Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Representations of Numbers in Linear Recurrent Systems

Reprezentace čísel v lineárních rekurentních systémech

Master's Thesis

Author: **Bc. Hynek Peřina**

Supervisor: **Ing. Petr Ambrož, PhD.**

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ZADÁNÍ DIPLOMOVÉ PRÁCE

Student: Bc. Hynek Peřina

Studijní program: Aplikace přírodních věd

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Pokyny pro vypracování:

1) Proveďte rešerši známých výsledků týkajících se vlastností funkce R(n), která udává počet rozvojů čísla n ve Fibonacciho číselné soustavě.
2) Zobecněte metodu pro výpočet hodnoty R(n) pro jiné lineární rekurentní systémy (s bází zadanou rekurencí 2. řádu) než je Fibonacciho num. systém.
3) Pro zkoumané lin. rekurentní systémy nalezněte a implementujte vhodné algoritmy pro napočítání funkce R(n) pro dostatečně velký interval hodnot čísla n.
4) Na základě napočítaných hodnot funkce R(n) zkoumejte její vlastnosti: symetrie, minima a maxima na podintervalech, posloupnosti argmin R(n) a argmax R(n), atd.

Doporučená literatura:

1) J. Berstel, An excercise on Fibonacci representations, RAIRO Theor. Inform. Appl. 35, 2002, 491–498.

2) P. Kocábová, Z. Masáková, E. Pelantová, Integers with maximal number of Fibonacci representations, RAIRO Theor. Inf. Appl. 39, 2005, 343–358.

3) P. Kocábová, Z. Masáková, E. Pelantová, Ambiguity in the m-Bonacci numeration system, Discrete Math. Theor. Comput. Sci. 9, 2007, 109–124.

Jméno a pracoviště vedoucího diplomové práce:

Ing. Petr Ambrož, Ph.D.
KM FJFI, Trojanova 13, 120 00 Praha 2

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Author's declaration:
I declare that this Master's Thesis is entirely my own work and I have listed all the used sources in the bibliography.

Prague, May 3, 2021 Hynek Peřina

Název práce:

Reprezentace čísel v lineárních rekurentních systémech

Autor: Hynek Peřina

Obor: Matematická informatika

Druh práce: Diplomová práce

Vedoucí práce: Ing. Petr Ambrož, PhD., katedra matematiky, FJFI ČVUT v Praze

Abstrakt: Nechť $t_1, \dots, t_m \in \mathbb{N}_0$, $t_m \neq 0$ jsou koeficienty lineárně rekurentní posloupnosti $B_k = \sum_{i=1}^{m} t_i B_{k-i}$ s počátečními podmínkami $B_0 = 1$, $B_1 = t_1 + 1$, $\dots$, $B_{m-1} = \sum_{j=1}^{m-1} t_j B_{m-1-j} + 1$. Každá taková posloupnost určuje numerační systém, kde každému $n \in \mathbb{N}_0$ je přiřazeno slovo $w_{N-1} \cdots w_0$ z celočíselných cifer splňující rovnost $n = \sum_{i=0}^{N-1} w_i B_i$. Dané číslo n může mít více takových reprezentací, označíme $R(n)$ počet reprezentací n nad kanonickou abecedou. Zkoumáme vlastnosti $R(n)$ v konfluentních numeračních systémech a zobecňujeme výsledky P. Kocábové, Z. Masákové a E. Pelantové týkající se $R(n)$ v soustavách založených na Fibonacciho a m-bonacciho posloupnostech. Dokazujeme maticový vzorec pro $R(n)$ v konfluentních systémech a určujeme maxima funkce $R(n)$ ve všech konfluentních systémech. Dále ukazujeme, že v soustavách založených na posloupnostech, které mají všechny koeficienty rekurence stejné, se maximální hodnoty $R(n)$ shodují s těmi ve Fibonacciho a m-bonacciho soustavách.

Klíčová slova: konfluentní numerační systémy, lineární numerační systémy, redundance

Title:

Representations of Numbers in Linear Recurrent Systems

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Abstract: Given coefficients $t_1, \dots, t_m \in \mathbb{N}_0$, $t_m \neq 0$, the linear recurrent sequence $B_k = \sum_{i=1}^{m} t_i B_{k-i}$ with initial conditions $B_0 = 1$, $B_1 = t_1 + 1$, $\dots$, $B_{m-1} = \sum_{j=1}^{m-1} t_j B_{m-1-j} + 1$ defines a numeration system. Every $n \in \mathbb{N}_0$ can be represented by a word $w_{N-1} \cdots w_0$ consisting of integer digits that is defined by the equality $n = \sum_{i=0}^{N-1} w_i B_i$. A given n can have several such representations. Let $R(n)$ be the function that counts the number of distinct representations of n over the canonical alphabet. We study the properties of the function $R(n)$ in confluent numeration systems and extend the results of P. Kocábová, Z. Masáková, and E. Pelantová for $R(n)$ in the Fibonacci and m-bonacci systems. We prove a matrix formula for $R(n)$ in confluent systems and determine the maxima of $R(n)$ in all confluent systems. Namely, we show that in systems based on sequences whose recurrence coefficients are all identical, the maximal values of $R(n)$ equal those in the Fibonacci and m-bonacci systems.

Key words: confluent numeration systems, linear numeration systems, redundancy

Contents

Introduction

Numbers are intrinsincally linked with the way we write them. In our daily life, the decimal system is the most practical, whereas computers store numbers using the binary system. A *numeration system* is a set of rules that we use to assign strings of digits to values and vice versa.

The most commonly used numeration system is the standard b-ary system. In this numeration system, we construct a representation of an integer x by first finding the largest power of b that is smaller than x, then dividing x by this power of b and storing the result as the most significant digit, then diving the remainder by the next smaller power of b, recording that as the next digit and repeatedly dividing the remainder by smaller and smaller powers of b until we construct the whole representation. However, we do not necessarily have to use a geometric sequence. It is easy to prove that any strictly increasing sequence starting by 1 can be used to represent natural numbers. We will call numeration systems based on such sequences the *B-systems*. In literature [5], the name U-systems is also used.

Of particular interest are numeration systems based on a sequence satisfying a linear recurrence with integer coefficients. The most famous example is the *Fibonacci representation*, also called *Zeckendorf representation* [14] after its discoverer, which uses the Fibonacci sequence. For example, in the Zeckendorf representation, the number six has the representation 1001, since $6 = 5 + 1$ and 5 and 1 are the first and fourth Fibonacci numbers, respectively. Much has been done in the study of the Fibonacci system and B-systems in general and the language of normal representations. For example, Hollander [8] studied the conditions needed for a B-system's language of normal representations (obtained by the usual greedy algorithm) to be regular.

B-systems based on a linear recurrent sequence have the property that they are redundant, i.e. a given number may have multiple representations in such a system. The number six has another valid representation in the Fibonacci numeration system, namely 111, because $6 = 3 + 2 + 1$. The focus of this work will be quantifying the degree of this ambiguity for a selected class of linear numeration systems, namely the *confluent systems*. Denote by $R(n)$ the number of representations of the number $n \in \mathbb{N}_0$ in a given B-system. Kocábová, Masáková, and Pelantová studied the properties of the function $R(n)$ in the systems based on the Fibonacci and m-bonacci sequences [12, 11]. By the m-bonacci sequence we mean the sequence whose every element is the sum of $m > 2$ consecutive preceding elements. We will expand on their work and study the properties of $R(n)$ in their generalisation, the confluent systems. Established by Frougny in [3], they are linear numeration systems which generate a rewriting system that is confluent. We will specify this in more detail in Chapter 2.

In Chapter 1, we introduce some basic terminology from combinatorics on words, since that will be needed for working with representations of numbers (which are strings of digits, i.e. words).

In Chapter 2 we establish linear numeration systems and verify some of their properties. Namely, we will introduce the (F) systems and confluent systems and derive a way how to recognise a greedy representation in an (F) system.

In Chapter 3 we present the algorithm for calculating $R(n)$ and the computational results of our survey of the function $R(n)$ in several confluent systems. Our data suggests that confluent systems are a close generalisation of the Fibonacci and m-bonacci systems, since in two large subclasses of confluent systems the function $R(n)$ displays substantially similar behaviour to the Fibonacci and m-bonacci systems. In this section we also conjecture expressions for the values of the maxima of $R(n)$ and the number of arguments of the maxima of $R(n)$.

In Chapter 4 we study the theoretical properties of the function $R(n)$ and derive a closed-form matrix formula for the calculation of $R(n)$ in confluent systems. We then use this matrix formula to verify our hypotheses from Chapter 3 and show that confluent systems with all recurrence coefficients equal behave identically to the Fibonacci and m-bonacci systems as well as show the difference to the confluent systems where the last recurrence coefficient is strictly smaller.

Lastly, in the Appendix, we describe in detail our program for calculating $R(n)$.

Chapter 1

Preliminaries

The focus of this work will be representations of numbers. Numbers are represented by words, i.e. sequences of characters (digits) from a a finite set. Therefore, in this section we will establish some basic terminology related to combinatorics on words.

An *alphabet* is any finite set A. Its elements are known as *letters* or *symbols*. In our case A will be typically a finite subset of integers. A *word* or *string over* A is some sequence of letters from A. Formally, a word w is defined as $w = w_n w_{n-1} \cdots w_0$, where $w_i \in A$, $n \in \mathbb{N}$. The length of a word $w = w_n w_{n-1} \cdots w_0$ is denoted $|w| = n+1$. The *set of all finite words over* A is denoted by

$$A^* = \{\varepsilon\} \cup \bigcup_{n \in \mathbb{N}_0, w_i \in A} w_n w_{n-1} \cdots w_0.$$

where ε is the *empty word*, i.e. a sequence of length zero. The set A^* is endowed with the binary operation *concatenation of words* $\circ : A^* \times A^* \to A^*$ which is defined followingly: For $u = u_n u_{n-1} \cdots u_0$, $v = v_m v_{m-1} \cdots v_0 \in A^*$ set

$$u \circ v = u_n u_{n-1} \cdots u_0 v_m v_{m-1} \cdots v_0.$$

The circle operator $\circ$ is however usually left out and we write $w = uv$. The structure $(A^*, \circ)$ is a free monoid, ε being the neutral element. The concatenation of w with itself is defined recursively as

$$w^0 = \varepsilon, \qquad w^{n+1} = w^n w.$$

We say that u is a *prefix of* w if w can be factorised as $w = uv, v \in A^*$. Analogically, u is a *suffix of* w if w can be factorised as $w = vu, v \in A^*$. Additionally, u is said to be a *proper prefix* or *proper suffix* if v from the above factorisations is non-empty. Lastly, u is a *factor* of w if there exists a factorisation of w such that $w = xuv, x, v \in A^*$. Likewise, if x or v are not equal to ε then u is a *proper factor*. Note: In all cases the words u, v, x can equal ε (ε is the prefix, suffix and factor of every word).

A *language* is any subset of A^*. We say that a word $w \in A^*$ *avoids* a set $X \subset A^*$ if no word $x \in X$ is a factor of w. By extension we say that a language L avoids X if all $w \in L$ avoid X.

We define two canonical orderings on the set A^*.

Definition 1.1. Consider the two words $x = x_N x_{N-1} \cdots x_0$, $y = y_M y_{M-1} \cdots y_0$ over a totally ordered alphabet A. Then x is said to be *lexicographically greater than* y (denoted $x \succ_{\text{lex}} y$), when one of the following conditions holds:

- $N > M$ (i.e. x is longer than y) and y is a prefix of x.

- There exists an index $r \leq N$ such that $x_r > y_r$ and $x_i = y_i$ for all $r < i \leq N$.

Definition 1.2. Again let $x = x_N x_{N-1} \cdots x_0, y = y_M y_{M-1} \cdots y_0$ be words over a totally ordered alphabet A. Then x is said to be *radix greater than* y (denoted $x \succ y$) when one of the following conditions holds:

- $N > M$, i.e. x is longer than y.
- $N = M$ and there exists an index $r \leq N$ such that $x_r > y_r$ and $x_i = y_i$ for all $r < i \leq N$.

The lexicographic ordering is equivalent to the alphabetic ordering whilst the radix order is equivalent to ordering by value. Consider for example numbers written in the decimal representation. The string 42 is lexicographically greater than 107, even though the value it represents is smaller, whereas by radix order $107 \succ 42$. The radix order can also be understood followingly: align x and y to the least significant digit (to the right), pad the shorter word with zeroes on the left until both words have the same length, and then compare them lexicographically. On the other hand, in the lexicographic order we align the two words to the most significant digit (to the left), pad the shorter word with zeroes on the right and then compare them by radix order. Lastly, it is evident that for two words of the same length the lexicographic and radix order are equivalent.

In later sections, we will use terminology from abstract rewriting systems, which we will define here. We will largely follow the notation and terms used in [3], as that will suffice for our needs. More on the theory of abstract rewriting systems may be found in [9] and [10].

A *rewriting system* ρ over A^* is a set of rewriting rules $s \to t$, where $s, t \in A^*$. The regular closure of ρ is denoted $\underset{\rho}{\to}$ and defined followingly:

$$x \underset{\rho}{\to} y \quad \text{if and only if} \quad x = fsg, y = ftg \text{ and } (s \to t) \in \rho.$$

This relation can be called „x is rewritten to y using rule $(s \to t)$“.

The reflexive and transitive closure of $\to$ is denoted $\overset{*}{\underset{\rho}{\to}}$. In other words, $x \overset{*}{\underset{\rho}{\to}} y$ if $y = x$ or there exists a sequence of rewritings $x \underset{\rho}{\to} t_1 \underset{\rho}{\to} t_2 \cdots \underset{\rho}{\to} y$.

The relation $\underset{\rho}{\to}$ is called *confluent* if for every three words z, s, t such that $z \overset{*}{\underset{\rho}{\to}} s$ and $z \overset{*}{\underset{\rho}{\to}} t$ there exists a word v satisfying $s \overset{*}{\underset{\rho}{\to}} v$ and $t \overset{*}{\underset{\rho}{\to}} v$. A rewriting system ρ is *confluent* if the relation $\underset{\rho}{\to}$ is confluent.

If no word $t \neq s$ exists such that $s \overset{*}{\underset{\rho}{\to}} t$ we say that s is *irreducible modulo* ρ. If $v \overset{*}{\underset{\rho}{\to}} s$ where s is irreducible, we say that v *reduces* to s or that s is the *result of reduction* of v. Furthermore, if ρ is confluent, then there exists a *reduction function* $\rho^* : A^* \to A^*$ which maps every word $w \in A^*$ to the irreducible word $\rho^*(w)$ (which is the result of reduction of w). Let us confirm that ρ^* is truly a function. Let $w \in A^*$ be reducible modulo ρ. Then rewrite w using rules from ρ until an irreducible word t is reached. Take w and start rewriting it again and if possible, select in each step a different rewriting rule than that used in deriving t. Continue this process until an irreducible word s is reached. Because ρ is confluent, s must equal t, otherwise we would have words w, s, t such that $w \overset{*}{\underset{\rho}{\to}} s$ and $w \overset{*}{\underset{\rho}{\to}} t$, but no v would exist such that $s \overset{*}{\underset{\rho}{\to}} v$ and $t \overset{*}{\underset{\rho}{\to}} v$, which is in contradiction with the confluence property of ρ. Thus the irreducible word $\rho^*(w)$ is uniquely defined for every w and ρ^* is indeed a function.

Chapter 2

Linear Numeration Systems

In this section we shall introduce linear numeration systems, the focus of our study. Informally, a *numeration system* is the set of rules that we use to assign a word (a representation) to a given value. More formally, a numeration system for the integers can be understood as a map from $\mathbb{N}_0$ to some subset of A^*, where A is a finite alphabet. For example, the standard b-ary system for integers is a map $\mathcal{N} : \mathbb{N}_0 \to \{0, 1, \ldots, b-1\}^*$ such that $\mathcal{N}(x) = x_{N-1} \cdots x_1 x_0$, where $b \in \mathbb{N}$, $b \geq 2$, N is defined by $b^N > x \geq b^{N-1}$ and the digits are defined as $x_k = \lfloor \frac{x}{b^k} \rfloor - \sum_{i=0}^{k-1} x_i b^i$ for all $k \in \{0, 1, \ldots, N-1\}$.

Along with the standard b-ary representation, multiple other numeration systems exist. In the standard b-ary system, a letter represents how many times a given power of b is included in the number that it represents. For example, the string 203, understood as a decimal expansion, represents the value composed by adding $2 \cdot 10^2$, $0 \cdot 10^1$, and $3 \cdot 10^0$. Thus, in the decimal system, we represent numbers using the geometric sequence $B_n = 10^n$, $n \in \mathbb{N}_0$.

However, this is not the only type of sequence that can be used. It is easy to show that $(B_n)_{n=0}^{\infty}$ can be any strictly increasing sequence of positive integers. Numeration systems based on such a sequence are known as *B-systems* (also called U-systems in literature, see Frougny [6]). Such systems can be used to represent all natural numbers, and with a slight modification all integers. In this work we will focus solely on representing natural numbers.

It must be noted that not all sequences generate a numeration system whose language of *normal representations* (which we will define later) is well-behaved. However, a class of sequences that generate numeration systems with reasonable properties are the linearly recurrent sequences with natural coefficients. We will call any such sequence a *basis*.

Definition 2.1. Let $(B_n)_{n=0}^{\infty}$ be a sequence of positive integers satisfying

$$B_n = t_1 B_{n-1} + t_2 B_{n-2} + \cdots + t_m B_{n-m}, \tag{2.1}$$

where $t_1, t_2, \ldots, t_m \in \mathbb{N}_0$, $t_m \neq 0$. Set the m initial conditions equal to

$$\begin{aligned} B_0 &= 1, \\ B_1 &= t_1 + 1, \\ B_2 &= t_1 B_1 + t_2 + 1, \\ &\vdots \\ B_{m-1} &= t_1 B_{m-2} + t_2 B_{m-3} + \cdots + t_{m-1} + 1. \end{aligned} \tag{2.2}$$

Then $(B_n)_{n=0}^{\infty}$ is a *basis* and $m \geq 1$ is its *basis order*.

We can see that this is indeed a generalisation of the b-ary system, since for any natural $b \geq 2$ the sequence $(B_n)_{n=0}^{\infty} = (b^n)_{n=0}^{\infty}$ satisfies the recurrence $B_n = b \cdot B_{n-1}$ and so it is a basis of order 1.

A basis can be used to assign a value to a word over an alphabet of integers followingly.

Definition 2.2. Let $(B_n)_{n=0}^{\infty}$ be a basis. Then a *B-representation* of the number $x \in \mathbb{Z}$ is any string $x_N x_{N-1} \cdots x_0$ over a subset of $\mathbb{Z}$ with $N \in \mathbb{N}_0$ such that $x = \sum_{i=0}^{N} x_i B_i$. The empty word ε is understood as a B-representation of zero.

When it will be necessary to differentiate a B-representation from other representations (typically a decimal representation), we will label the B-representation with a subscript B, as in the following example.

Example 2.3. Let $(B_n)_{n=0}^{\infty}$ be the Fibonacci sequence, i.e. $B_n = B_{n-1} + B_{n-2}$, $B_0 = 1$, $B_1 = 2$. Then $(B_n)_{n=0}^{\infty} = \{1, 2, 3, 5, 8, \dots\}$ and 1001_B is a B-representation of the number six, since $1 \cdot 5 + 0 \cdot 3 + 0 \cdot 2 + 1 \cdot 1 = 6$. Another possible B-representation of six is 111_B, since $1 \cdot 3 + 1 \cdot 2 + 1 \cdot 1 = 6$.

Remark 2.4. Even though in Definition 2.2 we allowed digits to be from $\mathbb{Z}$, we will focus solely on representations of non-negative integers, where non-negative digits will suffice. Therefore, from now on a B-representation is understood to be a word consisting of non-negative digits only (unless specified otherwise).

Remark 2.5. Consider a basis $(B_n)_{n=0}^{\infty}$ with coefficients $t_1, t_2, \dots, t_m$. For brevity, we will often use the expression $(t_1, t_2, \dots, t_m)$-*B-system* or just $(t_1, t_2, \dots, t_m)$-*system* when speaking about the numeration system generated by this basis. For example, the Fibonacci numeration system from Example 2.3 would be known as the $(1, 1)$-system.

Using Definition 2.2 a numeric value can be assigned to any string of integer digits, but typically we want to do the opposite – that is to generate a string representing a given value. To prove that this is possible for any non-negative integer x, we will use the greedy algorithm. However, first, we need to prove a technical lemma.

Lemma 2.6. *Every polynomial of the form*

$$f(x) = x^m - t_1 x^{m-1} - t_2 x^{m-2} - \cdots - t_{m-1} x - t_m,$$

where $m \geq 1$, $t_1, t_2, \dots, t_m \in \mathbb{N}_0$, $t_m \neq 0$, has exactly one real positive root β. Furthermore, all other roots of $f(x)$ lie in the circle $|x| \leq \beta$ and if $\sum_{i=1}^{m} t_i > 1$, then $\beta > 1$.

This lemma is required to show that the alphabet of digits will be finite. In its proof we will utilise the following theorem (from Marden [13] (Theorem 27.2), originally by Cauchy).

Theorem 2.7. *Given a polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where $a_n \neq 0$, define the polynomial*

$$Q(x) = |a_n| x^n - |a_{n-1}| x^{n-1} - \cdots - |a_1| x - |a_0|. \tag{2.3}$$

By Descartes' rule of signs, $Q(x)$ has precisely one positive real root R. Then all zeroes of $p(x)$ lie in the circle $|x| \leq R$.

Proof of Lemma 2.6. Suppose that $f(x)$ has no real root larger than 1. Clearly $f(x)$ is the characteristic polynomial of the basis $B_n = t_1 B_{n-1} + t_2 B_{n-2} + \cdots + t_m B_{n-m}$. Set the m initial conditions equal to

$$B_0 = 1,\ B_1 = t_1 + 1,\ B_2 = t_1 B_1 + t_2 + 1,\ \ldots,\ B_{m-1} = t_1 B_{m-2} + \cdots + t_{m-1} + 1.$$

Since $\sum_{i=1}^{m} t_i > 1$, this yields a basis $(B_n)_{n=0}^{\infty}$ that is strictly increasing. Therefore, if we solve the recurrence for B_n, at least one root of $f(x)$ must have absolute value larger than 1. Take the root maximum in modulus and label it $\beta_{\max}$. By assumption, $\beta_{\max}$ cannot be real and positive.

From Descartes' rule of signs we can see that $f(x)$ has exactly one positive real root. Label this root β_{pos}. Evidently by assumption $\beta_{\text{pos}} \leq 1$. Additionally, we can see that $f(x)$ is in the same form as the right-hand side of (2.3), i.e. $Q(x) = f(x)$. Therefore, by Theorem 2.7 all roots of $f(x)$ lie in the circle $|x| \leq R = \beta_{\text{pos}} \leq 1$. However, at least one root outside this circle exists $(\beta_{\max})$, leading to a contradiction. □

Now we can prove that every number can be represented using a given basis.

Theorem 2.8 (B-Representation.)**.** *Let $(B_n)_{n=0}^{\infty}$ be a basis with coefficients satisfying $\sum_{i=1}^{m} t_i > 1$ and $t_m \neq 0$. Then for every $x \in \mathbb{N}_0$ there exists an $N \in \mathbb{N}_0$ and coefficients $a_i \in A = \{0, 1, 2, ..., a\}$, $i = 0, 1, 2, ..., N$ such that*

$$x = a_N B_N + a_{N-1} B_{N-1} + \cdots + a_1 B_1 + a_0 B_0,$$

where $a \in \mathbb{N}$ is a constant satisfying $\left\lfloor \sup_{N \in \mathbb{N}_0} \frac{B_{N+1}}{B_N} \right\rfloor \geq a$. In other words, every $x \in \mathbb{N}_0$ has a B-representation $a_N a_{N-1} \cdots a_1 a_0$ over the canonical alphabet *A.*

Proof. We will prove the existence of the B-representation of x by constructing it. Given an $x \in \mathbb{N}_0$ and a basis $(B_n)_{n=0}^{\infty}$ we proceed by the following greedy algorithm. Set

$$N := \max\{n|\, x \geq B_n\}$$

and let initially $R := x, i := N$. Then in the i-th iteration of the algorithm do the following:

1. Set $a_i := \lfloor R/B_i \rfloor$.
2. Set R equal to the remainder of the division by B_i, i.e. $R := R - a_i B_i$
3. If $i = 0$, terminate, otherwise lower i by one and repeat from step 1).

The representation is generated from the most significant digit. It is evident that this algorithm always terminates, since the number of iterations is finite. Also, the resulting word $a_N a_{N-1} \cdots a_1 a_0$ is clearly a B-representation of x. What remains is to verify that all digits belong to the set $A = \{0, 1, 2, ..., a\}$.

Evidently, for every non-zero x an N exists such that $B_{N+1} > x \geq B_N$. Dividing by B_N yields the inequality

$$B_{N+1}/B_N > x/B_N \geq 1. \tag{2.4}$$

Since B_n is linearly recurrent and real, we can write it as a linear combination of m real base sequences $\zeta_1^n, \ldots \zeta_m^n$, which we will construct from the m roots $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{C}$ of the characteristic polynomial of the basis B_n.

The characteristic polynomial of the basis B_n is of the form

$$f(x) = x^m - t_1 x^{m-1} - t_2 x^{m-2} - \ldots - t_{m-1} x - t_m.$$

The polynomial $f(x)$ has real coefficients, therefore for every complex root β_i of $f(x)$ its conjugate $\overline{\beta_i}$ will also be a root of $f(x)$. Suppose, without loss of generality, that the roots are ordered by modulus and multiplicity, that is $|\beta_1| \geq |\beta_2| \geq \cdots \geq |\beta_m|$ and repeated roots are ordered adjacently. That is for every root β_r with multiplicity ν_r there exists exactly one $1 \leq k \leq m$ such that $\beta_k = \beta_{k+1} = \cdots = \beta_r = \cdots = \beta_{k+\nu_r-1}$.

Let us now construct the sequences ζ_k^n. For every root $\beta_k \in \mathbb{R}$ with multiplicity ν_k, where k is such that $\beta_k = \beta_{k+1} = \cdots = \beta_r = \cdots = \beta_{k+\nu_r-1}$, set for all $0 \leq j \leq \nu_k - 1$

$$\zeta_{k+j}^n = n^j \beta_k^n.$$

By Lemma 2.6, $f(x)$ will have exactly one positive real root larger than 1 and all other roots will be smaller in modulus, denote this root $\beta = \beta_1$.

For every $\beta_k \in \mathbb{C}\backslash\mathbb{R}$ with multiplicity ν_k denote its complex conjugate $\beta_l = \overline{\beta_k}$, $\nu_l = \nu_k$, where the index $1 \leq l \leq m$ is again minimal. For unambiguity let $k < l$. Then for all $0 \leq j \leq \nu_k - 1$ set

$$\zeta_{k+j}^n := n^j \frac{{\beta_k}^n + \beta_l^n}{2} = n^j |\beta_k|^n \cos\left(n \frac{\mathrm{Re}(\beta_k)}{|\beta_k|}\right),$$
$$\zeta_{l+j}^n := n^j \frac{{\beta_k}^n - \beta_l^n}{2} = n^j |\beta_k|^n \sin\left(n \frac{\mathrm{Im}(\beta_k)}{|\beta_k|}\right).$$

In this way the m basic real sequences $\zeta_1^n, \ldots, \zeta_m^n$ are constructed. Substituting initial conditions for B_n yields coefficients $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ such that

$$B_n = \sum_{i=1}^{m} \alpha_i \zeta_i^n. \tag{2.5}$$

Substituting (2.5) into (2.4) and the fact that $\zeta_1^{N+1} = \beta_1^{N+1} = \beta^{N+1}$ yields for all $N \in \mathbb{N}_0$

$$1 < \frac{B_{N+1}}{B_N} = \frac{\sum_{i=1}^m \alpha_i \zeta_i^{N+1}}{\sum_{i=1}^m \alpha_i \zeta_i^N} = \frac{\alpha_1 \beta^{N+1} + \sum_{i=2}^m \alpha_i \zeta_i^{N+1}}{\alpha_1 \beta^N + \sum_{i=2}^m \alpha_i \zeta_i^N} = \beta \frac{\alpha_1 + \sum_{i=2}^m \alpha_i \frac{\zeta_i^{N+1}}{\beta^{N+1}}}{\alpha_1 + \sum_{i=2}^m \alpha_i \frac{\zeta_i^N}{\beta^N}}.$$

By Lemma 2.6, it is evident that

$$\lim_{N \to +\infty} \frac{B_{N+1}}{B_N} = \beta > 1.$$

The ratio of consecutive elements of B_n is therefore for all $N \in \mathbb{N}_0$ bounded by some constant $K = \sup_{N \in \mathbb{N}} \frac{B_{N+1}}{B_N}$, $K \in \mathbb{Q}$. Together with (2.4) this results in

$$K \geq B_{N+1}/B_N > x/B_N \geq 1.$$

The digit $a_N = \lfloor x/B_N \rfloor$ can therefore have only a finite number of values. Consider now other digits a_i, $0 \leq i < N$. As the remainder of the division by B_{i+1} will always be smaller than B_{i+1} (step 2. of the greedy algorithm), in the i-th iteration the following will hold:

$$B_{i+1} > R,$$

dividing by B_i this leads to

$$K \geq \frac{B_{i+1}}{B_i} > \frac{R}{B_i} \geq 0.$$

After rounding we can see that $a_i = \lfloor R/B_i \rfloor$ can also have only a finite number of values. The digit a_i is non-negative because R is non-negative and B_i is positive.

Clearly, a maximum digit $a \in \mathbb{N}$ exists such that $a_i \leq a$ for all $i = 0, 1, \ldots, N$ and

$$a \leq \left\lfloor \sup_{N \in \mathbb{N}_0} \frac{B_{N+1}}{B_N} \right\rfloor = \lfloor K \rfloor ,$$

The *canonical alphabet* $A = \{0, 1, \ldots, a\}$ for the basis $(B_n)_{n=0}^{\infty}$ is therefore well-defined. □

In general the value of a can be known only by calculating the elements of the basis. However, if the basis coefficients satisfy the inequality $t_1 \geq t_2 \geq \cdots \geq t_m \geq 1$, it is possible to deduce a immediately from the coefficients of the recurrence, which we will prove shortly. This type of basis will also have certain other practical properties, so we will give it a name.

Definition 2.9. Let $(B_n)_{n=0}^{\infty}$ be a basis of order m whose coefficients satisfy

$$t_1 \geq t_2 \geq \cdots \geq t_m \geq 1. \tag{2.6}$$

Then we say that the basis $(B_n)_{n=0}^{\infty}$ has the *(F) property* or that $(B_n)_{n=0}^{\infty}$ is an *(F) basis.* By extension a B-system is said to have the (F) property if its basis has the (F) property.

The above definition is carried over from numeration systems with a non-integer base, the so called *β-systems*, which are studied for example in [5, 6]. The F stands for *finite*, as a real base $\beta > 1$ is said to have the (F) property if every member of $\mathbb{Z}\left[\beta^{-1}\right] \cap \mathbb{R}^{+}$ has a finite β-expansion. Take the polynomial

$$\chi(x) = x^m - t_1 x^{m-1} - \cdots - t_{m-1} x - t_m \tag{2.7}$$

whose coefficients satisfy (2.6) and denote $\beta > 1$ its root greatest in modulus. In [4] Frougny and Solomyak prove that β is a Pisot number and that it has the (F) property. A *Pisot number* is an algebraic integer whose conjugates are all less than one in modulus. Notice that the polynomial $\chi(x)$ is the characteristic polynomial of a basis satisfying Definition 2.9, which is why we use this name.

Moreover, β-systems are closely tied to B-systems. Every B-system can be uniquely associated with a β-system whose language of greedy representations shares many combinatoric properties with the language of greedy B-representations. More on this association can be found in [5, 6].

We proceed with determining the canonical alphabet of B-systems with the (F) property.

Lemma 2.10. *Let $(B_n)_{n=0}^{\infty}$ be an (F) basis. Then the canonical alphabet of the numeration system generated by $(B_n)_{n=0}^{\infty}$ is equal to $A = \{0, 1, \ldots, t_1\}$.*

Proof. Take $x \in \mathbb{N}$ and an (F) basis $(B_n)_{n=0}^{\infty}$. Suppose that we are generating the B-representation of x using the algorithm from Theorem 2.8. Let $N \in \mathbb{N}_0$ such that $B_{N+1} > x \geq B_N$. Substituting for B_{N+1} from the recurrence (2.1) and dividing by B_N yields

$$\begin{aligned} t_1 B_N + t_2 B_{N-1} + \cdots + t_m B_{N-m+1} &> x \geq B_N, \\ t_1 + \frac{t_2 B_{N-1} + \cdots + t_m B_{N-m+1}}{B_N} &> x/B_N \geq 1. \end{aligned} \tag{2.8}$$

Let us now focus on the fraction on the left hand side of (2.8). Substituting for B_N in the denominator yields

$$\frac{t_2 B_{N-1} + \cdots + t_m B_{N-m+1}}{t_1 B_{N-1} + \cdots + t_{m-1} B_{N-m+1} + t_m B_{N-m}} < 1. \tag{2.9}$$

The inequality holds thanks to the (F) property $t_1 \geq t_2 \geq \cdots \geq t_m \geq 1$, which implies that the denominator is strictly larger than the numerator. Let us now return to the inequality (2.8). Due to inequality (2.9), by rounding we get

$$t_1 \geq \lfloor x/B_N \rfloor \geq 1.$$

Consider now the other digits. Because the remainder after division by B_{i+1} will always be smaller than B_{i+1}, in the i-th iteration of the algorithm we will have

$$B_{i+1} > R.$$

Following the same steps as above we arrive at

$$t_1 + \frac{t_2 B_{i-1} + \cdots + t_m B_{i-m+1}}{B_i} > R/B_i,$$

and thus the digit $a_i = \lfloor R/B_i \rfloor$ is bounded by

$$t_1 \geq \lfloor R/B_i \rfloor \geq 0.$$

All digits $a_N, a_{N-1}, \ldots, a_1, a_0$ are contained in the finite alphabet $A = \{0, 1, \ldots, t_1\}$. □

For further study of B-representations, we define the *value of a word.*

Definition 2.11. Given a basis $(B_n)_{n=0}^{\infty}$ and some alphabet $C \subset \mathbb{Z}$, the *evaluator function* $\pi : C^* \to \mathbb{Z}$ is defined for every word $w = w_N w_{N-1} \cdots w_1 w_0 \in C^*$ as

$$\pi(w) = \sum_{i=0}^{N} w_i B_i,$$

for the empty word we set $\pi(\varepsilon) = 0$. More often we will say that $\pi(w)$ is the *value of word w (in the numeration system with basis $(B_n)_{n=0}^{\infty}$).*

As is evident from Definition 2.2, several words can represent the same value. In other words, in general the map π is not injective. The only case when it is injective is when a B-system coincides with the standard b-ary system, that is, the basis is of the form $B_n = bB_{n-1}$ for some $b \in \mathbb{N}, b \geq 2$.

Example 2.12. Consider the basis $B_n = 3B_{n-1} + B_{n-2}$. Then $(B_n)_{n=0}^{\infty} = \{1, 4, 13, 43, 142, \ldots\}$, the canonical alphabet is equal to $A = \{0, 1, 2, 3\}$ and the number $x = 286_{\mathrm{DEC}}$ has three different B-representations (over A).

$$\begin{aligned} 286_{\mathrm{DEC}} &= 20002_B \qquad (2.10) \\ &= 13102_B \\ &= 13033_B. \end{aligned}$$

Only the first representation (2.10) could have been constructed by the algorithm in Theorem 2.8, since it has the largest possible most significant digit. Accordingly, the representation (2.10) is largest by the radix order. This representation will be known as the *greedy representation.*

Definition 2.13. Let $x \in \mathbb{N}_0$. Then the B-representation constructed by the algorithm from Theorem 2.8 is called the *greedy representation,* or equivalently, the *normal representation.* It will be denoted $\langle x \rangle_B$. The set of all greedy B-representations will be known as the *language of greedy representations*, denoted $L(B)$.

Theorem 2.14 (Properties of Greedy Representations.)**.** *Let $x_N x_{N-1} \cdots x_1 x_0 = \langle x \rangle_B$ be the greedy B-representation of some $x \in \mathbb{N}$. Then the following holds:*

1. $x_N \neq 0$.
2. *$\langle x \rangle_B$ is the greatest by radix order among all B-representations of x.*
3. *$\langle x \rangle_B \succ \langle y \rangle_B \Leftrightarrow x > y$ for every two $x, y \in \mathbb{N}_0$.*

Proof. Property 1. Evident from the proof of Theorem 2.8.

Property 2. The greedy representation is the longest among all B-representations of x, since the most significant digit x_N is obtained by dividing x by the greatest element of the basis B that is smaller than x. Additionally, this digit will be the greatest possible:

$$x_N = \max \{ k \mid kB_N < x \} .$$

Suppose now a different representation of x, denote it $\tilde{x} = \tilde{x}_{\tilde{N}} \tilde{x}_{\tilde{N}-1} \cdots \tilde{x}_1 \tilde{x}_0$. Evidently no $\tilde{x}$ can have $\tilde{x}_N > x_N$, so one of the following must occur:

a) $|\langle x \rangle_B| > |\tilde{x}|$

b) $|\langle x \rangle_B| = |\tilde{x}|$ and $\tilde{x}_N < x_N$.

c) $|\langle x \rangle_B| = |\tilde{x}|$ and $\tilde{x}_N = x_N$.

In cases a), b) we immediately obtain $\langle x \rangle_B \succ \tilde{x}$. In case c) the words $\langle x \rangle_B$ and $\tilde{x}$ share a common prefix beginning with (but not limited to) the digit x_N. Removing this prefix yields two words $\langle x \rangle_B^*$, $\tilde{x}^*$ of length $M+1 < N+1$ that start with digits $x_M \neq \tilde{x}_M$. Because the greedy algorithm always selects the greatest digit, the inequality $x_M > \tilde{x}_M$ must hold and so case b) applies.

Property 3. $(\Rightarrow)$:
Let $x_N x_{N-1} \cdots x_0 = \langle x \rangle_B \succ \langle y \rangle_B = y_M y_{M-1} \cdots y_0$. Then from the definition of the radix ordering one of the following holds:

a) $|\langle x \rangle_B| > |\langle y \rangle_B|$, i.e. $N > M$.

b) $N = M$ and an $r \leq N$ exists such that $x_r > y_r$ and $x_i = y_i$ for all $N \geq i > r$.

In case a) the inequality $x > y$ is evident from the fact that $B_{N+1} > x \geq B_N$, $B_{M+1} > y \geq B_M$ and $N > M$ implies $N \geq M + 1$. In total $x \geq B_N > y$.

Case b) warrants a more thorough analysis. Clearly

$$x - y = \sum_{i=0}^{N} x_i B_i - \sum_{i=0}^{N} y_i B_i = \sum_{i=0}^{r} x_i B_i - \sum_{i=0}^{r} y_i B_i.$$

Because $x_r - y_r \geq 1$, this can be bounded from below by

$$\sum_{i=0}^{r} x_i B_i - \sum_{i=0}^{r} y_i B_i \geq B_r + \sum_{i=0}^{r-1} (x_i - y_i) B_i \geq B_r - \sum_{i=0}^{r-1} y_i B_i.$$

The word $y_{r-1}y_{r-2}\dots y_0$ is a greedy representation of some value $\tilde{y} < y$ (due to the already proven case a) and also due to the fact that any suffix of a greedy representation is a greedy representation). Since its length is precisely r, the value $\tilde{y}$ must satisfy $B_r > \tilde{y} \geq B_{r-1}$. Therefore

$$B_r - \sum_{i=0}^{r-1} y_i B_i > 0$$

from which $x > y$ follows.

($\Leftarrow$) :
The reverse implication is a corollary of the greedy algorithm. Let $x > y$. If an N exists such that $x \geq B_N > y$, the greedy representation $\langle x \rangle_B$ will be longer than $\langle y \rangle_B$ and so $\langle x \rangle_B \succ \langle y \rangle_B$. If $B_{N+1} > x > y \geq B_N$, then an $0 \leq r \leq N$ exists such that in the r-th step of the greedy algorithm run simultaneously for x and y we will have remainders $R^{(x)}$ a $R^{(y)}$ which will satisfy

$$\left\lfloor R^{(x)}/B_{N-r-1} \right\rfloor > \left\lfloor R^{(y)}/B_{N-r-1} \right\rfloor .$$

For the $(N-r)$-th digits this will result in $x_{N-r} > y_{N-r}$ and $x_i = y_i$ for all $N - r < i \leq N$, therefore $\langle x \rangle_B \succ \langle y \rangle_B$, by the definition of the radix order. □

2.1 Combinatorics of Linear Numeration Systems

In this section we will explore some further combinatorial properties of B-systems, most importantly factors of value zero and rewriting rules generated by B-systems. This will be followed by establishing the *confluent* B-systems. Finally, we will show a way how to recognise greedy representations in (F) systems. More on other properties of B-systems (for example, the regularity of $L(B)$) can be found in [5, 6, 8].

2.1.1 Abstract Rewriting and Confluent B-Systems

In this section we will introduce the confluent numeration systems, first established and studied by Frougny [3]. The first step is to realise that all linear numeration systems implicitly generate a rewriting system that is given by the basis recurrence.

Definition 2.15. Consider some alphabet $C = \{0, 1, \dots, c\}$. Then the *rewriting system generated by the rule* $0t_1t_2 \cdots t_m \to 10^m$ is defined as

$$\begin{aligned} \rho_C = \{ & x_m x_{m-1} \cdots x_0 \to (x_m{+}1)\,(x_{m-1}{-}t_1) \cdots (x_0{-}t_m) | \\ & 0 \leq x_m < c \text{ and } x_{m-i} \geq t_i \text{ for all } i \geq 1 \} . \end{aligned}$$

Every B-system with coefficients $t_1, t_2, \dots, t_m$ thus defines a rewriting system generated by the rule $0t_1t_2 \cdots t_m \to 10^m$ on its canonical alphabet A, which we denote ρ_A. We will call ρ_A the *rewriting system associated with the* $(t_1, t_2, \dots, t_m)$*-system.* Because of the basis recurrence, this rewriting system has the practical property that it preserves the numerical value of words. More formally, given a basis, every two words w and v over the canonical alphabet A satisfy

$$w \xrightarrow[\rho_A]{*} v \text{ iff } \pi(w) = \pi(v).$$

Fact 2.16. *Take a B-system with canonical alphabet A and its associated rewriting system ρ_A. Then evidently all words $w \neq v$ for which $w \xrightarrow[\rho_A]{*} v$ satisfy $w \succ v$.*

We will now show an example of a confluent rewriting system in the context of B-systems.

Example 2.17. Consider the basis $B_n = 3B_{n-1}+3B_{n-2}+2B_{n-3}$. Then $B_n = \{1, 4, 15, 59, \dots\}$, the canonical alphabet is $A = \{0, 1, 2, 3\}$, and ρ_A is generated by the rule $0332 \to 1000$:

$$\rho_A = \left\{ \begin{array}{llll} 0332 & \to 1000, & 0333 & \to 1001, \\ 1332 & \to 2000, & 1333 & \to 2001, \\ 2332 & \to 3000, & 2333 & \to 3001. \end{array} \right\}$$

Let $w = 0332333$. Then there are two possible reductions of w:

$$0332333 \underset{\rho_A}{\longrightarrow} 1000333 \underset{\rho_A}{\longrightarrow} 1001001,$$

and

$$0332333 \underset{\rho_A}{\longrightarrow} 0333001 \underset{\rho_A}{\longrightarrow} 1001001.$$

Both reductions lead to the same result, because the rewriting system ρ_A is confluent. (See Theorem 2.19).

Compare this with an example of a rewriting system that is not confluent:

Example 2.18. Consider the basis $B_n = 3B_{n-1} + 2B_{n-2} + B_{n-3}$. Then $B_n = \{1, 4, 14, 51, \dots\}$, the canonical alphabet is $A = \{0, 1, 2, 3\}$, since B_n satisfies the (F) property and ρ_A is generated by the rule $0332 \to 1000$:

$$\rho_A = \left\{ \begin{array}{lllllll} 0321 & \to 1000, & 0322 & \to 1001, & \cdots & 0333 & \to 1012, \\ 1321 & \to 2000, & 1322 & \to 2001, & \cdots & 1333 & \to 2012, \\ 2321 & \to 3000, & 2322 & \to 3001, & \cdots & 2333 & \to 3012. \end{array} \right\}$$

Let $w = 032333$. Then there are two possible reductions of w. Either

$$0\underline{323}33 \underset{\rho_A}{\longrightarrow} 100232,$$

or

$$032\underline{333} \underset{\rho_A}{\longrightarrow} 033011.$$

Both 100232 are 033011 are irreducible modulo ρ_A, thus ρ_A is not a confluent rewriting system.

Naturally, we are led to ask for what B-systems is the associated rewriting system confluent. Frougny showed in [3] that confluent systems can be characterised by the coefficients of their basis.

Theorem 2.19 (Frougny)**.** *Suppose a basis $(B_n)_{n=0}^{\infty}$ of order m with coefficients $t_1, t_2, \dots, t_m \in \mathbb{N}_0$ and canonical alphabet A. Then the rewriting system ρ_A associated with the B-system is confluent if and only if the coefficients of $(B_n)_{n=0}^{\infty}$ satisfy*

$$t_1 = t_2 = \cdots = t_{m-1} = a, \qquad t_m = b, \tag{2.11}$$

where $a \geq b \geq 1$.

The above theorem justifies the following definition.

Definition 2.20. A basis $(B_n)_{n=0}^{\infty}$ of order m is called *confluent* if its coefficients satisfy (2.11). By extension, a B-system is confluent if its basis is confluent.

Remark 2.21. Evidently, all confluent B-systems are also (F) systems. The opposite inclusion does not hold, as was illustrated by Example 2.18. However, if we limit ourselves to only B-systems of order 2, the confluent and (F) systems coincide.

Importantly, the confluence property allows us to perform *normalisation* by means of a finite transducer, a result due to Frougny [3]. This was one of the initial motivations of the study of such systems. We define what is meant by normalisation.

Definition 2.22. Take a B-system with canonical alphabet A and some other alphabet $C \supset A$. Then *normalisation* is the map $\nu : C^* \to A^*$ that assigns to a word w the greedy (normal) representation of the value represented by w, i.e.

$$\nu(w) = \langle \pi(w) \rangle_B .$$

In effect, when reducing using the rewriting system ρ_A associated with a confluent B-system, we are performing normalisation. This can be restated as the following theorem, also from [3]:

Theorem 2.23 (Frougny)**.** *Suppose a confluent B-system with canonical alphabet A. Then normalisation in this system is equivalent to reduction in the associated rewriting system ρ_A. Formally for every $w \in A^*$*

$$\nu(w) = \rho_A^*(w).$$

Recall now Example 2.17.

$$\begin{array}{ccc} 0332333 & \xrightarrow[\rho_A]{} & 1000333 \\ {\scriptstyle\rho_A}\downarrow & & {\scriptstyle\rho_A}\downarrow \\ 0333001 & \xrightarrow[\rho_A]{} & 1001001 \end{array}$$

We can say that the three words 0332333, 0333001, and 1000333 normalise to 1001001.

The interesting property of confluent numeration systems is that in order to perform normalisation, it suffices to use only rules from ρ_A. In other numeration systems, sometimes we have to go backwards, i.e. use a rule from ρ_A^{-1}. Take the representation 033011 from Example 2.18. Then 033011 normalises by the use of one backward rule and one forward rule:

$$033011 \xleftarrow[\rho_A]{} 032332 \xrightarrow[\rho_A]{} 100232.$$

Another interesting property of confluent systems is that the rewriting system consisting of rules applied in reverse is also confluent. We will call this system the *inverse (reverse) rewriting system* ρ^{-1}. Hence, for a given B-system with coefficients $t_1, t_2, \ldots, t_m$ and canonical alphabet $A = \{0, 1, \ldots, a\}$, the *associated inverse rewriting system* ρ_A^{-1} is defined as

$$\begin{aligned} \rho_A^{-1} = \{ & x_m x_{m-1} \cdots x_0 \to (x_m - 1)\,(x_{m-1} + t_1) \cdots (x_0 + t_m) | \\ & 1 \le x_m < a \text{ and } 0 \le x_{m-i} \le a - t_i \text{ for all } i \ge 1 \} . \end{aligned}$$

For completeness, we shall establish the concept of the *lazy representation* of an integer.

Definition 2.24. Suppose a B-system with canonical alphabet A. Then for every $n \in \mathbb{N}_0$ we define the *lazy representation of* n as the word $\rangle n \langle_B \in A^*$ that is radix smallest among all the representations of x over the alphabet A.

The lazy representation is well-defined, since the set of representations of a number $n \in \mathbb{N}_0$ is finite and two different words $w \neq v$ cannot have the same radix value (because that occurs only when w and v are identical). Clearly, if a given number n has only one representation, its lazy and greedy representation coincide.

Example 2.25. Take the $(2,1)$-system. Then the canonical alphabet is $A = \{0,1,2\}$, the basis is equal to $B_n = \{1,3,7,17,41,99,\dots\}$ and the associated rewriting system consists of the rules $\rho_A = \{021 \to 100, 121 \to 200, 022 \to 101, 122 \to 201\}$. The greedy representation of 49 is 10101_B, whereas its lazy representation is 02122_B.

Unfortunately, no direct algorithm for constructing lazy representations is known. The only way they can be obtained is by constructing the greedy representation and reducing it using the associated reverse rewriting system until the lazy representation is reached.

We will now move on to introduce another practical concept for dealing with B-systems, which we will utilise in proofs. It is the concept of the so-called *factors of value zero*, a subset of the B-representations of zero. For a given B-system, they are easy to determine.

Definition 2.26. Consider a B-system of order m with coefficients $t_1, t_2, \dots, t_m \in \mathbb{N}_0$ and canonical alphabet A. Denote by the overline a minus sign, i.e. $\overline{t_1} = -t_1$. Then the *factors of value zero* are the words

$$1\overline{t_1}\overline{t_2}\cdots\overline{t_{m-1}}\overline{t_m}, \quad \overline{1}t_1t_2\cdots t_{m-1}t_m.$$

Furthermore, the $m-1$ *initial representations of zero (initial factors of value zero)* are the B-representations

$$\begin{array}{cccccc}
1 & \overline{t_1} & \overline{t_2} & \cdots & \overline{t_{m-2}} & \overline{t_{m-1}+1}, \\
 & 1 & \overline{t_1} & \cdots & \overline{t_{m-3}} & \overline{t_{m-2}+1}, \\
 & & \ddots & \ddots & \ddots & \vdots \\
 & & & 1 & \overline{t_1} & \overline{t_2+1}, \\
 & & & & 1 & \overline{t_1+1},
\end{array}
\qquad
\begin{array}{cccccc}
\overline{1} & t_1 & t_2 & \cdots & t_{m-2} & t_{m-1}+1, \\
 & \overline{1} & t_1 & \cdots & t_{m-3} & t_{m-2}+1, \\
 & & \ddots & \ddots & \ddots & \vdots \\
 & & & \overline{1} & t_1 & t_2+1, \\
 & & & & \overline{1} & t_1+1.
\end{array}$$

If we denote the canonical alphabet $A = \{0,1,\dots,a\}$, then the factors of value zero are the only words z over $\{-a,\dots,0,\dots,a\}$ such that $\pi(z) = 0$ and $|z| = m+1$. We call them factors of value zero because they satisfy this property regardless of how many zeros we write to their right.

On the other hand, the initial representations of zero in general need not have digits contained in $\{-a,\dots,0,\dots,a\}$. This will in occur systems with the (F) property whenever a recurrence coefficient t_r, $r \geq 2$ exists such that $t_r = t_1$. Also, all the confluent systems have this property. Moreover, they satisfy $\pi(z) = 0$ only if their least significant digit is in the place of B_0. That is why we will refer to them as initial representations of zero. Note that they are needed because of the initial conditions that we adopted in Definition 2.1.

In further sections, we will use these representations in proofs when we will need to rewrite a word to another one representing the same value and to ensure that the resulting word has its digits contained in the canonical alphabet A. In effect, adding the factor $1\overline{t_1}\overline{t_2}\cdots\overline{t_{m-1}t_m}$ digit by digit to some B-representation w over an alphabet C containing A corresponds to using one of the rules from the rewriting system ρ_C generated by the rule $0t_1t_2\cdots t_m \to 10^m$. On the other hand, the initial representations of value zero will serve their purpose at the end of representations, where the standard factor of value zero does not fit.

2.1.2 Recognising Greedy Representations

Definition 2.27. Let $(B_n)_{n=0}^{\infty}$ be an (F) basis with coefficients $t_1, t_2, \ldots, t_m$. Then the *maximal factor* is the word

$$t_1 t_2 \cdots t_{m-1}(t_m - 1).$$

This factor will be pivotal to recognising greedy representations in B-systems with the (F) property. Namely, we will prove that a word over the canonical alphabet is a greedy representation if and only if it avoids factors which are lexicographically greater than the maximal factor and whose length is smaller than or equal m. To ensure that the B-system posesses this behaviour is the reason why in Definition 2.1 the basis initial conditions are chosen in the form (2.2). In a way, the initial conditions (2.2) are optimal, which we show in the following example.

Example 2.28. Consider the basis $B_n = 3B_{n-1} + 2B_{n-2} + B_{n-3}$. Then $A = \{0, 1, 2, 3\}$ is the canonical alphabet and the maximal factor is equal to $t_2 t_1 (t_0 - 1) = 320$. Suppose three sets of initial conditions:

$$\text{(A)} \begin{cases} B_0 &= 1, \\ B_1 &= 3B_0 = 3, \\ B_2 &= 3B_1 + 2B_0 = 11. \end{cases}$$

$$\text{(B)} \begin{cases} B_0 &= 1, \\ B_1 &= 3B_0 + 1 = 4, \\ B_2 &= 3B_1 + 2B_0 + 1 = 15. \end{cases}$$

$$\text{(C)} \begin{cases} B_0 &= 1, \\ B_1 &= 3B_0 + 2 = 5, \\ B_2 &= 3B_1 + 2B_0 + 2 = 19. \end{cases}$$

Then in case (A) there are words that are not a greedy representation, but do not contain a factor lexicographically greater than the maximal factor. It is for example $\pi(3_B) = 3_{\text{DEC}} = \pi(10_B)$ but $3 \not\succ_{\text{lex}} 320$.

On the other hand, in case (C) there exist values which cannot be represented by a word above the canonical alphabet – for example there is no word $w \in A^*$ such that $\pi(w) = 4$.
In case (B) none of these occur. Once a word contains a factor greater than the maximal factor, the word is not a greedy representation. Compare:

$$\begin{aligned} \pi(32_B) &= 14_{\text{DEC}}, & 32 &\prec_{\text{lex}} 320. \\ \pi(100_B) = \pi(33_B) &= 15_{\text{DEC}}, & 33 &\succ_{\text{lex}} 320. \end{aligned}$$

With initial conditions in the form (2.2), we will now proceed with proving that greedy representations avoid factors greater than the maximal factor. For that we will require the following technical lemma.

Lemma 2.29. *Let $(B_n)_{n=0}^{\infty}$ be an (F) basis of order m with canonical alphabet A. Then every word $w \in A^*$ that has length $|w| \le N$ and value $\pi(w) \ge B_N$ where $N \in \mathbb{N}$, contains a factor of length less than or equal to m that is lexicographically greater than the maximal factor.*

Proof. We will prove the claim by induction on the length of the word w.

1. $N \in \{1, 2, \ldots, m-1\}$:
Suppose that $w = w_M w_{M-1} \ldots w_0$, where $|w| = M+1 \leq N$ and $\pi(w) \geq B_N$. If $M+1 < N$, extend w to length N by adding zeroes to the left.
Then

$$\pi(w) = \sum_{i=1}^{N} w_{N-i} B_{N-i} \geq B_N = \sum_{i=1}^{N} t_i B_{N-i} + 1.$$

The equality on the right hand side follows from the recurrence relation and initial conditions for B_n. Together this implies

$$\sum_{i=1}^{N} (w_{N-i} - t_i) B_{N-i} - 1 \geq 0. \tag{2.12}$$

Suppose now that

$$w_{N-1} w_{N-2} \cdots w_1 w_0 \preceq_{\text{lex}} t_1 t_2 \cdots t_{N-1} t_N \prec_{\text{lex}} t_1 t_2 \cdots t_{m-1} (t_m - 1),$$

i.e. that there exists an $1 \leq r \leq N$ such that $w_{N-r} < t_r$ and $w_{N-j} = t_j$ for all $1 \leq j < r$. Then, we can rewrite inequality (2.12) as

$$\sum_{i=r+1}^{N} (w_{N-i} - t_i) B_{N-i} - 1 \geq -(w_{N-r} - t_r) B_{N-r}. \tag{2.13}$$

Because $t_q \geq 1$ for all $q \in \{1, 2, \ldots, m\}$ and $0 \leq w_{N-i} \leq t_1$ for all $1 \leq i \leq N$, the coefficients $(w_{N-i} - t_i)$ in the sum on the left hand side of (2.12) (and (2.13)) are at most equal to $t_1 - 1$ for all $r < i \leq N$. We can therefore bound the left hand side of (2.13) by

$$\sum_{i=r+1}^{N} (w_{N-i} - t_i) B_{N-i} - 1 < \sum_{i=r+1}^{N} (t_1 - 1) B_{N-i},$$

which after reindexing is equivalent to

$$\sum_{i=1}^{N-r} (w_{N-r-i} - t_{r+i}) B_{N-r-i} - 1 < \sum_{i=1}^{N-r} (t_1 - 1) B_{N-r-i}. \tag{2.14}$$

On the other hand, $(w_{N-r} - t_r)$ is smaller than or equal to -1, so we bound the right hand side of (2.13) followingly:

$$-(w_{N-r} - t_r) B_{N-r} \geq B_{N-r} = \sum_{i=1}^{N-r} t_i B_{N-r-i} + 1. \tag{2.15}$$

Lastly, we will verify that

$$\sum_{i=1}^{N-r} t_i B_{N-r-i} + 1 \geq \sum_{i=1}^{N-r} (t_1 - 1) B_{N-r-i}. \tag{2.16}$$

We can rewrite this inequality as

$$\sum_{i=1}^{N-r} t_i B_{N-r-i} - \sum_{i=1}^{N-r} (t_1 - 1) B_{N-r-i} + 1 \geq 0,$$

and write it as the digit by digit sum

$$\begin{array}{ccccc} \overline{t_1} & \overline{t_2} & \cdots & \overline{t_{N-r-1}} & \overline{t_{N-r}+1} \\ \underline{\overline{t_1-1}} & \underline{\overline{t_1-1}} & \underline{\cdots} & \underline{\overline{t_1-1}} & \underline{\overline{t_1-1}} \\ * & * & \cdots & * & *. \end{array}$$

We want to prove that all digits marked $*$ will be non-negative. If $N-r \leq 1$, the proof is completed. Hence suppose now that $N-r > 1$.
Then using the initial conditions of the basis we may add the first $N-r-1$ initial representations of zero, resulting in

$$\begin{array}{cccccc} \overline{t_1} & t_2 & t_3 & \cdots & t_{N-r-1} & t_{N-r}+1 \\ 1 & \overline{t_1} & t_2 & \cdots & t_{N-r-2} & t_{N-r-1}+1 \\ & 1 & \overline{t_1} & \cdots & t_{N-r-3} & t_{N-r-2}+1 \\ & & 1 & \cdots & t_{N-r-4} & t_{N-r-3}+1 \\ & & & \ddots & \vdots & \vdots \\ & & & & 1 & \overline{t_1+1} \\ \underline{\overline{t_1-1}} & \underline{\overline{t_1-1}} & \underline{\overline{t_1-1}} & \underline{\cdots} & \underline{\overline{t_1-1}} & \underline{\overline{t_1-1}} \\ * & * & * & \cdots & * & * \end{array}$$

Due to the (F) property, all digits marked $*$ will be non-negative, thus proving inequality (2.16). Together with the previous inequalities (2.13), (2.14), and (2.15) we have derived the contradiction

$$\sum_{i=1}^{N-r}(t_1-1)B_{N-r-i} \overset{(2.14)}{>} \sum_{i=r+1}^{N}(w_{N-i}-t_i)B_{N-i}-1$$
$$\overset{(2.13)}{\geq} -(w_{N-r}-t_r)B_{N-r} \overset{(2.15)}{\geq} \sum_{i=1}^{N-r} t_iB_{N-r-i}+1 \overset{(2.16)}{\geq} \sum_{i=1}^{N-r}(t_1-1)B_{N-r-i}.$$

In other words, if there is some digit $w_r < t_r$ and $w_{m-j} = t_j$ for all $1 \leq j < r$, then regardless of how large the digits $w_{r-1}, w_{r-2}, \ldots, w_0 \in A$ are, they will not be sufficient to satisfy the inequality (2.12) and ensure that the expression on the left hand side is non-negative, which is a contradiction.
Finally, if $w_{N-j} = t_j$ for all $1 \leq j \leq N$, then the sum $\sum_{i=1}^{N} w_{N-i}B_{N-i}$ is equal to $\sum_{i=1}^{N} t_iB_{N-i}$, which is again in contradiction with (2.12).
Therefore, there must exist a $1 \leq r \leq N$ such that $w_{N-r} > t_r$ and $w_{N-j} = t_j$ for all $1 \leq j < r$, from which by definition

$$w_{N-1}w_{N-2}\cdots w_1w_0 \succ_{\text{lex}} t_1t_2\cdots t_{m-1}(t_m-1).$$

2. $\{1,2,\ldots,N\} \longrightarrow N+1$, where $N \geq m$:
Consider a word $z = z_Mz_{M-1}\cdots z_1z_0 \in A^*$ such that $|z| = M+1 \leq N+1$ and $\pi(z) \geq B_{N+1}$, where $N \geq m$. If $M+1 < N+1$, extend z to length $N+1$ by adding zeroes to the left. Then clearly

$$\sum_{j=0}^{N} z_jB_j = \pi(z) \geq B_{N+1} = \sum_{i=1}^{m} t_iB_{N+1-i}.$$

Subtracting $z_N B_N$ yields

$$\sum_{j=0}^{N-1} z_j B_j \geq (t_1 - z_N) B_N + \sum_{i=2}^{m} t_i B_{N+1-i}.$$

If $z_N < t_1$, then the word $z_{N-1} \cdots z_1 z_0$ of length N represents a value larger than B_N. By induction it contains a factor larger than the maximal factor and since $z_{N-1} \cdots z_1 z_0$ is a suffix of z, z also contains this factor.
Suppose now that $z_N = t_1$. We know that

$$\sum_{j=0}^{N-1} z_j B_j \geq \sum_{i=2}^{m} t_i B_{N+1-i} \tag{2.17}$$

and one of the following three cases occurs:

(a) $z_{N+1-r} > t_r$ holds for some $r \in \{2, \ldots, m\}$ and $z_{N+1-l} = t_l$ for all $2 \leq l < r$. Then evidently the prefix $z_N z_{N-1} \cdots z_{N+1-r}$ of the word z is lexicographically greater than the maximal factor.

(b) $z_{N+1-l} = t_l$ for all $2 \leq l \leq m$. Then clearly

$$z_N z_{N-1} \cdots z_{N+2-m} z_{N+1-m} = t_1 t_2 \cdots t_{m-1} t_m \succ_{\text{lex}} t_1 t_2 \cdots t_{m-1}(t_m - 1).$$

(c) $z_{N+1-r} < t_r$ for some $r \in \{2, \ldots, m\}$ and $z_{N+1-l} = t_l$ for all $2 \leq l \leq r$. The word z and maximal factor have the common prefix $t_1 t_2 \cdots t_{r-1}$. Consider the word $z^{(1)}$ obtained by removing this prefix from z and the word $y^{(1)}$ formed from the coefficients of the sum on the right hand side of (2.17) also with the same prefix removed:

$$\begin{array}{rcccccccccc} z^{(1)} & := & z_{N+1-r} & z_{N-r} & \cdots & z_{N+1-m} & z_{N-m} & \cdots & z_1 & z_0, \\ y^{(1)} & := & t_r & t_{r+1} & \cdots & t_m & 0 & \cdots & 0 & 0. \end{array}$$

Surely $\pi(z^{(1)}) \geq \pi(y^{(1)})$, as the inequality (2.17) cannot change by removing the same prefix (which corresponds to subtracting the same value from both sides). Then the inequality $\pi(z^{(1)}) \geq \pi(y^{(1)})$ is equivalent to

$$z_{N+1-r} B_{N+1-r} + \sum_{j=0}^{N-r} z_j B_j \geq t_r B_{N+1-r} + \sum_{i=2}^{m} t_i B_{N+1-r-i}.$$

After subtracting $z_{N+1-r} B_{N+1-r}$ we obtain

$$\sum_{j=0}^{N-r} z_j B_j \geq \underbrace{(t_r - z_{N+1-r})}_{\geq 1} B_{N+1-r}.$$

Hence the word $z^{(2)} = z_{N-r} z_{N-r-1} \cdots z_1 z_0$ has length $N+1-r$, but represents a value greater than B_{N+1-r}. By induction $z^{(2)}$ contains a factor greater than the maximal factor and so does the word z.

□

We will now use the above lemma to prove the fundamental theorem about the language of greedy representations.

Theorem 2.30 (Language of Greedy Representations)**.** *Let $(B_n)_{n=0}^{\infty}$ be an (F) basis of order m with canonical alphabet A. Then $L(B)$ is equal to*

$$L(B) = \{w \in A^* \mid \text{ no factor } u \text{ of } w \text{ of length } |u| = d \leq m \\ \text{is lexicographically greater than the maximal factor.}\}. \tag{2.18}$$

Proof. We will prove two inclusions. Denote X the set on the right hand side of (2.18).

$\mathbf{L(B) \subseteq X}$:

Consider some $x \in L(B)$ and suppose that x contains a factor u of length $2 \leq d \leq m$ such that

$$u = x_{i-1}x_{i-2}\cdots x_{i-d+1}x_{i-d} \succ_{\text{lex}} t_1 t_2 \cdots t_{m-1}(t_m{-}1),$$

where $i \geq m$. Let i be the maximal index with this property. From the definition of the lexicographic order this means that an $r \in \{1, 2, \ldots, d\}$ exists such that $x_{i-r} > t_r$ and $x_{i-s} = t_s$ for all $1 \leq s < r$, or that $d = m$, and $x_{i-q} = t_q$ for all $1 \leq q \leq m$. In the latter case, we can surely rewrite x by adding a factor of value zero starting at the digit x_i.

$$\begin{array}{cccccccc} \cdots & x_i & x_{i-1} & x_{i-2} & \cdots & x_{i-m+1} & x_{i-m} & \cdots \\ & 1 & \overline{t_1} & \overline{t_2} & \cdots & \overline{t_{m-1}} & \overline{t_m} & \\ \hline \cdots & x_i{+}1 & x_{i-1}{-}t_1 & x_{i-2}{-}t_2 & \cdots & x_{i-m+1}{-}t_{m-1} & x_{i-m}{-}t_m & \cdots \end{array} \tag{2.19}$$

In the former case, we have to proceed more carefully. There can be a digit $x_{i-p} < t_p$ for some $s < p \leq m$, thus we have to add another factor of value zero (but with the opposite sign) in place of x_{i-r}. We can do this because we know that $x_{i-r} > t_r$:

$$\begin{array}{ccccccccccccc} \cdots & x_i & x_{i-1} & \cdots & x_{i-r} & x_{i-r-1} & \cdots & x_{i-p} & x_{i-p} & x_{i-p} & \cdots & x_{i-m} & x_{i-m-1} & \cdots \\ & 1 & \overline{t_1} & \cdots & \overline{t_r} & \overline{t_{r+1}} & \cdots & \overline{t_{p-1}} & \overline{t_p} & \overline{t_{p+1}} & \cdots & \overline{t_m} & & \\ & & & & \overline{1} & t_1 & \cdots & t_{p-r-1} & t_{p-r} & t_{p-r+1} & \cdots & t_{m-r} & t_{m-r+1} & \cdots \\ \hline \cdots & x_i{+}1 & * & \cdots & * & ? & \cdots & ? & * & ? & \cdots & ? & ? & \cdots \end{array} \tag{2.20}$$

Since the basis has the (F) property, all the resulting digits marked with an asterisk $*$ are non-negative and contained in the alphabet A. Digits marked ? will be also non-negative, but not necessarily contained in A. There can be an index $q \in \{r+1, \ldots, m\}$, $q \neq p$ such that $x_{i-q} - t_q + t_{q-r} > t_1$ or an index $s \in \{m-r+1, \ldots, m\}$ such that $x_{i-s+r} + t_s > t_1$. In that case we can again add another factor of value zero and reduce the value of the digit concerned whilst keeping the value of the whole representation unchanged. This may again introduce digits that are not contained in A, but since the representation is finite, this rewriting process will always end and yield a representation with digits contained in A. It is because we will never create digits strictly larger than t_1 to the left of the digit x_{i-r} and because the index r will be strictly smaller in each subsequent addition. If we encounter digits not contained in A close to x_0 (the digit at B_0), we proceed as in the following paragraph.

Suppose now that $i < m$. We will use the same approach as above, the only difference is that we will use the initial representations of zero instead of the factors of value zero. Analogically to (2.19), if $x_{i-j} \geq t_j$ for all $1 \leq j < i$ and $x_0 > t_i$, we add a representation of zero:

$$\begin{array}{cccccc} \cdots & x_i & x_{i-1} & \cdots & x_1 & x_0 \\ & 1 & \overline{t_1} & \cdots & \overline{t_{i-1}} & \overline{t_i+1} \\ \hline \cdots & x_i{+}1 & x_{i-1}{-}t_1 & \cdots & x_1{-}t_{i-1} & x_0{-}t_i{-}1. \end{array} \tag{2.21}$$

On the other hand, if $x_{i-r} > t_r$ and $x_{i-j} = t_j$ for all $1 \leq j \leq r-1$ and there exists a $r+1 \leq p \leq i$ such that $x_{i-p} < t_p$, we add two initial representations of zero, analogically to what was done in (2.20):

$$\begin{array}{cccccccccccc} \cdots & x_i & x_{i-1} & \cdots & x_{i-r} & x_{i-r-1} & \cdots & x_{i-p} & x_{i-p} & x_{i-p} & \cdots & x_1 & x_0 \\ & 1 & \overline{t_1} & \cdots & \overline{t_r} & \overline{t_{r+1}} & \cdots & \overline{t_{p-1}} & \overline{t_p} & \overline{t_{p+1}} & \cdots & \overline{t_{i-1}} & \overline{t_i+1} \\ & & & & \overline{1} & t_1 & \cdots & t_{p-r-1} & t_{p-r} & t_{p-r+1} & \cdots & t_{i-s-1} & t_{i-s}+1 \\ \hline \cdots & x_i{+}1 & * & \cdots & * & ? & \cdots & ? & * & ? & \cdots & ? & ? \end{array} \tag{2.22}$$

Again, due to the (F) property, all digits marked $*$ will be non-negative and contained in A. Digits marked ? are also non-negative, but not necessarily contained in A. If there is a digit among them that is not contained in A (i.e. there exists an index $s+1 \leq q < i$, $q \neq p$ such that $x_{i-q} - t_q + t_{q-s} > t_1$), we repeat adding initial representations of zero as in (2.21) and (2.22) until all the resulting digits are contained in A. This process must end because of three reasons: we will never create digits strictly larger than t_1 to the left of the digit x_{i-r}, r will be smaller in each subsequent addition, and the representation is finite.

Lastly, notice that $x_i + 1 \leq t_1$, because if $x_i + 1 > t_1$, then $x_i = t_1$ and we would have

$$x_i x_{i-1} x_{i-2} \cdots x_{i-d+2} x_{i-d+1} x_{i-d} \succ_{\text{lex}} t_1 t_1 t_2 \cdots t_{m-2} t_{m-1} (t_m{-}1). \tag{2.23}$$

From the (F) property we can see that

$$t_1 t_1 t_2 \cdots t_{m-2} t_{m-1} (t_m{-}1) \succeq_{\text{lex}} t_1 t_2 t_3 \cdots t_{m-1} (t_m{-}1)(t_m{-}1),$$

using this with (2.23) and removing the last digit $(t_m{-}1)$ from both strings (which does not change the inequality) yields

$$x_i x_{i-1} \cdots x_{i-d+2} x_{i-d+1} \succ_{\text{lex}} t_1 t_2 t_3 \cdots t_{m-1} (t_m{-}1),$$

which is a contradiction with our definition of i.

In all four cases (2.19), (2.20), (2.21), (2.22) we have constructed a representation $\hat{x} \in A^*$ which satisfies $\pi(\hat{x}) = \pi(x)$ and $\hat{x} \succ x$, which is a contradiction with property 2 of greedy representations – if x is greedy, then it must be the greatest among all representations of $\pi(x)$ in the radix order. Therefore $L(B) \subseteq X$.

$\mathbf{L(B) \supseteq X}$:

Let $w = w_M w_{M-1} \cdots w_1 w_0 \in A^*$ and suppose that $w \notin L(B)$. We will show that w contains a factor greater than the maximal factor.

Take the greedy representation of the value $\pi(w)$ and denote its digits $\langle \pi(w) \rangle_B = x_N \cdots x_0$. From property 2 of greedy representations we have $\langle \pi(w) \rangle_B \succ w$. By definition of the radix order, precisely one of the following occurs:

a) $|\langle \pi(w) \rangle_B| > |w|$.

b) $|\langle \pi(w) \rangle_B| = |w| = N+1$ and an index r, $N+1 \geq r \geq 0$ exists such that $x_r > w_r$ and $x_l = w_l$ for all $N+1 \geq l > r$.

Case a): From the greedy algorithm we know that $\pi(w) \geq B_N$. At the same time $|w| \leq N$. Hence, by Lemma 2.29 the word w contains a factor greater than the maximal factor.
Case b): Let $|\langle \pi(w) \rangle_B| = |w| = N+1$ and $x_r > w_r$ for some $N+1 \geq r \geq 0$ and $N+1 \geq l > i$

for all $x_l = w_l$. We will modify the words w and $\langle \pi(w) \rangle_B$ and then apply the Lemma 2.29. By removing the common prefix $w_1 w_2 \cdots w_{r-1}$ we obtain the words

$$\begin{aligned} x^{(1)} &= x_{N-r} x_{N-r-1} \cdots x_1 x_0, \\ w^{(1)} &= w_{N-r} w_{N-r-1} \cdots w_1 w_0. \end{aligned}$$

Evidently $\pi(x^{(1)}) = \pi(w^{(1)})$, therefore

$$x_{N+1-r} B_{N+1-r} + \sum_{j=0}^{N-r} x_j B_j = w_{N+1-r} B_{N+1-r} + \sum_{j=0}^{N-r} w_j B_j,$$

subtracting $w_{N+1-r} B_{N+1-r}$ yields

$$\sum_{j=0}^{N-r} w_j B_j \geq \underbrace{(x_{N+1-r} - w_{N+1-r})}_{\geq 1} B_{N+1-r}.$$

The word $w^{(2)} = w_{N-r} w_{N-r-1} \cdots w_1 w_0$ has length $N + 1 - r$, but represents a value greater than B_{N+1-r}. Thus by Lemma 2.29 $w^{(2)}$ contains a factor greater than the maximal factor, and since it is a suffix of w, w contains this factor too. $\square$

Chapter 3

Ambiguity of Linear Numeration Systems

As has been noted in the previous chapter, B-systems are redundant. In a given B-system, most natural numbers have more than one representation over the canonical alphabet. For the Fibonacci and m-bonacci systems, much has been done to describe and quantify the ambiguity of such systems [1, 2, 11, 12]. Our main contribution consists in generalising these results to all confluent B-systems.

In this and next chapter we study the redundancy of confluent B-systems in terms of the redundancy function $R(n)$.

Definition 3.1. Consider a B-system. Then the *redundancy function* $R(n)$ is defined as the number of all B-representations of the natural number n over the canonical alphabet A. Formally,

$$R(n) := \#\left\{v \in A^* \mid \pi(v) = n\right\}.$$

Similarly, for a greedy representation $w = \langle n \rangle_B$, denote by $R(w)$ the number of possible B-representations of the number n.

In this chapter we will mostly use the function R in the latter notation. In Section 3.1 we will introduce the algorithm for calculating $R(n)$ and lay out its technical requirements. A more detailed description of the C++ program can be found in the Appendix along with instructions on its usage. We will follow this in Section 3.2 with computational results of our algorithm and statement of claims that can be inferred from the data, which will be later verified and proved in Chapter 4. We start with a motivational example.

Example 3.2. Consider the B-system with basis $B_n = 2B_{n-1} + B_{n-1}$. Then the first elements of the basis are $(B_n)_{n=0}^{\infty} = \{1, 3, 7, 17, 41, 99, 239, \dots\}$, and the associated rewriting system ρ_A conists of four rules: $100 \to 021$, $101 \to 022$, $200 \to 121$, and $201 \to 122$. Let $w = 1020100$. Then $R(w) = 6$, since all the possible B-representations representing the value $\pi(w) = 1 \cdot 239 + 2 \cdot 41 + 1 \cdot 7 = 328$ are 1020100, 1020021, 1012200, 1012121, 0222200, and 0222121, and the representations are related to each other by the following rewritings:

$$\begin{array}{ccccc} 1020100 & \underset{\rho_A}{\longleftarrow} & 1012200 & \underset{\rho_A}{\longleftarrow} & 0222200 \\ \rho_A\uparrow & & \rho_A\uparrow & & \rho_A\uparrow \\ 1020021 & \underset{\rho_A}{\longleftarrow} & 1012121 & \underset{\rho_A}{\longleftarrow} & 0222121 \end{array}$$

On the other hand, let $u = 1020202$. Then $R(u) = 1$, since there is no way any factor of u can be rewritten using the four rules from the associated rewriting system ρ_A. Another possible view can be that the addition of the factor of value zero $\bar{1}21$ to some factor of u would result in a string that has digits not contained in the canonical alphabet $\{0, 1, 2\}$.

3.1 Calculating $R(n)$

This section lays out the technical requirements for the practical calculation of $R(n)$.

To be able to calculate the $R(n)$ function and to study its properties in all (F) systems, we have to bound the interval on which it will be calculated. Suppose we have chosen some bounds $n_{\min}$, $n_{\max}$. Then the calculation of $R(n)$ for all $n_{\min} \leq n \leq n_{\max}$ is done by a simple algorithm:

Algorithm:

Denote by $\mathtt{R}(n)$ the intermediate values of $R(n)$. Initialise $\mathtt{R}(n)$ to zero for all $n_{\min} \leq n \leq n_{\max}$.

Then, for all words $w \in A^*$ that satisfy $\langle n_{\min} \rangle_B \preceq w \preceq \langle n_{\max} \rangle_B$ do:

1. Set $\mathtt{R}(\pi(w)) := \mathtt{R}(\pi(w)) + 1$.
2. Increment w by one in the radix order, i.e. increment it as if it was a standard b-ary representation.

After the algorithm terminates, $\mathtt{R}(n)$ will equal $R(n)$ for all $n_{\min} \leq n \leq n_{\max}$. However, this simple algorithm can fail to compute correct values of $R(n)$ for n that are close to the bound $n_{\min}$. Since we are counting all representations and not just the greedy representations, there can surely be a representation u such that $u \prec \langle n_{\min} \rangle_B$ but $\pi(u) > n_{\min}$, as in the following example:

Example 3.3. Consider the $(2, 1)$-B-system. Then $B_n = (1, 3, 7, 17, \dots)$. Let $\langle n_{\min} \rangle_B = 100_B$ and $u = 022_B$. Then $u \prec \langle n_{\min} \rangle_B$ but $\pi(w) = 8 > 7 = n_{\min}$.

Therefore, if we set $n_{\min} := 7$ and proceeded with counting $R(n)$ as in the above algorithm, we would come to the false result that $R(8) = 1$, because we would have omitted the non-greedy representation 022_B.

The converse case, i.e. a representation v such that $v \succ \langle n_{\max} \rangle_B$ but $\pi(v) < n_{\max}$ cannot occur due to property 3 of greedy representations (see Theorem 2.14). We thus have to replace the bound $\langle n_{\min} \rangle_B$ with $\rangle n_{\min} \langle_B$, counting all the words $\rangle n_{\min} \langle_B \preceq w \preceq \langle n_{\max} \rangle_B$.

Unfortunately, in general there is no way to obtain $\rangle n_{\min} \langle_B$ other than determining all the possible B-representations of $n_{\min}$ and selecting the radix smallest one. To rectify this, we have to select the bound $n_{\min}$ such that every representation $u \prec \langle n_{\min} \rangle_B$ has a value $\pi(u)$ strictly smaller than $n_{\min}$. Thankfully, such bounds are easy to find, which we do in the following technical lemma.

Lemma 3.4. *Consider an (F) basis $(B_n)_{n=0}^{\infty}$. Then for every $k \in \mathbb{N}_0$ the following holds:*

1. *$R(B_k - 1) = 1$.*
2. *For every word $w \in A^*$, where A is the canonical alphabet, $w \prec \langle B_k - 1 \rangle_B$ iff $\pi(w) < B_k - 1$.*
3. *Likewise, for every word $w \in A^*$, $w \succ \langle B_k - 1 \rangle_B$ iff $\pi(w) > B_k - 1$.*

Proof. Let $(B_n)_{n=0}^{\infty}$ be an (F) basis of order m, denote its coefficients $t_1, t_2, \ldots, t_m$. Let $k \in \mathbb{N}_0$. Then the greedy representation of $B_k - 1$ will have the form

$$\langle B_k - 1\rangle_B = (t_1 t_2 \cdots t_{m-1}(t_m - 1))^{\lfloor \frac{k}{m} \rfloor} t_1 t_2 \cdots t_{q-1} t_q, \tag{3.1}$$

where q is the remainder of the division of k by m, formally $q = k - \lfloor \frac{k}{m} \rfloor m$. $\langle B_k - 1\rangle_B$ will be equal to (3.1) because of two reasons.

Firstly, $\langle B_k - 1\rangle_B$ must be the largest greedy representation of length k, since the greedy representation of B_k is $\langle B_k\rangle_B = 10^k_B$, which has length $k+1$ (and it is the radix smallest representation of length $k+1$).

Secondly, due to Theorem 2.30, a greedy representation in a B-system with the (F) property avoids any factor larger than the maximal factor. The representation on the right hand side of (3.1) is precisely the (radix) largest possible B-representation of length k that does not contain a maximal factor, so it must be equal to $\langle B_k - 1\rangle_B$.

Let us now prove statement 1 of the lemma. The case $k = 0$ is trivial, since $B_0 - 1 = 0$, which has only one representation over the canonical alphabet A. Hence, let $k > 0$ and suppose now that $B_k - 1$ has another B-representation w. Surely w can be reached by one or more additions of the factor of value zero to $\langle B_k - 1\rangle_B$. However, we will show that any word created this way will have digits that are not contained in A. Because the B-system has the (F) property, the canonical alphabet is equal to $A = \{0, 1, \ldots, t_1\}$. As the greedy representation $\langle B_k - 1\rangle_B$ is equal to (3.1), there is no valid location to add the factor of value zero, because at least one resulting digit will be strictly greater than t_1. Furthermore, if we try to subsequently shrink this digit by adding t_{r+1} factors of zero, we will again introduce at least one digit that is greater than t_1, as seen below in (3.2).

$$\begin{array}{cccccccc}
\langle B_k - 1\rangle_B = & t_1 & \cdots & t_r & t_{r+1} & t_{r+2} & t_{r+3} & \cdots \\
 & & & \bar{1} & t_1 & t_2 & t_3 & \cdots \\
\hline
 & & & t_r - 1 & t_{r+1} + t_1 & t_{r+2} + t_2 & t_{r+3} + t_3 & \cdots \\
 & & & & \bar{1} & t_1 & t_2 & \cdots \\
 & & & & \bar{1} & t_1 & t_2 & \cdots \\
 & & & & \vdots & \vdots & \vdots & \cdots \\
 & & & & \bar{1} & t_1 & t_2 & \cdots \\
\hline
w = & w_{k-1} & \cdots & t_r - 1 & t_1 & t_{r+2} + t_2 + t_{r+1} \cdot t_1 & t_{r+3} + t_3 + t_{r+1} \cdot t_2 & \cdots
\end{array} \tag{3.2}$$

This applies also if we add the factor of value zero in the place of t_{m-1} (thus adding t_1 in the place of $t_m - 1$). If $t_m - 1 = 0$, then $t_m - 1 + t_1 \in A$, but we will again introduce digits that are not contained in A in other locations, as seen below in (3.3).

$$\begin{array}{cccccccc}
\langle B_k - 1\rangle_B = & t_1 & t_2 & \cdots & t_{m-1} & t_m - 1 & t_1 & t_2 & \cdots \\
 & & & & \bar{1} & t_1 & t_2 & t_3 & \cdots \\
\hline
 & t_1 & t_2 & \cdots & t_{m-1} - 1 & t_1 & t_1 + t_2 & t_2 + t_3 & \cdots \\
 & & & & & 1 & \bar{t_1} & \bar{t_2} & \cdots \\
\hline
w = & t_1 & t_2 & \cdots & t_{m-1} - 1 & t_1 + 1 & t_2 & t_3 & \cdots
\end{array} \tag{3.3}$$

The same argument holds also at the end of the representation (i.e. close to the digit at B_0). Therefore, w cannot have its digits contained in A, which is a contradiction.

Statement 2; $(\Rightarrow)$: Take a word $w \in A^*$ such that $w \prec \langle B_k - 1\rangle_B$. Then by definition of the radix order, it is either a) shorter than $\langle B_k - 1\rangle_B$, denote $|w| = l < k$, or b) it has the same length as $\langle B_k - 1\rangle_B$ and there exists an index $s \in \{0, 1, \ldots, k-1\}$ such that $w_{k-s} < (\langle B_k - 1\rangle_B)_{k-s}$ and w and $\langle B_k - 1\rangle_B$ share a common prefix of length s. We will treat both cases simultaneously.

In case b) first remove the common prefix from both words. This yields the words

$$\begin{aligned} w^* &= w_{k-s-1}w_{k-s-2}\cdots w_1 w_0, \\ \langle B_k - 1\rangle_B^* &= t_p t_{p+1}\cdots t_{m-1}(t_m{-}1)(t_1 t_2 \cdots t_{m-1}(t_m{-}1))^q t_1 t_2 \cdots t_{r-1} t_r, \end{aligned} \tag{3.4}$$

where $1 \le p \le m-1$, and $0 \le q \le \lfloor \frac{k}{m} \rfloor$.

Case a) can be converted to case b) by setting $s = 0$ and $w_{k-1} = w_{k-2} = \cdots = w_l = 0$, which yields $w^* = 0\cdots 0w$. Lastly, set $\langle B_k - 1\rangle_B^* = \langle B_k - 1\rangle_B$.

To prove $B_k - 1 > \pi(w)$ we will evaluate the sign of $\pi(\langle B_k - 1\rangle_B^*) - \pi(w^*)$, because clearly $\pi(\langle B_k - 1\rangle_B^*) - \pi(w^*) > 0$ implies $B_k - 1 > \pi(w)$. Evidently, we can bound $\pi(w^*)$ followingly:

$$\pi(w^*) \le w_{k-s-1}B_{k-s-1} + \sum_{i=0}^{k-s-2} t_1 B_i.$$

That is, we replace every digit of w^* other than the most significant one with t_1.

Using (3.4), we can write the expression

$$\pi(\langle B_k - 1\rangle_B^*) - w_{k-s-1}B_{k-s-1} - \sum_{i=0}^{k-s-2} t_1 B_i$$

digit by digit as

$$\begin{array}{cccc|ccc|c|cccc|cccc} t_p & t_{p+1} & \cdots & t_{m-1} & t_m{-}1 & t_1 & t_2 & \cdots & \cdots & \cdots & t_{m-1} & t_m{-}1 & t_1 & \cdots & t_{r-1} & t_r \\ \overline{w_{k-s-1}} & \overline{t_1} & \cdots & \overline{t_1} & \overline{t_1} & \overline{t_1} & \overline{t_1} & \cdots & \cdots & \cdots & \overline{t_1} & \overline{t_1} & \overline{t_1} & \cdots & \overline{t_1} & \overline{t_1} \\ \hline\hline \end{array},$$

where by vertical lines we delimit each repetition of the factor $t_1 t_2 \cdots t_{m-1}(t_m{-}1)$ in $\langle B_k - 1\rangle_B^*$. We know that $t_p > w_{k-s-1}$, and since the basis has the (F) property, $t_i \ge 1$ holds for all $i = 1, 2, \ldots, m$. We can thus add the factor $\overline{1}t_1 t_2 \cdots t_{m-2}t_{m-1}t_m$ (that has numeric value zero) at every digit. This results in adding t_1 to the next digit to the right, which cancels out with $\overline{t_1}$ and together this always yields a digit that is non-negative. The only location in which we cannot subtract 1 is in the location of the digit $t_m{-}1$, because if $t_m = 1$, then $t_m - 1 = 0$. However, as can be seen from (3.5) and due to the fact that m is at least 2 and $t_i \ge 1$, we will still obtain a non-negative digit at that location. Also, this results in an addition of at least 1 to the digit to the right of $t_m - 1$, allowing us to subtract 1 again and cancel out $\overline{t_1}$ in every location in the following appearance of the $t_1 t_2 t_3 \cdots t_{m-1}t_m{-}1$ factor.

$$
\begin{array}{ccccc|cccccccc|cc}
t_p & t_{p+1} & \cdots & t_{m-1} & t_m{-}1 & t_1 & t_2 & \cdots & t_p & t_{p+1} & \cdots & t_{m-1} & t_m{-}1 & t_1 & \cdots \\
\overline{1} & t_1 & \cdots & t_{m-p-2} & t_{m-p-1} & t_{m-p} & t_{m-p+1} & \cdots & t_m & & & & & & \\
 & \overline{1} & \cdots & t_{m-p-3} & t_{m-p-2} & t_{m-p-1} & t_{m-p} & \cdots & t_{m-1} & t_m & & & & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\
 & & & \overline{1} & t_1 & t_2 & t_3 & \cdots & t_{p+1} & t_{p+2} & \cdots & t_m & & & \\
 & & & & & \overline{1} & t_1 & \cdots & t_{p-1} & t_{p-2} & \cdots & t_{m-2} & t_{m-1} & t_m & \\
 & & & & & & \overline{1} & \cdots & t_{p-2} & t_{p-3} & \cdots & t_{m-3} & t_{m-2} & t_{m-1} & \cdots \\
 & & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
 & & & & & & & & \overline{1} & t_1 & \cdots & t_{m-p-2} & t_{m-p-1} & t_{m-p} & \cdots \\
 & & & & & & & & & \overline{1} & \cdots & t_{m-p-3} & t_{m-p-2} & t_{m-p-1} & \cdots \\
 & & & & & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots \\
 & & & & & & & & & & & \overline{1} & t_1 & t_2 & \cdots \\
 & & & & & & & & & & & & & \overline{1} & \cdots \\
 & & & & & & & & & & & & & & \ddots \\
\overline{w_{k-s-1}} & \overline{t_1} & \cdots & \overline{t_1} & \overline{t_1} & \overline{t_1} & \overline{t_1} & \cdots & \overline{t_1} & \overline{t_1} & \cdots & \overline{t_1} & \overline{t_1} & \overline{t_1} & \cdots \\
\hline\hline
* & * & \cdots & * & * & * & * & \cdots & * & * & \cdots & * & * & * & \cdots
\end{array}
\tag{3.5}
$$

At the start and in the middle of the representation, the subtraction will proceed as shown above in (3.5). In the middle, $\overline{1}t_1t_2\cdots t_{m-2}t_{m-1}t_m$ is repeatedly shifted and added at every digit except for the digit that has value $t_m{-}1$.

At the end of the representation, if $r \geq 1$, the subtraction is as in (3.6). In total, all digits marked with an asterisk ($*$) will be non-negative, and the digit at B_0 (marked with a plus sign $+$) will be always positive:

$$
\begin{array}{ccc|ccccc}
\cdots & t_{m-1} & t_m{-}1 & t_1 & t_2 & \cdots & t_{r-1} & t_r \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \overline{1} & t_1 & t_2 & t_3 & \cdots & t_{m-1} & t_m \\
 & & & \overline{1} & t_1 & \cdots & t_{m-2} & t_{m-1}{+}1 \\
 & & & & \overline{1} & \cdots & \vdots & \vdots \\
 & & & & & \cdots & t_1 & t_2{+}1 \\
 & & & & & & \overline{1} & t_1{+}1 \\
\cdots & \overline{t_1} & \overline{t_1} & \overline{t_1} & \overline{t_1} & \cdots & \overline{t_1} & \overline{t_1} \\
\hline\hline
\cdots & * & * & * & * & * & * & +
\end{array}
\tag{3.6}
$$

Otherwise, if $r = 0$, then the addition is as follows:

$$
\begin{array}{||ccc|ccccc}
\cdots & t_{m-1} & t_m{-}1 & t_1 & t_2 & \cdots & t_{m-1} & t_m{-}1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \overline{1} & t_1 & t_2 & t_3 & \cdots & t_{m-1} & t_m \\
 & & & \overline{1} & t_1 & \cdots & t_{m-2} & t_{m-1}{+}1 \\
 & & & & \overline{1} & \cdots & \vdots & \vdots \\
 & & & & & \cdots & t_1 & t_2{+}1 \\
 & & & & & & \overline{1} & t_1{+}1 \\
\cdots & \overline{t_1} & \overline{t_1} & \overline{t_1} & \overline{t_1} & \cdots & \overline{t_1} & \overline{t_1} \\
\hline\hline
\cdots & * & * & * & * & * & * & +
\end{array}
$$

and again the digit at B_0 is positive.

Together, this yields the desired inequality $\pi\left(\langle B_k - 1\rangle_B^*\right) - \pi\left(w^*\right) > 0$ and so $B_k - 1 > \pi(w)$.

Part 2; ($\Leftarrow$) : Let $\pi(w) < B_k - 1$. Then $w \prec \langle B_k - 1\rangle_B$ follows from properties 2 and 3 of the greedy representation, i.e. every representation w such that $\pi(w) < B_k - 1$ must satisfy $w \preceq \langle \pi(w)\rangle_B \prec \langle B_k - 1\rangle_B$.

Part 3; ($\Rightarrow$) : Let $w \succ \langle B_k - 1\rangle_B$. Then either w is longer than $\langle B_k - 1\rangle_B$ and so $\pi(w) \geq B_k$, from which $\pi(w) > B_k - 1$ clearly follows, or $|w| = |\langle B_k - 1\rangle_B|$ and there exists an $s \in \{0, 1, \ldots, k-1\}$ such that w and $\langle B_k - 1\rangle_B$ share a common prefix of length s and $w_{k-s-1} > (\langle B_k - 1\rangle_B)_{k-s-1}$. In this case, remove the common prefix, which yields the representations

$$
\begin{aligned}
\widetilde{w} &= w_{k-s-1} w_{k-s-2} \cdots w_1 w_0, \\
\widetilde{\langle B_k - 1\rangle_B} &= t_p t_{p+1} \cdots t_{m-1} (t_m{-}1)(t_1 t_2 \cdots t_{m-1}(t_m{-}1))^q t_1 t_2 \cdots t_{r-1} t_r,
\end{aligned}
\tag{3.7}
$$

where again $1 \leq p \leq m-1$, and $0 \leq q \leq \left\lfloor \frac{k}{m} \right\rfloor$. The number $\pi\left(\widetilde{w}\right)$ can clearly be bounded followingly:

$$\pi\left(\widetilde{w}\right) \geq w_{k-s-1} B_{k-s-1},$$

and so we can write

$$\pi\left(\widetilde{w}\right) - \pi\left(\widetilde{\langle B_k - 1\rangle_B}\right) \geq w_{k-s-1} B_{k-s-1} - \pi\left(\widetilde{\langle B_k - 1\rangle_B}\right). \tag{3.8}$$

The expression on the right of (3.8) can be written digit by digit as

$$
\begin{array}{ccccc|ccc||ccc|cccc}
w_{k-s-1} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\overline{t_p} & \overline{t_{p+1}} & \cdots & \overline{t_{m-1}} & \overline{t_m - 1} & \overline{t_1} & \overline{t_2} & \cdots & \cdots & \overline{t_{m-1}} & \overline{t_m - 1} & \overline{t_1} & \cdots & \overline{t_{r-1}} & \overline{t_r}. \\
\hline\hline
\end{array}
$$

We will now proceed as in part 2. Clearly the digit w_{k-s-1} satisfies $w_{k-s-1} > t_p$, so we can add the factor $\overline{1}t_1 t_2 \cdots t_{m-2} t_{m-1} t_m$. Then, since the basis has the (F) property, $t_1 \geq t_2 \geq \cdots \geq t_m$ and so we will obtain a non-negative digit (marked $*$) in every column:

$$
\begin{array}{cccc|ccccccc|cc}
w_{k-s-1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\
\overline{1} & t_1 & \cdots & t_{m-p-2} & t_{m-p-1} & \cdots & t_m & & & & & & \\
 & & & \overline{1} & t_1 & \cdots & t_{p+1} & t_{p+2} & \cdots & t_{m-1} & t_m & & \\
 & & & & & & & & & & \overline{1} & t_1 & \cdots \\
 & & & & & & & & & & & & \vdots \\
\overline{t_p} & \overline{t_{p+1}} & \cdots & \overline{t_m - 1} & \overline{t_1} & \cdots & \overline{t_p} & \overline{t_{p+1}} & \cdots & \overline{t_{m-1}} & \overline{t_m - 1} & \overline{t_1} & \cdots \\
\hline\hline
* & * & \cdots & * & * & * & \cdots & * & * & * & * & * & \cdots
\end{array}
\tag{3.9}
$$

The subtraction is much simpler than in part 2, and like in part 2 we are left with a positive digit at B_0:

$$
\begin{array}{||ccc|ccccc}
\cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & t_{m-1} & t_m & & & & & \\
 & & \overline{1} & t_1 & t_2 & \cdots & t_{r-1} & t_r + 1 \\
\cdots & \overline{t_{m-1}} & \overline{t_m - 1} & \overline{t_1} & \overline{t_2} & \cdots & \overline{t_{r-1}} & \overline{t_r}. \\
\hline\hline
\cdots & * & * & * & * & * & * & +
\end{array}
$$

Together this yields

$$
w_{k-s-1} B_{k-s-1} - \pi\left(\widetilde{\langle B_k - 1 \rangle_B}\right) > 0
$$

and so the desired inequality $\pi(w) > B_k - 1$.

Part 3; $(\Leftarrow)$: Let $\pi(w) > B_k - 1$. In case when w is longer than $\langle B_k - 1 \rangle_B$, the inequality $w \succ \langle B_k - 1 \rangle_B$ is evident from definition, so let w be such that $|w| = |\langle B_k - 1 \rangle_B|$. Then since $\langle B_k - 1 \rangle_B$ is the radix largest greedy representation of this length, w is not a greedy representation. Therefore, w contains a factor that is larger than the maximal factor and so $w \succ \langle B_k - 1 \rangle_B$. □

With Lemma 3.4 in hand, we can proceed to calculate $R(n)$. In effect, part 2 of Lemma 3.4 would be sufficient for our needs. In counting $R(n)$ for all $n \in \{n_{\min}, n_{\min} + 1, \ldots, n_{\max}\}$ we will traverse all the words$\langle n_{\min} \rangle_B \preceq w \preceq \langle n_{\max} \rangle_B$, and since the greedy representation is the radix greatest among all representations of a given number, we will not omit any representation of $n_{\max}$. If we encounter a w such that $\pi(w) > n_{\max}$, we can ignore it.

Thus, if we want to determine $R(n)$ on the interval $n \in \{n_{\min}, n_{\min}+1, \ldots, n_{\max}\}$ for arbitrary $n_{\min}$ and $n_{\max}$, we have to find the largest basis element B_N such that $B_N - 1 \leq n_{\min}$, traverse all the words $\langle B_N - 1 \rangle_B \preceq w \preceq \langle n_{\max} \rangle_B$ and then discard the values of $R(n)$ for all n in the interval $\{B_N - 1, B_N, \ldots, n_{\min} - 1\}$.

However, we will still usually set $n_{\min} = B_k - 1$, $n_{\max} = B_{k+1} - 1$, because this precisely delimits all representations of length $k + 1$. Calculating $R(n)$ on such intervals allows us to uncover the palindromic structure of $R(n)$, as well as trends in the number of its maxima and the sequence of numbers that have a unique representation.

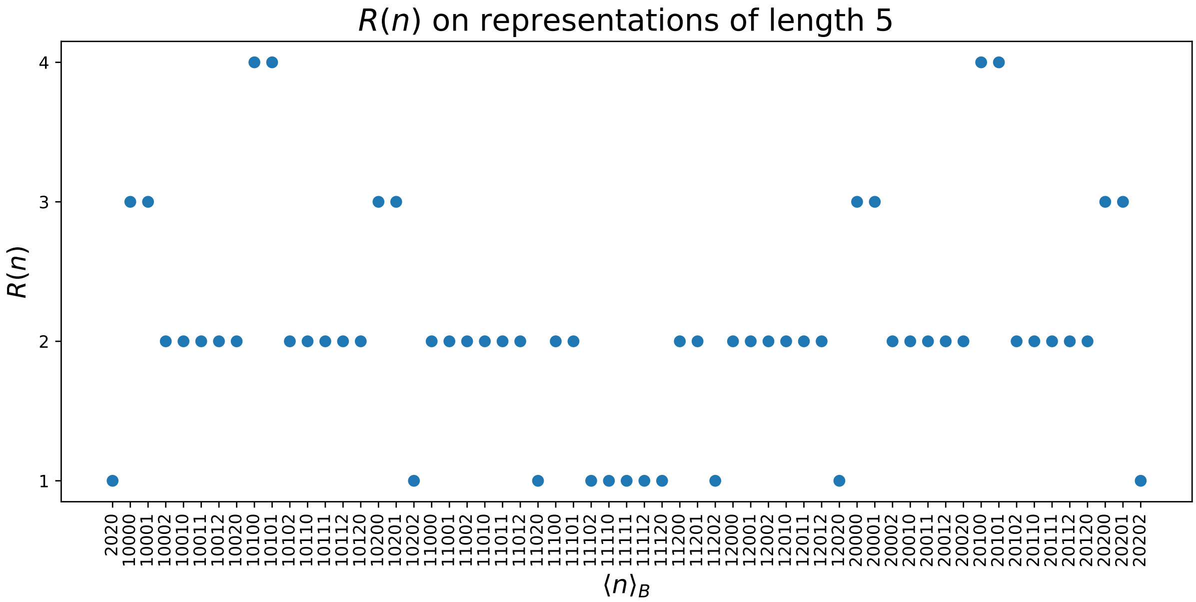

Figure 3.1: $R(n)$ in the $(2,1)$-B-system on all n whose greedy representation has length 5.

3.2 Computational Results

In this section we will present computational results of our survey of $R(n)$ in various confluent systems. We will begin by presenting results for the $(2,1)$-B-system as a model example. In this system, $R(n)$ was calculated for all representations with lengths up to 23 (i.e. up to $B_{23} - 1 \approx 7{,}68 \cdot 10^8$). In Figures 3.1, 3.2, 3.3, 3.4 see the graph of $R(n)$ for all n whose representations have lengths 5-8. We can see that $R(n)$ is symmetric on the interval $B_{l-1} - 1$ to $B_l - 1$. This precisely delimits representations of length l (plus the element $B_{l-1} - 1$), the representation $\langle n \rangle_B = 1^l$ being the center of symmetry. We will later prove that the $R(n)$ function displays such a palindromic structure in all B-systems with the (F) property. We will also show this palindrome is precisely aligned with this interval (i.e. the numbers $B_k - 1 \leq n \leq B_{k+1} - 1$) in all confluent systems of order 2 with coefficients $a, 1$, where a is some natural number.

For further study of the maxima of $R(n)$, we will establish some notation. The value of $R(n)$ depends on the length of representation, thus it suffices to restrict our analysis of $R(n)$ to representations of a given length. Denote

$$\psi(l) := \max_{|\langle n \rangle_B| = l} R(n) = \max \left\{ R(n) \,|\, B_{l-1} - 1 < n \leq B_l - 1 \right\},$$

and

$$\Psi(l) := \left\{ \arg \max_{|\langle n \rangle_B| = l} R(n) \right\}.$$

Note: We count 0_B, the representation of zero, among the representation of length 1.

In Table 3.1, the maxima of $R(n)$ with respect to the length of the representation are displayed, along with the first 4 members of the set $\Psi(l)$. We notice that the value of $\psi(l)$ satisfies the following relation

$$\psi(l) = 2^{\lceil \frac{l}{2} \rceil - 1}, \tag{3.10}$$

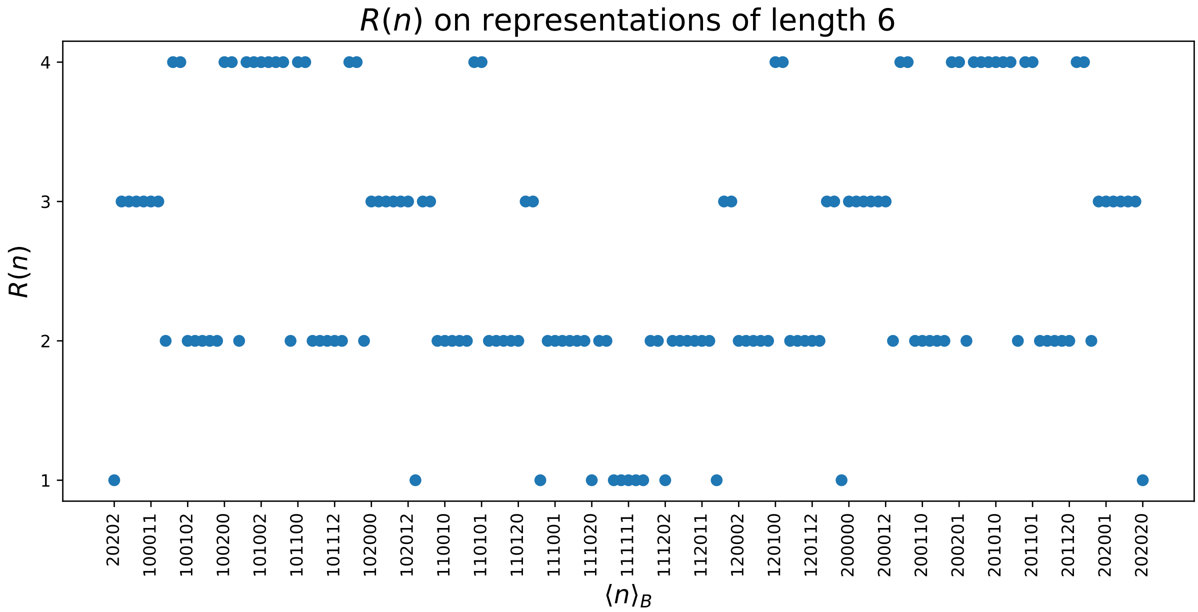

Figure 3.2: $R(n)$ in the $(2, 1)$-B-system on all n whose greedy representation has length 6.

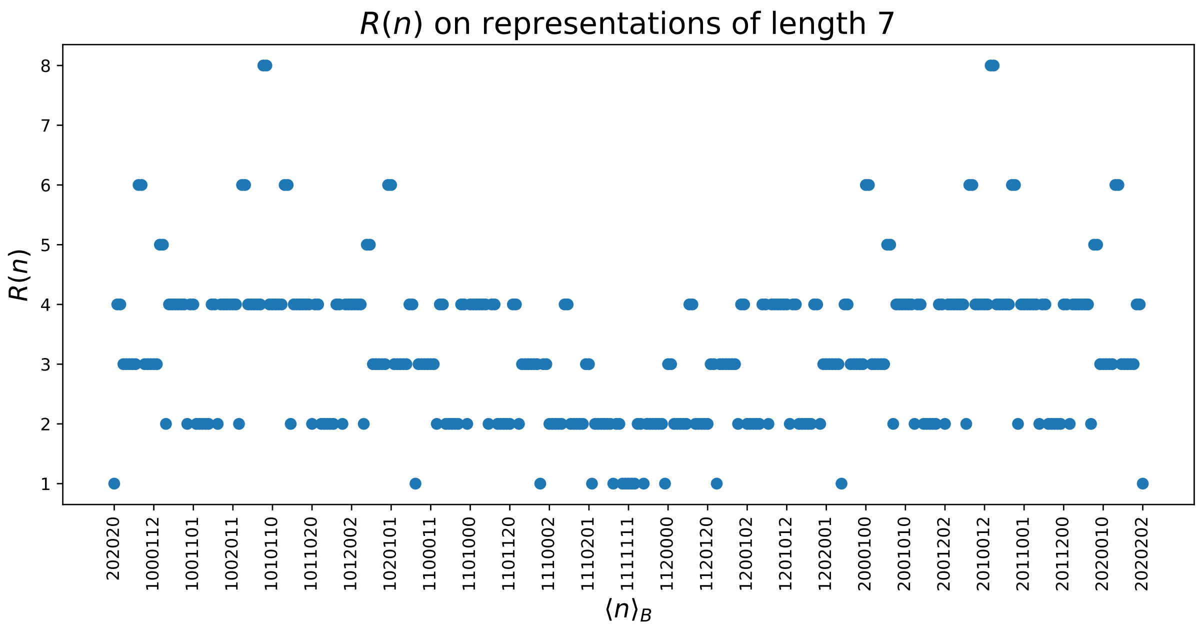

Figure 3.3: $R(n)$ in the $(2, 1)$-B-system on all n whose greedy representation has length 7.

l	$\psi(l)$	$\#\Psi(l)$	First 4 elements of $\Psi(l)$			
1	1	3	0	1	2	
2	1	4	10	11	12	20
3	2	4	100	101	200	201
4	2	16	1000	1001	1002	1010
5	4	4	10100	10101	20100	20101
6	4	32	100100	100101	100200	100201
7	8	4	1010100	1010101	2010100	2010101
8	8	48	10010100	10010101	10020100	10020101
9	16	4	$101(01)^2 00$	$101(01)^2 01$	$201(01)^2 00$	$201(01)^2 01$
10	16	64	$1001(01)^2 00$	$1001(01)^2 01$	$1002(01)^2 00$	$1002(01)^2 01$
11	32	4	$101(01)^3 00$	$101(01)^3 01$	$201(01)^3 00$	$201(01)^3 01$
12	32	80	$1001(01)^3 00$	$1001(01)^3 01$	$1002(01)^3 00$	$1002(01)^3 01$
13	64	4	$101(01)^4 00$	$101(01)^4 01$	$201(01)^4 00$	$201(01)^4 01$
14	64	96	$1001(01)^4 00$	$1001(01)^4 01$	$1002(01)^4 00$	$1002(01)^4 01$
15	128	4	$101(01)^5 00$	$101(01)^5 01$	$201(01)^5 00$	$201(01)^5 01$
16	128	112	$1001(01)^5 00$	$1001(01)^5 01$	$1002(01)^5 00$	$1002(01)^5 01$
17	256	4	$101(01)^6 00$	$101(01)^6 01$	$201(01)^6 00$	$201(01)^6 01$
18	256	128	$1001(01)^6 00$	$1001(01)^6 01$	$1002(01)^6 00$	$1002(01)^6 01$
19	512	4	$101(01)^7 00$	$101(01)^7 01$	$201(01)^7 00$	$201(01)^7 01$
20	512	144	$1001(01)^7 00$	$1001(01)^7 01$	$1002(01)^7 00$	$1002(01)^7 01$
21	1024	4	$101(01)^8 00$	$101(01)^8 01$	$201(01)^8 00$	$201(01)^8 01$
22	1024	160	$1001(01)^8 00$	$1001(01)^8 01$	$1002(01)^8 00$	$1002(01)^8 01$
23	2048	4	$101(01)^9 00$	$101(01)^9 01$	$201(01)^9 00$	$201(01)^9 01$

Table 3.1: Maxima of $R(n)$ in relation to the length of representation in the $(2, 1)$-system.

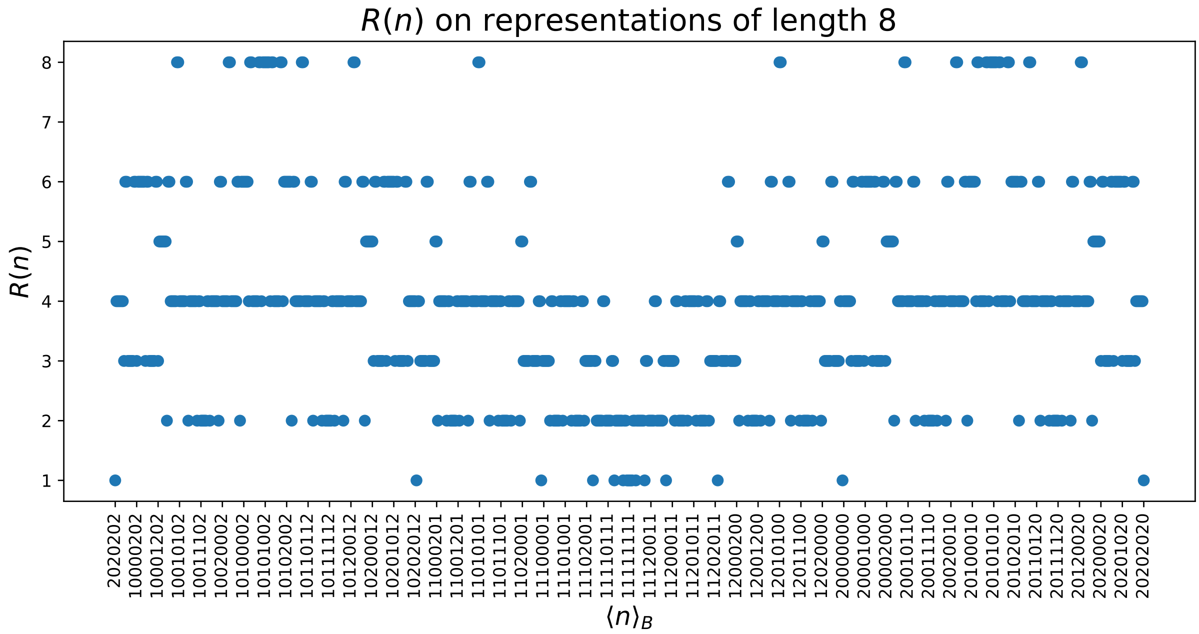

Figure 3.4: $R(n)$ in the $(2,1)$-B-system on all n whose greedy representation has length 8.

and that for the size of the set $\Psi(l)$ the following holds – for every $l \geq 3$:

$$\#\Psi(l) = \begin{cases} 4 & \text{for } l \text{ odd}, \\ 16\left(\frac{l}{2} - 1\right) & \text{for } l \text{ even}. \end{cases}$$

We will later prove these two claims using the formula for $R(n)$ that will be introduced in Chapter 4.

For other confluent B-systems, closed-form expressions for $\psi(l)$ and $\#\Psi(l)$ can be found as well, which we will show in the following tables. Our survey of confluent systems revealed that confluent numeration systems can be divided into three groups according to the behaviour of $R(n)$. These groups are distinguished by whether the last coefficient is equal or strictly less than the other coefficients and the order of recurrence. More precisely, the confluent systems with $a = b$ and order $m = 2$ show very similar behaviour to the Fibonacci system, those with and $a = b$ and order $m > 2$ behave analogically to the m-bonacci systems, whilst confluent systems with $a > b$ can be grouped together as they all satisfy (3.10). We will verify these statements in Chapter 4.

3.2.1 Confluent Systems with $a = b$ and order $m = 2$

Confluent systems with $a = b$ and order $m = 2$ show analogous behaviour to the Fibonacci system. See Table 3.2 where we present the values of $\psi(l)$ and $\#\Psi(l)$ as well as the first four elements of $\Psi(l)$ for the $(2,2)$-system. In Figures 3.7 and 3.8 see the graph of $R(n)$ for the systems with coefficients $(2,2)$ and $(3,3)$ on all n whose representation has length 7. Lastly, in Table 3.3 we have the sizes of the set $\Psi(l)$ for all surveyed systems of this type.

Notice that for the value of $\psi(l)$ the following holds:

$$\psi(2l+1) = F_l \quad \text{for } l \geq 0,$$
$$\psi(2l+2) = 2F_{l-1} \text{ for } l \geq 1.$$

Furthermore, except for the initial cases $l = 1, 2, 3, 4$ and $l = 6, 9, 12$ we can see from Table 3.3 that the sizes of the set $\Psi(l)$ satisfy

$$\#\Psi(2k+1) = 2 \cdot a \quad \text{for } k \geq 1, k \neq 4,$$
$$\#\Psi(2k) = 4 \cdot a^2 \quad \text{for } k \geq 4, k \neq 6.$$

For lengths $l = 1$ and $l = 2$ the set $\Psi(l)$ is simply composed of all numbers with greedy representations over the alphabet $A = \{0, 1, \dots, a\}$ (where we count 0 among the representations of length 1), of which there are $\#\Psi(1) = a + 1$ and $\#\Psi(2) = (a + 1) \cdot a - 1$. For example for the $(3, 3)$-system we have $\Psi(2) = \{10, 11, 12, 13, 20, 21, 22, 23, 30, 31, 32\}$.

For $l = 3$ the set $\Psi(l)$ consists solely of numbers whose greedy representation has the form $x00$, because then we can perform one interchange $x00 \leftrightarrow (x-1)aa$. Hence $\#\Psi(3) = a$ because the most significant digit x can be any nonzero digit from A.

The situation for $l = 4$ is similar, $\Psi(l)$ will consist of numbers with greedy representations $x00y$ and $xy00$. In the first case, $x \in \{1, 2, \dots, a\}$ and $y \in A$, so we obtain $a \cdot (a + 1)$ possible representations. In the latter case, $x \in \{1, 2, \dots, a\}$ again but the situation for y is more complicated. The digit y cannot be zero, since $y = 0$ has been counted as part of the first string $x00y$. Then, if $x = a$, then y can only be from the set $\{1, \dots, a - 1\}$ because the representation $xy00$ is greedy. Thus $xy00$ accounts for $(a - 1) \cdot a + (a - 1) = a^2 - 1$ representations. In total we obtain $\#\Psi(4) = a \cdot (a + 1) + a^2 - 1 = 2a^2 + a - 1$ possible representations.

The case $l = 6$ can be solved by a similar analysis. The value $\psi(6) = 4$ is reached on representations of the form $x00y00$, where $x, y \in \{1, 2, \dots, a\}$, because that allows two independent interchanges $x00 \leftrightarrow (x-1)aa$ and $y00 \leftrightarrow (y-1)aa$. Hence $\#\Psi(6) = a^2$.

The case $l = 9$ is more complicated. There are three basic forms of words w on which the value $R(w) = 8 = \psi(9)$ is reached. They are $x00010000$, $x00y00z00$, and $x01000100$, where $x, y, z \in \{1, 2, \dots, a\}$. The string $x00y00z00$ allows three independent interchanges $*00 \leftrightarrow (*-1)aa$, this corresponds to a^3 elements of $\Psi(9)$. The words $x01000100$, $x00010000$ contribute another $2a$ elements of $\Psi(9)$.

Lastly, all the maximal representations of length $l = 12$ are precisely the words with one of the forms

$$x00y00z00v00,\ x00y00010000,\ x00y01000100,\ x01000100y00,\ \text{and } x00010000y00,$$

where again $x, y, z, v \in \{1, 2, \dots, a\}$, thus we obtain $\#\Psi(12) = a^4 + 4 \cdot a^2$.

We will revisit the cases $\Psi(9)$ and $\Psi(12)$ in Chapter 4 when we have proven the matrix formula for $R(n)$ and proven the expression for the value of $\psi(l)$.

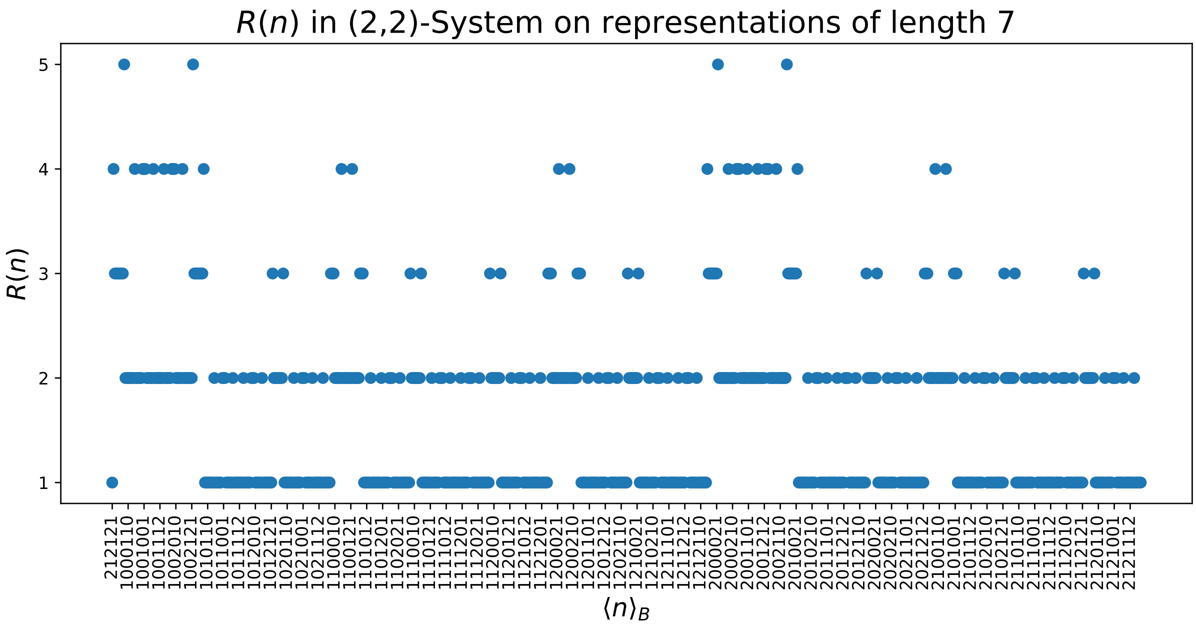

Figure 3.5: $R(n)$ in the $(2, 2)$-B-system for all n whose greedy representation has length 7.

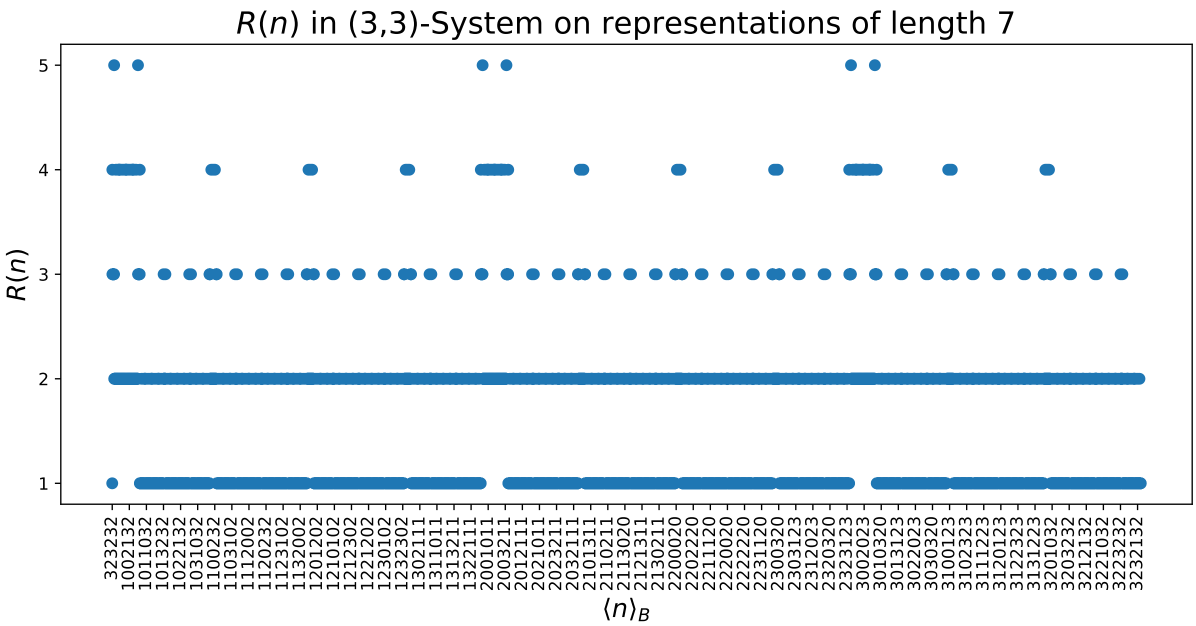

Figure 3.6: $R(n)$ in the $(3, 3)$-B-system for all n whose greedy representation has length 7.

l	$\psi(l)$	$\#\Psi(l)$	First four elements of $\Psi(l)$			
1	1	3	0	1	2	
2	1	6	2	10	11	12
3	2	2	100	200		
4	2	9	1000	1001	1002	1100
5	3	4	10000	10100	20000	20100
6	4	4	100100	100200	200100	200200
7	5	4	1000100	1010000	2000100	2010000
8	6	16	10000100	10000200	10010000	10010100
9	8	12	100010000	100100100	100100200	100200100
10	10	16	1000100100	1000100200	1001000100	1001010000
11	13	4	10001000100	10100010000	20001000100	20100010000
12	16	32	100010000100	100010000200	100100010000	100100100100
13	21	4	1000100010000	1010001000100	2000100010000	2010001000100
14	26	16	10001000100100	10001000100200	10010001000100	10010100010000
15	34	4	100010001000100	101000100010000	200010001000100	201000100010000
16	42	16	1000100010000100	1000100010000200	1001000100010000	1001010001000100
17	55	4	10001000100010000	10100010001000100	20001000100010000	20100010001000100
18	68	16	100010001000100000	100010001000100000	100100010001000000	100101000100010000
19	89	4	1000100010001000000	1010001000100010000	2000100010001000000	2010001000100010000
20	110	16	10001000100010000000	10001000100010000000	10010001000100000000	10010100010001000000
21	144	4	100010001000100000000	101000100010001000000	200010001000100000000	201000100010001000000
22	178	16	1000100010001000000000	1000100010001000000000	1001000100010000000000	1001010001000100000000

Table 3.2: Maxima of $R(n)$ in relation to the length of representation in the $(2,2)$-system.

		#Ψ(l)						
l	$\psi(l)$	1, 1	2, 2	3, 3	4, 4	5, 5	6, 6	7, 7
1	1	2	3	4	5	6	7	8
2	1	1	5	11	19	29	41	55
3	2	1	2	3	4	5	6	7
4	2	2	9	20	35	54	77	104
5	3	2	4	6	8	10	12	14
6	4	1	4	9	16	25	36	49
7	5	2	4	6	8	10	12	14
8	6	4	16	36	64	100	144	196
9	8	3	12	33	72	135	228	357
10	10	4	16	36	64	100	144	196
11	13	2	4	6	8	10	12	14
12	16	5	32	117	320	725	1440	2597
13	21	2	4	6	8	10	12	
14	26	4	16	36	64			
15	34	2	4	6	8			
16	42	4	16	36				
17	55	2	4	6				
18	68	4	16	36				
19	89	2	4					
20	110	4	16					
21	144	2	4					
22	178	4	16					
23	233	2	4					
24	288	4	16					

Table 3.3: Sizes of the set $\Psi(l)$ for all surveyed systems with coefficients $a = b$ and order $m = 2$.

3.2.2 Confluent Systems with $a = b$ and order $m > 2$

Confluent systems with $a = b$ and order $m > 2$ show analogous behaviour to the m-bonacci systems. See Table 3.4, where we display the values of $\psi(l)$ and the first four elements of the set $\Psi(l)$ for the $(2, 2, 2)$-system. In Figures 3.7 and 3.8, see $R(n)$ on representations of length 7 in B-systems with coefficients $(2, 2, 2)$ and $(3, 3, 3)$. Lastly, we present the values of $\psi(l)$ for surveyed B-systems of order $m = 3$ and $m = 4$ in Tables 3.5. 3.6.

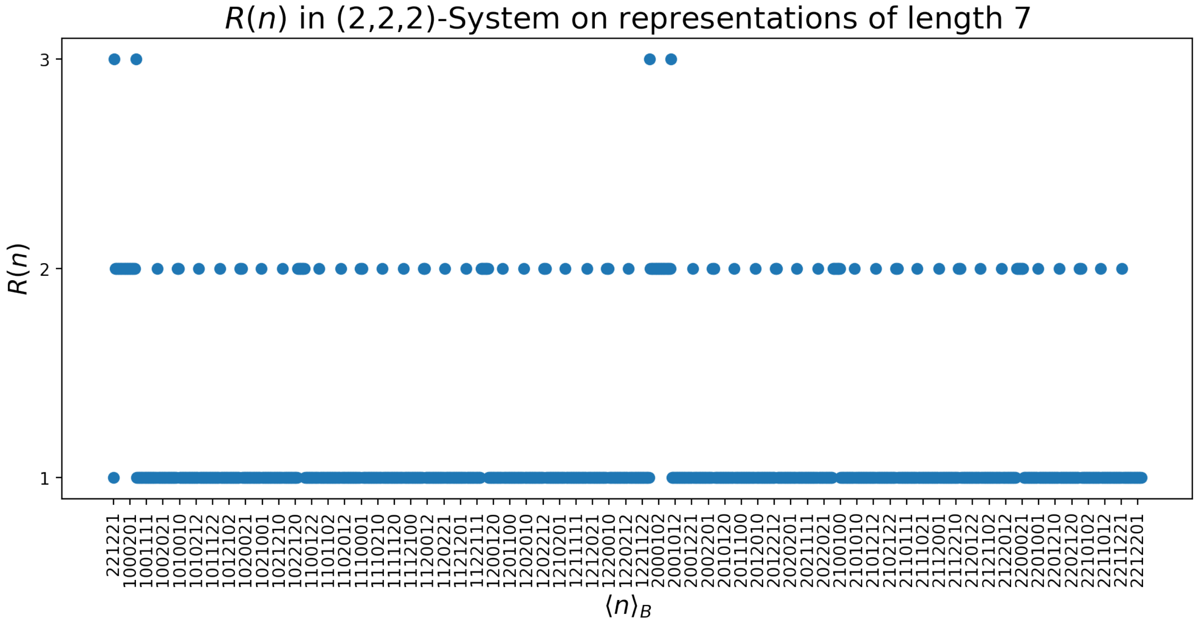

Figure 3.7: $R(n)$ in the $(2,2,2)$-B-system on all n whose greedy representation has length 7.

The expression for $\psi(l)$ is more difficult to uncover than in the previous case $a = b$ and $m = 2$. However, as we will show in Chapter 4.4, the values of $\psi(l)$ in relation to $l = p(m+1)+q$ satisfy

$$\begin{aligned}
\psi\left(p\left(m+1\right)+q\right) &= 2^p && \text{for } q \in \{0,1,\ldots,m-2\},\\
\psi\left(p\left(m+1\right)+m-1\right) &= 2^p + 2^{p-2} && \text{if } p \geq 2,\\
\psi\left(p\left(m+1\right)+m\right) &= 2^p + 2^{p-1}.
\end{aligned}$$

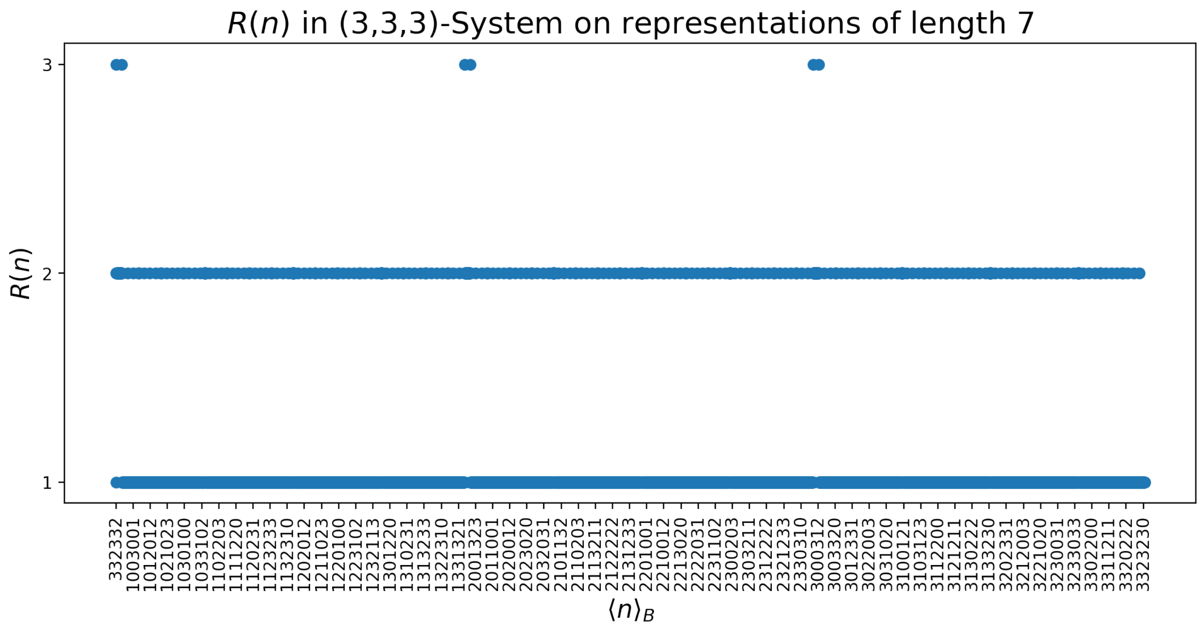

Figure 3.8: $R(n)$ in the $(3,3,3)$-B-system on all n whose greedy representation has length 7.

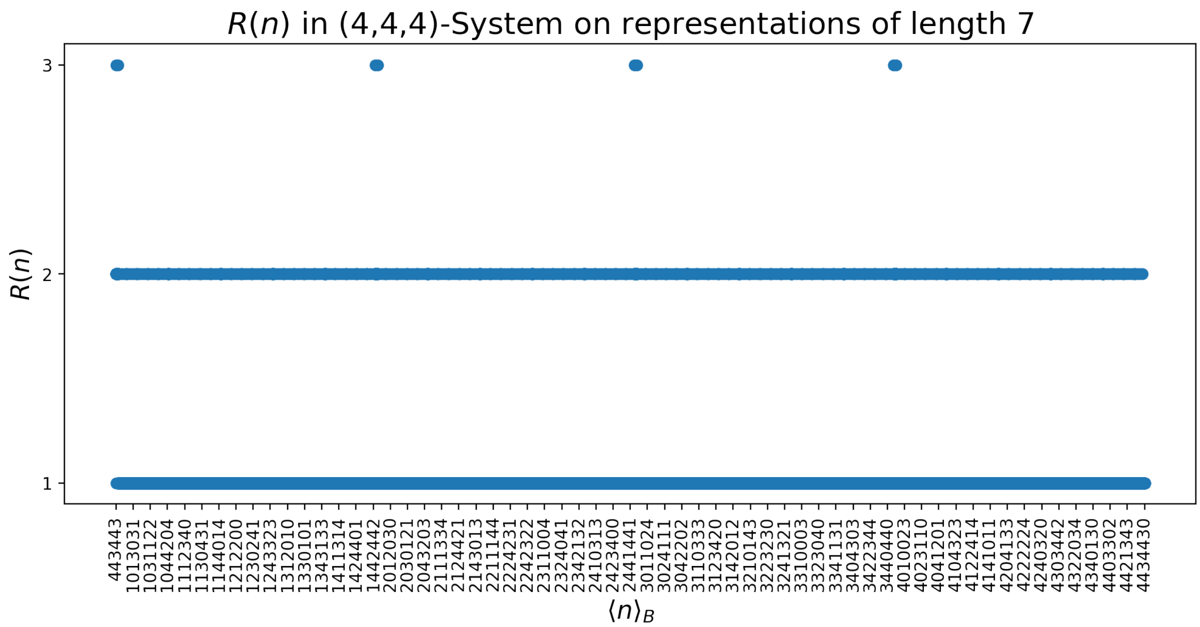

Figure 3.9: $R(n)$ in the $(4,4,4)$-B-system on all n whose greedy representation has length 7.

l	$\psi(l)$	$\#\Psi(l)$	First four elements of $\Psi(l)$			
1	1	3	0	1	2	
2	1	6	10	11	12	20
3	1	17	100	101	102	110
4	2	2	1000	2000		
5	2	10	10000	10001	10002	11000
6	2	41	100000	100001	100002	100010
7	3	4	1000000	1001000	2000000	2001000
8	4	4	10001000	10002000	20001000	20002000
9	4	32	100001000	100002000	100010000	100010001
10	5	4	1000001000	1001000000	2000001000	2001000000
11	6	16	10000001000	10000002000	10001000000	10001001000
12	8	8	100010001000	100010002000	100020001000	100020002000
13	8	92	1000001000000	1000010001000	1000010002000	1000020001000
14	10	16	10000010001000	10000010002000	10001000001000	10001001000000
15	12	48	100000010001000	100000010002000	100000020001000	100000020002000
16	16	16	1000100010001000	1000100010002000	1000100020001000	1000100020002000
17	16	240	10000010000001000	10000010000002000	10000100010001000	10000100010002000
18	20	48	100000100010001000	100000100010002000	100000100020001000	100000100020002000
19	24	128	1000000100010000000	1000000100010000000	1000000100020000000	1000000100020000000
20	32	32	10001000100010000000	10001000100010000000	10001000100020000000	10001000100020000000

Table 3.4: Maxima of $R(n)$ in relation to the length of representation in the $(2, 2, 2)$-system.

		#Ψ(l)				
l	$\psi(l)$	$1,1,1$	$2,2,2$	$3,3,3$	$4,4,4$	$5,5,5$
1	1	2	3	4	5	6
2	1	2	6	12	20	30
3	1	3	17	47	99	179
4	2	1	2	3	4	5
5	2	3	10	21	36	55
6	2	7	41	119	259	479
7	3	2	4	6	8	10
8	4	1	4	9	16	25
9	4	5	32	99	224	425
10	5	2	4	6	8	10
11	6	4	16	36	64	100
12	8	1	8	27	64	125
13	8	9	92	411	1224	2885
14	10	4	16	36	64	
15	12	6	48	162	384	
16	16	1	16	81		
17	16	13	240	1575		
18	20	6	48	162		
19	24	8	128			
20	32	1	32			
21	32	17	592			
22	40	8	128			

Table 3.5: Sizes of the set $\Psi(l)$ for all surveyed systems with coefficients $a = b$ and order $m = 3$
.

		$\#\Psi(l)$			
l	$\psi(l)$	$1,1,1,1$	$2,2,2,2$	$3,3,3,3$	$4,4,4,4$
1	1	2	3	4	5
2	1	2	6	12	20
3	1	4	18	48	100
4	1	7	53	191	499
5	2	1	2	3	4
6	2	3	10	21	36
7	2	8	42	120	260
8	2	19	161	623	1699
9	3	2	4	6	8
10	4	1	4	9	16
11	4	5	32	99	224
12	4	18	180	756	2160
13	5	2	4	6	8
14	6	4	16	36	64
15	8	1	8	27	64
16	8	7	88	405	
17	8	34	628	3894	
18	10	4	16		
19	12	6	48		
20	16	1	16		
21	16	9			
22	16	54			

Table 3.6: Sizes of the set $\Psi(l)$ for all surveyed systems with coefficients $a = b$ and order $m = 4$
.

3.2.3 Confluent Systems with $a > b$

Confluent systems with $a > b$ differ from the m-bonacci systems. Besides the $(2, 1)$-system that served as our introductory example, we also looked at further systems of this kind. The values of $\psi(l)$ and $\#\Psi(l)$ can be seen in Table 3.7 for systems of order $m = 2$ and in Table 3.8 for systems of order $m = 3$ and in Table 3.9 for systems with $m = 4$. As in the case of the systems with coefficients $a = b$, the maximal value of $R(n)$ is independent of the recurrence coefficients and depends solely on the order of recurrence and length of representation. In Chapter 4 we will prove that for all confluent systems with $a > b$ it is in fact equal to

$$\psi(l) = 2^{\lceil \frac{l}{m} \rceil - 1}.$$

Furthermore, the maxima are always concentrated in a clusters, as could be seen in the $(2,1)$-system and in Figures 3.10, 3.11, and 3.13, where we show $R(n)$ in the systems with coefficients $(3,1)$, $(3,2)$ and $(2,2,1)$ on all representations of length 7. Notice that only in the $(3,1)$-system the graph is symmetric as in the $(2,1)$-systems.

Looking at Tables 3.7 & 3.8 further, we may uncover a pattern in the size of the set $\Psi(l)$. Whenever $l \equiv 1 \mod m$, the size of the set $\Psi(l)$ is equal to

$$\#\Psi(l) = a \cdot (a-b)^{\lfloor \frac{l}{m} \rfloor - 1}(a-b+1).$$

This is due to the fact that in all confluent systems with $a > b$ the representations on which the value $\psi(pm+1)$ is reached are of the form

$$w = w_{l-1} \underbrace{(0^{m-1}c_p) \cdots (0^{m-1}c_1)}_{p \text{ times}} 0^{m-1} w_0,$$

where $p = \lfloor \frac{l}{m} \rfloor - 1$, $w_{l-1} \in \{1, 2, \ldots, a\}$, $w_0 \in \{0, 1, \ldots, a-b\}$ and $c_i \in \{1, 2, \ldots, a-b\}$ for all $i \in \{1, 2, \ldots, p\}$.

		#Ψ(l)													
l	$\psi(l)$	2, 1	3, 1	3, 2	4, 1	4, 2	4, 3	5, 1	5, 2	5, 3	5, 4	6, 1	6, 2	6, 3	6, 4
1	1	3	4	4	5	5	5	6	6	6	6	7	7	7	7
2	1	4	9	10	16	17	18	25	26	27	28	36	37	38	39
3	2	4	9	6	16	12	8	25	20	15	10	36	30	24	18
4	2	16	54	38	128	99	68	250	204	156	106	432	365	296	225
5	4	4	18	6	48	24	8	100	60	30	10	180	120	72	36
6	4	32	189	74	640	342	132	1625	1012	537	206	3456	2360	1464	774
7	8	4	36	6	144	48	8	400	180	60	10	900	480	216	72
8	8	48	540	110	2688	972	196	9000	4236	1524	306	23760	13040	6120	2196
9	16	4	72	6	432	96	8	1600	540	120	10	4500	1920	648	144
10	16	64	1404	146	10368	2520	260	46000	16308	3948	406				
11	32	4	144	6	1296	192	8	6400	1620	240	10				
12	32	80	3456	182	38016	6192	324								
13	64	4	288	6	3888	384									
14	64	96	8208	218											
15	128	4	576												
16	128	112													
17	256	4													
18	256	128													

Table 3.7: Sizes of the set $\Psi(l)$ for all surveyed systems with order $m = 2$ and coefficients satisfying $a > b$.

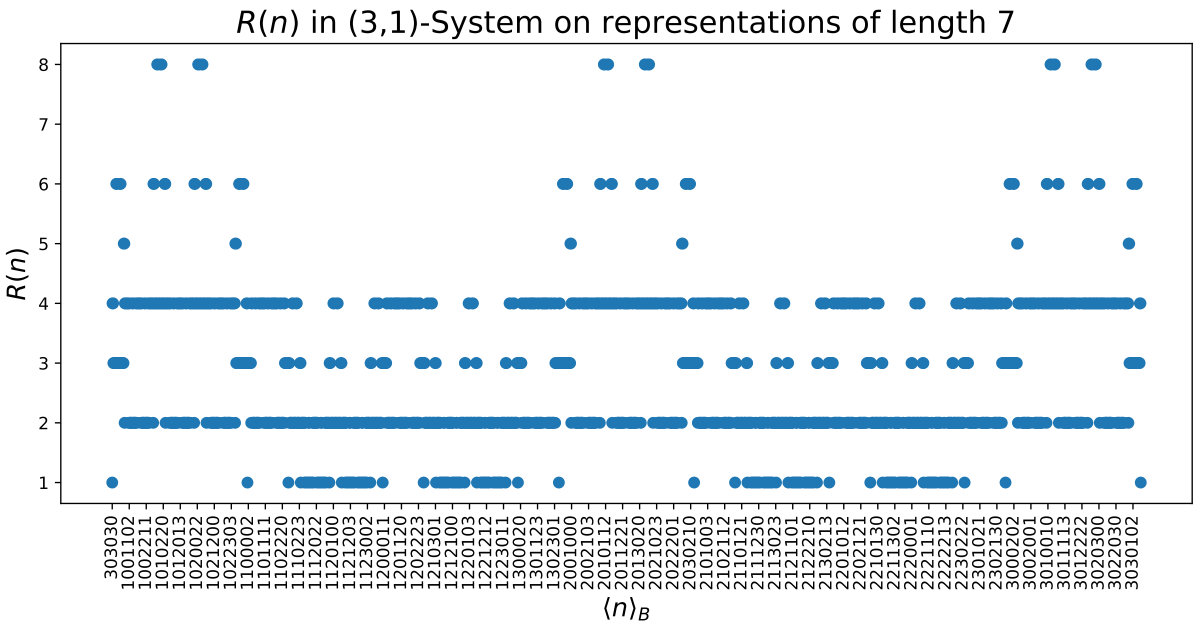

Figure 3.10: $R(n)$ in the $(3,1)$-B-system on all n whose greedy representation has length 7.

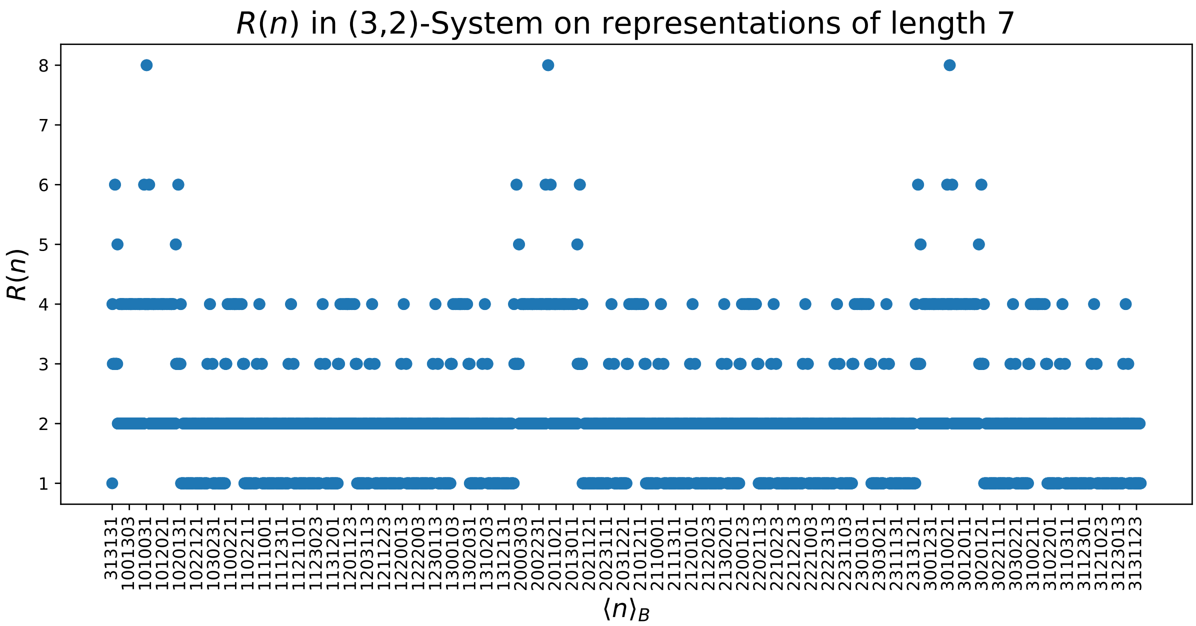

Figure 3.11: $R(n)$ in the $(3,2)$-B-system on all n whose greedy representation has length 7.

		#Ψ(l)					
l	$\psi(l)$	$2,2,1$	$3,3,1$	$3,3,2$	$4,4,1$	$4,4,2$	$4,4,3$
1	1	3	4	4	5	5	5
2	1	6	12	12	20	20	20
3	1	16	45	46	96	97	98
4	2	4	9	6	16	12	8
5	2	20	63	42	144	108	72
6	2	80	351	236	1024	771	516
7	4	4	18	6	48	24	8
8	4	36	207	78	688	360	136
9	4	208	1593	632	6656	3558	1412
10	8	4	36	6	144	48	8
11	8	52	576	114	2832	1008	200
12	8	400	5697	1244	34816	12876	2820
13	16	4	72	6	432	96	8
14	16	68	1476	150			
15	16	656	17874	2072			
16	32	4	144	6			
17	32	84	3600	186			
18	32	976					
19	64	4					
20	64	100					

Table 3.8: Sizes of the set $\Psi(l)$ for all surveyed systems with $a > b$ and order $m = 3$.

		$\#\Psi(l)$		
l	$\psi(l)$	$2,2,2,1$	$3,3,3,1$	$3,3,3,2$
1	1	3	4	4
2	1	6	12	12
3	1	18	48	48
4	1	52	189	190
5	2	4	9	6
6	2	20	63	42
7	2	84	360	240
8	2	320	1863	1244
9	4	4	18	6
10	4	36	207	78
11	4	212	1611	636
12	4	1040	10530	4268
13	8	4	36	6
14	8	52	576	114
15	8	404	5733	1248
16	8	2464	45603	10532
17	16	4	72	6
18	16	68		
19	16	660		
20	16	4848		
21	32	4		
22	32	84		
23	32	980		

Table 3.9: Sizes of the set $\Psi(l)$ for all surveyed systems with $a > b$ and order $m = 4$.

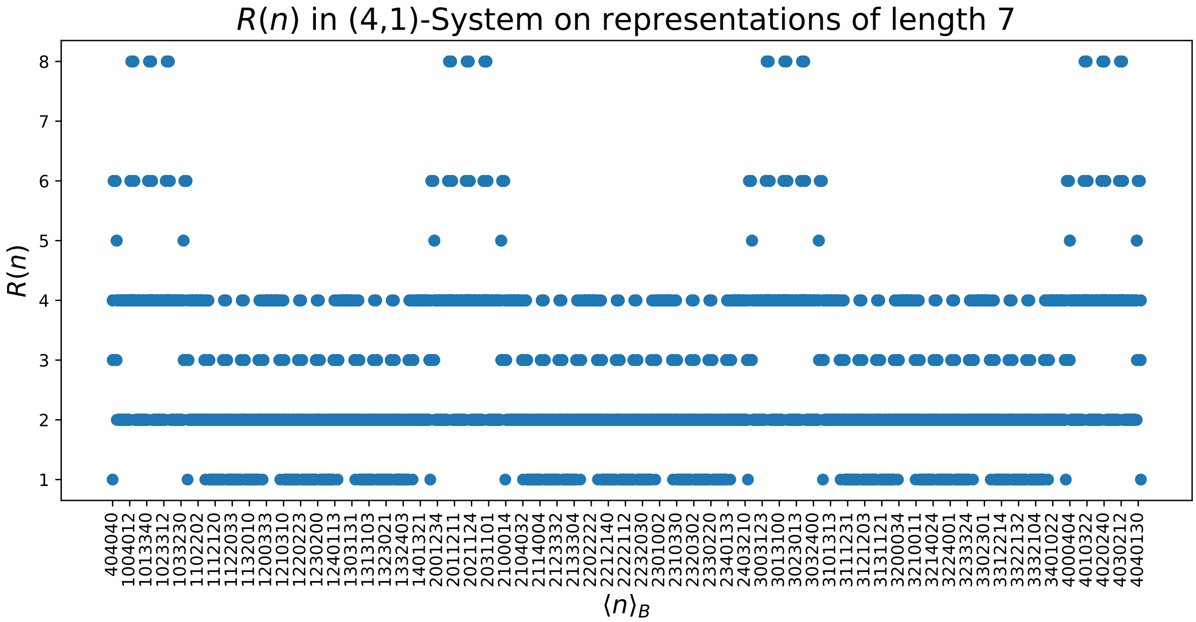

Figure 3.12: $R(n)$ in the $(4,1)$-B-system on all n whose greedy representation has length 7.

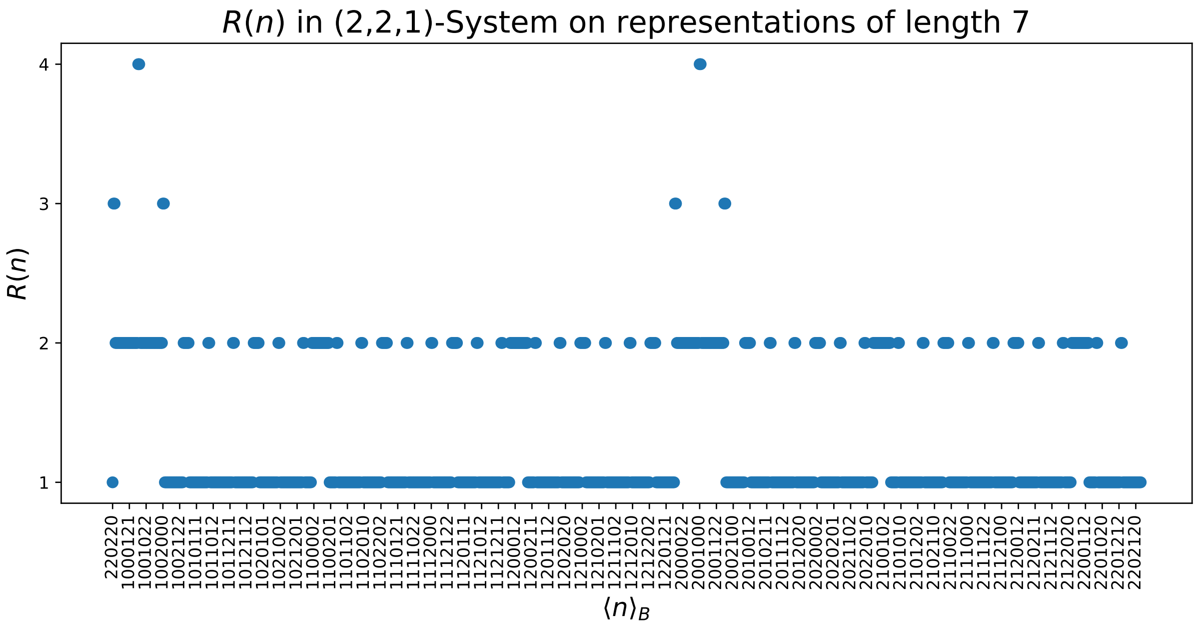

Figure 3.13: $R(n)$ in the $(2,2,1)$-B-system on all n whose greedy representation has length 7.

Chapter 4

Properties of $R(n)$ in Confluent B-systems

4.1 Palindromic Structure of $R(n)$

As noted before, $R(n)$ displays a piecewise palindromic structure. It is not difficult to realise that this is true in all systems with the (F) property. Take an $n \in \mathbb{N}_0$ with greedy representation $\langle n \rangle_B = x = x_l x_{l-1} \cdots x_1 x_0$. The word $x^C = (a{-}x_l)(a{-}x_{l-1}) \cdots (a{-}x_1)(a{-}x_0)$, where $a = t_1$ is the largest digit of the canonical alphabet, is called the *complement* of x. The word x^C is a representation of some value $\pi(x^C) = \tilde{n}$. The value of $R(n)$ depends solely on the number of possible interchanges generated by the rule $0t_1t_2 \cdots t_{m-1}t_m \to 10^m$, which form the rewriting system consisting of the rules

$$\begin{aligned} 0t_1t_2 \cdots t_{m-1}t_m &\to 10^m, \\ &\vdots \\ 0t_1t_1 \cdots t_1t_1 &\to 10(t_1{-}t_2) \cdots (t_1{-}t_{m-1})(t_1{-}t_m), \\ 1t_1t_2 \cdots t_{m-1}t_m &\to 20^m, \\ &\vdots \\ (t_1{-}1)t_1t_1 \cdots t_1t_1 &\to t_10(t_1{-}t_2) \cdots (t_1{-}t_{m-1})(t_1{-}t_m), \end{aligned}$$

along with all the interchanges generated from the initial representations of zero (i.e. the rules $0t_1t_2 \cdots (t_{m-1}{+}1) \to 10^{m-1}$, $\ldots$, $0(t_1{+}1) \to 10$ that can be used at the end of a B-representation). Clearly, the complement of every rewritable factor is rewritable, however, not necessarily by the same rule. Take for example the $(3,2,2)$-system. Then the associated rewriting system ρ_A is generated by the rule $0322 \to 1000$ and consists of a total of 12 rules, written below:

$$\begin{aligned} &0322 \to 1000, && 1322 \to 2000, && 2322 \to 3000, \\ &0323 \to 1001, && 1323 \to 2001, && 2323 \to 3001, \\ &0332 \to 1010, && 1332 \to 2010, && 2332 \to 3010, \\ &0333 \to 1011, && 1333 \to 2011, && 2333 \to 3011. \end{aligned}$$

Notice that the complement of every string on the right hand side of a rule appears on the left hand side of a rule and vice versa. For example, the complement of the right side string 3010 is 0323, which appears on the left side of the rule $0323 \to 1001$. It is easy to prove that this in fact holds in general. Take an arbitrary rewritable string $u = u_m u_{m-1} u_{m-2} \cdots u_1 u_0$ that is on the left side of a rule. Then $u_m < t_1$ and clearly $u_{m-i} \geq t_i$ for all $i \in \{1, 2, \dots, m\}$. Then we get $u^C = (t_1 - u_m)0(t_1 - u_{m-2}) \cdots (t_1 - u_1)(t_1 - u_0)$ and u^C is a rewritable string on the right side of a rule, since $u^C_m < t_1$ and $u^C_{m-i} \leq t_1 - t_i$ for all $i \in \{1, 2, \dots, m\}$. The same statement can be proved for rewriting rules using the initial representations of zero. However, only if those initial representations of zero have digits contained in the canonical alphabet A. Omitting those rewriting rules that have digits not contained in A is not a problem, since we are only interested in complements of greedy representations.

Let us now return to the number $\tilde{n}$ represented by the string x^C. Clearly if a factor of x was rewritable, then the factor of its complement will be also rewritable. Therefore $R(n) = R(\tilde{n})$. Since

$$n + \tilde{n} = a \cdot \sum_{i=0}^{l} B_i,$$

the centre of symmetry will correspond to the value which we denote

$$C(l+1) = \frac{a}{2} \cdot \sum_{i=0}^{l} B_i,$$

where we use $l+1$ as the argument because that is the length of the word x. Thus, the sequence $(R(n))_{n=0}^{\infty}$ contains a palindrome ending in the value $R(B_{l+1} - 1)$ and beginning in the value $a \cdot \sum_{i=0}^{l} B_i - B_{l+1} + 1$. As we noted before in Chapter 3, in $(a, 1)$-systems the palindrome spans precisely the numbers whose representation has length $l + 1$ (plus the largest number whose representation has length l, which is $B_l - 1$). This is a consequence of the fact that in $(a, 1)$-systems, the greedy representation of $B_l - 1$ has the form

$$\langle B_l - 1 \rangle_B = \begin{cases} (a0)^{\frac{l}{2}} & \text{for } l \text{ even,} \\ (a0)^{\lfloor \frac{l}{2} \rfloor} a & \text{for } l \text{ odd.} \end{cases}$$

The complement of $\langle B_l - 1 \rangle_B$ is thus the word $\langle B_{l-1} - 1 \rangle_B$. In all other systems, the complement of $\langle B_l - 1 \rangle_B$ is a word with value strictly smaller than $B_{l-1} - 1$, thus the palindrome does not align with representations of a given length. We can say that the palindrome with centre $C(l+1)$ overlaps with the palindromes with centres $C(l)$ and $C(l+2)$, and possibly others in certain systems.

4.2 Matrix Formula for $R(n)$

In this section we will formalise our findings from the previous chapter and derive a closed-form formula for the function $R(n)$. Throughout this section, we will use the word *gap* to refer to factors consisting of consecutive zeroes.

Analogically to the approach used in [11], we will derive a matrix formula for the function $R(n)$. The formula for $R(n)$ in the Fibonacci system is originally due to Berstel [1]. Kocábová, Masáková and Pelantová [11] then derived a matrix formula for $R(n)$ in the m-bonacci numeration systems. We will generalise their results to all confluent B-systems. During the time of writing, we did not know that Edson [2] derived the formula as well, as part of her study of confluent systems of order two (i.e. the (a, b)-systems, where $a \geq b \geq 1$).

Firstly, we will explore in detail the proof the matrix formula for $R(n)$ for the B-system with basis $B_n = 2B_{n-1} + B_{n-1}$, then generalise our findings to all confluent systems.

In the second half of this chapter, we will use the matrix formula to verify our observations about the properties of the function $R(n)$ from Chapter 3. We will begin by a trivial observation.

Lemma 4.1. *Suppose some representation of the form $x_1 0^{r_1}$, where x_1 is non-zero, i.e. $x_1 \in \{1, 2\}$ and $r_1 \in \mathbb{N}_0$. Then*

$$R(x_1 0^{r_1}) = 1 + \left\lfloor \frac{r_1}{2} \right\rfloor.$$

Proof. The representation $x_1 0^{r_1}$ is certainly greedy. Using the rewriting rules $100 \to 021, 200 \to 121$ we can generate new (non-greedy) representations

$$x_1 0^{r_1} \to (x_1{-}1)210^{r_1-2} \to (x_1{-}1)20210^{r_1-4} \to \cdots$$

until the end of the string is reached – i.e. until we cannot apply a rewriting rule any further:

$$\cdots \to \begin{cases} (x_1{-}1)2020\cdots 20210 & \text{if } r_1 \text{ is odd,} \\ (x_1{-}1)2020\cdots 2021 & \text{if } r_1 \text{ is even.} \end{cases}$$

Evidently, the string $x_1 0^{r_1}$ can be rewritten only $\lfloor \frac{r_1}{2} \rfloor$ times in total, since there are r_1 zeroes available to be rewritten and each rewriting replaces two zeroes in the suffix 0^{r_1}. We can therefore write

$$R(x_1 0^{r_1}) = 1 + \left\lfloor \frac{r_1}{2} \right\rfloor,$$

where we count the original representation $x_1 0^{r_1}$ plus the $\lfloor \frac{r_1}{2} \rfloor$ representations generated by subsequent rewritings of $x_1 0^{r_1}$. □

Similarly to the terminology established in [11], we will distinguish *long* and *short* representations. This will be a key concept for deriving the matrix formula of $R(n)$.

Definition 4.2. Let $w = x_1 0^{r_1} u$ be some greedy representation, where x_1 is non-zero, $r_1 \in \mathbb{N}_0$ and the suffix u is either empty or has a non-zero initial digit. Then a *long representation of w (with respect to x_1)* is any B-representation v such that

- $\pi(v) = \pi(w)$,
- $v \in A^*$, where A is the canonical alphabet of the B-system,
- w and v share the prefix $x_1 0^{r_1}$.

Conversely, a *short representation of w (with respect to x_1)* is any B-representation u such that

- $\pi(u) = \pi(w)$,
- $u \in A^*$, where A is the canonical alphabet of the B-system,
- $u = (x_1{-}1)u_{N-2}u_{N-3}\cdots$. I.e. the digit x_1 was rewritten to $x_1 - 1$ using some rewriting rule generated by the B-system.

The number of long and short representations will be denoted $\overline{R}(x_1 0^{r_1} u)$ and $\underline{R}(x_1 0^{r_1} u)$, respectively.

Note: The naming is based on the m-bonacci B-systems, where indeed, every short representation of w is shorter than a long representation of w. For example, among the two Fibonacci representations $w = 100$ and $v = 011$ of the number three, v is shorter than w (where the length is understood as to be counted to the first non-zero digit). However, in other B-systems, such as the $(2,1)$-system, this need not be the case, since $w = 200$ is a long representation, while $u = 121$ is called a short representation of the same value, even if it has the same length.

A trivial observation is that every greedy representation is a long representation. Recall the representations from Example 3.2:

$$\begin{aligned} &1020100, \quad 1002200, \quad 0212200, \\ &1020021, \quad 1002121, \quad 0212121. \end{aligned}$$

Then representations in the left and centre columns are long representations of $w = 1020100$ with respect to the initial 1, whereas those in the right column are short representations. We can therefore write $\overline{R}(1020100) = 4$ and $\underline{R}(1020100) = 2$.

From the above definition and example it is apparent that $R(x_1 0^{r_1} u) = \overline{R}(x_1 0^{r_1} u) + \underline{R}(x_1 0^{r_1} u)$ for every greedy representation $x_1 0^{r_1} u$. In matrix form:

$$R(x_1 0^{r_1} u) = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \overline{R}(x_1 0^{r_1} u) \\ \underline{R}(x_1 0^{r_1} u) \end{pmatrix}.$$

Let us now consider a more complex example. Suppose that we have a greedy representation with two non-zero digits, one that can be written as $w = x_2 0^{r_2} x_1 0^{r_1}$, where $x_1, x_2 \in \{1,2\}$, $r_1, r_2 \in \mathbb{N}_0$. Then the following holds – if $x_2 = 2$, then $r_2 \geq 1$. This condition is equivalent to the normality (greediness) of the representation. We will now assess short and long representations separately before synthesizing our findings into the matrix formula for $R(w)$.

Lemma 4.3. *Let $x_2 0^{r_2} x_1 0^{r_1}$, where $x_1, x_2 \in \{1,2\}$, $r_1, r_2 \in \mathbb{N}_0$, be a greedy representation in the B-system satisfying the recurrence $B_n = 2B_{n-1} + B_{n-2}$, where $x_1, x_2 \in \{1,2\}$, $r_1, r_2 \in \mathbb{N}_0$. Then the number of long representations of $x_2 0^{r_2} x_1 0^{r_1}$ is equal to*

$$\overline{R}(x_2 0^{r_2} x_1 0^{r_1}) = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \overline{R}(x_1 0^{r_1}) \\ \underline{R}(x_1 0^{r_1}) \end{pmatrix}.$$

Proof. The number of long representations of $x_2 0^{r_2} x_1 0^{r_1}$ is equal to the total number of representations of $x_1 0^{r_1}$ because the only allowed rewritings can be done on the suffix $x_1 0^{r_1}$, the prefix $x_2 0^{r_2}$ must be kept unchanged. Then, the total number of representations that can be generated by rewriting the suffix $x_1 0^{r_1}$ is equal precisely to $R(x_1 0^{r_1}) = \overline{R}(x_1 0^{r_1}) + \underline{R}(x_1 0^{r_1})$. □

Lemma 4.4. *Let $x_2 0^{r_2} x_1 0^{r_1}$, where $x_1, x_2 \in \{1,2\}$, $r_1, r_2 \in \mathbb{N}_0$, be a greedy representation in the B-system satisfying the recurrence $B_n = 2B_{n-1} + B_{n-2}$. Then the number of short representations of $x_2 0^{r_2} x_1 0^{r_1}$ is equal to*

$$\underline{R}(x_2 0^{r_2} x_1 0^{r_1}) = \begin{cases} \begin{pmatrix} \lfloor \frac{r_2+1}{2} \rfloor & \lfloor \frac{r_2+1}{2} \rfloor \end{pmatrix} \begin{pmatrix} \overline{R}(x_1 0^{r_1}) \\ \underline{R}(x_1 0^{r_1}) \end{pmatrix} & \text{if } x_1 = 1, \\ \\ \begin{pmatrix} \lfloor \frac{r_2}{2} \rfloor & \lfloor \frac{r_2+1}{2} \rfloor \end{pmatrix} \begin{pmatrix} \overline{R}(x_1 0^{r_1}) \\ \underline{R}(x_1 0^{r_1}) \end{pmatrix} & \text{if } x_1 = 2. \end{cases} \tag{4.1}$$

Proof. Suppose initially that $r_2 = 0$. Then evidently $x_2 = 1$, otherwise $x_2 0^{r_2} x_1 0^{r_1}$ would not be greedy. The number of short representations of $1x_1 0^{r_1}$ is equal to zero, since even after rewriting x_1 we are left with

$$1x_1 0^{r_1} \to 1(x_1{-}1)210^{r_1-2},$$

where the prefix $1(x_1{-}1)2$ cannot be rewritten further, so $1x_1 0^{r_1}$ has no short representations. Accordingly, both expressions on the right-hand side of (4.1) are equal to zero when $r_2 = 0$.

Consider now the case when $r_2 \geq 1$. Additionally, let r_2 be an even integer. Then apparently we can rewrite the prefix $x_2 0^{r_2}$ precisely $r_2/2$ times, until we arrive at the string

$$x_2 0^{r_2} x_1 0^{r_1} \to \cdots \to (x_2{-}1)\,\underbrace{20\cdots 2021}_{r_2}\,x_1 0^{r_1},$$

where the prefix $(x_2{-}1)20\cdots 2021$ cannot be rewritten further. All of these rewritings have no effect on the suffix $x_1 0^{r_1}$, so the number of short representations of $x_2 0^{r_2} x_1 0^{r_1}$ is equal simply to $\frac{r_2}{2}$ multiplied by the total number of representations of $x_1 0^{r_1}$. Furthermore, since $\lfloor \frac{r_2}{2} \rfloor = \lfloor \frac{r_2+1}{2} \rfloor = \frac{r_2}{2}$ for even r_2, the formula (4.1) holds regardless of the value of the digit x_1.

Lastly, take an odd $r_2 \geq 1$. Then suppose the prefix $x_2 0^{r_2}$ was already rewritten $\lfloor \frac{r_2}{2} \rfloor$ times, yielding the string

$$(x_2{-}1)\,\underbrace{20\cdots 20210}_{r_2}\,x_1 0^{r_1}.$$

Now, if $x_1 = 1$, we can rewrite the string once more using the rule $101 \to 022$ (whose usage is highlighted in bold), which yields the representation

$$(x_2{-}1)20\cdots 202\mathbf{101}0^{r_1} \to (x_2{-}1)20\cdots 202\mathbf{022}0^{r_1}. \tag{4.2}$$

Alternatively, x_1 could have been rewritten earlier to 0, which would then using the rule $100 \to 021$ yield

$$(x_2{-}1)20\cdots 202\mathbf{021}210^{r_1-2}. \tag{4.3}$$

Again, in both cases, all rewritings of the prefix $x_2 0^{r_2}$ can be done independently of whether the suffix $x_1 0^{r_1}$ was rewritten. Therefore, the total number of short representations of $x_2 0^{r_2} x_1 0^{r_1}$ is equal to $\lfloor \frac{r_2+1}{2} \rfloor$ times the number of long representations of $x_1 0^{r_1}$ (string (4.2)) plus $\lfloor \frac{r_2+1}{2} \rfloor$ times the number of short representations of $x_1 0^{r_1}$ (string (4.3) plus its $\lfloor \frac{r_1-2}{2} \rfloor$ subsequent rewritings). Formally

$$\underline{R}(x_2 0^{r_2} x_1 0^{r_1}) = \left\lfloor \frac{r_2+1}{2} \right\rfloor (\overline{R}(x_1 0^{r_1}) + \underline{R}(x_1 0^{r_1})),$$

which is precisely the first row of equation (4.1).

Consider now the case when $x_1 = 2$. Then after $\lfloor \frac{r_2}{2} \rfloor$ rewritings of the prefix $x_2 0^{r_2}$ the following string is reached:

$$(x_2{-}1)\,\underbrace{20\cdots 202\mathbf{10}}_{r_2}\,\mathbf{2}0^{r_1}.$$

The factor 102 (bold) cannot be rewritten further, so the number of short representations of $x_2 0^{r_2} x_1 0^{r_1}$ that are reachable without rewriting the suffix 20^{r_1} is equal to $\lfloor \frac{r_2}{2} \rfloor \overline{R}(x_1 0^{r_1})$. However, if the digit $x_1 = 2$ is rewritten using the $200 \to 121$ rule (which is possible only if $r_1 \geq 2$), the factor 102 is replaced by 101, which we can rewrite:

$$\begin{aligned}\cdots \to &(x_2{-}1)20\cdots 202102000^{r_1-2} \to \\ \to &(x_2{-}1)20\cdots 20210\mathbf{121}0^{r_1-2} \to \\ \to &(x_2{-}1)20\cdots 202\mathbf{022}210^{r_1-2}.\end{aligned}$$

Therefore, the number of short representations of $x_2 0^{r_2} x_1 0^{r_1}$ that are obtained if we also rewrite the suffix $x_1 0^{r_1}$ at least once (making it a short representation of $x_1 0^{r_1}$) is equal to $\lfloor \frac{r_2+1}{2} \rfloor \underline{R}(x_1 0^{r_1})$.

The total number of short representations of $x_2 0^{r_2} x_1 0^{r_1}$ is therefore equal to

$$\underline{R}(x_2 0^{r_2} x_1 0^{r_1}) = \left\lfloor \frac{r_2}{2} \right\rfloor \overline{R}(x_1 0^{r_1}) + \left\lfloor \frac{r_2+1}{2} \right\rfloor \underline{R}(x_1 0^{r_1}),$$

which is precisely the second row of equation (4.1). □

In Lemmas 4.3 and 4.4 we saw that the contribution of the factor $x_2 0^{r_2}$ to the value of $R(w)$ depends on r_2 (the length of the gap) and the digit x_1. This is a key difference to the m-bonacci systems, where we have to only consider the digits 0 and 1. The different contribution of 1 and 2 can be expressed in the form of two matrices:

Definition 4.5. Consider the B-system with basis $B_n = 2B_{n-1} + B_{n-1}$. Then for all $r \in \mathbb{N}_0$ the *redundancy matrices* are defined as

$$M_1(r) = \begin{pmatrix} 1 & 1 \\ \lfloor \frac{r+1}{2} \rfloor & \lfloor \frac{r+1}{2} \rfloor \end{pmatrix}, \qquad M_2(r) = \begin{pmatrix} 1 & 1 \\ \lfloor \frac{r}{2} \rfloor & \lfloor \frac{r+1}{2} \rfloor \end{pmatrix}.$$

With this definition, we can then synthesise the findings from Lemmas 4.1, 4.3, and 4.4 into the following theorem.

Theorem 4.6. *Consider the B-system with basis $B_n = 2B_{n-1} + B_{n-1}$. Then every greedy representation can be written in the form*

$$w = x_s 0^{r_s} x_{s-1} 0^{r_{s-1}} \cdots x_1 0^{r_1},$$

where $x_i \in \{1,2\}$,$r_i \in \mathbb{N}_0$ for all i, and the following holds: If $x_i = 2$ for some index $i > 1$, then $r_i \geq 1$. Consider some greedy representation w thus written. Then $R(w)$ has the closed form

$$R(x_s 0^{r_s} \cdots x_1 0^{r_1}) = \begin{pmatrix} 1 & 1 \end{pmatrix} M_{x_{s-1}}(r_s) \cdots M_{x_1}(r_2) \begin{pmatrix} 1 \\ \lfloor \frac{r_1}{2} \rfloor \end{pmatrix}. \tag{4.4}$$

Proof. By induction on $s \in \mathbb{N}_0$. Case $s = 1$ is treated in Remark 4.1, case $s = 2$ is an immediate corollary of Lemmas 4.3 and 4.4.

Suppose now that $s > 2$ and that the formula (4.4) holds for some s. To prove that it holds for $s+1$, recall the proofs of Lemmas 4.3 and 4.4. We never evaluated $\overline{R}(x_1 0^{r_1})$ and $\underline{R}(x_1 0^{r_1})$, so the factor $x_1 0^{r_1}$ can be replaced by any $x_s u$, where $x_s \in \{1,2\}$, $u \in A^*$ and $x_s u$ is a greedy representation. Setting $x_s u = x_s 0^{r_s} x_{s-1} 0^{r_{s-1}} \cdots x_1 0^{r_1}$ and using Lemmas 4.3, 4.4 yields

$$R(x_{s+1} 0^{r_{s+1}} x_s 0^{r_s} \cdots x_1 0^{r_1}) = \begin{pmatrix} 1 & 1 \end{pmatrix} M_{x_s}(r_{s+1}) M_{x_{s-1}}(r_s) \cdots M_{x_1}(r_2) \begin{pmatrix} 1 \\ \lfloor \frac{r_1}{2} \rfloor \end{pmatrix},$$

which proves the theorem. □

The above theorem proven for the $(2,1)$-system can be generalised to all confluent B-systems. However, another, third case of a digit ending the factor of consecutive zeroes has to be proven first. We will first introduce some notation.

Definition 4.7. Consider a confluent B-system of order m with coefficients $a \geq b \geq 1$. Then for all $r \in \mathbb{N}_0$ the three *redundancy matrices* are defined as

$$M_c(r) = \begin{pmatrix} 1 & 1 \\ \lfloor \frac{r+1}{m} \rfloor & \lfloor \frac{r+1}{m} \rfloor \end{pmatrix}, \quad M_{a-b+1}(r) = \begin{pmatrix} 1 & 1 \\ \lfloor \frac{r}{m} \rfloor & \lfloor \frac{r+1}{m} \rfloor \end{pmatrix}, \quad M_d(r) = \begin{pmatrix} 1 & 1 \\ \lfloor \frac{r}{m} \rfloor & \lfloor \frac{r}{m} \rfloor \end{pmatrix},$$

for all digits $1 \leq c < a - b + 1$ and $a - b + 1 < d \leq a$.

Evidently, not all three matrices are defined for all possible pairs of coefficients a, b. For example, in the case when $a = b$, only the matrices $M_{a-b+1}(r)$, $M_d(r)$, $d \in \{2, \dots a\}$ are defined, since there is no digit c in the canonical alphabet that would satisfy $1 \leq c < a - b + 1$. On the other hand, all three matrices are defined for example in the $(3, 2)$-system. Note that $M_{a-b+1}(r)$ is the same matrix as in the matrix formula for the m-bonacci systems [11]. We will prove three propositions establishing the origin of these matrices, from which the matrix formula for $R(n)$ will follow.

Proposition 4.8. *Suppose a confluent B-system of order m with coefficients $a \geq b \geq 1$. Consider a greedy representation of the form $w = x0^r cu$, where $x \in \{1, \dots, a\}$, $r \in \mathbb{N}_0$, c is a digit satisfying $1 \leq c < a-b+1$, and u is either the empty word or a word such that cu is a greedy representation. Then*

$$\begin{pmatrix} \overline{R}(x0^r cu) \\ \underline{R}(x0^r cu) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \lfloor \frac{r+1}{m} \rfloor & \lfloor \frac{r+1}{m} \rfloor \end{pmatrix} \begin{pmatrix} \overline{R}(cu) \\ \underline{R}(cu) \end{pmatrix}.$$

Proof. A generalisation of Lemmas 4.3 and 4.4. The number of long representations $\overline{R}(x0^r cu)$ is clearly equal to the total number of representations of cu, hence $\overline{R}(x0^r cu) = \overline{R}(cu) + \underline{R}(cu)$.

Let us now determine the number of short representations. Suppose p, q such that $r = pm+q$ and $q \in \{0, 1, \dots m-1\}$. Clearly the gap 0^r may be rewritten $p = \lfloor \frac{r}{m} \rfloor$ times, after rewriting we are left with the string

$$(x{-}1)\, a^{m-1}\, (b{-}1) \cdots a^{m-1} b 0^q cu. \tag{4.5}$$

If $q = m-1$, then since $1 \leq c < a-b+1$ we can rewrite (4.5) once more, which yields the string

$$(x{-}1)\, a^{m-1}\, (b{-}1) \cdots a^{m-1}\, (b{-}1)\, a^{m-1}\, (b{+}c)\, u.$$

If $q < m-1$, this rewriting is not possible, so we obtain

$$\underline{R}(x0^r cu) = \left\lfloor \frac{r+1}{m} \right\rfloor \left(\overline{R}(cu) + \underline{R}(cu) \right),$$

which proves the second row of the matrix. □

Proposition 4.9. *Consider a confluent B-system of order m with coefficients $a \geq b \geq 1$. Consider a greedy representation of the form $w = x0^r(a{-}\,b{+}1)u$, where $x \in \{1, \dots, a\}$, $r \in \mathbb{N}_0$, and u is either the empty word or a word such that $(a{-}\,b{+}1)u$ is a greedy representation. Then*

$$\begin{pmatrix} \overline{R}(x0^r(a{-}\,b{+}1)u) \\ \underline{R}(x0^r(a{-}\,b{+}1)u) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \lfloor \frac{r}{m} \rfloor & \lfloor \frac{r+1}{m} \rfloor \end{pmatrix} \begin{pmatrix} \overline{R}((a{-}\,b{+}1)u) \\ \underline{R}((a{-}\,b{+}1)u) \end{pmatrix}.$$

Proof. Suppose p, q such that $r = pm + q$ and $q \in \{0, 1, \dots m-1\}$. Again, the gap 0^r may be rewritten $p = \lfloor \frac{r}{m} \rfloor$ times, after which we are left with the string

$$(x{-}1)\, a^{m-1}\, (b{-}1) \cdots a^{m-1}\, (b{-}1)\, 0^q (a{-}\,b{+}1)u. \tag{4.6}$$

Let $q = m - 1$. Then we can rewrite (4.6) once more only if we rewrite the digit $(a{-}\,b{+}1)$ first. This yields the string

$$(x{-}1)\, a^{m-1}\, (b{-}1) \cdots a^{m-1}\, (b{-}1)\, a^{m-1} a\widetilde{u},$$

where $\widetilde{u}$ is the suffix of the result of the rewriting $(a{-}\,b{+}1)u \to (a{-}b)\widetilde{u}$. This corresponds to $\lfloor \frac{r+1}{m} \rfloor \underline{R}((a{-}\,b{+}1)u)$ representations. If we do not rewrite the digit $(a{-}b{+}1)$, we do not gain this extra rewriting, thus we count another $\lfloor \frac{r}{m} \rfloor \overline{R}((a{-}\,b{+}1)u)$ possible representations. If $q < m-1$, this rewriting is not possible, so in total we obtain

$$\underline{R}(x0^r(a{-}\,b{+}1)u) = \left\lfloor \frac{r}{m} \right\rfloor \overline{R}((a{-}\,b{+}1)u) + \left\lfloor \frac{r+1}{m} \right\rfloor \underline{R}((a{-}\,b{+}1)u),$$

which proves the claim. □

Proposition 4.10. *Suppose a confluent B-system of order m with coefficients $a \geq b \geq 1$. Consider a greedy representation of the form $w = x0^r du$, where $x \in \{1, \ldots, a\}$, $r \in \mathbb{N}_0$, the digit d satisfies $d > a - b + 1$, and u is either the empty word or a word such that du is a greedy representation. Then*

$$\begin{pmatrix} \overline{R}(x0^r du) \\ \underline{R}(x0^r du) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \lfloor \frac{r}{m} \rfloor & \lfloor \frac{r}{m} \rfloor \end{pmatrix} \begin{pmatrix} \overline{R}(du) \\ \underline{R}(du) \end{pmatrix}.$$

Proof. Suppose p, q such that $r = pm + q$ and $q \in \{0, 1, \ldots m-1\}$. Clearly the gap 0^r may be rewritten $p = \lfloor \frac{r}{m} \rfloor$ times, after which we are left with the string

$$(x{-}1)\, a^{m-1}\, (b{-}1) \cdots a^{m-1} b 0^q du. \tag{4.7}$$

Because $d > a - b + 1$, no more rewritings are possible, which leads us to

$$\underline{R}(x0^r du) = \left\lfloor \frac{r}{m} \right\rfloor \left(\overline{R}(du) + \underline{R}(du) \right),$$

thus proving the claim. □

Theorem 4.11. *Consider a confluent B-system of order m with coefficients $a \geq b \geq 1$. Then every greedy representation can be written in the form*

$$w = x_n 0^{r_n} x_{n-1} 0^{r_{n-1}} \cdots x_1 0^{r_1},$$

where $x_i \in \{1, 2, \ldots, a\}$,$r_i \in \mathbb{N}_0$ for all i, and the following holds: If $x_i x_{i-1} \cdots x_{i-m+1} = a^{m-1}$ for some index $i > m-1$, then either $x_{i-m} < b$ or $r_k \geq 1$ for some $k \in \{i, i-1, \ldots, i-m\}$.

Then $R(w)$ has the closed form

$$R\left(x_n 0^{r_n} x_{n-1} 0^{r_{n-1}} \cdots x_1 0^{r_1}\right) = \begin{pmatrix} 1 & 1 \end{pmatrix} M_{x_{n-1}}(r_n) M_{x_{n-2}}(r_{n-1}) \cdots M_{x_1}(r_2) \begin{pmatrix} 1 \\ \lfloor \frac{r_1}{m} \rfloor \end{pmatrix}. \tag{4.8}$$

Proof. Corollary of Propositions 4.8, 4.9, and 4.10. □

We will now use Theorem 4.11 to verify our observations from Section 3.2.

4.3 Maxima of $R(n)$ in General Confluent Systems

As said before, when discussing the maxima of $R(n)$, confluent B-systems' behaviour can be split into three groups based on their recurrence coefficients and order. For the first two groups consisting of systems with coefficients satisfying $a = b$, we will simply generalise findings for the Fibonacci and m-bonacci systems [11, 12]. The third group with coefficients that satisfy $a > b$ displays simpler behaviour.

For all three classes of confluent systems we will use the same approach as in [11, 12]. Firstly, we shall derive a lower bound for $\psi(l)$ by finding representations w of a given length on which the maximal value of $R(w)$ is reached. Secondly, we will derive an upper bound on $\psi(l)$ and prove that it is indeed equal to the value of $R(w)$ that is reached on the representations derived in the first step. We will do this by showing which factors elements of $\Psi(l)$ must avoid and prove the expression for $\psi(l)$ by induction on l, the length of representation.

We will need the following terminology (that is again adapted from [11, 12]) that will simplify our analysis. We shall establish a partial ordering on matrices and prove that this ordering implies an ordering on the values of $R(w)$.

Definition 4.12. Let $\mathbb{X} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $\tilde{\mathbb{X}} = \left(\begin{smallmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{smallmatrix}\right)$ be integer matrices with non-negative elements. Then we say that $\mathbb{X}$ *majores* $\tilde{\mathbb{X}}$ (written as $\mathbb{X} \succ \tilde{\mathbb{X}}$) if

$$a \geq \tilde{a}, \quad b \geq \tilde{b}, \quad b + d \geq \tilde{b} + \tilde{d}, \quad \text{and} \quad a + c > \tilde{a} + \tilde{c}. \tag{4.9}$$

Furthermore, we say that $\mathbb{X}$ *weakly majores* $\tilde{\mathbb{X}}$ (written as $\mathbb{X} \succsim \tilde{\mathbb{X}}$) if

$$a \geq \tilde{a}, \quad b \geq \tilde{b}, \quad a + c \geq \tilde{a} + \tilde{c}, \quad \text{and} \quad b + d > \tilde{b} + \tilde{d}. \tag{4.10}$$

Lemma 4.13. *Let* $\alpha = \left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right) \mathbb{A}\mathbb{X}\mathbb{B} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \tilde{\alpha} = \left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right) \mathbb{A}\tilde{\mathbb{X}}\mathbb{B} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, *where*

$$\mathbb{A} = \mathbb{I}_2 \quad \text{or} \quad \mathbb{A} = M_{x_{s-1}}(r_s) M_{x_{s-2}}(r_{s-1}) \cdots M_{x_1}(r_2),$$
$$\mathbb{B} = \mathbb{I}_2 \quad \text{or} \quad \mathbb{B} = M_{y_{t-1}}(p_t) M_{y_{t-2}}(p_{t-1}) \cdots M_{y_1}(p_2),$$

where $r_{i+1}, p_{j+1} \in \mathbb{N}_0$, $x_i, y_j \in \{1, \ldots, a\}$ *for all* $i = 1, \ldots, s-1$, $j = 1, \ldots, t-1$, *and* $\mathbb{X}$, $\tilde{\mathbb{X}}$ *are non-negative integer matrices. If* $\mathbb{X} \succ \tilde{\mathbb{X}}$, *then* $\alpha > \tilde{\alpha}$. *Furthermore, if* $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) = \mathbb{B} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, *where* $u \geq 0$ *and* $v \geq 1$, *then if* $\mathbb{X} \succsim \tilde{\mathbb{X}}$, *then* $\alpha > \tilde{\alpha}$.

Proof. Denote $\begin{pmatrix} f & g \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbb{A}$ and $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) = \mathbb{B} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. It is easy to see that $g \geq f \geq 1$, $u \geq 1$ and $v \geq 0$. Let $\mathbb{X} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tilde{\mathbb{X}} = \left(\begin{smallmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{smallmatrix}\right)$ satisfy (4.9). Then

$$\begin{aligned}
\alpha - \tilde{\alpha} &= \begin{pmatrix} f & g \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} f & g \end{pmatrix} \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\
&= \begin{pmatrix} (a - \tilde{a})f + (c - \tilde{c})g & (b - \tilde{b})f + (d - \tilde{d})g \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\
&\geq \begin{pmatrix} (a - \tilde{a} + c - \tilde{c})f & (b - \tilde{b} + d - \tilde{d})f \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \geq \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1.
\end{aligned}$$

Suppose now that $u \geq 0$, $v \geq 1$ and that $\mathbb{X}$, $\tilde{\mathbb{X}}$ satisfy (4.10). Then

$$\begin{aligned}
\alpha - \tilde{\alpha} &= \begin{pmatrix} (a - \tilde{a})f + (c - \tilde{c})g, & (b - \tilde{b})f + (d - \tilde{d})g \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\
&\geq \begin{pmatrix} (a - \tilde{a} + c - \tilde{c})f, & (b - \tilde{b} + d - \tilde{d})f \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \geq \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1.
\end{aligned}$$

□

With Lemma 4.13 in hand, it is now much easier to find representations w on which the maximal value of $R(w)$ is reached. We will aim to eliminate factors that are suboptimal for maximising $R(w)$, i.e. they can be replaced by factors of the same length that contribute more to $R(w)$. Again, adapting the terminology from [11, 12], we will call these the *factors forbidden for maximality:*

Definition 4.14. Suppose a confluent B-system with canonical alphabet $A = \{0, 1, \ldots, a\}$. We say that the string $x_s 0^{r_s} x_{s-1} 0^{r_{s-1}} \cdots x_1 0^{r_1} x_0$, where $x_i \in \{1, 2, \ldots, a\}$ for all $i \in \{1, \ldots, s\}$ and $x_0 \in A$ is *forbidden for maximality* if there exists a word $y_t 0^{p_t} y_{t-1} 0^{p_{t-1}} \cdots y_1 0^{p_1} y_0$, where $p_1, p_t \geq 0$, $y_j \in \{1, 2, \ldots, a\}$ for all $j \in \{1, \ldots, t\}$, and $y_0 \in A$ such that

$$r_s + r_{s-1} + \cdots + r_1 + s + 1 = p_t + p_{t-1} + \cdots + p_1 + t + 1,$$
$$M_{x_{s-1}}(r_s) M_{x_{s-2}}(r_{s-1}) \cdots M_{x_0}(r_1) \prec M_{y_{t-1}}(p_t) M_{y_{t-2}}(p_{t-1}) \cdots M_{y_0}(p_1).$$

We will sometimes say that the word $y_t 0^{p_t} \cdots y_1 0^{p_1} y_0$ *improves the factor* $x_s 0^{r_s} \cdots x_1 0^{r_1} x_0$.

We have now everything ready for determining the expressions for $\psi(l)$ in all three groups of confluent systems. We will end this section by stating an evident fact about the ordering of redundancy matrices:

Fact 4.15. *Suppose a confluent B-system with coefficients $a \geq b \geq 1$. Then for all $r \geq 1$ such that $r \equiv m - 1 \mod m$ the redundancy matrices satisfy the inequality*

$$M_c(r) \succ M_{a-b+1}(r) \succsim M_d(r),$$

for all digits $1 \leq c < a - b + 1$ and $a - b + 1 < d \leq a$. For other values of r they satisfy

$$M_c(r) = M_{a-b+1}(r) = M_d(r).$$

4.3.1 Confluent Systems with $a = b$ and order $m = 2$

As shown in Section 3.2, systems with $a = b$ and order $m = 2$ display analogous behaviour to the Fibonacci system. We will show why the values of $\psi(l)$ in such systems equal those in the Fibonacci system, which has been studied in [12]. We will follow their approach in this section.

In short, the reason why the maxima of $R(n)$ have the same value as in the Fibonacci system is due to Fact 4.15 and the fact that in confluent systems with $a = b$ only the redundancy matrices $M_1(r) = M_{a-b+1}(r)$ and $M_d(r)$ are defined (where $d \in \{2, \ldots, a\}$). $M_d(r)$ does not increase the value $R(n)$ since it is either weakly majored by or equal to $M_1(r)$ – if $r > 1$ and $r \equiv m - 1 \mod m$ then $M_1(r) \succsim M_f(r)$, otherwise $M_1(r) = M_f(r)$.

We shall first derive a lower bound on the value of $\psi(l)$ by evaluating $R(w)$ on some chosen representations w. The following lemma is taken from [12] (Lemma 3.1) and adapted to our notation.

Lemma 4.16. *Suppose a confluent B-system with coefficients $a = b$ and order $m = 2$. Let $x \in \{1, 2, \ldots, a\}$ and let either $y \in \{1, 2, \ldots, a\}$ or $y = \varepsilon$. Then*

$$R\left(x \left(0^3 1\right)^{k-1} 0^4 y\right) = R\left(x01 \left(0^3 1\right)^{k-1} 0^2 y\right) = F_{2k} \text{ for } k \geq 1. \tag{4.11}$$

$$R\left(x \left(0^3 1\right)^{k} 0^2 y\right) = R\left(x01 \left(0^3 1\right)^{k-1} 0^4 y\right) = F_{2k+1} \text{ for } k \geq 1. \tag{4.12}$$

Proof. We have chosen representations of the forms (4.11) and (4.12), because $a - b + 1 = 1$. Hence the expression for $R(n)$ from the matrix formula will include prevailingly the matrix $M_1(r)$. Recall the initial conditions for the Fibonacci sequence – we set $F_0 = 1$, $F_1 = 2$, plus we additionally define $F_{-1} = 1$. Then we can prove by induction the following

$$(M_1(3))^q = \begin{pmatrix} 1 & 1 \\ \lfloor \frac{3}{2} \rfloor & \lfloor \frac{3+1}{2} \rfloor \end{pmatrix}^q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^q = \begin{pmatrix} F_{2q-3} & F_{2q-2} \\ F_{2q-2} & F_{2q-1} \end{pmatrix} \text{ for all } q \in \mathbb{N}. \tag{4.13}$$

The case $q = 1$ is evident, hence suppose the equality holds for some $q > 1$. Then using the induction hypothesis and the fact that $2F_p + F_{p-1} = F_{p+2}$ for all $p \in \mathbb{N}_0$ we obtain

$$(M_1(3))^{q+1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} F_{2q-3} & F_{2q-2} \\ F_{2q-2} & F_{2q-1} \end{pmatrix} = \begin{pmatrix} F_{2q-1} & F_{2q} \\ F_{2q} & F_{2q+1} \end{pmatrix}.$$

Let us now consider the contribution of the suffixes 10^4y and 10^2y. Suppose first that $y \in \{1, 2, 3, \ldots, a\}$. Then evidently

$$\begin{pmatrix} \overline{R}(10^4y) \\ \underline{R}(10^4y) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \overline{R}(10^4) \\ \underline{R}(10^4) \end{pmatrix},$$

and

$$\begin{pmatrix} \overline{R}(10^2y) \\ \underline{R}(10^2y) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \overline{R}(10^2) \\ \underline{R}(10^2) \end{pmatrix},$$

thus both the cases $y \in \{1, 2, \ldots, a\}$ and $y = \varepsilon$ are equivalent for determining the values of $R(w)$ for words w from (4.11) and (4.12). Using (4.13), we determine

$$\begin{aligned} R\left(x\,(0^31)^{k-1}\,0^4y\right) &= (1 \quad 1) \begin{pmatrix} F_{2k-5} & F_{2k-4} \\ F_{2k-4} & F_{2k-3} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= (F_{2k-3} \quad F_{2k-2}) \begin{pmatrix} 1 \\ 2 \end{pmatrix} = F_{2k}, \end{aligned}$$

$$\begin{aligned} R\left(x01\,(0^31)^{k-1}\,0^2y\right) &= (1 \quad 1) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} F_{2k-5} & F_{2k-4} \\ F_{2k-4} & F_{2k-3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= (F_{2k-3} \quad 2F_{2k-2}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = F_{2k}, \end{aligned}$$

$$\begin{aligned} R\left(x\,(0^31)^{k}\,0^2y\right) &= (1 \quad 1) \begin{pmatrix} F_{2k-3} & F_{2k-2} \\ F_{2k-2} & F_{2k-1} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= (F_{2k-1} \quad F_{2k}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = F_{2k+1}, \end{aligned}$$

$$\begin{aligned} R\left(x01\,(0^31)^{k-1}\,0^4y\right) &= (1 \quad 1) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} F_{2k-5} & F_{2k-4} \\ F_{2k-4} & F_{2k-3} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= (F_{2k-2} \quad F_{2k-1}) \begin{pmatrix} 1 \\ 2 \end{pmatrix} = F_{2k+1}. \end{aligned}$$

□

Thus we can now derive lower bounds on the value of $\psi(l)$.

Corollary 4.17. *Suppose a confluent B-system with coefficients $a = b$ and order 2. Then for all $l \geq 1$*

$$\psi(2l+1) \geq F_l, \quad \textit{and} \quad \psi(2l+2) \geq 2F_{l-1}.$$

Proof. The bound on the maxima of $R(w)$ on representations of odd length $\psi(2l+1) \geq F_l$ is evident from Lemma 4.16 if we set $y = \varepsilon$ and relate the coefficent k to l. According to the factorisation of the representation in (4.11) we obtain

$$2l + 1 = 1 + 4(k-1) + 4 = 4k + 1,$$

hence $l = 2k$. From the factorisation of representation (4.12) we obtain

$$2l + 1 = 1 + 4k + 2 = 4k + 3$$

thus we derive $l = 2k + 1$. In both cases this implies $\psi(2l+1) \geq F_l$ from the proof of Lemma 4.16.

The bound for representations of even length is a consequence of the fact that for all $x \in \{1, 2, \ldots, a\}$, $y \in \{1, 2, \ldots, a\}$ and $u \in A^*$ where yu is a greedy representation, the following holds:

$$R(x0^2yu) \geq 2R(yu).$$

This is a consequence of the fact that

$$\begin{pmatrix} 1 & 1 \end{pmatrix} M_y(2) = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 2 \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

Likewise, for $r_1 > 0$ also

$$R(yu0^{r_1}x0^2) \geq 2R(yu0^{r_1}),$$

since for all integer matrices with $a, b \geq 1$ and $c, d \geq 0$ we have

$$\begin{aligned} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} M_x(r_1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} &\geq \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \lfloor \frac{r_1}{2} \rfloor & \lfloor \frac{r_1}{2} \rfloor \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} 2 \begin{pmatrix} 1 \\ \lfloor \frac{r_1}{2} \rfloor \end{pmatrix}. \end{aligned}$$

Thus we have derived the following lower bound on the value of $\psi(2l+2)$:

$$\psi(2l+2) \geq 2\psi(2l-1) \geq 2F_{l-1}.$$

□

With the lower bounds on the value of $\psi(l)$ established, we will now prove that $\psi(l)$ is in fact equal to these lower bounds. We will first establish some factors that are forbidden in maximal representations. For that purpose we have to include and slightly adapt results of Kocábová, Masáková, and Pelantová [12] about the factors present in representations on which the maxima of $R(n)$ are reached.

Proposition 4.18 (Kocábová, Masáková, Pelantová). *Suppose a confluent B-system with coefficients $a = b$ and order 2. Take the greedy representation of length l on which the value of $\psi(l)$ is reached, i.e.*

$$\psi(l) = R(x_s 0^{r_s} x_{s-1} 0^{r_{s-1}} \cdots x_1 0^{r_1}),$$

where $r_i \in \mathbb{N}_0$ and $x_i \in \{1, 2, \cdots, a\}$ for all $i \in \{1, 2, \ldots, s\}$. Then clearly

$$l = s + r_s + \cdots + r_2 + r_1$$

and all the following hold for the values of l, s, $x_s, x_{s-1}, \ldots, x_1$ and $r_s, r_{s-1}, \ldots, r_1$:

1. *If r_i is odd for some $i \in \{2, \ldots, s\}$, then $x_{i-1} = 1$.*
2. *$s \geq 2$ or $l \leq 5$.*
3. *If $l \geq 6$, then r_1 is even.*
4. *$r_i \leq 5$ for all $i \in \{1, 2, \ldots, s\}$.*
5. *Suppose that $l \geq 6$ and r_i are odd for all $i \in \{2, 3, \ldots, s\}$. Then $r_s \in \{1, 3\}$, $r_{s-1} = \cdots = r_2 = 3$, and $r_1 \in \{2, 4\}$.*

Proof. Statement 1 is the only new claim compared to those in the Fibonacci system. Parts 2, 3, 4, and 5 are originally proven in [12] (Propositions 4.1, 4.2, 4.5, and 4.6). The proofs of Statements 2 and 4 for the Fibonacci case can be applied without modification to all confluent systems with $a = b$ and order $m = 2$. Furthermore, we do not include the proof of Statement 5, because thanks to Statement 1 it would be identical to the proof in the Fibonacci system. We will include the proofs of Statement 1 as well as an adaptation of the proof of Statement 3, because that requires slightly different treatment to that in the Fibonacci system.
Statement 1.
Suppose that an $i \in \{2, \ldots, s\}$ exists such that r_i is odd and $x_{i-1} > 1$. Then $\lfloor \frac{r_i+1}{2} \rfloor$ is strictly greater than $\lfloor \frac{r_i}{2} \rfloor$, hence the matrix $M_{x_{i-1}}(r_i)$ is weakly majored by $M_1(r_i)$:

$$M_{x_{i-1}}(r_i) = \begin{pmatrix} 1 & 1 \\ \lfloor \frac{r_i}{2} \rfloor & \lfloor \frac{r_i}{2} \rfloor \end{pmatrix} \precsim \begin{pmatrix} 1 & 1 \\ \lfloor \frac{r_i}{2} \rfloor & \lfloor \frac{r_i+1}{2} \rfloor \end{pmatrix} = M_1(r_i).$$

Thus, if some r_i is odd, the digit x_{i-1} must be equal to 1.
Statement 3.
Since $l > 5$, Part 2 implies $s \geq 2$, thus it is sufficient to prove that for odd r_1 we have

$$R(x_s 0^{r_s} x_{s-1} 0^{r_{s-1}} \cdots x_1 0^{r_1}) < R(x_s 0^{r_s+1} x_{s-1} 0^{r_{s-1}} \cdots x_1 0^{r_1-1}). \tag{4.14}$$

Suppose first that r_s is even. Then $\lfloor \frac{r_s}{2} \rfloor = \lfloor \frac{r_s+1}{2} \rfloor$, thus $M_{x_{s-1}}(r_s) = M_1(r_s)$ for all $x_{s-1} \in \{1, 2, \ldots, a\}$, and we have

$$\begin{pmatrix} 1 & 1 \end{pmatrix} M_{x_{s-1}}(r_s) = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \lfloor \frac{r_s}{2} \rfloor & \lfloor \frac{r_s}{2} \rfloor \end{pmatrix} = \left(\frac{r_s}{2} + 1 \right) \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

Hence we can write

$$R(x_s 0^{r_s} x_{s-1} 0^{r_{s-1}} \cdots x_1 0^{r_1}) = \left(\frac{r_s}{2} + 1 \right) R(x_{s-1} 0^{r_{s-1}} \cdots x_1 0^{r_1}).$$

However, because r_s+1 is odd, then from Statement 1 the digit x_{s-1} must be equal to 1 in order for the value $R(x_s 0^{r_s+1} x_{s-1} 0^{r_{s-1}} \cdots x_1 0^{r_1-1})$ to be maximal. From the matrix formula we obtain

$$R(x_s 0^{r_s+1} 1 0^{r_{s-1}} \cdots x_1 0^{r_1-1}) = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \lfloor \frac{r_s+1}{2} \rfloor & \lfloor \frac{r_s+2}{2} \rfloor \end{pmatrix} \begin{pmatrix} \overline{R}(10^{r_{s-1}} \cdots x_1 0^{r_1-1}) \\ \underline{R}(10^{r_{s-1}} \cdots x_1 0^{r_1-1}) \end{pmatrix}$$
$$= \left(\frac{r_s}{2}+1\right) \overline{R}(10^{r_{s-1}} \cdots x_1 0^{r_1-1}) + \left(\frac{r_s}{2}+2\right) \underline{R}(10^{r_{s-1}} \cdots x_1 0^{r_1-1}).$$

To obtain (4.14), we have to show that $\underline{R}(10^{r_{s-1}} \cdots x_1 0^{r_1-1}) > 0$. Clearly, $\underline{R}(10^{r_{s-1}} \cdots x_1 0^{r_1-1}) = 0$ together with r_1 odd implies either

a) that there exists an index $s-1 \geq q \geq 2$ such that $x_{q-1} > 1$ and $x_{s-2} = \cdots = x_q = 1$ and $r_{s-1} = r_{s-2} = \cdots = r_q = 1$, or

b) that $r_{s-1} = r_{s-2} = \cdots = r_1 = 1$.

In both cases there is no way to perform a sequence of rewritings by which we would create a short representation of $x_s 0^{r_s+1} 1 0^{r_{s-1}} \cdots x_1 0^{r_1-1}$. Let us treat case a) first. First note that

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \lfloor \frac{r_s+1}{2} \rfloor & \lfloor \frac{r_s+2}{2} \rfloor \end{pmatrix} = \left(\tfrac{r_s}{2}+1, \quad \tfrac{r_s}{2}+2\right),$$

and that $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)^p = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ for all $p \in \mathbb{N}$. Suppose first that $r_{q-1} \geq 2$. Then clearly we have $\underline{R}(x_{q-1} 0^{r_{q-1}} \cdots x_1 0^{r_1-1}) > 0$ and from the fact that

$$M_{x_{q-1}}(1) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \precsim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = M_1(1)$$

we derive

$$R(x_s 0^{r_s+1} 10 \cdots 10 x_{q-1} 0^{r_{q-1}} \cdots x_1 0^{r_1-1}) =$$
$$= \left(\tfrac{r_s}{2}+1, \quad \tfrac{r_s}{2}+2\right) \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}^{s-q} \begin{pmatrix} \overline{R}(x_{q-1} 0^{r_{q-1}} \cdots x_1 0^{r_1-1}) \\ \underline{R}(x_{q-1} 0^{r_{q-1}} \cdots x_1 0^{r_1-1}) \end{pmatrix}$$
$$= \left(\tfrac{r_s}{2}+1, \quad \tfrac{r_s}{2}+2\right) \begin{pmatrix} \overline{R}(x_{q-1} 0^{r_{q-1}} \cdots x_1 0^{r_1-1}) \\ \underline{R}(x_{q-1} 0^{r_{q-1}} \cdots x_1 0^{r_1-1}) \end{pmatrix},$$

$$R(x_s 0^{r_s+1} 10 \cdots 1010^{r_{q-1}} \cdots x_1 0^{r_1-1}) =$$
$$= \left(\tfrac{r_s}{2}+1, \quad \tfrac{r_s}{2}+2\right) \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}^{s-q-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \overline{R}(10^{r_{q-1}} \cdots x_1 0^{r_1-1}) \\ \underline{R}(10^{r_{q-1}} \cdots x_1 0^{r_1-1}) \end{pmatrix}$$
$$= \left(\tfrac{r_s}{2}+1, \quad r_s+3\right) \begin{pmatrix} \overline{R}(10^{r_{q-1}} \cdots x_1 0^{r_1-1}) \\ \underline{R}(10^{r_{q-1}} \cdots x_1 0^{r_1-1}) \end{pmatrix},$$

hence

$$R(x_s 0^{r_s+1} 10 \cdots 10 x_{q-1} 0^{r_{q-1}} \cdots x_1 0^{r_1-1}) < R(x_s 0^{r_s+1} 10 \cdots 1010^{r_{q-1}} \cdots x_1 0^{r_1-1}),$$

which is a contradiction with the maximality of $x_s0^{r_s+1}10\cdots 10x_{q-1}0^{r_{q-1}}\cdots x_10^{r_1-1}$. Suppose now that $r_{q-1}=1$. Then the prefix $x_s0^{r_s+1}(10)^{s-q}x_{q-1}$ is forbidden for maximality. Consider the prefix $x_s0^{r_s+1+2(s-q)}x_{q-1}$ with the same length. Then

$$M_1(r_s+1)\begin{pmatrix}1&1\\0&0\end{pmatrix}^{s-q}=\begin{pmatrix}1&1\\ \lfloor\frac{r_s+1}{2}\rfloor&\lfloor\frac{r_s+2}{2}\rfloor\end{pmatrix}\begin{pmatrix}1&1\\0&0\end{pmatrix}$$
$$=\begin{pmatrix}1&1\\ \lfloor\frac{r_s+1}{2}\rfloor&\lfloor\frac{r_s+1}{2}\rfloor\end{pmatrix}\prec\begin{pmatrix}1&1\\ \left\lfloor\frac{r_s+1+2(s-q)}{2}\right\rfloor&\left\lfloor\frac{r_s+1+2(s-q)}{2}\right\rfloor\end{pmatrix}=M_{x_{q-1}}(r_s+1+2(s-q)).$$

The inequality $\left\lfloor\frac{r_s+1+2(s-q)}{2}\right\rfloor>\lfloor\frac{r_s+1}{2}\rfloor$ holds because $s-q\geq 1$. We have thus derived a contradiction.

Let us now treat case b). Then, from Statement 2 it follows that $s\geq 2$, because by assumption $l\geq 6$. Similarly to the end of case a), consider the string $x_s0^{r_s+1+2(s-1)}$, which has the same length as $x_s0^{r_s+1}10x_{s-2}0\cdots x_10$. Again, we obtain the inequality

$$M_{x_{q-1}}(r_s+1)\begin{pmatrix}1&1\\0&0\end{pmatrix}^{s-1}=\begin{pmatrix}1&1\\ \lfloor\frac{r_s+1}{2}\rfloor&\lfloor\frac{r_s+1}{2}\rfloor\end{pmatrix}$$
$$\begin{pmatrix}1&1\\ \lfloor\frac{r_s+1}{2}\rfloor&\lfloor\frac{r_s+1}{2}\rfloor\end{pmatrix}\prec\begin{pmatrix}1&1\\ \left\lfloor\frac{r_s+1+2(s-1)}{2}\right\rfloor&\left\lfloor\frac{r_s+1+2(s-1)}{2}\right\rfloor\end{pmatrix}=M_{x_{q-1}}(r_s+1+2(s-1)),$$

and thus a contradiction with the maximality of $x_s0^{r_s+1}10x_{s-2}0\cdots x_10$. In both cases a) and b) we have shown that $\underline{R}(10^{r_{s-1}}\cdots x_10^{r_1-1})$ cannot be equal to zero, proving the inequality (4.14) when r_s is even.

Suppose now that r_s is odd. Since $M_1(r_s)\succsim M_d(r_s)$ for all $d>1$, the digit x_{s-1} must equal 1. Then from the matrix formula we obtain

$$R(x_s0^{r_s}10^{r_{s-1}}\cdots x_10^{r_1})=\left(1+\left\lfloor\frac{r_s}{2}\right\rfloor\right)\overline{R}(10^{r_{s-1}}\cdots x_10^{r_1})+\left(1+\left\lfloor\frac{r_s+1}{2}\right\rfloor\right)\underline{R}(10^{r_{s-1}}\cdots x_10^{r_1}),$$

$$R(x_s0^{r_s+1}10^{r_{s-1}}\cdots x_10^{r_1-1})=R(10^{r_s+1}10^{r_{s-1}}\cdots x_10^{r_1})$$
$$=\left(1+\left\lfloor\frac{r_s+1}{2}\right\rfloor\right)\left(\overline{R}(10^{r_{s-1}}\cdots x_10^{r_1})+\underline{R}(10^{r_{s-1}}\cdots x_10^{r_1})\right).$$

thus proving (4.14), because $\lfloor\frac{r_s}{2}\rfloor<\lfloor\frac{r_s+1}{2}\rfloor$. □

Using Proposition 4.18 we can now prove the formula for $\psi(l)$. The proof is almost identical to that for the Fibonacci system ([12], Theorem 4.7). We include the proof because we will utilise it for determining the arguments of the maxima of $R(n)$, i.e. in Section 4.4.1, where we will determine the greedy representations that form the set $\Psi(l)$ in all confluent systems with $a=b$ and order $m=2$.

Theorem 4.19 (Kocábová, Masáková, Pelantová). *Suppose a confluent B-system with coefficients $a=b$ and order $m=2$. Then*

$$\psi(2k+1)=F_k \qquad \text{for } k\geq 0,$$
$$\psi(2k+2)=2F_{k-1} \qquad \text{for } k\geq 1.$$

Proof. In the proof we shall use the following inequalities for the Fibonacci numbers, which are simple to demonstrate. Recall that $F_0 = 1$, $F_1 = 2$, and $F_{-1} = 1$.Then

$$F_p F_q \leq 2F_{p+q-1} \quad \text{for } p, q \geq 0, \tag{4.15}$$

where the equality holds only if $p = 1$ or $q = 1$.

$$2F_p F_q \leq F_{p+q+2} \quad \text{for } p, q \geq 0, \tag{4.16}$$

where the equality holds only if $p = q = 1$.

We shall prove the statement by induction on k, the length of representation. From Corollary 4.17 we already know that $\psi(2k+1) \geq F_k$ and $\psi(2k+2) \geq 2F_{k-1}$, so it suffices to show that these lower bounds are also upper bounds, i.e. we shall prove that

$$\psi(2k+1) \leq F_k \quad \text{and} \quad \psi(2k+2) \leq 2F_{k-1}. \tag{4.17}$$

The initial values of $\psi(k)$ are clearly $\psi(1) = 1$ and $\psi(2) = 1$, since no interchange $x00 \leftrightarrow (x-1)aa$, where $x \in \{1, 2, \ldots, a\}$ and a is the greatest digit of the canonical alphabet, is possible on these lengths. Continuing further, $\psi(3) = 2$, since into a greedy representation of length 3 we can fit precisely one rewriteable factor $x00$. We conclude $\psi(4) = 2$ by the same argument.

Furthermore, notice that for r_i even we have $M_x(r_i) = M_x(r_i)\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix}$ and $M_x(r_i)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\ \lfloor \frac{r_i}{2} \rfloor\end{pmatrix}$ for all $x \in \{1, 2, \ldots, a\}$. This means that for r_i even we can say

$$R(x_s 0^{r_s} \cdots x_i 0^{r_i} x_{i-1} 0^{r_i-1} \cdots x_1 0^{r_1}) = R(x_s 0^{r_s} \cdots x_i 0^{r_i}) R(x_{i-1} 0^{r_{i-1}} \cdots x_1 0^{r_1}).$$

We are now ready to prove (4.17).

1. Let us first show that $\psi(2k+2) \leq 2F_{k-1}$.
   Let $w = x_s 0^{r_s} x_{s-1} 0^{r_{s-1}} \cdots x_1 0^{r_1}$, where $r_i \in \mathbb{N}_0$, $x_i \in \{1, 2, \ldots, a\}$, be a greedy representation such that $R(w) = \psi(2k+2)$, where $k \geq 2$. Statement 3 of Proposition 4.18 implies that r_1 is even. Because $r_s + r_{s-1} + \cdots + r_1 + s = 2k+2$, there must exist an $s \geq i > 1$ such that r_i is even. Let i be the minimal index with this property. The number $r_{i-1} + \cdots + r_1 + (s-i)$ is odd, e.g. $2p+1$. Then $r_s + \cdots + r_i + i = 2k+2-(2p+1)$. Using the inequality (4.15) and the induction hypothesis we then obtain

$$\begin{aligned}\psi(2k+2) &= R(x_s 0^{r_s} x_{s-1} 0^{r_{s-1}} \cdots x_1 0^{r_1}) = R(x_s 0^{r_s} \cdots x_i 0^{r_i}) R(x_{i-1} 0^{r_{i-1}} \cdots x_1 0^{r_1}) \\ &\leq \psi\,(2k-2p+1)\,\psi\,(2p+1) = F_{k-p} F_p \leq 2F_{k-1}.\end{aligned} \tag{4.18}$$

2. Let us now prove that $\psi(2k+1) \leq F_k$.
   Let $w = x_s 0^{r_s} x_{s-1} 0^{r_{s-1}} \cdots x_1 0^{r_1}$, where $r_i \in \mathbb{N}_0$, $x_i \in \{1, 2, \ldots, a\}$, be a greedy representation such that $R(w) = \psi(2k+1)$, where $k \geq 2$. Statement 3 of Proposition 4.18 implies that r_1 is even. Suppose that there exists an index $s \geq i > 1$ such that r_i is even. Let i be the minimal index with this property. Again, denote $r_{i-1} + \cdots + r_1 + (s-i) = 2p+1$. Then $r_s + \cdots + r_i + i = 2k+1-(2p+1) = 2k-2p$. Using the inequality (4.15) and the induction hypothesis we then obtain

$$\begin{aligned}\psi(2k+2) &= R(x_s 0^{r_s} x_{s-1} 0^{r_{s-1}} \cdots x_1 0^{r_1}) = R(x_s 0^{r_s} \cdots x_i 0^{r_i}) R(x_{i-1} 0^{r_{i-1}} \cdots x_1 0^{r_1}) \\ &\leq \psi\,(2k-2p)\,\psi\,(2p+1) = \psi\,(2(k-p-1)+2)\,\psi\,(2p+1) \\ &= 2F_{k-p-2} F_p \leq 2F_k.\end{aligned} \tag{4.19}$$

It remains to consider the case when r_i is odd for all $i \in \{2, \ldots, s\}$. Then from Statement 1 of Proposition 4.18 clearly $x_{i-1} = 1$ for all i and according to Statement 5 of Proposition 4.18, the only allowed s-tuples $(r_s, r_{s-1}, \ldots, r_1)$ are of the form $(1, 3, \ldots, 3, 4)$, $(3, \ldots, 3, 4)$, $(1, 3, \ldots, 3, 2)$, or $(3, \ldots, 3, 2)$. For a given length l only two of these are possible. Namely, for $l \equiv 1 \mod 4$ we can have either $(1, 3, \ldots, 3, 2)$ or $(3, \ldots, 3, 4)$, because $s + 3(s-2) + 1 + 2 = 4s - 3$ and $s + 3(s-1) + 4 = 4s + 1$, whereas for $l \equiv 3 \mod 4$ we can have either $(1, 3, \ldots, 3, 4)$ or $(3, \ldots, 3, 2)$ because $s + 3(s-2) + 1 + 4 = 4s + 3$ and $s + 3(s-1) + 2 = 4s - 1$. The values of R on representations constructed from such s-tuples were determined in Lemma 4.16. Thus the theorem is proved.

□

4.3.2 Confluent Systems with $a = b$ and order $m > 2$

For confluent systems whose coefficients satisfy $a = b$ and whose order is greater than 2, findings about the maximal values of $R(n)$ for m-bonacci systems [11] largely carry over. As in the previous group $a = b$ and $m = 2$, we will first determine the value of $R(w)$ on chosen greedy representations w, thus deriving a lower bound for $\psi(l)$ in such systems. We will then establish some strings that are forbidden for maximality. Finally, we will use these forbidden strings to find an upper bound on $\psi(l)$ and thus prove that it is equal to the values that we observed in Chapter 3.

We will first estabilish some notation which we will also utilise in Sections 4.3.3 and 4.4.

Definition 4.20. Let A be some finite alphabet and let $\alpha, \beta \in A^*$. Then for every finite alphabet X and every $p \in \mathbb{N}$ we define the *wildcard concatenation symbol* $[\alpha x_* \beta]_X^p$ which we set equal to

$$[\alpha x_* \beta]_X^p := (\alpha x_p \beta)(\alpha x_{p-1} \beta) \cdots (\alpha x_1 \beta),$$

where $x_p, x_{p-1}, \ldots, x_1 \in X$. For completeness and consistency, we set $[\alpha x_* \beta]_X^0 := \varepsilon$.

In essence, the notation $[\alpha x_* \beta]_X^p$ could be read as *"repeat the word $\alpha\beta$ precisely p times and insert between every α and β a digit from X".* The wildcard concatenation symbol will allow us to be efficient when talking about repeating a given factor and inserting a different digit into each repetition. We will utilise this most when analysing the maxima of $R(w)$ in confluent systems, since for many different representations w the value $R(w)$ is identical. Similarly to what we saw in the case $a = b$, $m = 2$, in all confluent systems the value $R(w)$ depends largely on the lengths of factors of consecutive zeros and not so much on the values of non-zero digits. The wildcard concatenation symbol will allow us to talk more efficiently in general about a set of factors that include the same number of consecutive zeros but differ in the values of the nonzero digits.

With this notation, we can now determine lower bounds on $\psi(l)$. However, first, let us determine the value of $\psi(l)$ for initial values of l. Clearly $\psi(1) = \psi(2) = \cdots = \psi(m) = 1$, since no rewritable factor $x0^m$, where $x \in \{1, 2, \ldots, a-1\}$, fits into a word of this length. All numbers n smaller than B_m thus have a unique representation.

The next case to consider is $\psi(m+1) = \psi(m+2) = \cdots = \psi(2m) = 2$, which holds because only one rewritable factor $x0^m$ fits into a representation of length $m + 1 \leq l \leq 2m$.

The next case is $l = 2m + 1$. We have $\psi(2m + 1) = 3$ because the maximal representations w will be of the form $x0^{m-1}10^m$. After rewriting the suffix 10^m to $0a^m$ we gain one more zero for the rewriting $x0^m \to (x-1)a^m$. Together this can be written as

$$x0^{m-1}10^m \to x0^{m-1}0a^m \to (x-1)a^{m-1}aa^m.$$

Lemma 4.21. *Suppose a confluent B-system with coefficients $a = b$ and order $m > 2$. Then the maxima of the function R defined for this system satisfy:*

$$\begin{aligned}\psi\left(p\left(m+1\right)+q\right) &\geq 2^p && \text{for } q \in \{0,1,\ldots,m-2\},\\ \psi\left(p\left(m+1\right)+m-1\right) &\geq 2^p + 2^{p-2} && \text{if } p \geq 2,\\ \psi\left(p\left(m+1\right)+m\right) &\geq 2^p + 2^{p-1}.\end{aligned}$$

Proof. Denote by A the canonical alphabet of the B-system. Then denote $C = \{1,2,\ldots,a\}$. For the first case we determine the value $R(w)$ on greedy representations of the form $w \in \left\{[x_*0^m]_C^p, y\,[x_*0^m]_C^p, y0\,[x_*0^m]_C^p, \ldots, y0^{m-2}\,[x_*0^m]_C^p\right\}$, where $y \in C$. Clearly, such representations have lengths $l = p(m+1), p(m+1)+1, \ldots, p(m+1)+q$. Then, because $M_x(m) = \left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$ for all $x \in C$ and $\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$, we obtain from the matrix formula

$$\psi\left(p(m+1)+q\right) \geq R(w) = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}^p \begin{pmatrix}1\\0\end{pmatrix} = 2^p \quad \text{for all } q \in \{0,1,\ldots,m-2\}.$$

For $l = p(m+1)+m-1$ we evaluate $R(w)$ on the greedy representation $w = y0^{2m-1}10^m\,[x_*0^m]_C^{p-2}$, where again $y \in C$:

$$\psi\left(p(m+1)+m-1\right) \geq R(w) = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}^{p-1}\begin{pmatrix}1\\0\end{pmatrix} = 2^p + 2^{p-2}.$$

Lastly, for $l = p(m+1)+m$ consider the value of $R(w)$ on the greedy representation $w = y0^{m-1}10^m\,[x_*0^m]_C^{p-1}$, where again $y \in C$:

$$\psi\left(p(m+1)+m\right) \geq R(w) = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}^{p}\begin{pmatrix}1\\0\end{pmatrix} = 2^p + 2^{p-1}.$$

□

We will now show some factors forbidden for maximality and restrict the set of possible representations w on which $R(w) = \psi(l)$ is reached. Again, most results carry over (with a slight modification) from the m-bonacci case [11], so we will prove the following claims only if there is a substantial difference.

Proposition 4.22. *Suppose a B-system with coefficients $a = b$ and order $m > 2$. Suppose a greedy representation $w = x_s0^{r_s}\cdots x_10^{r_1}$ of length l such that it is maximal, i.e. $R(w) = \psi(l)$. Then for every $i = 1,2,\ldots,s$ it holds that $r_i \leq 2m$ or that $r_i = 3m-1$ and $x_{i-1} = 1$.*

Proof. Analogous to the m-bonacci case ([11], Claim 5.4). Suppose for some i that $r_i > 2m$ and $r_i \neq 3m-1$. Then the string $x_i0^{r_i}x_{i-1}$ is forbidden for maximality for all $x_{i-1} \in \{1,2,\ldots,a\}$. First, note that if $x_{i-1} > 1$, then either $M_{x_{i-1}}(r_i) \precsim M_1(r_i)$ or $M_{x_{i-1}}(r_i) = M_1(r_i)$, so it suffices to treat the case $x_{i-1} = 1$ only. Consider now the string $x_i0^{r_i-m-1}10^mx_{i-1}$ that has the same length as $x_i0^{r_i}x_{i-1}$. In order to verify

$$\begin{aligned}M_1(r_i) = \begin{pmatrix}1 & 1\\ \lfloor\frac{r_i}{m}\rfloor & \lfloor\frac{r_i+1}{m}\rfloor\end{pmatrix} &\prec \begin{pmatrix}2 & 2\\ \lfloor\frac{r_i}{m}\rfloor + \lfloor\frac{r_i-1}{m}\rfloor - 2 & \lfloor\frac{r_i}{m}\rfloor + \lfloor\frac{r_i-1}{m}\rfloor - 2\end{pmatrix}\\ &= \begin{pmatrix}1 & 1\\ \lfloor\frac{r_i-m-1}{m}\rfloor & \lfloor\frac{r_i-m}{m}\rfloor\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = M_1(r_i-m-1)M_{x_{i-1}}(m).\end{aligned}$$

we use the fact that

$$\left\lfloor \frac{r_i+1}{m} \right\rfloor \le \left\lfloor \frac{r_i}{m} \right\rfloor + \left\lfloor \frac{r_i-1}{m} \right\rfloor - 2$$

holds for all $r_i > 2m$ and $r \neq 3m-1$. □

Proposition 4.23 (Kocábová, Masáková, Pelantová). *Suppose a B-system with coefficients $a = b$ and order $m > 2$. Then the following factors are forbidden for maximality for all $x, y, z, v \in \{1, 2, \dots, a\}$:*

1. $x0^{m-1}y0^{3m-1}z$,
2. $x0^{m-1}y0^{m-1}z$,
3. $x0^{m-1}y0^{2m-1}z0^{m-1}v$,
4. $x0^{m-1}y0^{2m-1}z0^{2m-1}v$, *whenever* $m \geq 4$.
5. $x0^{m-1}y0^{2m-1}z0^{3m-1}v$.

Proof. Since $M_1(r) \succsim M_x(r)$ for all $x > 1$ whenever $r \equiv m-1 \mod m$, it suffices to consider the above factors with $y = z = v = 1$. Furthermore, since the contributions of any of the factors (i.e. the matrices $M_y(m-1)$, $M_z(3m-1)$, etc.) do not depend on the initial digit x, we can consider only factors with $x = 1$, fully reducing this statement to the m-bonacci case. Refer thus to Claims 5.5–5.9 in [11] for the full proofs that these factors are forbidden. For completeness, we will include the factors which improve upon the factors 1.–5.

1. $x0^m y0^m z0^{2m-3}v$ improves $x0^{m-1}y0^{3m-1}z$ because
$$M_y(m)M_z(m)M_v(2m-3) = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \succ \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} = M_1(m-1)M_z(3m-1).$$
2. $x0^{2m-1}1$ improves $x0^{m-1}y0^{m-1}z$ because
$$M_1(2m-1) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \succ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = M_1(m-1)M_1(m-1).$$
3. $x0^m y0^m z0^{2m-3}v$ improves $x0^{m-1}y0^{2m-1}z0^{m-1}v$ because
$$M_y(m)M_z(m)M_v(2m-3) = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \succ \begin{pmatrix} 1 & 3 \\ 5 & 2 \end{pmatrix} = M_1(m-1)M_1(2m-1)M_1(m-1).$$
4. Let $m \geq 4$. Then $[x_*0^m]^3_C\, y0^{2m-4}z$, where $C = \{1, 2, \dots, a\}$, improves $x0^{m-1}y0^{2m-1}z0^{2m-1}v$, because $M_x(m) = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ for all $x \in C$ and thus
$$\left(M_y(m)\right)^3 M_v(2m-3) = \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix} \succ \begin{pmatrix} 5 & 3 \\ 8 & 5 \end{pmatrix} = M_1(m-1)M_1(2m-1)M_1(2m-1).$$
for all $x \in C$.

5. $x0^{2m}y0^{2m}z0^{2m-3}v$ improves $x0^{m-1}y0^{2m-1}z0^{3m-1}v$ because

$$M_y(2m)M_z(2m)M_v(2m-3) = \begin{pmatrix} 6 & 6 \\ 12 & 12 \end{pmatrix} \succ \begin{pmatrix} 8 & 11 \\ 5 & 7 \end{pmatrix} = M_1(m-1)M_1(2m-1)M_1(3m-1).$$

□

With these forbidden factors ready, we can state the central theorem for the value of $\psi(l)$. The proof is identical to that in the m-bonacci system (see [11], Theorem 5.11), thus we will not include it in this work. The only difference is that the greedy representations w that form the set $\Psi(l)$ may have the most significant digit larger than 1. Also, because $M_1(r) \succsim M_x(r)$ for all $x > 1$ only if $r \equiv m-1 \mod m$, any factors of even length may end with a digit larger than 1. Otherwise every step of the proof of Theorem 4.24 is identical to the m-bonacci case.

Theorem 4.24 (Kocábová, Masáková, Pelantová)**.** *Consider a confluent B-system with coefficients $a = b$ and order $m > 2$. Then for every $p \geq 1$ the maxima of the function R in this system satisfy*

$$\begin{aligned} \psi\left(p\left(m+1\right)+q\right) &= 2^p && \text{for } q \in \{0,1,\dots,m-2\}, \\ \psi\left(p\left(m+1\right)+m-1\right) &= 2^p + 2^{p-2} && \text{if } p \geq 2, \\ \psi\left(p\left(m+1\right)+m\right) &= 2^p + 2^{p-1}. \end{aligned}$$

4.3.3 Confluent Systems with $a > b$

To explain the behaviour of confluent systems with coefficients $a > b$, we can use what we found for the $(2,1)$-system as a model. Let us start by noting that in representations that have length l smaller than or equal to m, no rewriting rule from the associated rewriting system ρ_A can be applied, therefore $\psi(l) = 1$ for all $l = 1,2,\dots,m$. For representations of length $l > m$ we will follow the approach used for the other two groups of numeration systems. Firstly, we will determine the value of $R(w)$ on some chosen B-representations w and use these for deriving a lower bound for $\psi(l)$. Secondly, we will show that the value $\psi(l)$ is indeed equal to $R(w)$.

Lemma 4.25. *Suppose a confluent B-system with coefficients $a > b$ and order m. Denote its canonical alphabet A. Then*

$$\begin{aligned} R\left(z\left[0^{m-1}c_*\right]_C^p 0^{m-1}x_1\right) &= 2^{p+1}, \\ R\left(z\left[0^{m-1}c_*\right]_C^p 0^{m-1}x_1x_2\right) &= 2^{p+1}, \\ &\vdots \\ R\left(z\left[0^{m-1}c_*\right]_C^p 0^{m-1}x_1x_2\cdots x_m\right) &= 2^{p+1}, \end{aligned}$$

where $p \in \mathbb{N}_0$ and $z \in \{1,2,\dots a\}$, $C = \{1,2,\dots,a-b\}$, $x_1 \in \{0,1,\dots,a-b\}$ and $x_j \in A$ for all $j = 2,3,\dots,m-1$.

Proof. Let us first realise that for all $q \in \{1,2,\dots,m-1\}$ the suffix $x_1\cdots x_q$ contributes the same value to $R(w)$. Since no rewriting rule from ρ_A can be used in $x_1\cdots x_q$, we obtain

$$\begin{pmatrix} \overline{R}(c0^{m-1}x_1\cdots x_q) \\ \underline{R}(c0^{m-1}x_1\cdots x_q) \end{pmatrix} = M_{x_1}(m-1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

for every $c \in \{1,2,\dots,a\}$. If $q \geq 1$ and $x_1\cdots x_q$ has a proper prefix consisting of zeroes, i.e. if there exists an $r < q$ such that $x_1 = x_2 = \cdots = x_r = 0$ then the equality holds as well because

$M_{x_{1+r}}(m-1+r) = M_c(m-1)$ for all $x_r \in A$ and $c \leq a-b$. Lastly, in the case when $x_1 \cdots x_q = 0^q$ we obtain the equivalent result

$$\begin{pmatrix} \overline{R}(c0^{m-1+q}) \\ \underline{R}(c0^{m-1+q}) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We can now evaluate $R(w)$ on the whole representation:

$$R\left(w_{N-1}\left[0^{m-1}c_*\right]_C^p 0^{m-1}x_1\cdots x_q\right) = \begin{pmatrix} 1 & 1 \end{pmatrix} (M_c(m-1))^p \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Finally, because

$$(M_c(m-1))^p = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^p = \begin{pmatrix} 2^{p-1} & 2^{p-1} \\ 2^{p-1} & 2^{p-1} \end{pmatrix} \quad \text{for all } p \in \mathbb{N},$$

this results in

$$R\left(w_{N-1}\left[0^{m-1}c_*\right]_C^p 0^{m-1}x_1\cdots x_q\right) = 2^{p+1}.$$

□

Now let us show that the values from Lemma 4.25 are in fact a lower bound on $\psi(l)$.

Corollary 4.26. *Suppose a confluent B-system with coefficients $a > b$ and order m. Then for all $l \geq 1$*

$$\psi(l) \geq 2^{\lceil \frac{l}{m} \rceil - 1}.$$

Proof. Clearly for every $l \geq 1$ there exist $p \in \mathbb{N}_0$ and $q \in \{1, 2, \ldots, m-1, m\}$ such that $l = pm+q$. Denote $C = \{1, 2, \ldots, a-b\}$. Then clearly

$$\psi(l) = \psi(pm+q) \geq R(w_{N-1}\left[0^{m-1}c_*\right]_C^{p-1} 0^{m-1}x_1\cdots x_q) = 2^p = 2^{\lceil \frac{l}{m} \rceil - 1},$$

because $\lceil \frac{l}{m} \rceil - 1 = \lceil \frac{pm+q}{m} \rceil - 1 = p$. □

We will now establish some strings forbidden for maximality before proving that $\psi(l) = 2^{\lceil \frac{l}{m} \rceil - 1}$.

Proposition 4.27. *Let $r \geq 2m-1$. Then the string $y0^r x$ is forbidden for maximality for all $x, y \in \{1, 2, \ldots, a\}$.*

Proof. Take any digit $c \in \{1, 2, \ldots, a-b\}$. Then matrix $M_x(r_i)$ is majored by or equal to $M_c(r_i)$, which is majored by $M_c(m-1)M_c(r_i-m)$:

$$M_c(m-1)M_c(r_i-m) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \lfloor \frac{r_i-m+1}{m} \rfloor & \lfloor \frac{r_i-m+1}{m} \rfloor \end{pmatrix}$$
$$= \begin{pmatrix} \lfloor \frac{r_i+1}{m} \rfloor & \lfloor \frac{r_i+1}{m} \rfloor \\ \lfloor \frac{r_i+1}{m} \rfloor & \lfloor \frac{r_i+1}{m} \rfloor \end{pmatrix} \succ \begin{pmatrix} 1 & 1 \\ \lfloor \frac{r_i+1}{m} \rfloor & \lfloor \frac{r_i+1}{m} \rfloor \end{pmatrix} = M_c(r_i).$$

□

Proposition 4.28. *Take the greedy representation of length l on which the value of $\psi(l)$ is reached, i.e. the word $w = x_s 0^{r_s} x_{s-1} 0^{r_{s-1}} \cdots x_1 0^{r_1}$ such that $x_i \in \{1, 2, \ldots, a\}$, $r_i \in \mathbb{N}_0$ for all $i = 1, 2, \ldots, s$, and*

$$\psi(l) = R(x_s 0^{r_s} x_{s-1} 0^{r_{s-1}} \cdots x_1 0^{r_1}).$$

Then $r_1 < 2m$.

Proof. Suppose that w is maximal (i.e. that $\psi(l) = R(w)$) and that $r_1 \geq 2m$. Let c be a digit $c \in \{1, \ldots, a-b\}$. Then

$$M_c(m-1) \begin{pmatrix} 1 \\ \lfloor \frac{r_1 - m}{m} \rfloor \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \lfloor \frac{r_1 - m}{m} \rfloor \end{pmatrix} = \begin{pmatrix} \lfloor \frac{r_1}{m} \rfloor \\ \lfloor \frac{r_1}{m} \rfloor \end{pmatrix}.$$

hence for the value of $R(w)$ we obtain

$$R(x_s 0^{r_s} \cdots x_1 0^{m-1} c 0^{r_1 - m}) = \begin{pmatrix} u & v \end{pmatrix} \begin{pmatrix} \lfloor \frac{r_1}{m} \rfloor \\ \lfloor \frac{r_i}{m} \rfloor \end{pmatrix} > \begin{pmatrix} u & v \end{pmatrix} \begin{pmatrix} 1 \\ \lfloor \frac{r_1}{m} \rfloor \end{pmatrix} = R(x_s 0^{r_s} \cdots x_1 0^{r_1}),$$

where $u, v \in \mathbb{N}$, which is a contradiction with the maximality of w. □

Proposition 4.29. *If $r_\alpha, r_\beta < m-1$ and $r_\alpha + r_\beta \geq m-2$, then the string $z 0^{r_\alpha} x 0^{r_\beta} y$ is forbidden for maximality for all nonzero digits z, x, y from the canonical alphabet A.*

Proof. This is a consequence of the fact that $M_x(r_\alpha) M_y(r_\beta)$ is majored by the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$:

$$M_x(r_\alpha) M_y(r_\beta) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}^2 \prec \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

If $r_\alpha + r_\beta = m-2$, then $M_c(r_\alpha + r_\beta + 1)$, where $c \in \{1, 2, \ldots, a-b\}$ is equal to $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Thus in this case the factor $z 0^{r_\alpha} x 0^{r_\beta} y$ is improved by the factor $z 0^{r_\alpha + r_\beta + 1} c$. On the other hand, if $r_\alpha + r_\beta > m-2$ then using the assumption $r_\alpha, r_\beta < m-1$ we derive $m - 1 < r_\alpha + r_\beta + 1 < 2m - 1$, hence $M_f(r_\alpha + r_\beta + 1) = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, where f is any nonzero digit from A. Hence in this case the factor $z 0^{r_\alpha} x 0^{r_\beta} y$ is improved by the factor $z 0^{r_\alpha + r_\beta + 1} f$. □

Theorem 4.30. *Consider a confluent B-system with coefficients $a > b$ and order m. Then for the maxima of the function $R(w)$ defined in this B-system the following holds:*

$$\psi(l) = 2^{\lceil \frac{l}{m} \rceil - 1}.$$

Proof. We prove the theorem by induction on the length l of the greedy representation. First write $l = pm + q$, where $p \in \mathbb{N}_0$ and $q \in \{1, 2, \ldots, m-1, m\}$. For initial values of l, i.e. for $l = 1, 2, \ldots, m$ we have since shown that $\psi(l) = 1$. We have also shown in Corollary 4.26 that $2^{\lceil \frac{l}{m} \rceil - 1}$ is a lower bound on the value of $\psi(l)$, so it suffices to show that it is also an upper bound.

Take the greedy representation of length l on which the value of $\psi(l)$ is reached, i.e. the word $w = x_s 0^{r_s} x_{s-1} 0^{r_{s-1}} \cdots x_1 0^{r_1}$ such that $s \in \mathbb{N}$ and $x_i \in \{1, 2, \ldots, a\}$, $r_i \in \mathbb{N}_0$ for all $i = 1, 2, \ldots, s$, and

$$\psi(l) = R(x_s 0^{r_s} x_{s-1} 0^{r_{s-1}} \cdots x_1 0^{r_1}).$$

Suppose first that $\underline{R}(w) = 0$. Then clearly

$$\psi(l) = R(x_s 0^{r_s} x_{s-1} 0^{r_{s-1}} \cdots x_1 0^{r_1}) = R(x_{s-1} 0^{r_{s-1}} \cdots x_1 0^{r_1}) \leq \psi(l - r_s - 1)$$

and the statement follows from the induction hypothesis. Consider now the case $\underline{R}(w) \geq 1$. Then either $r_s = m-1$ and $x_{s-1} \leq a-b$ or $r_s \geq m$, otherwise we would not be able to rewrite the prefix $x_s 0^{r_s} x_{s-1}$ to obtain a short representation of w with respect to the digit x_s. Let us show that the coefficients r_i and digits x_i can only take certain values. Suppose that there is an index $2 \leq i \leq s-1$ such that $0 \leq r_i \leq m-2$. Then since $M_{x_{i-1}}(r_i) = \left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)$, we have

$$M_{x_i}(r_{i+1})M_{x_{i-1}}(r_i) = \begin{cases} \begin{pmatrix} 1 & 1 \\ \left\lfloor \frac{r_{i+1}+1}{m} \right\rfloor & \left\lfloor \frac{r_{i+1}+1}{m} \right\rfloor \end{pmatrix} & \text{if } x_i \leq a-b, \\ \begin{pmatrix} 1 & 1 \\ \left\lfloor \frac{r_{i+1}}{m} \right\rfloor & \left\lfloor \frac{r_{i+1}}{m} \right\rfloor \end{pmatrix} & \text{if } x_i = a-b+1, \\ \begin{pmatrix} 1 & 1 \\ \left\lfloor \frac{r_{i+1}}{m} \right\rfloor & \left\lfloor \frac{r_{i+1}}{m} \right\rfloor \end{pmatrix} & \text{if } x_i \geq a-b+2. \end{cases}$$

In all three cases this implies

$$\psi(l) = R(x_s 0^{r_s} \cdots x_1 0^{r_1}) \leq R(x_s 0^{r_s} \cdots x_{i+1} 0^{r_{i+1}} x_i 0^{r_{i-1}} \cdots x_1 0^{r_1}) \leq \psi(l - r_i - 1),$$

and the statement follows from the induction hypothesis. Similarly, if there exists an $2 \leq i \leq s$ such that $m+1 \leq r_i \leq 2m-2$, then $M_{x_{i-1}}(r_i) = M_{x_{i-1}}(m)$, and thus $\psi(l) \leq \psi(l - r_i + m)$ and again the statement follows from the induction hypothesis. Lastly, if an index $2 \leq i \leq s$ exists such that $r_i = m$ and $x_i \geq a-b+1$, then for all $1 \leq c \leq a-b$ we obtain $M_{x_i-1}(r_i) = M_c(m-1)$, which implies $\psi(l) \leq \psi(l-1)$ and again the statement follows from the induction hypothesis. Therefore, using Propositions 4.27, 4.28 and 4.29 it is sufficient to consider only coefficients $r_s = r_{s-1} = \cdots = r_2 = m-1$ and $r_1 \in \{0, 1, \ldots, 2m-1\}$ and digits $x_{s-1}, x_{s-2}, \ldots, x_1 \in \{1, 2, \ldots, a-b\}$. We will now determine $R(w)$ for this combination of coefficients. Since

$$\begin{pmatrix} 1 & 1 \end{pmatrix} M_{x_{s-1}}(m-1) = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 2 \begin{pmatrix} 1 & 1 \end{pmatrix},$$

then using the induction hypothesis we obtain

$$\begin{aligned} \psi(l) &= R(x_s 0^{r_s} x_{s-1} 0^{r_{s-1}} \cdots x_1 0^{r_1}) = 2R(x_{s-1} 0^{r_{s-1}} \cdots x_1 0^{r_1}) \\ &\leq 2\psi(l-m) \leq 2 \cdot 2^{\lceil \frac{l-m}{m} \rceil - 1} = 2^{\lceil \frac{l}{m} \rceil - 1}, \end{aligned}$$

which proves the theorem. □

4.4 Arguments of the Maxima of $R(n)$ in Confluent Systems

In this section we will verify our observations from Chapter 3 about the sizes of the set $\Psi(l)$ in the surveyed B-systems.

4.4.1 Confluent Systems with $a = b$ and order $m = 2$

In Chapter 3 we found that except for the initial cases $l = 1, 2, 3, 4$ and $l = 6, 9, 12$, the following relationship for the size of the set $\Psi(l)$ holds:

$$\begin{aligned} \#\Psi(2k+1) &= 2 \cdot a && \text{for } k \geq 1, k \neq 4, \\ \#\Psi(2k) &= 4 \cdot a^2 && \text{for } k \geq 4, k \neq 6. \end{aligned}$$

We shall prove these relations (as well as derive the ones for $l = 6, 9, 12$) by determining the greedy representations w of length l on which the maximal values of $\psi(l) = R(w)$ are reached. The proof of Theorem 4.19 will serve for this purpose. We will again closely follow the approach taken in [12].

Denote by $w = x_s 0^{r_s} x_{s-1} 0^{r_{s-1}} \cdots x_1 0^{r_1}$, where $x_i \in \{1, 2, \ldots, a\}$ and $r_i \in \mathbb{N}_0$ for all $i = 1, 2, \ldots, s$, a greedy representation of length $l = s + r_s + r_{s-1} + \cdots + r_1$ such that $\psi(l) = R(w)$.

Let us first suppose that l is odd and that equality is reached in (4.19). Recall that the relation (4.16) implies that for the equality to be reached, i.e. that

$$\psi(2k+1) = \psi(2(k-p-1)+2)\psi(2p+1) = 2F_{k-p-2}F_p = F_k,$$

it is required that $p = 1$ and $k = 4$. Then $F_{k-p-2} = F_p = F_1 = 2$ and $F_k = F_4 = 8$. From (4.19) we then obtain

$$\psi(2(k-p-1)+2) = \psi(6) = R(x_s 0^{r_s} \cdots x_i 0^{r_i}),$$

and

$$\psi(2p+1) = \psi(3) = R(x_{i-1} 0^{r_{i-1}} \cdots x_1 0^{r_1}).$$

Hence we have determined one of the forms of the maximal representations for length $l = 9$. They will have $r_3 = r_2 = r_1 = 2$, thus they will be of the form $w = x_3 00 x_2 00 x_1 00$, where $x_3, x_2, x_1 \in \{1, 2, \ldots, a\}$. In total, this yields a^3 representations.

Let us now suppose that l is odd and that equality is not reached in (4.19). The proof of Theorem 4.19 suggests in this case that unless equality holds in (4.19), all the coefficients r_i are odd and thus as a consequence of Proposition 4.18, they will have a very specific form, which we show below.

Corollary 4.31. *Suppose a confluent B-system with $a = b$ and order $m = 2$. Then*

1. *$\#\Psi(4k+3)$ is equal to a for $k = 0$ and to $2 \cdot a$ for $k \geq 1$. We have $\psi(3) = R(x00)$, where $x \in \{1, \ldots, a\}$, thus a different possible greedy representations, and for $k \geq 1$ we have*

$$\psi(4k+3) = R\left(x0\left(10^3\right)^{k-1}10^4\right) = R\left(x0^3\left(10^3\right)^{k-1}10^2\right).$$

   Again, $x \in \{1, \ldots, a\}$, thus we obtain $a + a$ possible representations on which $\psi(4k+3)$ is reached.

2. *$\#\Psi(4k+1)$ is equal to $a+1$ for $k = 0$, since $\psi(1)$ is reached on all representations of length 1. Then $\#\Psi(4k+1)$ is equal to $a^3 + 2a$ for $k = 2$, since*

$$\psi(9) = R(x_3 00 x_2 00 x_1 00) = R(x_3 01000100) = R(x_2 00010000),$$

   where $x_3, x_2, x_1 \in \{1, 2, \ldots, a\}$. Lastly, $\#\Psi(4k+1) = 2 \cdot a$ for $k \geq 3$ or $k = 1$, because then

$$\psi(4k+1) = R\left(x0\left(10^3\right)^{k-1}10^2\right) = R\left(x0^3\left(10^3\right)^{k-1}10^4\right).$$

   The digit x belongs to the set $\{1, 2, \ldots, a\}$, thus there are $a + a$ possible representations on which $\psi(4k+1)$ is reached.

Consider now the case when the length l is even. Take the greedy representation $w = x_s 0^{r_s} x_{s-1} 0^{r_{s-1}} \cdots x_1 0^{r_1}$ of length $l = 2k+2$ such that the maximum $\psi(2k+2)$ is reached on w. Then the proof of Theorem 4.19 requires that equality is reached in (4.18), i.e. that

$$\psi(2k+2) = \psi(2k-2p+1)\psi(2p+1) = F_{k-p}F_p = 2F_{k-1}.$$

Relation (4.15) for the Fibonacci numbers then implies that either $k-p=1$ or that $p=1$. This further implies that the maximal representation $x_s0^{r_s}x_{s-1}0^{r_{s-1}}\cdots x_10^{r_1}$ of length $2k+2$ is split at $i=1$ or $i=s-1$, namely that either $r_s=2$ and $r_{s-1},\ldots,r_1$ are odd, and that $x_{s-1}0^{r_{s-1}}\cdots x_10^{r_1}$ is maximal, i.e. $R(x_{s-1}0^{r_{s-1}}\cdots x_10^{r_1})=\psi(2k-1)$, or that $r_1=2$ and that $x_s0^{r_s}\cdots x_20^{r_2}$ is maximal.

Corollary 4.32. *Let $k\geq 3$ and let $r_s,\ldots,r_1$ satisfy $\sum_{i=1}^{s} r_i+s=2k+2$. Then*

$$R(x_s0^{r_s}x_{s-1}0^{r_{s-1}}\cdots x_10^{r_1})=\psi(2k+2)$$

if and only if

$$r_s=2 \quad \text{and} \quad R(x_{s-1}0^{r_{s-1}}\cdots x_10^{r_1})=\psi(2k-1), \tag{4.20}$$

or

$$r_1=2 \quad \text{and} \quad R(x_s0^{r_s}\cdots x_20^{r_2})=\psi(2k-1). \tag{4.21}$$

From Corollary 4.32 we thus obtain all the possible greedy representations of even length.

Proposition 4.33. *Let $k\geq 3$. Then*

$$\#\Psi(2k+2)=4\cdot a^2 \quad \text{for } k\neq 5,$$

and

$$\#\Psi(2k+2)=a^4+2a^2 \quad \text{for } k=5.$$

Proof. Let $k\neq 5$. Then we construct elements of $\Psi(2k+2)$ using the recipe from Corollary 4.32. Denote by u an element of $\Psi(2k-1)$ and let x be a nonzero digit from the canonical alphabet. Then $\Psi(2k+2)$ will consist of strings of the forms $x00u$ and $ux00$ corresponding to (4.20) and (4.21), respectively. Both $x00u$ and $ux00$ can have $2\cdot a^2$ different instances, since $x\in\{1,2,\ldots,a\}$ and in Corollary 4.31 we counted that $\Psi(2k-1)=\Psi(2p+1)$ has $2\cdot a$ elements for $p\neq 4$. In total we obtain $\Psi(2k+2)=4\cdot a^2$.

Suppose now that $k=5$. Then representations from the set $\Psi(2k+2)=\Psi(12)$ will be constructed by concatenating elements of $\Psi(3)$ with those from $\Psi(9)$. Again, denote u an element of $\Psi(9)$. The set $\Psi(2k+2)$ will again consist of strings of the forms $x00u$ and $ux00$. However, in this case, there are a^3+2a possible instances of u. Therefore in total we obtain $\#\Psi(12)=a^4+2a^2$. □

4.4.2 Confluent Systems with $a>b$

In Chapter 3, we found that the number of maxima at representation of odd length is constant and equal to 4 in the $(2,1)$-system. In this section we will use the matrix formula to explain the number of arguments of the maxima of $R(n)$, or in other words, the size of the set $\Psi(l)$ for all studied confluent systems with $a>b$. We will determine $\Psi(l)$ based on the residue class of l modulo m, where m is the basis order. First let us state some general observations. An immediate corollary of Theorem 4.30 is that the maxima of $R(w)$ are reached on greedy representations

$$w=x_n0^{r_n}x_{n-1}0^{r_{n-1}}\cdots x_10^{r_1}$$

where as many r_j are equal to $m-1$ as possible. Furthermore, when $r_j=m-1$, the digit x_{j-1} is forced to be from the set $\{1,2,\ldots,a-b\}$. Then $M_{x_{j-1}}(r_j)=\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$, which majores $M_y(r_j)$ for any $y\geq a-b+1$.

We will start with the simplest case, when $l\equiv 1 \mod m$, i.e. $l=km+1$ for some $k\in\mathbb{N}_0$.

Theorem 4.34. *Consider a confluent B-system with coefficients $a > b$ and order m. Then the number of arguments of the maximum of $R(n)$ on all n whose greedy representation has length $l = km + 1$ is equal to*

$$\#\Psi(km+1) = a \cdot (a-b)^{k-1} \cdot (a-b+1) \qquad \textit{for all } k \geq 1.$$

Proof. From Theorem 4.30 we know that the value $R(w) = \psi(km+1)$ is reached on representations w of the form

$$w = w_{km-1} \left[0^{m-1} c_*\right]_C^{k-1} 0^{m-1} w_0,$$

where $w_{km-1} \in \{1, 2, \ldots, a\}$, $C = \{1, 2, \ldots, a-b\}$ for all $j = 1, 2, \ldots, k-1$, and $w_0 \in \{0, 1, \ldots, a-b\}$. We can see that truly $|w| = km + 1$ and that w includes as many factors 0^{m-1} as possible on this length. Let us now count the number of possible instances of w. We have a choices for w_{km-1}, then $a - b$ choices for c_* for every $j = 1, 2, \ldots, k-1$, and finally $a - b + 1$ choices for w_0. Thus we obtain the result $\#\Psi(km+1) = a \cdot (a-b)^{k-1} \cdot (a-b+1)$. □

Let us now move on to the case $l = km+2$. This residue class requires a much more technical proof, hence for simplicity we will start by determining the elements of the set $\Psi(km+2)$ in the $(2,1)$-system.

Since l is even, the number of repetitions of the factor 01 will be $\left\lfloor \frac{l}{2} \right\rfloor - 1$ because the most significant digit w_{l-1} cannot be equal to zero. Thus we can construct maximal representations of length $km + 2$ by taking the maximal representations for $km + 1$, which will have the form

$$w = w_{l-1}0101 \cdots 01010w_0, \tag{4.22}$$

and extend them by one more digit to length $km + 2$. We will denote this extra digit x. We can place x to the left of every 01 factor, to the left of $0w_0$ and to the right of w_0. All possible locations are shown below.

$$w_{l-1}x01x01 \cdots x01x01x0w_0x. \tag{4.23}$$

Other locations of x are either equivalent or would lead to a decrease of $R(w)$, as we would break apart one of the 01 factors. Let us now evaluate $R(w)$ depending on the value of x and verify that we will not change it by introducing the new digit x.

If $x = 0$ and we place x in front of a zero, then since $M_1(1) = M_1(2)$, the value of $R(w)$ does not change. Note that because $M_1(2) = M_2(2)$ this further allows us to change the digit that ends this gap to 2, i.e. we obtain the two possible factors $x01 \in \{001, 002\}$.

Suppose the other case, i.e. that we place x at the end of the representation. This yields the suffix $10w_00$, which can either contribute to $R(w)$ as 1000 or 1010. Both cases are equivalent, since

$$\begin{pmatrix} \overline{R}(1010) \\ \underline{R}(1010) \end{pmatrix} = M_1(1) \begin{pmatrix} 1 \\ \left\lfloor \frac{1}{2} \right\rfloor \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \left\lfloor \frac{3}{2} \right\rfloor \end{pmatrix} = \begin{pmatrix} \overline{R}(1000) \\ \underline{R}(1000) \end{pmatrix}.$$

If x is a non-zero digit, then its placement anywhere except the end of the representation introduces the matrix

$$M_x(0) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

into the product for $R(w)$, because that is the contribution of the factors $w_{l-1}x$ and $1x$. In other words, we introduced a gap of zero length into the representation w. Since the equality

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^p \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^q = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^r \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^s \tag{4.24}$$

holds for all $p+q=r+s$, $p,q,r,s \in \mathbb{N}_0$, the value of $R(w)$ does not depend on the placement of x and we can write

$$\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}^{p}\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}^{q} = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}^{p+q} = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}^{p+q}. \quad (4.25)$$

Placing x at the end of the representation yields the suffix $10w_0x$, which is realised as $100x$ or $101x$. Since $M_1(2) = M_2(2)$, the contribution of both suffixes to $R(w)$ is identical:

$$\begin{pmatrix}\overline{R}(100x)\\ \underline{R}(100x)\end{pmatrix} = M_x(2)\begin{pmatrix}1\\ \lfloor\frac{0}{2}\rfloor\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix},$$

$$\begin{pmatrix}\overline{R}(101x)\\ \underline{R}(101x)\end{pmatrix} = M_1(1)M_x(0)\begin{pmatrix}1\\ \lfloor\frac{0}{2}\rfloor\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix}.$$

Hence

$$\begin{pmatrix}\overline{R}(101x)\\ \underline{R}(101x)\end{pmatrix} = \begin{pmatrix}\overline{R}(100x)\\ \underline{R}(100x)\end{pmatrix} = \begin{pmatrix}\overline{R}(10w_0)\\ \underline{R}(10w_0)\end{pmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix}.$$

Using the results for the case when l is odd and setting $p+q = \lfloor\frac{l}{2}\rfloor - 2$ in (4.25) we obtain

$$R(w) = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}^{\lfloor\frac{l}{2}\rfloor-2}\begin{pmatrix}1\\ 1\end{pmatrix} = 2^{\lceil\frac{l}{2}\rceil-1},$$

which proves that the value of $R(w)$ does not change by introducing x.

We can now prove why

$$\#\Psi(l) = 16\left(\frac{l}{2} - 1\right) \quad (4.26)$$

when l is even and greater than or equal to 3. Suppose that $l = 4$. Then $\psi(l) = 2$, using the factorisation introduced above we observe that there are two possible placements of x:

$$w_3x0w_0x,$$

hence elements of $\Psi(4)$ must have one of the following forms:

1. $w = w_30w_0x$, of which there are $2\cdot 2\cdot 3 = 12$ variants, since $w_3 \in \{1,2\}$, $w_0 \in \{0,1\}$ and $x \in \{0,1,2\}$.

2. $w = w_3x0w_0$, of which there are $1\cdot 2\cdot 2 = 4$ variants, since $w_3 = 1$ because w is a greedy representation, $x \in \{1,2\}$ and $w_0 \in \{0,1\}$, because $x = 0$ is included in case 1.

Together, we have $\#\Psi(4) = 16$, which agrees with formula (4.26) and results in Table 3.1.

Let us now consider $l = 6$. There are three possible placements of x:

$$w_5x01x0w_0x,$$

hence elements of $\Psi(6)$ must one of the following forms:

1. $w = w_5010w_0x$, of which there are again $2\cdot2\cdot3 = 12$ variants, since $w_5 \in \{1,2\}$, $w_0 \in \{0,1\}$ and $x \in \{0,1,2\}$.

2. $w = w_501x0w_0$, of which there are again $2\cdot2\cdot2 = 8$ instances, because $w_5 \in \{1,2\}$ $x \in \{1,2\}$ and $w_0 \in \{0,1\}$, $x = 0$ is included in case 1.

3. $w = w_5x010w_0$, amounts to 12 representations in total. If $x \in \{1,2\}$, then we gain $1{\cdot}2{\cdot}2 = 4$ representations, since $w_5 = 1$ (due to w being a greedy representation) and $w_0 \in \{0,1\}$. However, if $x = 0$, we obtain another $2 \cdot 1 \cdot 2 \cdot 2 = 8$ representations, due to the fact we can exchange $w_2 = 1$ for $w_2 = 2$, because the rewriting of $w_5x0 = w_500 \to (w_5{-}1)21$ is independent of the digit w_2.

In total, we obtain $\#\Psi(6) = 32$ as expected.

Finally, let us treat $\Psi(8)$ separately. There are now four possible placements of x:

$$w_7x01x01x0w_0x,$$

1. $w = w_701010w_0x$, of which there are $2 \cdot 2 \cdot 3 = 12$ possible variants as in $\Psi(6)$.
2. $w = w_70101x0w_0$, of which there are again $2 \cdot 2 \cdot 2 = 8$ possible instances as in $\Psi(6)$.
3. $w = w_701x010w_0$ amounts to 16 representations. If $x \in \{1,2\}$, $w_7 \in \{1,2\}$ and $w_0 \in \{0,1\}$ together yield $2 \cdot 2 \cdot 2 = 8$ representations. If $x = 0$, we can exchange the following 1 for a 2, i.e. we write w as $w_70100y0w_0$, where $y \in \{1,2\}$ and this yields another $2 \cdot 2 \cdot 2 = 8$ representations.
4. $w = w_7x01010w_0$ contributes 12 representations, as in $\Psi(6)$.

Together this yields $\#\Psi(8) = 48$.

Now let us explore the general case $\Psi(2k+2)$, where $k \geq 3$: There are $k+1$ possible placements of x:

$$w_{2k-1}x01(x01)^{k-2}x0w_0x,$$

where by $(x01)^{k-2}$ we denote the placement of x in front of *precisely one* of the $k-3$ repetitions of the 01 factor. As shown in the previous cases,

1. $w_{2k-1}01(01)^{k-2}0w_0x$ contributes 12 different representations,
2. $w_{2k-1}01(01)^{k-2}x0w_0$ contributes 8 different representations,
3. $w_{2k-1}01(01)^{k-2-i}x(01)^i0w_0$ contributes 16 different representations for each $i \in \{1,2,\ldots,k-2\}$, in total $(k-2) \cdot 16$ representations.
4. $w_{2k-1}x01(01)^{k-2}0w_0$ contributes 12 different representations.

In total, we obtain

$$\#\Psi_{2,1}(2k+2) = 12 + 8 + 16(k-2) + 12 = 16k$$

different representations, which corresponds with (4.26). We will now determine $\Psi(2k+2)$ in all confluent B-systems with $a > b$.

Theorem 4.35. *Consider a confluent B-system with coefficients $a > b$ and order m. Then the number of arguments of the maximum of $R(n)$ on all n whose greedy representation has length $l = km+2$ is for all $k \geq 1$ equal to*

$$\#\Psi(km+2) = \begin{cases} (a-b)^{k-2}(a-b+1)\left((k-1)a^2 + (a-b)\left((k+1)a^2 + b - 1\right)\right) & \text{if } m = 2, \\ (a-b)^{k-2}(a-b+1)\left((k-1)a^2 + (a-b)\left((k+1)a^2 + a\right)\right) & \text{if } m > 2. \end{cases}$$

Proof. In order to determine $\#\Psi(km+2)$ we will follow the same approach as in the above analysis for the $(2,1)$-system. Take a greedy representation w from the set $\Psi(km+1)$. Clearly w will have the form

$$w = w_{mk-1}\left[0^{m-1}c_*\right]_C^{k-1} 0^{m-1} w_0,$$

where $w_{mk-1} \in \{1,2,\ldots,a\}$, $C = \{1,2,\ldots,a-b\}$ and $w_0 \in \{0,1,\ldots,a-b\}$. We shall now insert a new digit x into w, denote $\tilde{w}$ this new representation of length $km+2$. There are $k+1$ possible locations for x.

$$w_{mk-1}x\left[0^{m-1}c_*x\right]_C^{k-1} 0^{m-1} w_0 x.$$

Other locations are again equivalent or would reduce the value of $R(\tilde{w})$. We shall now determine how many representations correspond to each placement of x.

1. $\tilde{w} = w_{mk-1}\left[0^{m-1}c_*\right]_C^{k-1} 0^{m-1} w_0 x$:
   In this case, there are evidently a possible values for w_{mk-1} and $a-b$ possible values for each c_* in each of the $k-1$ repetitions of $0^{m-1}c_*$. The digit w_0 is from the set $\{0,1,\ldots,a-b\}$, thus $a-b+1$ possible values and finally, since there are no restrictions on x, the digit x can be any digit from the alphabet A, thus $a+1$ possible values of x. In total we obtain $a\cdot(a-b)^{k-1}\cdot(a-b+1)\cdot(a+1)$ representations.

2. $\tilde{w} = w_{mk-1}\left[0^{m-1}c_*\right]_C^{k-2-i} 0^{m-1}c_{i+1}x\left[0^{m-1}c_*\right]_C^{i} 0^{m-1} w_0$ for all $0 \le i \le k-2$:
   We again count a possible values for w_{mk-1}. Then, we have to place x into precisely one of the repetitions of the factor $0^{m-1}c_*$, thus we multiply by the coefficient $k-1$. The factor $0^{m-1}c_{i+1}x$ has $(a-b+1)\cdot a$ possible realisations, since c_{i+1} can now also be zero whenever x is nonzero. This follows from the fact that $M_{c_{i+1}}(m-1) = M_x(m)$ for all $c_{i+1} \in \{1,2,\ldots,a-b\}$ and $x \in \{1,2,\ldots,a\}$, thus the contribution of the factor $0^m x$ towards $R(\tilde{w})$ is identical to $0^{m-1}c_{i+1}$. Evidently, both c_{i+1} and x cannot be zero simultaneously, but even if $c_{i+1} \neq 0$, then x cannot be zero. This is because we would count the same word $\tilde{w}$ twice. Compare the two placements of x into two consecutive factors $0^{m-1}c_{i+1}$ and $0^{m-1}c_i$:

(a)	$\cdots$	0^{m-1}	c_{i+1}	x	0^{m-2}	0	c_i	$\cdots$
(b)	$\cdots$	0^{m-1}	c_{i+1}	0	0^{m-2}	c_i	x	$\cdots$

   Setting $x=0$ in case (a) yields the same string as setting $c_i = 0$ in case (b). We have already counted the case $c_i = 0$, thus we forbid the case $x = 0$. There remain $k-2$ factors $0^{m-1}c_{i+1}$ where we have not placed x, and in these, again, c_{i+1} can have the values $\{1,2,\ldots,a-b\}$, thus we multiply by $(a-b)^{k-2}$. Lastly, the digit w_0 can again have $a-b+1$ possible values. In total we count $a\cdot(k-1)\cdot(a-b+1)\cdot a\cdot(a-b)^{k-2}\cdot(a-b+1) = a^2\cdot(k-1)\cdot(a-b+1)^2\cdot(a-b)^{k-2}$ possible representations.

3. $\tilde{w} = w_{mk-1}x\left[0^{m-1}c_*\right]_C^{k-1} 0^{m-1} w_0$:
   In the third possible placement of x we have to split our analysis according to the order of the basis. We again forbid $x=0$ because that is included above in case 2. If $m>2$, then the prefix $w_{mk-1}x$ has a^2 possible values. On the other hand, if $m=2$, then the prefix $w_{mk-1}x$ has $1\cdot(b-1)+(a-1)\cdot a$ possible realisations, because $\tilde{w}$ is a greedy representation. In other words, whenever $w_{mk-1} = a$, the digit x must be smaller than b. Lastly, we again count $(a-b)^{k-1}$ as the contribution of the $k-1$ factors $0^{m-1}c_*$ and $(a-b+1)$ as all the

possible values of w_0. In total we obtain

$$\begin{aligned}(b-1+(a-1)\cdot a)\cdot(a-b)^{k-1}\cdot(a-b+1) &\text{ if } m=2,\\ a^2\cdot(a-b)^{k-1}\cdot(a-b+1) &\text{ if } m>2.\end{aligned}$$

possible representations.

We will now add the above cases 1., 2., and 3. together and simplify. Suppose first that $m=2$. Then

$$\begin{aligned}\#\Psi(km+2) =& a(a-b)^{k-1}(a-b+1)(a+1)\\ &+a^2(k-1)(a-b+1)^2(a-b)^{k-2}\\ &+(b-1+a(a-1))\,(a-b)^{k-1}(a-b+1),\end{aligned}$$

factoring out $(a-b)^{k-2}(a-b+1)$ yields

$$\#\Psi(km+2)=(a-b)^{k-2}(a-b+1)\cdot\Delta, \tag{4.27}$$

where we denote

$$\Delta := a(a+1)(a-b)+a^2(k-1)(a-b+1)+((a-1)a+b-1)\,(a-b).$$

The expression Δ can be further simplified by factoring out $(a-b)$:

$$\begin{aligned}\Delta &= a(a+1)(a-b)+a^2(k-1)(a-b+1)+((a-1)a+b-1)\,(a-b),\\ &=(a^2+a+a^2-a+b-1)(a-b)+a^2(k-1)(a-b+1),\\ &=(2a^2+b-1)(a-b)+(k-1)a^2(a-b)+(k-1)a^2,\end{aligned}$$

which finally simplifies to

$$\Delta=(k-1)a^2+(a-b)\left((k+1)a^2+b-1\right).$$

Returning to (4.27), we obtain the desired result

$$\#\Psi(km+2)=(a-b)^{k-2}(a-b+1)\left((k-1)a^2+(a-b)\left((k+1)a^2+b-1\right)\right).$$

The case $m>2$ is derived by the same steps. □

Let us demonstrate our formula for $\#\Psi(km+2)$ on an example. In Table (3.7) we may find the value $\#\Psi(8)=540$ for the $(3,1)$-system. Thus we have $a=3$, $b=1$, and $k=3$, since $8=2\cdot 3+2$. Inputting these values yields

$$\begin{aligned}\#\Psi_{3,1}(8) &= (3-1)^{3-2}(3-2+1)\left((3-1)3^2+(3-1)\left((3+1)3^2+1-1\right)\right),\\ &=2\cdot 3\cdot(2\cdot 9+2\cdot 4\cdot 9),\\ &=6\cdot(18+72),\\ &=540.\end{aligned}$$

Expressions could be derived for $\#\Psi(km+3)$, $\#\Psi(km+4)$, etc., but they would be even more technical and complex.

Conclusion

In this work we studied linear numeration systems and focused on their ambiguity. In Chapter 2 we introduced and verified basic properties of linear numeration systems. We then derived and implemented an algorithm for calculating $R(n)$ in a general B-system, which we used to calculate $R(n)$ on a chosen subclass of B-systems, the confluent systems. Based on our data, we conjectured that $R(n)$ in confluent systems with $a = b$ behaves very similarly to $R(n)$ defined in the Fibonacci and m-bonacci systems and gave an expression for the maxima of $R(n)$ in all confluent systems. Using the matrix formula derived in Chapter 4 we then verified that this is true. Furthermore we showed that the confluent systems can be split into precisely three classes. The function $R(n)$ in confluent systems with $a = b$ and order $m = 2$ displayed analogous behaviour to the Fibonacci system, those with with $a = b$ and order $m > 2$ behaved identically to the m-bonacci systems, and finally the confluent systems with $a > b$ behaved entirely differently. We have thus generalised the work of Kocábová, Masáková and Pelantová to all confluent systems.

What remains is to derive an expression for the arguments of the maxima in the confluent systems with order $a = b$ and order $m > 2$. Unfortunately, in this case the results from the m-bonacci systems cannot be easily generalised. Furthermore, we did not study the numbers that have a unique representation in a given confluent system.

Further work could focus on trying to derive a closed-form formula for $R(n)$ (which will not be a matrix formula) in general (F) systems and on studying the properties of $R(n)$ in thsese systems, as they are a close generalisation of confluent systems.

Appendix

In the Appendix we will describe in detail our program for calculating the function $R(n)$ in arbitrary B-systems. The source code may be found in the following GitHub repository:
https://github.com/hypernek/Redundance-Calculator,
while a compiled and runnable version program can be found in the same repository here:
https://github.com/hypernek/Redundance-Calculator/releases/tag/v1.0.

The program runs on Windows. The source code and compiled program as well as sample data may be also found on the CD attached with the physical copy of this work.

Usage

Two programs are included – the first calculates $R(n)$ on bounds $n_{\min}$ and $n_{\max}$ entered by the user and the other calculates the maxima of $R(n)$ on a range of lengths entered by the user. The output of both programs is saved as a .csv file to the directory where the program was run.

Both programs are console applications that on initialisation ask the user to enter coefficients of the basis of the B-system. After entering these coefficients, the basis of the B-system is initialised. Afterwards, each of the two programs behaves differently.

The first program (`Rn_calculator.exe`) asks the user to enter the lower and upper bound $n_{\min}$ and $n_{\max}$ of the values $n_{\min} \leq n \leq n_{\max}$ for which the function $R(n)$ is to be calculated. Alternatively, the user may enter an asterisk $*$ and then enter the bounds $l_{\min}$ and $l_{\max}$ of the range of lengths on which they desire $R(n)$ to be calculated. In effect, this sets $n_{\min}$ and $n_{\max}$ equal to $B_{l_{\min}-1}$ (the smalest n whose representation has length $l_{\min}$) and $B_{l_{\max}} - 1$ respectively (the largest n whose representation has length $l_{\max}$). Hence, the user does not have to know the values of the elements of the basis sequence. After the bounds are set, the calculation commences and the console displays progress information. The values $R(n_{\min})$, $R(n_{\min}+1)$, …, $R(n_{\max})$ are first stored into the computer's memory, and after they are all calculated, the program starts writing them to disc. Each row of the resultant .csv file is the triplet $(n, \langle n \rangle_B, R(n))$ – i.e. for every $n = n_{\min}, n_{\min}+1, \ldots, n_{\max}$ the greedy representation of n is stored along with the value of $R(n)$. After writing out all the calculated values of $R(n)$, the program writes the time needed for calculation and writing to disc. A sample output of the program for the $(3,2,1)$-system may be found in Table 4.1.

The second program (`Maxima_of_Rn_calculator.exe`) asks the user to input the bounds $l_{\min}$ and $l_{\max}$ of the range of lengths $l = l_{\min}, l_{\min}+1, \ldots, l_{\max}$ for which the values $\psi(l)$ and $\#\Psi(l)$ are to be determined. Then the program asks the user to enter a number M, which will be the maximum number of elements of $\Psi(l)$ they desire to save for a given length l. As shown in Chapters 3 and 4, the size of $\Psi(l)$ can be very large and the user does not necessarily need to save all members of $\Psi(l)$. For example, the data in Table 3.2 corresponds to entering coefficients $2, 1$, lengths $l_{\min} = 1$ and $l_{\max} = 22$ and setting M equal to 4. For every $l = l_{\min}, l_{\min}+1, \ldots, l_{\max}$

n	$\langle n\rangle_B$	$R(n)$	n	$\langle n\rangle_B$	$R(n)$
9355	10000000	3	9371	10000101	3
9356	10000001	3	9372	10000102	3
9357	10000002	3	9373	10000103	3
9358	10000003	3	9374	10000110	3
9359	10000010	3	9375	10000111	3
9360	10000011	3	9376	10000112	3
9361	10000012	3	9377	10000113	3
9362	10000013	3	9378	10000120	3
9363	10000020	3	9379	10000121	3
9364	10000021	3	9380	10000122	3
9365	10000022	3	9381	10000123	3
9366	10000023	4	9382	10000130	2
9367	10000030	3	9383	10000131	2
9368	10000031	3	9384	10000132	2
9369	10000032	3	9385	10000200	4
9370	10000100	5	9386	10000201	2

Table 4.1: Sample output data for $R(n)$ calculated in the $(3,2,1)$-system for $n = 9355, \ldots, 9386$, i.e. the first 32 numbers whose greedy representation has length 8. Note that along with the rule $1000 \leftrightarrow 0321$ we may also perform interchanges utilising the rule $100 \leftrightarrow 033$ at the end of the representation (i.e. only at the suffix $x_2x_1x_0$), which corresponds to the addition/subtraction of the initial representation of zero $\bar{1}33$.

the program outputs $\psi(l)$, $\#\Psi(l)$ and the first M elements of $\Psi(l)$. The algorithm by which the values $\psi(l)$ and $\#\Psi(l)$ are determined is as follows.

***Algorithm for Determining the Maxima of R(n)*:**

Initialisation: Set `number_of_maximal_representations:=0`.
Then for every $l = l_{\min}, l_{\min}+1, \ldots, l_{\max}$ do:

1. Calculate $R(n)$ for every n from the range $B_{l-1} \leq n \leq B_l - 1$ and store the values into memory in the array `Rn_array`.

2. Find the maximal value in `Rn_array` and for every n such that $R(n)$ is maximal (i.e. $R(n) = \psi(l)$), increment by one the counter `number_of_maximal_representations` and store its greedy representation $\langle n\rangle_B$ into the list `representation_list` (but only if it contains less than M representations).

3. Write the triplet (max(`Rn_array), number_of_maximal_representations, representation_list`) into the output .csv file (I.e. write the triplet $\psi(l)$, $\#\Psi(l)$, $\Psi(l)$).

4. Empty `Rn_array, representation_list` and set `number_of_maximal_representations` equal to zero.

The program also creates a second file recording the time needed for calculation and memory usage for each l.

Lastly, in both programs, whenever the user is asked for input, they can enter a percent sign (%) to reset the basis and enter new coefficients, and then running the calculation of $R(n)$ in a different B-system.

Note on Systems not Possessing the (F) Property:

When entering the basis coefficients, any sequence of integers separated by commas is a valid input, thus the program is not limited only to (F) systems. However, the correctness of the values of $R(n)$ is not guaranteed for non-(F) systems, because we do not a priori know the size of the canonical alphabet. In the case that a non-(F) system is entered, the program sets the largest digit of the canonical alphabet to the recurrence coefficient that is maximal. For example, in the $(1,5)$-system, it would set the canonical alphabet to $\{0,1,2,3,4,5\}$, which is too large, as in this system greedy representations contain only digits $\{0,1,2,3\}$. Hence the resultant values of $R(n)$ will be incorrect. For (F) systems however, the correctness of the calculated values of $R(n)$ is always guaranteed.

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