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HIGHLIGHTS FROM THE LHC

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Abstract. The Large Hadron Collider (LHC) and the two multi-purpose detectors, ATLAS and CMS, have been operated successfully at record centre-of-mass energies of $7 \div 8$ TeV. This paper presents the main physics results from proton–proton collisions based on a total luminosity of $2 \times 5\,\mathrm{fb}^{-1}$. The most recent results from Standard Model measurements, Standard Model and MSSM Higgs searches, as well as searches for supersymmetric and exotic particles are reported. Prospects for ongoing and future data taking are presented.

Keywords: LHC, ATLAS, CMS, Standard Model measurements, SM Higgs boson, MSSM Higgs boson, supersymmetry, top quark, exotic particle searches.

1. Introduction

The Large Hadron Collider (LHC) [1] is a proton–proton (pp) and heavy-ion collider built in an underground tunnel at CERN. Four detectors are installed at the interaction points: the multi-purpose experiments ATLAS [2] and CMS [3], the b-physics experiment LHCb [4], and the ALICE detector [5] designed to study heavy ion physics. This paper concentrates on pp collision measurements with the ATLAS and CMS detectors, and represents the status at the time of the conference (June 2012).

In 2011 and 2012, pp centre-of-mass energies, $\sqrt{s}$, of 7 TeV and 8 TeV were reached. Maximum instantaneous luminosities of $6.5 \times 10^{33}\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}$ and an average LHC operating time fraction of about 35 % with stable beams [6] allowed LHC to deliver about $5.5\,\mathrm{fb}^{-1}$ and $3\,\mathrm{fb}^{-1}$ of data at 7 and 8 TeV, respectively, to the ATLAS and CMS detectors. The prospects for the data-taking period of 2012 are to collect about $12\,\mathrm{fb}^{-1}$ of additional pp collision data per experiment at 8 TeV.

Following the current LHC run, an 18-month period is foreseen to repair the LHC superconducting magnet splices and to prepare for operation at centre-of-mass energies of 13 TeV or more. In addition, the peak luminosity is planned to be increased to $2 \times 10^{34}\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}$, which corresponds to twice the nominal value.

The ATLAS and CMS detectors ran in 2011 and 2012 with more than 97 % of the detector channels operational. High data taking efficiencies of about 94 % were achieved. The detectors were calibrated using control data samples. Similarly, trigger and detector efficiencies were determined directly from data or derived from simulations which were verified with data.

The high instantaneous luminosities lead to a pile-up of simultaneous pp collisions in one LHC proton bunch crossing. Interesting hard-scattering reactions are thus overlaid with additional inelastic pp collisions, as shown in Fig. 1. The average number of pile-up events reached mean values of ≈ 12 in 2011 and ≈ 20 in 2012. Simulations to which the ATLAS and CMS data are compared include event pile-up effects as measured in data.

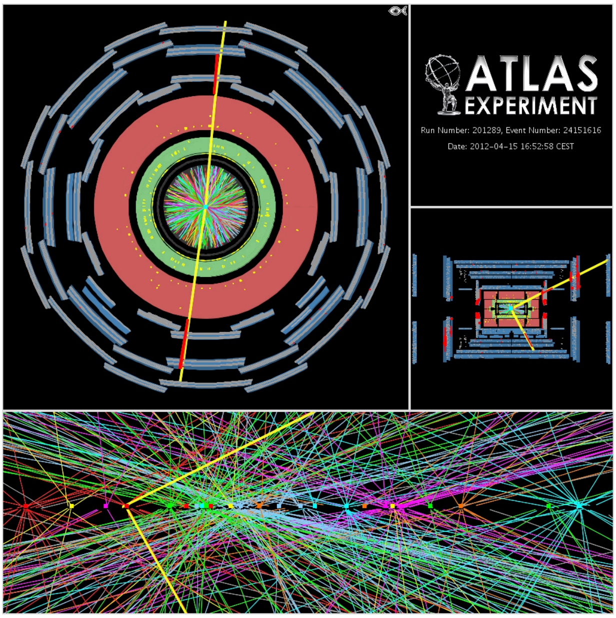

Figure 1. Display of a $Z \to \mu^+\mu^-$ candidate event recorded by the ATLAS detector with 25 additional pp collision vertices reconstructed in the tracking detectors.

Since the longitudinal momentum of the partonic process in a pp collision is a priori unknown, kinetic quantities like the transverse momentum, p_T, and the transverse energy, E_T, of electrons, muons, tau leptons, and jets are often used in the analyses. Momentum conservation in the plane transverse to the beam is applied to measure missing transverse energy, $E_\mathrm{T}^\mathrm{miss}$. Event triggers and selection algorithms apply thresholds to these quantities in order to identify hard scattering processes.

2. Standard Model Measurements

The LHC experiments have performed an extensive measurement program of Standard Model (SM) pp collision processes. The aim is to establish whether SM reactions take place at the expected rate also at centre-of-mass energies of $7 \div 8$ TeV, to improve on precision measurements of SM parameters, and to develop a good understanding of those SM processes which form the background to searches for New Physics.

An important study is the measurement of gauge boson production, $\mathrm{pp} \to \mathrm{W}^{\pm} + \mathrm{X}$, $\mathrm{Z} + \mathrm{X}$, accompanied by additional quark and gluon jets. The corresponding final states with leptons, possibly missing transverse energy, and jets are typical background signatures for new particle searches. As an example, the measured p_{T} spectra of the jet with the largest p_{T} in $\mathrm{W} + \mathrm{jets}$ events are shown in Fig. 2 [7]. The data are well described by leading-order (LO) multi-parton event generators [8, 9], after normalisation to the next-to-next-to-leading-order (NNLO) total cross-section of inclusive W-boson production, and by NLO calculations [10].

The leptonic decays of W and Z bosons, $\mathrm{W} \to \mathrm{e}\nu_{\mathrm{e}}$, $\mu\nu_{\mu}$ and $\mathrm{Z} \to \mathrm{e}^+\mathrm{e}^-$, $\mu^+\mu^-$, are also studied to derive a detailed understanding of the parton density functions (PDF) of the colliding protons. In particular, the $\mathrm{W}^{\pm}$ leptonic charge asymmetry and the Z rapidity distributions are sensitive to the strange-to-down sea quark ratio, $r_{\mathrm{s}} = 0.5(\mathrm{s} + \bar{\mathrm{s}})/\bar{\mathrm{d}}$, which is derived to be $1.00^{+0.25}_{-0.28}$ at a momentum transfer of $Q^2 = 1.9\,\mathrm{GeV}^2$, and a parton momentum fraction of $x = 0.023$ in an NNLO pQCD analysis [11]. This is significantly above previous PDF analyses. A similar measurement is done by studying $\mathrm{W} + \mathrm{c}$ final states [12], which are dominantly produced from sea s-quarks. The $\sigma(\mathrm{W} + \mathrm{c} + \mathrm{X})/\sigma(\mathrm{W} + \mathrm{jets} + \mathrm{X})$ cross-section ratio is found to be in agreement with NLO PDF predictions.

In general, the production cross-sections of W and Z bosons are measured in all leptonic final sates, including tau leptons, and are found to agree with (N)NLO predictions, as illustrated in Fig. 3. Similarly, the production of two gauge bosons, WW, WZ, ZZ, Wγ, Zγ, shows no deviations from SM expectations [14]. Thus, anomalous contributions to triple gauge boson vertices are further constrained.

In the area of tests of the electroweak SM, the LHC collaborations will attempt to achieve precisions for electroweak parameters, like W and top quark masses, m_{W}, m_{top}, competitive with the current LEP and Tevatron results [15]. In preparation of the LHC W-mass measurement from the reconstructed lepton p_{T} and W transverse-mass spectra, the p_{T} distributions of W and Z bosons is compared to NLO calculations [13]. The good agreement observed will enable the LHC experiments to control one important source of systematic uncertainty of the future W mass measurement.

Top quarks are produced at the LHC individually or in pairs, and decay nearly exclusively to a W boson and a b-quark. Top pairs are detected using information from b-tagging of jets in single-lepton, di-lepton, tau + lepton, tau + jets, and all-hadronic final states. The measured cross-sections [16], summarized in Fig. 4, reach relative precisions of $6 \div 8\,\%$, mostly dominated by systematic uncertainties. The main sources of channel-dependent systematics are jet energy scales, b-tag uncertainties, pile-up description, signal and background modeling. Full NNLO theory calculations will soon be necessary to meet the experimental precision.

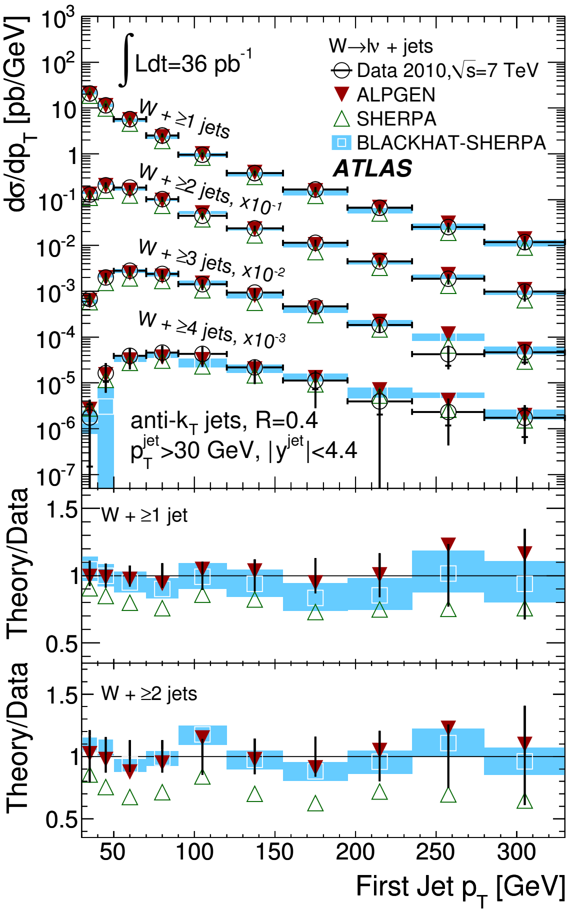

Figure 2. Differential cross-section of W+jets events as a function of the p_{T} of the jet with the highest p_{T} in the event as measured by ATLAS [7], compared with LO predictions by ALPGEN [8] and SHERPA [9], normalized to NNLO inclusive cross-section, and to NLO predictions using BLACKHAT-SHERPA [10].

The top quark pair events in the di-lepton, lepton + jet, and fully hadronic channels are used further to derive the mass of the top quark. Mass-dependent event samples, so-called templates, are simulated and the underlying top-quark mass is varied until the mass-sensitive distributions fit the data. Important systematic uncertainties, like light-quark jet energy scale, b-jet energy scale, signal and background modeling are determined directly from data or by comparing data control samples with simulations. The results [17] obtained by ATLAS and CMS are summarized in Tab. 1.

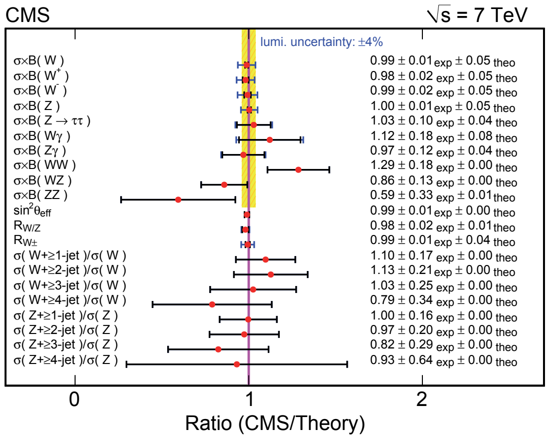

Figure 3. Ratios of cross-sections and quantities for single- and pair-production of gauge bosons with decay into leptons, as measured by CMS, compared to (N)NLO predictions.

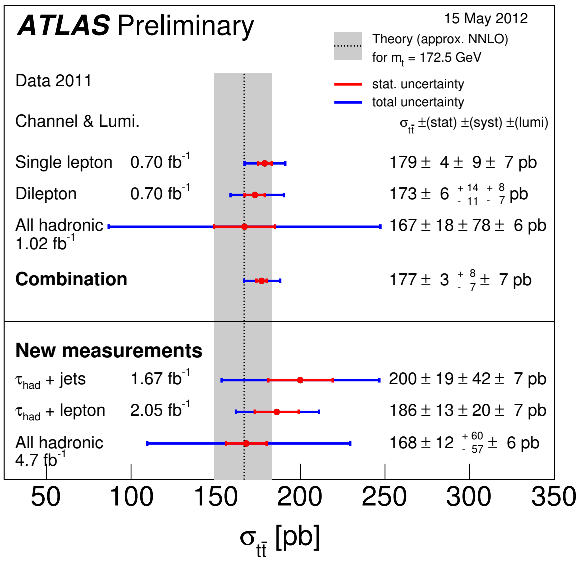

Figure 4. Measurements of the top-pair production cross-section in various final states by ATLAS in comparison with theoretical predictions. CMS has obtained comparable results [16].

They are in good agreement with each other and with the top-quark mass measurements performed at the Tevatron [15]. The precision of the LHC measurements is currently limited by systematic uncertainties, which are expected to improve with further understanding of the details of soft-QCD effects in top-pair production and decay, of the relevant background, and with more refined calibration procedures.

The physics of b-flavoured hadrons is being analysed in detail at the LHC, and new resonances like $\chi_b(3P)$ [18] and Ξ_b [19] have been discovered. This gives an insight into the description of bound quark states, e.g. by lattice QCD [20]. Furthermore, flavour-changing B-hadron decays are loop-suppressed in the SM and thus appear with very small branching

Channel	m_{top} [GeV]
CMS: di-lepton, lepton+jets	$172.6 \pm 0.4 \pm 1.2$
ATLAS: all-hadronic	$174.9 \pm 2.1 \pm 3.8$
ATLAS: lepton+jets	$174.5 \pm 0.6 \pm 2.3$

Table 1. Top-quark mass measurements performed at LHC in different top-pair final states. The statistical and systematic uncertainties are given separately. The CMS results do not yet include systematic uncertainties due to so-called colour reconnection and underlying event effects.

ratios. New Physics may increase these decay rates by orders of magnitude. As an example, the decay $B_s \to \mu^+\mu^-$ is predicted to appear at a relative rate of $BR_{SM}(B_s \to \mu^+\mu^-) = (3.5 \pm 0.3) \times 10^{-9}$ [21], and is searched for by ATLAS, CMS, and LHCb. Isolated di-muon pairs in the relevant B_s mass range are triggered and selected. Background from hadronic B_s decays and combinatorial background remains, but still allows upper limits to be put on the branching ratio, normalized by the $B^+ \to J/\psi K^+ \to \mu^+\mu^- K^+$ decay, of 2.2×10^{-8} (ATLAS [22]), 7.7×10^{-9} (CMS [23]), and 4.5×10^{-9} (LHCb [24]). All results are in agreement with SM expectations, and no significant excess of events is observed.

3. Searches for the Standard Model Higgs Boson

One of the major goals of the LHC physics program is to understand the source of electroweak symmetry breaking. Within the SM, this symmetry breaking is introduced by an SU(2) Higgs doublet field, which gives mass to the gauge bosons and fermions. Its excitations can be measured as a scalar Higgs boson. At the LHC, the SM Higgs boson is produced by gluon–gluon fusion, $gg \to H$, vector–boson fusion, $qqV^*V^* \to qqH$, and Higgs–Strahlung, $W/Z \to W/Z+H$, processes, as well as associated production with top-quarks, where the first two dominate the production cross-section [25].

ATLAS and CMS search mainly for decays into photon pairs, $H \to \gamma\gamma$, and into gauge boson pairs with subsequent decay into leptons, neutrinos or quarks, $H \to ZZ^* \to 4\ell$, $\ell\ell qq$, $\ell\ell\nu\nu$, $H \to WW^* \to \ell\nu\ell\nu$, $\ell\nu qq$. The $\gamma\gamma$ and 4ℓ final states provide excellent mass resolution. Low-mass Higgs bosons decay mostly into pairs of b-quarks, which is also searched for in the associated production with W or Z bosons. At low masses, $H \to \tau\tau$ decays are analysed, though with limited mass resolution due to unmeasured neutrinos in the final state. The background in each channel is estimated either from data using one- and two-dimensional side-band methods and control samples, or by simulation after verifying its consistency with data in phase space regions with negligible contributions from an SM Higgs signal.

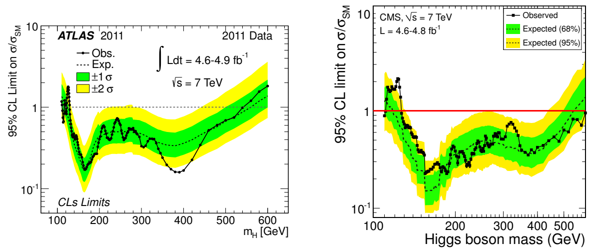

FIGURE 5. Upper limit at 95 % CL for the production cross-section of an SM Higgs boson relative to the SM expectation for ATLAS (left) and CMS (right).

Combining all search channels that were analysed [26], a 95 % CL upper limit on the SM Higgs production cross-section is derived by both ATLAS and CMS, and is displayed in Fig. 5. With the data of 2011 of in total $5\,\mathrm{fb}^{-1}$ at $\sqrt{s} = 7\,\mathrm{TeV}$, ATLAS excludes an SM Higgs boson in the mass ranges $110 < m_\mathrm{H} < 117.5\,\mathrm{GeV}$, $118.5 < m_\mathrm{H} < 122.5\,\mathrm{GeV}$ and $129 < m_\mathrm{H} < 539\,\mathrm{GeV}$, while CMS excludes $127.5 < m_\mathrm{H} < 600\,\mathrm{GeV}$. At masses of about $120 \div 127\,\mathrm{GeV}$, both experiments observe an excess above background expectation. The local probability p_0 for a background-only experiment to be more signal-like than the observation corresponds to significances of about 3σ, for both ATLAS and CMS. The total production rate observed would be compatible with the production of an SM Higgs boson. However, the additional data taken in 2012 will have to be analysed to further study the mass region where the excess is observed.

4. Searches for Supersymmetry and supersymmetric Higgs Bosons

Supersymmetric (SUSY) extensions of the SM predict scalar partners of the SM fermions and fermionic partners of the SM bosons, however at higher mass scales due to supersymmetry breaking. In R-parity conserving models, like the Constrained Minimal Supersymmetric Standard Model (CMSSM/mSUGRA), the lightest SUSY particle (LSP) is stable. In the CMSSM in particular, the weakly interacting neutralino represents this LSP and thus escapes undetected with a typical signature of missing transverse energy, $E_\mathrm{T}^\mathrm{miss}$. Furthermore, gluino–squark initiated SUSY particle production involves SUSY decay chains with multiple leptons and jets measured in the detector. SUSY particle production is thus being searched for with these signatures. Background from SM processes is estimated directly from data, using multi-dimensional side-band methods. Further signatures with $E_\mathrm{T}^\mathrm{miss}$ are photons + jets, disappearing tracks, hadronic multijets, same-sign di-leptons, multi-leptons, same-sign dileptons with b-jets, etc. Recent analyses also look for exclusive production, e.g. direct gaugino or 3rd generation sparticle production with leptons and b-jets in the final state. R-parity violating scenarios are also studied. However, no significant excess of data is observed, either by ATLAS or by CMS. As an example, the excluded parameter ranges in the CMSSM/mSUGRA scenario are shown in Fig. 6 [28]. Within this model, gluino masses of $m_\mathrm{gluino} < 800\,\mathrm{GeV}$ (CMS), $850\,\mathrm{GeV}$ (ATLAS) can be excluded at 95 % CL, and for the assumption $m_\mathrm{squark} = m_\mathrm{gluino}$, the exclusion limits are further improved to $1.2\,\mathrm{TeV}$ (ATLAS) and $1.35\,\mathrm{TeV}$ (CMS) at 95 % CL, respectively.

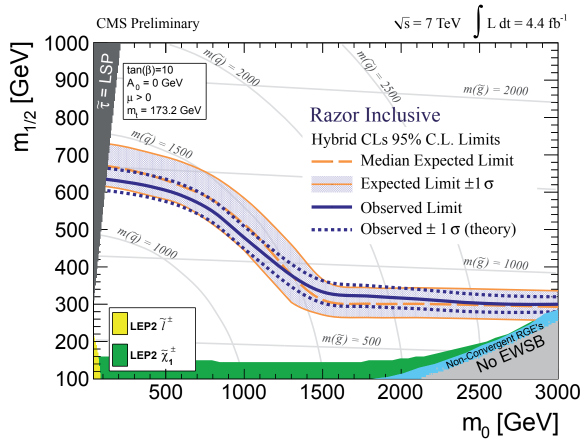

FIGURE 6. LHC results of searches for SUSY particles within the CMSSM/mSUGRA framework. The excluded parameter range for the universal scalar and gaugino masses, m_0 and $m_{1/2}$, at the GUT scale is shown for CMS, while ATLAS obtained very similar results [28].

Even if the excess in the search for a light neutral Higgs boson with SM properties will be confirmed with more data, this boson may be part of a larger set of Higgs bosons, as predicted by supersymmetric extensions of the SM. In the MSSM, two Higgs doublets give rise to 5 physical Higgs bosons: two CP-even, h, H, one CP-odd, A, two charged Higgs bosons, $\mathrm{H}^\pm$. The MSSM Higgs sector is described to first order by the mass of the CP-odd Higgs, m_A, and the ratio of the vacuum expectation values of the two Higgs doublets, $\tan\beta$. The latter parameter also modifies the coupling strength of up- and down-type fermions to the Higgs bosons. For large $\tan\beta$, down-type fermion couplings are enhanced and thus increasing production mechanisms involving b-quarks. Final states like $\mathrm{h/H/A} \to \tau\tau$ and $\mathrm{H}^\pm \to \tau\nu$, cs are studied by ATLAS and CMS. In the absence of a significant excess in the 2011 data, upper limits on the relevant production cross-sections are derived and interpreted as exclusions in the $\tan\beta$ and m_A parameter plane in the $m_\mathrm{h}^\mathrm{max}$ scenario at 95 % CL, as displayed in Fig. 7 [27]. Assuming the absence of a possible MSSM Higgs boson signal, the gap in the $m_\mathrm{A} \lesssim 180\,\mathrm{GeV}$ range between LHC and LEP results is expected to be closed with the additional LHC data of 2012.

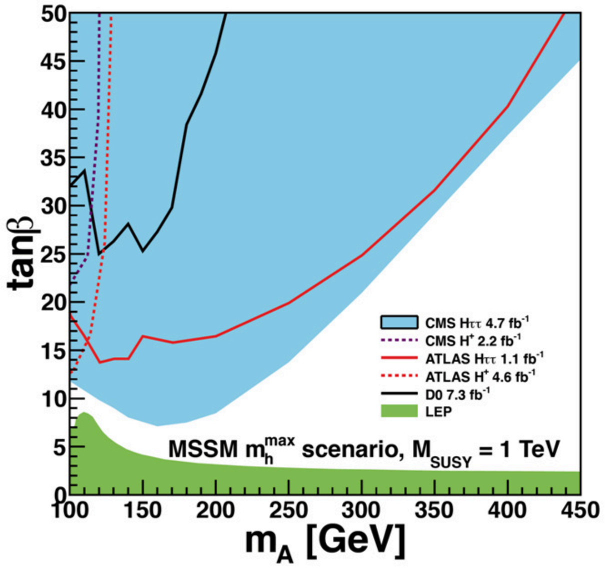

Figure 7. LHC results [27] of searches for MSSM Higgs bosons in the $\tan\beta$-m_A parameter space in the $m_\mathrm{h}^\mathrm{max}$ scenario.

5. Searches for Exotic Particles

ATLAS and CMS have performed searches for New Physics beyond the SM in a wide range of possible theoretical frameworks, but without observing any signal. As an example, searches for pair-produced down-type 4$^\mathrm{th}$ generation quarks decaying to a top quark and a W boson yielded limits in the order of $400 \div 600$ GeV [29] at 95 % CL. Masses of heavy gauge bosons, W$'$ and Z$'$, were excluded by ATLAS up to 2.2 TeV at 95 % CL in the Sequential Standard Model [30]. In searches for signatures of large extra dimensions, the scale for the onset of quantum gravity, M_S, was extracted to be larger than $2.5 \div 3.8$ TeV [31] at 95 % CL, depending on the number of extra dimensions, and the fundamental Planck scale, M_D, was found to be larger than $2.0 \div 3.2$ TeV [32] at 95 % CL, again depending on the number of extra dimensions. These results represent only a few examples, and ATLAS and CMS typically obtained similar exclusions in the TeV range for mass and energy scales of New Physics signatures.

6. Summary and Conclusion

The LHC experiments ATLAS and CMS have measured SM processes in generally good agreement with (N)NLO calculations, including gauge-boson and top-quark physics. Improved determination on SM parameters like m_W and m_top is progressing. Searches for new particles in the SUSY and exotic sector, including MSSM Higgs bosons, have not shown significant deviations from SM background expectations in the data analysed up to now.

In the search for the SM Higgs boson, an excess of signal candidates is observed by both ATLAS and CMS at the 3σ level. More data will have to be analysed to verify if this excess will stay consistent with the production of a light Higgs boson and possibly to confirm the Higgs field as the source of electroweak symmetry breaking.

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Discussion

A. Antonelli — If new data confirms a neutral Higgs boson at $\approx 125\,\mathrm{GeV}$ and no signal for supersymmetric particles is found, which of the different SUSY models will be favoured?

A. Straessner — A low-mass Higgs boson with properties predicted by the Standard Model Higgs is not in contradiction even with the Minimal Supersymmetric Standard Model (MSSM). However, the currently observed absence of signatures of super-symmetric particles up to the TeV scale places challenging constraints on the SUSY models. Further discussions of the MSSM Higgs sector in the light of the LHC measurements can be found e.g. in Reference [33].