Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Parallel and online arithmetics in imaginary quadratic fields

Paralelní a online aritmetika v imaginárních kvadratických tělesech

Master's Thesis

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Academic year: 2020/2021

Katedra: matematiky | Akademický rok: 2020/2021

ZADÁNÍ DIPLOMOVÉ PRÁCE

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Studijní program:	Aplikace přírodních věd
Studijní obor:	Matematická informatika
Název práce (česky):	Paralelní a online aritmetika v imaginárních kvadratických tělesech
Název práce (anglicky):	Parallel and online arithmetics in imaginary quadratic fields

Pokyny pro vypracování:

1) Nastudujte podmínky na komplexní bázi beta v absolutní hodnotě větší než 1 a konečnou abecedu A komplexních cifer tak, aby číselný systém (beta,A) umožňoval a) reprezentaci všech komplexních čísel, b) paralelní algoritmus pro sčítání a odčítání, c) on-line algoritmus pro násobení a dělení.
2) Soustřeďte se na číselné systémy, v nichž báze i abeceda leží v okruhu celých čísel imaginárního kvadratického tělesa, podle článku [1].
3) Zjistěte, které z těchto systémů splňují vlastnost OL nezbytnou pro on-line algoritmy.
4) Zjistěte, u kterých z těchto systémů je možné navrhnout algoritmus pro paralelní sčítání. U systémů, kde toto nelze, studujte možnost rozšíření abecedy tak, aby paralelní algoritmus existoval.
5) U zadaných systémů proveďte přípravu na předzpracování dělitelů, tj. najděte seznam přepisovacích pravidel a odhad na předzpracovaného dělitele.

Doporučená literatura:

1) T. Safer, Polygonal radix representations of complex numbers, Theoret. Comput. Sci. 210, (1999), 159-171.

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Datum zadání diplomové práce: 31.10.2020

Datum odevzdání diplomové práce: 3.5.2021

Doba platnosti zadání je dva roky od data zadání.

Acknowledgement:
I would like to express my gratitude to prof. Ing. Zuzana Masáková, Ph.D for her valuable advice and many useful suggestions concerning this thesis as well as all my previous research work. I have been lucky to have a supervisor who cared so much about my work and who guided me so patiently.

This work was supported by the Grant Agency of the Czech Technical University in Prague, grant No. SGS20/183/OHK4/3T/14

Author's declaration:
I declare that this Master's Thesis is entirely my own work and I have listed all the used sources in the bibliography.

Prague, May 3, 2021 Pavla Veselá

Název práce:
Paralelní a online aritmetika v imaginárních kvadratických tělesech

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Abstrakt: Nestandardní číselné systémy jsou určené svou bází $\beta \in \mathbb{C}, |\beta| > 1$, a svou abecedou cifer $A \subset \mathbb{C}$. Zabýváme se *polygonálními číselnými systémy* s abecedou ve tvaru $A_n = \{0, 1, \xi, \ldots, \xi^{n-1}\}$, kde $\xi = e^{\frac{2\pi i}{n}}$. Navíc požadujeme, aby báze i abeceda byly v okruhu celých čísel nějakého imaginárního kvadratického tělesa. Pro efektivní počítání základních aritmetických operací v těchto číselných systémech lze využít algoritmy pro paralelní sčítání a on-line násobení a dělení. V této práci charakterizujeme polygonální číselné systémy v imaginárních kvadratických tělesech. Pro tyto systémy využíváme metodu na konstrukci algoritmů na paralelní sčítání [20]. Dále rozhodneme, zda je splněná OL vlastnost pro počítání on-line aritmetiky a použijeme implementaci předzpracování pro on-line dělení [29].

Klíčová slova: on-line dělení, on-line násobení, paralelní sčítání, polygonální číselné systémy

Title of the Work:
Parallel and online arithmetics in imaginary quadratic fields

Author: Pavla Veselá

Abstract: Non-standard numeration systems are given by their base $\beta \in \mathbb{C}, |\beta| > 1$, and their alphabet of digits $A \subset \mathbb{C}$. We focus on the so-called *polygonal numeration systems* where the alphabet is of the form $A_n = \{0, 1, \xi, \ldots, \xi^{n-1}\}$ where $\xi = e^{\frac{2\pi i}{n}}$ and both the base and the alphabet are in the ring of algebraic integers of some imaginary quadratic field. Feasibility of several arithmetic operations including parallel addition and on-line division and multiplication is discussed. We characterize the complete polygonal numeration systems in imaginary quadratic fields. The Extending Window Method [20] is used to find the algorithms for parallel addition. Then the decision whether the numeration systems satisfy OL property follows along with computation of preprocessing for on-line division using the implementation from [29].

Key words: on-line division, on-line multiplication, parallel addition, polygonal numeration systems

Contents

List of Symbols

Symbol	Description	Page
ξ	n-th root of 1	16
$\Phi_n(x)$	n-th cyclotomic polynomial	17
$\mathcal{O}_K$	ring of algebraic integers of the field K	17
$\mathbb{Q}(\beta)$	the smallest field containing $\mathbb{Q}$ and complex number β	17
$\mathbb{Z}[\omega]$	the smallest ring containing $\mathbb{Z}$ and algebraic number ω	23
(β, A)	numeration system with base β and alphabet A	18
$\mathrm{Fin}_A(\beta)$	set of all complex numbers with finite (β, A)-representation	18
$W^A(\beta)$	set of fractions of numeration system (β, A)	18
$X^A(\beta)$	spectrum of numeration system (β, A)	19
ρ	sixth root of 1, $\rho = \frac{1+i\sqrt{3}}{2}$	27
A_1	alphabet $\{0, 1\}$	27
A_2	alphabet $\{-1, 0, 1\}$	27
A_3	alphabet $\{0, 1, \rho^2, \rho^4\}$	27
A_4	alphabet $\{0, \pm 1, \pm i\}$	27
A_6	alphabet $\{0, 1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$	27
p	parameter of p-local function for parallel addition	42
q_j	weight coefficient on position j	45
$\mathrm{cl}(I)$	closure of the set I	57
$\mathrm{int}(I)$	interior of the set I	57
δ	delay of on-line algorithm	56
D_{min}	the smallest value of the denominator of preprocessing for on-line division	56
$\mathcal{L}$	list of rewriting rules of preprocessing for on-line division	59
H	the maximal absolute value of a sequence from the set $W^A(\beta)$	60

Introduction

Arithmetic operations such as addition, multiplication and division are the essence of any computation. The algorithms for computing such operations are well known for numeration systems with integer base $q \geq 2$ and canonical alphabet $\{0, \ldots, q-1\}$. The situation differs for non-standard numeration systems with non-integer base or alphabet. To decide whether every real/complex number has a representation in a non-standard numeration system is generally a difficult question which was studied for example by Thurston [27] and Daróczy and Kátai [3].

We are interested in redundant systems where any number has generally multiple representations in (β, A). Only in such numeration systems we can perform arithmetic operations using effective algorithms, in particular, parallel addition and on-line multiplication and division. Parallel algorithms were first given for integer bases by Avizienis [2]. In [11] the possibility of parallel addition was shown for systems with algebraic base. On-line algorithms were introduced for classical numeration systems by Trivedi and Ercegovac [28] and later modified for non-standard numeration systems in [9].

In redundant numeration systems an algorithm for parallel addition where the digit of the result depends only on fixed number of neighbouring digits can be designed. Such an algorithm allows us to perform addition in constant time. On-line arithmetic is a mode of computation where operands and results are processed in a digit serial manner, starting with the most significant digit. In order to execute on-line division and multiplication in linear time, we need to ensure that the numeration system satisfies conditions for parallel addition [11].

The aim of this work is to study numeration systems where both the base and the alphabet are in the ring of algebraic integers in some imaginary quadratic field. Moreover we focus only on polygonal numeration systems where the alphabet consists of 0 and all powers of an n-th root of unity for $n \in \mathbb{N}$. Such alphabets are closed under multiplication. In [26] several conditions on completeness of these numeration systems were introduced. These numeration systems for a real base β were studied in [16].

The work is organised as follows. Chapter 1 is dedicated to basic definitions and concepts of combinatorics on words, algebraic numbers, number systems and their spectra.

In Chapter 2 numeration systems in the ring of algebraic integers of an imaginary quadratic fields and their completeness are discussed. We describe the relation among different polygonal numeration systems with the same alphabet of digits.

Chapters 3 contains basic definitions and concepts of parallel addition. The Extending Window Method for construction of an algorithms for parallel addition from [22] is discussed along with the description of the implementation borrowed from [20].

Basic definitions and concepts of on-line multiplication and division can be found in Chapter 4. We define the so-called OL property and introduce a necessary condition for computation of on-line algorithms. The preprocessing of divisors for on-line division which we implemented in the previous work [29] is described.

Chapter 5 provides an application of the previous theory on complete polygonal numeration systems in imaginary quadratic fields and a discussion whether these numeration systems satisfy OL property. We then apply two computer programs in order to construct the algorithm for parallel addition and perform preprocessing for on-line division. The resulting parameters of the algorithms for effective arithmetic operations are presented.

Chapter 1

Preliminaries

1.1 Combinatorics on words

Numbers are represented by finite or infinite strings (words) of digits. In order to work with such objects, we introduce basic definitions and concepts of combinatorics on words.

A non-empty finite set A is called *alphabet*. Elements of the alphabet are called *letters*. *A word over the alphabet* A is defined as a finite sequence of letters from A. We denote the empty word ε.

Let $u = u_0u_1 \cdots u_m$ and $v = v_0v_1 \cdots v_n$ be words, where $u_i, v_j \in A$ for $i = 0, \dots, m$ and $j = 0, \dots, n$. Then *concatenation of words* is defined as $uv = u_0 \cdots u_m v_0 \cdots v_n$. the set of words over the alphabet A with operation concatenation of words is a monoid of words A^*. Concatenation of words is obviously associative and the empty word ε is the unit of the monoid A^*.

We further define:

- *The length of a word* $u = u_0 \cdots u_{n-1} \in A^*$ is defined as the number of its letters, we write $|u| = n$. The number of occurrences of a letter $a \in A$ in the word $u \in A^*$ is denoted as $|u|_a$.

- Let $u, v, v^{(1)}, v^{(2)} \in A^*$ be words such that $u = v^{(1)}vv^{(2)}$. Then the word $v^{(1)}$ is a *prefix*, v is a *factor* and $v^{(2)}$ is a *suffix* of the word u.

- A sequence $\mathbf{u} = (u_i)_{i=0}^{\infty}$ where $u_i \in A$ for $i \in \mathbb{N}$ is called an *infinite word over the alphabet A*. The set of all infinite words over A is denoted $A^{\mathbb{N}}$.

The definition of prefix, factor and suffix can be extended to $\mathbf{u} \in A^{\mathbb{N}}$. Then prefix and factor are finite words and suffix is an infinite word. The infinite concatenation $uuu\cdots$ for a finite word u is denoted u^ω.

1.2 Algebraic numbers

In this section we recall basic definitions concerning algebraic numbers and give some examples of important algebraic numbers.

A number $\beta \in \mathbb{C}$ is called *algebraic* if β is a root of a monic polynomial with rational coefficients

$$P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad \text{where } a_0, \ldots, a_{n-1} \in \mathbb{Q}.$$

Let β be an algebraic number. Then among all polynomials from $\mathbb{Q}[x]$ (polynomials with rational coefficients) with root β, there exists one monic polynomial f of minimal degree. Moreover, f divides all polynomials $g \in \mathbb{Q}[x]$ such that $g(\beta) = 0$.

The polynomial f defined above is called *the minimal polynomial of the number* β. The degree of the polynomial is the *degree of the number* β. Two algebraic numbers are *algebraic conjugates* if they have the same minimal polynomial. A number $\beta \in \mathbb{C}$ is called *algebraic integer* if there exists a monic polynomial $f \in \mathbb{Z}[x]$ such that $f(\beta) = 0$.

We consider two special cases of algebraic integers, Pisot and complex Pisot numbers.

Definition 1.1. A real algebraic integer $\beta > 1$ whose algebraic conjugates are less than 1 in absolute value is called a *Pisot number*.

Definition 1.2. Let $\beta \in \mathbb{C} \setminus \mathbb{R}$ be an algebraic integer such that $|\beta| > 1$. If all algebraic conjugates of β except for $\overline{\beta}$ are less than 1 in absolute value, then β is called a *complex Pisot number*.

Example 1.3. Here are some examples of Pisot and complex Pisot numbers:

- rational Pisot number: every $m \in \mathbb{N}, m \geq 2$ with minimal polynomial of the form $x - m$.
- quadratic Pisot number:
  - root of a monic polynomial of the form $x^2 - ax - b$ where $a, b \in \mathbb{Z}$ and $a > |b - 1|$.
  - e.g. golden ratio $\tau = \frac{1+\sqrt{5}}{2}$ with minimal polynomial of the form $x^2 - x - 1$.
- d-Bonacci number, $d \geq 2$, which is a root of $x^d - x^{d-1} - \cdots - x - 1$.
  - for $d = 2$ the d-Bonacci number is equal to the golden ratio τ.
  - for $d = 3$ the d-Bonacci number is called Tribonacci number.
- quadratic complex Pisot number:
  - root of a monic polynomial $x^2 - ax - b$ where $a, b \in \mathbb{Z}, |b| \geq 2$ and $a^2 + 4b < 0$.
  - Eisenstein number $\frac{-3+i\sqrt{3}}{2}$ with minimal polynomial $x^2 + 3x + 3$.
  - Penney number $-1 + i$ with minimal polynomial $x^2 + 2x + 2$.
  - Knuth number $-2i$ with minimal polynomial $x^2 + 4$.

Example 1.4. Another example of algebraic integer is determined by the so-called *cyclotomic polynomials*. Let $\xi \in \mathbb{C}$ be an n-th root of 1, i.e. a number such that $\xi^n = 1$ for some $n \in \mathbb{N}$. The number ξ is of the form $e^{\frac{2\pi ik}{n}}$ where $k \in \{0, 1, \ldots, n-1\}$. Let ξ be a primitive n-th root of 1, i.e. $\xi^j \neq 1$ for $0 < j < n$.

Then ξ^k is also a primitive n-th root of 1, if and only if k and n are coprime numbers which we denote by $k \perp n$. The *n-th cyclotomic polynomial* is defined as

$$\Phi_n(x) = \prod_{k \leq n, k \perp n} (x - \xi^k).$$

It can be shown that Φ_n has integer coefficients and is irreducible over $\mathbb{Q}$. It is therefore the minimal polynomial of ξ. Notice that the degree of the n-th root of one is $\varphi(n)$ where φ is Euler's totient function, which counts the number of positive integers smaller than n, coprime to n.

We define for number $\beta \in \mathbb{C}$ the set

$$\mathbb{Q}(\beta) = \bigcap \{T : T \text{ is subfield of } \mathbb{C}, \beta \in T\}.$$

The field $\mathbb{Q}(\beta)$ for β an algebraic number is called *algebraic number field* and it is the minimal subfield of $\mathbb{C}$ which contains β. It can be shown that if β is an algebraic number of degree n then $\mathbb{Q}(\beta)$ is equal to

$$\mathbb{Q}(\beta) = \left\{c_0 + c_1\beta + \cdots + c_{n-1}\beta^{n-1} : c_0, \ldots, c_{n-1} \in \mathbb{Q}\right\}.$$

The field $\mathbb{Q}(\beta)$ is isomorphic to $\mathbb{Q}(\beta^{(j)})$ where $\beta^{(j)}$ is an algebraic conjugate of β. We denote the image of an element

$$x = c_0 + c_1\beta + \cdots + c_{n-1}\beta^{n-1}$$

under the isomorphism by $x^{(j)}$. In particular, we have

$$x^{(j)} = c_0 + c_1\beta^{(j)} + \cdots + c_{n-1}\left(\beta^{(j)}\right)^{n-1}.$$

Definition 1.5. Let β be an algebraic number. The *ring of all algebraic integers* in the field $K = \mathbb{Q}(\beta)$ is denoted by O_K.

The ring of all algebraic integers O_K is an integral domain. This means that one can define naturally the notion of divisibility. For $\alpha, \beta \in O_K$ we say that α divides $\beta, \alpha \mid \beta$, if there exists $\delta \in O_K$ such that $\beta = \alpha\delta$. Similarly as in rational integers, we say that α is congruent to δ modulo $\beta, \alpha \equiv \delta \bmod \beta$, if $\beta \mid (\delta - \alpha)$.

The question of divisibility in O_K can be converted into questions of divisibility in $\mathbb{Z}$ through the notion of a *norm over the field* K, which is a function $N : K \to \mathbb{Q}, N(x) := \prod_{j=1}^{n} x^{(j)}$ that satisfies $N(\alpha) \in \mathbb{Z}$ for every $\alpha \in O_K$. We have $N(\alpha) \mid N(\beta)$ in $\mathbb{Z}$ whenever $\alpha \mid \beta$ in $O_{\mathcal{K}}$.

Note that in imaginary quadratic fields K which are of interest in our work, the norm is equal to the square of the absolute value of the complex number.

1.3 Number representations and number systems

Definition 1.6. Let β be a complex number, $|\beta| > 1$, and $A \subseteq \mathbb{C}$ be an alphabet of digits. By a *numeration system* we understand the ordered pair (β, A). A (β, A)-*representation* of a complex number z is a convergent series $\sum_{j=-\infty}^{m} z_j \beta^j$ where $z_j \in A$.

If $\beta \in \mathbb{R}$ and $A \subset \mathbb{R}$ we say that the numeration system (β, A) is *real*. Otherwise we say that the numeration system (β, A) is *complex*.

The representation is usually identified with the infinite word of digits. We denote

$$z = z_m z_{m-1} \cdots z_1 z_0 \bullet z_{-1} z_{-2} \cdots, \quad \text{or} \quad z = 0 \bullet z_{-1} z_{-2} \cdots,$$

respectively. If the digits in the representation form an eventually periodic sequence with a period $u \in A^*$, we write $(u)^\omega$ for the infinite repetition.

We say that the (β, A)-representation of z is finite, if only finitely many digits are non-zero. In the notation, we usually omit the suffix 0^ω. The set of numbers with finite (β, A)-representation is denoted by

$$\mathrm{Fin}_A(\beta) = \left\{ x \in \mathbb{C} : x = \sum_{j=k}^{l} x_j \beta^j, k, l \in \mathbb{Z}, x_j \in A \right\}.$$

Let us introduce another important property of a numeration system.

Definition 1.7. A numeration system (β, A) in which every complex number is representable is called *complete*. We say that a real, resp. complex numeration system is *complete in* $\mathbb{R}$, resp. *in* $\mathbb{C}$.

Two different meanings of completeness need to be distinguished depending on the type of the numeration system. If the field is not specified it is obvious from the context.

A general condition for completeness of the numeration system follows.

Proposition 1.8. *Let $\beta \in \mathbb{C}, \beta > 1$ and let $A \subset \mathbb{C}$. If the set of fractions*

$$W^A(\beta) = \left\{ \sum_{j=-\infty}^{-1} a_j \beta^j : a_j \in A \right\}$$

contains an open neighbourhood of 0*, then the numeration system* (β, A) *is complete.*

Proof. If for a numeration system (β, A) there exists an open set $B \subset W^A(\beta)$ such that $0 \in B$, where for every $x \in B$ the number x has a (β, A)-representation, then the representation of any $z \in \mathbb{C}$ can be found. We simply divide z by β^k for appropriate k in order to $z \cdot \beta^{-k} \in B$. We know that $z \cdot \beta^{-k}$ has a (β, A)-representation. Finally we shift the fractional point back. □

In [27] Thurston formulated a sufficient condition for completeness of the numeration system in the following terms.

Theorem 1.9 ([27]). *Let β be a complex number such that $|\beta| > 1$ and let $A \subset \mathbb{C}$ be a finite set satisfying $0 \in A$. If there exists $V \subset \mathbb{C}$ such that*

1. *V is bounded,*
2. *$0 \in V$,*
3. *$\beta V \subseteq \bigcup_{a \in A}(V + a)$,*

then all complex numbers have a representation in the numeration system (β, A).

A necessary condition for completeness of numeration system in $\mathbb{C}$ was provided in [3].

Proposition 1.10 ([3]). *Let β be a complex number and let $A \subset \mathbb{C}$ be an alphabet. If the numeration system (β, A) is complete in $\mathbb{C}$, then*

$$|\beta|^2 \leq \#A. \tag{1.1}$$

This work deals with a special case of numeration systems and with the question whether they enable various arithmetic algorithms to be performed.

Definition 1.11. A *polygonal numeration system* is a numeration system (β, A_n) for $n \geq 4$, where $\beta = se^{i\theta} \in \mathbb{C}$, $s > 1$ real and $\theta \in [0, 2\pi]$ in general and $A_n = \{0, 1, \xi, \xi^2, \ldots, \xi^{n-1}\}$ where $\xi = e^{\frac{2\pi i}{n}}$.

In [16],[26] completeness of a polygonal numeration system was studied. The following result was proven in [16] for real bases and then extended in [26] for complex bases.

Theorem 1.12 ([16],[26]). *Let $\beta = se^{i\theta} \in \mathbb{C}$ where $\theta \in [0, 2\pi], s > 1$ is real number and let $A_n = \{0, 1, \xi, \xi^2, \ldots, \xi^{n-1}\}$, with $n \geq 4$ and $\xi = e^{\frac{2\pi i}{n}}$. If $1 < s \leq 1 + 2\cos\left(\frac{2\pi}{n}\right)$, then the polygonal numeration system (β, A_n) is complete, and if $s > 1 + 2\cos\left(\frac{\pi}{n}\right)$, then there exist complex numbers with no representation in (β, A_n).*

The question whether every complex number has a (β, A_n)-representation for s satisfying

$$1 + 2\cos\left(\frac{\pi}{n}\right) \geq s > 1 + 2\cos\left(\frac{2\pi}{n}\right)$$

remains to be an open problem.

1.4 Spectra of real and complex numbers

As will be seen later, certain features of (β, A)-numeration systems depend on the properties of the corresponding spectrum of the number β.

Definition 1.13. Let β be a complex number, $|\beta| > 1$, and let $A \subset \mathbb{C}$ be a finite alphabet of digits. The A-*spectrum* of β is the set

$$X^A(\beta) = \left\{ \sum_{k=0}^{n} a_k \beta^k : n \in \mathbb{N}, a_k \in A \right\}.$$

The spectrum was first considered by Erdös [5] in the case that β is a real number, $\beta > 1$, and $A = \{0, 1, 2, \ldots, m\}$. Then $X^A(\beta)$ is discrete (has no accumulation point) and its elements can be arranged into an increasing sequence $0 = x_0 < x_1 < \cdots$. Many authors have been studying the value $l_m(\beta) = \liminf_{k\longrightarrow\infty}(x_{k+1} - x_k)$.

We will discuss several geometric properties of the spectrum, relative density and discreteness in particular. The first property we are going to focus on is closely linked with completeness of the numeration system.

Definition 1.14. Let $X \subset \mathbb{C}$. We say that X is *relatively dense* if there exists some $r > 0$ such that for all $z \in \mathbb{C}$ the condition $B_r(z) \cap X \neq \emptyset$ holds.

Relative density of the spectrum has the following implication for the numeration system.

Theorem 1.15 ([15]). *Let β be a complex number, $|\beta| > 1$, and let $A \subset \mathbb{C}$ be finite. If $X^A(\beta)$ is discrete, then the following are equivalent:*

1. *$X^A(\beta)$ is relatively dense in $\mathbb{C}$,*
2. *(β, A) is complete,*
3. *$0 \in \operatorname{int}\left(W^A(\beta)\right)$ where $W^A(\beta) = \left\{\sum_{j=-\infty}^{-1} a_j\beta^j : a_j \in A\right\}$.*

For an alphabet composed of any complex numbers, a necessary condition for relative density (and thus completeness of the corresponding numeration system) is formulated using the number of letters in the alphabet. Note that the same can be derived using Proposition 1.10 and Theorem 1.15.

Theorem 1.16 ([15]). *Let β be a complex number, $|\beta| > 1$, and let $A \subset \mathbb{C}$ be finite. If $\#A < |\beta|^2$, then the set $X^A(\beta)$ is not relatively dense.*

In case of a real base β and a real alphabet A, one can state an analogy of Theorem 1.15, namely that relative density of the spectrum in $\mathbb{R}$ is equivalent to completeness of the system in $\mathbb{R}$. Necessary condition for completeness of a real numeration system is then given as $\#A \geq |\beta|$.

From the result of Pedicini [24], one can derive a sufficient condition for completeness in case that β is positive.

Theorem 1.17 ([24]). *Let $\beta > 1$ be a real number and let $A = \{a_1, a_2, \ldots, a_m\} \subset \mathbb{R}$ be an alphabet where $a_1 < a_2 < \cdots < a_m$ and $a_1 < 0 < a_m$. If*

$$\max_{1\leq j\leq m-1}\{a_{j+1} - a_j\} < \frac{a_m - 1}{\beta - 1},$$

then every $x \in \left(\frac{a_1}{\beta-1}, \frac{a_m}{\beta-1}\right]$ has a representation $x = 0\bullet x_1x_2\cdots$, and thus the numeration system (β, A) is complete in $\mathbb{R}$.

Let us focus on another property of the spectrum called discreteness. Usually, discreteness is a topological property, i.e. that any element of the discrete set is isolated. In the Euclidean topology of $\mathbb{C}$, it is equivalent to the following definition.

Definition 1.18. A point set $X \subset \mathbb{C}$ is *discrete* in $\mathbb{C}$, if it has no accumulation point in $\mathbb{C}$. Analogically we define a set discrete in $\mathbb{R}$.

The property of the spectrum $X^A(\beta)$ to have an accumulation point is linked to (β, A)-representations of 0. Discreteness of the spectrum is also necessary and sufficient for the possibility of performing on-line division in the corresponding numeration system.

In case of a positive base $\beta > 1$ and a non-negative alphabet, the spectrum is always discrete in $\mathbb{R}$. The question of discreteness of the spectrum is much less obvious if the alphabet contains both positive and negative digits, or in the case of a complex base β and $A \subset \mathbb{C}$.

Let us cite several results. The case for β real and a symmetric alphabet is completely answered.

Theorem 1.19 ([1], [6])**.** *Let $\beta > 1$ be a real number and $A = \{-M, \dots, M\}$ be an alphabet. Then the spectrum $X^A(\beta)$ has an accumulation point if and only if $\beta < M + 1$ and β is not Pisot.*

If the alphabet is not symmetric or the base is not real, one can state at least a sufficient condition for discreteness of the spectrum.

Theorem 1.20 ([10], [13])**.** *Let β be a complex number, $|\beta| > 1$, and let $A \subset \mathbb{Q}(\beta)$ be an alphabet, $0 \in A$. If*

1. *β is a real number and if β or $-\beta$ is Pisot,*
2. *or $\beta \in \mathbb{C} \backslash \mathbb{R}$ is complex Pisot,*

then $X^A(\beta)$ has no accumulation point.

Chapter 2

Numeration systems in imaginary quadratic fields

In this chapter we present description of all complete polygonal numeration systems in imaginary quadratic fields. Such description was given in [26], however, it turned out that the characterisation given there contains mistakes. Therefore we provide a correction with a detailed proof (Theorem 2.5).

2.1 Number systems in rings

For the proof of Theorem 2.5 we need to prepare an auxiliary statement. The idea is taken from Kovács [18] who studies the so-called "number systems" in rings of algebraic fields and shows that such number systems give rise to complete numeration systems. The "number systems" are in focus of for example [14], [19]. In all cases, the alphabet considered was a subset of $\mathbb{Z}$. Combining the results of [14] and [18], we can already state the following.

Proposition 2.1. *The numeration systems for bases*

$$\beta \in \left\{-1 \pm i, \pm i\sqrt{2}, \frac{\pm 1 \pm i\sqrt{7}}{2}\right\}$$

and alphabet $A = \{0, 1\}$ *are complete.*

In order to adapt the idea to polygonal numeration systems, we had to extend the result given as Theorem 3 in [19] to non-integer alphabets. We only needed the sufficient condition of this theorem.

Theorem 2.2. *Let* $\alpha \in O_K$ *where* K *is a field of degree* n. *Let* $\beta \in \mathbb{Z}[\alpha]$, $|\beta| > 1$ *and* $A \subset \mathbb{Z}[\alpha]$. *Assume that*

1. $|\beta^{(j)}| > 1$ *where* $j = 1, \ldots, n$,

2. $\mathbb{Z}[\alpha] \subset A + \beta \cdot \mathbb{Z}[\alpha]$,

3. *every* $y \in \mathbb{Z}[\alpha]$ *satisfying*

$$|y^{(j)}| \leq \frac{\max_{a\in A} |a^{(j)}|}{|\beta^{(j)}| - 1} \quad \textit{for } j = 1, \dots, n$$

has a (β, A)*-representation of the form* $y = \sum_{l=0}^{N} b_l \beta^l, b_l \in A, N \in \mathbb{N}_0$.

Then the numeration system (β, A) *is complete.*

First let us prove the following lemma.

Lemma 2.3. *Let the conditions of the previous theorem be satisfied. Then for all* $x \in \mathbb{Z}[\alpha]$ *and for all* $k \in \mathbb{N}$ *there exist* $a_0, \dots, a_{k-1} \in A$ *and* $y \in \mathbb{Z}[\alpha]$ *such that*

$$x = \sum_{l=0}^{k-1} a_l \beta^l + y\beta^k \qquad \textit{and} \qquad |y^{(j)}| < \frac{|x^{(j)}|}{|\beta^{(j)}|^k} + \frac{\max_{a\in A} |a^{(j)}|}{|\beta^{(j)}| - 1} \qquad \textit{for } j = 1, \dots, n. \tag{2.1}$$

Proof. We use mathematical induction on k to show the first part of (2.1). First let $k = 0$. Clearly $x = y$. Now we assume that

$$x = \sum_{l=0}^{k-1} a_l \beta^l + y_k \beta^k \tag{2.2}$$

is satisfied for $k \geq 1$ and we will prove that it is satisfied also for $k + 1$. Since $y_k \in \mathbb{Z}[\alpha]$, using the assumption that $\mathbb{Z}[\alpha] \subset A + \beta \cdot \mathbb{Z}[\alpha]$, it can be written as

$$y_k = a_k + \beta \cdot y_{k+1}$$

where $y_{k+1} \in \mathbb{Z}$ and $a_k \in A$. After substitution in (2.2) we obtain

$$x = \sum_{l=0}^{k-1} a_l \beta^l + \beta^k (a_k + \beta \cdot y_{k+1}) = \sum_{l=0}^{k} a_l \beta^l + \beta^{k+1} \cdot y_{k+1}.$$

Thus the first part of (2.1) is proven. Let us verify the second part of (2.1). From (2.2) we have

$$y_k = \frac{1}{\beta^{k-1}} \left(x - \sum_{l=0}^{k-1} a_l \beta^l \right).$$

When we consider the isomorphism j for $j = 1, \dots, n$:

$$|y_k^{(j)}| \leq \frac{|x^{(j)}|}{|\beta^{(j)}|^k} + \frac{\sum_{l=0}^{k-1} |a_l^{(j)}||\beta^{(j)}|^l}{|\beta^{(j)}|^k} < \frac{|x^{(j)}|}{|\beta^{(j)}|^k} + \max_{a\in A} |a^{(j)}| \cdot \sum_{l=1}^{+\infty} |\beta^{(j)}|^{-l} = \frac{|x^{(j)}|}{|\beta^{(j)}|^k} + \frac{\max_{a\in A} |a^{(j)}|}{|\beta^{(j)}| - 1}.$$

The lemma is hereby proven. □

Proof of Theorem 2.2. Clearly for a given $x \in \mathbb{Z}[\alpha]$ and for all $\varepsilon > 0$ there exists $K \in \mathbb{N}_0$ such that

$$|x^{(j)}| < \varepsilon \cdot |\beta^{(j)}|^K \qquad \text{for } j = 1, \dots, n.$$

From Lemma 2.3 we know that x can be rewritten to the form (2.1) for the same K. Moreover

$$|y_K^{(j)}| < \frac{|x^{(j)}|}{|\beta^{(j)}|^K} + \frac{\max_{a\in A}|a^{(j)}|}{|\beta^{(j)}|-1} < \varepsilon + \frac{\max_{a\in A}|a^{(j)}|}{|\beta^{(j)}|-1}.$$

Since there is only finitely many $y_K \in \mathbb{Z}[\alpha]$ which satisfy this inequality, we can choose ε very small and therefore

$$|y_K^{(j)}| \leq \frac{\max_{a\in A}|a^{(j)}|}{|\beta^{(j)}|-1}.$$

By the assumptions of Theorem 2.2, such y_K has a representation $y_K = \sum_{m=0}^{N} b_m\beta^m$. Substituting this into (2.2) we have

$$x = \sum_{l=0}^{k-1} a_l\beta^l + \sum_{m=0}^{N} b_m\beta^{m+k} = \sum_{l=0}^{N+k} a_l\beta^l$$

where we have denoted $a_{k+m} = b_m$ for $m = 0,\dots,N$. We have thus shown that the spectrum $X^A(\beta)$ contains a relatively dense set $\mathbb{Z}[\alpha]$ and thus it is relatively dense itself. By Theorem 1.15, the numeration system (β, A) is complete. □

2.2 Quadratic fields and their rings of algebraic integers

Quadratic fields are algebraic number fields of the form $\mathbb{Q}(\sqrt{d}) = \{r + s\sqrt{d} : r, s \in \mathbb{Q}\}$ where $d \in \mathbb{Z}$ is square-free. If d is a negative integer then $\mathbb{Q}(\sqrt{d})$ is called *imaginary quadratic field*. The set of all algebraic integers O_K of the number field $\mathbb{Q}(\beta) = K$ was defined in Definition 1.5. We can say more about this set for a quadratic field $\mathbb{Q}(\sqrt{d})$.

Proposition 2.4 ([23])**.** *Let d be a square-free integer. Then the set of all algebraic integers in the quadratic field $K = \mathbb{Q}(\sqrt{d})$ is for $d \not\equiv 1 \bmod 4$ equal to*

$$O_K = \left\{a + b\sqrt{d} : a, b \in \mathbb{Z}\right\} = \left\{\frac{a + b\sqrt{d}}{2} : a, b \in 2\mathbb{Z}\right\}$$

and for $m \equiv 1 \bmod 4$ equal to

$$O_K = \left\{a + b\frac{1+\sqrt{d}}{2} : a, b \in \mathbb{Z}\right\} = \left\{\frac{a + b\sqrt{d}}{2} : a, b \in \mathbb{Z} \text{ and } a \equiv b \bmod 2\right\}.$$

Proof. First we want to prove that O_K is a subset of our desired sets from the theorem. Assume $z = a + b\sqrt{d} \in O_K$ for $a, b \in \mathbb{Q}$. Its complex conjugate $\bar{z}$ is also in O_K since z and $\bar{z}$ share the same minimal polynomial $x^2-(z+\bar{z})x+z\bar{z}$. Since the minimal polynomial of an algebraic integer has integer coefficients, we need to have

$$z + \bar{z} = 2a \in \mathbb{Z}$$

and also

$$z\bar{z} = a^2 - db^2 \in \mathbb{Z}.$$

The requirement that d is square-free combined with the fact that $4db^2 = d(2b)^2 \in \mathbb{Z}$ implies $2b \in \mathbb{Z}$. Otherwise it would mean that $2b = \frac{p}{q}$ for some coprime $p, q \in \mathbb{Z}$ where $q \geq 2$ which after substitution in the previous equation would mean that $d(2b)^2 = m\frac{p^2}{q^2} \in \mathbb{Z}$ and thus $q^2|d$ which is a contradiction.

So far we have proven that

$$O_K \subset \left\{ \frac{x + y\sqrt{d}}{2} : x, y \in \mathbb{Z} \right\}.$$

In order to prove the opposite inclusion we need to find pairs of integers x, y for which the condition $\left|\frac{x+y\sqrt{d}}{2}\right|^2 \in \mathbb{Z}$ is satisfied. This would imply that $\frac{x+y\sqrt{d}}{2}$ is algebraic integer from $\mathbb{Q}(\sqrt{d})$ and thus $\frac{x+y\sqrt{d}}{2} \in O_K$. We know that

$$\left|\frac{x + y\sqrt{d}}{2}\right|^2 = \frac{x^2 - dy^2}{4} \in \mathbb{Z}$$

if and only if $x^2 - dy^2 \equiv 0 \bmod 4$. If both numbers x and y are even, the congruence stands. Assume that at least one of x, y is odd. When we square an odd number, we obtain a number of the following form

$$(2k + 1)^2 = 4k^2 + 4k + 1 \equiv 1 \bmod 4.$$

Therefore the equivalence $x^2 - dy^2 \equiv 0 \bmod 4$ has a solution only for $d \equiv 1 \bmod 4$ for x and y both odd numbers. □

Two of the imaginary quadratic fields are particularly interesting, namely $\mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$, which are in the same time cyclotomic fields. The ring of integers in these fields are the well known *Gaussian integers*

$$\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\} = O_{\mathbb{Q}(\sqrt{-1})}$$

and *Eisenstein numbers* where $\omega = \frac{1+i\sqrt{3}}{2}$

$$\mathbb{Z}[\omega] = \{a + b\omega : a, b \in \mathbb{Z}\} = O_{\mathbb{Q}(\sqrt{-3})},$$

respectively.

2.3 Complete polygonal numeration systems

We are interested in complete polygonal numeration systems of imaginary quadratic fields in particular. The question of whether a general polygonal numeration system is complete was treated in [26]. We recalled it in Theorem 1.12. Recall the notation: A polygonal numeration system has for base a complex number

$$\beta = se^{i\xi} \quad \text{and alphabet is} \quad A_n = \{0, 1, \xi, \xi^2, \ldots, \xi^{n-1}\}$$

where $\xi = e^{\frac{2\pi i}{n}}, n \in \mathbb{N}, s \in \mathbb{R}$ and $s > 1$.

The results from [26] should have completely answered the question of which polygonal numeration systems in imaginary quadratic fields are complete. Unfortunately, the description contained mistakes both in the formulation and in the proof. The majority of the proof was based on Lemma 3 in [26] which was not used correctly. This resulted in numeration systems in Table 2.1. We state here the correct version and include the proof with all the details.

d	$O_K = \mathbb{Z}[\rho]$	n	A_n	β
-1	$\mathbb{Z}[i]$	1	$\{0,1\}$	$\pm 1 \pm i$
		2	$\{0,\pm 1\}$	$\pm 1 \pm i$
		4	$\{0,\pm 1,\pm i\}$	$\pm 1 \pm i, \pm 2 \pm 2i, \pm 1 \pm 2i, \pm 2$
-2	$\mathbb{Z}[i\sqrt{2}]$	1	$\{0,1\}$	$\pm i\sqrt{2}$
		2	$\{0,\pm 1\}$	$\pm i\sqrt{2}, \pm 1 \pm i\sqrt{2}$
-3	$\mathbb{Z}[\frac{1+i\sqrt{3}}{2}]$	2	$\{0,\pm 1\}$	$\pm i\sqrt{3}, \frac{\pm 3 \pm i\sqrt{3}}{2}$
		3	$\{0,1,\rho^2,\rho^4\}$	$\pm i\sqrt{3}$
		6	$\{0,1,\rho,\rho^2,\rho^3,\rho^4,\rho^5\}$	$\pm 2, \pm 2(1 \pm i\sqrt{3}), \pm(2 + i\sqrt{3}),$ $\pm i\sqrt{3}, \frac{\pm 3 \pm i\sqrt{3}}{2}$
-7	$\mathbb{Z}[\frac{1+i\sqrt{7}}{2}]$	1	$\{0,1\}$	$\frac{\pm 1 \pm i\sqrt{7}}{2}$
		2	$\{0,\pm 1\}$	$\frac{\pm 1 \pm i\sqrt{7}}{2}$
-11	$\mathbb{Z}[\frac{1+i\sqrt{11}}{2}]$	2	$\{0,\pm 1\}$	$\frac{\pm 1 \pm i\sqrt{11}}{2}$

Table 2.1: The original table from [26] which should have contained complete polygonal numeration systems (β, A_n) of imaginary quadratic fields $\mathbb{Q}(\sqrt{d}) = K$ where $\beta \in O_K$ and $A_n \subset O_K$ where $d < 0$.

Theorem 2.5. *The only complete polygonal numeration systems (β, A_n) of imaginary quadratic fields $\mathbb{Q}(\sqrt{d}) = K$ where $d < 0$, that satisfy $\beta \in O_K$, and $A \subset O_K$, are given in Table 2.2.*

Notice how Table 2.2 is different from the original table in [26]. The missing cases were

$$(\beta, A) \in \left\{ (\pm 2i, A_4), (\pm 2 \pm i, A_4), \left(\frac{\pm 3 \pm i\sqrt{3}}{2}, A_3 \right), (-2, A_3), (\pm 1 \pm i\sqrt{3}, A_3), \right.$$
$$\left. \left(\pm(2 - i\sqrt{3}), A_6\right), \left(\frac{\pm 5 \pm i\sqrt{3}}{2}, A_6 \right), \left(\frac{\pm 1 \pm i\, 3\sqrt{3}}{2}, A_6 \right) \right\}.$$

On the other hand, Table 2.1 contains numeration systems

$$(\beta, A) \in \left\{ (+1 \pm i, A_1), (\pm 2 \pm 2i, A_4), \left(\frac{\pm 3 \pm i\sqrt{3}}{2}, A_2 \right), \left(\pm 2(1 \pm i\sqrt{3}), A_6\right) \right\}$$

which are not complete.

All combinations of bases and alphabets of complete polygonal numeration systems in imaginary quadratic fields are shown in Figure 2.12 below.

d	$O_K = \mathbb{Z}[\rho]$	n	A_n	β
-1	$\mathbb{Z}[i]$	1	$\{0,1\}$	$-1 \pm i$
		2	$\{0,\pm 1\}$	$\pm 1 \pm i$
		4	$\{0,\pm 1,\pm i\}$	$\pm 1 \pm i, \pm 2, \pm 2i, \pm 1 \pm 2i, \pm 2 \pm i$
-2	$\mathbb{Z}[i\sqrt{2}]$	1	$\{0,1\}$	$\pm i\sqrt{2}$
		2	$\{0,\pm 1\}$	$\pm i\sqrt{2}, \pm 1 \pm i\sqrt{2}$
-3	$\mathbb{Z}[\frac{1+i\sqrt{3}}{2}]$	2	$\{0,\pm 1\}$	$\pm i\sqrt{3}$
		3	$\{0,1,\rho^2,\rho^4\}$	$\pm i\sqrt{3}, \frac{\pm 3\pm i\sqrt{3}}{2}, -2, +1 \pm i\sqrt{3}$
		6	$\{0,1,\rho,\rho^2,\rho^3,\rho^4,\rho^5\}$	$\pm 2, \pm 1 \pm i\sqrt{3}, \pm i\sqrt{3}, \frac{\pm 3\pm i\sqrt{3}}{2}$
				$\pm 2 \pm i\sqrt{3}, \frac{\pm 5\pm i\sqrt{3}}{2}, \frac{\pm 1\pm i 3\sqrt{3}}{2}$
-7	$\mathbb{Z}[\frac{1+i\sqrt{7}}{2}]$	1	$\{0,1\}$	$\frac{\pm 1\pm i\sqrt{7}}{2}$
		2	$\{0,\pm 1\}$	$\frac{\pm 1\pm i\sqrt{7}}{2}$
-11	$\mathbb{Z}[\frac{1+i\sqrt{11}}{2}]$	2	$\{0,\pm 1\}$	$\frac{\pm 1\pm i\sqrt{11}}{2}$

Table 2.2: The complete polygonal numeration systems (β, A_n) of imaginary quadratic fields $\mathbb{Q}(\sqrt{d}) = K$ where $\beta \in O_K$ and $A_n \subset O_K$ where $d < 0$.

In order to prove Theorem 2.5, we first have to cite several results. We accompany them with detailed proofs.

Lemma 2.6 ([26]). *Let $d < 0$ be a square-free integer and let $K = \mathbb{Q}(\sqrt{d})$ and O_K be the set of all algebraic integers in K. Let $A_n \subset O_K$ where $n \in \mathbb{N}, n \neq 0$. Then $A_1 = \{0,1\}$ and $A_2 = \{0,\pm 1\}$ are subsets of O_K for all values of d, and for $n \geq 3$, the set A_n satisfies the following:*

- *if $d \equiv 2$ or $3 \bmod 4$, then $A_n \subset O_K$ if and only if $d = -1$ and $n = 4$,*
- *if $d \equiv 1 \bmod 4$, then $A_n \subset O_K$ if and only if $d = -3$ and $n \in \{3,6\}$.*

Proof. The first part of the statement for $n = 1,2$ is obvious. Let $n \geq 3$. In order that $A_n \subset O_K$, necessarily $\xi = e^{\frac{2\pi i}{n}}$ must be a quadratic number. The minimal polynomial of the algebraic number $e^{\frac{2\pi i}{n}}$ is the n-th cyclotomic polynomial Φ_n, which is of degree $\varphi(n)$ where φ is Euler's totient function. It can be easily shown that $\varphi(n) = 2$ only for $n = 3,4,6$.

In order to complete the proof, realize that according to Proposition 2.4, if $d < 0$ is a square-free integer, then the set of all algebraic integers in $\mathbb{Q}(\sqrt{d})$ is

$$O_K = \begin{cases} \mathbb{Z}[\sqrt{d}] & \text{if } d \equiv 2 \text{ or } 3 \bmod 4, \\ \mathbb{Z}[\frac{1+\sqrt{d}}{2}] & \text{if } d \equiv 1 \bmod 4. \end{cases}$$

Thus for $d = -1, n = 4$ we have $O_K = \mathbb{Z}[i]$ and obviously $A_4 \subset O_K$ and for $d = -3, n \in \{-3,-6\}$ we have $O_K = \mathbb{Z}[\frac{1+i\sqrt{3}}{2}]$ and $A_3, A_6 \subset O_K$. □

Lemma 2.7. *For all $n \geq 1, n \in \mathbb{N}$, the set A_n is closed under multiplication by ξ^k for all $k \in \mathbb{Z}$, and under complex conjugation, in other words $\xi^k A_n = A_n$ and $\bar{A}_n = A_n$. Thus if (β, A_n) is a complete numeration system, then $(\bar{\beta}, A_n)$ and $(\xi^k\beta, A_n)$ are also complete numeration systems for all $k \in \mathbb{N}$.*

Proof. Let the numeration system (β, A_n) be complete. The completeness of the numeration system $(\bar{\beta}, A_n)$ is clear from the fact that $\bar{A}_n = A_n$. In order to prove that the numeration system $(\xi^k\beta, A_n)$ is complete, we only have to realize that if $z \in \mathbb{C}$ such that

$$z = \sum_{j=-\infty}^{N} a_j \beta^j$$

is a (β, A_n)-representation of z where $a_i \in A_n$ and $N \in \mathbb{Z}$, then we obtain the $(\xi^k\beta, A_n)$-representation in the following form

$$z = \sum_{j=-\infty}^{N} a_j \beta^j = \sum_{j=-\infty}^{N} \underbrace{a_j \xi^{-kj}}_{\in A_n} \left(\xi^k \beta\right)^j .$$

The digits $a_j\xi^{-kj}$ belong to A_n, since the alphabet A_n is closed under multiplication by ξ. Therefore we can obtain a $(\xi^k\beta, A_n)$-representation of every complex number and the numeration system $(\xi^k\beta, A_n)$ is complete. □

During the proof of Theorem 2.5 we will also use the obvious fact that if (β, A) is complete and $A' \supset A$, then (β, A') is also complete.

Proof of Theorem 2.5. We will divide the analysis into two cases depending on whether the parameter d is equal to 1 modulo 4 or if it is $d \equiv 2$ or $3 \bmod 4$. Each part is then subdivided into several cases depending on $n \in \mathbb{N}$.

Case d ≡ 2 or 3 mod 4

According to Proposition 2.4 the set of all algebraic integers of quadratic field $K = \mathbb{Q}(\sqrt{d})$ is $O_K = \mathbb{Z}[\sqrt{d}]$. Let us consider $\beta = a + b\sqrt{d}$ where $a, b \in \mathbb{Z}$. This implies that $|\beta|^2 = a^2 - db^2$. From Lemma 2.6 we know that $n \in \{1, 2, 4\}$.

Case n = 1: If (β, A_n) is complete, then by Proposition 1.10 we know that β must satisfy $1 < |\beta|^2 = a^2 - db^2 \leq \#A_1 = 2$ where $a, b \in \mathbb{Z}$ which implies that d is either -1 or -2. This gives $\beta \in \{\pm 1 \pm i, \pm i\sqrt{2}\}$. From Proposition 2.1 we know that for

$$\beta \in \{-1 \pm i, \pm i\sqrt{2}\}$$

the numeration systems (β, A_1) are complete. For $\beta = +1 \pm i$ the numeration system is not complete. It can be seen from Theorem 1.15, since the set $W^{A_1}(\beta)$ does not contain 0 in its interior, see Figure 2.1. By Lemma 2.7 this implies non-completeness of $(+1 - i, A_1)$.

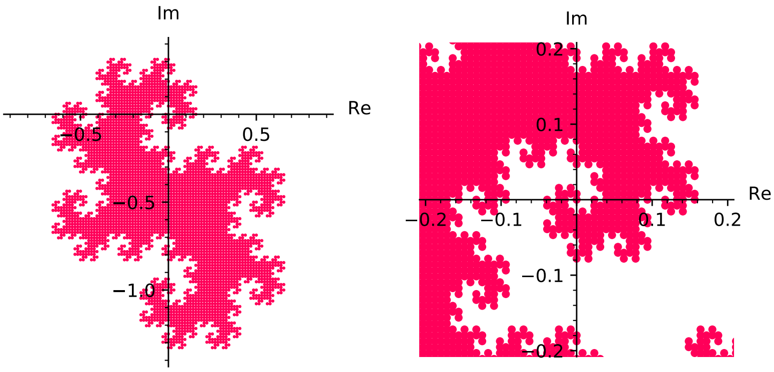

Figure 2.1: The set $W^{A_1}(\beta)$ for $\beta = +1 + i$ and its close-up.

Case n = 2: For $n = 2$, β must satisfy $1 < |\beta|^2 \leq \#A_2 = 3$. The bases from the previous case still verify this condition and moreover we obtain additional values $\pm 1 \pm i\sqrt{2}$. Together

$$\beta \in \{\pm 1 \pm i, \pm i\sqrt{2}, \pm 1 \pm i\sqrt{2}\}.$$

The numeration systems (β, A_2) for $\beta \in \{-1 \pm i, \pm i\sqrt{2}\}$ are complete, since even smaller alphabet $A_1 \subset A_2$ is sufficient for completeness. For the bases $\beta = +1 \pm i$ the systems are also complete using Lemma 2.7.

In order to prove completeness for $\beta = \pm 1 \pm i\sqrt{2}$ we only have to verify that (β, A_2) satisfies the conditions from Theorem 2.2 for $\alpha = i\sqrt{2}$. Figure 2.2 displays the method to verify that

$$\mathbb{Z}[i\sqrt{2}] \subset A_2 + \beta \cdot \mathbb{Z}[i\sqrt{2}].$$

It remains to check that all $x \in \mathbb{Z}[i\sqrt{2}]$ such that $|x^{(j)}| \leq \frac{1}{\sqrt{3}-1} = 1.366025\cdots$, in particular $\{0, \pm 1\}$, have a (β, A_2) representation. This is obviously true. Therefore the numeration system with base β and its

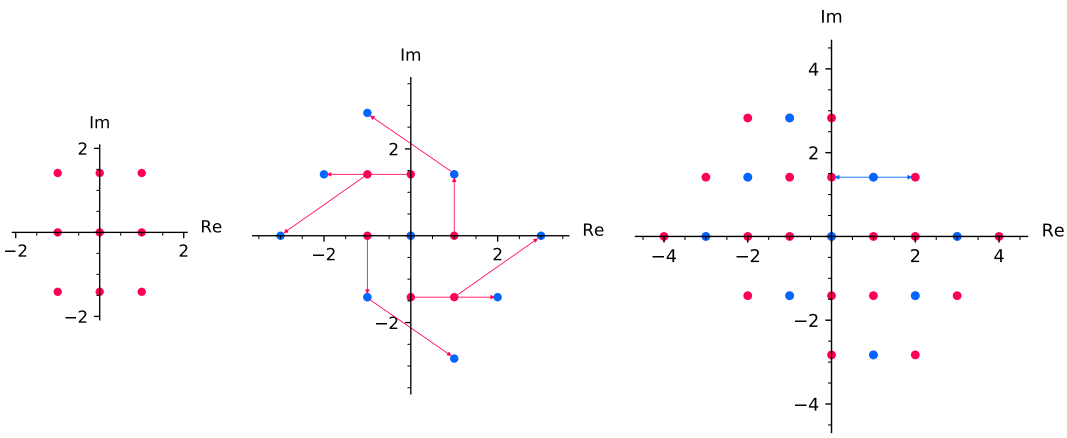

Figure 2.2: Verification of condition $\mathbb{Z}[i\sqrt{2}] \subset A_2 + \beta \cdot \mathbb{Z}[i\sqrt{2}]$ from Theorem 2.2 where $\beta = 1 + i\sqrt{2}$. Each row shows respectively a small part of the grid $\mathbb{Z}[i\sqrt{2}]$, the grid multiplied by β (the blue points) and all points obtained by adding a digit from A_2 (the red points).

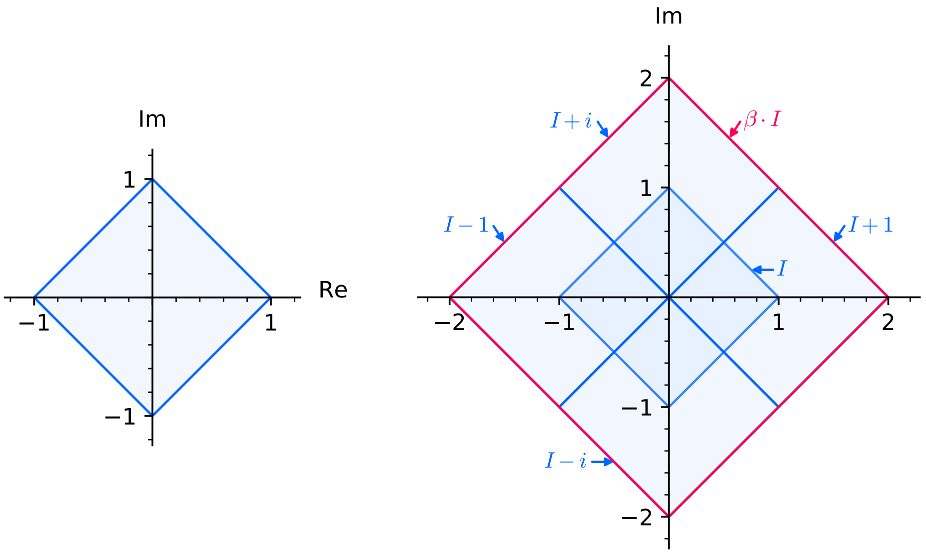

Figure 2.3: The set showing the sufficient condition from Theorem 1.9 for $\beta = 2$ and $A_4 = \{0, \pm 1, \pm i\}$.

conjugate are complete.

Case n = 4: If $n = 4$, then from Lemma 2.6 we know that $d = -1$. According to Theorem 1.12 the numeration system (β, A) with $\beta = se^{i\theta}$ where $\theta \in [0, 2\pi]$ can be complete only if

$$1 < s \leq 1 + 2\cos\left(\frac{\pi}{4}\right) = 1 + \sqrt{2}.$$

This gives the following candidates

$$\beta \in \{\pm 1 \pm i, \pm 2, \pm 2i, \pm 1 \pm 2i, \pm 2 \pm i\}.$$

Since $A_2 \subset A_4$, we have completeness for $\beta \in \{\pm 1 \pm i\}$.

The case where $\beta \in \{\pm 2, \pm 2i\}$ follows from [4] and Lemma 2.7. The set I which proves the sufficient condition from Theorem 1.9 can be seen in Figure 2.3.

Now we consider $\beta \in \{\pm 1 \pm 2i, \pm 2 \pm i\}$. By Lemma 2.7, it is sufficient to verify for only one of these values of β that the system is complete since A_4 is closed under multiplication by i and complex conjugation.

Let us focus on $\beta = +2 + i$ in particular. We will check the conditions of Theorem 2.2 for $\alpha = i$. All conjugates of β , in our case $2 \pm i$, satisfy $|\beta^{(j)}| = \sqrt{5} > 1$. The second condition

$$\mathbb{Z}[i] \subset A_4 + \beta \cdot \mathbb{Z}[i]$$

is verified in Figure 2.4. The last condition is to check whether every $x \in \mathbb{Z}[i]$ such that

$$|x^{(j)}| \leq \frac{1}{\sqrt{5} - 1} = 0.809016 \cdots$$

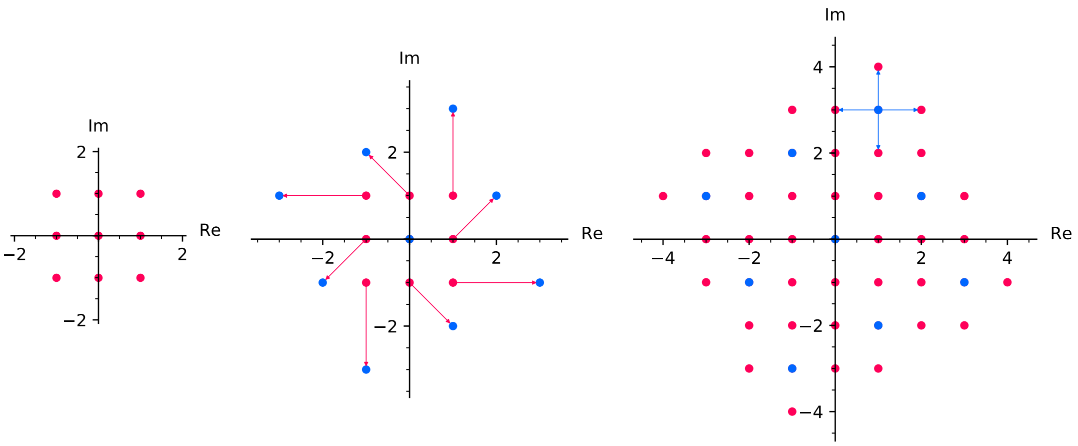

Figure 2.4: Verification of condition $\mathbb{Z}[i] \subset A_4 + \beta \cdot \mathbb{Z}[i]$ from Theorem 2.2 where $\beta = 2 + i$. Each row shows respectively a small part of the grid $\mathbb{Z}[i]$, the grid multiplied by β (the blue points) and all points obtained by adding a digit from A_4 (the red points).

has a (β, A_4)-representation. The only possible value of x is 0 which is in the alphabet. Therefore the numeration system $(2 + i, A_4)$ is complete. Hence all other corresponding numeration systems are complete as well.

Case d ≡ 1 mod 4

From Proposition 2.4 we know that the set of all algebraic integers of quadratic field $\mathbb{Q}(\sqrt{d}) = K$ is $O_K = \mathbb{Z}[\frac{1+\sqrt{d}}{2}] = \left\{\frac{a+b\sqrt{d}}{2} : a, b \in \mathbb{Z}, a \equiv b \bmod 2\right\}$. Since we consider the base of the form $\beta = \frac{a+b\sqrt{d}}{2}$, we have $|\beta|^2 = \frac{a^2-db^2}{4}$. In Lemma 2.6 it was proven that if $A_n \subset O_K$, the only possible values for n are $n \in \{1, 2, 3, 6\}$.

Case n = 1: The base β has to satisfy $1 < |\beta|^2 \leq \#A_1 = 2$. Let us check which values the parameter d can take. Since $d \equiv 1 \bmod 4$ and $d < 0$, then $d = -3, -7$ since these are the only two values for which it is possible that $|\beta|^2 \leq 2$.

There is no possible combination of coefficients a and b when $d = -3$. Therefore the only possibility is $d = -7$ and

$$\beta = \frac{\pm 1 \pm i\sqrt{7}}{2}.$$

By Proposition 2.1 the numeration systems (β, A_1) are complete.

Case n = 2: If $n = 2$, then we have $1 < |\beta|^2 \leq \#A_2 = 3$. Since $d \equiv 1 \bmod 4$ and $d < 0$, the possible values of d are $-3, -7, -11$ for the same reason as for $n = 1$. Then

$$\beta = \begin{cases} \pm i\sqrt{3}, \frac{\pm 3 \pm i\sqrt{3}}{2} & \text{for } d = -3, \\ \frac{\pm 1 \pm i\sqrt{7}}{2} & \text{for } d = -7, \\ \frac{\pm 1 \pm i\sqrt{11}}{2} & \text{for } d = -11. \end{cases}$$

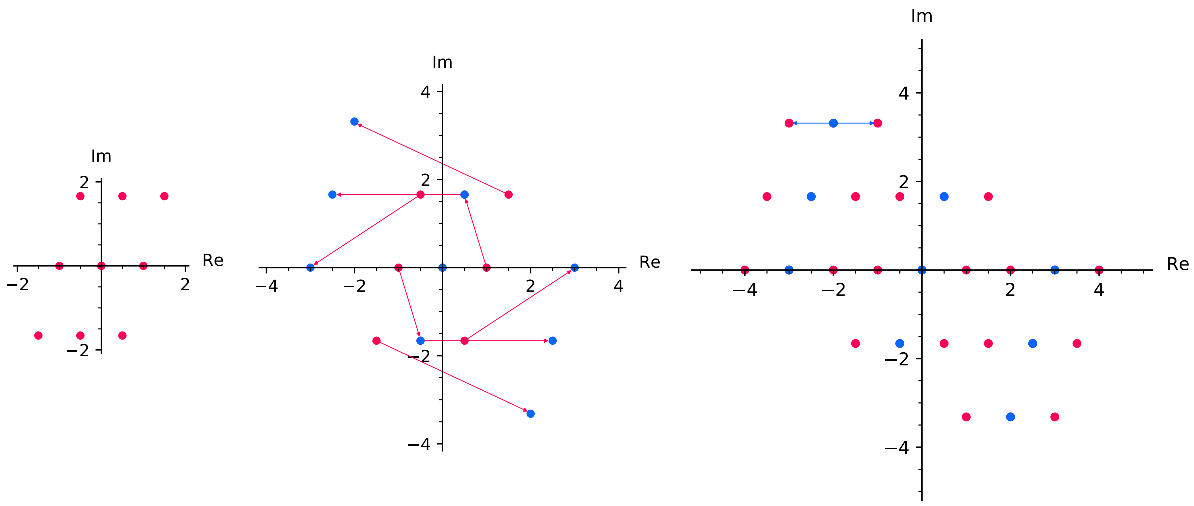

Figure 2.5: Verification of condition $\mathbb{Z}[\frac{1+i\sqrt{11}}{2}] \subset A_2 + \beta \cdot \mathbb{Z}[\frac{1+i\sqrt{11}}{2}]$ from Theorem 2.2 where $\beta = \frac{1+i\sqrt{11}}{2}$. Each row shows respectively a small part of the grid $\mathbb{Z}[\frac{1+i\sqrt{11}}{2}]$, the grid multiplied by β (the blue points) and all points obtained by adding a digit from A_2 (the red points).

For $\beta = \frac{\pm 1 \pm i\sqrt{7}}{2}$ the completeness follows from the previous case, since $A_2 \subset A_4$.

For $\beta = \frac{\pm 1 \pm i\sqrt{11}}{2}$ we use Theorem 2.2 with $\alpha = \frac{1+i\sqrt{11}}{2}$. The first condition is clearly fulfilled. In Figure 2.5 we show an illustration of the second condition

$$\mathbb{Z}\left[\frac{1+i\sqrt{11}}{2}\right] \subset A_2 + \beta \cdot \mathbb{Z}\left[\frac{1+i\sqrt{11}}{2}\right].$$

Finally we have to verify that all $x \in \mathbb{Z}\left[\frac{1+i\sqrt{11}}{2}\right]$ for which

$$|x^{(j)}| \leq \frac{1}{\sqrt{3}-1} = 1.366025\cdots$$

have a (β, A_2)-representation. The only possible values of x are $\{-1, 0, 1\}$ which are in the alphabet. Therefore $(\frac{1+i\sqrt{11}}{2}, A_2)$ is complete which implies completeness for all bases $\frac{\pm 1 \pm i\sqrt{11}}{2}$ according to Lemma 2.7.

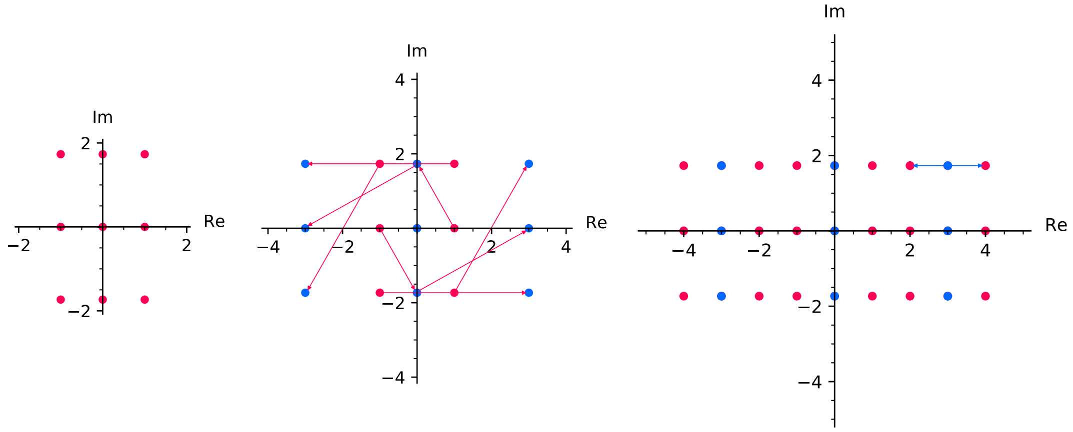

Figure 2.6: Verification of condition $\mathbb{Z}[i\sqrt{3}] \subset A_2 + i\sqrt{3} \cdot \mathbb{Z}[i\sqrt{3}]$ from Theorem 2.2. Each row shows respectively a small part of the grid $\mathbb{Z}[i\sqrt{3}]$, the grid multiplied by β (the blue points) and all points obtained by adding a digit from A_2 (the red points).

The case where $\beta = \pm i\sqrt{3}$ satisfies the conditions of Theorem 2.2 for $\alpha = i\sqrt{3}$. See Figure 2.6 for verification of the second part of the sufficient condition

$$\mathbb{Z}[i\sqrt{3}] \subset A_2 + i\sqrt{3} \cdot \mathbb{Z}[i\sqrt{3}].$$

Since the threshold for the third part is the same as in the previous case (the bases are of the same absolute value) the only values of x are $\{-1, 0, 1\} = A_2$. Using Lemma 2.7 the numeration systems with bases $\beta = \pm i\sqrt{3}$ and alphabet A_2 are complete.

When considering $\beta = \frac{+3+i\sqrt{3}}{2}$, the numeration system does not satisfy the sufficient condition from Theorem 2.2. In Figure 2.7 it can be seen that 0 is not in the interior of $W^{A_2}(\beta)$ and thus by Theorem 1.15, the numeration system for $\beta = \frac{+3+i\sqrt{3}}{2}$ is not complete. This also implies non-completeness for $\beta = \frac{\pm 3 \pm i\sqrt{3}}{2}$ by Lemma 2.7.

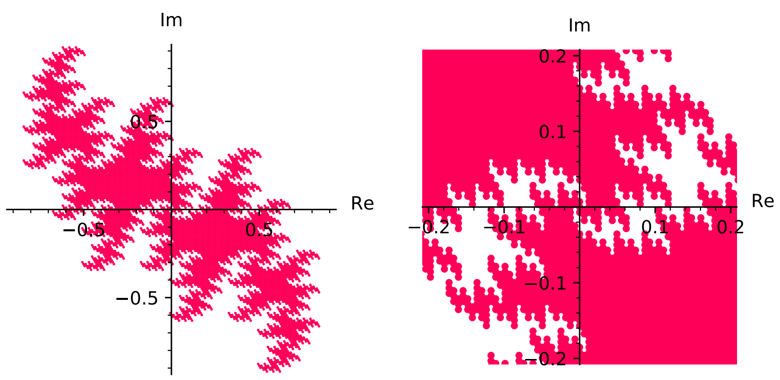

Figure 2.7: The set $W^{A_2}(\beta)$ for $\beta = \frac{+3+i\sqrt{3}}{2}$ and its close-up.

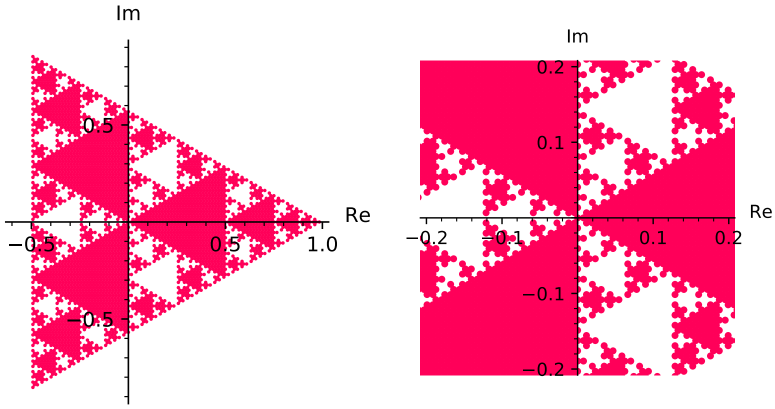

Figure 2.8: The set $W^{A_3}(\beta)$ for $\beta = 2$ and its close-up.

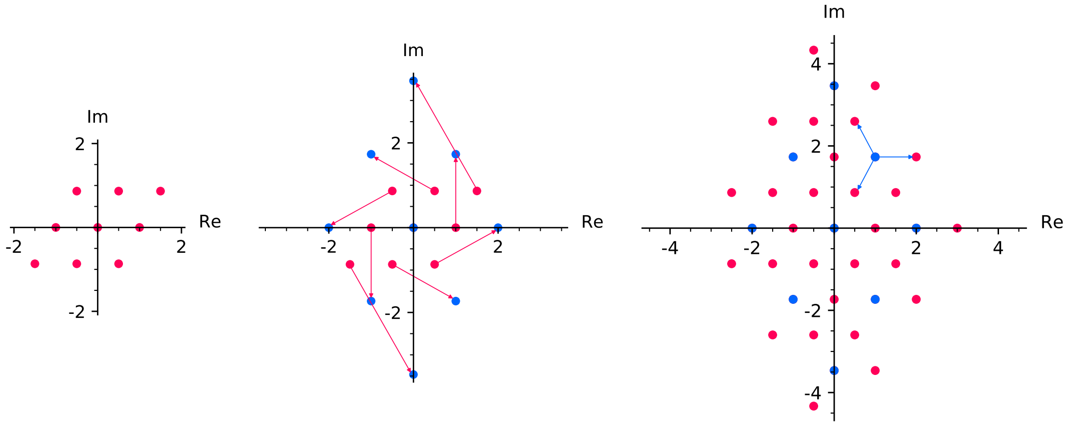

Figure 2.9: Verification of condition $\mathbb{Z}[\frac{1+i\sqrt{3}}{2}] \subset A_3 + \beta \cdot \mathbb{Z}[\frac{1+i\sqrt{3}}{2}]$ from Theorem 2.2 where $\beta = 1 + i\sqrt{3}$. Each row shows respectively a small part of the grid $\mathbb{Z}[\frac{1+i\sqrt{3}}{2}]$, the grid multiplied by β (the blue points) and all points obtained by adding a digit from A_3 (the red points).

Case n = 3: Using Lemma 2.6 we know that for $n = 3$ the parameter d has to be equal to -3. In order that the numeration system (β, A_3) be complete when considering the base of the form $\beta = se^{i\theta}$, from the necessary condition (1.1) we obtain $1 < s = \frac{\sqrt{a^2 - db^2}}{2} \leq \sqrt{\#A_3} = 2$. Thus

$$\beta \in \left\{ \pm 2, \pm i\sqrt{3}, \pm 1 \pm i\sqrt{3}, \frac{\pm 3 \pm i\sqrt{3}}{2} \right\}.$$

However in [16] it was proven that in the numeration system $(2, A_3)$, the point 0 is not in the interior of $W^{A_3}(2)$, see Figure 2.8. Thus by Theorem 1.15 the numeration system $(2, A_3)$ is not complete. Lemma 2.7 and the fact that A_3 is closed under multiplication by ρ^2 where $\rho = \frac{1+i\sqrt{3}}{2}$ implies that $(-1 \pm i\sqrt{3}, A_3)$ are not complete as well.

The numeration system $(-2, A_3)$ is complete if and only if numeration systems with bases $+1 \pm i\sqrt{3}$ are also complete according to Lemma 2.7. We verify that $(+1 + i\sqrt{3}, A_3)$ satisfies the sufficient condition

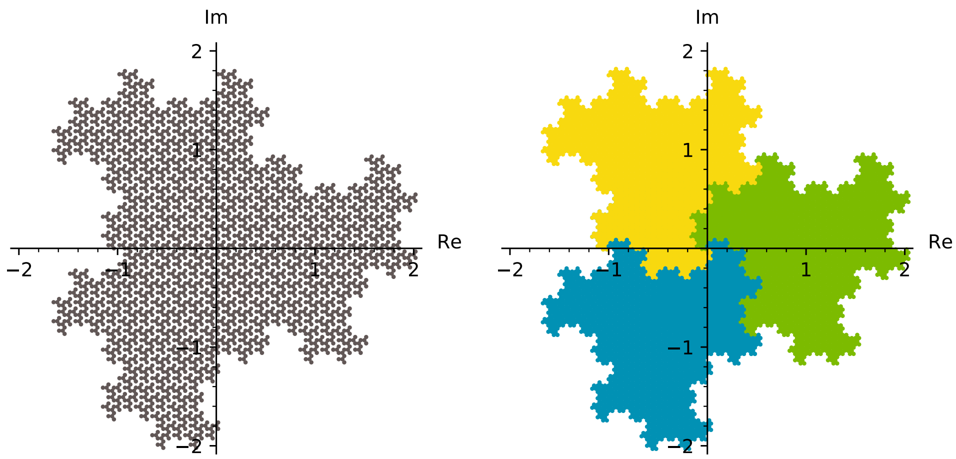

Figure 2.10: The set I ensuring the sufficient condition from Theorem 1.9 for $\beta = i\sqrt{3}$ and alphabet A_3. The left figure displays the set $\beta \cdot I$. On the right the sets $I+1, I+\rho^2$ and $I+\rho^4$ where $\rho = \frac{1+i\sqrt{3}}{3}$ can be seen. The sets I and $\beta \cdot I$ are covered.

from Theorem 2.2 for $\alpha = \frac{1+i\sqrt{3}}{2}$. The verification of the condition

$$\mathbb{Z}[\frac{1+i\sqrt{3}}{2}] \subset A_3 + \beta \cdot \mathbb{Z}[\frac{1+i\sqrt{3}}{2}]$$

can be seen in Figure 2.9. The last condition we have to verify is whether every $x \in \mathbb{Z}[\frac{1+i\sqrt{3}}{2}]$ such that

$$|x^{(j)}| \leq \frac{1}{2-1} = 1$$

has a (β, A_3)-representation. Among them, the only possible values of x are in $\{0, 1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$. Only ρ, ρ^3 and ρ^5 are not in the alphabet A_3. Nevertheless they have the following (β, A_3)-representations:

$$\begin{aligned} \rho &= \beta \cdot 1 + \rho^4, \\ \rho^3 &= \beta \cdot \rho^2 + 1, \\ \rho^5 &= \beta \cdot \rho^4 + \rho^2. \end{aligned}$$

The spectrum of the numeration system (β, A_3) can be seen in the appendix. Therefore the numeration system with base $1 + i\sqrt{3}$ and alphabet A_3 is complete along with the bases $-2, +1 - i\sqrt{3}$.

The last case is for $\beta \in \left\{\pm i\sqrt{3}, \frac{\pm 3 \pm i\sqrt{3}}{2}\right\}$. In Figure 2.10 the set ensuring the sufficient condition from Theorem 1.9 is rendered.

Case n = 6: By Lemma 2.6, it remains to treat the case when $n = 6$ and $d = -3$. By Theorem 1.12, if

$$1 < s = \frac{\sqrt{a^2 - db^2}}{2} \leq 1 + 2\cos\left(\frac{2\pi}{6}\right) = 2,$$

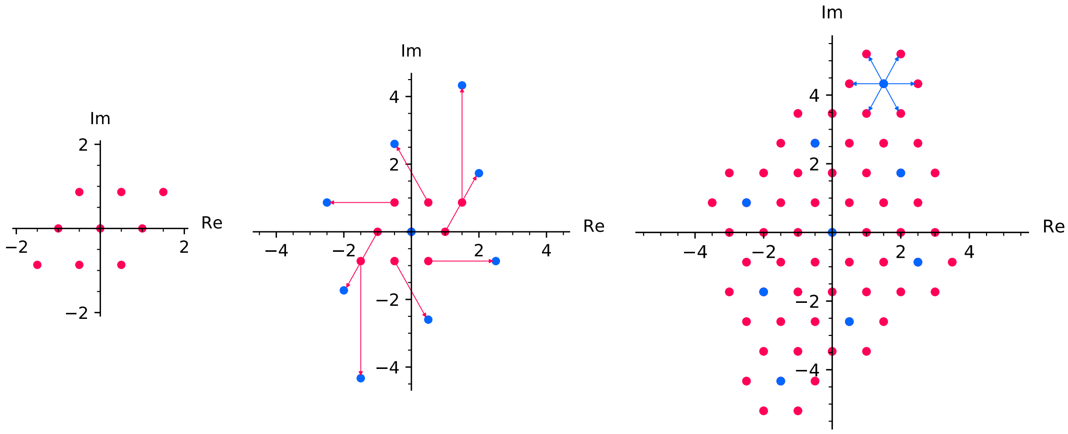

Figure 2.11: Verification of condition $\mathbb{Z}[\frac{1+i\sqrt{3}}{2}] \subset A_6 + \beta \cdot \mathbb{Z}[\frac{1+i\sqrt{3}}{2}]$ from Theorem 2.2 where $\beta = 2 + i\sqrt{3}$. Each row shows respectively a small part of the grid $\mathbb{Z}[\frac{1+i\sqrt{3}}{2}]$, the grid multiplied by β (the blue points) and all points obtained by adding a digit from A_6 (the red points).

then the numeration system is complete. This condition is satisfied for all bases

$$\beta \in \left\{ \pm 2, \pm 1 \pm i\sqrt{3}, \pm i\sqrt{3}, \frac{\pm 3 \pm i\sqrt{3}}{2} \right\}.$$

We obtain more candidates for β by considering the necessary condition from Theorem 1.12, i.e. bases $\beta = \frac{a+b\sqrt{d}}{2}$ such that

$$2 < s = \frac{\sqrt{a^2 - db^2}}{2} < 1 + 2\cos\left(\frac{\pi}{6}\right) = 1 + \sqrt{3}.$$

We obtain the following candidates

$$\beta \in \left\{ \pm 2 \pm i\sqrt{3}, \frac{\pm 5 \pm i\sqrt{3}}{2}, \frac{\pm 1 \pm i\,3\sqrt{3}}{2} \right\}.$$

In order to prove that these numeration systems are complete, we only have to prove this fact for one value of β. This will imply the completeness for all values of β using Lemma 2.7 and closure of A_6 to multiplication by $\rho = \frac{1+i\sqrt{3}}{2}$ and complex conjugation.

Let us verify that $\beta = +2 + i\sqrt{3}$ satisfies the conditions from Theorem 2.2 for $\alpha = \rho$. Figure 2.11 shows that the condition

$$\mathbb{Z}[\rho] \subset A_6 + \beta \cdot \mathbb{Z}[\rho]$$

is satisfied. Finally every $x \in \mathbb{Z}[\rho]$ such that $|x^{(j)}| \leq \frac{1}{\sqrt{7}-1} = 0.607625\cdots$ has to have a (β, A_6)-representation. This is trivial since the only $x \in \mathbb{Z}[\rho]$ with such property is 0. $\square$

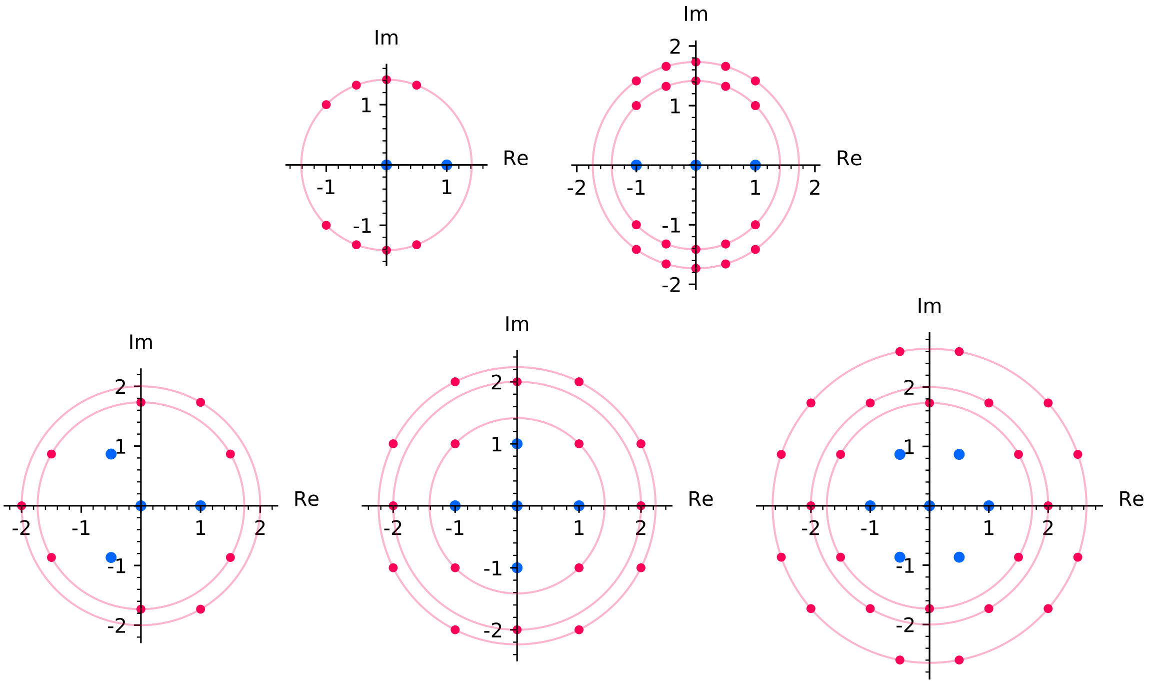

Figure 2.12: Alphabets A_1, A_2, A_3, A_4 and A_6, respectively, and corresponding bases β for which the numeration systems (β, A) are complete. The elements of the alphabet are blue, the bases are red.

2.4 Groups of similar numeration systems

In the proof of Theorem 2.5 we have used Lemma 2.7 which shows certain symmetry of numeration systems (β, A_n) and (β', A_n) when $\beta' = \bar{\beta}$ or $\beta' = \xi^j \beta$, $\xi = e^{\frac{2\pi i}{n}}$. Indeed the bases can be divided into groups which give similar numeration systems with the same alphabet A_n, see Figure 2.12. One example of a group of numeration systems is for alphabet $A_6 = \{0, 1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$ where $\rho = \frac{1+i\sqrt{3}}{2}$ and bases $\beta \in \{\pm 2, \pm 1 \pm i\sqrt{3}\}$. If we have a (β, A_6)-representation of a number x, we can always easily obtain representation of x in the numeration system considering another base from this group.

Consider $x = \sum_{j=-\infty}^{n} a_j 2^j$ where $a_j \in A_6$ is the $(2, A_6)$-representation of number x. The representation in base $\beta \in \{-2, \pm 1 \pm i\sqrt{3}\}$ of the form $x = \sum_{j=-\infty}^{n} b_j \beta^j$ is given by the rules in Table 2.3. It is clear that for every β from the group it holds that $\beta^6 = 64$. Therefore the formula for the digit b_j will be the same for position in the same class modulo 6.

$j \bmod 6$	$(2, A_6)$	$\to (2\rho, A_6)$	$\to (2\rho^2, A_6)$	$\to (-2, A_6)$	$\to (-2\rho, A_6)$	$\to (-2\rho^2, A_6)$
0	a_j	a_j	a_j	a_j	a_j	a_j
1	a_j	$a_j \cdot \rho^5$	$a_j \cdot \rho^4$	$a_j \cdot \rho^3$	$a_j \cdot \rho^2$	$a_j \cdot \rho$
2	a_j	$a_j \cdot \rho^4$	$a_j \cdot \rho^2$	a_j	$a_j \cdot \rho$	$a_j \cdot \rho^2$
3	a_j	$a_j \cdot \rho^3$	a_j	$a_j \cdot \rho^3$	a_j	$a_j \cdot \rho^3$
4	a_j	$a_j \cdot \rho^2$	$a_j \cdot \rho^4$	a_j	$a_j \cdot \rho^2$	$a_j \cdot \rho^4$
5	a_j	$a_j \cdot \rho$	$a_j \cdot \rho^2$	$a_j \cdot \rho^3$	$a_j \cdot \rho$	$a_j \cdot \rho^5$

Table 2.3: Transformation of $(2, A_6)$-representation into representations in (β, A_6) where $\beta \in \{-2, \pm 1 \pm i\sqrt{3}\}$ and $A_6 = \{0, 1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$ with $\rho = \frac{1+i\sqrt{3}}{2}$ for each position j.

These transformations are possible since the alphabet A_6 is closed under multiplication. Similar rules can be applied on every group of numeration systems with similar properties. In Table 2.2 one finds the following groups of numeration systems:

$$
\begin{array}{lll}
A_2 = \{0, \pm 1\} & \text{groups:} & \{\pm(1+i)\}, \{\pm(1-i)\}, \left\{\pm i\sqrt{2}\right\}, \left\{\pm(1+i\sqrt{2})\right\}, \\
 & & \left\{\pm(1-i\sqrt{2})\right\}, \left\{\pm i\sqrt{3}\right\}, \left\{\dfrac{\pm(1+i\sqrt{7})}{2}\right\}, \left\{\dfrac{\pm(1-i\sqrt{7})}{2}\right\}, \\
 & & \left\{\dfrac{\pm(1+i\sqrt{11})}{2}\right\}, \left\{\dfrac{\pm(1-i\sqrt{11})}{2}\right\}, \\
A_3 = \{0, 1, \rho^2, \rho^4\} & \text{groups:} & \left\{i\sqrt{3}, \dfrac{\pm 3 - i\sqrt{3}}{2}\right\}, \left\{-i\sqrt{3}, \dfrac{\pm 3 + i\sqrt{3}}{2}\right\}, \left\{-2, +1 \pm i\sqrt{3}\right\} \\
A_4 = \{0, \pm 1, \pm i\} & \text{groups:} & \{\pm 1 \pm i\}, \{\pm 2, \pm 2i\}, \{\pm(1+2i), \pm(2-i)\}, \{\pm(-1+2i), \pm(2+i)\}, \\
A_6 = \{0, 1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\} & \text{groups:} & \left\{\pm 2, \pm 1 \pm i\sqrt{3}\right\}, \left\{\pm i\sqrt{3}, \dfrac{\pm 3 \pm i\sqrt{3}}{2}\right\}, \\
 & & \left\{\dfrac{\pm(5+i\sqrt{3})}{2}, \dfrac{\pm(1+i3\sqrt{3})}{2}, \pm(2-i\sqrt{3})\right\}, \\
 & & \left\{\dfrac{\pm(5-i\sqrt{3})}{2}, \dfrac{\pm(1-i3\sqrt{3})}{2}, \pm(2+i\sqrt{3})\right\}.
\end{array}
$$

Note that every numeration system from Table 2.2 with alphabet A_1 is a group itself, since the size of the groups for alphabet A_k is equal to k. Moreover, using the closeness of A_n on complex conjugation, one can compactly rewrite these groups as

$$
\begin{array}{lll}
A_1 = \{0, 1\} & \text{groups:} & \{-1 \pm i\}, \left\{\pm i\sqrt{2}\right\}, \left\{\dfrac{1 \pm i\sqrt{7}}{2}\right\}, \left\{\dfrac{-1 \pm i\sqrt{7}}{2}\right\}, \\
A_2 = \{0, \pm 1\} & \text{groups:} & \{\pm 1 \pm i\}, \left\{\pm i\sqrt{2}\right\}, \left\{\pm 1 \pm i\sqrt{2}\right\}, \left\{\pm i\sqrt{3}\right\}, \\
 & & \left\{\dfrac{\pm 1 \pm i\sqrt{7}}{2}\right\}, \left\{\dfrac{\pm 1 \pm i\sqrt{11}}{2}\right\}, \\
A_3 = \{0, 1, \rho^2, \rho^4\} & \text{groups:} & \left\{\pm i\sqrt{3}, \dfrac{\pm 3 \pm i\sqrt{3}}{2}\right\}, \left\{-2, +1 \pm i\sqrt{3}\right\} \\
A_4 = \{0, \pm 1, \pm i\} & \text{groups:} & \{\pm 1 \pm i\}, \{\pm 2, \pm 2i\}, \{\pm 1 \pm 2i, \pm 2 \pm i\}, \\
A_6 = \{0, 1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\} & \text{groups:} & \left\{\pm 2, \pm 1 \pm i\sqrt{3}\right\}, \left\{\pm i\sqrt{3}, \dfrac{\pm 3 \pm i\sqrt{3}}{2}\right\}, \\
 & & \left\{\dfrac{\pm 5 \pm i\sqrt{3}}{2}, \dfrac{\pm 1 \pm i3\sqrt{3}}{2}, \pm 2 \pm i\sqrt{3}\right\}.
\end{array}
$$

Not only we can easily transform representations among numeration systems in a certain group, we will later show that other properties of these systems are the same, for example the parameters of algorithms for parallel addition, the set ensuring OL property or the results of preprocessing for on-line division.

The groups of bases for each alphabet A_1, A_2, A_3, A_4 and A_6 can be seen in Figure 2.12 where the bases are rendered in the complex plane. The circles highlight the absolute value of the bases since all the bases in one group are of the same size.

For each of the complete polygonal numeration systems we give an illustration of the spectrum $X^A(\beta)$ and the set of fractions $W^A(\beta)$ in the appendix.

In Chapter 5 we will discuss whether the numeration systems from Table 2.2 enable us to compute effective arithmetic algorithms including parallel addition and on-line multiplication and division.

Chapter 3

Parallel addition

In usual algorithm for addition performed by hand, the most significant digit of the sum may depend on the least significant digit of the input, as can be seen on the simple example

$$\begin{array}{r} 999\cdots 9 \\ +1 \\ \hline 1000\cdots 0 \end{array}$$

This is the case in numeration systems which are not redundant. On the other hand in redundant numeration systems one may design algorithm for addition in which the digit on position j of the result depends only on digits $j+t, \dots, j-r$ of the inputs for fixed r, t. The addition is then performed in constant time.

Let us formalize the above description of parallel addition and its properties according to [8]. First we introduce several definitions and conditions on feasibility of parallel addition in a given numeration system. A description of a general algorithm for parallel addition together with the so-called neighbour-free algorithm follows. At the end of this chapter, the Extending Window Method is explained introducing a possible approach in search for the algorithm for parallel addition [20],[22].

3.1 Local functions and parallel addition

When performing addition of numbers written in base β with digits in a given alphabet A, $x = \sum_{l\geq 0} x_l\beta^l$, $y = \sum_{j\geq 0} y_j\beta^j$ where $x_l, y_j \in A$, one aims to obtain the sum

$$z = x + y = \sum_{j\geq 0}(x_j + y_j)\beta^j$$

again in the form $z = \sum_{k\geq 0} z_k\beta^k$, $z_k \in A$. In fact, the task consists in transforming a string of digits over the alphabet $A + A$ into a string of digits over A, so that the value of the corresponding number remains unchanged. Such a transformation is called *digit set conversion*.

Definition 3.1. Let $\beta \in \mathbb{C}$ be a base, $|\beta| > 1$ and let A and B be two alphabets of complex numbers containing 0. A *digit set conversion* in base β from B to A is a function $\varphi : B^{\mathbb{Z}} \to A^{\mathbb{Z}}$ such that

1. for any $u = (u_j)_{j\in\mathbb{Z}} \in B^{\mathbb{Z}}$ with a finite number of non-zero digits, the image $v = \varphi(u) = (v_j)_{j\in\mathbb{Z}} \in A^{\mathbb{Z}}$ has only a finite number of non-zero digits as well,
2. $\sum_{j\in\mathbb{Z}} v_j\beta^j = \sum_{j\in\mathbb{Z}} u_j\beta^j$.

In order that the digit set conversion is performed in constant time, it should be computable using a local function.

Definition 3.2. A function $\varphi : B^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is said to be *p-local* if there exist two non-negative integers r and t satisfying $p = r+t+1$, and a function $\psi : B^p \to A$ such that for any $u = (u_j)_{j\in\mathbb{Z}} \in B^{\mathbb{Z}}$ and its image $v = \varphi(u) = (v_j)_{j\in\mathbb{Z}} \in A^{\mathbb{Z}}$, we have $v_j = \phi(u_j + t \cdots u_j - r)$ for every $j \in \mathbb{Z}$.

Definition 3.3. A digit set conversion φ in base $\beta \in \mathbb{C}, |\beta| > 1, \varphi : B^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is said to be *computable in parallel* if it is a p-local function for some $p \in \mathbb{N}$.

Sometimes, a numeration system does not allow parallel addition in the sense of Definition 3.3. The problem of addition in constant time can, however, be still solved by the so-called block parallel addition. The idea stems in dividing the (β, A)-representation of a number x into groups of digits, which represent x in base β^k. In [17] the notation

$$A_{(k)} = \left\{a_0 + a_1\beta + \cdots + a_{k-1}\beta^{k-1} : a_i \in A, i \in \{0, 1, \ldots, k-1\}\right\}$$

was introduced, where A is an alphabet and $k \in \mathbb{N}$. Obviously for $k = 1$ the set $A_{(1)}$ is equal to the original alphabet A.

Also, a digit set conversion in base β from B to A is said to be *block parallel computable* if there exists some $k \in \mathbb{N}$ such that the digit set conversion in base β^k from $B_{(k)}$ to $A_{(k)}$ is computable in parallel. When the specification of k is needed, we say *k-block parallel computable.*

In this terminology, the original parallel addition is 1-block parallel addition.

Example 3.4. Let $\beta = 2$ and let $A = \{-1, 0, 1\}$ be an alphabet. Then the alphabet $A_{(2)}$ is obtained in the following form

$$A_{(2)} = \left\{a_0 + a_1\beta : a_0, a_1 \in A\right\} = \{-3, \ldots, 0, \ldots, 3\}.$$

Definition 3.5. The algorithm for parallel addition is *neighbour-sensitive* if the decision how to compute the digit on position j depends on the digit at position $j-1$. Otherwise we say that the algorithm for parallel addition is *neighbour-free.*

In [11] a sufficient condition for existence of a parallel addition algorithm in base β was given.

Theorem 3.6 ([11])**.** *Let β be an algebraic number such that $|\beta| > 1$ and all its conjugates in modulus differ from* 1*. Then there exists an alphabet A of consecutive integers containing* 0 *such that addition on* $\mathrm{Fin}_A(\beta)$ *can be performed in parallel.*

Parallel addition is possible only in numeration systems with sufficiently large alphabet with respect to the base. An estimate on the necessary cardinality of the alphabet was formulated in [12] for numeration systems where the alphabet is a finite set of consecutive integers containing 0.

Theorem 3.7 ([12]). *Let β, with $|\beta| > 1$, be an algebraic integer of degree d with minimal polynomial*

$$f(x) = x^d - a_{d-1}x^{d-1} - a_{d-2}x^{d-2} - \cdots - a_1 x - a_0.$$

Let A be an alphabet of consecutive integers containing 0 and 1. If addition in $\mathrm{Fin}_A(\beta)$ is computable in parallel, then $\#A \geq |f(1)|$. If, moreover, β is a positive real number, $\beta > 1$, then $\#A \geq |f(1)| + 2$.

From the following proposition it can be seen that for some cases of β a positive number, $\beta > 1$, the lower bound on the size of the alphabet A is acquired.

Proposition 3.8 ([12]). *Let $\beta = \sqrt[k]{b}, b \in \mathbb{Z}, |b| \geq 2$ and $k \geq 1$ integer. Any alphabet A of consecutive integers containing 0 with cardinality $\#A = b + 1$ ensures that addition in $\mathrm{Fin}_A(\beta)$ is computable in parallel.*

In [21] the author treats more general numeration systems where the digits take values in the ring generated by the base β. This is useful for example when considering k-block parallel addition.

Theorem 3.9. *Let (β, A) be a numeration system such that $\beta \in \mathbb{C}, |\beta| > 1$ is an algebraic integer with minimal polynomial $f(x)$, and let A be an alphabet of complex digits such that $A[\beta] = \mathbb{Z}[\beta]$. If addition in $\mathrm{Fin}_A(\beta)$ is computable in parallel, then β is expanding and*

$$\#A \geq \max\{|f(0)|, |f(1)|\}.$$

Moreover, if β has a positive real conjugate, then

$$\#A \geq \max\{|f(0)|, |f(1)| + 2\}.$$

Let us show what the assumption of the theorem actually mean. We presume that

$$A[\beta] = \left\{ \sum_{j=0}^{n} x_j \beta^j : n \in \mathbb{N}, x_j \in A \right\} = \left\{ \sum_{j=0}^{n} x_j \beta^j : n \in \mathbb{N}, x_j \in \mathbb{Z} \right\} = \mathbb{Z}[\beta].$$

Hence the set of all numbers with a finite (β, A)-representation with only non-negative powers of β is equal to the same set when considering the whole $\mathbb{Z}$ as the alphabet.

The condition $A[\beta] = \mathbb{Z}[\beta]$ could be replaced by a slightly weaker condition of $A[\beta]$ being closed under addition. The stronger condition is used since it is required in applications using parallel addition, e.g. in on-line multiplication and division.

3.2 Algorithms for parallel addition

In this section we will describe how an algorithm for parallel addition works in general. Then a description of a neighbour-free parallel addition will be covered.

When we want to calculate addition of two numbers, the easiest part of the addition is to simply obtain a digit-wise sum of the two input (β, A)-representations which gives a number w of the form

$$w = w_{n'} \cdots w_1 w_0 \bullet w_{-1} \cdots w_{-m'}$$

which is a $(\beta, A + A)$ representation of the result. The desired form of the result of the addition is a (β, A) representation of w which we denote as

$$z = z_n \cdots z_1 z_0 \bullet z_{-1} \cdots z_{-m}$$

satisfying our requirements, i.e. $z_j \in A$ for every j and

$$\sum_{l=-m'}^{n'} w_l = \sum_{j=-m}^{n} z_j.$$

Notice that the indices n and m are not necessarily equal to the indices n' and m' because in general the representation of a number over a smaller alphabet results in a longer representation, i.e. $m \geq m'$ and $n \geq n'$.

The main part of the algorithm for parallel addition is therefore the conversion from the alphabet $A+A$ to the alphabet A. The conversion we are performing is based on a digit-wise addition with a convenient representation of zero in the base β which in this context is any polynomial $R(x) = b_s x^s + \cdots + b_1 + b_0$ where $b_j \in \mathbb{Z}[\beta]$ such that $R(\beta) = 0$. The polynomial $R(x)$ is also called a *rewriting rule*.

There are definitely multiple representations of zero which can be used. In Section 3.3 and 3.4 we show two different examples of representation of zero which can be used in the algorithm for parallel addition. Both of these examples have one thing in common which is crucial for the algorithm. Specifically the representation of zero needs to have the so-called *dominant coefficient* which is the greatest coefficient of R in modulus in order to be suitable for computation of the conversion.

Further we consider a general rewriting rule in the form

$$R(x) = b_k x^k + b_{k-1} x^{k-1} + \cdots + b_1 x + b_0 + b_{-1} x^{-1} + \cdots + b_{-h} x^{-h} \tag{3.1}$$

where b_0 is the dominant coefficient of the rewriting rule R, $b_j \in \mathbb{Z}[\beta]$. Since

$$\begin{aligned} R(\beta) &= 0 \\ &= \beta^j \cdot R(\beta) = b_k \beta^{j+k} + b_{k-1} \beta^{j+k-1} + \cdots + b_1 \beta^{j+1} + b_0 \beta^j + b_{-1} \beta^{j-1} + \cdots + b_{-h} \beta^{j-h} \end{aligned}$$

for every $j \in \mathbb{N}$, the rewriting rule R can be then written in its representation

$$b_k b_{k-1} \cdots b_1 b_0 b_{-1} \cdots \cdots b_{-h} \underbrace{00 \cdots 0}_{j-h} \bullet = (0)_\beta.$$

Moreover any rewriting rule can be also multiplied by any number from $\mathbb{C}$ and its value is still equal to 0. Therefore we can multiply all digits of the representation of zero by the so-called *weight coefficient* $q_j \in \mathbb{Z}[\beta]$ in order to obtain the following representation of zero

$$(q_j b_k)(q_j b_{k-1}) \cdots (q_j b_1)(q_j b_0)(q_j b_{-1}) \cdots (q_j b_{-h}) \underbrace{00 \cdots 0}_{j-h} \bullet = 0. \tag{3.2}$$

This rewriting rule in particular is used in the digit-wise conversion. The idea of the algorithm is the following. We have the rewriting rule R with parameters k, h which are the greatest and the smallest powers of β in $R(\beta)$. In order to compute the resulting digit z_j on position j we need to look at digits of w on positions

$$l \in \{j+h, j+h-1, \ldots, j-k+1, j-k\}.$$

For each of these positions we select the weight coefficient q_l and then compute the sum of w_j and the corresponding powers of selected representations of zero given by (3.2). The correctness of the conversion from $A + A$ to A is ensured by the selection of particular weight coefficients q_l which is crucial. Illustration of how this process works for position j can be seen in the following column notation.

w	$=$	$\cdots$	w_{j+k}	$\cdots$	w_{j+1}	w_j	w_{j-1}	$\cdots$	w_{j-h}	$\cdots$
$q_{j-k} \cdot 0$	$=$					$q_{j-k}b_k$	$q_{j-k}b_{k-1}$	$\cdots$	$\cdots$	$q_{j-k}b_{-h}$
	$\vdots$				$\iddots$	$\vdots$				$\iddots$
$q_{j-1} \cdot 0$	$=$			$q_{j-1}b_k$	$\cdots$	$q_{j-1}b_1$	$q_{j-1}b_0$	$\cdots$	$\cdots$	
$q_j \cdot 0$	$=$		q_jb_k	$\cdots$	q_jb_1	q_jb_0	q_jb_{-1}	$\cdots$	q_jb_{-h}	
$q_{j+1} \cdot 0$	$=$		$\cdots$	$\cdots$	$q_{j+1}b_0$	$q_{j+1}b_{-1}$	$\cdots$	$\iddots$		
	$\vdots$	$\iddots$				$\vdots$	$\iddots$			
$q_{j+h} \cdot 0$	$=$	$q_{j+h}b_k$	$\cdots$	$\cdots$	$\cdots$	$q_{j+h}b_{-h}$				
z	$=$	$\cdots$	z_{j+k}	$\cdots$	z_{j+1}	z_j	z_{j-1}	$\cdots$	z_{j-h}	$\cdots$

Such conversion of the digit on the position j causes a carry q_j onto the positions

$$l = j+k, \ldots, j+1, j-1, j-2, \ldots, j-h.$$

This process can be calculated for every position $j \in \mathbb{N}$ independently of calculation for other positions, therefore it can be computed in parallel. The resulting formula for the digit conversion from $A + A$ to A on the position j is

$$z_j = w_j + \sum_{l=-h}^{k} q_{j-l} b_l. \tag{3.3}$$

The presented conversion preserves value of w since only representations of zero are added. A formal proof of this fact follows.

$$
\begin{aligned}
z = \sum_{j=m-h}^{n+k} z_j\beta^j &= \sum_{j=m-h}^{n+k}\left(w_j + \sum_{l=-h}^{k} q_{j-l}b_l\right)\beta^j = \sum_{j=m-h}^{n+k} w_j\beta^j - \sum_{j=m-h}^{n+k}\sum_{l=-h}^{k} q_{j-l}b_l\beta^j \\
&= w - \sum_{j=m-h}^{n+k}\sum_{l=-h}^{k}\left(b_l\beta^l\right)\left(q_{j-l}\beta^{j-l}\right) = w - \sum_{s\in\mathbb{Z}}\underbrace{\left(\sum_{l=-h}^{k} b_l\beta^l\right)}_{=0} q_s\beta^s = w - 0 = w.
\end{aligned}
$$

The bracket is equal to 0 since β is a root of the polynomial $R(x)$ which determines our chosen rewriting rule.

The difficult part is to find for a given numeration system (β, A) the weight coefficients ensuring correctness of the algorithm. The choice of the weight coefficient q_j for a neighbour-free algorithm for parallel addition is dependent only on digit on position j, i.e. $q_j = q_j(w_j)$. Then the resulting weight function is quite simple. The case is much different for q_j depending also on digits w_l on other positions than j. The resulting weight function then can be considered as a look-up-table where based on the digits w_l around position j you can find corresponding weight coefficient q_j. This is one of the outputs of the algorithm for Extending Window Method discussed in Section 3.4.

3.3 Neighbour-free parallel addition

Given a base β, it may be possible to design several different algorithms for parallel addition with different alphabets of digits. One is of course interested in algorithms in numeration systems with small alphabets. However, the digit set conversion is then performed using p-local function with large p, or, one is even forced to use k-block algorithms with $k \geq 2$. On the other hand, most simple parallel algorithms are 1-block neighbour free, on the expense of taking a larger alphabet.

In the following we present a method from [11] for finding a neighbour-free algorithm for any algebraic base β without conjugates on the unit circle, the alphabet of digits being composed of consecutive integers. We will consider base β an algebraic number, $|\beta| > 1$, and search for an alphabet A so that the numeration system (β, A) allows a neighbour-free algorithm to be performed.

We define a property which will ensure that the algorithms used for parallel addition are neighbour-free.

Definition 3.10. Let β be a complex number, $|\beta| > 1$. We say that β satisfies *t-representation of zero property* where $t \in \mathbb{N}$ if there exist coefficients $b_k, b_{k-1}, \dots, b_1, b_0, b_{-1}, \dots, b_{-h}$ with $b_j \in \mathbb{Z}$ for some $k, h \in \mathbb{N}_0$ such that β is a root of the polynomial

$$T(x) = b_k x^k + b_{k-1}x^{k-1} + \cdots + b_1 x + b_0 + b_{-1}x^{-1} + \cdots + b_{-h}x^{-h} \tag{3.4}$$

and

$$b_0 > t \sum_{j\in\{-h,\ldots,k\}\setminus\{0\}} |b_j|.$$

We say that β satisfies the *strong representation of zero property* (SRZ) if $t \geq 2$ and β satisfies the *weak representation of zero property* (WRZ) if $t \geq 1$.

Notice that the coefficient b_0 has to satisfy $b_0 > 0$. If it does not satisfy this requirement, we simply take the polynomial $-T(x)$ instead of the polynomial $T(x)$.

Notation. We set $P = b_0$ and $S = \sum_{j\in\{-h,\ldots,k\}\setminus\{0\}} |b_j|$.

The proof of the following theorem can be found in [11].

Theorem 3.11. *Let β be a complex number, $|\beta| > 1$, and let A be a symmetric alphabet of the form $A = \{-a, \ldots, 0, \ldots, a\}$. If β satisfies SRZ, then there exists a neighbour-free algorithm which realizes addition in constant time in parallel in $\mathrm{Fin}_A(\beta)$ where $a = \left\lceil \frac{P-1}{2} \right\rceil + \left\lceil \frac{P-1}{2(P-2S)} \right\rceil S$.*

If β satisfies WRZ, then there exists a neighbour-free algorithm which realizes addition in constant time in parallel in $\mathrm{Fin}_A(\beta)$ where $a = \left\lceil \frac{P-1}{2} \right\rceil + S$.

We can find strong and weak polynomials of β using a constructive proof of the following proposition, again taken from [11]. We present its proof here, since it is a base for one of the programs we have implemented.

Proposition 3.12. *Let β be an algebraic number of degree d with algebraic conjugates $\beta_1, \beta_2, \ldots, \beta_d$ (including β itself). Let $|\beta_i| \neq 1$ for all $i \in \{1, 2, \ldots, d\}$ and $|\beta| > 1$. Then for any $t \geq 1$ there exists a polynomial*

$$Q(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots a_1 x + a_0 \in \mathbb{Z}[x]$$

and an index $i_0 \in \{1, \ldots, m\}$ such that

$$Q(\beta) = 0 \quad \textit{and} \quad |a_{i_0}| > t \sum_{\substack{i=0 \\ i\neq i_0}}^{m} |a_i|.$$

Proof. Let us denote the minimal polynomial of β as

$$F(x) = \prod_{i=1}^{d} (x - \beta_i) = x^d + f_{d-1} x^{d-1} + \cdots + f_1 x + f_0 \in \mathbb{Q}[x].$$

Let M be the companion matrix of the polynomial $F(x)$, i.e.

$$M = \begin{pmatrix} -f_{d-1} & 1 & 0 & 0 & \dots & 0 \\ -f_{d-2} & 0 & 1 & 0 & \dots & 0 \\ -f_{d-3} & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -f_1 & 0 & 0 & 0 & \dots & 1 \\ -f_0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \in \mathbb{Q}^{d\times d}.$$

We can easily prove that

$$\det(M - xI) = (-1)^d F(x),$$

where I denotes the unit matrix of the size $d \times d$. We can also notice that the numbers β_i are eigenvalues of M. We define for any $n \in \mathbb{N}$ the following polynomial:

$$F_n(x) = \prod_{i=1}^{d}(x - \beta_i^n) = f_d(n)x^d + f_{d-1}(n)x^{d-1} + \cdots + f_1(n)x + f_0(n).$$

As the matrix M^n has eigenvalues $\beta_1^n, \dots, \beta_d^n$, it can be also easily proven that

$$\det(M^n - xI) = (-1)^d \prod_{i=1}^{d}(x - \beta_i^n) = (-1)^d F_n(x).$$

Since $M \in \mathbb{Q}^{d\times d}$, the matrix M^n is also from $\mathbb{Q}^{d\times d}$ and therefore the determinant $\det(M^n - xI)$ is a polynomial with rational coefficients and consequently, for all $n \in \mathbb{N}$, the polynomial $F_n(x)$ has also rational coefficients. We can see that for every $n \in \mathbb{N}$ and for every $i \leq d$ the following equality stands:

$$f_i(n) = (-1)^{d-i} \sum_{\{j_1, j_2, \dots, j_i\} \in S_i} \beta_{j_1}^n \beta_{j_2}^n \cdots \beta_{j_i}^n, \tag{3.5}$$

where S_i denotes the set of all subsets of $\{1, 2, \dots, d\}$ with i elements.

Without loss of generality, we assume that $|\beta_1| \geq |\beta_2| \geq \cdots \geq |\beta_d|$ and denote j_0 the greatest index for which the algebraic conjugate $|\beta_{j_0}| > 1$, i.e. $j_0 = \max\{j : |\beta_j| > 1\}$. Our choice of the index j_0 ensures that

$$\left| \frac{\beta_{j_1}\beta_{j_2} \cdots \beta_{j_r}}{\beta_1\beta_2 \cdots \beta_{j_0}} \right| < 1$$

for every subset $\{j_1, j_2, \dots, j_r\} \subset \{1, 2, \dots, d\}$ such that $\{j_1, j_2, \dots, j_r\} \neq \{1, 2, \dots, j_0\}$. Therefore the following stands

$$\lim_{n\to\infty} \frac{\beta_{j_1}^n \beta_{j_2}^n \cdots \beta_{j_r}^n}{\beta_1^n \beta_2^n \cdots \beta_{j_0}^n} = 0.$$

Since the coefficients $f_i(n)$ of the polynomial $F_n(x)$ satisfy (3.5) for every $n \in \mathbb{N}$, we can deduce that the limit

$$\lim_{n\to\infty} \frac{f_i(n)}{\beta_1^n \beta_2^n \cdots \beta_{j_0}^n} = \begin{cases} 0 & \text{for every } i \in \{1,\ldots,d\}, i \neq j_0, \\ (-1)^{d-i} & \text{for } i = j_0. \end{cases}$$

This implies that for every $t \geq 1$ there exists some $n_0 = n_0(t) \in \mathbb{N}$ such that

$$|f_{j_0}(n_0)| > t \sum_{\substack{i=1 \\ i\neq j_0}}^{m} |f_i(n_0)|.$$

We can equivalently write

$$\frac{|f_{j_0}(n_0)|}{|\beta_1^{n_0}\beta_2^{n_0}\cdots\beta_{j_0}^{n_0}|} > t \sum_{\substack{i=1 \\ i\neq j_0}}^{m} \frac{|f_i(n_0)|}{|\beta_1^{n_0}\beta_2^{n_0}\cdots\beta_{j_0}^{n_0}|}.$$

Such $n_0 = n_0(t)$ exists, because the right side of the inequality has the limit 0 and the left side has the limit 1.

Therefore, to construct the polynomial $Q(x)$, we fix n_0 and denote K the least common multiple of the denominators of the rational numbers $f_1(n_0), \ldots, f_{d-1}(n_0), f_d(n_0)$. Then we get the polynomial $Q(x) = KF_{n_0}(x^{n_0})$ and the index $i_0 = n_0 j_0$. Thus the obtained polynomial has the required properties. □

Let us present one example of the construction given by the proof of Proposition 3.12.

Example 3.13. Let β be a root of the polynomial $f(x) = x^2 + 3x + 3$, i.e. the Eisenstein number. First we want to find a weak polynomial using the construction from Proposition 3.12. The companion matrix of the polynomial f is

$$M = \begin{pmatrix} -3 & 1 \\ -3 & 0 \end{pmatrix}.$$

Set $n = 2$ as the parameter of the construction. We compute the characteristic polynomial of the matrix M on the power of n:

$$\left|M^2 - xI\right| = \begin{vmatrix} 6-X & -3 \\ 9 & -3-X \end{vmatrix} = (6-X)(-3-X) + 27 = X^2 - 3X + 9$$

which already has a dominant coefficient $P = 9$ which satisfies the condition $P > 2S$ where $S = 4$. In other words, it already satisfies both weak and strong representation of zero property. The weak polynomial is

$$W(x) = x^4 - 3x^2 + 9$$

and according to Theorem 3.11 the alphabet enabling neighbour-free parallel addition is of the form $\mathcal{A} = \{-a, \ldots, 0, \ldots, a\}$ for $a = \left\lceil \frac{P-1}{2} \right\rceil + S = 8$. According to (3.4) the rewriting rule derived from $W(x)$ has the coefficients $b_0 = 9, b_1 = 0, b_2 = -3, b_3 = 0$ and $b_4 = 1$.

The algorithm for neighbour-free parallel addition uses the rewriting rule given by the polynomial $W(x)$, see [11]. Assume that $x = \sum_{j=m}^{n} x_j\beta^j, y = \sum_{j=m}^{n} y_j\beta^j$ are the inputs of the algorithm where $x_j, y_j \in A$.

After computation of the digit-wise addition we obtain

$$w = x + y = \sum_{j=m}^{n}(x_j + y_j)\beta^j = \sum_{j=m}^{n} w_j\beta^j$$

where $w_j \in A+A = \{-16, \dots, 0, \dots, 16\}$. In order to make the description of the algorithm more clear, we define an auxiliary alphabet $A' = \{-a', \dots, 0, \dots, a'\}$ where $a' = \left\lceil \frac{P-1}{2} \right\rceil = 4$. The conversion is performed in a cycle of the length $s = \left\lceil \frac{a}{P-S} \right\rceil$, in this case $s = 2$. Then for each position $j \in \{m, \dots, 0, \dots, n\}$ at the same time we perform the following steps written in the pseudocode:

For $l = 1, 2, \dots, s$ do:

set the weight coefficient $q_j \in \{-1, 0, 1\}$:

$$q_j = \begin{cases} 1 & w_j > a', \\ 0 & w_j \in A', \\ -1 & w_j < -a', \end{cases}$$

$z_j = w_j - 9q_j + 3q_{j-2} - q_{j-4}$,

set $w_j = z_j$.

One can check that if we run the conversion for our β and alphabet $A = \{-8, \dots, 8\}$, we start with $-16 \le w_j \le 16$, and thus the partial result z_j is after the first run of the cycle in the following alphabet:

$$z_j = \underbrace{w_j - 9q_j}_{\in\{-7,\dots,0,\dots,7\}} - \underbrace{(q_{j-4} - 3q_{j-2})}_{\in\{-4,\dots,0,\dots,4\}} \in \{-11, \dots, 0, \dots, 11\}.$$

For the second and final run we start with $-11 \le w_j \le 11$ and therefore we obtain the result for position j:

$$z_j = \underbrace{w_j - 9q_j}_{\in\{-4,\dots,0,\dots,4\}} - \underbrace{(q_{j-4} - 3q_{j-2})}_{\in\{-4,\dots,0,\dots,4\}} \in \{-8, \dots, 0, \dots, 8\} = A.$$

The output of the conversion is therefore $z = \sum_{j=m-h}^{n+k} z_j\beta^j$ where $z_j \in A$. The fact that the value is preserved ($w = z$) is proven for a general algorithm for parallel addition based on some rewriting rule at the end of Section 3.2.

For every position j the choice of the weight coefficient is dependent only on a digit w_j, i.e. $q_j = q_j(w_j)$. Thus for one run of the cycle is

$$z_j = z_j(q_j, q_{j-2}, q_{j-4}) = z_j(w_j, w_{j-2}, w_{j-4}).$$

The whole conversion is then p-local function where $p = s \cdot 4 + 1 = 9$.

For performing the algorithm for parallel addition using the strong representation of zero, we still can use the polynomial $W(x)$ since it already satisfies $P = 9 > 8 = 2 \cdot S$. According to Theorem 3.11 the symmetric integer alphabet enabling the algorithm for parallel addition is $A = \{-a, \dots, 0, \dots, a\}$ where $a = \left\lceil \frac{P-1}{2} \right\rceil + \left\lceil \frac{P-1}{2(P-2S)} \right\rceil S = 20$. The size of the alphabet A is then $\#A = 41$ which is much greater that the alphabet enabling parallel addition using weak representation of zero. On the other hand, the resulting the result for position j is obtained by one run of the algorithm, we do not have to compute the digit z_j in a cycle. More details about this algorithm can be found in [11].

In the previous example a very simple neighbour-free conversion function was shown. If we want to compute parallel addition with a smaller alphabet, we would have to change the dependency $q_j = q_j(w_j)$ and choose the weight coefficient based on multiple digits.

It is not possible to find a neighbour-free algorithm for a smaller alphabet than A using a different rewriting rule, since we obtained the representation of zero with the smallest P and S possible. In other iterations (for $n > 2$) of the construction from Proposition 3.12 these two parameters would only increase.

Example 3.14. Let us compute a 2-block algorithm for parallel addition for numeration system from Example 3.4 with base 2 and alphabet $A = \{-1, 0, 1\}$ which is essentially an algorithm for parallel addition with base $\beta^2 = 2^2 = 4$ and alphabet $A_{(2)} = \{-3, \dots, 0, \dots, 3\}$.

When we sum two inputs x and y we obtain the number $w = x + y = \sum_{j=-m}^{n} (x_j + y_j)\beta^j = \sum_{j=-m}^{n} w_j \beta^j$ where all digits $w_j \in \{-6, \dots, 0, \dots, 6\}$. Therefore to obtain the result z whose digits satisfy $z_j \in A_{(2)}$ we use the identity

$$\beta^2 - 4 = 0.$$

Depending on the digit w_j on position j we perform the following adjustments:

$$q_j = \begin{cases} 1 & w_j \in \{4, 5, 6\}, \\ 0 & w_j \in A_{(2)}, \\ -1 & w_j \in \{-6, -5, -4\}, \end{cases}$$
$$z_j = w_j - 4q_j + q_{j-1}.$$

This can be done for each position independently and the resulting digit on position j depends only on inputs on positions j and $j - 1$. The last step is to transform the digit $z_j \in A_{(2)}$ to a pair of digits from alphabet A. This operation is not unique since for example $\bar{1}1 = 0\bar{1}$.

3.4 Extending window method

In this section we will describe the main idea of a method to construct algorithms for parallel addition in a (β, A)-numeration system. The so-called Extending Window Method was introduced in [22], from now on abbreviated as EWM. This algorithm works not only for the ring generated by β but we can choose some algebraic number $\omega \in \mathbb{Q}(\beta)$. Then we consider the numeration system (β, A) such that $\beta \in \mathbb{Z}[\omega]$ is an algebraic integer and $A \subset \mathbb{Z}[\omega]$ is an input alphabet where $A \subsetneq B \subset A + A$. We select the alphabet B because the conversion from $A + A$ to A can sometimes be replaced by a conversion from a smaller alphabet B applied multiple times. In default we set $B = A + A$.

The EWM is a proposed approach to construct algorithms for digit set conversion in the base β from input alphabet B to alphabet A which can be performed in parallel. This method uses the simplest rewriting rule determined by the following polynomial

$$R(x) = x - \beta \in (\mathbb{Z}[\omega])[x]. \tag{3.6}$$

Notice that the parameters $k, h \in \mathbb{N}_0$ from (3.1) are set as $k = 1$ and $h = 0$ and thus carry from position $j \in \mathbb{N}$ is only to position $j + 1$.

The main issue of the construction of the algorithm for parallel addition based on the principle described in Section 3.2 is to find a suitable weight coefficients $q_j \in \mathbb{Z}[\omega]$ such that

$$z_j = w_j + q_{j-1} - q_j\beta \in A, \quad w_j \in B \quad \text{for all } j \geq 0$$

for any $w \in \mathrm{Fin}_B(\beta)$ with (β, B)-representation of the form $w = w_n \cdots w_1 w_0 \bullet$. This formulation was created by substituting for our rewriting rule in (3.3).

The aim of any parallel algorithm for addition is for the digit z_j of the result to be dependent only on a fixed number of digits of the inputs, i.e. $z_j = z_j(w_j, \ldots, w_{j-r})$ for some fixed $r \in \mathbb{N}$. Since we are using the rewriting rule (3.6), the digit z_j is dependent only on information from the right, i.e. we have $q_j = q_j(w_j, \ldots, w_{j-(r-1)})$. The goal is to find a weight coefficient $q_j \in \mathbb{Z}[\omega]$ satisfying

$$z_j = w_j + q_{j-1} - q_j\beta \in A. \tag{3.7}$$

Notice that the above formula was obtained simply by substitution of our rewriting rule (3.6) in the general formulation (3.3).

We introduce two definitions which will simplify the description of the Extending Window Method, [22].

Definition 3.15. Let (β, A) be a numeration system, let $B \subset \mathbb{Z}[\omega]$ be a digit set such that $A \subsetneq B \subset A + A$. Any finite set $F \subset \mathbb{Z}[\omega]$ containing 0 such that

$$B + F \subset A + \beta F$$

is called a *weight coefficients set* for the numeration system (β, A) and input digit set B.

The idea of EWM is to design a weight coefficient set $F \subset \mathbb{Z}[\omega]$ so that for any carry $q_{j-1} \in F$ and for any digit $w_j \in B$ it contains a weight coefficient $q_j \in F$ such that $z_j = w_j + q_{j-1} - q_j\beta$ is in the alphabet A.

Definition 3.16. Let $F \subset \mathbb{Z}[\omega]$ be a weight coefficients set for numeration system (β, A) and let $B \subset \mathbb{Z}[\omega]$ be an input digit set. Let $r \in \mathbb{N}$ and let $q : B^r \to F$ be a mapping such that $q(0, \ldots, 0) = 0$ and

$$w_j + q(w_{j-1}, \ldots, w_{j-r}) - \beta q(w_j, \ldots, w_{j-(r-1)}) \in A$$

for any $w_j, w_{j-1}, \dots, w_{j-r} \in B$. Such mapping q is called *weight function of length r* for (β, A)-numeration system and input digit set B.

We also define another mapping $\varphi : B^{r+1} \to A$ by the following formula

$$\varphi(w_j, \dots, w_{j-r}) = w_j + \underbrace{q(w_{j-1}, \dots, w_{j-r})}_{q_{j-1}} - \beta \underbrace{q(w_j, \dots, w_{j-(r-1)})}_{q_j} =: z_j \in A. \tag{3.8}$$

where $q : B^r \to F$ is the weight function. Using the function φ, we can directly calculate the digit set conversion from the input digit set B to the alphabet A in base β as a p-local function where $p = r + 1$.

Notice the requirement $q(0, \dots, 0) = 0$ in Definition 3.16. This ensures that if the inputs of the conversion (i.e. digits of w) are zero, the result of the conversion and of the weight function is also going to be zero. Therefore the construction as presented above satisfies both conditions of the definition of the digit set conversion in Definition 3.1.

3.4.1 Algorithm for EWM

The so-called *Extending Window Method* implementing the above described approach to search for an algorithm for parallel addition was introduced in [20],[22]. EWM is organized in two phases, both of them performable by several different methods. The first phase finds for a given numeration system (β, A) and given input digit set B some weight coefficients set $F \subset \mathbb{Z}[\omega]$ according to Definition 3.15. This set is not uniquely determined. For different methods for Phase 1 we can obtain different weight coefficients sets F.

The weight coefficients set F is then considered as input of the second phase. In this phase the expected length r is gradually incremented until the weight function $q : B^r \to F$ is completely defined for each $(w_j, \dots, w_{j-(r-1)}) \in B^r$.

Finally, the algorithm is successful if both phases are completed and the desired local conversion function is then completely determined by the function found in Phase 2. We use the weight function outputs q as the weight coefficients in the formula (3.8).

The algorithm for both phases are in detail described in [20] together with the description of their convergence.

3.4.2 Computer program

The algorithm for Extending Window Method was implemented in programming language SageMath, which is a computer algebra system written in Python, and described in detail in [20] and can be found in the following GitHub repository

`https://github.com/Legersky/ParallelAddition.git` .

We applied this algorithm on several numeration systems in Chapter 5.

From the previous section it can be seen that if the algorithm finds a weight function, then we obtain a working algorithm for parallel addition. However, it does not imply anything if the construction does not find a weight function. The algorithm for parallel addition may still exist even if the search was not successful.

Let us describe how the computer program is applied on a given numeration system (β, A). First we need to set the inputs of the program which can be done either directly in the file of the computer program, or in a Google spreadsheet where we can choose multiple numeration systems to apply the program on. The required inputs of the EWM are:

- minimal polynomial of ring generator ω (not necessary equal to β),
- embedding of the ring generator (the closest root of the minimal polynomial to this value is taken as the ring generator)
- alphabet $A \subset \mathbb{Z}[\omega]$,
- input alphabet $B \subset \mathbb{Z}[\omega]$ (optional, default input alphabet is set as $B = A + A$),
- base $\beta \in \mathbb{Z}[\omega]$ of the numeration system for which we search for an algorithm for parallel addition,
- parameter $k \in \mathbb{N}$ of the k-block parallel addition.

We can also choose whether we want to save some output files. For example we can save general information about the process, file containing the weight function, a local conversion function or a log file. We can also save images including the image of the alphabet and the input alphabet in the complex plane, step-by-step images of Phase 1, image of the weight coefficient set or step-by-step images of Phase 2.

A very important part of the input is to choose methods to compute each phase. More than one method can be chosen at the same time from several methods implemented. There are 5 recommended methods for both phases and several experimental methods which may also lead to obtaining the algorithm for parallel addition.

All the methods implemented are described in [20]. If there exists an algorithm for parallel addition it does not necessarily mean that the computer program will converge. Even the convergence of different methods for each phase often vary. Therefore it is necessary to run the program multiple times while selecting different methods for each phase.

Chapter 4

On-line arithmetics

On-line arithmetic is a mode of computation where operands and results are processed in a digit serial manner, starting with the most significant digit. We will focus on the algorithms for on-line multiplication and division in particular.

The algorithms for on-line multiplication and division were introduced by Trivedi and Ercegovac for computation in integer bases with symmetric integer alphabet [28]. In [9] a modification was presented for non-standard numeration systems for arbitrary base $\beta, |\beta| > 1$ (in general a complex number) and alphabet A (in general a finite set of complex numbers).

4.1 Algorithm for on-line multiplication

The algorithm for on-line multiplication in numeration system (β, A) has only one parameter, the delay $\delta \in \mathbb{N}$. We work with (β, A)-representations of:

- the number $X = \sum_{j=1}^{+\infty} x_j \beta^{-j}$,
- the number $Y = \sum_{j=1}^{+\infty} y_j \beta^{-j}$,
- their product $P = \sum_{j=1}^{+\infty} p_j \beta^{-j}$.

The inputs of the algorithm are two strings of arbitrary length

$$
\begin{aligned}
&0\bullet x_1 x_2 \cdots x_\delta x_{\delta+1} x_{\delta+2} \cdots && \text{with } x_j \in A \text{ and } x_1 = x_2 = \cdots = x_\delta = 0, \text{ and} \\
&0\bullet y_1 y_2 \cdots y_\delta y_{\delta+1} y_{\delta+2} \cdots && \text{with } y_j \in A \text{ and } y_1 = y_2 = \cdots = y_\delta = 0.
\end{aligned}
$$

The output of the algorithm is a string of arbitrary length $0\bullet p_1 p_2 p_3 \cdots$ corresponding to a (β, A)-representation of the product $P = X \cdot Y = \sum_{j=1}^{+\infty} p_j \beta^{-j}$. The settings of the algorithm ensure that the representation of P indeed starts only on the right of the fractional point.

The on-line multiplication is carried out in iterative steps. First set $W_0 = X_0 = Y_0 = p_0 = 0$. For $k \geq 1$, the k-th step of the iteration process computes:

$$W_k = \beta(W_{k-1} - p_{k-1}) + (x_k Y_{k-1} + y_k X_k), \quad \text{and} \quad p_k = \mathsf{Select}_M(W_k) \in A.$$

The selection function is chosen so to ensure correctness of the algorithm for on-line multiplication. The particular properties of the algorithm and its parameters can be found in [9].

4.2 Algorithm for on-line division

The algorithm for on-line division in (β, A) numeration system has two parameters:

- the delay $\delta \in \mathbb{N}$,
- the minimal value (in modulus) of the denominator $D_{min} > 0$.

We will work with (β, A)-representations of:

- the numerator $N = \sum_{j=1}^{+\infty} n_j \beta^{-j}$,
- the denominator $D = \sum_{j=1}^{+\infty} d_j \beta^{-j}$,
- their quotient $Q = \sum_{j=1}^{+\infty} q_j \beta^{-j}$.

Partial sums are denoted by $N_k = \sum_{j=1}^{k} n_j \beta^{-j}, D_k = \sum_{j=1}^{k} d_j \beta^{-j}$, and $Q_k = \sum_{j=1}^{k} q_j \beta^{-j}$.

The inputs of the algorithm are two (possibly infinite) strings representing the nominator N and the denominator D, namely

$$\begin{aligned} &0\bullet n_1 n_2 \cdots n_\delta n_{\delta+1} n_{\delta+2} \cdots \quad \text{where } n_j \in A \text{ and } n_1 = n_2 = \cdots = n_\delta = 0, \text{ and} \\ &0\bullet d_1 d_2 d_3 \cdots \quad \text{where } d_j \in A \text{ satisfying } |D_k| \geq D_{min} \text{ for all } k \in \mathbb{N}, k \geq 1. \end{aligned} \tag{4.1}$$

The output is an arbitrarily long string $0\bullet q_1 q_2 q_3 \cdots$ corresponding to a (β, A)-representation of the quotient

$$Q = \frac{N}{D} = \sum_{j=1}^{+\infty} q_j \beta^{-j}.$$

Note that the representation of Q starts behind the fractional point. This is ensured by the setting of the algorithm.

We realize the on-line division in iterative steps. In the beginning, we set $W_0 = q_0 = Q_0 = 0$. For $k \geq 1$, the step of the iteration proceeds by calculation of

$$W_k = \beta(W_{k-1} - q_{k-1} D_{k-1+\delta}) + (n_{k+\delta} - Q_{k-1} d_{k+\delta})\beta^{-\delta}. \tag{4.2}$$

The k-th digit q_k of the representation of the quotient $Q = \frac{N}{D}$ is evaluated by Select, a function of the values of the auxiliary variable W_k and the interim representation $D_{k+\delta}$, so that

$$q_k = \mathsf{Select}(W_k, D_{k+\delta}) \in A.$$

The selection function is chosen so to ensure correctness of the algorithm for on-line division. For more details about the setting of the function Select see [9].

4.3 OL Property

Both algorithms for on-line multiplication and division work for the numeration systems which satisfy the so-called OL property.

Definition 4.1. A numeration system (β, A) has the *OL Property* if there exists a bounded set I such that $0 \in I$ and

$$\beta \operatorname{cl}(I) \subset \bigcup_{a \in A} (\operatorname{int}(I) + a)$$

where $\operatorname{cl}(I)$ and $\operatorname{int}(I)$ are *closure* and *interior* of the set I, respectively.

We need to be careful about the field in which we construct the set I. If the numeration system is real, then $I \subset \mathbb{R}$ is an interval. Otherwise for complex numeration systems $I \subset \mathbb{C}$ is two-dimensional.

Notice the similarities in the definition of OL property and the property of numeration systems, namely completeness, cf. Theorem 1.9. It is easily seen that the OL property is a sufficient condition for the numeration system to be complete.

Whether the on-line division can be performed depends not only on the OL Property of the numeration system, but also on Condition (4.1) imposed on the string representing the denominator . This fact is stated in the following theorem.

Theorem 4.2 ([9])**.** *Assume that the OL Property is satisfied in the (β, A)-numeration system. Then on-line multiplication in (β, A) can be performed by the Trivedi-Ercegovac algorithm. Moreover let strings $0\bullet n_1 n_2 \cdots$ and $0\bullet d_1 d_2 \cdots$ satisfying condition* (4.1) *represent numbers N and D respectively. Then computing $\frac{N}{D}$ can be performed by the Trivedi-Ercegovac algorithm.*

In this section, we will discuss cases in which the OL Property is satisfied. In order to prove that the set ensuring OL property can not be found, a necessary condition on the size of the alphabet can be formulated using the volume of sets.

Proposition 4.3. *Let $\beta \in \mathbb{C}$ and let $A \subset \mathbb{C}$ be an alphabet. Assume that the numeration system (β, A) has OL property. If (β, A) is real, then*

$$\#A > |\beta|.$$

If (β, A) is complex, then

$$\#A > |\beta|^2.$$

Proof. Let I be the set ensuring OL property of the numeration system (β, A), i.e.

$$\beta \operatorname{cl}(I) \subset \bigcup_{a \in A} (\operatorname{int}(I) + a). \tag{4.3}$$

Let n be the dimension of the field we are in (for (β, A) real $n = 1$, otherwise $n = 2$). We want to cover something slightly larger than $\operatorname{cl}(I)$ by the set $\{\operatorname{int}(I) + a : a \in A\}$. Hence there exists some small

$\varepsilon \in \mathbb{R}, \varepsilon > 0$ which satisfies

$$\beta(I + \varepsilon) \subseteq \bigcup_{a \in A} (\mathrm{int}(I) + a).$$

and $\mathrm{cl}(I) \subsetneq I + \varepsilon$. If we compute volume of the closure of I multiplied by β

$$\mathrm{vol}(\beta\, \mathrm{cl}(I)) = |\beta|^n \mathrm{vol}(I) < \mathrm{vol}((1 + \varepsilon)\beta\, I) = (1 + \varepsilon)^n |\beta|^n \mathrm{vol}(I)$$

and compare it with volume of the right side of (4.3):

$$\mathrm{vol}\Big(\bigcup_{a \in A} (\mathrm{int}(I) + a)\Big) \leq \#A\, \mathrm{vol}(I)$$

we obtain the following inequality

$$|\beta|^n\, \mathrm{vol}(I) < (1 + \varepsilon)^n |\beta|^n \mathrm{vol}(I) \leq \#A\, \mathrm{vol}(I).$$

Therefore $\#A > |\beta|^n$. In the proof we have used that the volume of the set I is defined. We can set it to be the Lebesgues measure of the measurable set $\mathrm{cl}(I)$, resp. $\mathrm{int}(I)$. □

For a special case of numeration system with real base and integer alphabet, a sufficient condition for OL Property is given by the following proposition.

Proposition 4.4 ([9]). *Let β be a real number with $|\beta| > 1$ and $A = \{m, \ldots, 0, \ldots, M\} \subset \mathbb{Z}$. Let us assume that $m \leq 0 < M$ for $\beta > 1$, and $m \leq 0 \leq M$ for $\beta < -1$. If*

$$|\beta| < \#A = M - m + 1,$$

then division in the numeration system (β, A) is on-line performable by the Trivedi-Ercegovac algorithm.

For complex bases and integer alphabets, we cite only a general result for systems with a symmetric alphabet [9].

Theorem 4.5. *Let $\beta \in \mathbb{C} \backslash \mathbb{R}$ with $|\beta| > 1$ and let $A = \{-M, \ldots, M\} \subset \mathbb{Z}$ be an alphabet where $M \geq 1$. If*

$$\beta\overline{\beta} + |\beta + \overline{\beta}| < \#A = 2M + 1,$$

then the numeration system (β, A) has the OL Property.

Besides that, redundant Eisenstein system and redundant Penney system were treated in [9]. Corresponding sets proving OL property for each numeration system can be seen in Figure 4.1.

Proposition 4.6. *Let $\beta = -1 + \omega$ where $\omega = \exp \frac{2\pi i}{3}$ is the third root of unity and $A = \{0, \pm 1, \pm\omega, \pm\omega^2\}$ (redundant Eisenstein system). The (β, A)-numeration system satisfies the OL Property.*

Proposition 4.7. *Let $\beta = -1 + i$ and $A = \{0, \pm 1, \pm i\}$ (redundant Penney system). The (β, A)-numeration system satisfies the OL Property.*

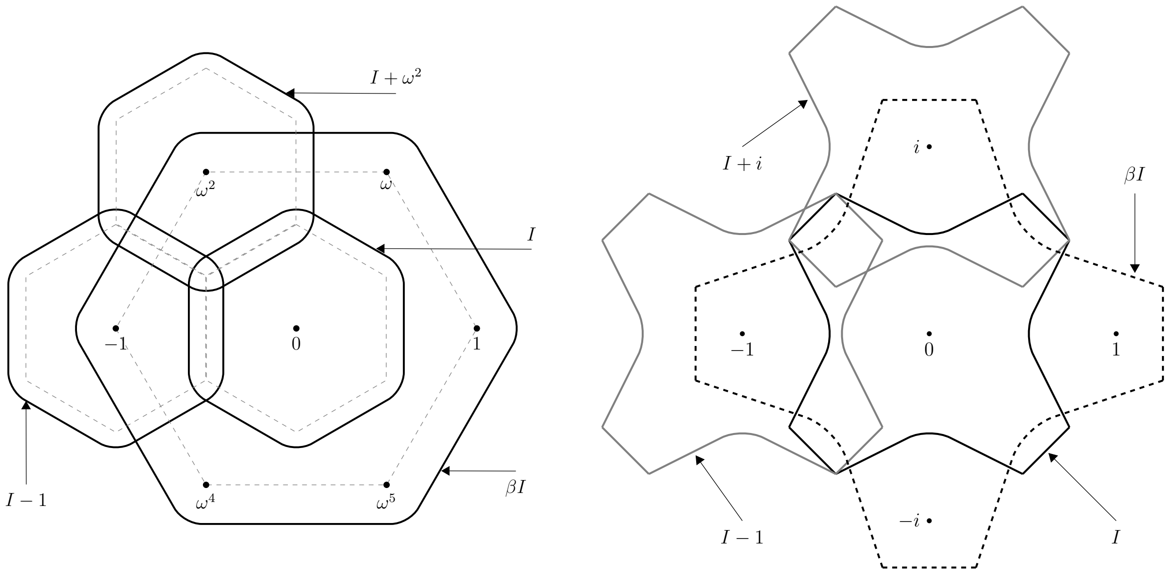

Figure 4.1: Sets I satisfying OL property for redundant Eisenstein system and redundant Penney system from article [9], respectively.

4.4 Preprocessing of divisors

When making division, we need that the divisor stays away from 0. By definition of the on-line algorithm, this means that the value of all the prefixes of the divisor $d_1 d_2 \cdots$ must be greater in absolute value than some $D_{min} > 0$. So the divisor must be preprocessed before making the division.

Example 4.8. Let us take $\beta = 2$ and the alphabet $A = \{\bar{1}, 0, 1\}$. Then 0 has two non-trivial representations in (β, A)-numeration system:

$$0 = 0\bullet 1\bar{1}\bar{1}\bar{1}\cdots = 0\bullet\bar{1}111\cdots .$$

This means that even if a divisor has a representation with infinitely many non-zero digits, its numerical value can be still 0, which we have to avoid.

So we consider only divisors not starting on $1\bar{1}\ldots$ because they can be rewritten to $01\ldots$. For the same reason we do not consider divisors of the type $\bar{1}1\ldots$. It can be shown that any divisor without prefix $0^k 1\bar{1}$ or $0^k\bar{1}1$ for $k \in \mathbb{N}$ has a non-zero value.

Definition 4.9. We say that a complex numeration system (β, A) *allows preprocessing* if there exists $D_{min} > 0$ and a finite list $\mathcal{L}$ of identities of the type $0\bullet w_k \cdots w_0 = 0\bullet 0u_{k-1}\cdots u_0$ with digits in A such that any string $d_1 d_2 \cdots$ on A without prefix $w_k \cdots w_0$ from $\mathcal{L}$ satisfies $|0\bullet d_1 d_2 \cdots d_j| > D_{min}$ for all $j \in \mathbb{N}$.

In order to have $d_1 \neq 0$ the preprocessing starts by shifting the fractional point to the most significant non-zero digit of the (β, A)-representation of the divisor. After such transformation the value of the original divisor v has been changed into a new divisor d which is just a shift of the original one, so $d = v\beta^k$ for some $k \in \mathbb{Z}$.

If zero has only the trivial (β, A)-representation, we can equivalently rewrite this fact as

$$\inf \mathcal{R} > 0, \quad \text{where } \mathcal{R} = \left\{ \left| \sum_{i\geq 1} z_i \beta^{-i} \right| : z_1 \neq 0, z_i \in A \right\}.$$

In this case the numeration system (β, A) allows preprocessing, since D_{min} is trivially equal to $\inf \mathcal{R}$ and the list of rewriting rules is empty.

The question of existence of a non-trivial (β, A)-representation of zero is quite complicated in general. The following proposition taken from [10] provides a necessary and sufficient condition in case of a real numeration system with integer alphabet.

Proposition 4.10 ([10])**.** *Let $\beta > 1$ be a real number and let $\{-1, 0, 1\} \subset A = \{m, \dots, 0, \dots, M\} \subset \mathbb{Z}$ be an alphabet. Zero has a non-trivial (β, A)-representation if and only if β satisfies*

$$\beta \leq \max\{M + 1, -m + 1\}. \tag{4.4}$$

Example 4.11. Let $\beta = \frac{3+\sqrt{5}}{2}$ and the alphabet $A = \{-1, 0, 1\}$. According to Proposition 4.10 the numeration system (β, A) does not have non-trivial representation of 0, since $\beta > 2$. Therefore the parameter $D_{min} = \inf \mathcal{R} = \frac{\beta - 1 + \min A}{\beta(\beta - 1)} = 0.145898\cdots$ and the list of rewriting rules is empty.

The following theorem links the property of the spectrum $X^A(\beta)$ of having an accumulation point to the preprocessing in on-line division in the (β, A)-numeration system. Since this theorem is directly linked to our research problem, we include it with its proof, which is based on several lemmas, see [10].

Theorem 4.12 ([10])**.** *Let β be a complex number and let $A \subset \mathbb{C}$ be an alphabet. The numeration system (β, A) allows preprocessing if and only if the spectrum $X^A(\beta)$ has no accumulation point.*

Notation. In the following three lemmas we will use notation

$$H = \max \left\{ \left| \sum_{i\geq 1} d_i \beta^{-i} \right| : d_i \in A \text{ for all } i \in \mathbb{N} \right\}. \tag{4.5}$$

Lemma 4.13. *If $X^A(\beta)$ has an accumulation point then the numeration system (β, A) does not allow preprocessing.*

Proof. In Theorem 3.5 of [10] it is proven that $X^A(\beta)$ has an accumulation point if and only if there exists $z_1 z_2 \cdots$ a β-representation of $z = 0$ in the alphabet A, i.e. $\sum_{i\geq 1} z_i \beta^{-i} = 0$ with $z_i \in A$, such that

$$0 \bullet z_1 z_2 \cdots z_j \neq 0 \bullet 0 z_2' \cdots z_j' \tag{4.6}$$

for all $j \geq 2$ and for all $z_2' \cdots z_j'$ in A^*. Assume that

$$0 = 0 \bullet z_1 z_2 z_3 \cdots , \tag{4.7}$$

is such a representation of 0. Also assume that preprocessing is possible with the value of parameter $D_{min} > 0$. First we find $j \in \mathbb{N}$ such that

$$\frac{H}{|\beta|^j} < D_{min}.$$

Then we consider the string $0 \bullet z_1 z_2 z_3 \cdots z_j 000 \cdots$. Due to (4.6) no prefix of the string $z_1 z_2 z_3 \cdots z_j$ is contained in the list of rewriting rules. But from (4.7) we know that

$$|0 \bullet z_1 z_2 z_3 \cdots z_j| = |0 \bullet \underbrace{000 \cdots 0}_{j\text{-times}} z_{j+1} z_{j+2} \cdots| < \frac{H}{|\beta|^j} < D_{min},$$

which is a contradiction. □

Lemma 4.14. *Let us assume that $X^A(\beta)$ has no accumulation point and fix $K > 0$. Then there exists $m \in \mathbb{N}$ such that any string $x_{m-1} x_{m-2} \cdots x_1 x_0$ of length m over A satisfies either*

$$|x_{m-1}\beta^{m-1} + x_{m-2}\beta^{m-2} + \cdots + x_1\beta + x_0| \geq K$$

or there exists a string $y_{k-1} y_{k-2} \cdots y_1 y_0$ of length $k < m$ over A such that

$$\begin{aligned} & x_{m-1}\beta^{m-1} + x_{m-2}\beta^{m-2} + \cdots + x_1\beta + x_0 \\ & = y_{k-1}\beta^{k-1} + y_{k-2}\beta^{k-2} + \cdots + y_1\beta + y_0. \end{aligned}$$

Proof. Since the spectrum $X^A(\beta)$ has no accumulation point, the set of points in $X^A(\beta)$ in absolute value smaller then K is finite, i.e. the set $S = \{z \in X^A(\beta) : |z| < K\}$ is finite.

We will denote for every $x \in X^A(\beta)$ the number

$$\rho(z) = \min \left\{ n \in \mathbb{N} : z = \sum_{j=0}^{n} z_j \beta^j \text{ where } z_j \in A \right\}$$

and $m = 2 + \max\{\rho(z) : z \in S\}$. Let us take some x with (β, A)-representation

$$x = x_{k-1}\beta^{k-1} + x_{k-2}\beta^{k-2} + \cdots + x_1\beta + x_0, \ x_j \in A.$$

Obviously, $x \in X^A(\beta)$. Then either $|x| \geq K$, which is the first case, or $x \in S$. In that case $k \leq \max\{\rho(z) : z \in S\} \leq m - 1$. □

Lemma 4.15. *If the spectrum $X^A(\beta)$ has no accumulation point, then there exists $D_{min} > 0$ and $m \in \mathbb{N}$ such that for all infinite strings $d_1 d_2 \cdots$ over A one has*

(i) either $|0 \bullet d_1 d_2 \cdots d_j| \geq D_{min}$ for all $j \in \mathbb{N}$,

(ii) or $0 \bullet d_1 d_2 \cdots d_m \neq 0 \bullet 0 d'_2 d'_3 \cdots d'_m$ for some string $d'_2 d'_3 \cdots d'_m \in A^*$.

Proof. Let $\varepsilon > 0$ and let us apply Lemma 4.14 where $K = H + \varepsilon$ to get the parameter $m \in \mathbb{N}$. We will denote the set

$$\mathcal{D} = \left\{|0\bullet d_1 d_2 \cdots d_j| : j < m \text{ and } 0\bullet d_1 d_2 \cdots d_j \neq 0\bullet 0 d'_2 \cdots d'_j\right\}$$

which does not contain 0. Since $X^A(\beta)$ has no accumulation point, the set $\mathcal{D}$ is finite. Thus $D' = \min \mathcal{D} > 0$.

We consider an infinite string $d_1 d_2 \cdots$ and assume that $0\bullet d_1 d_2 \cdots d_m \neq 0\bullet 0 d'_2 d'_3 \cdots d'_m$ for all $d'_2 d'_3 \cdots d'_m \in A^*$. There may be two cases:

- $j < m$ where $j \in \mathbb{N}$. Then $0\bullet d_1 d_2 \cdots d_j \neq 0\bullet 0 d'_2 \cdots d'_j$, otherwise it would mean that

  $$0\bullet d_1 d_2 \cdots d_m = 0\bullet 0 d'_2 d'_3 \cdots d'_j d_{j+1} d_m$$

  which is a contradiction. Hence $|0\bullet d_1 d_2 \cdots d_j| \geq D'$.

- $j \geq m$ where $j \in \mathbb{N}$. In this case

  $$\begin{aligned} |0\bullet d_1 d_2 \cdots d_j| &\geq |0\bullet d_1 d_2 \cdots d_m| - \frac{1}{|\beta|^m}|0\bullet d_{m+1} d_{m+2} \cdots d_j| \\ &\geq \frac{1}{|\beta|^m}K - \frac{1}{|\beta|^m}H = \frac{\varepsilon}{|\beta|^m}. \end{aligned}$$

Therefore, we can set the value

$$D_{min} = \min\left\{D', \frac{\varepsilon}{|\beta|^m}\right\} \tag{4.8}$$

and the theorem is hereby proved. □

Proof of Theorem 4.12. In case that the spectrum $X^A(\beta)$ has an accumulation point, Lemma 4.13 implies that the numeration system (β, A) does not allow preprocessing.

Assume that the spectrum does not have an accumulation point. Then Lemma 4.15 enables us to find a list $\mathcal{L}$ of identities for preprocessing. □

Using the constructive proof of Lemma 4.15 we can perform the algorithm for preprocessing for on-line division in the numeration system (β, A).

Every preprocessing consist of three steps:

1. Given a complex base β where $|\beta| > 1$ and an alphabet A, check the OL property, so that on-line division in the numeration system (β, A) is possible.

2. Check if the numeration system (β, A) has a non-trivial 0 representation.

3. If there exists a non-trivial 0 representation, find the minimal list of rewriting rules and the value of parameter D_{min}.

We have implemented the preprocessing algorithm in the SageMath programming language, which is a computer algebra system written in Python. The details of the implementation are described in our previous work [29].

4.5 Time complexity of on-line algorithms

In the algorithms for on-line multiplication and division, the inputs can be infinite strings. Therefore, by time complexity we understand the number of elementary operations needed to get n digits of the output, the product or quotient of given numbers, respectively.

The time complexity depends on the number of steps needed to compute the auxiliary value W_k and on the Select function. If both these operations can be performed in constant time, the time complexity of computing the first n digits of the result is $\mathcal{O}(n)$.

As we can see from (4.2), the time complexity of computation of W_k heavily depends on whether we can perform addition and subtraction fast. If we can compute addition in parallel, the time complexity of this operation is $\mathcal{O}(1)$.

Conditions on the numeration system $(\beta, \mathcal{A})$ so that it allows parallel addition were discussed in Chapter 3.

Chapter 5

Arithmetics in imaginary quadratic fields

In this chapter we apply all the previously developed theory to complete polygonal numeration systems of imaginary quadratic fields which are listed in Table 2.2. For each numeration system we will examine whether OL property is satisfied and what is the set ensuring this property. Then we check whether the necessary condition for parallel addition is fulfilled and we try to find an algorithm for parallel addition using a computer program implementing the Extending Window Method. If the numeration system has OL property, we also run our computer program in order to compute the preprocessing for on-line division and find the minimal list of rewriting rules and the corresponding parameter D_{min}.

When examining whether the numeration system satisfies OL property first we have to check if the necessary condition formulated in Proposition 4.3 stands. Table 5.1 contains a list of complete polygonal numeration systems of imaginary quadratic fields which satisfy the necessary condition on the size of the alphabet. We will consider only these numeration systems from now on.

The question whether addition is computable in parallel considering numeration systems in Table 5.1 which does not satisfy necessary condition for OL property is not completely answered. Each numeration system either does not satisfy a necessary condition for parallel addition from Theorem 3.9 as well or there was not found an algorithm using the Extending Window Method described in Section 3.4. This does not necessarily mean that there does not exist any algorithm for parallel addition using these numeration systems. The fact that they do not satisfy the necessary condition for OL property mean that the redundancy of these numeration systems is too small to compute on-line algorithms. We believe that parallel addition will not be possible either. Although even if the necessary condition is not satisfied, a k-block algorithm for parallel addition may exist. It remains an open question.

This chapter is organised as follows. Each section is dedicated to a particular quadratic field $\mathbb{Q}(\sqrt{d})$ for integer $d < 0$. Specifically we consider complete polygonal numeration systems of quadratic fields $\mathbb{Q}(\sqrt{d}) = K$ for $d = -1, -2, -3, -7$. The title of each section is the set of all algebraic integers O_K in the particular quadratic field which is given by Proposition 2.4 and is also stated in Table 5.1. These rings are very important for our numeration systems (β, A_n) since $\beta \in O_K$ and $A_n \subset O_K$.

Sections are further divided according to the numeration systems. We will consider only those numeration systems which either have OL property and therefore allows to compute on-line algorithms or allows to compute addition in parallel. This is verified by applying the computer program described

$\mathcal{O}_K$	β	A	$\lvert\beta\rvert^2 < \#A$
$\mathbb{Z}[i]$	$-1 \pm i$	$\{0, 1\}$	$\times$
	$\pm 1 \pm i$	$\{0, \pm 1\}$	$\checkmark$
		$\{0, \pm 1, \pm i\}$	$\checkmark$
	± 2	$\{0, \pm 1, \pm i\}$	$\checkmark$
	$\pm 2i$	$\{0, \pm 1, \pm i\}$	$\checkmark$
	$\pm 2 \pm i$	$\{0, \pm 1, \pm i\}$	$\times$
	$\pm 1 \pm 2i$	$\{0, \pm 1, \pm i\}$	$\times$
$\mathbb{Z}[i\sqrt{2}]$	$\pm i\sqrt{2}$	$\{0, \pm 1\}$	$\checkmark$
	$\pm 1 \pm i\sqrt{2}$	$\{0, \pm 1\}$	$\times$
$\mathbb{Z}[\frac{1+i\sqrt{3}}{2}]$	$\pm i\sqrt{3}$	$\{0, \pm 1\}$	$\times$
		$\{0, 1, \rho^2, \rho^4\}$	$\checkmark$
		$\{0, 1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$	$\checkmark$
	$\frac{\pm 3 \pm i\sqrt{3}}{2}$	$\{0, 1, \rho^2, \rho^4\}$	$\checkmark$
		$\{0, 1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$	$\checkmark$
	-2	$\{0, 1, \rho^2, \rho^4\}$	$\times$
	± 2	$\{0, 1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$	$\checkmark$
	$+1 \pm i\sqrt{3}$	$\{0, 1, \rho^2, \rho^4\}$	$\times$
	$\pm 1 \pm i\sqrt{3}$	$\{0, 1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$	$\checkmark$
	$\pm 2 \pm i\sqrt{3}$	$\{0, 1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$	$\times$
	$\frac{\pm 5 \pm i\sqrt{3}}{2}$	$\{0, 1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$	$\times$
	$\frac{\pm 1 \pm i3\sqrt{3}}{2}$	$\{0, 1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$	$\times$
$\mathbb{Z}[\frac{1+i\sqrt{7}}{2}]$	$\frac{\pm 1 \pm i\sqrt{7}}{2}$	$\{0, 1\}$	$\times$
		$\{0, \pm 1\}$	$\checkmark$
$\mathbb{Z}[\frac{1+i\sqrt{11}}{2}]$	$\frac{\pm 1 \pm i\sqrt{11}}{2}$	$\{0, \pm 1\}$	$\times$

Table 5.1: Numeration systems from Table 2.2 which satisfy necessary condition for OL property from Proposition 4.3. The first column lists the ring of all algebraic integers $\mathcal{O}_K$ containing particular polygonal numeration systems, i.e. $\beta \in \mathcal{O}_K$ and $A \subset \mathcal{O}_K$. The constant ρ is $\rho = \frac{1+i\sqrt{3}}{2}$.

in Section 3.4.2. The resulting weight functions for the algorithms for parallel addition found by the program are saved in `.csv` files which are available in the following GitHub repository:

`https://github.com/pajav7/weight_functions` .

Then we apply our computer program implementing preprocessing for on-line division, which was written as part of research assignment [29], to numeration systems with OL property. The implementation uses programming language SageMath as well and can be seen in the GibHub repository:

`https://github.com/pajav7/preprocessing.git` .

We use this program to compute the minimal list of rewriting rules and minimal parameter D_{min} but one can also use it to find greater and more appropriate parameter D_{min} and corresponding bigger list of identities.

The algorithm for parallel addition and results of preprocessing for on-line division are the same for the numeration systems which have the same alphabet and the bases are mutually complex conjugates since both are obtained considering the minimal polynomial and not the particular root of this polynomial. Therefore in every section we will show the results for bases with positive imaginary part.

5.1 Ring $\mathbb{Z}[i]$

Let us consider the quadratic field $\mathbb{Q}(\sqrt{-1})$ and the corresponding set of all algebraic integers $\mathbb{Z}[i]$. We describe 12 complete polygonal numeration systems in $\mathbb{Z}[i]$.

We will discuss the numeration systems (β, A) where β is $\pm 1 \pm i$ and $A_2 = \{0, \pm 1\}$ or $\beta \in \{\pm 1 \pm i, \pm 2, \pm 2i\}$ and the alphabet is $A_4 = \{0, \pm 1, \pm i\}$. The results of preprocessing for on-line division and the parameters of the algorithms for parallel addition are very similar for these numeration systems and can be seen in Tables 5.2 and 5.3, respectively.

5.1.1 Base $+1 \pm i$

First base we are going to discuss is $\beta = +1 + i$ with minimal polynomial $x^2 - 2x + 2$. The base $+1 - i$ is very similar since it has the same minimal polynomial. Therefore the algorithm for parallel addition and results of preprocessing for on-line division are the same. We consider two alphabets $A_2 = \{0, \pm 1\}$ and $A_4 = \{0, \pm 1, \pm i\}$.

In order to find an algorithm for parallel addition we applied the Extending Window Method and found a 2-block parallel addition algorithm for (β, A_2) and 1-block algorithm for numeration system (β, A_4). Other parameters of the algorithms can be seen in Table 5.3.

The set I ensuring OL property of the system (β, A_4) is shown for illustration in Figure 5.1. We did not find a set I for numeration system (β, A_2).

Although we have an algorithm for 2-block parallel addition even for smaller alphabet A_2, we are not sure if this numeration would satisfy OL property. Therefore we compute preprocessing for on-line division only with the larger alphabet A_4. Results of preprocessing are listed in Table 5.2 and the list of rewriting rules follows.

$$
\begin{array}{llll}
\mathcal{L}: & 1\bar{1} \to 0i & 1\bar{i} \to 01 & 10\bar{i} \to 00i \\
& 101 \to 01i & 10\bar{1} \to 0ii & 11\bar{i} \to 01i \\
& 1\bar{i}\bar{i} \to 0ii & 1001 \to 00ii & 100\bar{i} \to 00i\bar{1}
\end{array}
$$

Unlike the parameter D_{min}, H or number of identities, the particular rewriting rules are very dependent on the minimal polynomial of β and therefore they are slightly different for each numeration system in the given ring of all algebraic integers.

5.1.2 Base $-1 \pm i$ - Penney number

Let us consider the base with minimal polynomial of the form $x^2 - 2x + 2$, i.e. the Penney number and its complex conjugate. Let $\beta = -1 \pm i$. We use the same alphabets $A_2 = \{0, \pm 1\}$ and $A_4 = \{0, \pm 1, \pm i\}$ as in the previous section.

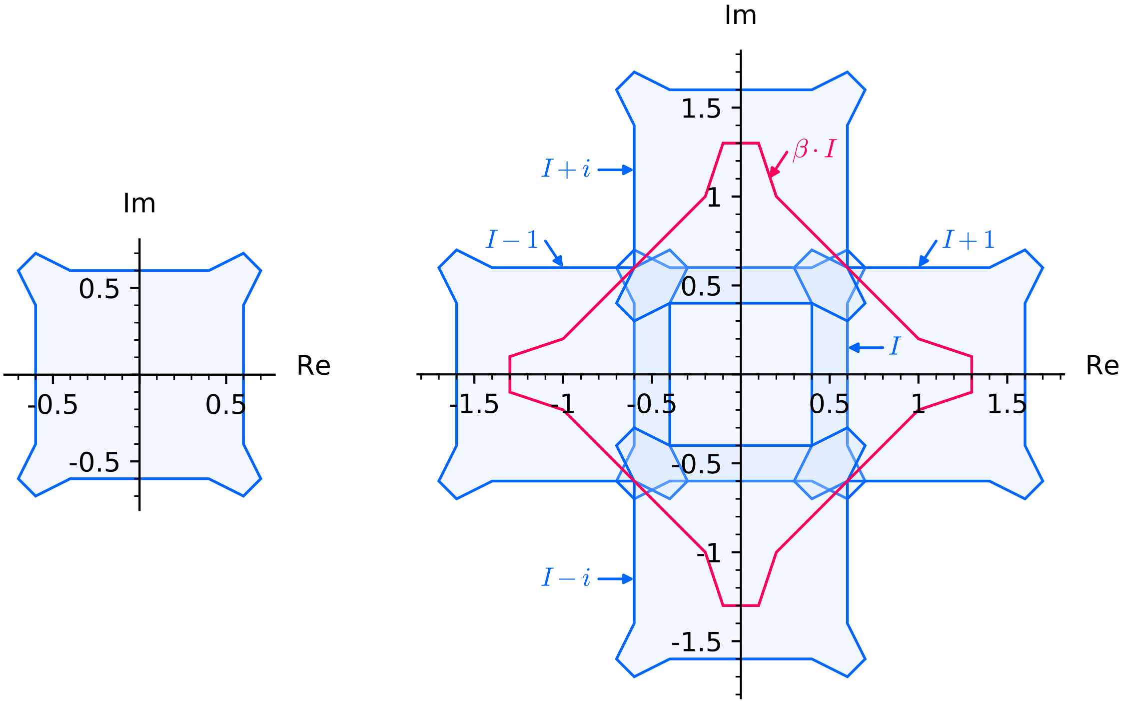

Figure 5.1: OL property for numeration system (β, A) with base $\beta = +1 + i$ and alphabet A_4.

The program implementing the EWM was used to find a 2-block and 1-block algorithms for parallel addition for numeration systems (β, A_2) and (β, A_4), respectively. Details of these algorithms may be seen in Table 5.3 and the weight functions are available in the GitHub repository. Note that these algorithms were already mentioned in [22].

The question whether the numeration system (β, A_4) satisfies OL property was already treated in [9]. Notice that we can use the same set I as we used in Figure 5.1 in order to prove that OL property is satisfied because the set is symmetrical to multiplying by -1 and i. We did not find a set ensuring OL property for the numeration system (β, A_2).

The results of preprocessing and the parameters of the algorithm for parallel addition can be seen in Tables 5.2 and 5.3. We can compare the list of rewriting rules with the results obtained in the previous section:

$$\begin{array}{llll} \mathcal{L}: & 11 \to 0i & 1\bar{i} \to 0\bar{1} & 10i \to 00\bar{i} \\ & 101 \to 0\bar{1}\bar{i} & 10\bar{1} \to 0i\bar{i} & 1\bar{1}i \to 0\bar{1}\bar{i} \\ & 1ii \to 0i\bar{i} & 100\bar{1} \to 00\bar{i}i & 100\bar{i} \to 00\bar{i}\bar{1} \end{array}$$

5.1.3 Base $\pm 2, \pm 2i$

The question whether OL property is satisfied is not yet answered for four numeration systems in $\mathbb{Z}[i]$, namely numeration systems with bases $\beta \in \{\pm 2, \pm 2i\}$ and alphabet $A_4 = \{0, \pm 1, \pm i\}$. Since the necessary condition

$$\#A_4 = 5 > |\beta|^2 = 4$$

is valid, it is possible that some set I ensuring OL property of these numeration systems exists. Therefore we did not perform preprocessing for on-line division since it is not clear whether necessary condition is satisfied. Also, the EWM did not find an algorithm for parallel addition for either numeration system.

5.1.4 Summary

In Tables 5.2 and 5.3 we provide the results of preprocessing for on-line division and parameters of algorithms for parallel addition for complete polygonal numeration systems in $\mathbb{Z}[i]$.

D_{min}	=	$0.148089\cdots$
$\#\mathcal{L}$	=	36
H	=	$2.236067\cdots$

Table 5.2: Results of preprocessing for polygonal numeration system (β, A_4) in $\mathbb{Z}[i]$, namely for bases $\beta = \pm 1 \pm i$.

A	$\{0, \pm 1\}$	$\{0, \pm 1, \pm i\}$
k-block	2	1
$p = 1 + r$	5	6
# LuT	60721	2165713

Table 5.3: Parameters of algorithms for parallel addition for complete polygonal numeration systems (β, A) with bases $\beta = \pm 1 \pm i$ and alphabets A_2 and A_4.

5.2 Ring $\mathbb{Z}[i\sqrt{2}]$

Let us consider the quadratic field $\mathbb{Q}(\sqrt{-2})$ and the corresponding ring of all algebraic integers $\mathbb{Z}[i\sqrt{2}]$. The only two numeration systems we are going to discuss are for $\beta = \pm i\sqrt{2}$ with minimal polynomial $x^2 + 2$ and alphabet $A_2 = \{0, \pm 1\}$.

The necessary condition for parallel addition from Theorem 3.9 is satisfied. The program implementing the Extending Window Method found a 1-block parallel addition algorithm. The properties of this algorithm can be seen in Table 5.4. Note that algorithm for parallel addition for the numeration systems $(\pm i\sqrt{2}, A_2)$ was already mentioned in [7]. In [22] the Extending Window Method was already used to find an algorithm for parallel addition for numeration system (β, A_2).

k-block	=	1
p	=	4
# LuT	=	289

Table 5.4: Parameters of algorithms for parallel addition for complete polygonal numeration systems (β, A_2) with bases $\beta = \pm i\sqrt{2}$.

OL property for the numeration system (β, A_2) is satisfied due to Theorem 4.5. The corresponding set I can be found in Figure 5.2. This figure shows the set I for $\beta = i\sqrt{2}$ but since I is symmetrical to complex conjugation, the set for $-i\sqrt{2}$ is the same.

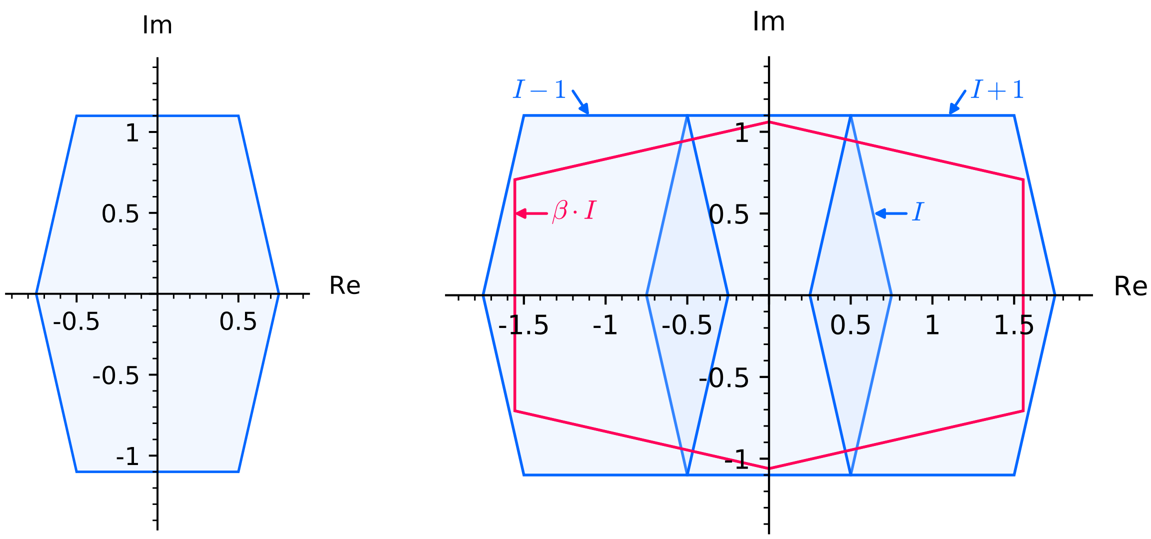

Figure 5.2: OL property for numeration system (β, A_2) with base $\beta = i\sqrt{2}$ and alphabet A_2.

The results of preprocessing for numeration system (β, A_2) and the corresponding list of rewriting rules $\mathcal{L}$ are the following.

D_{min}	$=$	$0.274094\cdots$
$\#\mathcal{L}$	$=$	6
H	$=$	$1.732050\cdots$

$$\mathcal{L}: \quad \begin{array}{l} 101 \to 00\bar{1} \\ 111 \to 01\bar{1} \\ 1\bar{1}1 \to 0\bar{1}\bar{1} \end{array}$$

Thus effective algorithms for parallel addition and on-line multiplication and division can be performed. Notice that the parameters of the algorithms are very convenient, namely both the weight function for parallel addition and the list of rewriting rules are very small especially considering how the minimal parameter D_{min} is large compared to other numeration systems treated in this chapter.

5.3 Ring $\mathbb{Z}[\frac{1+i\sqrt{3}}{2}]$

We will discuss the quadratic field $\mathbb{Q}(\sqrt{-3})$. The set of all algebraic integers is according to Proposition 2.4 equal to $\mathbb{Z}[\frac{1+i\sqrt{3}}{2}]$. We denote $\rho = \frac{1+i\sqrt{3}}{2}$. The complete polygonal numeration systems (β, A) in $\mathbb{Z}[\rho]$ which satisfy necessary condition for OL property are for base $\beta = \pm i\sqrt{3}$ and alphabet $A_3 = \{0, 1, \rho^2, \rho^4\}$ or for bases $\beta \in \left\{\pm i\sqrt{3}, \frac{\pm 3 \pm i\sqrt{3}}{2}, \pm 2, \pm(1 + i\sqrt{3})\right\}$ and alphabet $A_6 = \{0, 1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$.

Notice that the numeration systems with alphabet A_6 can be divided into two groups by the size of the base since

$$|\beta| = \begin{cases} \sqrt{3} & \text{for } \beta \in \left\{\pm i\sqrt{3}, \frac{\pm 3 \pm i\sqrt{3}}{2}\right\}, \\ 2 & \text{for } \beta \in \left\{\pm 2, \pm(1 + i\sqrt{3})\right\}. \end{cases}$$

This is an important fact since the results of preprocessing and the algorithm for parallel addition are the same for numeration systems in each group as can be seen in Tables 5.5, 5.6 and 5.7.

5.3.1 Base $\pm i\sqrt{3}$

Let us focus on two numeration systems with bases $\beta = \pm i\sqrt{3}$, whose minimal polynomial is of the form x^2+3. In [25] the author examined the possibility of expanding any complex number in the base $+i\sqrt{3}$ and alphabet $\{0, 1, \rho\}$ which is however not a polygonal system. We do not know much about the numeration system (β, A_3) with $A_3 = \{0, 1, \rho^2, \rho^4\}$ where $\rho = \frac{1+i\sqrt{3}}{2}$ since we did not find a set I proving that OL property is satisfied, nor did the Extending Window Method find an algorithm for parallel addition.

From now on we will consider the numeration system $(i\sqrt{3}, A_6)$ with alphabet A_6. An algorithm for 1-block parallel addition was found for this numeration system. The parameters of the algorithm can be seen in Table 5.5.

The set I ensuring that OL property was already found in [9], see Figure 5.3. Preprocessing for on-line division was performed using our computer program and the obtained results can be seen in Table 5.6. The corresponding list of identities follows. Only two rules are listed bellow since the other 10 rules can be obtained by simply multiplying by ρ, ρ^2 or -1.

$$\begin{aligned} \mathcal{L}: \qquad & 1\rho^3 \to 0\rho^2 \\ & 1\rho^5 \to 0\rho \end{aligned}$$

5.3.2 Base $\frac{+3 \pm i\sqrt{3}}{2}$

Let us focus on numeration system with base $\beta = \frac{+3 \pm i\sqrt{3}}{2}$ with minimal polynomial x^2+3x+3. We did not find a set ensuring OL property for the numeration systems (β, A_3) where $A_3 = \{0, 1, \rho^2, \rho^4\}$ where $\rho = \frac{1+i\sqrt{3}}{2}$, nor did the EWM find an algorithm for parallel addition. Hence we focus on numeration systems with alphabet $A_6 = \{0, 1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$.

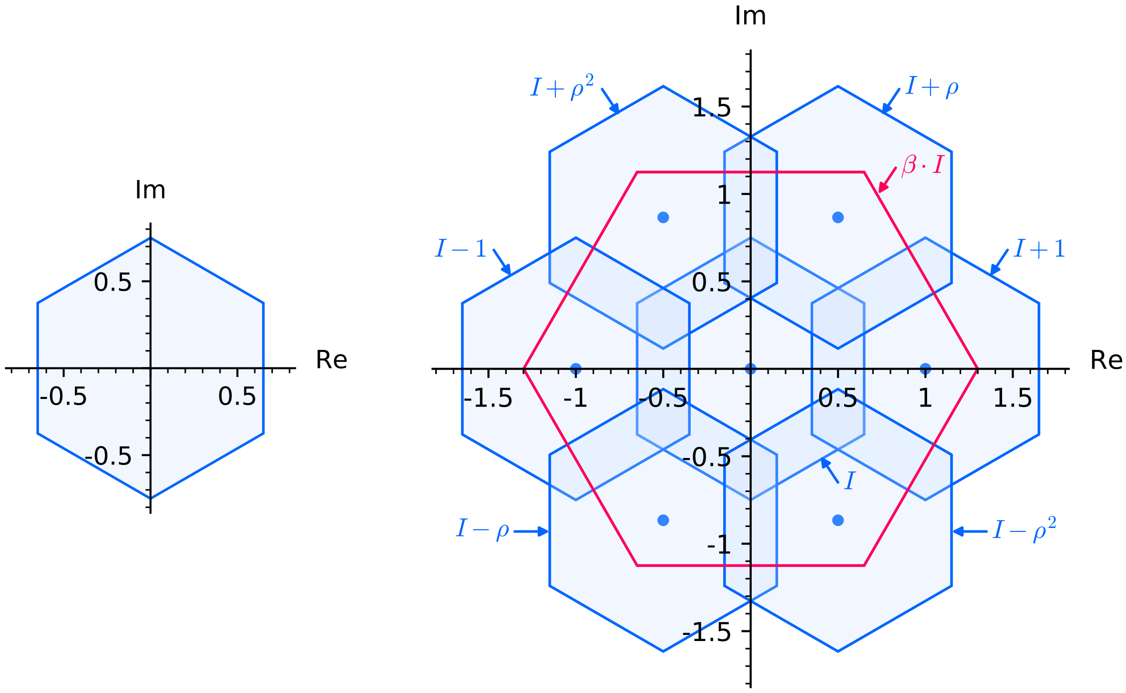

Figure 5.3: OL property for numeration system (β, A_6) with base $\beta = i\sqrt{3}$ and alphabet A_6.

This numeration system is very similar to the numeration system with base $\pm i\sqrt{3}$. The algorithm for parallel addition was found using the Extending Window Method. Its parameters can be seen in Table 5.5.

In order to prove that OL property is satisfied, we can use the same set I as in Figure 5.3 since

$$i\sqrt{3} \cdot I = \frac{+3 \pm i\sqrt{3}}{2} \cdot I$$

and the alphabet is the same for both numeration systems. Therefore we can use our program to compute preprocessing for on-line division. Results of the preprocessing are in Table 5.6 and the list of identities is the following.

$$\begin{aligned} \mathcal{L}: \qquad & 1\rho^3 \to 0\rho \\ & 1\rho^4 \to 01 \end{aligned}$$

The other 10 identities can be obtained by multiplying these rules by ρ, ρ^2 and -1 since the alphabet is closed under multiplication by these numbers.

5.3.3 Base $\frac{-3\pm i\sqrt{3}}{2}$ - Eisenstein number

The case for alphabet $A_3 = \{0, 1, \rho^2, \rho^4\}$ is the same as in the previous section for bases $\frac{3\pm i\sqrt{3}}{2}$. Let us discuss the so-called redundant Eisenstein system, i.e. numeration system with the base $\beta = \frac{-3\pm i\sqrt{3}}{2}$ with minimal polynomial $x^2 + 3x + 3$ and the alphabet $A_6 = \{0, 1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$ where $\rho = \frac{1+i\sqrt{3}}{2}$.

The EWM found a 1-block algorithm for parallel addition. Details of this algorithm are in Table 5.5. Note that the EWM method was already used in [22] to find an algorithm for parallel addition for this numeration system. According to Proposition 4.6 the redundant Eisenstein system satisfies OL property, see [9]. It was proven using a set I from Figure 4.1 which is similar to the set proving the OL property for bases $\pm i\sqrt{3}$ and $\frac{+3\pm i\sqrt{3}}{2}$ in Figure 5.3. The results of preprocessing are listed in Table 5.6 and the list of rewriting rules considering the symmetry of the alphabet A_6 follows.

$$\begin{aligned} \mathcal{L}: \qquad 11 &\to 0\rho^2 \\ 1\rho^5 &\to 0\bar{1} \end{aligned}$$

The previous six numeration systems we discussed with bases $\pm i\sqrt{3}$, $\frac{+3\pm i\sqrt{3}}{2}$ and $\frac{-3\pm i\sqrt{3}}{2}$ and alphabet A_6 are very similar when you compare the results for parallel addition and preprocessing for on-line division. The only difference among them are the rewriting rules for preprocessing since their minimal polynomial is different. But symmetry can be seen even for the rewriting rules.

5.3.4 Base +2

Let us discuss the second group of numeration systems in $\mathbb{Z}[\frac{1+i\sqrt{3}}{2}]$ where the base is either ± 2 or $\pm(1 + i\sqrt{3})$ and the alphabet is $A_6 = \{0, 1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$ where $\rho = \frac{1+i\sqrt{3}}{2}$. First let $\beta = +2$.

The necessary condition for parallel addition from Theorem 3.9 is satisfied but the algorithm itself was not yet found using the Extending Window Method even though we search up to the $k = 5$ for k-block algorithm for parallel addition. In [4] it was proven that addition is computable in parallel in the numeration system (β, A_6). The proof of existence of the algorithm uses a possibility of representation of any number in the spectrum $X^{A_6}(\beta)$ by an integer alphabet.

The numeration system (β, A_6) has OL property as can be seen in Figure 5.4. Thus we can compute preprocessing for on-line division. The results of preprocessing are in Table 5.9 and the list of rewriting rules (considering symmetry of the alphabet A_6) is the following.

$$\mathcal{L}: \qquad 1\rho^3 \to 01$$

5.3.5 Base −2

Let the base be $\beta = -2$. We did not find the set I ensuring OL property for the numeration system with alphabet $A_3 = \{0, 1, \rho^2, \rho^4\}$ where $\rho = \frac{+1+i\sqrt{3}}{2}$ and the Extending Window Method did not find an algorithm for parallel addition. Therefore we focus on the alphabet $A_6 = \left\{0, 1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\right\}$.

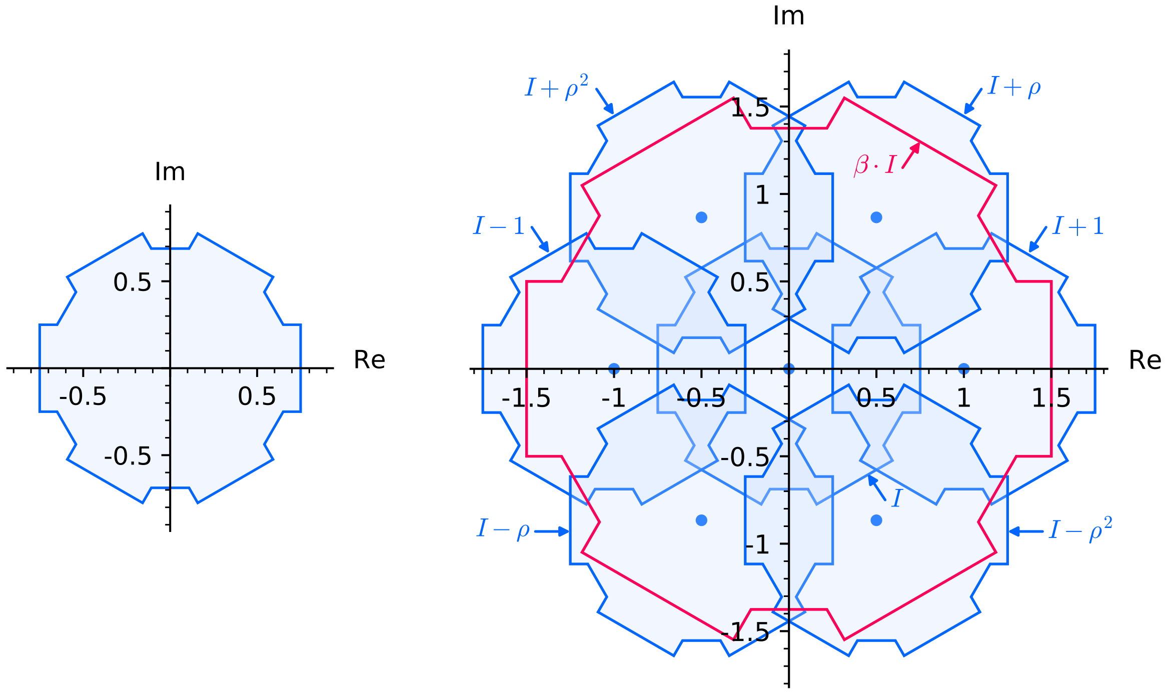

Figure 5.4: OL property for the numeration system with base $\beta = 2$ and alphabet A_6.

The necessary condition for parallel addition is satisfied but the algorithm itself was not found using the Extending Window Method. However we know that the algorithm for parallel addition exists since we can use the similarity with numeration system $(2, A_6)$ and the transformation rules from Table 2.3.

The idea of the algorithm follows. We take two $(-2, A_6)$-representation of the inputs of the algorithm and transform them both into $(2, A_6)$-representation. Then we compute the algorithm for parallel addition which we know exists from the previous section. The last step is to transform the result back to $(-2, A_6)$ numeration system. Since both the transformations are dependent only on particular digit, the resulting algorithm for parallel addition with additional steps is still computable in parallel. There is one detail to keep in mind and it is that we have to know on which position the digit is in order to transform it into representation in different numeration system.

OL property is also satisfied for the same set I as in Figure 5.4 since the set is symmetrical to multiplying by -1.

We can perform preprocessing for on-line division whose results are shown in Table 5.7. The list of rewriting rules while considering that the alphabet is symmetric to multiplying by ρ, ρ^2 and -1 is the following.

$$\mathcal{L} : \qquad 11 \to 0\rho^3$$

5.3.6 Base $+1 \pm i\sqrt{3}$

Let $\beta = +1 + i\sqrt{3}$ with minimal polynomial of the form $x^2 - 2x + 4$. The case for alphabet $A_3 = \{0, 1, \rho^2, \rho^4\}$ where $\rho = \frac{1+i\sqrt{3}}{2}$ is similar to the previous section for base -2. Let $A_6 = \left\{0, 1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\right\}$ be the alphabet. In order to prove that OL property is satisfied for the numeration system (β, A_6) we can use the set I from Figure 5.4 since

$$2 \cdot I = (1 + i\sqrt{3}) \cdot I$$

and both these numeration systems have the same alphabet A_6. The results of the preprocessing are similar to results of the numeration systems with bases ± 2, see Table 5.7. The list containing rewriting rules can be summarized by the following due to symmetry of the alphabet A_6.

$$\mathcal{L}: \qquad 1\rho^4 \to 0\rho$$

The Extending Window Method did not find an algorithm for parallel addition. Since there exists the algorithm for the base 2 and these numeration systems are very similar (e.g. the results of preprocessing) we believe there might exist a similar algorithm for numeration system (β, A_6). The same idea for creating parallel addition using the transformation from $(2, A_6)$-numeration system from Table 2.3 described in the previous section can be also applied on numeration systems $(1 \pm i\sqrt{3}, A_6)$.

5.3.7 Base $-1 \pm i\sqrt{3}$

Let us consider the base $\beta = -1 - i\sqrt{3}$ with minimal polynomial $x^2 + 2x + 4$ and the alphabet $A_6 = \left\{0, 1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\right\}$. This numeration system is similar not only to the base $+1 + i\sqrt{3}$ but also to bases ± 2.

The OL property is satisfied for the same set I as the numeration system with base $+1+i\sqrt{3}$ which can be seen in Figure 5.4, the set I is also symmetrical to complex conjugation. The results of preprocessing can be found in Table 5.7 and the list of identities follows. The other rules can be derived from symmetry of the alphabet A_6.

$$\mathcal{L}: \qquad 1\rho^5 \to 0\rho^2$$

The situation for parallel addition is the same as in the previous section. The Extending Window Method did not find an algorithm for parallel addition which is something that all the numeration systems in the group have in common. There is a possibility that there exists a similar algorithm for parallel addition as was described in [4] since all the numeration systems in this group are very similar and we can obtain a $(-1 \pm i\sqrt{3}, A_6)$-representation of number using the transformation rules from Table 2.3.

5.3.7.1 Summary

This section contains the results of preprocessing for on-line division and parameters of algorithms for parallel addition for complete polygonal numeration systems in $\mathbb{Z}[\frac{1+i\sqrt{3}}{2}]$ which satisfy OL property.

k-block	=	1
p	=	3
# LuT	=	6085

Table 5.5: Parameters of algorithms for parallel addition for complete polygonal numeration systems (β, A_6) where $\beta \in \left\{\pm i\sqrt{3}, \frac{\pm 3\pm i\sqrt{3}}{2}\right\}$ and $A_6 = \{0, 1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$.

D_{min}	=	$0.136391\cdots$
$\#\mathcal{L}$	=	12
H	=	$1.322875\cdots$

Table 5.6: Results of preprocessing for polygonal numeration system (β, A_6) in $\mathbb{Z}[\frac{1+i\sqrt{3}}{2}]$, namely for bases $\beta \in \left\{\pm i\sqrt{3}, \frac{\pm 3\pm i\sqrt{3}}{2}\right\}$ and alphabet $A_6 = \{0, 1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$.

D_{min}	=	$0.183012\cdots$
$\#\mathcal{L}$	=	6
H	=	$1.000000\cdots$

Table 5.7: Results of preprocessing for polygonal numeration system (β, A_6) in $\mathbb{Z}[\frac{1+i\sqrt{3}}{2}]$, namely for bases $\beta \in \left\{\pm 2, \pm(1+i\sqrt{3})\right\}$ and alphabet $A_6 = \{0, 1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$.

5.4 Ring $\mathbb{Z}[\frac{1+i\sqrt{7}}{2}]$

Let us focus on the quadratic field $\mathbb{Q}(\sqrt{-7})$ and its set of all algebraic integers $\mathbb{Z}[\frac{1+i\sqrt{7}}{2}]$. We will discuss four complete polygonal numeration systems in $\mathbb{Z}[\frac{1+i\sqrt{7}}{2}]$ with bases $\beta = \frac{\pm 1\pm i\sqrt{7}}{2}$ and alphabet $A_2 = \{0, \pm 1\}$. In order to prove the OL property of these systems, we cannot use Theorem 4.5, since the condition $\#A_2 > |\beta|^2 + |\beta + \bar{\beta}|$ is not satisfied. Nevertheless, we have found the set I explicitly, see Figures 5.5a and 5.5b.

5.4.1 Base $\frac{+1\pm i\sqrt{7}}{2}$

Let the base be $\beta = \frac{+1+i\sqrt{7}}{2}$ with minimal polynomial of the form $x^2 - x + 2$ and the alphabet $A_2 = \{0, \pm 1\}$. The necessary condition for parallel addition is satisfied. The EWM found a 2-block algorithm for parallel addition whose parameters are given in Table 5.8.

The set I proving the OL property was found explicitly, see Figure 5.5a. Note that for the base $\frac{1-i\sqrt{7}}{2}$ we have to use the set I from Figure 5.5b since I is not closed under complex conjugation. Hence we can use our program to compute preprocessing for on-line division. Results of the preprocessing are in Table 5.9 and the list containing rewriting rules follows. We list only the rules starting on positive number since the rest can be obtained by multiplying by -1.

$$\begin{aligned} \mathcal{L}: \qquad & 101 \to 01\bar{1} \\ & 1\bar{1}1 \to 00\bar{1} \end{aligned}$$

Note also that the value of parameter H is chosen as a rough estimate of its exact value, since the argument of the base β is an irrational multiple of π and therefore the value β^j for $j \in \mathbb{N}$ never has imaginary part equal to 0.

5.4.2 Base $\frac{-1\pm i\sqrt{7}}{2}$

Let us focus on the base $\frac{-1+i\sqrt{7}}{2}$ with minimal polynomial $x^2 + x + 2$ and the alphabet $A_2 = \{0, \pm 1\}$. The necessary condition for parallel addition from Theorem 3.9 is not satisfied but the Extending Window Method found a 2-block algorithm for parallel addition. All parameters of the algorithm are given in Table 5.8.

The set ensuring OL property can be seen in Figure 5.5b. For the base $\frac{-1-i\sqrt{7}}{2}$ we have to use the set I from Figure 5.5b since I is not closed under complex conjugation as in the previous section.

If we compute preprocessing for on-line division, we obtain the following list of rewriting rules:

$$\begin{aligned} \mathcal{L}: \qquad & 101 \to 0\bar{1}\bar{1} \\ & 111 \to 00\bar{1} \end{aligned}$$

and other parameters of the algorithm are the same as for the previous numeration system and can be seen in Table 5.9.

5.4.3 Summary

This section provides the results of preprocessing for on-line division and parameters of algorithms for parallel addition for complete polygonal numeration systems in $\mathbb{Z}[\frac{1+i\sqrt{7}}{2}]$ which satisfy OL property.

k-block	=	2
p	=	4
# LuT	=	11809

Table 5.8: Parameters of algorithms for parallel addition for complete polygonal numeration systems (β, A_2) where $\beta = \frac{\pm 1 \pm i\sqrt{7}}{2}$ and $A_2 = \{0, \pm 1\}$.

D_{min}	=	$0.085665\cdots$
$\#\mathcal{L}$	=	4
H	=	$1.757700\cdots$

Table 5.9: Results of preprocessing for polygonal numeration systems (β, A_2) in $\mathbb{Z}[\frac{1+i\sqrt{7}}{2}]$, namely for bases $\beta = \frac{\pm1\pm i\sqrt{7}}{2}$ and alphabet $A_2 = \{0, \pm1\}$.

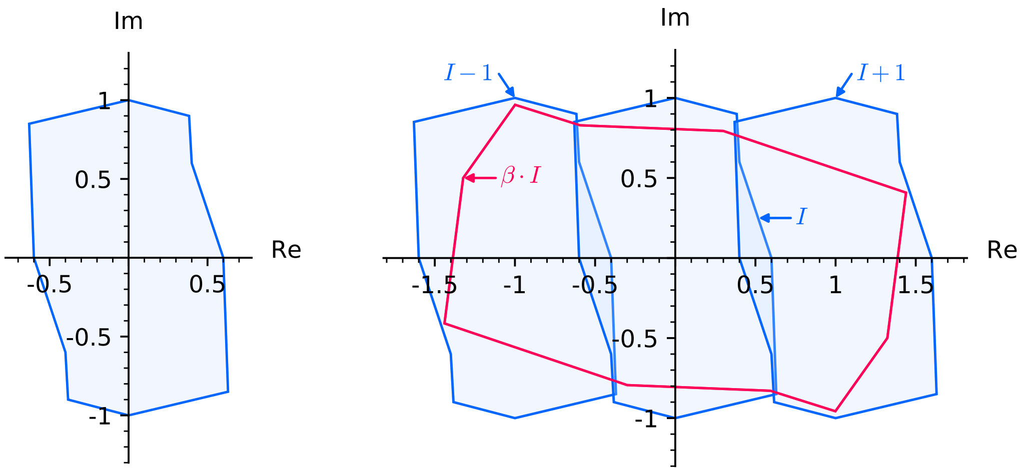

Figure 5.5a: OL property for numeration system with bases $\beta = \frac{\pm(1+i\sqrt{7})}{2}$ and alphabet A_2.

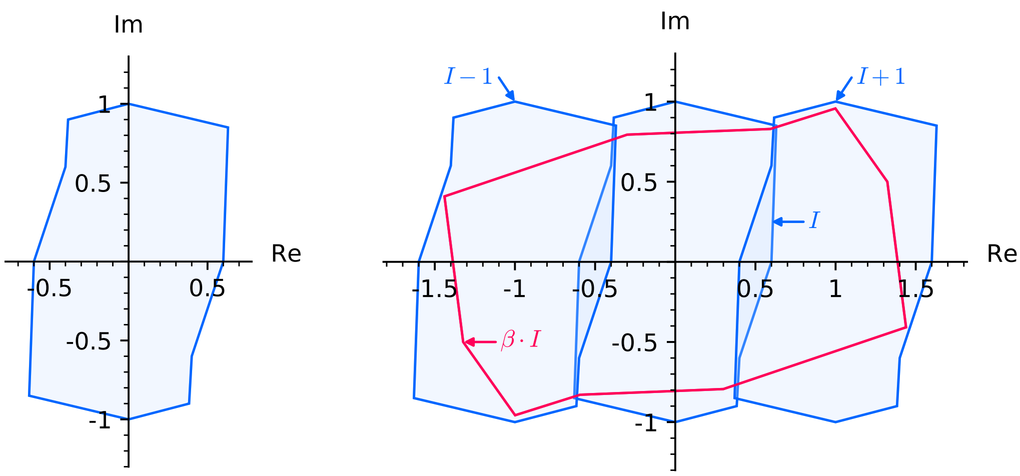

Figure 5.5b: OL property for numeration system with bases $\beta = \frac{\pm(1-i\sqrt{7})}{2}$ and alphabet A_2.

Conclusion

This work deals with polygonal numeration systems in imaginary quadratic fields. We focused on completeness and on feasibility of arithmetic operations including parallel addition and on-line division and multiplication in these non-standard numeration systems.

First we fully characterized the complete polygonal numeration systems in imaginary quadratic fields by generalizing the results from [19] to non-integer alphabets. Later we introduced a necessary condition for OL property, i.e. for feasibility of on-line arithmetics. For several previously unresolved cases of polygonal numeration systems, we explicitly found the set I ensuring OL property.

We used several computer programs. The first program by Legerský [20] implemented the Extending Window Method in order to find an algorithm for parallel addition for a given numeration system. Another program implemented within the research assignment [29] computes preprocessing of divisors for on-line division.

We applied the above mentioned computer programs on complete polygonal numeration systems in imaginary quadratic fields which did satisfy the necessary condition for OL property. The results can be seen in Table 5.10. The first two columns specify the numeration system. The following three columns show the parameters of the algorithms found for parallel addition. Finally, the result of preprocessing for on-line division including the number of rewriting rules and corresponding parameter D_{min} are displayed.

β	A	k-block	$p = 1 + r$	# LuT	$\#\mathcal{L}$	D_{min}
$\pm 1 \pm i$	$\{0, \pm 1\}$	2	5	60721	-	-
	$\{0, \pm 1, \pm i\}$	1	6	2165713	36	$0.148089\cdots$
$\pm i\sqrt{2}$	$\{0, \pm 1\}$	1	4	289	6	$0.274094\cdots$
$i\sqrt{3} \cdot \rho^j$	$\{0, 1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$	1	3	6085	12	$0.136391\cdots$
$2 \cdot \rho^j$	$\{0, 1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$	*	*	*	6	$0.183012\cdots$
$\frac{\pm 1 \pm i\sqrt{7}}{2}$	$\{0, \pm 1\}$	2	4	11809	4	$0.085665\cdots$

Table 5.10: Results for preprocessing for on-line division and parameters of algorithm for parallel addition for numeration systems from Table 5.1. We consider the constant $\rho = \frac{1+i\sqrt{3}}{2}$ and $j \in \{0, 1, 2, 3, 4, 5\}$. The symbol * means that we know that the algorithm for parallel addition exists from [16] but we did not find it using the EWM.

Among the questions remaining open is the problem of describing the OL property for the following polygonal numeration systems in imaginary quadratic fields:

- bases $\beta = \pm 1 \pm i$ and alphabet $A_2 = \{0, \pm 1\}$,
- bases $\beta \in \{\pm 2, \pm 2i\}$ and alphabet $A_4 = \{0, \pm 1, \pm i\}$,
- bases $\beta \in \left\{\pm i\sqrt{3}, \frac{\pm 3 \pm i\sqrt{3}}{2}\right\}$ and alphabet $A_3 = \{0, 1, \rho^2, \rho^4\}$.

More generally, OL property is rarely answered for any polygonal numeration systems in algebraic fields of degree $n \geq 3$.

Some of the complete polygonal numeration systems may not satisfy OL property but we can modify them in order to obtain this property. Consider the base $\pm(1 + i\sqrt{2})$ and the alphabet $A_4 = \{0, \pm 1, \pm i\}$ which is a polygonal extension of the original alphabet $A_2 = \{0, \pm 1\}$. This numeration system is still polygonal, but the field containing this numeration system is no longer quadratic. In Figure 5.7 the set I ensuring OL property is shown.

A similar set I can be applied for the numeration system with the base $\pm(1 - i\sqrt{2})$ and the alphabet $A_4 = \{0, \pm 1, \pm i\}$ and it is rendered in Figure 5.8.

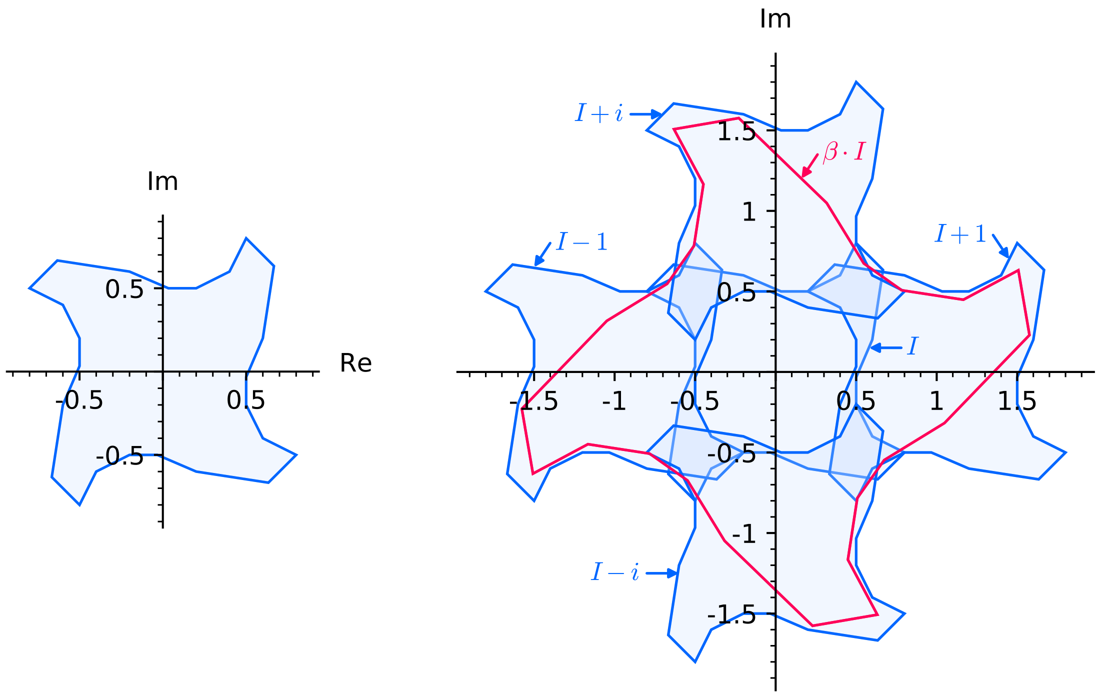

Figure 5.7: OL property for numeration system with base $\beta = +1 + i\sqrt{2}$ and alphabet $A_4 = \{0, \pm 1, \pm i\}$.

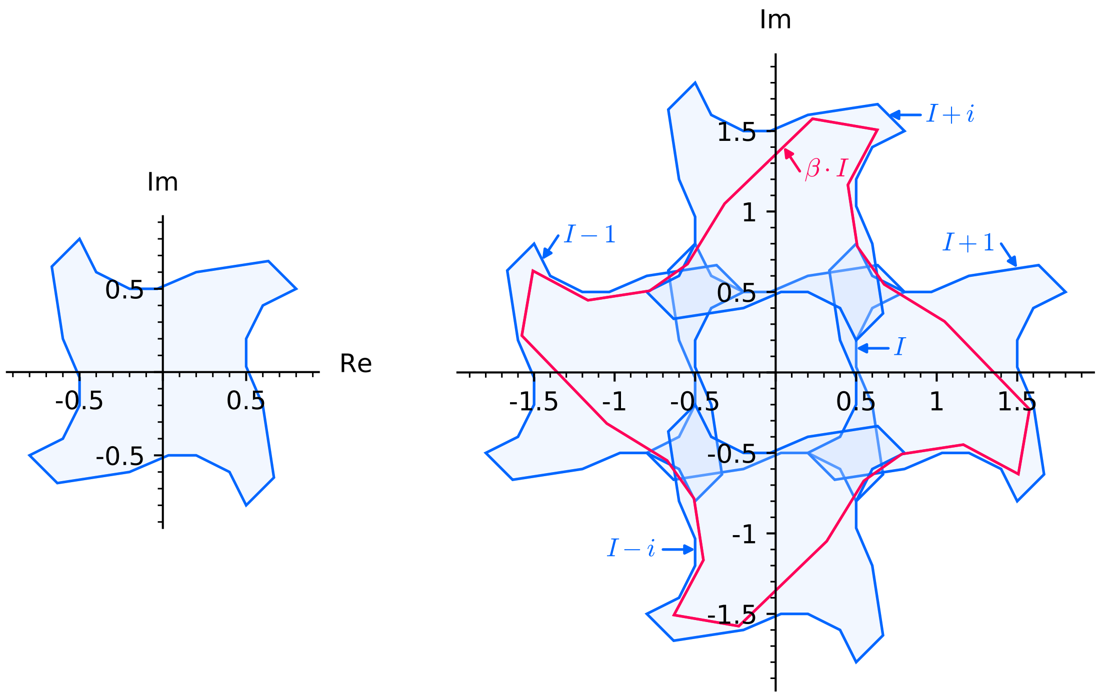

Figure 5.8: OL property for numeration system with base $\beta = -1 + i\sqrt{2}$ and alphabet $A_4 = \{0, \pm 1, \pm i\}$.

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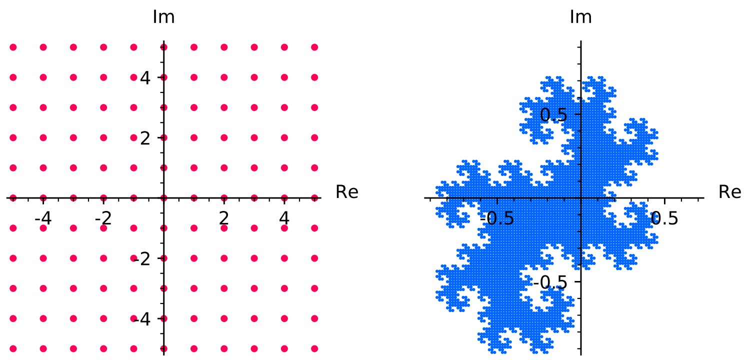
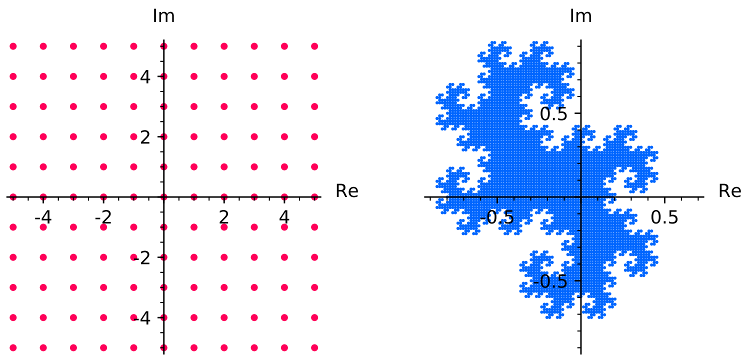
Appendices

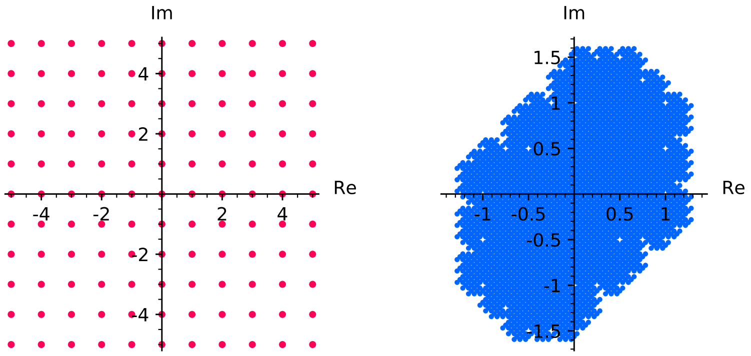
In each row, the first figure shows the spectrum of the numeration system $X^A(\beta)$ and the second figure displays the set of fractions $W^A(\beta)$.



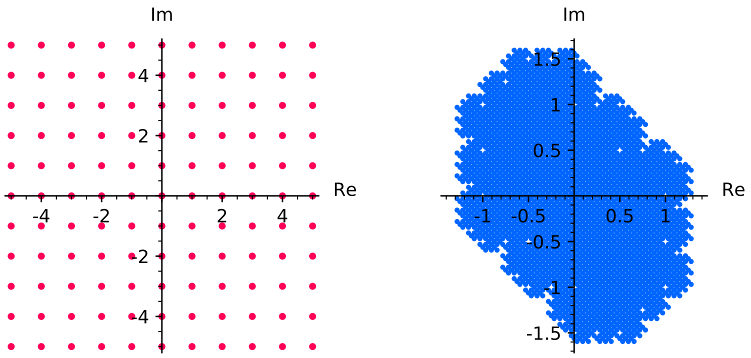
$\beta = -1 + i$,
$A_1 = \{0, 1\}$,
$m = 12$



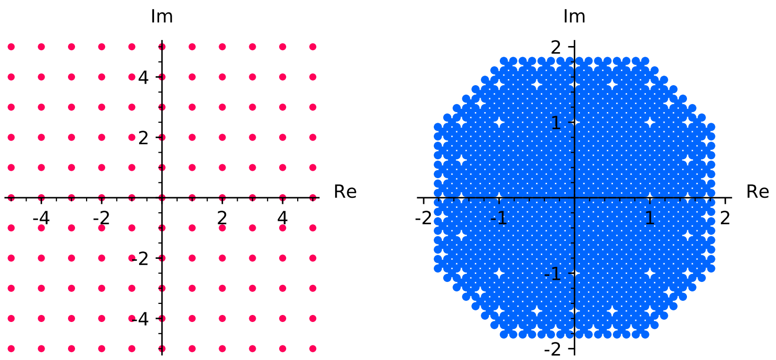
$\beta = -1 - i$,
$A_1 = \{0, 1\}$,
$m = 12$



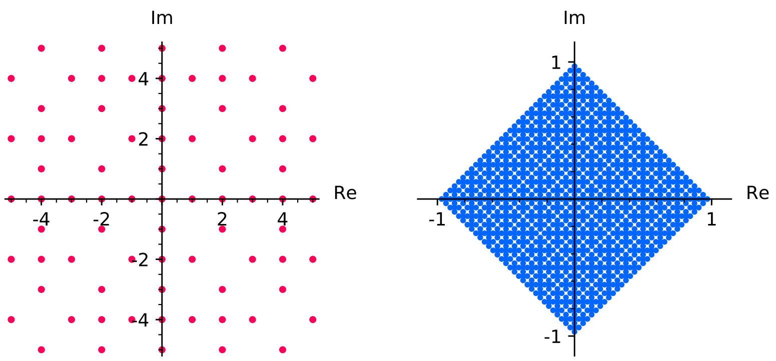
$\beta = \pm(-1+i)$,
$A_2 = \{0, \pm 1\}$,
$m = 9$



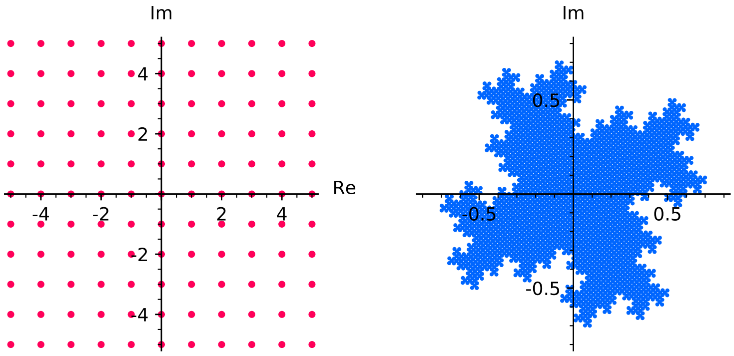
$\beta = \pm(-1-i)$,
$A_2 = \{0, \pm 1\}$,
$m = 9$



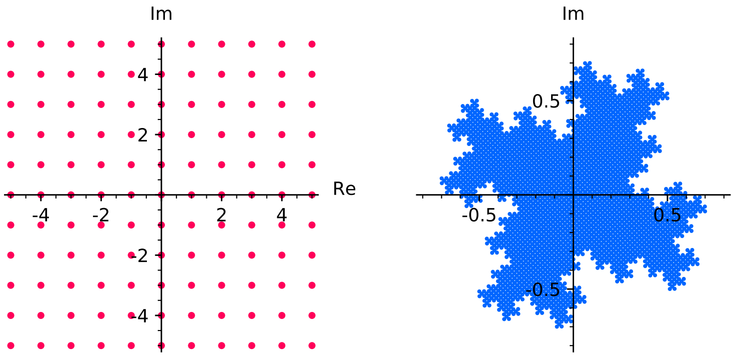
$\beta = \pm 1 \pm i$,
$A_4 = \{0, \pm 1, \pm i\}$,
$m = 7$



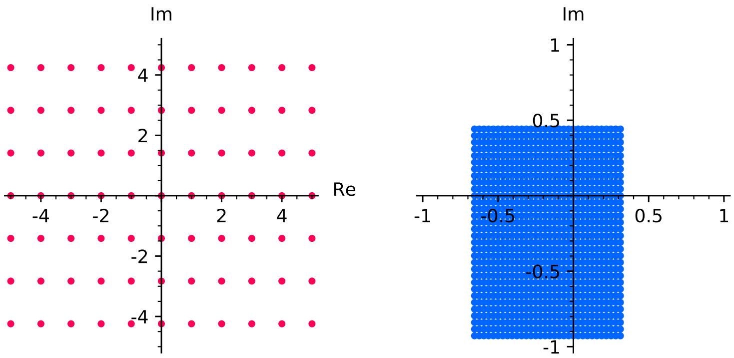
$\beta \in \{\pm 2, \pm 2i\}$,
$A_4 = \{0, \pm 1, \pm i\}$,
$m = 5$



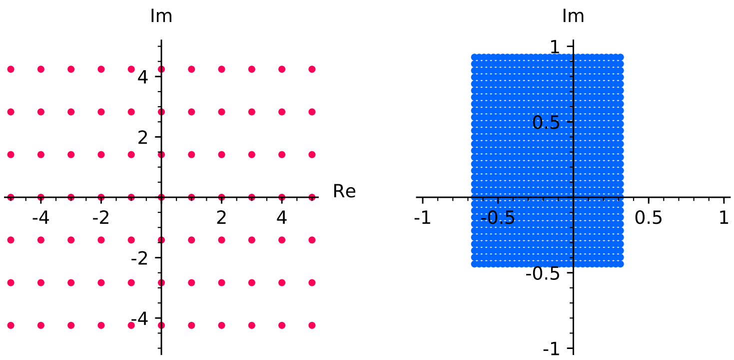
$\beta \in \{\pm(1+2i), \pm(2-i)\}$,
$A_4 = \{0, \pm 1, \pm i\}$,
$m = 5$



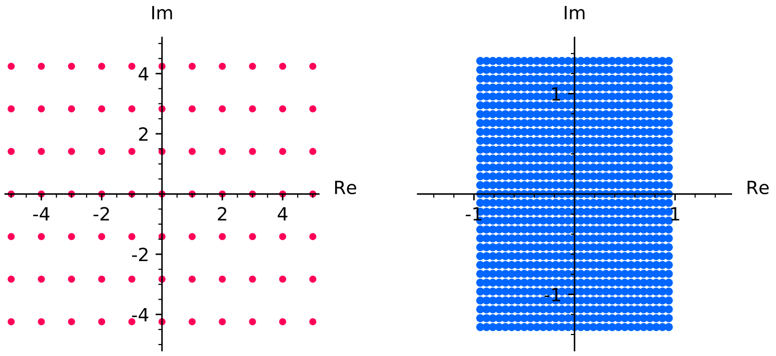
$\beta \in \{\pm(+1-2i), \pm(2+i)\}$,
$A_4 = \{0, \pm 1, \pm i\}$,
$m = 5$



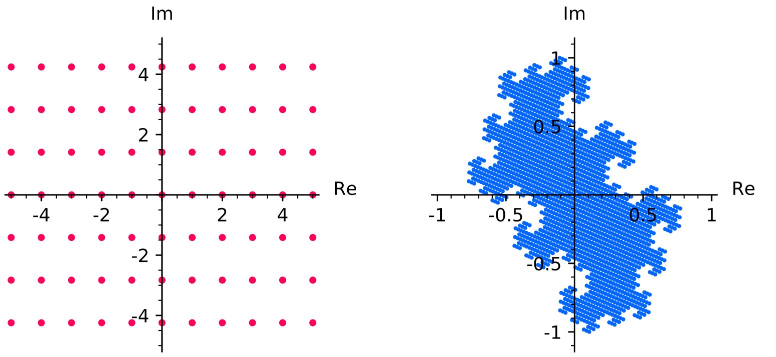
$\beta = i\sqrt{2}$,
$A_1 = \{0, 1\}$,
$m = 10$



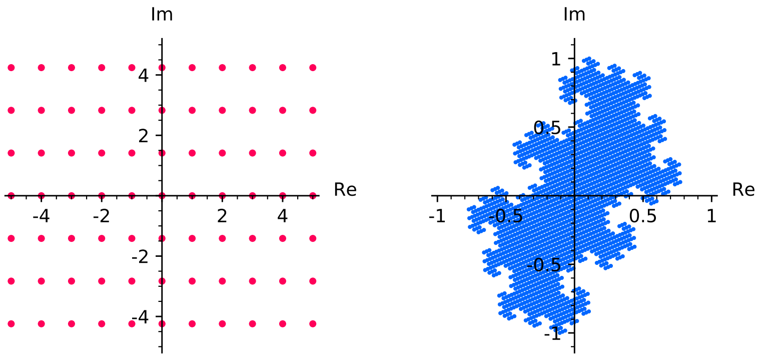
$\beta = -i\sqrt{2}$,
$A_1 = \{0, 1\}$,
$m = 10$



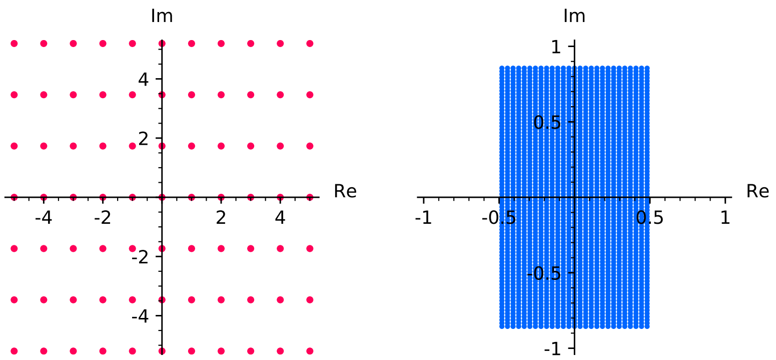
$\beta = \pm i\sqrt{2}$,
$A_2 = \{0, \pm 1\}$,
$m = 8$



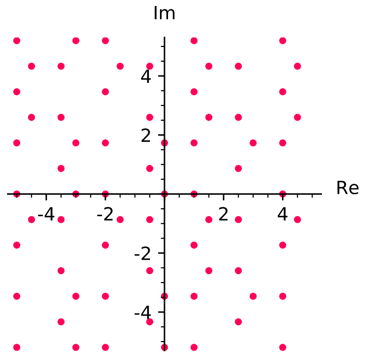
$\beta = \pm(1 + i\sqrt{2})$,
$A_2 = \{0, \pm 1\}$,
$m = 7$

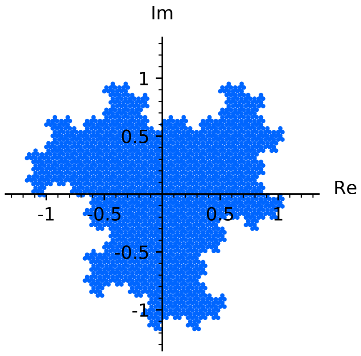


$\beta = \pm(1 - i\sqrt{2})$,
$A_2 = \{0, \pm 1\}$,
$m = 7$

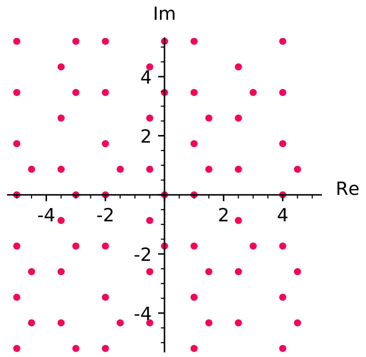


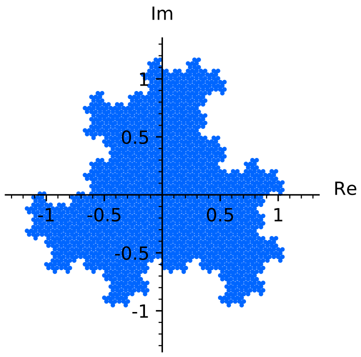
$\beta = \pm i\sqrt{3}$,
$A_2 = \{0, \pm 1\}$,
$m = 7$



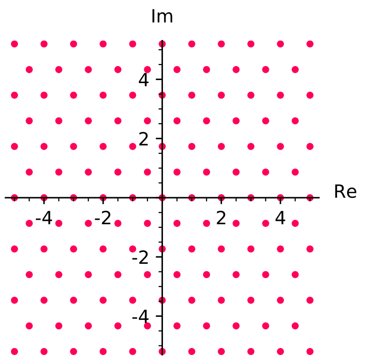


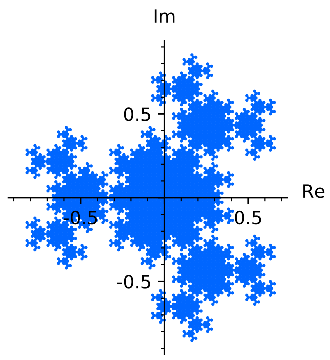
$\beta \in \left\{ i\sqrt{3}, \frac{\pm 3 - i\sqrt{3}}{2} \right\}$,
$A_3 = \{0, 1, \rho^2, \rho^4\}$,
$m = 7$



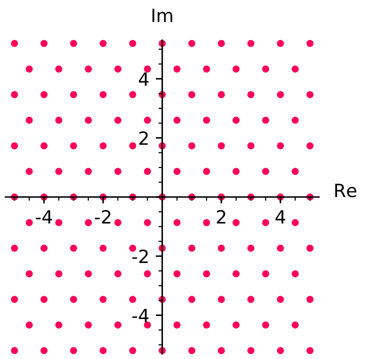


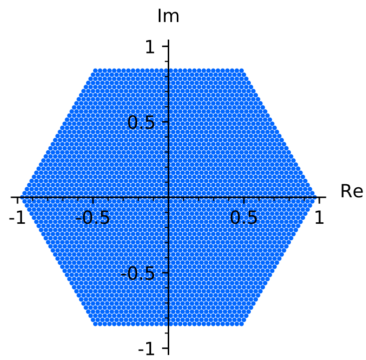
$\beta \in \left\{ -i\sqrt{3}, \frac{\pm 3 + i\sqrt{3}}{2} \right\}$,
$A_3 = \{0, 1, \rho^2, \rho^4\}$,
$m = 7$



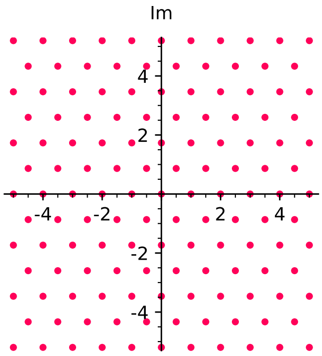


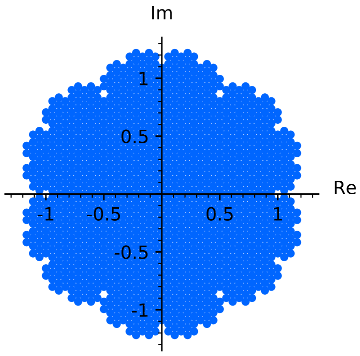
$\beta \in \left\{ -2, 1 \pm i\sqrt{3} \right\}$,
$A_3 = \{0, 1, \rho^2, \rho^4\}$,
$m = 6$



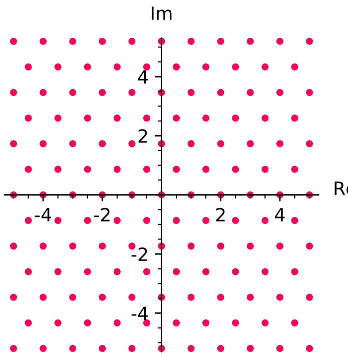


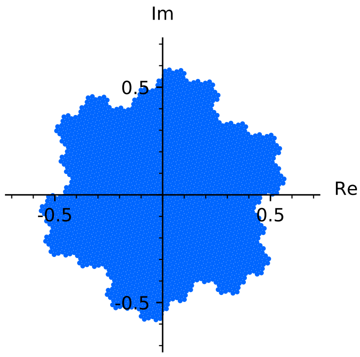
$\beta \in \left\{ \pm 2, \pm 1 \pm i\sqrt{3} \right\}$,
$A_6 = \{0, \pm 1, \pm\rho, \pm\rho^2\}$,
$m = 5$



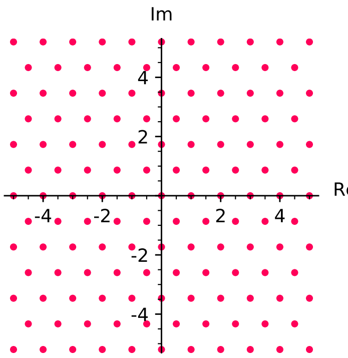


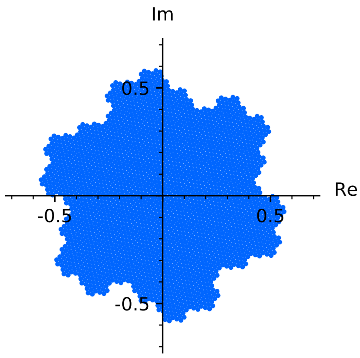
$\beta \in \left\{\pm i\sqrt{3}, \frac{\pm 3 \pm i\sqrt{3}}{2}\right\}$,
$A_6 = \{0, \pm 1, \pm\rho, \pm\rho^2\}$,
$m = 5$



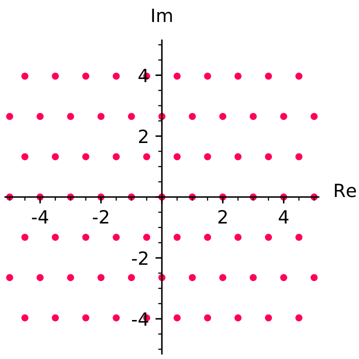


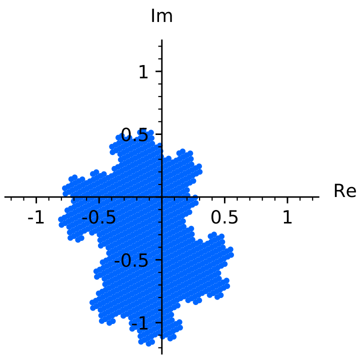
$\beta \in \left\{\pm(2 + i\sqrt{3}), \frac{\pm(1 - i\,3\sqrt{3})}{2}, \frac{\pm(5 - i\sqrt{3})}{2}\right\}$,
$A_6 = \{0, \pm 1, \pm\rho, \pm\rho^2\}$,
$m = 4$



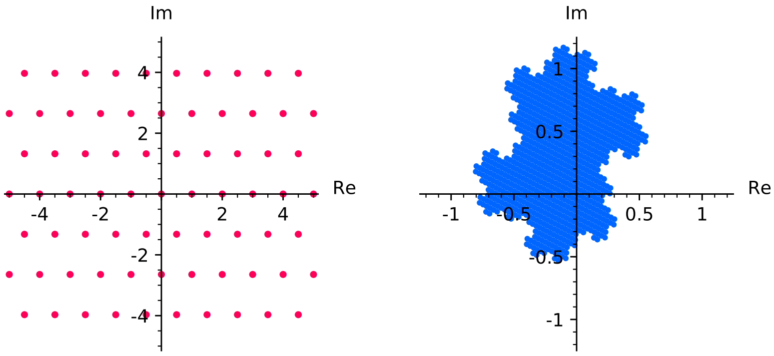


$\beta \in \left\{\pm(2 - i\sqrt{3}), \frac{\pm(1 + i\,3\sqrt{3})}{2}, \frac{\pm(5 + i\sqrt{3})}{2}\right\}$,
$A_6 = \{0, \pm 1, \pm\rho, \pm\rho^2\}$,
$m = 4$

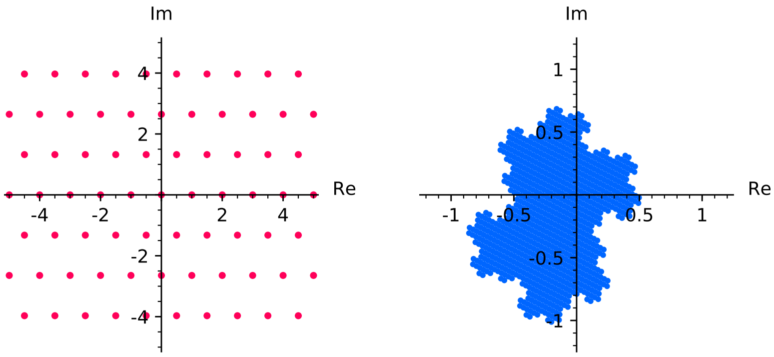




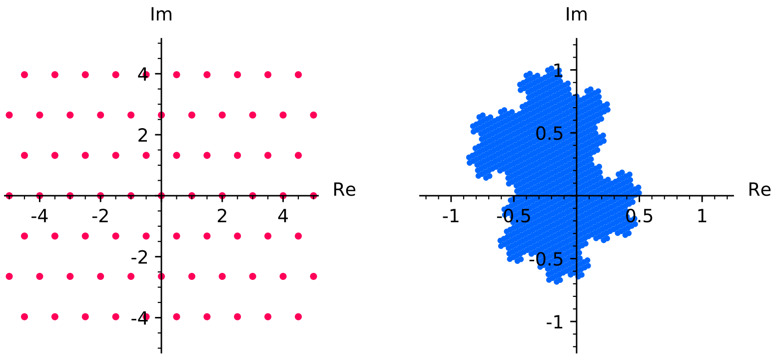
$\beta = \frac{1 + i\sqrt{7}}{2}$,
$A_1 = \{0, 1\}$,
$m = 10$



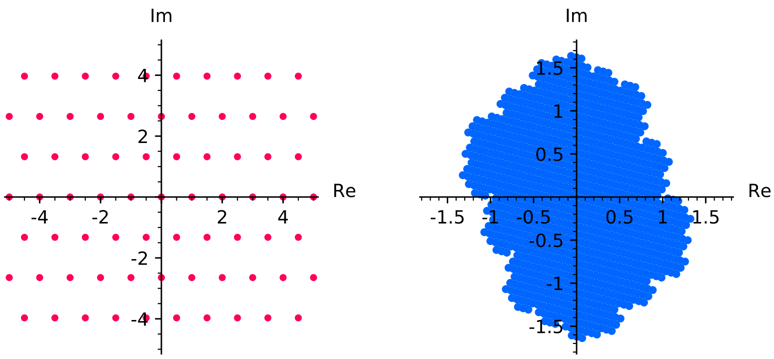
$\beta = \frac{1-i\sqrt{7}}{2}$,
$A_1 = \{0.1\}$,
$m = 10$



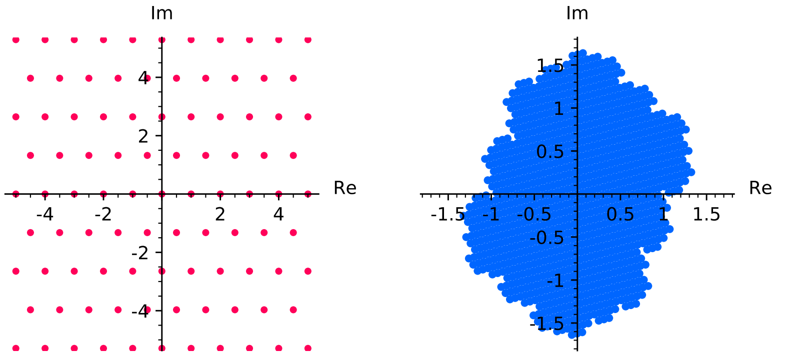
$\beta = \frac{-1+i\sqrt{7}}{2}$,
$A_1 = \{0.1\}$,
$m = 10$



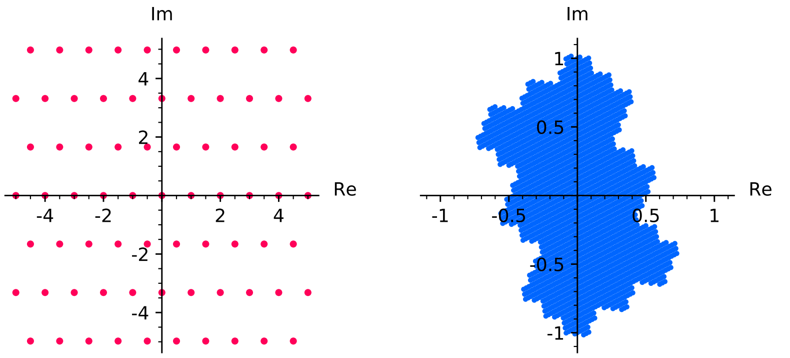
$\beta = \frac{-1-i\sqrt{7}}{2}$,
$A_1 = \{0.1\}$,
$m = 10$



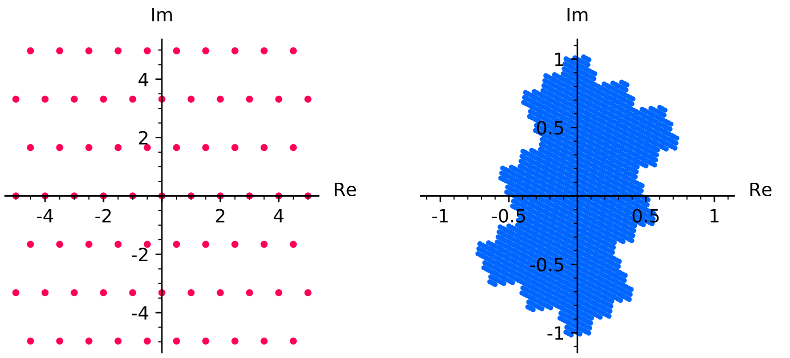
$\beta = \frac{\pm(1+i\sqrt{7})}{2}$,
$A_2 = \{0, \pm 1\}$,
$m = 8$



$\beta = \frac{\pm(1-i\sqrt{7})}{2}$,
$A_2 = \{0, \pm 1\}$,
$m = 8$



$\beta = \frac{\pm(1+i\sqrt{11})}{2}$,
$A_2 = \{0, \pm 1\}$,
$m = 7$



$\beta = \frac{\pm(1-i\sqrt{11})}{2}$,
$A_2 = \{0, \pm 1\}$,
$m = 7$