we trivially have  $\pi_3(T_2(n)) = 2\lfloor n/2 \rfloor \lceil n/2 \rceil$ , giving  $n^2/2$  for even n and  $(n^2-1)/2$ 

for odd n. In order to determine  $\pi_3(K_n)$ , we have to determine the maximum number of edgedisjoint triangles in  $K_n$ . Clearly, the graph made of their edges is triangle-divisible, that is, each vertex has even degree and the total number of edges is divisible by three. It is routine to see that the minimum size of a graph H on n vertices whose complement  $\overline{H}$  is triangle-divisible is attained by taking at most one copy of the claw  $K_{1,3}$  and a perfect matching on the remaining vertices for even n, and isolated vertices plus at most one copy of the 4-cycle  $K_{2,2}$  for odd n. (Note that  $\binom{n}{2}$  is never equal to 2 modulo 3.) In fact, this gives the value of  $\pi_3(K_n)$  for all large n by the following general result (which we will use also inside our proof).

**Theorem 4** (Barber, Kuhn, Lo and Osthus [2]). For every  $\varepsilon > 0$ , if G is a triangle-divisible graph of large order n and minimum degree at least  $(0.9 + \varepsilon)n$ , then G has a perfect triangle decomposition.

The constant 0.9 in the minimum degree condition in Theorem 4 comes from the result of Dross [6] on fractional triangle decompositions, and it was conjectured by Nash-Williams [21] that it can be replaced by 3/4. Very recently, Dukes and Horsley [7] and Delcourt and Postle [5] improved the constant to 0.852 and  $(7 + \sqrt{21})/14 = 0.8273...$ , respectively.

Let us list the values of  $\pi_3$  for the graphs  $K_n$  and  $T_2(n)$  for large n.

$n \bmod 6$	$K_2$ 's in an optimal decomposition of $K_n$	$\pi_3(K_n)$	$\pi_3(T_2(n))$
0	perfect matching	$\frac{n^2}{2}$	$\frac{n^2}{2}$
1	none	$\binom{n}{2}$	$\frac{n^2-1}{2}$
2	perfect matching	$\frac{n^2}{2}$	$\frac{n^2}{2}$ $\frac{n^2 - 1}{2}$ $\frac{n^2}{2}$ $n^2 - 1$
3	none	$\binom{n}{2}$	$\frac{n^2-1}{2}$
4	$K_{1,3}$ + perfect matching	$\frac{n^2}{2} + 1$	$\frac{n^2}{2}$ $\frac{n^2}{2}$ $n^2 - 1$
5	$C_4$	$\binom{n}{2} + 4$	$\frac{n^2-1}{2}$

Table 1: Values of  $\pi_3(K_n)$  and  $\pi_3(T_2(n))$  for large n.

Let us define

$$\mathcal{E}_n := \begin{cases} \{T_2(n), K_n\}, & \text{if } n \equiv 0, 2 \pmod{6}, \\ \{T_2(n)\}, & \text{if } n \equiv 1, 3, 5 \pmod{6}, \\ \{K_n\}, & \text{if } n \equiv 4 \pmod{6}, \end{cases}$$

and

$$\{K_n\}, \qquad \text{if } n \equiv 4 \pmod{6},$$

$$\ell(n) := \begin{cases} n^2/2, & \text{for } n \equiv 0, 2 \pmod{6}, \\ (n^2 - 1)/2, & \text{for } n \equiv 1, 3, 5 \pmod{6}, \\ n^2/2 + 1, & \text{for } n \equiv 4 \pmod{6}. \end{cases}$$

Thus, by the calculations of Table 1, we have for all large n that  $\mathcal{E}_n$  consists of those graphs in  $\{T_2(n), K_n\}$  which maximise  $\pi_3$  while  $\ell(n)$  is this maximum value.

Clearly,  $\ell(n)$  is a lower bound on  $\pi_3(n)$  for large n. Our main result is that this is equality.

**Theorem 5.** There exists  $n_0 \in \mathbb{N}$  such that for all  $n \ge n_0$ , we have  $\pi_3(n) = \ell(n)$  and the set of  $\pi_3(n)$ -extremal graphs up to isomorphism is exactly  $\mathcal{E}_n$ .

A simple corollary of Theorem 5 is an affirmative answer to a question of Pyber [23], see also [27, Problem 45], for sufficiently large n. A covering of a graph G is a collection of subgraphs of G such that every edge of G appears in at least one subgraph. (For comparison, a decomposition requires that every edge appears in exactly one subgraph.)

**Corollary 6.** There exists  $n_0 \in \mathbb{N}$  such that for all  $n \ge n_0$ , the edge set of every n-vertex graph can be covered with triangles and edges so that the sum of their orders is at most  $\lfloor n^2/2 \rfloor$ .

Proof. Theorem 5 directly implies the corollary unless  $n \equiv 4 \pmod{6}$  and the graph under consideration is  $K_n$ . So assume that  $n \equiv 4 \pmod{6}$ . Denote the vertices of  $K_n$  by  $v_1, \ldots, v_n$ . Recall that an optimal decomposition for  $K_n$  is obtained by taking edges  $v_1v_2, v_1v_3, v_1v_4$  and  $v_iv_{i+1}$  for all odd i with  $1 \leq i \leq n-1$ . The rest of the graph becomes triangle-divisible and Theorem 4 can be applied. This gives a decomposition of cost  $n^2/2 + 1$ . A covering of cost at most  $n^2/2$  can be obtained from this decomposition by replacing edges  $v_1v_2$  and  $v_1v_3$  by a triangle  $v_1v_2v_3$ . (Notice that the pair  $v_2v_3$  is covered by two triangles in the resulting covering.)

We also study an extension of Theorem 5, where we consider decompositions into  $K_2$ 's and  $K_3$ 's but we modify the cost of  $K_3$ 's to be  $\alpha$  (with the cost of  $K_2$  still being 2). The minimum over all costs of such decompositions of a graph G is denoted by  $\pi_3^{\alpha}(G)$ . The maximum value of  $\pi_3^{\alpha}(G)$  over all n-vertex graphs G is denoted by  $\pi_3^{\alpha}(n)$ . Notice that  $\pi_3^3(G) = \pi_3(G)$  and  $\pi_3^3(n) = \pi_3(n)$ . Denote  $K_n$  without one edge by  $K_n^-$  and  $K_n$  without a matching of size two by  $K_n^-$ . Then the following result holds.

**Theorem 7.** For every real  $\alpha$  exists  $n_0 \in \mathbb{N}$  such that every  $\pi_3^{\alpha}$ -extremal graph G with  $n \geq n_0$  vertices satisfies the following (up to isomorphism).

- If  $\alpha < 3$ , then  $G = T_2(n)$ ;
- if  $\alpha = 3$  then Theorem 5 applies;
- if  $3 < \alpha < 4$  and  $n \equiv 0, 2, 4, 5 \pmod{6}$ , then  $G = K_n$ ;
- if  $3 < \alpha < 4$  and  $n \equiv 1, 3 \pmod{6}$ , then  $G = K_n^=$ ;
- if  $\alpha = 4$  and  $n \equiv 1, 3 \pmod{6}$ , then  $G \in \{K_n, K_n^-, K_n^=\}$  and, moreover, the three listed graphs are all  $\pi_3^{\alpha}$ -extremal;
- if  $\alpha = 4$  and  $n \equiv 0, 2, 4, 5 \pmod{6}$ , then  $G = K_n$ ;

• if  $4 < \alpha$ , then  $G = K_n$ .

This paper is organised as follows. In Section 2 we give an outline of the proof of Theorem 3 from [19] that we build on. Theorem 5 is proved in Section 3. Extension for other weights of triangles is in Section 4. Some related results are mentioned in Section 5.

**Notation.** We follow standard graph theory notation (see e.g. [3]).

For a graph G, we denote the set neighbours of  $x \in V(G)$  by  $\Gamma_G(x)$  (or just  $\Gamma(x)$  when G is understood) and the number of edges in a set  $B \subseteq E(G)$  incident with x by  $d_B(x)$ . We denote by  $K[V_1, V_2]$  the complete bipartite graph with vertex partition  $(V_1, V_2)$ . The term [X, Y]-edges refers to edges  $xy \in E(G)$  such that  $x \in X$  and  $y \in Y$ . We write [x, Y]-edges as a short-hand for  $[\{x\}, Y]$ -edges.

Let  $t_2(n) := |E(T_2(n))|$  be the number of edges in the Turán graph  $T_2(n)$ . Recall that  $t_2(n) = |n^2/4|$ . By a *cherry* we mean a path with 2 edges.

We consider graphs up to isomorphism; in particular, we write G = H to denote that G and H are isomorphic graphs.

## 2 Outline of the proof of Theorem 3 from [19]

In this section we give a short outline of the proof of [19, Lemma 5], which was a key step in proving  $\pi_3(n) \leq n^2/2 + o(n^2)$  and is a starting point of our argument towards Theorem 5. For an *n*-vertex graph G and each  $i \in \mathbb{N}$ , let  $K_i(G)$  be the set of all i-cliques in G. Let  $\pi_{3,f}(G)$  be the minimum of

$$2\sum_{xy \in K_2(G)} c(xy) + 3\sum_{xyz \in K_3(G)} c(xyz)$$

over fractional  $\{K_2, K_3\}$ -decompositions c of E(G), that is, over maps  $c: K_2(G) \cup K_3(G) \to [0, 1]$  such that for every edge  $xy \in E(G)$  we have  $c(xy) + \sum_{z:xyz \in K_3(G)} c(xyz) \ge 1$ . Of course,  $\pi_{3,f}(G) \le \pi_3(G)$ . By a result of Haxell and Rödl [17] or a more general version by Yuster [28], it also holds that  $\pi_3(G) \le \pi_{3,f}(G) + o(n^2)$ . So, to show that  $\pi_3(G) \le n^2/2 + o(n^2)$ , it suffices to consider the fractional equivalent  $\pi_{3,f}(G)$ .

**Lemma 8.** Let G be an n-vertex graph. Then

$$\binom{n}{7}^{-1} \sum_{W \in \binom{V(G)}{7}} \pi_{3,f}(G[W]) \leqslant 21 + o(1)$$

where the sum is taken over 7-vertex subsets W of V(G).

Outline of proof. Let M be the following positive semi-definite matrix

$$M := \frac{1}{12 \cdot 10^9} \begin{pmatrix} 180000000 & 2444365956 & 640188285 & -1524146769 & 1386815580 & -732139362 & -129387078 \\ 2444365956 & 4759879134 & 1177441152 & -1783771230 & 2546923788 & -1397639394 & -143552208 \\ 640188285 & 1177441152 & 484273772 & -317303211 & 1038156300 & -591902130 & -6783162 \\ -1524146769 & -1783771230 & -317303211 & 1558870290 & -651906630 & 305728704 & 154602378 \\ 1386815580 & 2546923788 & 1038156300 & -651906630 & 2285399634 & -1283125950 & -10755036 \\ -732139362 & -1397639394 & -591902130 & 305728704 & -1283125950 & 734039016 & -1621938 \\ -129387078 & -143552208 & -6783162 & 154602378 & -10755036 & -1621938 & 23860164 \end{pmatrix} \rightleftharpoons 0$$

and let  $\overrightarrow{F} := (F_1, \dots, F_7)$  be the following vector of rooted graphs, each having 4 vertices with the root denoted by the white square:

Take any graph G of order  $n \to \infty$ . For  $w \in V(G)$ , let  $\mathbf{v}_{G,w} \in \mathbb{R}^7$  denote the column vector whose i-th component is  $p(F_i, (G, w))$ , the density of the 1-flag  $F_i$  in the rooted graph (G, w), which is G with the vertex w designated as the root.

It was shown in [19] that

$$\frac{1}{\binom{n}{7}} \sum_{W \in \binom{V(G)}{7}} \pi_{3,f}(G[W]) + \frac{1}{n} \sum_{w \in V(G)} \mathbf{v}_{G,w}^T M \mathbf{v}_{G,w} \leqslant 21 + o(1). \tag{1}$$

Namely, if we re-write the left-hand size as a linear combination  $\sum_{H} c_{H} p(H, G)$ , where H ranges over all 7-vertex unlabelled graphs and p(H, G) is the density of H in G, then each coefficient  $c_{H}$  is at most 21. Since  $\sum_{H} p(H, G) = 1$ , the claimed inequality (1) follows.

In particular, since M is positive semi-definite, the quantity  $\frac{1}{n} \sum_{w \in V(G)} \mathbf{v}_{G,w}^T M \mathbf{v}_{G,w}$  is always non-negative, yielding the result.

The main result of [19] that  $\pi_3(n) \leq n^2/2 + o(n^2)$  now follows directly from Lemma 8.

Proof of Theorem 3. Let G be any graph of order  $n \to \infty$ . As mentioned before,  $\pi_3(G) \leq \pi_{3,f}(G) + o(n^2)$ . Also, we have

$$\binom{n}{2}^{-1} \pi_{3,f}(G) \leqslant \binom{7}{2}^{-1} \binom{n}{7}^{-1} \sum_{W \in \binom{V(G)}{7}} \pi_{3,f}(G[W]),$$

by averaging optimal fractional decompositions of all 7-vertex induced subgraphs. Combining this inequality with Lemma 8 immediately gives that  $\pi_3(G) \leq (1/2 + o(1))n^2$ .

## 3 Proof of Theorem 5

We use the so-called *stability approach*, where the first step is to describe the approximate structure of all almost  $\pi_3$ -extremal graphs of order  $n \to \infty$  within  $o(n^2)$  adjacencies. Namely, our Corollary 10 will show that every such graph is close to  $K_n$  or  $T_2(n)$ .

For this purpose, we start by showing that all almost  $\pi_3$ -extremal graphs contain almost no copies of the three graphs in Figure 1 (which are obtained by taking the unlabelled versions of the corresponding graphs in  $\overrightarrow{F}$ ). This is achieved by the following lemma that builds on the results from [19].

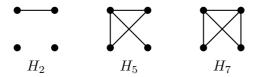


Figure 1: Graphs  $H_2$ ,  $H_5$ , and  $H_7$ .

**Lemma 9.** For every c > 0 there exist  $\varepsilon > 0$  and  $n_0 \in \mathbb{N}$  such that for all  $n \ge n_0$ , if G is a graph of order n with  $\pi_3(G) \ge (1/2 - \varepsilon)n^2$ , then G has at most  $c\binom{n}{4}$  copies of each of the graphs  $H_2 := (\{a, b, c, d\}, \{ab\})$ ,  $H_5 := (\{a, b, c, d\}, \{ab, bc, ac, ad\})$  and  $H_7 := (\{a, b, c, d\}, \{ab, bc, ac, bd, ad\})$  from Figure 1.

*Proof.* Given c > 0, let  $\varepsilon \gg 1/n_0 > 0$  be sufficiently small. Let G be a graph as in the lemma. Let M and  $\overrightarrow{F}$  be as in the proof of Lemma 8.

First, the rank of the matrix M is 6 with  $\mathbf{v} = (1, 0, 3, 1, 0, 3, 0)$  being the only zero eigenvector. (Thus all others eigenvalues of M are strictly positive by  $M \geq 0$ .)

Second, by the almost optimality of G and the fact that each term in the left-hand side of (1) is non-negative, we have that

$$\sum_{w \in V(G)} \mathbf{v}_{G,w}^T M \mathbf{v}_{G,w} = o_{\varepsilon}(n). \tag{2}$$

We now show that G must contain few copies of the graphs  $H_2$ ,  $H_5$  and  $H_7$ . Suppose, for contradiction, that G contains at least  $c\binom{n}{4}$  copies of  $H_2$ . Then, by a simple double-counting argument we have that at least cn/4 vertices in G contain at least  $c\binom{n}{3}/4$  copies of the rooted flag  $F_2$ . In particular, the second coordinate of at least cn/4 of the vectors  $\mathbf{v}_{G,w}$  is at least c/4. For each such vector  $\mathbf{u}$ , let  $\mathbf{u}' := \mathbf{u}/\|\mathbf{u}\|_2$  be the scalar multiple of  $\mathbf{u}$  of  $\ell^2$ -norm 1. Since  $\|\mathbf{u}\|_2 \leqslant \sqrt{7}$ , we have that its second coordinate  $\mathbf{u}'_2$  is at least  $c/4\sqrt{7}$ . The scalar product of  $\mathbf{u}'$  and the  $\ell^2$ -normalised zero eigenvector  $\mathbf{v}/\sqrt{20}$  (whose second coordinate is 0) is at most  $\sqrt{1-(c/4\sqrt{7})^2}$ . Thus the projection of  $\mathbf{u}$  on the orthogonal complement  $L=\mathbf{v}^\perp$  of the zero eigenspace of M has  $\ell^2$ -norm at least  $c/4\sqrt{7}$ . Thus  $\mathbf{u}^T M \mathbf{u} \geqslant \lambda_2 (c/4\sqrt{7})^2$ , where  $\lambda_2 > 0$  is the smallest positive eigenvalue of M (in fact, one can check with computer that  $\lambda_2 = 0.0005228...$ ). Thus, we have that the left-hand side of (2) in which each term is non-negative by  $M \succcurlyeq 0$  is at least  $(cn/4) \times \lambda_2 (c/4\sqrt{7})^2 = \Omega(n)$ , a contradiction.

The analogous argument shows that the densities of  $H_5$  and  $H_7$  in G are also at most c.

Let us say that two graphs  $G_1$  and  $G_2$  of the same order are k-close in the edit distance (or simply k-close) if there is a relabelling of the vertices of  $G_2$  so that  $|E(G_1)\triangle E(G_2)| \leq k$ . In other words, we can make  $G_1$  and  $G_2$  isomorphic by changing at most k adjacencies.

Corollary 10. For every  $\delta > 0$  there exists  $n_1 \in \mathbb{N}$  such that if G is a graph of order  $n \ge n_1$  with  $\pi_3(G) \ge \ell(n) - n^2/n_1$ , then G is  $\delta n^2$ -close in edit distance to  $K_n$  or to  $T_2(n)$ .

*Proof.* Given any  $\delta > 0$ , choose sufficiently small constants  $\delta \gg c \gg 1/n_1 > 0$ . Take any graph G on  $n \geqslant n_1$  vertices such that  $\pi_3(G) \geqslant \ell(n) - n^2/n_1$ .

By Lemma 9 and the Induced Removal Lemma [1], G can be made  $\{H_2, H_5, H_7\}$ -free by changing at most  $cn^2$  adjacencies. Denote this new graph by G' and note that  $\pi_3(G') \ge \pi_3(G) - 2cn^2$ . By  $c \ll \delta$ , it is enough to show that G' is  $\delta n^2/2$ -close to  $K_n$  or  $T_2(n)$ .

Let us show that G' is either triangle-free, or the disjoint union of at most two cliques. Indeed, if some vertices a, b, c span a triangle in G' then, by the  $\{H_5, H_7\}$ -freeness of G, all the remaining vertices of G' have either no or three neighbours among  $\{a, b, c\}$ . Let  $A_0$  be the set of vertices in  $G' \setminus \{a, b, c\}$  which see none of  $\{a, b, c\}$ , and let  $A_3$  be the set of vertices which see all of  $\{a, b, c\}$ . Then  $A_3$  is a clique because G' is  $H_7$ -free. The set  $A_0$  is also a clique because G' is  $H_2$ -free. Also, no pair xy in  $A_3 \times A_0$  can be an edge as otherwise e.g. the 4-set  $\{a, b, x, y\}$  spans a copy of  $H_5$  in G. It follows that G is the disjoint union of the cliques on  $A_0$  and  $A_3 \cup \{a, b, c\}$ , as required.

Now, if G' is triangle-free, then  $e(G') = \pi_3(G')/2 \ge \ell(n)/2 - n^2/n_1 - 2cn^2 \ge t_2(n) - 3cn^2$ . Thus, by the stability result for Mantel's theorem by Erdős [8] and Simonovits [26], the graph G' must indeed be  $\delta n^2/2$ -close in edit distance to  $T_2(n)$ .

Otherwise, G' is the disjoint union of two cliques. Let us show that one of them has size at most  $\delta n/2$ . Indeed, otherwise G' has a triangle packing covering all but at most n/2+2 edges by Theorem 4, meaning that  $\pi_3(G') \leq e(G') + n/2 + 2$ . Also, e(G') is maximum when clique sizes are as far apart as possible. Thus, by the lower bound on  $\pi_3(G) \leq \pi_3(G') + 2cn^2$ , we conclude that e.g.  $\ell(n) - 3cn^2 \leq {\delta n/2 \choose 2} + {(1-\delta/2)n \choose 2}$ , leading to a contradiction to our choice of constants. Therefore, G' is at most  $n \cdot \delta n/2$  adjacency edits away from  $K_n$ , as desired.

The key steps in proving Theorem 5 are Lemmas 11–13.

**Lemma 11.** There exist constants  $\delta > 0$  and  $n_1 \in \mathbb{N}$  such that, among all graphs on  $n \ge n_1$  vertices which are  $\delta n^2$ -close to  $T_2(n)$ , the maximiser of  $\pi_3$  is  $T_2(n)$ .

Proof. Choose sufficiently small  $\varepsilon \gg \delta \gg 1/n_1 > 0$ . Let G be an arbitrary graph with  $n \geqslant n_1$  vertices which is  $\delta n^2$ -close to  $T_2(n)$ . We will show that  $\pi_3(G) \leqslant \pi_3(T_2(n))$  with equality if and only if  $G = T_2(n)$ . In fact, this claim can be directly derived from the result of Győri [12, Theorem 1] that a graph with n vertices and  $t_2(n) + k$  edges, where  $n \to \infty$  and  $k = o(n^2)$ , has at least  $k - O(k^2/n^2)$  edge-disjoint triangles. More specifically, for each  $\varepsilon > 0$  there exists  $\delta > 0$ 

and  $n_0 \in \mathbb{N}$  such that every graph with  $n \ge n_0$  vertices and  $t_2(n) + k$  edges, where  $k \le \delta n^2$ , has at least  $k - \varepsilon k^2/n^2$  edge-disjoint triangles. (See also [13, Theorem 1] for a generalisation of this to r-cliques for any fixed  $r \ge 3$ .) Since G is  $\delta n^2$ -close to  $T_2(n)$ , it must have at most  $t_2(n) + \delta n^2$  edges. From this and  $1/n \ll \delta \ll \varepsilon \ll 1$ , we have that, for  $k := e(G) - t_2(n)$ ,

$$\pi_3(G) \le 2(t_2(n)) + k - 3(k - \varepsilon k^2/n^2) = 2t_2(n) - k(1 - 3\varepsilon k/n^2) \le 2t_2(n).$$

Clearly, if equality is achieved then k = 0, that is,  $e(G) = t_2(n)$ ; furthermore, G must be triangle-free and thus  $G = T_2(n)$ , as required.

Next, we need to analyse graphs that are close to  $K_n$ . If  $n \equiv 1, 3 \pmod{6}$ , then let  $\mathcal{E}'_n$  consist of those graphs which are obtained from  $K_n$  by removing a matching of size  $m \equiv 2 \pmod{3}$ ; otherwise let  $\mathcal{E}'_n := \{K_n\}$ . Also, define

$$w(n) := \begin{cases} n/2, & n \equiv 0, 2 \pmod{6}, \\ 2, & n \equiv 1, 3 \pmod{6}, \\ n/2 + 1, & n \equiv 4 \pmod{6}, \\ 4, & n \equiv 5 \pmod{6}. \end{cases}$$

Using Theorem 4 and the calculation for  $K_n$  described in Table 1, one can show that  $\pi_3(G) = \binom{n}{2} + w(n)$  for all large n and every  $G \in \mathcal{E}'_n$ . We are going to show that these are exactly the extremal graphs among those close to  $K_n$ . It is more convenient to do first the case when we have some bound on the minimum degree of a graph and then derive the general case (in a separate Lemma 13).

**Lemma 12.** There exist constants  $\delta > 0$  and  $n_0 \in \mathbb{N}$  such that the following holds. Let G be a graph on  $n \geq n_0$  vertices with minimum degree at least n/8 such that G is  $\delta n^2$ -close to  $K_n$  and  $\pi_3(G) \geq \binom{n}{2} + w(n)$ . Then  $G \in \mathcal{E}'_n$ .

*Proof.* Choose small constants in the following order:  $c \gg \delta \gg 1/n_0 > 0$ . Suppose that G is a graph of order  $n \geqslant n_0$  as in the statement of the lemma. Let w := w(n).

Let  $U := \{ v \in V(G) : d_G(v) \leq (1 - c)n \}$ . Then

$$\frac{|U|cn}{2} \leqslant e(\overline{G}) \leqslant \delta n^2,$$

and so  $|U| \leq \frac{2\delta}{c}n$ . Denote  $W := V(G) \setminus U$ , and let  $S := \{v \in W : d_G(v) \text{ is odd}\}$ . Let M be a set of edges forming a maximum matching in G[S], and denote  $X := S \setminus V(M)$ . Then X is an independent set and thus  $\binom{|X|}{2} \leq \delta n^2$ , which implies that rather roughly

$$|X| < cn. (3)$$

Moreover, for every edge  $yz \in M$  and any two distinct vertices  $y', z' \in X$ , at most one of yy' and zz' can be an edge of G (otherwise y'yzz' is an augmenting path contradicting the maximality

of M). It follows that, if  $|X| \neq 1$ , then for every edge  $yz \in M$  there are at least |X| edges missing between yz and X. Let  $Y_W$  denote the set of missing edges in G[W]. Thus

$$|Y_W| \geqslant {|X| \choose 2} + |M|(|X| - \mathbb{1}_{|X|=1}),$$
 (4)

where the indicator function  $\mathbb{1}_{|X|=1}$  is 1 if |X|=1 and is 0 otherwise. Moreover, the set  $Y_U$  of missing edges in G with at least one endpoint in U satisfies

$$|Y_U| \geqslant cn|U| - \binom{|U|}{2} \tag{5}$$

by the definition of U. Note that  $e(G) = \binom{n}{2} - |Y_W| - |Y_U|$ . See Figure 2 for a sketch of  $Y_W$  and  $Y_U$ .

We now build a decomposition  $\mathcal{D}$  of G into edges and triangles, starting with  $\mathcal{D} = \emptyset$ . If we add edges/triangles to  $\mathcal{D}$ , we regard them as removed from E(G). It is convenient to split our argument into the following two cases.

Case 1:  $U \neq \emptyset$  or  $S = \emptyset$ .

In this case, our procedure for constructing  $\mathcal{D}$  is as follows.

- **Step 1:** Add the following to  $\mathcal{D}$  as  $K_2$ 's: the edges of the matching M and the edges of some  $\lfloor |X|/2 \rfloor$  cherries with distinct endpoints in X such that their middle points are pairwise distinct.
- Step 2: For each  $u \in U$ , one at a time, add to  $\mathcal{D}$  a maximum set of edge-disjoint  $K_3$ 's containing u and two vertices from W. Add all remaining edges incident to vertices in U as  $K_2$ 's to  $\mathcal{D}$ .
- **Step 3:** (a) Let  $S' \subseteq V(G)$  be the set of vertices with odd degree after Step 2. Add to  $\mathcal{D}$  the edges of some |S'|/2 cherries with distinct endpoints in S' such that their middle points are pairwise distinct.
  - (b) If the number of remaining edges is not divisible by 3, then fix this by adding to  $\mathcal{D}$  (as single edges) the edge set of some cycle of length 4 or 5.
- **Step 4:** Add a perfect triangle decomposition of the remaining edges to  $\mathcal{D}$ .

For  $i \in \{1, 2, 3\}$ , let  $Z_i$  be the set of edges that are added to  $\mathcal{D}$  in Step i as copies of  $K_2$ . See Figure 2 for some illustrations of the above steps.

**Claim.** The above steps can be carried out as stated. Moreover, the obtained decomposition  $\mathcal{D}$  of G has at most  $|M| + |X| + {|U| \choose 2} + 2|U| + 6$  copies of  $K_2$ .

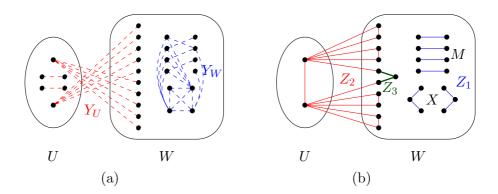


Figure 2: (a) Missing edges in  $Y_W$  are colored blue and edges in  $Y_U$  are red. (b) Edges in  $Z_1$  are colored blue, edges in  $Z_2$  are red and in  $Z_3$  green. The same vertices are on the right, where dashed are some of the missing edges. Note that this is a sketch and vertices in W can incident to both blue and red (dashed) edges.

Proof of Claim. In order to do Step 1 as stated, we can iteratively pick any two new vertices  $x, y \in X$  and then an arbitrary vertex z which is suitable as the middle point for a cherry on xy. Note that the number of choices for z is at least n-2-2cn, the number of common neighbours of  $x, y \in X \subseteq W$ , minus |X|-1, the number of vertices previously used as middle points. This is positive by (3) and  $c \ll 1$ , so we can always proceed. Note for future reference that every vertex is incident to at most 3 edges removed in Step 1. Also, Step 1 adds  $|Z_1| = |M| + 2(\lfloor |X|/2 \rfloor) \leqslant |M| + |X|$  copies of  $K_2$  to  $\mathcal{D}$ .

Clearly, Step 2 can always be processed. Consider the moment when we apply Step 2 to some  $u \in U$ . In the current graph, the induced subgraph  $G[\Gamma(u) \cap W]$  has minimum degree at least  $|\Gamma(u) \cap W| - cn - 3$ , which is at least  $|\Gamma(u) \cap W|/2$  since  $|\Gamma(u)| \ge n/8 - 3$ . So by Dirac's theorem, this subgraph has a matching covering all but at most one vertex, that is, all edges between u and W except at most one are decomposed as triangles in Step 2. Let U' be the set of those  $u \in U$  for which an exceptional edge occurs. Thus we have  $|U'| \le |U|$  copies of  $K_2$  connecting U to W that are added to  $\mathcal{D}$  in Step 2. There are trivially at most  $\binom{|U|}{2}$  edges with both endpoints in U. So Step 2 adds  $|Z_2| \le \binom{|U|}{2} + |U|$  copies of  $K_2$  to  $\mathcal{D}$ . Note that all edges incident to U are decomposed after Step 2.

Since all vertices of W but at most one had even degrees before Step 2, we have that S' has at most  $|U'| + 1 \le |U| + 1$  vertices. Similarly as in Step 1, a simple greedy algorithm finds all cherries as stated Step 3(a). (Note that S', as the set of all odd-degree vertices, has even size.)

The minimum degree of G[W] after Step 3(a) is at least 0.99n, since each  $w \in W$  has at most 2|U| + 6 incident edges removed (at most 2|U| from Step 2 and at most 3 in each of Steps 1 and 3(a)). Thus, we can find the required 4- or 5-cycle in Step 3(b).

Clearly, we add  $|Z_3| \leq |S'| + 5 \leq |U| + 6$  copies of  $K_2$  to  $\mathcal{D}$  in Step 3.

Note that, at the end of Step 3, the graph G[W] has minimum degree at least, say, 0.98n while

all its degrees are even. By Theorem 4, all remaining edges can be decomposed using only triangles, so Step 4 indeed removes all remaining edges.

Step 4 adds no additional  $K_2$ 's, so the total number of  $K_2$ 's in  $\mathcal{D}$  is

$$|Z_1| + |Z_2| + |Z_3| \le |M| + |X| + {|U| \choose 2} + 2|U| + 6,$$

finishing the proof of the claim.

Now we compute the cost of  $\mathcal{D}$ . Using the notation from above, we have

$$w \leqslant \pi_3(G) - \binom{n}{2} \leqslant -|Y_U| - |Y_W| + |Z_1| + |Z_2| + |Z_3|$$
  
$$\leqslant -|Y_U| - |Y_W| + |M| + |X| + \binom{|U|}{2} + 2|U| + 6.$$
 (6)

Substituting the bounds from (4) and (5) and rearranging the terms, we get

$$w \leqslant \left(2\binom{|U|}{2} + 2|U| - cn|U| + 6\right) + (3 - |X|)\left(\frac{|X|}{2} + |M|\right) + \left(\mathbb{1}_{|X|=1} - 2\right)|M|. \tag{7}$$

First, suppose that |U| > 0. Then, the estimate  $|U| \leq 2\delta n/c$  yields that

$$2\binom{|U|}{2} + 2|U| - cn|U| + 6 \leqslant -cn|U|/2 \leqslant -cn/2.$$

Since  $w \ge 2$ , we must have that  $|X| \le 1$ . Observe that n is odd as otherwise  $w \ge n/2$  and, by  $|M| \le n/2$ , the cases  $|X| \in \{0,1\}$  also contradict (7). So every vertex of degree n-1 has even degree, meaning that every vertex of S is in some pair from  $Y_W$  or  $Y_U$ . Hence,  $2|M| \le 2|Y_W| + |Y_U|$ . Substituting this into the right-hand size of (6) and using our bound on  $|Y_U|$  from (5), we obtain

$$w \leqslant -\frac{|Y_U|}{2} + |X| + {|U| \choose 2} + 2|U| + 6 \leqslant \frac{3}{2} {|U| \choose 2} + 2|U| - \frac{cn|U|}{2} + 7,$$

which again is negative for |U| > 0 and large n, contradicting  $w \ge 2$ .

Thus U is empty and, by the assumption of Case 1, S is also empty (and so are X and M). This gives that the initial graph G has minimum degree at least (1-c)n,  $|Z_1|=|Z_2|=0$ ,  $S'=\emptyset$ , and no  $K_2$ 's are added to  $\mathcal{D}$  in Step 3(a).

If n is even, then every vertex of G has at least one missing edge,  $e(G) \leq {n \choose 2} - \frac{n}{2}$  and

$$\pi_3(G) \leqslant \binom{n}{2} - \frac{n}{2} + |Z_3| \leqslant \binom{n}{2} - \frac{n}{2} + 5,$$

which is strictly less than  $\pi_3(K_n)$ , a contradiction.

Let n be odd and let  $r := \binom{n}{2} - e(G)$  be the number of missing edges in G. Suppose that r > 0, as otherwise  $G = K_n$  and we are done. The upper bound on  $\pi_3(G)$  given by  $\mathcal{D}$  is  $\rho_r + \binom{n}{2} - r$ ,

where we define  $\rho_r$  as the unique element of  $\{0,4,5\}$  with  $\binom{n}{2} - \rho_r - r \equiv 0 \pmod{3}$ . Therefore,  $r \leqslant 3$  as otherwise  $\pi_3(G) \leqslant \binom{n}{2} + 1$  contradicting  $w \geqslant 2$ . On the other hand, all the degrees of  $\overline{G}$  are even so r = 3 and the only non-empty component of  $\overline{G}$  is a triangle. However, this contradicts  $w \geqslant 2$  because

$$\pi_3(G) = \begin{cases} \binom{n}{2} - 1, & n \equiv 1, 3 \pmod{6}, \\ \binom{n}{2} + 1, & n \equiv 5 \pmod{6}. \end{cases}$$

Case 2:  $U = \emptyset$  and  $S \neq \emptyset$ .

Some things simplify in this case (as we do not need to deal with U). On the other hand, we have to be a bit more careful with calculations, as the new extremal graphs ( $K_n$  minus a matching) fall into this case. In particular, removing a 4- or 5-cycle may be too wasteful here. So we construct a decomposition  $\mathcal{D}$  of G as follows. Recall that M is a maximum matching in G[S] and X is the set of vertices of S not matched by M.

Step 1: Make the graph triangle-disivible by removing the following as  $K_2$ 's. If  $X = \emptyset$ , then remove all but one edge  $xy \in M$  and a path of length  $\rho + 1 \in \{1, 2, 3\}$  whose endpoints are x and y (thus, for  $\rho = 0$ , we remove just the matching M). If X is non-empty, then remove M and the edge sets of some |X|/2 - 1 paths of length 2 and one path of length  $\rho + 2 \in \{2, 3, 4\}$  so that their degree-1 vertices partition X and their degree-2 vertices are pairwise distinct.

**Step 2:** Decompose the rest perfectly into triangles.

Note that S, the set of all odd-degree vertices of G, has even size (and also |X| = |S| - 2|M| is even). Since the minimal degree of G is at least (1 - c)n, a simple greedy algorithm achieves Step 1 (and Theorem 4 takes care of Step 2).

The decomposition  $\mathcal{D}$  has exactly  $|M| + |X| + \rho$  copies of  $K_2$ . Also,  $e(G) = \binom{n}{2} - |Y_W|$ . Thus

$$w \le \pi_3(G) - \binom{n}{2} \le -|Y_W| + |M| + |X| + \rho.$$
 (8)

Using (4) and that  $|X| \neq 1$  (since |X| is even), we obtain that

$$w \le (3 - |X|) \left(\frac{|X|}{2} + |M|\right) - 2|M| + \rho.$$
 (9)

Moreover,  $|X| \leq 2$  as otherwise  $2 \leq w \leq \rho - 2 - 3|M|$  contradicting  $\rho \leq 2$ . Thus X has either 0 or 2 elements.

Suppose that  $X = \emptyset$ . First, let n be even. Then every vertex not in S is incident to at least one non-edge of G,  $|Y_W| \ge (n-2|M|)/2$ , and by (8),

$$n/2 \leqslant w \leqslant 2|M| + \rho - n/2.$$

If  $2|M| \leqslant n-2$ , then all inequalities here become equalities and thus  $|M| = \frac{n-2}{2}$ ,  $|Y_W| = 1$ ,  $\rho = 2$ ,  $w = \frac{n}{2}$ , and  $n \equiv 0, 2 \pmod{6}$ . However, then the graph after Step 1 has exactly  $\binom{n}{2} - 1 - \frac{n-2}{2} - 2$  edges, which is not divisible by 3, a contradiction. Thus 2|M| = n, the copies of  $K_2$  in the decomposition contains a perfect matching of G, and  $\pi_3(G) \leqslant \pi_3(K_n)$  with equality only if  $G = K_n$ , giving the desired. So suppose that n is odd. Since every vertex of S has to be incident to a missing edge of G, we have  $|Y_W| \geqslant |S|/2 = |M|$  and the bound in (8) becomes  $w \leqslant \rho$ . It follows that we have equality throughout,  $|Y_W| = |M|$ ,  $w = \rho = 2$ ,  $n \equiv 1, 3 \pmod{6}$ , and  $\binom{n}{2} - |M| - \rho \equiv 0 \pmod{3}$ ; the last gives that  $|M| \equiv 2 \pmod{3}$ . Thus G is as required.

Finally, it remains to consider the case when |X|=2. This time, (9) yields that

$$2 \leqslant w \leqslant \rho - |M| + 1 \leqslant 3.$$

Therefore,  $|M| \leq 1$ , and  $n \equiv 1,3 \pmod 6$  as otherwise  $w \geq 4$ . If |M| = 1, then we have equality everywhere, giving that  $w = \rho = 2$ , |S| = 4 and  $|Y_W| = 3$ . However, then the graph after Step 1 has  $\binom{n}{2} - |Y_W| - |M| - |X| - \rho = \binom{n}{2} - 8$  edges, which is not divisible by 3, a contradiction. Thus M is empty,  $\rho \in \{1,2\}$  and S = X. By (8),  $|Y_W| \leq 2$  and hence  $|Y_W| = 1$ . In other words,  $G = K_n^-$ . However, then the graph after Step 1 has  $\binom{n}{2} - 1 - (2 + \rho)$  edges, which is not divisible by 3. (Alternatively, Theorem 4 gives that  $\pi_3(K_n^-) - \binom{n}{2} < 2 = w$ .) This contradiction finishes Case 2 and the proof of the lemma.

**Lemma 13.** There exist constants  $\delta > 0$  and  $n_1 \in \mathbb{N}$  such that the following holds. Let G be a graph on  $n \geq n_1$  vertices maximizing  $\pi_3(G)$  among all graphs that are  $\delta n^2$ -close to  $K_n$ . Then  $G \in \mathcal{E}'_n$ .

Proof. Let  $n_0$  and  $\delta$  be the constants from Lemma 12. We claim that, for example,  $n_1 := 2n_0$  is enough for the conclusion of Lemma 13 to hold. Indeed, take any extremal graph G of order  $n \ge n_1$ . If G satisfies the assumption on minimum degree of Lemma 12, then we are done. Hence assume that the minimum degree of G is less than n/8. Let  $G_n := G$ , and iteratively define a sequence of graphs  $G_{n-1}, G_{n-2}, \ldots$  as follows. Given a graph  $G_i$  of order i, if it has a vertex x of degree less than i/8, let  $G_{i-1} := G_i - x$  be obtained from  $G_i$  by removing the vertex x; otherwise stop. Note that the process does not reach i < n/2 for otherwise G has roughly at least  $(n/2) \times (n/4)$  non-edges, which is a contradiction to G being  $\delta n^2$ -close to  $K_n$ .

Let  $G_s$  with  $|G_s| = s \ge n/2 \ge n_0$  be the graph for which the above process terminates. By Lemma 12, we have that  $\pi_3(G_s) \le \frac{s^2}{2} + 1$ . By decomposing all edges in  $E(G) \setminus E(G_s)$  as  $K_2$ 's, we obtain that

$$\pi_3(G_n) \leqslant \pi_3(G_s) + 2(n-s) \cdot \frac{n}{8} \leqslant \frac{s^2}{2} + 1 + (n-s) \cdot \frac{n}{4}.$$

This is a convex function in s so it is maximized on the boundary of  $\frac{n}{2} \le s \le n-1$ . If s = n/2, we get  $\pi_3(G_n) \le n^2/4 + 2 < \binom{n}{2} \le \pi_3(K_n)$ . If s = n-1, we get

$$\pi_3(G_n) \leqslant \pi_3(G_s) + 2(n-s) \cdot \frac{n}{8} \leqslant \frac{(n-1)^2}{2} + 1 + \frac{n}{4} \leqslant \binom{n}{2} - \frac{n}{4} + 2 < \pi_3(K_n).$$

In both cases, we get a contradiction to  $G_n$  being extremal.

Proof of Theorem 5. Choose sufficiently small constants in this order  $1 \gg \delta \gg 1/n_0 > 0$ . In particular,  $n_0$  is sufficiently large to satisfy Corollary 10 for this  $\delta$  as well as Lemmas 11 and 13. Let G be an arbitrary graph of order  $n \geqslant n_0$  with  $\pi_3(G) \geqslant \ell(n)$ . By Corollary 10, G is  $\delta n^2$ -close to either  $T_2(n)$  or  $K_n$ .

If G is close to  $T_2(n)$  then it must be  $T_2(n)$  by Lemma 11. If G is close to  $K_n$  then it must be in  $\mathcal{E}'_n$  by Lemma 13. By comparing the costs of optimal decompositions, we conclude that  $G \in \mathcal{E}_n$ .

## 4 Extension to an arbitrary cost $\alpha$

The goal of this section is to prove Theorem 7. Everywhere in this section, let n be sufficiently large.

First, note that the case  $\alpha \geqslant 6$  is trivial. Indeed, the cost of a triangle is not better than a cost of three edges. Thus for every graph G an optimal decomposition is to decompose all edges of G as  $K_2$ 's. The unique graph maximizing the number of edges is  $K_n$ , so it is also the unique maximizer of  $\pi_3^{\alpha}$  for every  $\alpha \geqslant 6$ .

Next, let us make some easy general observations which apply when  $\alpha < 6$ . First,

$$\pi_3^{\alpha}(G) = \alpha \nu(G) + 2(e(G) - 3\nu(G)) = 2e(G) - (6 - \alpha)\nu(G),$$

where  $\nu(G)$  denotes the maximum number of edge-disjoint triangles contained in G. Also, if  $\alpha_1 \leqslant \alpha_2 < 6$ ,  $\nu(G_1) \geqslant \nu(G_2)$  and  $\pi_3^{\alpha_1}(G_1) > \pi_3^{\alpha_1}(G_2)$  for some graphs  $G_1$  and  $G_2$ , then

$$\pi_3^{\alpha_2}(G_1) - \pi_3^{\alpha_2}(G_2) = \pi_3^{\alpha_1}(G_1) - \pi_3^{\alpha_1}(G_2) + (\alpha_2 - \alpha_1)(\nu(G_1) - \nu(G_2)) > 0.$$
 (10)

In particular, if  $K_n$  is the maximizer of  $\pi_3^{\alpha_1}$ , it is also a maximizer for  $\pi_3^{\alpha_2}$ .

#### 4.1 The case $\alpha < 3$

Next, we discuss the case  $\alpha < 3$ . Let n be large and let G be a  $\pi_3^{\alpha}(n)$ -extremal graphs. Since

$$\pi_3^3(G) \geqslant \pi_3^{\alpha}(G) \geqslant \pi_3^{\alpha}(T_2(n)) = \pi_3^3(T_2(n)) = (1/2 + o(1))n^2,$$

Corollary 10 gives that G is  $o(n^2)$ -close to  $K_n$  or  $T_2(n)$ . Since  $\alpha < 3$ , we have that  $\pi_3^{\alpha}(T_2(n)) \ge (1 + \Omega(1))\pi_3^{\alpha}(K_n)$  and thus G is close to  $T_2(n)$ . Now, Lemma 11 implies that  $\pi_3^{\alpha}(G) \le \pi_3^3(T_2(n)) = \pi_3^{\alpha}(T_2(n))$ , with equality if and only if  $G = T_2(n)$ , giving the desired.

#### **4.2** The case $3 < \alpha < 4$

This subsection proves Theorem 7 in case  $3 < \alpha < 4$ .

First, let us show that every  $\pi_3^{\alpha}$ -maximiser G is in  $K_n$  or  $K_n^{=}$ . Suppose for a contradiction that G violates this. In particular, we have  $\pi_3^{\alpha}(G) \geqslant \pi_3^{\alpha}(K_n)$ . By (10), we have that  $\pi_3^3(G) \geqslant \pi_3^3(K_n)$ . For  $n \to \infty$ , it holds by Table 1 that  $\pi_3^{\alpha}(K_n) \geqslant (1 + \Omega(1)) \pi_3^{\alpha}(T_2(n))$ . Hence G needs to be close to  $K_n$  and Lemma 13 applies to G. In particular, this means that  $n \equiv 1, 3 \pmod{6}$ . Lemma 13 gives that all  $\pi_3^3$ -extremal graphs are obtained from  $K_n$  by removing a matching of size congruent to 2 modulo 3. It follows from (10) that, among these graphs,  $\pi_3^{\alpha}$  is strictly maximized by  $K_n^{=}$  since this graph has the largest  $\nu$ .

Theorem 4 gives that  $3\nu(K_n^=) = \binom{n}{2} - 6$ . Since  $\pi_3^{\alpha}(G) \geqslant \pi_3^{\alpha}(K_n^=)$  and  $\pi_3^3(G) < \pi_3^3(K_n^=)$ , this implies by (10) that  $\nu(G) > \nu(K_n^=)$ . Since also  $\nu(G) < \nu(K_n)$  (otherwise  $\pi_3^{\alpha}(G) < \pi_3^{\alpha}(K_n)$ ), we conclude that  $3\nu(G) = \binom{n}{2} - 3$ , that is, exactly three pairs of vertices of G are not included into some triangle from an optimal decomposition of G. This implies that G is a complete graph without one edge, or a path on three vertices, or a triangle. Among these three candidates (that have the same  $\nu$ ),  $K^-$  has the largest size and thus maximizes  $\pi_3^{\alpha}$ . So  $K^-$  is the only possible candidate for G. However,  $\pi_3^{\alpha}(K_n^=) > \pi_3^{\alpha}(K_n^-)$  if  $\alpha < 4$ . This contradiction finishes the proof in case  $3 < \alpha < 4$ .

Thus, every  $\pi_3^{\alpha}$ -maximiser is in  $\{K_n, K_n^{=}\}$ . It remains to compare these two graphs. Calculations based on Theorem 4 show that

$$\frac{\pi_3^{\alpha}(K_n^{=}) - \pi_3^{\alpha}(K_n) + 4}{6 - \alpha} = \nu(K_n) - \nu(K_n^{=}) = \begin{cases} 0, & n \equiv 0, 2, 4, 5 \pmod{6}, \\ 2, & n \equiv 1, 3 \pmod{6}. \end{cases}$$

Thus  $\pi_3^{\alpha}(K_n) > \pi_3^{\alpha}(K_n^{=})$  if  $n \equiv 0, 2, 4, 5 \pmod{6}$  and  $\pi_3^{\alpha}(K_n^{=}) > \pi_3^{\alpha}(K_n)$  otherwise, as required.

### 4.3 The case $4 \leqslant \alpha < 6$

In this case we provide a direct proof, without using flag algebras or fractional decompositions. Let n be large and let G be any graph of order n such that  $\pi_3^{\alpha}(G) = \pi_3^{\alpha}(n)$ . Let  $\mathcal{D}$  be a decomposition of G with minimum weight consisting of t triangles and  $\ell$  edges.

If G is a complete graph, then we are done. Hence we assume there exists some pair of vertices  $x, y \in G$  such that  $xy \notin E(G)$ . Let G' be obtained from G by adding the edge xy. Let  $\mathcal{D}'$  be an optimal decomposition of G' containing t' triangles and  $\ell'$  edges. Recall that finding an optimal decomposition is equivalent to maximizing a triangle packing, that is,  $t' = \nu(G')$ . Hence  $t' \ge t$ .

If xy is used as an edge in  $\mathcal{D}'$ , then removing xy from  $\mathcal{D}'$  gives a decomposition of G with cost  $\pi_3^{\alpha}(G') - 2$ , contradicting the maximality of G. Therefore xy must appear in a triangle  $xyz \in \mathcal{D}'$ . We now construct a decomposition  $\mathcal{D}^*$  of G by removing xyz from  $\mathcal{D}'$  and adding the edges xz and yz. Since the total cost of  $\mathcal{D}^*$  is  $\alpha(t'-1) + 2(\ell'+2)$  we have

$$\pi_3^{\alpha}(G) \leqslant \cot(\mathcal{D}^*) = \alpha(t'-1) + 2(\ell'+2) = \alpha t' + 2\ell' - \alpha + 4 \leqslant \alpha t' + 2\ell' = \pi_3^{\alpha}(G'),$$

which contradicts the maximality of  $\pi_3^{\alpha}(G)$  if at least one of the inequalities is strict. Hence  $\alpha = 4$ , xy must be in a triangle in  $\mathcal{D}'$  and  $\pi_3^{\alpha}(G') = \pi_3^{\alpha}(n)$ .

This means that it is possible to keep adding edges to G, which results in a sequence of graphs  $G, G', ..., K_n$  where an optimal decomposition of each of these graphs has cost  $\pi_3^{\alpha}(n)$ , i.e. they all are  $\pi_3^{\alpha}$ -extremal graphs. Note that we can add missing edges to G in any order, always obtaining a sequence of extremal graphs.

This allows us to reverse the process and examine a sequence of edge removals from  $K_n$ .

Suppose that G is obtained from  $K_n$  by removing the edge xy, i.e. G' is  $K_n$ . Notice that if  $\ell' > 0$ , i.e. the optimal decomposition of  $K_n$  contains an edge, then there exist an option for  $\mathcal{D}'$  that contains the edge xy, which was already ruled out. This means that  $K_n$  is triangle-divisible, which is the case if and only if  $n \equiv 1, 3 \pmod{6}$ .

Now assume that G is missing more than one edge. Hence  $K_n^-$  must be also extremal. By above,  $n \equiv 1, 3 \pmod{6}$ ,  $K_n$  is triangle-divisible, and  $\pi_3^4(n) = 4\nu(K_n)$ , where  $\nu(K_n) = \frac{1}{3}\binom{n}{2}$ .

Suppose that G is obtained from  $K_n$  by removing two edges uv and xy. First, suppose that u = x. Let  $\mathcal{D}^*$  be a decomposition of G into triangles and one edge vy. This gives

$$\pi_3^4(G) \leqslant \cot(\mathcal{D}^*) = 4(\nu(K_n) - 1) + 2 < 4\nu(K_n) = \pi_3^4(n),$$

contradicting the maximality of  $\pi_3^4(G)$ . Hence xy and uv form a matching. Notice that x, y, u, and v have odd degrees in G, so  $\ell \geq 2$  for else we are unable to fix the parity of the vertices x, y, u, and v. Now  $\binom{n}{2} - \ell - 2$  needs to be divisible by 3, so  $\ell \geq 4$ . There indeed exists a decomposition with  $\ell = 4$  by taking edges xu, xv, yu, and yv and rest as triangles. This gives

$$\pi_3^4(G) = 4(\nu(K_n) - 2) + 2 \cdot 4 = \pi_3^4(n).$$

Therefore, G is extremal.

Suppose that G is obtained from  $K_n$  by removing three edges uv, xy, and zw. Since G' must be  $K_n$  without a matching, uv, xy, and zw also form a matching. Let  $\mathcal{D}^*$  be a decomposition of G into triangles and edges ux, yz, and vw. This gives

$$\pi_3^4(G) \leqslant \cot(\mathcal{D}^*) = 4(\nu(K_n) - 2) + 6 < 4\nu(K_n) = \pi_3^4(n),$$

contradicting the maximality of  $\pi_3^4(G)$ . This implies that G cannot be obtained from  $K_n$  by deleting three or more edges, thus finishing the proof of this case and of Theorem 7.

## 5 Related results

A related question of Erdős (see e.g., [9]) asks for the largest t = t(n, m) such that every graph with n vertices and  $t_2(n) + m$  edges has at least t edge-disjoint triangles. Of course,  $t \leq m$ . Győri [12] (see [14] for a correction) showed, for large n, that  $t \geq m - O(m^2/n^2)$  if  $m = o(n^2)$ , and t = m if n is odd and  $m \leq 2n - 10$  or n is even and  $m \leq 3n/2 - 5$ . Moreover, the last two bounds on m are sharp.

More recently, Győri and Keszegh [15] proved that every  $K_4$ -free graph with  $t_2(n) + m$  edges has m edge-disjoint triangles.

Theorem 5 shows that the maximum of  $\pi_3(G)$  is attained for  $G = T_2(n)$  or  $G = K_n$ . However, if we restrict the set of graphs under consideration to graphs of a particular edge density, the decomposition is perhaps cheaper. Note that if the optimal decomposition of a graph G contains t triangles and  $\ell$  edges, then  $\pi_3(G) = 2e(G) - 3t$ . That is, we have that  $\pi_3(G) = 2e(G) - 3\nu(G)$ , where as before  $\nu(G)$  denotes the maximum number of edge-disjoint triangles in G. Then Theorem 3 implies an inequality between the edge density of G and its triangle packing density which we denote by  $\nu_d(G) := 3\nu(G)/\binom{n}{2}$ :

Corollary 14 (of Theorem 3). Let G be a graph with  $d\binom{n}{2}$  edges. Then

$$\nu_d(G) \geqslant 2d - 1 + o(1).$$

We also have that  $\nu_d(G) \leq d$ , which is tight for all graphs which are the union of edge-disjoint triangles.

A question reminiscent of the seminal result of Razborov on the minimal triangle density in graphs [25] (see also [20, 22]) would be to determine the exact lower bound on  $\nu_d(G)$  in terms of d (answering asymptotically the question of Erdős stated above).

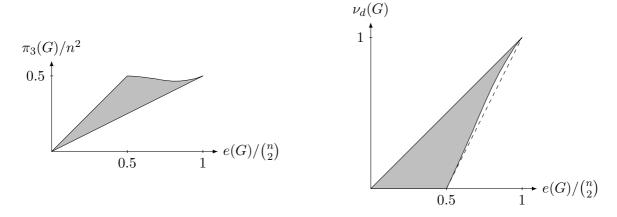


Figure 3: Asymptotic bounds on possible values of  $\pi_3(G)$  and  $\nu_d(G)$ . The dashed line is simply y = 2x - 1 for a better display of the shape.

Some flag algebra computations yield numerical asymptotic lower bounds on  $\nu_d(G)$  with different edge densities between 0.5 and 1. The result, depicted in Figure 3, suggests that the true asymptotic shape of the region  $\{(d,\nu_d(G)): 0 \leq d \leq 1, G \text{ graph}\}$  may indeed have a richer structure.

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