# Sharp bounds for decomposing graphs into edges and triangles 

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#### Abstract

For a real constant $\alpha$, let $\pi_{3}^{\alpha}(G)$ be the minimum of twice the number of $K_{2}$ 's plus $\alpha$ times the number of $K_{3}$ 's over all edge decompositions of $G$ into copies of $K_{2}$ and $K_{3}$, where $K_{r}$ denotes the complete graph on $r$ vertices. Let $\pi_{3}^{\alpha}(n)$ be the maximum of $\pi_{3}^{\alpha}(G)$ over all graphs $G$ with $n$ vertices.

The extremal function $\pi_{3}^{3}(n)$ was first studied by Győri and Tuza [Decompositions of graphs into complete subgraphs of given order, Studia Sci. Math. Hungar. 22 (1987), 315320]. In a recent progress on this problem, Král', Lidický, Martins and Pehova [Decomposing graphs into edges and triangles, Combin. Prob. Comput. 28 (2019) 465-472] proved via flag algebras that $\pi_{3}^{3}(n) \leqslant(1 / 2+o(1)) n^{2}$. We extend their result by determining the exact value of $\pi_{3}^{\alpha}(n)$ and the set of extremal graphs for all $\alpha$ and sufficiently large $n$. In particular, we show for $\alpha=3$ that $K_{n}$ and the complete bipartite graph $K_{\lfloor n / 2\rfloor,\lceil n / 2\rceil}$ are the only possible extremal examples for large $n$.


[^0]Keywords: maximum triangle packing, edge decomposition into cliques, stability property

## 1 Introduction

In a recent progress on a problem of Győri and Tuza [27], Král', Lidický, Martins and Pehova [19] proved via flag algebras that the edges of any $n$-vertex graph can be decomposed into copies of $K_{2}$ and $K_{3}$ whose total number of vertices is at most $(1 / 2+o(1)) n^{2}$, where $K_{r}$ denotes the clique on $r$ vertices. The origins of this problem can be traced back to Erdős, Goodman and Pósa [10] who considered the problem of minimising the total number of cliques in an edge decomposition of an arbitrary $n$-vertex graph. They showed the following:

Theorem 1 (Erdős, Goodman, Pósa [10]). The edges of every $n$-vertex graph can be decomposed into at most $\left\lfloor n^{2} / 4\right\rfloor$ complete graphs.

The only extremal example for this bound is the (bipartite) Turán graph $T_{2}(n):=K_{\lfloor n / 2\rfloor,\lceil n / 2\rceil}$, where $K_{a, b}$ denotes the complete bipartite graph with part sizes $a$ and $b$. Moreover, this result still holds if we restrict the sizes of the cliques used in the decomposition to 2 and 3 (that is, single edges and triangles). In a series of papers published independently by Chung [4], Győri and Kostochka [11, and Kahn [18], they proved that in fact something stronger than Theorem 1 is true, confirming a conjecture by Katona and Tarján:

Theorem 2 (Chung [4], Győri and Kostochka [11, Kahn [18]). Every n-vertex graph can be edge decomposed into cliques whose total number of vertices is at most $\left\lfloor n^{2} / 2\right\rfloor$.

For a given graph $G$ on $n$ vertices, let $\pi_{k}(G)$ be the minimum over all decompositions of the edges of $G$ into cliques $C_{1}, \ldots, C_{\ell}$ of size at most $k$ of the sum $\left|C_{1}\right|+\left|C_{2}\right|+\cdots+\left|C_{\ell}\right|$, where $|G|:=|V(G)|$ denotes the order of a graph $G$. Let $\pi_{k}(n)$ be the maximum of $\pi_{k}(G)$ over all graphs $G$ with $n$ vertices. With this notation, the conclusion of the above theorem is that $\min _{k \in \mathbb{N}} \pi_{k}(n) \leqslant\left\lfloor n^{2} / 2\right\rfloor$. In light of Theorem 2. Tuza [27] conjectured that $\pi_{3}(n) \leqslant n^{2} / 2+o\left(n^{2}\right)$, and in fact that $\pi_{3}(n) \leqslant n^{2} / 2+O(1)$. Győri and Tuza [16] showed that $\pi_{3}(n) \leqslant 9 n^{2} / 16$. This was the best known bound until recently, when using the celebrated flag algebra method of Razborov [24], Král', Lidický, Martins and Pehova [19] proved the asymptotic version of Tuza's conjecture:

Theorem 3 (Král', Lidický, Martins and Pehova [19). We have $\pi_{3}(n) \leqslant(1 / 2+o(1)) n^{2}$ as $n \rightarrow \infty$.

In this paper we show, by building upon the proof in [19], that for all large $n$ it holds in fact $\pi_{3}(n) \leqslant n^{2} / 2+1$. Moreover, if a graph $G$ of order $n$ attains $\pi_{3}(n)$ then $G$ is the complete graph $K_{n}$ or the Turán graph $T_{2}(n)$.

Which of these two graphs is extremal is a matter of divisibility of $n$ by 6 . In the case of the Turán graph, we trivially have $\pi_{3}\left(T_{2}(n)\right)=2\lfloor n / 2\rfloor\lceil n / 2\rceil$, giving $n^{2} / 2$ for even $n$ and $\left(n^{2}-1\right) / 2$
for odd $n$. In order to determine $\pi_{3}\left(K_{n}\right)$, we have to determine the maximum number of edgedisjoint triangles in $K_{n}$. Clearly, the graph made of their edges is triangle-divisible, that is, each vertex has even degree and the total number of edges is divisible by three. It is routine to see that the minimum size of a graph $H$ on $n$ vertices whose complement $\bar{H}$ is triangle-divisible is attained by taking at most one copy of the claw $K_{1,3}$ and a perfect matching on the remaining vertices for even $n$, and isolated vertices plus at most one copy of the 4 -cycle $K_{2,2}$ for odd $n$. (Note that $\binom{n}{2}$ is never equal to 2 modulo 3.) In fact, this gives the value of $\pi_{3}\left(K_{n}\right)$ for all large $n$ by the following general result (which we will use also inside our proof).

Theorem 4 (Barber, Kuhn, Lo and Osthus [2]). For every $\varepsilon>0$, if $G$ is a triangle-divisible graph of large order $n$ and minimum degree at least $(0.9+\varepsilon) n$, then $G$ has a perfect triangle decomposition.

The constant 0.9 in the minimum degree condition in Theorem 4 comes from the result of Dross [6] on fractional triangle decompostions, and it was conjectured by Nash-Williams [21] that it can be replaced by 3/4. Very recently, Dukes and Horsley [7] and Delcourt and Postle [5] improved the constant to 0.852 and $(7+\sqrt{21}) / 14=0.8273 \ldots$, respectively.

Let us list the values of $\pi_{3}$ for the graphs $K_{n}$ and $T_{2}(n)$ for large $n$.

| $n \bmod 6$ | $K_{2}$ 's in an optimal decomposition of $K_{n}$ | $\pi_{3}\left(K_{n}\right)$ | $\pi_{3}\left(T_{2}(n)\right)$ |
| :---: | :---: | :---: | :---: |
| 0 | perfect matching | $\frac{n^{2}}{2}$ | $\frac{n^{2}}{2}$ |
| 1 | none | $\binom{n}{2}$ | $\frac{n^{2}-1}{2}$ |
| 2 | perfect matching | $\frac{n^{2}}{2}$ | $\frac{n^{2}}{2}$ |
| 3 | none | $\binom{n}{2}$ | $\frac{n^{2}-1}{2}$ |
| 4 | $K_{1,3}+$ perfect matching | $\frac{n^{2}}{2}+1$ | $\frac{n^{2}}{2}$ |
| 5 | $C_{4}$ | $\binom{n}{2}+4$ | $\frac{n^{2}-1}{2}$ |

Table 1: Values of $\pi_{3}\left(K_{n}\right)$ and $\pi_{3}\left(T_{2}(n)\right)$ for large $n$.
Let us define

$$
\mathcal{E}_{n}:= \begin{cases}\left\{T_{2}(n), K_{n}\right\}, & \text { if } n \equiv 0,2 \quad(\bmod 6), \\ \left\{T_{2}(n)\right\}, & \text { if } n \equiv 1,3,5 \quad(\bmod 6), \\ \left\{K_{n}\right\}, & \text { if } n \equiv 4 \quad(\bmod 6),\end{cases}
$$

and

$$
\ell(n):= \begin{cases}n^{2} / 2, & \text { for } n \equiv 0,2 \quad(\bmod 6), \\ \left(n^{2}-1\right) / 2, & \text { for } n \equiv 1,3,5 \quad(\bmod 6), \\ n^{2} / 2+1, & \text { for } n \equiv 4 \quad(\bmod 6)\end{cases}
$$

Thus, by the calculations of Table $\mathbb{1}$, we have for all large $n$ that $\mathcal{E}_{n}$ consists of those graphs in $\left\{T_{2}(n), K_{n}\right\}$ which maximise $\pi_{3}$ while $\ell(n)$ is this maximum value.

Clearly, $\ell(n)$ is a lower bound on $\pi_{3}(n)$ for large $n$. Our main result is that this is equality.
Theorem 5. There exists $n_{0} \in \mathbb{N}$ such that for all $n \geqslant n_{0}$, we have $\pi_{3}(n)=\ell(n)$ and the set of $\pi_{3}(n)$-extremal graphs up to isomorphism is exactly $\mathcal{E}_{n}$.

A simple corollary of Theorem [5 is an affirmative answer to a question of Pyber [23], see also [27, Problem 45], for sufficiently large $n$. A covering of a graph $G$ is a collection of subgraphs of $G$ such that every edge of $G$ appears in at least one subgraph. (For comparison, a decomposition requires that every edge appears in exactly one subgraph.)

Corollary 6. There exists $n_{0} \in \mathbb{N}$ such that for all $n \geqslant n_{0}$, the edge set of every $n$-vertex graph can be covered with triangles and edges so that the sum of their orders is at most $\left\lfloor n^{2} / 2\right\rfloor$.

Proof. Theorem 5 directly implies the corollary unless $n \equiv 4(\bmod 6)$ and the graph under consideration is $K_{n}$. So assume that $n \equiv 4(\bmod 6)$. Denote the vertices of $K_{n}$ by $v_{1}, \ldots, v_{n}$. Recall that an optimal decomposition for $K_{n}$ is obtained by taking edges $v_{1} v_{2}, v_{1} v_{3}, v_{1} v_{4}$ and $v_{i} v_{i+1}$ for all odd $i$ with $5 \leqslant i \leqslant n-1$. The rest of the graph becomes triangle-divisible and Theorem 4 can be applied. This gives a decomposition of cost $n^{2} / 2+1$. A covering of cost at most $n^{2} / 2$ can be obtained from this decomposition by replacing edges $v_{1} v_{2}$ and $v_{1} v_{3}$ by a triangle $v_{1} v_{2} v_{3}$. (Notice that the pair $v_{2} v_{3}$ is covered by two triangles in the resulting covering.)

We also study an extension of Theorem [5 5 , where we consider decompositions into $K_{2}$ 's and $K_{3}$ 's but we modify the cost of $K_{3}$ 's to be $\alpha$ (with the cost of $K_{2}$ still being 2 ). The minimum over all costs of such decompositions of a graph $G$ is denoted by $\pi_{3}^{\alpha}(G)$. The maximum value of $\pi_{3}^{\alpha}(G)$ over all $n$-vertex graphs $G$ is denoted by $\pi_{3}^{\alpha}(n)$. Notice that $\pi_{3}^{3}(G)=\pi_{3}(G)$ and $\pi_{3}^{3}(n)=\pi_{3}(n)$. Denote $K_{n}$ without one edge by $K_{n}^{-}$and $K_{n}$ without a matching of size two by $K_{n}^{=}$. Then the following result holds.

Theorem 7. For every real $\alpha$ exists $n_{0} \in \mathbb{N}$ such that every $\pi_{3}^{\alpha}$-extremal graph $G$ with $n \geqslant n_{0}$ vertices satisfies the following (up to isomorphism).

- If $\alpha<3$, then $G=T_{2}(n)$;
- if $\alpha=3$ then Theorem 5 applies;
- if $3<\alpha<4$ and $n \equiv 0,2,4,5(\bmod 6)$, then $G=K_{n}$;
- if $3<\alpha<4$ and $n \equiv 1,3(\bmod 6)$, then $G=K_{n}^{=}$;
- if $\alpha=4$ and $n \equiv 1,3(\bmod 6)$, then $G \in\left\{K_{n}, K_{n}^{-}, K_{n}^{=}\right\}$and, moreover, the three listed graphs are all $\pi_{3}^{\alpha}$-extremal;
- if $\alpha=4$ and $n \equiv 0,2,4,5(\bmod 6)$, then $G=K_{n}$;
- if $4<\alpha$, then $G=K_{n}$.

This paper is organised as follows. In Section 2 we give an outline of the proof of Theorem 3 from [19] that we build on. Theorem [5 is proved in Section 3) Extension for other weights of triangles is in Section 4. Some related results are mentioned in Section 5 .

Notation. We follow standard graph theory notation (see e.g. [3]).
For a graph $G$, we denote the set neighbours of $x \in V(G)$ by $\Gamma_{G}(x)$ (or just $\Gamma(x)$ when $G$ is understood) and the number of edges in a set $B \subseteq E(G)$ incident with $x$ by $d_{B}(x)$. We denote by $K\left[V_{1}, V_{2}\right]$ the complete bipartite graph with vertex partition $\left(V_{1}, V_{2}\right)$. The term $[X, Y]$-edges refers to edges $x y \in E(G)$ such that $x \in X$ and $y \in Y$. We write $[x, Y]$-edges as a short-hand for $[\{x\}, Y]$-edges.

Let $t_{2}(n):=\left|E\left(T_{2}(n)\right)\right|$ be the number of edges in the Turán graph $T_{2}(n)$. Recall that $t_{2}(n)=$ $\left\lfloor n^{2} / 4\right\rfloor$. By a cherry we mean a path with 2 edges.

We consider graphs up to isomorphism; in particular, we write $G=H$ to denote that $G$ and $H$ are isomorphic graphs.

## 2 Outline of the proof of Theorem 3 from [19]

In this section we give a short outline of the proof of [19, Lemma 5], which was a key step in proving $\pi_{3}(n) \leqslant n^{2} / 2+o\left(n^{2}\right)$ and is a starting point of our argument towards Theorem 5 For an $n$-vertex graph $G$ and each $i \in \mathbb{N}$, let $K_{i}(G)$ be the set of all $i$-cliques in $G$. Let $\pi_{3, f}(G)$ be the minimum of

$$
2 \sum_{x y \in K_{2}(G)} c(x y)+3 \sum_{x y z \in K_{3}(G)} c(x y z)
$$

over fractional $\left\{K_{2}, K_{3}\right\}$-decompositions $c$ of $E(G)$, that is, over maps $c: K_{2}(G) \cup K_{3}(G) \rightarrow[0,1]$ such that for every edge $x y \in E(G)$ we have $c(x y)+\sum_{z: x y z \in K_{3}(G)} c(x y z) \geqslant 1$. Of course, $\pi_{3, f}(G) \leqslant \pi_{3}(G)$. By a result of Haxell and Rödl [17] or a more general version by Yuster [28], it also holds that $\pi_{3}(G) \leqslant \pi_{3, f}(G)+o\left(n^{2}\right)$. So, to show that $\pi_{3}(G) \leqslant n^{2} / 2+o\left(n^{2}\right)$, it suffices to consider the fractional equivalent $\pi_{3, f}(G)$.

Lemma 8. Let $G$ be an n-vertex graph. Then

$$
\binom{n}{7}^{-1} \sum_{W \in\binom{V(G)}{7}} \pi_{3, f}(G[W]) \leqslant 21+o(1)
$$

where the sum is taken over 7-vertex subsets $W$ of $V(G)$.
Outline of proof. Let $M$ be the following positive semi-definite matrix

$$
M:=\frac{1}{12 \cdot 10^{9}}\left(\begin{array}{ccccccc}
1800000000 & 2444365956 & 640188285 & -1524146769 & 1386815580 & -732139362 & -129387078 \\
2444365956 & 4759879134 & 1177441152 & -1783771230 & 2546923788 & -1397639394 & -143552208 \\
640188285 & 1177441152 & 484273772 & -317303211 & 1038156300 & -591902130 & -6783162 \\
-1524146769 & -1783771230 & -317303211 & 1558870290 & -651906630 & 305728704 & 154602378 \\
1386815580 & 2546923788 & 1038156300 & -651906630 & 2285399634 & -1283125950 & -10755036 \\
-732139362 & -1397639394 & -591902130 & 305728704 & -1283125950 & 734039016 & -1621938 \\
-129387078 & -143552208 & -6783162 & 154602378 & -10755036 & -1621938 & 23860164
\end{array}\right) \succcurlyeq 0
$$

and let $\vec{F}:=\left(F_{1}, \ldots, F_{7}\right)$ be the following vector of rooted graphs, each having 4 vertices with the root denoted by the white square:


Take any graph $G$ of order $n \rightarrow \infty$. For $w \in V(G)$, let $\boldsymbol{v}_{G, w} \in \mathbb{R}^{7}$ denote the column vector whose $i$-th component is $p\left(F_{i},(G, w)\right)$, the density of the 1-flag $F_{i}$ in the rooted graph $(G, w)$, which is $G$ with the vertex $w$ designated as the root.

It was shown in (19] that

$$
\begin{equation*}
\frac{1}{\binom{n}{7}} \sum_{W \in\binom{V(G)}{7}} \pi_{3, f}(G[W])+\frac{1}{n} \sum_{w \in V(G)} \boldsymbol{v}_{G, w}^{T} M \boldsymbol{v}_{G, w} \leqslant 21+o(1) \tag{1}
\end{equation*}
$$

Namely, if we re-write the left-hand size as a linear combination $\sum_{H} c_{H} p(H, G)$, where $H$ ranges over all 7-vertex unlabelled graphs and $p(H, G)$ is the density of $H$ in $G$, then each coefficient $c_{H}$ is at most 21. Since $\sum_{H} p(H, G)=1$, the claimed inequality (11) follows.

In particular, since $M$ is positive semi-definite, the quantity $\frac{1}{n} \sum_{w \in V(G)} \boldsymbol{v}_{G, w}^{T} M \boldsymbol{v}_{G, w}$ is always non-negative, yielding the result.

The main result of [19] that $\pi_{3}(n) \leqslant n^{2} / 2+o\left(n^{2}\right)$ now follows directly from Lemma 8 ,

Proof of Theorem 3. Let $G$ be any graph of order $n \rightarrow \infty$. As mentioned before, $\pi_{3}(G) \leqslant$ $\pi_{3, f}(G)+o\left(n^{2}\right)$. Also, we have

$$
\binom{n}{2}^{-1} \pi_{3, f}(G) \leqslant\binom{ 7}{2}^{-1}\binom{n}{7}^{-1} \sum_{W \in\binom{V(G)}{7}} \pi_{3, f}(G[W])
$$

by averaging optimal fractional decompositions of all 7-vertex induced subgraphs. Combining this inequality with Lemma 8 immediately gives that $\pi_{3}(G) \leqslant(1 / 2+o(1)) n^{2}$.

## 3 Proof of Theorem 5

We use the so-called stability approach, where the first step is to describe the approximate structure of all almost $\pi_{3}$-extremal graphs of order $n \rightarrow \infty$ within $o\left(n^{2}\right)$ adjacencies. Namely, our Corollary 10 will show that every such graph is close to $K_{n}$ or $T_{2}(n)$.

For this purpose, we start by showing that all almost $\pi_{3}$-extremal graphs contain almost no copies of the three graphs in Figure $\mathbb{1}$ (which are obtained by taking the unlabelled versions of the corresponding graphs in $\vec{F}$ ). This is achieved by the following lemma that builds on the results from 19.


Figure 1: Graphs $H_{2}, H_{5}$, and $H_{7}$.

Lemma 9. For every $c>0$ there exist $\varepsilon>0$ and $n_{0} \in \mathbb{N}$ such that for all $n \geqslant n_{0}$, if $G$ is a graph of order $n$ with $\pi_{3}(G) \geqslant(1 / 2-\varepsilon) n^{2}$, then $G$ has at most $c\binom{n}{4}$ copies of each of the graphs $H_{2}:=$ $(\{a, b, c, d\},\{a b\}), H_{5}:=(\{a, b, c, d\},\{a b, b c, a c, a d\})$ and $H_{7}:=(\{a, b, c, d\},\{a b, b c, a c, b d, a d\})$ from Figure 1 .

Proof. Given $c>0$, let $\varepsilon \gg 1 / n_{0}>0$ be sufficiently small. Let $G$ be a graph as in the lemma. Let $M$ and $\vec{F}$ be as in the proof of Lemma 8 .

First, the rank of the matrix $M$ is 6 with $\boldsymbol{v}=(1,0,3,1,0,3,0)$ being the only zero eigenvector. (Thus all others eigenvalues of $M$ are strictly positive by $M \succcurlyeq 0$.)

Second, by the almost optimality of $G$ and the fact that each term in the left-hand side of (1) is non-negative, we have that

$$
\begin{equation*}
\sum_{w \in V(G)} \boldsymbol{v}_{G, w}^{T} M \boldsymbol{v}_{G, w}=o_{\varepsilon}(n) . \tag{2}
\end{equation*}
$$

We now show that $G$ must contain few copies of the graphs $H_{2}, H_{5}$ and $H_{7}$. Suppose, for contradiction, that $G$ contains at least $c\binom{n}{4}$ copies of $H_{2}$. Then, by a simple double-counting argument we have that at least $c n / 4$ vertices in $G$ contain at least $c\binom{n}{3} / 4$ copies of the rooted flag $F_{2}$. In particular, the second coordinate of at least $c n / 4$ of the vectors $\boldsymbol{v}_{G, w}$ is at least $c / 4$. For each such vector $\boldsymbol{u}$, let $\boldsymbol{u}^{\prime}:=\boldsymbol{u} /\|\boldsymbol{u}\|_{2}$ be the scalar multiple of $\boldsymbol{u}$ of $\ell^{2}$-norm 1. Since $\|\boldsymbol{u}\|_{2} \leqslant \sqrt{7}$, we have that its second coordinate $\boldsymbol{u}_{2}^{\prime}$ is at least $c / 4 \sqrt{7}$. The scalar product of $\boldsymbol{u}^{\prime}$ and the $\ell^{2}$-normalised zero eigenvector $\boldsymbol{v} / \sqrt{20}$ (whose second coordinate is 0 ) is at most $\sqrt{1-(c / 4 \sqrt{7})^{2}}$. Thus the projection of $\boldsymbol{u}$ on the orthogonal complement $L=\boldsymbol{v}^{\perp}$ of the zero eigenspace of $M$ has $\ell^{2}$-norm at least $c / 4 \sqrt{7}$. Thus $\boldsymbol{u}^{T} M \boldsymbol{u} \geqslant \lambda_{2}(c / 4 \sqrt{7})^{2}$, where $\lambda_{2}>0$ is the smallest positive eigenvalue of $M$ (in fact, one can check with computer that $\lambda_{2}=0.0005228 \ldots$..). Thus, we have that the left-hand side of (2) in which each term is non-negative by $M \succcurlyeq 0$ is at least $(c n / 4) \times \lambda_{2}(c / 4 \sqrt{7})^{2}=\Omega(n)$, a contradiction.

The analogous argument shows that the densities of $H_{5}$ and $H_{7}$ in $G$ are also at most $c$.

Let us say that two graphs $G_{1}$ and $G_{2}$ of the same order are $k$-close in the edit distance (or simply $k$-close) if there is a relabelling of the vertices of $G_{2}$ so that $\left|E\left(G_{1}\right) \triangle E\left(G_{2}\right)\right| \leqslant k$. In other words, we can make $G_{1}$ and $G_{2}$ isomorphic by changing at most $k$ adjacencies.

Corollary 10. For every $\delta>0$ there exists $n_{1} \in \mathbb{N}$ such that if $G$ is a graph of order $n \geqslant n_{1}$ with $\pi_{3}(G) \geqslant \ell(n)-n^{2} / n_{1}$, then $G$ is $\delta n^{2}$-close in edit distance to $K_{n}$ or to $T_{2}(n)$.

Proof. Given any $\delta>0$, choose sufficiently small constants $\delta \gg c \gg 1 / n_{1}>0$. Take any graph $G$ on $n \geqslant n_{1}$ vertices such that $\pi_{3}(G) \geqslant \ell(n)-n^{2} / n_{1}$.

By Lemma 9 and the Induced Removal Lemma [1], $G$ can be made $\left\{H_{2}, H_{5}, H_{7}\right\}$-free by changing at most $c n^{2}$ adjacencies. Denote this new graph by $G^{\prime}$ and note that $\pi_{3}\left(G^{\prime}\right) \geqslant \pi_{3}(G)-2 c n^{2}$. By $c \ll \delta$, it is enough to show that $G^{\prime}$ is $\delta n^{2} / 2$-close to $K_{n}$ or $T_{2}(n)$.

Let us show that $G^{\prime}$ is either triangle-free, or the disjoint union of at most two cliques. Indeed, if some vertices $a, b, c$ span a triangle in $G^{\prime}$ then, by the $\left\{H_{5}, H_{7}\right\}$-freeness of $G$, all the remaining vertices of $G^{\prime}$ have either no or three neighbours among $\{a, b, c\}$. Let $A_{0}$ be the set of vertices in $G^{\prime} \backslash\{a, b, c\}$ which see none of $\{a, b, c\}$, and let $A_{3}$ be the set of vertices which see all of $\{a, b, c\}$. Then $A_{3}$ is a clique because $G^{\prime}$ is $H_{7}$-free. The set $A_{0}$ is also a clique because $G^{\prime}$ is $H_{2}$-free. Also, no pair $x y$ in $A_{3} \times A_{0}$ can be an edge as otherwise e.g. the 4 -set $\{a, b, x, y\}$ spans a copy of $H_{5}$ in $G$. It follows that $G$ is the disjoint union of the cliques on $A_{0}$ and $A_{3} \cup\{a, b, c\}$, as required.

Now, if $G^{\prime}$ is triangle-free, then $e\left(G^{\prime}\right)=\pi_{3}\left(G^{\prime}\right) / 2 \geqslant \ell(n) / 2-n^{2} / n_{1}-2 c n^{2} \geqslant t_{2}(n)-3 c n^{2}$. Thus, by the stability result for Mantel's theorem by Erdős [8] and Simonovits [26], the graph $G^{\prime}$ must indeed be $\delta n^{2} / 2$-close in edit distance to $T_{2}(n)$.
Otherwise, $G^{\prime}$ is the disjoint union of two cliques. Let us show that one of them has size at most $\delta n / 2$. Indeed, otherwise $G^{\prime}$ has a triangle packing covering all but at most $n / 2+2$ edges by Theorem 4 meaning that $\pi_{3}\left(G^{\prime}\right) \leqslant e\left(G^{\prime}\right)+n / 2+2$. Also, $e\left(G^{\prime}\right)$ is maximum when clique sizes are as far apart as possible. Thus, by the lower bound on $\pi_{3}(G) \leqslant \pi_{3}\left(G^{\prime}\right)+2 c n^{2}$, we conclude that e.g. $\ell(n)-3 c n^{2} \leqslant\binom{\delta n / 2}{2}+\binom{(1-\delta / 2) n}{2}$, leading to a contradiction to our choice of constants. Therefore, $G^{\prime}$ is at most $n \cdot \delta n / 2$ adjacency edits away from $K_{n}$, as desired.

The key steps in proving Theorem 5 are Lemmas 11 13,
Lemma 11. There exist constants $\delta>0$ and $n_{1} \in \mathbb{N}$ such that, among all graphs on $n \geqslant n_{1}$ vertices which are $\delta n^{2}$-close to $T_{2}(n)$, the maximiser of $\pi_{3}$ is $T_{2}(n)$.

Proof. Choose sufficiently small $\varepsilon \gg \delta \gg 1 / n_{1}>0$. Let $G$ be an arbitrary graph with $n \geqslant n_{1}$ vertices which is $\delta n^{2}$-close to $T_{2}(n)$. We will show that $\pi_{3}(G) \leqslant \pi_{3}\left(T_{2}(n)\right)$ with equality if and only if $G=T_{2}(n)$. In fact, this claim can be directly derived from the result of Györi [12, Theorem 1] that a graph with $n$ vertices and $t_{2}(n)+k$ edges, where $n \rightarrow \infty$ and $k=o\left(n^{2}\right)$, has at least $k-O\left(k^{2} / n^{2}\right)$ edge-disjoint triangles. More specifically, for each $\varepsilon>0$ there exists $\delta>0$
and $n_{0} \in \mathbb{N}$ such that every graph with $n \geqslant n_{0}$ vertices and $t_{2}(n)+k$ edges, where $k \leqslant \delta n^{2}$, has at least $k-\varepsilon k^{2} / n^{2}$ edge-disjoint triangles. (See also [13, Theorem 1] for a generalisation of this to $r$-cliques for any fixed $r \geqslant 3$.) Since $G$ is $\delta n^{2}$-close to $T_{2}(n)$, it must have at most $t_{2}(n)+\delta n^{2}$ edges. From this and $1 / n \ll \delta \ll \varepsilon \ll 1$, we have that, for $k:=e(G)-t_{2}(n)$,

$$
\left.\pi_{3}(G) \leqslant 2\left(t_{2}(n)\right)+k\right)-3\left(k-\varepsilon k^{2} / n^{2}\right)=2 t_{2}(n)-k\left(1-3 \varepsilon k / n^{2}\right) \leqslant 2 t_{2}(n)
$$

Clearly, if equality is achieved then $k=0$, that is, $e(G)=t_{2}(n)$; furthermore, $G$ must be triangle-free and thus $G=T_{2}(n)$, as required.

Next, we need to analyse graphs that are close to $K_{n}$. If $n \equiv 1,3(\bmod 6)$, then let $\mathcal{E}_{n}^{\prime}$ consist of those graphs which are obtained from $K_{n}$ by removing a matching of size $m \equiv 2(\bmod 3)$; otherwise let $\mathcal{E}_{n}^{\prime}:=\left\{K_{n}\right\}$. Also, define

$$
w(n):= \begin{cases}n / 2, & n \equiv 0,2 \quad(\bmod 6) \\ 2, & n \equiv 1,3 \quad(\bmod 6) \\ n / 2+1, & n \equiv 4 \quad(\bmod 6) \\ 4, & n \equiv 5 \quad(\bmod 6)\end{cases}
$$

Using Theorem 4 and the calculation for $K_{n}$ described in Table 1, one can show that $\pi_{3}(G)=$ $\binom{n}{2}+w(n)$ for all large $n$ and every $G \in \mathcal{E}_{n}^{\prime}$. We are going to show that these are exactly the extremal graphs among those close to $K_{n}$. It is more convenient to do first the case when we have some bound on the minimum degree of a graph and then derive the general case (in a separate Lemma 13).

Lemma 12. There exist constants $\delta>0$ and $n_{0} \in \mathbb{N}$ such that the following holds. Let $G$ be $a$ graph on $n \geqslant n_{0}$ vertices with minimum degree at least $n / 8$ such that $G$ is $\delta n^{2}$-close to $K_{n}$ and $\pi_{3}(G) \geqslant\binom{ n}{2}+w(n)$. Then $G \in \mathcal{E}_{n}^{\prime}$.

Proof. Choose small constants in the following order: $c \gg \delta \gg 1 / n_{0}>0$. Suppose that $G$ is a graph of order $n \geqslant n_{0}$ as in the statement of the lemma. Let $w:=w(n)$.

Let $U:=\left\{v \in V(G): d_{G}(v) \leqslant(1-c) n\right\}$. Then

$$
\frac{|U| c n}{2} \leqslant e(\bar{G}) \leqslant \delta n^{2}
$$

and so $|U| \leqslant \frac{2 \delta}{c} n$. Denote $W:=V(G) \backslash U$, and let $S:=\left\{v \in W: d_{G}(v)\right.$ is odd $\}$. Let $M$ be a set of edges forming a maximum matching in $G[S]$, and denote $X:=S \backslash V(M)$. Then $X$ is an independent set and thus $\binom{|X|}{2} \leqslant \delta n^{2}$, which implies that rather roughly

$$
\begin{equation*}
|X|<c n . \tag{3}
\end{equation*}
$$

Moreover, for every edge $y z \in M$ and any two distinct vertices $y^{\prime}, z^{\prime} \in X$, at most one of $y y^{\prime}$ and $z z^{\prime}$ can be an edge of $G$ (otherwise $y^{\prime} y z z^{\prime}$ is an augmenting path contradicting the maximality
of $M$ ). It follows that, if $|X| \neq 1$, then for every edge $y z \in M$ there are at least $|X|$ edges missing between $y z$ and $X$. Let $Y_{W}$ denote the set of missing edges in $G[W]$. Thus

$$
\begin{equation*}
\left|Y_{W}\right| \geqslant\binom{|X|}{2}+|M|\left(|X|-\mathbb{1}_{|X|=1}\right) \tag{4}
\end{equation*}
$$

where the indicator function $\mathbb{1}_{|X|=1}$ is 1 if $|X|=1$ and is 0 otherwise. Moreover, the set $Y_{U}$ of missing edges in $G$ with at least one endpoint in $U$ satisfies

$$
\begin{equation*}
\left|Y_{U}\right| \geqslant c n|U|-\binom{|U|}{2} \tag{5}
\end{equation*}
$$

by the definition of $U$. Note that $e(G)=\binom{n}{2}-\left|Y_{W}\right|-\left|Y_{U}\right|$. See Figure 2 for a sketch ot $Y_{W}$ and $Y_{U}$.

We now build a decomposition $\mathcal{D}$ of $G$ into edges and triangles, starting with $\mathcal{D}=\emptyset$. If we add edges/triangles to $\mathcal{D}$, we regard them as removed from $E(G)$. It is convenient to split our argument into the following two cases.

Case 1: $U \neq \emptyset$ or $S=\emptyset$.

In this case, our procedure for constructing $\mathcal{D}$ is as follows.

Step 1: Add the following to $\mathcal{D}$ as $K_{2}$ 's: the edges of the matching $M$ and the edges of some $\lfloor|X| / 2\rfloor$ cherries with distinct endpoints in $X$ such that their middle points are pairwise distinct.

Step 2: For each $u \in U$, one at a time, add to $\mathcal{D}$ a maximum set of edge-disjoint $K_{3}$ 's containing $u$ and two vertices from $W$. Add all remaining edges incident to vertices in $U$ as $K_{2}$ 's to $\mathcal{D}$.

Step 3: (a) Let $S^{\prime} \subseteq V(G)$ be the set of vertices with odd degree after Step 2. Add to $\mathcal{D}$ the edges of some $\left|S^{\prime}\right| / 2$ cherries with distinct endpoints in $S^{\prime}$ such that their middle points are pairwise distinct.
(b) If the number of remaining edges is not divisible by 3 , then fix this by adding to $\mathcal{D}$ (as single edges) the edge set of some cycle of length 4 or 5 .

Step 4: Add a perfect triangle decomposition of the remaining edges to $\mathcal{D}$.

For $i \in\{1,2,3\}$, let $Z_{i}$ be the set of edges that are added to $\mathcal{D}$ in Step $i$ as copies of $K_{2}$. See Figure 2 for some illustrations of the above steps.

Claim. The above steps can be carried out as stated. Moreover, the obtained decomposition $\mathcal{D}$ of $G$ has at most $|M|+|X|+\binom{|U|}{2}+2|U|+6$ copies of $K_{2}$.


Figure 2: (a) Missing edges in $Y_{W}$ are colored blue and edges in $Y_{U}$ are red. (b) Edges in $Z_{1}$ are colored blue, edges in $Z_{2}$ are red and in $Z_{3}$ green. The same vertices are on the right, where dashed are some of the missing edges. Note that this is a sketch and vertices in $W$ can incident to both blue and red (dashed) edges.

Proof of Claim. In order to do Step 1 as stated, we can iteratively pick any two new vertices $x, y \in X$ and then an arbitrary vertex $z$ which is suitable as the middle point for a cherry on $x y$. Note that the number of choices for $z$ is at least $n-2-2 c n$, the number of common neighbours of $x, y \in X \subseteq W$, minus $|X|-1$, the number of vertices previously used as middle points. This is positive by (3) and $c \ll 1$, so we can always proceed. Note for future reference that every vertex is incident to at most 3 edges removed in Step 1. Also, Step 1 adds $\left|Z_{1}\right|=$ $|M|+2(\lfloor|X| / 2\rfloor) \leqslant|M|+|X|$ copies of $K_{2}$ to $\mathcal{D}$.

Clearly, Step 2 can always be processed. Consider the moment when we apply Step 2 to some $u \in U$. In the current graph, the induced subgraph $G[\Gamma(u) \cap W]$ has minimum degree at least $|\Gamma(u) \cap W|-c n-3$, which is at least $|\Gamma(u) \cap W| / 2$ since $|\Gamma(u)| \geqslant n / 8-3$. So by Dirac's theorem, this subgraph has a matching covering all but at most one vertex, that is, all edges between $u$ and $W$ except at most one are decomposed as triangles in Step 2. Let $U^{\prime}$ be the set of those $u \in U$ for which an exceptional edge occurs. Thus we have $\left|U^{\prime}\right| \leqslant|U|$ copies of $K_{2}$ connecting $U$ to $W$ that are added to $\mathcal{D}$ in Step 2. There are trivially at most $\binom{|U|}{2}$ edges with both endpoints in $U$. So Step 2 adds $\left|Z_{2}\right| \leqslant\binom{|U|}{2}+|U|$ copies of $K_{2}$ to $\mathcal{D}$. Note that all edges incident to $U$ are decomposed after Step 2.

Since all vertices of $W$ but at most one had even degrees before Step 2, we have that $S^{\prime}$ has at most $\left|U^{\prime}\right|+1 \leqslant|U|+1$ vertices. Similarly as in Step 1, a simple greedy algorithm finds all cherries as stated Step 3(a). (Note that $S^{\prime}$, as the set of all odd-degree vertices, has even size.)

The minimum degree of $G[W]$ after Step $3(\mathrm{a})$ is at least $0.99 n$, since each $w \in W$ has at most $2|U|+6$ incident edges removed (at most $2|U|$ from Step 2 and at most 3 in each of Steps 1 and $3(\mathrm{a})$ ). Thus, we can find the required 4 - or 5 -cycle in Step 3(b).

Clearly, we add $\left|Z_{3}\right| \leqslant\left|S^{\prime}\right|+5 \leqslant|U|+6$ copies of $K_{2}$ to $\mathcal{D}$ in Step 3 .
Note that, at the end of Step 3, the graph $G[W]$ has minimum degree at least, say, $0.98 n$ while
all its degrees are even. By Theorem 4, all remaining edges can be decomposed using only triangles, so Step 4 indeed removes all remaining edges.

Step 4 adds no additional $K_{2}$ 's, so the total number of $K_{2}$ 's in $\mathcal{D}$ is

$$
\left|Z_{1}\right|+\left|Z_{2}\right|+\left|Z_{3}\right| \leqslant|M|+|X|+\binom{|U|}{2}+2|U|+6
$$

finishing the proof of the claim.

Now we compute the cost of $\mathcal{D}$. Using the notation from above, we have

$$
\begin{align*}
w & \leqslant \pi_{3}(G)-\binom{n}{2} \leqslant-\left|Y_{U}\right|-\left|Y_{W}\right|+\left|Z_{1}\right|+\left|Z_{2}\right|+\left|Z_{3}\right| \\
& \leqslant-\left|Y_{U}\right|-\left|Y_{W}\right|+|M|+|X|+\binom{|U|}{2}+2|U|+6 \tag{6}
\end{align*}
$$

Substituting the bounds from (4) and (5) and rearranging the terms, we get

$$
\begin{equation*}
w \leqslant\left(2\binom{|U|}{2}+2|U|-c n|U|+6\right)+(3-|X|)\left(\frac{|X|}{2}+|M|\right)+\left(\mathbb{1}_{|X|=1}-2\right)|M| \tag{7}
\end{equation*}
$$

First, suppose that $|U|>0$. Then, the estimate $|U| \leqslant 2 \delta n / c$ yields that

$$
2\binom{|U|}{2}+2|U|-c n|U|+6 \leqslant-c n|U| / 2 \leqslant-c n / 2
$$

Since $w \geqslant 2$, we must have that $|X| \leqslant 1$. Observe that $n$ is odd as otherwise $w \geqslant n / 2$ and, by $|M| \leqslant n / 2$, the cases $|X| \in\{0,1\}$ also contradict (7). So every vertex of degree $n-1$ has even degree, meaning that every vertex of $S$ is in some pair from $Y_{W}$ or $Y_{U}$. Hence, $2|M| \leqslant 2\left|Y_{W}\right|+\left|Y_{U}\right|$. Substituting this into the right-hand size of (6) and using our bound on $\left|Y_{U}\right|$ from (5), we obtain

$$
w \leqslant-\frac{\left|Y_{U}\right|}{2}+|X|+\binom{|U|}{2}+2|U|+6 \leqslant \frac{3}{2}\binom{|U|}{2}+2|U|-\frac{c n|U|}{2}+7
$$

which again is negative for $|U|>0$ and large $n$, contradicting $w \geqslant 2$.
Thus $U$ is empty and, by the assumption of Case $1, S$ is also empty (and so are $X$ and $M$ ). This gives that the initial graph $G$ has minimum degree at least $(1-c) n,\left|Z_{1}\right|=\left|Z_{2}\right|=0$, $S^{\prime}=\emptyset$, and no $K_{2}$ 's are added to $\mathcal{D}$ in Step 3(a).

If $n$ is even, then every vertex of $G$ has at least one missing edge, $e(G) \leqslant\binom{ n}{2}-\frac{n}{2}$ and

$$
\pi_{3}(G) \leqslant\binom{ n}{2}-\frac{n}{2}+\left|Z_{3}\right| \leqslant\binom{ n}{2}-\frac{n}{2}+5
$$

which is strictly less than $\pi_{3}\left(K_{n}\right)$, a contradiction.
Let $n$ be odd and let $r:=\binom{n}{2}-e(G)$ be the number of missing edges in $G$. Suppose that $r>0$, as otherwise $G=K_{n}$ and we are done. The upper bound on $\pi_{3}(G)$ given by $\mathcal{D}$ is $\rho_{r}+\binom{n}{2}-r$,
where we define $\rho_{r}$ as the unique element of $\{0,4,5\}$ with $\binom{n}{2}-\rho_{r}-r \equiv 0(\bmod 3)$. Therefore, $r \leqslant 3$ as otherwise $\pi_{3}(G) \leqslant\binom{ n}{2}+1$ contradicting $w \geqslant 2$. On the other hand, all the degrees of $\bar{G}$ are even so $r=3$ and the only non-empty component of $\bar{G}$ is a triangle. However, this contradicts $w \geqslant 2$ because

$$
\pi_{3}(G)= \begin{cases}\binom{n}{2}-1, & n \equiv 1,3 \quad(\bmod 6), \\ \binom{n}{2}+1, & n \equiv 5 \quad(\bmod 6) .\end{cases}
$$

Case 2: $U=\emptyset$ and $S \neq \emptyset$.
Some things simplify in this case (as we do not need to deal with $U$ ). On the other hand, we have to be a bit more careful with calculations, as the new extremal graphs ( $K_{n}$ minus a matching) fall into this case. In particular, removing a 4 - or 5 -cycle may be too wasteful here. So we construct a decomposition $\mathcal{D}$ of $G$ as follows. Recall that $M$ is a maximum matching in $G[S]$ and $X$ is the set of vertices of $S$ not matched by $M$.

Step 1: Make the graph triangle-disivible by removing the following as $K_{2}$ 's. If $X=\emptyset$, then remove all but one edge $x y \in M$ and a path of length $\rho+1 \in\{1,2,3\}$ whose endpoints are $x$ and $y$ (thus, for $\rho=0$, we remove just the matching $M$ ). If $X$ is non-empty, then remove $M$ and the edge sets of some $|X| / 2-1$ paths of length 2 and one path of length $\rho+2 \in\{2,3,4\}$ so that their degree- 1 vertices partition $X$ and their degree- 2 vertices are pairwise distinct.

Step 2: Decompose the rest perfectly into triangles.

Note that $S$, the set of all odd-degree vertices of $G$, has even size (and also $|X|=|S|-2|M|$ is even). Since the minimal degree of $G$ is at least $(1-c) n$, a simple greedy algorithm achieves Step 1 (and Theorem 4 takes care of Step 2).

The decomposition $\mathcal{D}$ has exactly $|M|+|X|+\rho$ copies of $K_{2}$. Also, $e(G)=\binom{n}{2}-\left|Y_{W}\right|$. Thus

$$
\begin{equation*}
w \leqslant \pi_{3}(G)-\binom{n}{2} \leqslant-\left|Y_{W}\right|+|M|+|X|+\rho . \tag{8}
\end{equation*}
$$

Using (44) and that $|X| \neq 1$ (since $|X|$ is even), we obtain that

$$
\begin{equation*}
w \leqslant(3-|X|)\left(\frac{|X|}{2}+|M|\right)-2|M|+\rho . \tag{9}
\end{equation*}
$$

Moreover, $|X| \leqslant 2$ as otherwise $2 \leqslant w \leqslant \rho-2-3|M|$ contradicting $\rho \leqslant 2$. Thus $X$ has either 0 or 2 elements.

Suppose that $X=\emptyset$. First, let $n$ be even. Then every vertex not in $S$ is incident to at least one non-edge of $G,\left|Y_{W}\right| \geqslant(n-2|M|) / 2$, and by (8),

$$
n / 2 \leqslant w \leqslant 2|M|+\rho-n / 2 .
$$

If $2|M| \leqslant n-2$, then all inequalities here become equalities and thus $|M|=\frac{n-2}{2},\left|Y_{W}\right|=1$, $\rho=2, w=\frac{n}{2}$, and $n \equiv 0,2(\bmod 6)$. However, then the graph after Step 1 has exactly $\binom{n}{2}-1-\frac{n-2}{2}-2$ edges, which is not divisible by 3 , a contradiction. Thus $2|M|=n$, the copies of $K_{2}$ in the decomposition contains a perfect matching of $G$, and $\pi_{3}(G) \leqslant \pi_{3}\left(K_{n}\right)$ with equality only if $G=K_{n}$, giving the desired. So suppose that $n$ is odd. Since every vertex of $S$ has to be incident to a missing edge of $G$, we have $\left|Y_{W}\right| \geqslant|S| / 2=|M|$ and the bound in (8) becomes $w \leqslant \rho$. It follows that we have equality throughout, $\left|Y_{W}\right|=|M|, w=\rho=2, n \equiv 1,3(\bmod 6)$, and $\binom{n}{2}-|M|-\rho \equiv 0(\bmod 3)$; the last gives that $|M| \equiv 2(\bmod 3)$. Thus $G$ is as required.
Finally, it remains to consider the case when $|X|=2$. This time, (9) yields that

$$
2 \leqslant w \leqslant \rho-|M|+1 \leqslant 3
$$

Therefore, $|M| \leqslant 1$, and $n \equiv 1,3(\bmod 6)$ as otherwise $w \geqslant 4$. If $|M|=1$, then we have equality everywhere, giving that $w=\rho=2,|S|=4$ and $\left|Y_{W}\right|=3$. However, then the graph after Step 1 has $\binom{n}{2}-\left|Y_{W}\right|-|M|-|X|-\rho=\binom{n}{2}-8$ edges, which is not divisible by 3 , a contradiction. Thus $M$ is empty, $\rho \in\{1,2\}$ and $S=X$. By (8), $\left|Y_{W}\right| \leqslant 2$ and hence $\left|Y_{W}\right|=1$. In other words, $G=K_{n}^{-}$. However, then the graph after Step 1 has $\binom{n}{2}-1-(2+\rho)$ edges, which is not divisible by 3. (Alternatively, Theorem 4 gives that $\pi_{3}\left(K_{n}^{-}\right)-\binom{n}{2}<2=w$.) This contradiction finishes Case 2 and the proof of the lemma.

Lemma 13. There exist constants $\delta>0$ and $n_{1} \in \mathbb{N}$ such that the following holds. Let $G$ be $a$ graph on $n \geqslant n_{1}$ vertices maximizing $\pi_{3}(G)$ among all graphs that are $\delta n^{2}$-close to $K_{n}$. Then $G \in \mathcal{E}_{n}^{\prime}$.

Proof. Let $n_{0}$ and $\delta$ be the constants from Lemma 12, We claim that, for example, $n_{1}:=2 n_{0}$ is enough for the conclusion of Lemma 13 to hold. Indeed, take any extremal graph $G$ of order $n \geqslant n_{1}$. If $G$ satisfies the assumption on minimum degree of Lemma 12, then we are done. Hence assume that the minimum degree of $G$ is less than $n / 8$. Let $G_{n}:=G$, and iteratively define a sequence of graphs $G_{n-1}, G_{n-2}, \ldots$ as follows. Given a graph $G_{i}$ of order $i$, if it has a vertex $x$ of degree less than $i / 8$, let $G_{i-1}:=G_{i}-x$ be obtained from $G_{i}$ by removing the vertex $x$; otherwise stop. Note that the process does not reach $i<n / 2$ for otherwise $G$ has roughly at least $(n / 2) \times(n / 4)$ non-edges, which is a contradiction to $G$ being $\delta n^{2}$-close to $K_{n}$.

Let $G_{s}$ with $\left|G_{s}\right|=s \geqslant n / 2 \geqslant n_{0}$ be the graph for which the above process terminates. By Lemma 12, we have that $\pi_{3}\left(G_{s}\right) \leqslant \frac{s^{2}}{2}+1$. By decomposing all edges in $E(G) \backslash E\left(G_{s}\right)$ as $K_{2}$ 's, we obtain that

$$
\pi_{3}\left(G_{n}\right) \leqslant \pi_{3}\left(G_{s}\right)+2(n-s) \cdot \frac{n}{8} \leqslant \frac{s^{2}}{2}+1+(n-s) \cdot \frac{n}{4}
$$

This is a convex function in $s$ so it is maximized on the boundary of $\frac{n}{2} \leqslant s \leqslant n-1$. If $s=n / 2$, we get $\pi_{3}\left(G_{n}\right) \leqslant n^{2} / 4+2<\binom{n}{2} \leqslant \pi_{3}\left(K_{n}\right)$. If $s=n-1$, we get

$$
\pi_{3}\left(G_{n}\right) \leqslant \pi_{3}\left(G_{s}\right)+2(n-s) \cdot \frac{n}{8} \leqslant \frac{(n-1)^{2}}{2}+1+\frac{n}{4} \leqslant\binom{ n}{2}-\frac{n}{4}+2<\pi_{3}\left(K_{n}\right)
$$

In both cases, we get a contradiction to $G_{n}$ being extremal.

Proof of Theorem 5 5 Choose sufficiently small constants in this order $1 \gg \delta \gg 1 / n_{0}>0$. In particular, $n_{0}$ is sufficiently large to satisfy Corollary 10 for this $\delta$ as well as Lemmas 11 and 13, Let $G$ be an arbitrary graph of order $n \geqslant n_{0}$ with $\pi_{3}(G) \geqslant \ell(n)$. By Corollary 10, $G$ is $\delta n^{2}$-close to either $T_{2}(n)$ or $K_{n}$.

If $G$ is close to $T_{2}(n)$ then it must be $T_{2}(n)$ by Lemma 11. If $G$ is close to $K_{n}$ then it must be in $\mathcal{E}_{n}^{\prime}$ by Lemma 13, By comparing the costs of optimal decompositions, we conclude that $G \in \mathcal{E}_{n}$.

## 4 Extension to an arbitrary cost $\alpha$

The goal of this section is to prove Theorem [7. Everywhere in this section, let $n$ be sufficiently large.

First, note that the case $\alpha \geqslant 6$ is trivial. Indeed, the cost of a triangle is not better than a cost of three edges. Thus for every graph $G$ an optimal decomposition is to decompose all edges of $G$ as $K_{2}$ 's. The unique graph maximizing the number of edges is $K_{n}$, so it is also the unique maximizer of $\pi_{3}^{\alpha}$ for every $\alpha \geqslant 6$.

Next, let us make some easy general observations which apply when $\alpha<6$. First,

$$
\pi_{3}^{\alpha}(G)=\alpha \nu(G)+2(e(G)-3 \nu(G))=2 e(G)-(6-\alpha) \nu(G),
$$

where $\nu(G)$ denotes the maximum number of edge-disjoint triangles contained in $G$. Also, if $\alpha_{1} \leqslant \alpha_{2}<6, \nu\left(G_{1}\right) \geqslant \nu\left(G_{2}\right)$ and $\pi_{3}^{\alpha_{1}}\left(G_{1}\right)>\pi_{3}^{\alpha_{1}}\left(G_{2}\right)$ for some graphs $G_{1}$ and $G_{2}$, then

$$
\begin{equation*}
\pi_{3}^{\alpha_{2}}\left(G_{1}\right)-\pi_{3}^{\alpha_{2}}\left(G_{2}\right)=\pi_{3}^{\alpha_{1}}\left(G_{1}\right)-\pi_{3}^{\alpha_{1}}\left(G_{2}\right)+\left(\alpha_{2}-\alpha_{1}\right)\left(\nu\left(G_{1}\right)-\nu\left(G_{2}\right)\right)>0 . \tag{10}
\end{equation*}
$$

In particular, if $K_{n}$ is the maximizer of $\pi_{3}^{\alpha_{1}}$, it is also a maximizer for $\pi_{3}^{\alpha_{2}}$.

### 4.1 The case $\alpha<3$

Next, we discuss the case $\alpha<3$. Let $n$ be large and let $G$ be a $\pi_{3}^{\alpha}(n)$-extremal graphs. Since

$$
\pi_{3}^{3}(G) \geqslant \pi_{3}^{\alpha}(G) \geqslant \pi_{3}^{\alpha}\left(T_{2}(n)\right)=\pi_{3}^{3}\left(T_{2}(n)\right)=(1 / 2+o(1)) n^{2}
$$

Corollary 10 gives that $G$ is $o\left(n^{2}\right)$-close to $K_{n}$ or $T_{2}(n)$. Since $\alpha<3$, we have that $\pi_{3}^{\alpha}\left(T_{2}(n)\right) \geqslant$ $(1+\Omega(1)) \pi_{3}^{\alpha}\left(K_{n}\right)$ and thus $G$ is close to $T_{2}(n)$. Now, Lemma 11implies that $\pi_{3}^{\alpha}(G) \leqslant \pi_{3}^{3}(G) \leqslant$ $\pi_{3}^{3}\left(T_{2}(n)\right)=\pi_{3}^{\alpha}\left(T_{2}(n)\right)$, with equality if and only if $G=T_{2}(n)$, giving the desired.

### 4.2 The case $3<\alpha<4$

This subsection proves Theorem 7 in case $3<\alpha<4$.

First, let us show that every $\pi_{3}^{\alpha}$-maximiser $G$ is in $K_{n}$ or $K_{n}^{=}$. Suppose for a contradiction that $G$ violates this. In particular, we have $\pi_{3}^{\alpha}(G) \geqslant \pi_{3}^{\alpha}\left(K_{n}\right)$. By (10), we have that $\pi_{3}^{3}(G) \geqslant \pi_{3}^{3}\left(K_{n}\right)$. For $n \rightarrow \infty$, it holds by Table $\square$ that $\pi_{3}^{\alpha}\left(K_{n}\right) \geqslant(1+\Omega(1)) \pi_{3}^{\alpha}\left(T_{2}(n)\right)$. Hence $G$ needs to be close to $K_{n}$ and Lemma 13 applies to $G$. In particular, this means that $n \equiv 1,3(\bmod 6)$. Lemma 13 gives that all $\pi_{3}^{3}$-extremal graphs are obtained from $K_{n}$ by removing a matching of size congruent to 2 modulo 3 . It follows from (10) that, among these graphs, $\pi_{3}^{\alpha}$ is strictly maximized by $K_{n}^{=}$since this graph has the largest $\nu$.
Theorem 4 gives that $3 \nu\left(K_{n}^{=}\right)=\binom{n}{2}-6$. Since $\pi_{3}^{\alpha}(G) \geqslant \pi_{3}^{\alpha}\left(K_{n}^{=}\right)$and $\pi_{3}^{3}(G)<\pi_{3}^{3}\left(K_{n}^{=}\right)$, this implies by (10) that $\nu(G)>\nu\left(K_{n}^{=}\right)$. Since also $\nu(G)<\nu\left(K_{n}\right)$ (otherwise $\pi_{3}^{\alpha}(G)<\pi_{3}^{\alpha}\left(K_{n}\right)$ ), we conclude that $3 \nu(G)=\binom{n}{2}-3$, that is, exactly three pairs of vertices of $G$ are not included into some triangle from an optimal decomposition of $G$. This implies that $G$ is a complete graph without one edge, or a path on three vertices, or a triangle. Among these three candidates (that have the same $\nu$ ), $K^{-}$has the largest size and thus maximizes $\pi_{3}^{\alpha}$. So $K^{-}$is the only possible candidate for $G$. However, $\pi_{3}^{\alpha}\left(K_{n}^{=}\right)>\pi_{3}^{\alpha}\left(K_{n}^{-}\right)$if $\alpha<4$. This contradiction finishes the proof in case $3<\alpha<4$.
Thus, every $\pi_{3}^{\alpha}$-maximiser is in $\left\{K_{n}, K_{n}^{=}\right\}$. It remains to compare these two graphs. Calculations based on Theorem 4 show that

$$
\frac{\pi_{3}^{\alpha}\left(K_{n}^{=}\right)-\pi_{3}^{\alpha}\left(K_{n}\right)+4}{6-\alpha}=\nu\left(K_{n}\right)-\nu\left(K_{n}^{=}\right)= \begin{cases}0, & n \equiv 0,2,4,5 \quad(\bmod 6) \\ 2, & n \equiv 1,3 \quad(\bmod 6)\end{cases}
$$

Thus $\pi_{3}^{\alpha}\left(K_{n}\right)>\pi_{3}^{\alpha}\left(K_{n}^{=}\right)$if $n \equiv 0,2,4,5(\bmod 6)$ and $\pi_{3}^{\alpha}\left(K_{n}^{=}\right)>\pi_{3}^{\alpha}\left(K_{n}\right)$ otherwise, as required.

### 4.3 The case $4 \leqslant \alpha<6$

In this case we provide a direct proof, without using flag algebras or fractional decompositions. Let $n$ be large and let $G$ be any graph of order $n$ such that $\pi_{3}^{\alpha}(G)=\pi_{3}^{\alpha}(n)$. Let $\mathcal{D}$ be a decomposition of $G$ with minimum weight consisting of $t$ triangles and $\ell$ edges.

If $G$ is a complete graph, then we are done. Hence we assume there exists some pair of vertices $x, y \in G$ such that $x y \notin E(G)$. Let $G^{\prime}$ be obtained from $G$ by adding the edge $x y$. Let $\mathcal{D}^{\prime}$ be an optimal decomposition of $G^{\prime}$ containing $t^{\prime}$ triangles and $\ell^{\prime}$ edges. Recall that finding an optimal decomposition is equivalent to maximizing a triangle packing, that is, $t^{\prime}=\nu\left(G^{\prime}\right)$. Hence $t^{\prime} \geqslant t$. If $x y$ is used as an edge in $\mathcal{D}^{\prime}$, then removing $x y$ from $\mathcal{D}^{\prime}$ gives a decomposition of $G$ with cost $\pi_{3}^{\alpha}\left(G^{\prime}\right)-2$, contradicting the maximality of $G$. Therefore $x y$ must appear in a triangle $x y z \in \mathcal{D}^{\prime}$. We now construct a decomposition $\mathcal{D}^{*}$ of $G$ by removing $x y z$ from $\mathcal{D}^{\prime}$ and adding the edges $x z$ and $y z$. Since the total cost of $\mathcal{D}^{*}$ is $\alpha\left(t^{\prime}-1\right)+2\left(\ell^{\prime}+2\right)$ we have

$$
\pi_{3}^{\alpha}(G) \leqslant \operatorname{cost}\left(\mathcal{D}^{*}\right)=\alpha\left(t^{\prime}-1\right)+2\left(\ell^{\prime}+2\right)=\alpha t^{\prime}+2 \ell^{\prime}-\alpha+4 \leqslant \alpha t^{\prime}+2 \ell^{\prime}=\pi_{3}^{\alpha}\left(G^{\prime}\right)
$$

which contradicts the maximality of $\pi_{3}^{\alpha}(G)$ if at least one of the inequalities is strict. Hence $\alpha=4, x y$ must be in a triangle in $\mathcal{D}^{\prime}$ and $\pi_{3}^{\alpha}\left(G^{\prime}\right)=\pi_{3}^{\alpha}(n)$.

This means that it is possible to keep adding edges to $G$, which results in a sequence of graphs $G, G^{\prime}, \ldots, K_{n}$ where an optimal decomposition of each of these graphs has cost $\pi_{3}^{\alpha}(n)$, i.e. they all are $\pi_{3}^{\alpha}$-extremal graphs. Note that we can add missing edges to $G$ in any order, always obtaining a sequence of extremal graphs.

This allows us to reverse the process and examine a sequence of edge removals from $K_{n}$.
Suppose that $G$ is obtained from $K_{n}$ by removing the edge $x y$, i.e. $G^{\prime}$ is $K_{n}$. Notice that if $\ell^{\prime}>0$, i.e. the optimal decomposition of $K_{n}$ contains an edge, then there exist an option for $\mathcal{D}^{\prime}$ that contains the edge $x y$, which was already ruled out. This means that $K_{n}$ is triangle-divisible, which is the case if and only if $n \equiv 1,3(\bmod 6)$.

Now assume that $G$ is missing more than one edge. Hence $K_{n}^{-}$must be also extremal. By above, $n \equiv 1,3(\bmod 6), K_{n}$ is triangle-divisible, and $\pi_{3}^{4}(n)=4 \nu\left(K_{n}\right)$, where $\nu\left(K_{n}\right)=\frac{1}{3}\binom{n}{2}$.

Suppose that $G$ is obtained from $K_{n}$ by removing two edges $u v$ and $x y$. First, suppose that $u=x$. Let $\mathcal{D}^{\star}$ be a decomposition of $G$ into triangles and one edge $v y$. This gives

$$
\pi_{3}^{4}(G) \leqslant \operatorname{cost}\left(\mathcal{D}^{\star}\right)=4\left(\nu\left(K_{n}\right)-1\right)+2<4 \nu\left(K_{n}\right)=\pi_{3}^{4}(n),
$$

contradicting the maximality of $\pi_{3}^{4}(G)$. Hence $x y$ and $u v$ form a matching. Notice that $x, y$, $u$, and $v$ have odd degrees in $G$, so $\ell \geqslant 2$ for else we are unable to fix the parity of the vertices $x, y, u$, and $v$. Now $\binom{n}{2}-\ell-2$ needs to be divisible by 3 , so $\ell \geqslant 4$. There indeed exists a decomposition with $\ell=4$ by taking edges $x u, x v, y u$, and $y v$ and rest as triangles. This gives

$$
\pi_{3}^{4}(G)=4\left(\nu\left(K_{n}\right)-2\right)+2 \cdot 4=\pi_{3}^{4}(n) .
$$

Therefore, $G$ is extremal.
Suppose that $G$ is obtained from $K_{n}$ by removing three edges $u v, x y$, and $z w$. Since $G^{\prime}$ must be $K_{n}$ without a matching, $u v, x y$, and $z w$ also form a matching. Let $\mathcal{D}^{\star}$ be a decomposition of $G$ into triangles and edges $u x, y z$, and $v w$. This gives

$$
\pi_{3}^{4}(G) \leqslant \operatorname{cost}\left(\mathcal{D}^{\star}\right)=4\left(\nu\left(K_{n}\right)-2\right)+6<4 \nu\left(K_{n}\right)=\pi_{3}^{4}(n),
$$

contradicting the maximality of $\pi_{3}^{4}(G)$. This implies that $G$ cannot be obtained from $K_{n}$ by deleting three or more edges, thus finishing the proof of this case and of Theorem 7 .

## 5 Related results

A related question of Erdős (see e.g., [9]) asks for the largest $t=t(n, m)$ such that every graph with $n$ vertices and $t_{2}(n)+m$ edges has at least $t$ edge-disjoint triangles. Of course, $t \leqslant m$. Győri [12] (see [14] for a correction) showed, for large $n$, that $t \geqslant m-O\left(m^{2} / n^{2}\right)$ if $m=o\left(n^{2}\right)$, and $t=m$ if $n$ is odd and $m \leqslant 2 n-10$ or $n$ is even and $m \leqslant 3 n / 2-5$. Moreover, the last two bounds on $m$ are sharp.

More recently, Győri and Keszegh [15] proved that every $K_{4}$-free graph with $t_{2}(n)+m$ edges has $m$ edge-disjoint triangles.

Theorem [5 shows that the maximum of $\pi_{3}(G)$ is attained for $G=T_{2}(n)$ or $G=K_{n}$. However, if we restrict the set of graphs under consideration to graphs of a particular edge density, the decomposition is perhaps cheaper. Note that if the optimal decomposition of a graph $G$ contains $t$ triangles and $\ell$ edges, then $\pi_{3}(G)=2 e(G)-3 t$. That is, we have that $\pi_{3}(G)=2 e(G)-3 \nu(G)$, where as before $\nu(G)$ denotes the maximum number of edge-disjoint triangles in $G$. Then Theorem 3 implies an inequality between the edge density of $G$ and its triangle packing density which we denote by $\nu_{d}(G):=3 \nu(G) /\binom{n}{2}$ :

Corollary 14 (of Theorem (3). Let $G$ be a graph with $d\binom{n}{2}$ edges. Then

$$
\nu_{d}(G) \geqslant 2 d-1+o(1) .
$$

We also have that $\nu_{d}(G) \leqslant d$, which is tight for all graphs which are the union of edge-disjoint triangles.

A question reminiscent of the seminal result of Razborov on the minimal triangle density in graphs [25] (see also [20, 22]) would be to determine the exact lower bound on $\nu_{d}(G)$ in terms of $d$ (answering asymptotically the question of Erdős stated above).



Figure 3: Asymptotic bounds on possible values of $\pi_{3}(G)$ and $\nu_{d}(G)$. The dashed line is simply $y=2 x-1$ for a better display of the shape.

Some flag algebra computations yield numerical asymptotic lower bounds on $\nu_{d}(G)$ with different edge densities between 0.5 and 1. The result, depicted in Figure 3, suggests that the true asymptotic shape of the region $\left\{\left(d, \nu_{d}(G)\right): 0 \leqslant d \leqslant 1, G\right.$ graph $\}$ may indeed have a richer structure.

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