Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Effective quantum Hamiltonian in thin domains with non-homogeneity

Efektivní kvantový hamiltonián v tenkých oblastech s nehomogenitou

Diploma Thesis

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Poděkování:
Chtěla bych zde poděkovat svému školiteli doc. Mgr. Davidovi Krejčiříkovi, Ph.D. DSc. za pečlivost, ochotu, vstřícnost a odborné i lidské zázemí při vedení mé diplomové práce.

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Prohlašuji, že jsem svou diplomovou práci vypracovala samostatně a použila jsem pouze podklady (literaturu, projekty, SW atd.) uvedené v přiloženém seznamu.

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V Praze dne......................... ....

podpis

Název práce:

Efektivní kvantový hamiltonián v tenkých oblastech s nehomogenitou

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Obor: Matematická fyzika

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Abstrakt: Cílem práce je odvození efektivního modelu pro laplacián s nehomogenní metrikou v tenkých oblastech s neumannovskými hraničními podmínkami. Nejprve rigorózně zavedeme Neumannův Laplaceův operátor s nehomogenní poruchou jako samosdružený operátor na Hilbertově prostoru pomocí přidružené kvadratické formy. Dále zkoumáme konvergenci tohoto operátoru k efektivnímu modelu, a to ve spektrálním, jakož i v silném, a dokonce i norm-rezolventním smyslu. Nakonec si odvodíme rychlost této konvergence a vše ilustrujeme konkrétním příkladem.

Klíčová slova: efektivní model, neumannovské hraniční podmínky, Neumannův Laplaceův operátor s nehomogenní poruchou, rezolventní konvergence, spektrální konvergence.

Title:

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Abstract: This work aims to derive an effective model of the Laplacian with a non-homogeneous metric in thin domains with Neumann boundary conditions. Firstly, the Neumann Laplace operator with a non-homogeneous failure will be defined as a self-adjoint operator on the Hilbert space by an associated quadratic form. Furthermore, this work shows the convergence of this operator to the effective model in the spectral, the strong resolvent, even in the norm-resolvent sense, all of which are illustrated with a concrete example. Finally, the rate of the convergence is derived.

Key words: Neumann boundary conditions, Neumann Laplace operator with a non-homogeneous failure, effective model, resolvent convergence, spectral convergence

Contents

Introduction

The primary motivation of the present study of the effective quantum dynamics in thin domains with a non-homogeneous metric is the article of Yachimura (see [17]) published in 2018, in which the spectral problem with piecewise constant coefficients in thin domains with Neumann boundary conditions is analysed. Let $D \subset \mathbb{R}^n$ be a bounded domain with a connected C^2 boundary ∂D. Define

$$\Omega_-(\varepsilon) = \{x \in D \mid \operatorname{dist}(x, \partial D) < \varepsilon\},$$
$$\Omega_+(\varepsilon) = \left\{x \in \mathbb{R}^n \setminus \overline{D} \,\middle|\, \operatorname{dist}(x, \partial D) < \varepsilon\right\},$$
$$\Omega(\varepsilon) = \Omega_-(\varepsilon) \cup \Omega_+(\varepsilon) \cup \partial D,$$

for $\varepsilon > 0$, $\operatorname{dist}(\cdot, \cdot)$ denotes the Euclidean distance. It can be seen in the following figures.

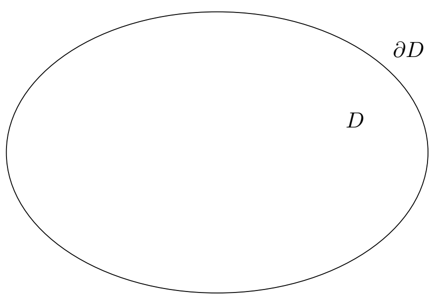

Figure 1: Domain D

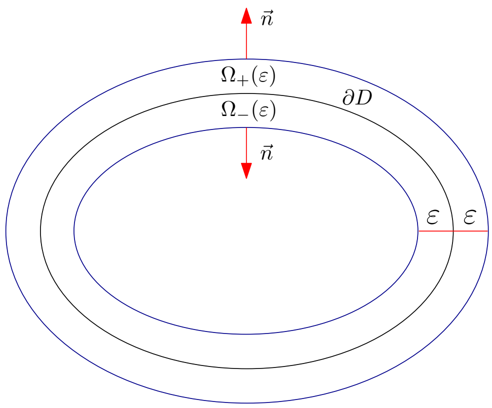

Figure 2: Domain $\Omega(\varepsilon)$

The mentioned spectral problem is

$$\begin{cases} -\nabla \cdot a\nabla\psi = \lambda\psi & \text{in } \Omega(\varepsilon), \\ \frac{\partial\psi}{\partial n} = 0 & \text{on } \partial\Omega(\varepsilon), \end{cases} \tag{1}$$

where n is the outward unit normal vector to the boundary $\partial\Omega(\varepsilon)$ and the function a satifies

$$a(x) = \begin{cases} a_- & \text{in } \Omega_-(\varepsilon) \cup \partial D, \\ a_+ & \text{in } \Omega_+(\varepsilon), \end{cases}$$

a_+, a_- are positive constants and $a_+ \neq a_-$.

Theorem 0.0.1 (see [17], Thm. 1.2)**.** Let $-\Delta^{\partial D}$ be the Laplace-Beltrami operator on the boundary ∂D. Then

$$(\lambda_\varepsilon)_k \underset{\varepsilon \mapsto 0}{\longrightarrow} \frac{a_- + a_+}{2} \lambda_k, \tag{2}$$

where $(\lambda_\varepsilon)_k$ is the kth eigenvalue satisfying the problem (1), λ_k is the kth eingenvalue of $-\Delta^{\partial D}$.

In this study is considered the following problem

$$\begin{cases} -\nabla \cdot a\nabla \tilde{v}_\varepsilon = \lambda_\varepsilon \tilde{v}_\varepsilon & \text{in } \Omega_\varepsilon, \\ \frac{\partial \tilde{v}_\varepsilon}{\partial \tilde{\boldsymbol{n}}} = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{3}$$

for $\varepsilon > 0$, where $\tilde{\boldsymbol{n}}$ is the outward unit normal vector to the boundary $\partial\Omega_\varepsilon$, a is a positive bounded function and Ω_ε is an ε-sorrounding of a connected orientable compact C^3 hypersurface Σ in $\mathbb{R}^d$, for $d \geq 2$, with the Riemannian metric g induced by the embedding.

Firstly, the problem (3) has to be formulated rigorously by the introducing the corresponding self-adjoint operator. Based on the Representation theorem, the self-adjoint operator can be defined via the associated quadratic form. Furthermore, the convergence of this operator to the $(d-1)$-dimensional effective model will be shown in the spectral, the strong resolvent, even in the norm-resolvent sense. Moreover, the rate of the norm-resolvent convergence will be derived.

Chapter 1

The Spectral Convergence

Firstly, the spectral problem with Neumann boundary conditions (3) has to be rigorously formulated. Define the self-adjoint operator $\tilde{H}_\varepsilon$ on $L^2(\Omega_\varepsilon)$ corresponding with the problem (3) in the following sense

$$\lambda_\varepsilon \in \sigma\left(\tilde{H}_\varepsilon\right) \qquad \longleftrightarrow \qquad \begin{cases} -\nabla \cdot a\nabla\tilde{v}_\varepsilon = \lambda_\varepsilon \tilde{v}_\varepsilon & \text{in } \Omega_\varepsilon, \\ \frac{\partial \tilde{v}_\varepsilon}{\partial \tilde{\boldsymbol{n}}} = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

where Ω_ε is an ε-surrounding of a connected orientable compact C^3 hypersurface Σ in $\mathbb{R}^d$, for $d \geq 2$, with the Riemannian metric g induced by the embedding, $\tilde{\boldsymbol{n}}$ is the outward unit normal vector to the boundary $\partial\Omega_\varepsilon$ and a is a positive function satisfying

$$(\exists C > 0)\,(\forall \varepsilon \in (0,1))\,(\text{almost every } x \in \Omega_\varepsilon)\left(\frac{1}{C} \leq a(x) < C\right). \tag{1.1}$$

The self-adjoint operator $\tilde{H}_\varepsilon$ can be introduced by the associated quadratic form $\tilde{Q}_\varepsilon$, which is defined by

$$\tilde{Q}_\varepsilon[\tilde{v}] = \int_{\Omega_\varepsilon} a\,|\nabla\tilde{v}|^2 \,\mathrm{d}\Omega_\varepsilon,$$

$$D(\tilde{Q}_\varepsilon) = W^{1,2}(\Omega_\varepsilon).$$

Remark. The parameter ε is considered to be always small and positive because the limit behaviour is studied.

Now focus on the geometry of the problem. The orientation can be determined by a unit normal vector field $\boldsymbol{n} : \Sigma \to S^{d-1}$ along the hypersurface Σ. Without loss of generality, the same orientation as the orientation of the ambient Euclidean space $\mathbb{R}^d$ can be considered. Let the Weingarten map be introduced as

$$L : T_s\Sigma \longrightarrow T_x\Sigma : \left\{\xi \mapsto -dn(\xi) = -\left(\mathrm{d}n^1(\xi), \ldots, \mathrm{d}n^d(\xi)\right) = -\left(\xi\left(n^1\right), \ldots, \xi\left(n^d\right)\right)\right\}$$

for any $s \in \Sigma$. Denote the local coordinate system of Σ by $s^1, \ldots, s^{d-1}$. The map L can be interpreted as a $(1,1)$ mixed tensor with the matrix representation with respect to the basis $\left(\frac{\partial}{\partial s^1}, \ldots, \frac{\partial}{\partial s^{d-1}}\right)$. Henceforth, the range of Greek indices is $1, \ldots, d-1$, the range of Latin indices is $1, \ldots, d$ and the Einstein summation convention is used. Define the second fundamental form $h(\xi, \eta) = \langle L(\xi), \eta\rangle$ for any $\xi, \eta \in T_s\Sigma$ and $s \in \Sigma$, where $\langle \cdot, \cdot \rangle$ is metric g, then the formula $L^\mu_\nu = g^{\mu\rho}h_{\rho\nu}$ is valid with the notation $(g^{\mu\nu}) = \left(g_{\mu\nu}\right)^{-1}$. It is discussed in more detail, e.g. in ([7] , Chap. 3).

The eigenvalues of L are known as the principal curvatures $\kappa_1, \ldots, \kappa_{d-1}$ of Σ. Now set the constant

$\rho \equiv \left(\max_\mu \left\{\left\|\kappa_\mu\right\|_{L^\infty(\Sigma)}\right\}\right)^{-1}$, with the convention, $\rho = \infty$ if all κ_μ are equal to zero on Σ and $\rho = 0$ if at least one of κ_μ is not bounded. The mean curvatures

$$K_\mu \equiv \binom{d-1}{\mu}^{-1} \sum_{\alpha_1 < \cdots < \alpha_\mu} \kappa_{\alpha_1} \ldots \kappa_{\alpha_\mu}$$

are invariants of L (see [9]). Furthermore, mean curvatures K_μ are globally defined C^1 functions.

Consider the domain $\Omega \equiv \Sigma \times (-1, 1)$ and the mapping $\mathcal{L}_\varepsilon$ satisfying

$$\mathcal{L}_\varepsilon : \Omega \longrightarrow \mathbb{R}^d : \{(s,t) \mapsto s + \varepsilon t \boldsymbol{n}\}, \tag{1.2}$$

then define $\Omega_\varepsilon \equiv \mathcal{L}(\Omega)$. Additionally, the non-self-overlapping of Ω_ε is required, i.e.

$$\varepsilon < \rho \qquad \text{and} \qquad \mathcal{L}_\varepsilon \text{ is injective.} \tag{1.3}$$

The parameter ε is small and positive. Hence, the former is satisfied if the principal curvatures are bounded. Now discuss the latter. Based on the inverse function theorem, the self-overlapping is excluded locally because $\mathcal{L}_\varepsilon$ is a local diffeomorphism. Since the injectivity is requiered, $\mathcal{L}_\varepsilon$ is a global diffeomorphism.

The mapping $\mathcal{L}_\varepsilon$ induces the metric G, which can be written in a block form

$$G = g \circ (Id - \varepsilon t L) + \varepsilon^2 \mathrm{d}t^2,$$

the map Id is the identity on $T_s\Sigma$. Consequently,

$$|G| \equiv \det(G) = \varepsilon^2 \, |g| \, [\det(1 - \varepsilon t L)]^2 = \varepsilon^2 \, |g| \prod_{\mu=1}^{d-1} (1 - \varepsilon t \kappa_\mu)^2 = \varepsilon^2 \, |g| \left(1 + \sum_{\mu=1}^{d-1} (-\varepsilon t)^\mu \binom{d-1}{\mu} K_\mu\right)^2,$$

where $|g| \equiv \det(g)$. From the following remark, it can be seen that G is regular if the first condition in (1.3) is satisfied. Therefore, Ω_ε can be understood as the Riemannian manifold (Ω, G).

Remark. For any $s \in \Sigma$ and $t \in (-1, 1)$, the assumption $\varepsilon < \rho$ implies

$$1 + \sum_{\mu=1}^{d-1} (-\varepsilon t)^\mu \binom{d-1}{\mu} K_\mu(s) \geq 1 - \sum_{\mu=1}^{d-1} \left(\frac{\varepsilon}{\rho}\right)^\mu \binom{d-1}{\mu} = 2 - \left(1 + \frac{\varepsilon}{\rho}\right)^{d-1} \equiv c_\varepsilon^- > 0 \tag{1.4}$$

$$1 + \sum_{\mu=1}^{d-1} (-\varepsilon t)^\mu \binom{d-1}{\mu} K_\mu(s) \leq 1 + \sum_{\mu=1}^{d-1} \left(\frac{\varepsilon}{\rho}\right)^\mu \binom{d-1}{\mu} = \left(1 + \frac{\varepsilon}{\rho}\right)^{d-1} \equiv c_\varepsilon^+ < +\infty \tag{1.5}$$

Remark. $L^2(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t) \sim L^2(\Omega, f_\varepsilon(s,t)\, \mathrm{d}\Sigma \wedge \mathrm{d}t)$ or $W^{1,2}(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t) \sim W^{1,2}(\Omega, f_\varepsilon(s,t)\, \mathrm{d}\Sigma \wedge \mathrm{d}t)$ follow from the remark mentioned above, i.e. any function f in $L^2(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)$ is also in $L^2(\Omega, f_\varepsilon(s,t)\, \mathrm{d}\Sigma \wedge \mathrm{d}t)$. Denote the norm in $L^2(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)$ as $\|\cdot\|_0$, the norm in $L^2(\Omega, f_\varepsilon(s,t)\, \mathrm{d}\Sigma \wedge \mathrm{d}t)$ as $\|\cdot\|_\varepsilon$ and the norm in $L^2((-1,1))$ as $\|\cdot\|_2$.

With the writing $x^d = t$ and $\frac{\partial}{\partial x^d} = \frac{\partial}{\partial t}$ for $t \in (-1, 1)$, the metric G can be represented by the matrix

$$\left(G_{ij}\right) = \begin{pmatrix} \left(G_{\mu\nu}\right) & 0 \\ 0 & \varepsilon^2 \end{pmatrix}, \qquad G_{\mu\nu} = g_{\mu\rho} \left(\delta_\sigma^\rho - \varepsilon t L_\sigma^\rho\right) \left(\delta_\nu^\sigma - \varepsilon t L_\nu^\sigma\right).$$

Then the following relations are valid

$$C_\varepsilon^-(g^{\mu\nu}) \le (G^{\mu\nu}) \le C_\varepsilon^+(g^{\mu\nu}), \qquad C_\varepsilon^\pm = \left(1 \mp \varepsilon\rho^{-1}\right)^{-2}, \tag{1.6}$$

where the notation $(G^{\mu\nu}) = \left(G_{\mu\nu}\right)^{-1}$ is used. Now introduce the volume element

$$\mathrm{d}\Omega_\varepsilon \equiv |G|^{1/2}|g|^{-1/2}\,\mathrm{d}\Sigma\wedge\mathrm{d}t = \varepsilon\left(1+\sum_{\mu=1}^{d-1}(-\varepsilon t)^\mu\binom{d-1}{\mu}K_\mu\right)\mathrm{d}\Sigma\wedge\mathrm{d}t = \varepsilon f_\varepsilon\,\mathrm{d}\Sigma\wedge\mathrm{d}t,$$

where $\mathrm{d}\Sigma = |g|^{1/2}\,\mathrm{d}s^1\wedge\cdots\wedge\mathrm{d}s^{d-1}$

Define the transformation

$$U : L^2(\Omega_\varepsilon) \longrightarrow L^2(\Omega, f_\varepsilon(s,t)\,\mathrm{d}\Sigma\wedge\mathrm{d}t) : \left\{\tilde{v}\mapsto v = \sqrt{\varepsilon}\tilde{v}\circ\mathcal{L}_\varepsilon\right\}. \tag{1.7}$$

The operator H_ε on $L^2(\Omega, f_\varepsilon(s,t)\,\mathrm{d}\Sigma\wedge\mathrm{d}t)$ is obtained by $H_\varepsilon = U\tilde{H}_\varepsilon U^{-1}$. Furthermore, $v_\varepsilon \equiv U\tilde{v}_\varepsilon$ are eigenfunctions of H_ε and the function $a_\varepsilon \equiv a\circ\mathcal{L}_\varepsilon$ satisfies

$$(\exists C>0)\,(\forall\varepsilon\in(0,1))\,(\text{almost every }(s,t)\in\Sigma\times(-1,1))\left(\frac{1}{C}\le a_\varepsilon(s,t) < C\right). \tag{1.8}$$

The operator H_ε is introduced via the associated quadratic form Q_ε defined as

$$\begin{aligned}\tilde{Q}_\varepsilon[U^{-1}v] &= \int_{\Omega_\varepsilon} a\left|\nabla\left(U^{-1}v\right)\right|^2\mathrm{d}\Omega_\varepsilon\\ &= \int_\Omega a\circ\mathcal{L}_\varepsilon\left[\left(\frac{\partial\bar{v}}{\partial s^\mu}G^{\mu\nu}\frac{\partial v}{\partial s^\nu}\right)+\frac{1}{\varepsilon^2}\left|\frac{\partial v}{\partial t}\right|^2\right]f_\varepsilon\mathrm{d}\Sigma\wedge\mathrm{d}t\\ &= \int_\Omega a_\varepsilon\left[\left(\frac{\partial\bar{v}}{\partial s^\mu}G^{\mu\nu}\frac{\partial v}{\partial s^\nu}\right)+\frac{1}{\varepsilon^2}\left|\frac{\partial v}{\partial t}\right|^2\right]f_\varepsilon\,\mathrm{d}\Sigma\wedge\mathrm{d}t \equiv Q_\varepsilon[v],\end{aligned} \tag{1.9}$$

$$D(Q_\varepsilon) = UW^{1,2}(\Omega_\varepsilon) = W^{1,2}(\Omega, f_\varepsilon(s,t)\mathrm{d}\Sigma\wedge\mathrm{d}t).$$

The following equation is obtained

$$H_\varepsilon v_\varepsilon = \lambda_\varepsilon v_\varepsilon. \tag{1.10}$$

The quadratic form Q_ε or the sesquilinear form h_ε are uniquely associated with the operator H_ε. Consequently, the equation (1.10) can be written in the form sense, i.e. (1.10) is equivalent to

$$(v_\varepsilon\in D(H_\varepsilon))\,(\forall v\in D(h_\varepsilon))\,(h_\varepsilon(v,v_\varepsilon) = \lambda_\varepsilon(v,v_\varepsilon)_\varepsilon),$$

where $h_\varepsilon(v,v_\varepsilon) = \lambda_\varepsilon(v,v_\varepsilon)_\varepsilon$ is explicitly

$$\int_\Omega a_\varepsilon\left[\frac{\partial\bar{v}}{\partial s^\mu}G^{\mu\nu}\frac{\partial v_\varepsilon}{\partial s^\nu}+\frac{1}{\varepsilon^2}\frac{\partial\bar{v}}{\partial t}\frac{\partial v_\varepsilon}{\partial t}\right]f_\varepsilon\,\mathrm{d}\Sigma\wedge\mathrm{d}t = \lambda_\varepsilon\int_\Omega\bar{v}v_\varepsilon f_\varepsilon\,\mathrm{d}\Sigma\wedge\mathrm{d}t. \tag{1.11}$$

Then put $v = v_\varepsilon$

$$\int_\Omega a_\varepsilon\left[\frac{\partial\overline{v_\varepsilon}}{\partial s^\mu}G^{\mu\nu}\frac{\partial v_\varepsilon}{\partial s^\nu}+\frac{1}{\varepsilon^2}\left|\frac{\partial v_\varepsilon}{\partial t}\right|^2\right]f_\varepsilon\,\mathrm{d}\Sigma\wedge\mathrm{d}t = \lambda_\varepsilon\int_\Omega|v_\varepsilon|^2 f_\varepsilon\,\mathrm{d}\Sigma\wedge\mathrm{d}t. \tag{1.12}$$

1.1 Estimations and the Convergence of Eigenfunctions

Remark. Henceforth, C denotes the generic positive constant and $\left|\nabla_g f\right|_g \equiv \sqrt{\frac{\partial f}{\partial s^\mu} g^{\mu\nu} \frac{\partial f}{\partial s^\nu}}$.

Lemma 1.1.1. Eigenvalues λ_ε are bounded.

Proof. The minimax principle has the following form

$$(\lambda_\varepsilon)_k = \min_{\mathcal{L}_k} \max_{v\in\mathcal{L}_k} \frac{Q_\varepsilon[v]}{\|v\|_\varepsilon^2} = \min_{\mathcal{L}_k} \max_{v\in\mathcal{L}_k} \frac{\int_\Omega a_\varepsilon \left[\frac{\partial \overline{v}}{\partial s^\mu} G^{\mu\nu} \frac{\partial v}{\partial s^\nu} + \frac{1}{\varepsilon^2}\left|\frac{\partial v}{\partial t}\right|^2\right] f_\varepsilon \, \mathrm{d}\Sigma \wedge \mathrm{d}t}{\int_\Omega |v|^2 f_\varepsilon \, \mathrm{d}\Sigma \wedge \mathrm{d}t}, \tag{1.13}$$

where subspace $\mathcal{L}_k \subset W^{1,2}\left(\Omega, f_\varepsilon(s,t)\, \mathrm{d}\Sigma \wedge \mathrm{d}t\right)$ satisfies $\dim\mathcal{L}_k = k$. The upper estimate for (1.13) is obtained by the selecting of the subspace $\mathcal{L}_k \subset \left\{\varphi \otimes \frac{1}{\sqrt{2}} \mid \varphi \in W^{1,2}(\Sigma, \mathrm{d}\Sigma)\right\}$

$$(\lambda_\varepsilon)_k \le \min_{\tilde{\mathcal{L}}_k} \max_{\varphi\in\tilde{\mathcal{L}}_k} \frac{\int_\Omega a_\varepsilon \left[\left(\frac{\partial \overline{\varphi}}{\partial s^\mu} G^{\mu\nu} \frac{\partial \varphi}{\partial s^\nu}\right) \otimes \frac{1}{2}\right] f_\varepsilon \, \mathrm{d}\Sigma \wedge \mathrm{d}t}{\int_\Omega \left|\varphi \otimes \frac{1}{\sqrt{2}}\right|^2 f_\varepsilon \, \mathrm{d}\Sigma \wedge \mathrm{d}t}$$

$$\le C \min_{\tilde{\mathcal{L}}_k} \max_{\varphi\in\tilde{\mathcal{L}}_k} \frac{\int_\Sigma \frac{\partial \overline{\varphi}}{\partial s^\mu} g^{\mu\nu} \frac{\partial \varphi}{\partial s^\nu} \mathrm{d}\Sigma}{\int_\Sigma |\varphi|^2 \, \mathrm{d}\Sigma},$$

$\tilde{\mathcal{L}}_k \subset W^{1,2}(\Sigma, \mathrm{d}\Sigma)$ is a subspace such that $\dim\tilde{\mathcal{L}}_k = k$.

The quadratic form Q defined as

$$Q[\varphi] = \int_\Sigma \left|\nabla_g \varphi\right|_g^2 \mathrm{d}\Sigma,$$
$$D(Q) = W^{1,2}(\Sigma, \mathrm{d}\Sigma)$$

is associated with the Laplce-Beltrami operator with the Neumann boundary conditions. It follows that

$$(\lambda_\varepsilon)_k \le C\nu_k,$$

ν_k is the kth eigenvalue of the operator mentioned above. □

Remark. Norms $\|\cdot\|_\varepsilon$ and $\|\cdot\|_0$ are not distinguished in the following lemmas because they are valid for $\|\cdot\|_\varepsilon$ and $\|\cdot\|_0$ equally.

Lemma 1.1.2. The eigenfunctions v_ε of the operator H_ε comply with

$$\left\|\left|\nabla_g v_\varepsilon\right|_g\right\| \le C \tag{1.14}$$

$$\left\|\frac{\partial v_\varepsilon}{\partial t}\right\| \le C\varepsilon \tag{1.15}$$

Proof. Begin with (1.12) and consider $\|v_\varepsilon\|_\varepsilon = 1$. Based on (1.4), the following estimation is valid.

$$\int_\Omega \left[\frac{\partial \overline{v_\varepsilon}}{\partial s^\mu} g^{\mu\nu} \frac{\partial v_\varepsilon}{\partial s^\nu} + \frac{1}{\varepsilon^2}\left|\frac{\partial v_\varepsilon}{\partial t}\right|^2\right] f_\varepsilon \, \mathrm{d}\Sigma \wedge \mathrm{d}t \le C\lambda_\varepsilon \int_\Omega |v_\varepsilon|^2 f_\varepsilon \, \mathrm{d}\Sigma \wedge \mathrm{d}t$$

It means

$$\left\|\left|\nabla_g v_\varepsilon\right|_g\right\|_\varepsilon + \frac{1}{\varepsilon^2}\left\|\frac{\partial v_\varepsilon}{\partial t}\right\|_\varepsilon^2 \le C.$$

It implies (1.14), (1.15) directly. □

The identical transformation

$$V : L^2(\Omega, f_\varepsilon(s,t)\,\mathrm{d}\Sigma \wedge \mathrm{d}t) \longrightarrow L^2(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t) : \{v \mapsto v\} \tag{1.16}$$

can be defined and any function in $W^{1,2}(\Omega, f_\varepsilon(s,t)\,\mathrm{d}\Sigma \wedge \mathrm{d}t)$ can be also understood as the function in $W^{1,2}(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)$. Take an arbitrary eigenfunction v_ε of the operator H_ε and consider v_ε as the function in $W^{1,2}(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)$. Let $\{\chi_n\}_{n=0}^\infty$ be the orthonormal base in $L^2((-1,1))$ given by

$$\chi_n(t) = \begin{cases} \frac{1}{\sqrt{2}} & \text{for } n = 0, \\ \cos\left(\frac{n\pi}{2}t\right) & \text{for } n \geq 1 \text{ even}, \\ \sin\left(\frac{n\pi}{2}t\right) & \text{for } n \text{ odd}. \end{cases} \tag{1.17}$$

Then the function v_ε can be expanded into the Fourier series

$$v_\varepsilon(s,t) = \sum_{n=0}^{\infty} (\chi_n, v_\varepsilon(s,\cdot))_2\,\chi_n(t).$$

Denote $\varphi_n(s) = (\chi_n, v_\varepsilon(s,\cdot))_2$ for any $n \in \mathbb{N}_0$, also $\varphi_\varepsilon(s,t) = \varphi_0(s)\chi_0(t) = \varphi_\varepsilon(s)$ and $\phi_\varepsilon(s,t) = \sum_{n=1}^\infty \varphi_n(s)\chi_n(t)$, then the function v_ε can be written as

$$v_\varepsilon = \varphi_\varepsilon + \phi_\varepsilon. \tag{1.18}$$

Due to the orthonormality of the base $\{\chi_n\}_{n=0}^\infty$, the function ϕ_ε satisfies for any μ

$$\int_{-1}^{1} \phi_\varepsilon\,\mathrm{d}t = 0 \quad \text{and} \quad \int_{-1}^{1} \frac{\partial \phi_\varepsilon}{\partial s^\mu}\,\mathrm{d}t = 0. \tag{1.19}$$

Lemma 1.1.3. The function ϕ_ε complies with the following estimates.

$$\left\| \frac{\partial \phi_\varepsilon}{\partial t} \right\| \leq C\varepsilon \tag{1.20}$$

$$\|\ \phi_\varepsilon\ \| \leq C\varepsilon \tag{1.21}$$

Proof. The decomposition (1.18) implies

$$\left\| \frac{\partial \phi_\varepsilon}{\partial t} \right\|_0^2 = \left\| \frac{\partial v_\varepsilon}{\partial t} \right\|_0^2 \leq C\varepsilon^2,$$

also Lemma 1.1.2 is used.

Take the Neumann Laplacian $-\Delta_N^{(-1,1)}$ and associate the operator with the quadratic form

$$Q[\varphi] = \int_{-1}^{1} \left| \frac{\mathrm{d}\varphi}{\mathrm{d}t} \right|^2 \mathrm{d}t,$$

$$D(Q) = W^{1,2}((-1,1)).$$

Eigenvalues of $-\Delta_N^{(-1,1)}$ are $\mu_n = \left(\frac{n\pi}{2}\right)^2$ and corresponding eigenfunctions χ_n are defined by (1.17) for all $n \in \mathbb{N}_0$. Prove, the minimax principle for $n = 1$ can be equivalently written as

$$\mu_1 = \min_{\substack{\varphi \in W^{1,2}((-1,1)) \\ \varphi \perp \chi_0}} \frac{Q[\varphi]}{\|\varphi\|_2^2}. \tag{1.22}$$

Since any function $\varphi \in W^{1,2}((-1,1))$ can be expanded into the orthonormal base $\{\chi_n\}_{n=0}^{\infty}$ using the Fourier series $\varphi = \sum_{n=0}^{\infty} (\chi_n, \varphi)_2 \chi_n$, the relations mentioned below are obtained

$$\min_{\substack{\varphi \in W^{1,2}((-1,1)) \\ \varphi \perp \chi_0}} \frac{Q[\varphi]}{\|\varphi\|_2^2} = \min_{\substack{\phi = \sum_{n=1}^{\infty} (\chi_n, \varphi)_2 \chi_n \\ \varphi \in W^{1,2}((-1,1))}} \frac{Q[\phi]}{\|\phi\|_2^2} = \min_{\varphi \in W^{1,2}((-1,1))} \frac{\sum_{n=1}^{\infty} \mu_n \left|(\chi_n, \varphi)_2\right|^2}{\sum_{n=1}^{\infty} \left|(\chi_n, \varphi)_2\right|^2} \geq \mu_1 .$$

Set $\varphi = \chi_1$, then

$$\min_{\substack{\varphi \in W^{1,2}((-1,1)) \\ \varphi \perp \chi_0}} \frac{Q[\varphi]}{\|\varphi\|_2^2} \leq \frac{\mu_1 \|\chi_1\|_2^2}{\|\chi_1\|_2^2} = \mu_1$$

is valid. It means, the relation (1.22) is proved. Consequently, (1.22) implies for any $\varphi \in W^{1,2}((-1,1))$ satisfying $\varphi \perp \chi_0$ in $L^2((-1,1))$

$$\mu_1 \leq \frac{\int_{-1}^{1} \left|\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right|^2 \mathrm{d}t}{\int_{-1}^{1} |\varphi|^2 \, \mathrm{d}t}. \tag{1.23}$$

Finally, it can be seen that the following relations are valid.

$$\|\phi_\varepsilon\|_0^2 = \int_\Omega |\phi_\varepsilon|^2 \, \mathrm{d}\Sigma \mathrm{d}t \leq C \int_\Omega \left|\frac{\partial \phi_\varepsilon}{\partial t}\right|^2 \mathrm{d}\Sigma \mathrm{d}t = C \left\|\frac{\partial \phi_\varepsilon}{\partial t}\right\|_0^2 \leq C\varepsilon^2$$

□

Lemma 1.1.4. Families of functions $\{\varphi_\varepsilon\}_{\varepsilon>0}$, $\{\phi_\varepsilon\}_{\varepsilon>0}$ are uniformly bounded in the parameter ε in $W^{1,2}(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)$ or $W^{1,2}(\Omega, f_\varepsilon(s,t)\, \mathrm{d}\Sigma \wedge \mathrm{d}t)$.

Proof. Due to (1.18) and (1.19), it can be written

$$\|v_\varepsilon\|_0^2 = \int_\Omega |\varphi_\varepsilon|^2 \, \mathrm{d}\Sigma \mathrm{d}t + \int_\Omega |\phi_\varepsilon|^2 \, \mathrm{d}\Sigma \mathrm{d}t + 2\mathrm{Re} \int_\Sigma \overline{\varphi_\varepsilon} \int_{-1}^{1} \phi_\varepsilon \mathrm{d}t \mathrm{d}\Sigma = \|\varphi_\varepsilon\|_0^2 + \|\phi_\varepsilon\|_0^2 .$$

Consider $\|v_\varepsilon\|_\varepsilon = 1$, then $\|v_\varepsilon\|_0 \underset{\varepsilon \mapsto 0}{\longrightarrow} 1$ can be seen from (1.4) and (1.5). It is known from (1.20) that $\|\phi_\varepsilon\|_0 \underset{\varepsilon \mapsto 0}{\longrightarrow} 0$, therefore $\|\varphi_\varepsilon\|_0 \underset{\varepsilon \mapsto 0}{\longrightarrow} 1$. The following equalities are obtained using (1.18) and (1.19).

$$\begin{aligned} \left\|\left|\nabla_g v_\varepsilon\right|_g\right\|_0^2 &= \int_\Omega \frac{\partial \overline{\varphi_\varepsilon}}{\partial s^\mu} g^{\mu\nu} \frac{\partial \varphi_\varepsilon}{\partial s^\nu} \mathrm{d}\Sigma \mathrm{d}t + \int_\Omega \frac{\partial \overline{\phi_\varepsilon}}{\partial s^\mu} g^{\mu\nu} \frac{\partial \phi_\varepsilon}{\partial s^\nu} \mathrm{d}\Sigma \mathrm{d}t + 2\mathrm{Re} \int_\Sigma \frac{\partial \overline{\varphi_\varepsilon}}{\partial s^\mu} g^{\mu\nu} \int_{-1}^{1} \frac{\partial \phi_\varepsilon}{\partial s^\nu} \mathrm{d}t \mathrm{d}\Sigma \\ &= \left\|\left|\nabla_g \varphi_\varepsilon\right|_g\right\|_0^2 + \left\|\left|\nabla_g \phi_\varepsilon\right|_g\right\|_0^2 \end{aligned}$$

In addition, Lemma 1.1.3 is proven. It means, the boundedness is shown. □

Remark. Let W be a continuously embedded subspace of a Hilbert space V. Let $\{\psi_\varepsilon\}_{\varepsilon>0}$ be a sequence satisfying $\psi_\varepsilon \underset{\varepsilon \mapsto 0}{\overset{w}{\longrightarrow}} \psi$ in W. Then $\psi_\varepsilon \underset{\varepsilon \mapsto 0}{\overset{w}{\longrightarrow}} \psi$ in V.

Proof. From assumptions, it is known, $(f, \psi_\varepsilon - \psi)_W \underset{\varepsilon \mapsto 0}{\longrightarrow} 0$ for any $f \in W$. Fix the element f in W and define functionals $F \in V^*$, $\tilde{F} \in W^*$

$$F : V \longrightarrow \mathbb{C} : \{v \longmapsto F(v) = (f, v)_V\},$$

$$\tilde{F} = F|_W : W \longrightarrow \mathbb{C} : \{w \longmapsto F(w)\}.$$

According to the Riesz representation theorem, there is $u \in W$ such that $F(w) = (u, w)_W$ for any $w \in W$. Therefore, the following equalities are valid

$$(f, \psi_\varepsilon - \psi)_V = F(\psi_\varepsilon - \psi) = (u, \psi_\varepsilon - \psi)_W \underset{\varepsilon \mapsto 0}{\longrightarrow} 0.$$

$\square$

Remark. It follows from Lemma 1.1.4, there is a weakly convergent subsequence, i.e. $\phi_{\varepsilon_k} \underset{k \mapsto \infty}{\overset{w}{\longrightarrow}} \phi$ in $W^{1,2}(\Omega, f_\varepsilon(s,t)\, \mathrm{d}\Sigma \wedge \mathrm{d}t)$. Lemma 1.1.3 implies $\phi_\varepsilon \underset{k \mapsto \infty}{\longrightarrow} 0$ in $L^2(\Omega, f_\varepsilon(s,t)\, \mathrm{d}\Sigma \wedge \mathrm{d}t)$. It means, $\phi_{\varepsilon_k} \underset{k \mapsto \infty}{\overset{w}{\longrightarrow}} 0$ in $L^2(\Omega, f_\varepsilon(s,t)\, \mathrm{d}\Sigma \wedge \mathrm{d}t)$. According to the remark mentioned above, it can be obtained, $\phi_{\varepsilon_k} \underset{k \mapsto \infty}{\overset{w}{\longrightarrow}} 0$ in $W^{1,2}(\Omega, f_\varepsilon(s,t)\, \mathrm{d}\Sigma \wedge \mathrm{d}t)$.

Lemma 1.1.5. Let $\{v_\varepsilon\}_{\varepsilon > 0}$ be a sequence of eigenfunctions of the operator H_ε. Then there is a subsequence $\{v_{\varepsilon_k}\}_{k=1}^{\infty}$ satisfying $v_{\varepsilon_k} \underset{k \mapsto \infty}{\longrightarrow} \varphi_0$ in $L^2(\Omega, f_\varepsilon(s,t)\, \mathrm{d}\Sigma \wedge \mathrm{d}t)$

Proof. Since any function in $L^2(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)$ can be understood as the function in $L^2(\Omega, f_\varepsilon(s,t)\, \mathrm{d}\Sigma \wedge \mathrm{d}t)$ and vice versa, the convergence can be shown only in $L^2(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)$. Based on Lemma 1.1.4, there is a weakly convergent subsequence $\varphi_{\varepsilon_k} \underset{\varepsilon \to 0}{\overset{w}{\longrightarrow}} \varphi_0$ in $W^{1,2}(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)$. The Hilbert space $W^{1,2}(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)$ is compactly embedded in $L^2(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)$ (see [10], Thm. 9.16), therefore the weakly convergent subsequence φ_{ε_k} in $W^{1,2}(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)$ can be mapped on the strongly convergent sequence in $L^2(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)$, i.e. $\varphi_{\varepsilon_k} \underset{\varepsilon \to 0}{\longrightarrow} \varphi_0$ in $L^2(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)$. Additionally, (1.21) implies $\phi_\varepsilon \underset{\varepsilon \mapsto 0}{\longrightarrow} 0$ in $L^2(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)$, then $v_{\varepsilon_k} \underset{k \mapsto \infty}{\longrightarrow} \varphi_0$ in $L^2(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)$. $\square$

1.2 The Convergence to the Effective Model

Consider the equations (1.11), (1.18). Denote $\mathcal{D} \equiv C^\infty(\Sigma)$. Then (1.11) is simplified into

$$\int_\Omega a_\varepsilon \left[\frac{\partial \overline{\varphi}}{\partial s^\mu} G^{\mu\nu} \frac{\partial \varphi_\varepsilon}{\partial s^\nu} + \frac{\partial \overline{\varphi}}{\partial s^\mu} G^{\mu\nu} \frac{\partial \phi_\varepsilon}{\partial s^\nu} \right] f_\varepsilon \,\mathrm{d}\Sigma \wedge \mathrm{d}t = \lambda_\varepsilon \int_\Omega \overline{\varphi} v_\varepsilon f_\varepsilon \,\mathrm{d}\Sigma \wedge \mathrm{d}t \tag{1.24}$$

for all $\varphi \in \mathcal{D}$.

Lemma 1.2.1. Let a_ε be a positive function satisfying (1.8) and

$$(\exists D > 0)\,(\forall \varepsilon \in (0,1))\,(\text{almost every } (s,t) \in \Sigma \times (-1,1))\left(\left| \nabla_g a_\varepsilon \right|_g \leq D \right). \tag{1.25}$$

Then the following convergence

$$\int_\Omega a_\varepsilon \frac{\partial \overline{\varphi}}{\partial s^\mu} G^{\mu\nu} \frac{\partial \phi_\varepsilon}{\partial s^\nu} f_\varepsilon \,\mathrm{d}\Sigma \wedge \mathrm{d}t \underset{\varepsilon \mapsto 0}{\longrightarrow} 0 \tag{1.26}$$

is valid for all $\varphi \in \mathcal{D}$.

Proof. Fix μ and ν. For any μ and ν, it is obtained using the integration by parts

$$\int_\Omega a_\varepsilon \frac{\partial \overline{\varphi}}{\partial s^\mu} G^{\mu\nu} \frac{\partial \phi_\varepsilon}{\partial s^\nu} f_\varepsilon \,\mathrm{d}\Sigma \wedge \mathrm{d}t = -\int_\Omega a_\varepsilon \frac{\partial^2 \overline{\varphi}}{\partial s^\nu \partial s^\mu} G^{\mu\nu} \phi_\varepsilon f_\varepsilon \,\mathrm{d}\Sigma \wedge \mathrm{d}t - \int_\Omega \frac{\partial \overline{\varphi}}{\partial s^\mu} G^{\mu\nu} \frac{\partial a_\varepsilon}{\partial s^\nu} \phi_\varepsilon f_\varepsilon \,\mathrm{d}\Sigma \wedge \mathrm{d}t$$

$$
-\int_{\Omega} a_{\varepsilon}\frac{\partial\overline{\varphi}}{\partial s^{\mu}}\frac{\partial G^{\mu\nu}}{\partial s^{\nu}}\phi_{\varepsilon}f_{\varepsilon}\,\mathrm{d}\Sigma\wedge\mathrm{d}t-\int_{\Omega} a_{\varepsilon}\frac{\partial\overline{\varphi}}{\partial s^{\mu}}G^{\mu\nu}\frac{\partial f_{\varepsilon}}{\partial s^{\nu}}\phi_{\varepsilon}\,\mathrm{d}\Sigma\wedge\mathrm{d}t
$$

$$
-\int_{\Omega} a_{\varepsilon}\frac{\partial\overline{\varphi}}{\partial s^{\mu}}G^{\mu\nu}\frac{\partial|g|}{\partial s^{\nu}}\frac{\phi_{\varepsilon}}{|g|}f_{\varepsilon}\,\mathrm{d}\Sigma\wedge\mathrm{d}t.
$$

Based on the Cauchy–Schwarz inequality, the estimate (1.21), the assumptions and properties of the metric G, the following relations are valid.

$$
\left|\int_{\Omega} a_{\varepsilon}\frac{\partial^2\overline{\varphi}}{\partial s^{\nu}\partial s^{\mu}}G^{\mu\nu}\phi_{\varepsilon}f_{\varepsilon}\,\mathrm{d}\Sigma\wedge\mathrm{d}t\right|\le\|a_{\varepsilon}\|_{L^{\infty}(\Omega)}\left\|G^{\mu\nu}\right\|_{L^{\infty}(\Omega)}\left\|\frac{\partial^2\varphi}{\partial s^{\nu}\partial s^{\mu}}\right\|_{\varepsilon}\|\phi_{\varepsilon}\|_{\varepsilon}\underset{\varepsilon\mapsto 0}{\longrightarrow}0 \tag{1.27}
$$

$$
\left|\int_{\Omega}\frac{\partial\overline{\varphi}}{\partial s^{\mu}}G^{\mu\nu}\frac{\partial a_{\varepsilon}}{\partial s^{\nu}}\phi_{\varepsilon}f_{\varepsilon}\,\mathrm{d}\Sigma\wedge\mathrm{d}t\right|\le\sqrt{\int_{\Omega}\left|\frac{\partial\overline{\varphi}}{\partial s^{\mu}}G^{\mu\nu}\frac{\partial a_{\varepsilon}}{\partial s^{\nu}}\right|^2 f_{\varepsilon}\,\mathrm{d}\Sigma\wedge\mathrm{d}t}\;\|\phi_{\varepsilon}\|_{\varepsilon}
$$

$$
\le\sqrt{\int_{\Omega}|\nabla_G\varphi|_G^2\,|\nabla_G a_{\varepsilon}|_G^2\,f_{\varepsilon}\,\mathrm{d}\Sigma\wedge\mathrm{d}t}\;\|\phi_{\varepsilon}\|_{\varepsilon}\le\left\||\nabla_G a_{\varepsilon}|_G\right\|_{L^{\infty}(\Omega)}\left\||\nabla_G\varphi|_G\right\|_{\varepsilon}\|\phi_{\varepsilon}\|_{\varepsilon} \tag{1.28}
$$

$$
\le C\left\|\left|\nabla_g a_{\varepsilon}\right|_g\right\|_{L^{\infty}(\Omega)}\left\|\left|\nabla_g\varphi\right|_g\right\|_{\varepsilon}\|\phi_{\varepsilon}\|_{\varepsilon}\underset{\varepsilon\mapsto 0}{\longrightarrow}0
$$

$$
\left|\int_{\Omega} a_{\varepsilon}\frac{\partial\overline{\varphi}}{\partial s^{\mu}}\frac{\partial G^{\mu\nu}}{\partial s^{\nu}}\phi_{\varepsilon}f_{\varepsilon}\,\mathrm{d}\Sigma\wedge\mathrm{d}t\right|\le\|a_{\varepsilon}\|_{L^{\infty}(\Omega)}\left\|\frac{\partial G^{\mu\nu}}{\partial s^{\nu}}\right\|_{L^{\infty}(\Omega)}\left\|\frac{\partial\varphi}{\partial s^{\mu}}\right\|_{\varepsilon}\|\phi_{\varepsilon}\|_{\varepsilon}\underset{\varepsilon\mapsto 0}{\longrightarrow}0 \tag{1.29}
$$

$$
\left|\int_{\Omega} a_{\varepsilon}\frac{\partial\overline{\varphi}}{\partial s^{\mu}}G^{\mu\nu}\frac{\partial f_{\varepsilon}}{\partial s^{\nu}}\phi_{\varepsilon}\,\mathrm{d}\Sigma\wedge\mathrm{d}t\right|\le\|a_{\varepsilon}\|_{L^{\infty}(\Omega)}\left\|G^{\mu\nu}\right\|_{L^{\infty}(\Omega)}\left\|\frac{\partial f_{\varepsilon}}{\partial s^{\nu}}\right\|_{L^{\infty}(\Omega)}\left\|\frac{\partial\varphi}{\partial s^{\mu}}\right\|_{0}\|\phi_{\varepsilon}\|_{0}\underset{\varepsilon\mapsto 0}{\longrightarrow}0 \tag{1.30}
$$

$$
\left|\int_{\Omega} a_{\varepsilon}\frac{\partial\overline{\varphi}}{\partial s^{\mu}}G^{\mu\nu}\frac{\partial|g|}{\partial s^{\nu}}\frac{\phi_{\varepsilon}}{|g|}f_{\varepsilon}\,\mathrm{d}\Sigma\wedge\mathrm{d}t\right|\le\frac{\|a_{\varepsilon}\|_{L^{\infty}(\Omega)}}{\|\,|g|\,\|_{L^{\infty}(\Omega)}}\left\|G^{\mu\nu}\right\|_{L^{\infty}(\Omega)}\left\|\frac{\partial|g|}{\partial s^{\nu}}\right\|_{L^{\infty}(\Omega)}\left\|\frac{\partial\varphi}{\partial s^{\mu}}\right\|_{\varepsilon}\|\phi_{\varepsilon}\|_{\varepsilon}\underset{\varepsilon\mapsto 0}{\longrightarrow}0 \tag{1.31}
$$

□

Remark. Furthermore, it can be seen from (1.27), (1.28), (1.29), (1.30) and (1.31),

$$
\left|\int_{\Omega} a_{\varepsilon}\frac{\partial\overline{\varphi}}{\partial s^{\mu}}G^{\mu\nu}\frac{\partial\phi_{\varepsilon}}{\partial s^{\nu}}f_{\varepsilon}\,\mathrm{d}\Sigma\wedge\mathrm{d}t\right|\le C\,\|\phi_{\varepsilon}\|\,. \tag{1.32}
$$

Define functions $\langle a_{\varepsilon}\rangle$, $\overline{a}$

$$
\langle a_{\varepsilon}\rangle=\int_{-1}^{1}a_{\varepsilon}f_{\varepsilon}\,\mathrm{d}t, \tag{1.33}
$$

$$
\overline{a}=\lim_{\varepsilon\mapsto 0}\langle a_{\varepsilon}\rangle\,. \tag{1.34}
$$

Lemma 1.2.2. Let functions $\langle a_{\varepsilon}\rangle$, $\overline{a}$ be defined by (1.33), (1.34) and a_{ε} is a positive function complying with (1.8) and (1.25). Let the following conditions (denoted as (CA))

1. $(\forall s\in\Sigma)\left(\langle a_{\varepsilon}\rangle\underset{\varepsilon\mapsto 0}{\longrightarrow}\overline{a}\right)$,

2. $\operatorname*{ess\,sup}_{s\in\Sigma}|\langle a_{\varepsilon}\rangle-\overline{a}|\underset{\varepsilon\mapsto 0}{\longrightarrow}0$

be satisfied. Then the equation (1.24) converges to

$$\int_\Sigma \overline{a} \frac{\partial \overline{\varphi}}{\partial s^\mu} g^{\mu\nu} \frac{\partial \varphi_0}{\partial s^\nu} \, \mathrm{d}\Sigma = 2\lambda_0 \int_\Sigma \overline{\varphi} \varphi_0 \, \mathrm{d}\Sigma \tag{1.35}$$

as $\varepsilon \longrightarrow 0$, for any $\varphi \in \mathcal{D}$.

Proof. Let $\{v_{\varepsilon_k}\}_{k=1}^{\infty}$ be the subsequence from Lemma 1.1.5. Take the equation (1.11). Based on the Cauchy-Schwarz inequality and the convergence of $\{v_{\varepsilon_k}\}_{k=1}^{\infty}$ in $L^2(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)$, the following equality is valid.

$$\lim_{k \mapsto \infty} \int_\Omega \overline{\varphi} v_{\varepsilon_k} f_{\varepsilon_k} \, \mathrm{d}\Sigma \wedge \mathrm{d}t = \lim_{k \mapsto \infty} \int_\Omega \overline{\varphi} v_{\varepsilon_k} \left(1 + \sum_{\mu=1}^{d-1} (-\varepsilon_k t)^\mu \binom{d-1}{\mu} K_\mu \right) \mathrm{d}\Sigma \wedge \mathrm{d}t = 2 \int_\Sigma \overline{\varphi} \varphi_0 \, \mathrm{d}\Sigma \tag{1.36}$$

Moreover, it is obtained,

$$\lim_{k \mapsto \infty} \int_\Omega a_{\varepsilon_k} \frac{\partial \overline{\varphi}}{\partial s^\mu} G^{\mu\nu} \frac{\partial \varphi_{\varepsilon_k}}{\partial s^\nu} f_{\varepsilon_k} \, \mathrm{d}\Sigma \wedge \mathrm{d}t = \lim_{k \mapsto \infty} \int_\Sigma \langle a_{\varepsilon_k} \rangle \frac{\partial \overline{\varphi}}{\partial s^\mu} G^{\mu\nu} \frac{\partial \varphi_{\varepsilon_k}}{\partial s^\nu} \, \mathrm{d}\Sigma$$

$$= \lim_{k \mapsto \infty} \left[\int_\Sigma \left(\langle a_{\varepsilon_k} \rangle - \overline{a} \right) \frac{\partial \overline{\varphi}}{\partial s^\mu} G^{\mu\nu} \frac{\partial \varphi_{\varepsilon_k}}{\partial s^\nu} \, \mathrm{d}\Sigma + \int_\Sigma \overline{a} \frac{\partial \overline{\varphi}}{\partial s^\mu} G^{\mu\nu} \frac{\partial \varphi_{\varepsilon_k}}{\partial s^\nu} \, \mathrm{d}\Sigma \right] = \lim_{k \mapsto \infty} \int_\Sigma \overline{a} \frac{\partial \overline{\varphi}}{\partial s^\mu} G^{\mu\nu} \frac{\partial \varphi_{\varepsilon_k}}{\partial s^\nu} \, \mathrm{d}\Sigma$$

$$= \lim_{k \mapsto \infty} \left[\int_\Sigma \overline{a} \frac{\partial \overline{\varphi}}{\partial s^\mu} \left(G^{\mu\nu} - C_\varepsilon^- g^{\mu\nu} \right) \frac{\partial \varphi_{\varepsilon_k}}{\partial s^\nu} \, \mathrm{d}\Sigma + C_\varepsilon^- \int_\Sigma \overline{a} \frac{\partial \overline{\varphi}}{\partial s^\mu} g^{\mu\nu} \frac{\partial \varphi_{\varepsilon_k}}{\partial s^\nu} \, \mathrm{d}\Sigma \right]$$

$$= \lim_{k \mapsto \infty} \int_\Sigma \overline{a} \frac{\partial \overline{\varphi}}{\partial s^\mu} g^{\mu\nu} \frac{\partial \varphi_{\varepsilon_k}}{\partial s^\nu} \, \mathrm{d}\Sigma = \int_\Sigma \overline{a} \frac{\partial \overline{\varphi}}{\partial s^\mu} g^{\mu\nu} \frac{\partial \varphi_0}{\partial s^\nu} \, \mathrm{d}\Sigma.$$

The third equality follows from the Cauchy-Schwarz inequality, the assumptions and the properties of the metric G. The fifth equality can be shown similarly, however the estimate $\left| G^{\mu\nu} - C_\varepsilon^- g^{\mu\nu} \right| \leq \left(C_\varepsilon^+ - C_\varepsilon^- \right) |g^{\mu\nu}|$ for any μ, ν has to be used. This estimate can be seen from (1.6). The subsequence $\{\varphi_{\varepsilon_k}\}_{k=1}^{\infty}$ is weakly convergent in $W^{1,2}(\Sigma, \mathrm{d}\Sigma)$, therefore the last equality is satisfied.

Generally, the sequence of eigenvalues $\{\lambda_{\varepsilon_k}\}_{k=1}^{\infty}$ does not converge. Therefore, denote $\lambda_{\overline{0}} = \limsup_{k \mapsto \infty} \lambda_{\varepsilon_k}$ and $\lambda_{\underline{0}} = \liminf_{k \mapsto \infty} \lambda_{\varepsilon_k}$. The limit superior and the limit inferior as $k \longrightarrow \infty$ of the equation (1.24) for the subsequence $\{v_{\varepsilon_k}\}_{k=1}^{\infty}$ leads to equations

$$\int_\Sigma \overline{a} \frac{\partial \overline{\varphi}}{\partial s^\mu} g^{\mu\nu} \frac{\partial \varphi_0}{\partial s^\nu} \, \mathrm{d}\Sigma = 2\lambda_i \int_\Sigma \overline{\varphi} \varphi_0 \, \mathrm{d}\Sigma,$$

where $i \in \left\{ \overline{0}, \underline{0} \right\}$. Limit values $\lambda_{\overline{0}}$ and $\lambda_{\underline{0}}$ satisfy the same equation. Hence, $\lambda_{\overline{0}}$ and $\lambda_{\underline{0}}$ are equal also $\lim_{k \mapsto \infty} \lambda_{\varepsilon_k} = \lambda_0$. Consequently,

$$\int_\Sigma \overline{a} \frac{\partial \overline{\varphi}}{\partial s^\mu} g^{\mu\nu} \frac{\partial \varphi_0}{\partial s^\nu} \, \mathrm{d}\Sigma = 2\lambda_0 \int_\Sigma \overline{\varphi} \varphi_0 \, \mathrm{d}\Sigma.$$

The equation in the same form can be derived for any convergent subsequence $\{v_{\varepsilon_l}\}_{l=1}^{\infty}$ of $\{v_\varepsilon\}_{\varepsilon > 0}$, i.e. there is $\tilde{\varphi}_0$ such that $v_{\varepsilon_l} \underset{l \mapsto \infty}{\longrightarrow} \tilde{\varphi}_0$ in $L^2(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)$. Denote $\lim_{l \mapsto \infty} \lambda_{\varepsilon_l} = \tilde{\lambda}_0$. Hence, $v_\varepsilon \underset{\varepsilon \mapsto 0}{\longrightarrow} \varphi_0$ in $L^2(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)$ and $\lambda_\varepsilon \underset{\varepsilon \mapsto 0}{\longrightarrow} \lambda_0$ follow. It directly implies the proposition of this lemma □

Remark. The function φ_0 is nonzero because $\|v_\varepsilon\|_\varepsilon = 1$, i.e. $\|v_\varepsilon\|_0 \underset{\varepsilon \mapsto 0}{\longrightarrow} 1$, is considered and also $v_\varepsilon \underset{\varepsilon \mapsto 0}{\longrightarrow} \varphi_0$ in $L^2(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)$. The solving of the spectral problem makes sense.

Remark. The latter condition of (CA) can be strengthened by the considering of the uniform convergence, then conditions (CA) are reduced to

1. $\langle a_\varepsilon \rangle \underset{\varepsilon\mapsto 0}{\overset{\Sigma}{\rightrightarrows}} \overline{a}.$

Using the integration by parts, the following equality can be obtained from the equation (1.35).

$$\int_\Sigma \overline{\varphi}\left[-|g|^{-1/2}\frac{\partial}{\partial s^\mu}\left(|g|^{1/2}\,\overline{a}g^{\mu\nu}\frac{\partial\varphi_0}{\partial s^\nu}\right)-2\lambda_0\varphi_0\right]\,\mathrm{d}\Sigma$$

Therefore, the operator h_0 on $L^2(\Sigma, \mathrm{d}\Sigma)$ such that satisfies the correspondence

$$\lambda_0\in\sigma(h_0)\qquad\longleftrightarrow\qquad -|g|^{-1/2}\frac{\partial}{\partial s^\mu}\left(|g|^{1/2}\,\frac{\overline{a}}{2}g^{\mu\nu}\frac{\partial\varphi_0}{\partial s^\nu}\right)=\lambda_0\psi\ \text{ in }\ \Sigma$$

is introduced by the associated quadratic form q_0, which is defined by

$$q_0[\varphi]=\int_\Sigma\frac{\overline{a}}{2}\frac{\partial\overline{\varphi}}{\partial s^\mu}g^{\mu\nu}\frac{\partial\varphi}{\partial s^\nu}\,\mathrm{d}\Sigma=\int_\Sigma\frac{\overline{a}}{2}\left|\nabla_g\varphi\right|_g^2\mathrm{d}\Sigma,$$

$$D(q_0)=W^{1,2}(\Sigma,\mathrm{d}\Sigma),$$

where the function $\overline{a}$ complies with (1.34). The respective sesquilinear form is $\tilde{h}_0$. It is also known, the subspace $\mathcal{D}$ is dense in $W^{1,2}(\Sigma, \mathrm{d}\Sigma)$, i.e. $\overline{\mathcal{D}}^{\|\cdot\|_{W^{1,2}(\Sigma,\mathrm{d}\Sigma)}} = W^{1,2}((\Sigma, \mathrm{d}\Sigma))$ (see in [10], Thm. 9.2). It implies

$$\tilde{h}_0(\varphi,\varphi_0)=\int_\Sigma\frac{\overline{a}}{2}\frac{\partial\overline{\varphi}}{\partial s^\mu}g^{\mu\nu}\frac{\partial\varphi_0}{\partial s^\nu}\,\mathrm{d}\Sigma=\lambda_0\int_\Sigma\overline{\varphi}\varphi_0\,\mathrm{d}\Sigma=\lambda_0\,(\varphi,\varphi_0)_{L^2(\Sigma,\mathrm{d}\Sigma)},\tag{1.37}$$

for any $\varphi \in W^{1,2}(\Sigma, \mathrm{d}\Sigma)$ and $\varphi_0 \in D(h_0)$. It is the equation $h_0\varphi_0 = \lambda_0\varphi_0$ written in the weak sense.

Now the extension of the operator h_0 on $L^2(\Omega, \mathrm{d}\Sigma\wedge \mathrm{d}t)$ is the aim. Let $\{\chi_n\}_{n=0}^\infty$ be the orthonormal base in $L^2((-1, 1))$ defined by (1.17), then any $\psi \in L^2(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)$ can be expanded into the Fourier series

$$\psi(s,t)=\sum_{n=0}^\infty\psi_n(s)\chi_n(t)=\psi_0(s)\chi_0(t)+\sum_{n=1}^\infty\psi_n(s)\chi_n(t),$$

where $\psi_n(s) = (\chi_n, \psi(s, .))_2$ for any $n \in \mathbb{N}_0$. Introduce the subspace $\mathcal{H}_0 \subset L^2(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)$

$$\mathcal{H}_0=\left\{\varphi\otimes\chi_0\;\middle|\;\varphi\in L^2(\Sigma,\mathrm{d}\Sigma)\right\}.\tag{1.38}$$

Lemma 1.2.3. The Hilbert space $L^2(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)$ satisfies the orthogonal decomposition

$$L^2(\Omega)=\mathcal{H}_0\oplus\mathcal{H}_0^\perp,$$

where $\mathcal{H}_0$ complies with (1.38).

Proof. Define the operator P_0

$$P_0:L^2(\Omega,\mathrm{d}\Sigma\wedge\mathrm{d}t)\to\mathcal{H}_0:\left\{\psi\mapsto\psi_0\otimes\chi_0=\left(\int_{-1}^1\psi\,\mathrm{d}t\right)\otimes\frac{1}{2}\right\},\tag{1.39}$$

where $\psi_0(s) = (\chi_0, \psi(s, .))_2$ for any $s \in \Sigma$. It is sufficient to show, the operator P_0 is an orthogonal projection. Since the following relation

$$P_0^2\psi=\left(\int_{-1}^1\left(\left(\int_{-1}^1\psi\,\mathrm{d}t\right)\otimes\frac{1}{2}\right)\mathrm{d}t\right)\otimes\frac{1}{2}=\left(\int_{-1}^1\psi\,\mathrm{d}t\right)\otimes\frac{1}{2}=P_0\psi$$

is valid, P_0 is the projection. The operator P_0 is bounded and also $(\psi, P_0\phi)_0 = (P_0\psi, \phi)_0$ for all ψ, $\phi \in L^2(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)$, which can be seen from

$$(\psi, P_0\phi)_0 = \int_\Omega \overline{\psi}\left(\phi_0 \otimes \frac{1}{\sqrt{2}}\right)\mathrm{d}\Sigma \wedge \mathrm{d}t = \int_\Sigma \overline{\psi}_0\phi_0\,\mathrm{d}\Sigma = \int_\Omega \left(\overline{\psi}_0 \otimes \frac{1}{\sqrt{2}}\right)\phi\,\mathrm{d}\Sigma \wedge \mathrm{d}t = (P_0\psi, \phi)_0.$$

Therefore, the projection P_0 is orthogonal.

□

Lemma 1.2.4. Let the map π be defined by

$$\pi : \mathcal{H}_0 \longrightarrow L^2(\Sigma, \mathrm{d}\Sigma) : \{\varphi \otimes \chi_0 \longmapsto \varphi\}. \tag{1.40}$$

Then π is an isometric isomorfism.

Proof. It follows from the definition (1.40), π is an isomorfism. Furthermore, π satisfies

$$\|\varphi \otimes \chi_0\|_0^2 = \int_\Omega \left|\varphi \otimes \frac{1}{\sqrt{2}}\right|^2 \mathrm{d}\Sigma \wedge \mathrm{d}t = \int_\Sigma |\varphi|^2\,\mathrm{d}\Sigma = \|\varphi\|^2_{L^2(\Sigma,\mathrm{d}\Sigma)} = \|\pi(\varphi \otimes \chi_0)\|_{L^2(\Sigma,\mathrm{d}\Sigma)}$$

for any $\varphi \in L^2(\Sigma, \mathrm{d}\Sigma)$. It means, the map π is isometric. □

The operator $\tilde{H}_0$ on $\mathcal{H}_0$ is introduced via $\tilde{H}_0 = \pi^{-1}h_0\pi$. The associated quadratic form $\tilde{Q}_0$ complies with

$$\tilde{Q}_0[\varphi \otimes \chi_0] = \int_\Omega \frac{\overline{a}}{2}\left\|\nabla_g\varphi\right|_g \otimes \chi_0\Big|^2 \mathrm{d}\Sigma \wedge \mathrm{d}t = q_0[\varphi],$$

$$D(\tilde{Q}_0) = \left\{\psi \otimes \chi_0 \,\middle|\, \psi \in W^{1,2}(\Sigma, \mathrm{d}\Sigma)\right\}.$$

The operator $H_0 = \tilde{H}_0 \oplus 0^\perp$ is the extension of $\tilde{H}_0$ on the whole Hilbert space $L^2(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)$, $0^\perp$ denotes the zero operator on $\mathcal{H}_0^\perp$. Therefore, the associated quadratic form Q_0 satisfies

$$Q_0[\psi] = \int_\Omega \frac{\overline{a}}{2}\left\|\nabla_g(P_0\psi)\right|_g \otimes \chi_0\Big|^2 \mathrm{d}\Sigma \wedge \mathrm{d}t = \tilde{Q}_0[P_0\psi \otimes \chi_0] = q_0[P_0\psi], \tag{1.41}$$

$$D(Q_0) = W^{1,2}(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t).$$

Theorem 1.2.5. Let the operator H_0 be defined by (1.41). The same assumptions as in Lemma 1.2.2 are required. Let v_ε be an eigenfunction and λ_ε the respective eigenvalue of H_ε, which is defined by (1.9). Then relations

$$v_\varepsilon \xrightarrow[\varepsilon\mapsto 0]{L^2(\Omega,\mathrm{d}\Sigma\wedge\mathrm{d}t)} \varphi_0 \qquad \text{and} \qquad \lambda_\varepsilon \underset{\varepsilon\mapsto 0}{\longrightarrow} \lambda_0$$

are satisfied for the eigenfunction φ_0 of H_0 and the respective eigenvalue λ_0.

Remark. Define the function a_ε by

$$a_\varepsilon(s,t) = \begin{cases} a_+ & t \in (0,1), \\ a_- & t \in (-1,0). \end{cases}$$

Using the Lebesgue theorem, the equality

$$\overline{a} = a_+ + a_-$$

is obtained and it is the same result as (2) (see [17]).

Chapter 2

The Resolvent Convergence

Definiton 2.0.1. Let A and A_k be self-adjoint operators on a Hilbert space $\mathcal{H}$ for any $k \in \mathbb{N}$. Then $\{A_k\}_{k=1}^{\infty}$ is said to converge to A in the **norm-resolvent sense** if

$$\lim_{k \mapsto \infty} \|R_k(z) - R(z)\|_{\mathcal{H} \to \mathcal{H}} = 0$$

for all $z \in \mathbb{C}$ with $\operatorname{Im} z \neq 0$. $\{A_k\}_{k=1}^{\infty}$ is said to converge to A in the **strong resolvent sense** if

$$\lim_{k \mapsto \infty} \|[R_k(z) - R(z)]F\|_{\mathcal{H}} = 0$$

for all $F \in \mathcal{H}$ and $z \in \mathbb{C}$ with $\operatorname{Im} z \neq 0$. R_k and R are resolvent operators of A_k and A.

Remark. Henceforth, the norm of operators $\|\cdot\|_{L^2(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t) \to L^2(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)}$ is denoted by $\|\cdot\|_{0 \to 0}$

2.1 The Strong Resolvent Convergence

R_ε and R_0 are resolvent operators of H_ε and H_0. R_0 complies with $R_0 = \tilde{R}_0 \oplus 0^\perp$, where $\tilde{R}_0$ is the resolvent operator of $\tilde{H}_0$, $\rho(H_0) = \rho(\tilde{H}_0)$ follows from the definition of H_0. Hence, the decomposition of R_0 makes sense on the whole $\rho(H_0)$. Since the resolvent operators R_ε and R_0 are on different spaces, the unitary transformation

$$U_\varepsilon : L^2(\Omega, f_\varepsilon \, \mathrm{d}\Sigma \wedge \mathrm{d}t) \longrightarrow L^2(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t) : \left\{ v \mapsto v f_\varepsilon^{1/2} \right\}$$

has to be introduced and the notation $R_\varepsilon^U = U_\varepsilon R_\varepsilon U_\varepsilon^{-1}$ will be used.

Lemma 2.1.1. Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$, then its resolvent operator R is self-adjoint for all $z \in \rho(A) \cap \mathbb{R}$.

Proof. There are unique F, $G \in \operatorname{Dom}(A)$ for any ϕ, $\psi \in \mathcal{H}$ such that equations mentioned below are valid for any $z \in \rho(A) \cap \mathbb{R}$

$$\phi = (A - z)F, \tag{2.1}$$

$$\psi = (A - z)G. \tag{2.2}$$

Then equalities

$$(\phi, R(z)\psi)_{\mathcal{H}} = ((A - z)F, G)_{\mathcal{H}} = (F, (A - z)G)_{\mathcal{H}} = (R(z)\phi, \psi)_{\mathcal{H}}$$

are satisfied for all ϕ, $\psi \in \mathcal{H}$ and for all $z \in \rho(A) \cap \mathbb{R}$. □

Theorem 2.1.2. Let the same assumptions as in Lemma 1.2.2 be considered. Then H_ε converges to H_0 in the strong resolvent sense.

Proof. Define $\{\psi_\varepsilon\}_{\varepsilon>0}$ in $L^2(\Omega, f_\varepsilon\,\mathrm{d}\Sigma\wedge\mathrm{d}t)$ and ψ_0 in $L^2(\Omega,\mathrm{d}\Sigma\wedge\mathrm{d}t)$ by

$$\psi_\varepsilon = R_\varepsilon(z)U_\varepsilon^{-1}F,$$

$$\psi = R_0(z)P_0F,$$

for all $z\in\mathbb{C}$ with $\operatorname{Im} z\neq 0$ or $\operatorname{Re} z<0$ and all $F\in L^2(\Omega,\mathrm{d}\Sigma\wedge\mathrm{d}t)$. It leads to $\left\|R_\varepsilon^U(z)F - R_0(z)P_0F\right\|_0 = \|U_\varepsilon\psi_\varepsilon-\psi_0\|_0$. Due to properties of R_0, the equality $R_0(z)P_0 = \tilde{R}_0(z)\oplus 0^\perp = R_0(z)$ is valid. It can be seen, the same estimates as in Chapter 1 are satisfied for ψ_ε. Then there is a subsequence $\{\psi_{\varepsilon_k}\}_{k=1}^\infty$ with the same properties as in Lemma 1.1.5 and the limit behaviour of the equation $(H_{\varepsilon_k}-z)\,\psi_{\varepsilon_k} = U_{\varepsilon_k}^{-1}F$ in the form sense is studied for an arbitrary $z<0$,

$$\lim_{k\mapsto\infty}\left(\int_\Omega a_{\varepsilon_k}\frac{\partial\overline{\varphi}}{\partial s^\mu}G^{\mu\nu}\frac{\partial\psi_{\varepsilon_k}}{\partial s^\nu}f_{\varepsilon_k}\,\mathrm{d}\Sigma\wedge\mathrm{d}t - z\int_\Omega\overline{\varphi}\psi_{\varepsilon_k}f_{\varepsilon_k}\,\mathrm{d}\Sigma\wedge\mathrm{d}t\right) = \lim_{k\mapsto\infty}\int_\Omega\overline{\varphi}F f_{\varepsilon_k}^{1/2}\,\mathrm{d}\Sigma\wedge\mathrm{d}t,$$

where $\varphi\in\mathcal{D}$. Now the left side is adjusted in the same way as in Lemma 1.2.2.

$$\int_\Sigma\overline{a}\frac{\partial\overline{\varphi}}{\partial s^\mu}g^{\mu\nu}\frac{\partial\psi}{\partial s^\nu}\,\mathrm{d}\Sigma - 2z\int_\Sigma\overline{\varphi}\psi\,\mathrm{d}\Sigma = \lim_{k\mapsto\infty}\int_\Omega\left(f_{\varepsilon_k}^{1/2}-\left(c_{\varepsilon_k}^-\right)^{1/2}\right)\overline{\varphi}F\,\mathrm{d}\Sigma\wedge\mathrm{d}t + \lim_{k\mapsto\infty}\left(c_{\varepsilon_k}^-\right)^{1/2}\int_\Omega\overline{\varphi}F\,\mathrm{d}\Sigma\wedge\mathrm{d}t$$

Based on the Cauchy-Schwarz inequality and relations (1.5), (1.4), the following equation is obtained.

$$\int_\Sigma\overline{a}\frac{\partial\overline{\varphi}}{\partial s^\mu}g^{\mu\nu}\frac{\partial\psi}{\partial s^\nu}\,\mathrm{d}\Sigma - 2z\int_\Sigma\overline{\varphi}\psi\,\mathrm{d}\Sigma = 2\int_\Sigma\overline{\varphi}P_0F\,\mathrm{d}\Sigma$$

Analogously as in Chapter 1, it can be shown that it is the equation $(H_0-z)\,\psi = P_0F$ in the form sense. The convergence $\psi_\varepsilon\underset{\varepsilon\to0}{\longrightarrow}\psi$ in $L^2(\Omega,\mathrm{d}\Sigma\wedge\mathrm{d}t)$ can be proven as in Lemma 1.2.2. Using (1.5) and (1.4), the convergence $U_\varepsilon\psi_\varepsilon\underset{\varepsilon\to0}{\longrightarrow}\psi$ in $L^2(\Omega,\mathrm{d}\Sigma\wedge\mathrm{d}t)$ is gained, therefore

$$\left\|\left[R_\varepsilon^U(z)-R_0(z)\right]F\right\|_0\underset{\varepsilon\mapsto0}{\longrightarrow}0 \tag{2.3}$$

for any $F\in L^2(\Omega,\mathrm{d}\Sigma\wedge\mathrm{d}t)$ and any $z<0$

The aim is to derive the convergence (2.3) for all z with $\operatorname{Im} z\neq0$. Following equations are obtained by the first resolvent identity (see [5], Thm. 5.13.)

$$R_\varepsilon^U(z)\left[I-(z+1)R_\varepsilon^U(-1)\right] = R_\varepsilon^U(-1), \tag{2.4}$$

$$R_0(z)\left[I-(z+1)R_0(-1)\right] = R_0(-1), \tag{2.5}$$

where $z\in\mathbb{C}$ with $\operatorname{Im} z\neq0$. The operator $A_\varepsilon = \left[I-(z+1)R_\varepsilon^U(-1)\right]$ is injective if and only if the equality $(z+1)R_\varepsilon^U(-1)\psi=\psi$ is valid only for $\psi=0$. It means, the equation

$$R_\varepsilon^U(-1)\psi = \frac{1}{z+1}\psi$$

is studied. It is known from Lemma 2.1.1 that $R_\varepsilon^U(-1)$ is self-adjoint and its spectrum is real. It implies $\frac{1}{z+1}\in\rho\left(R_\varepsilon^U(-1)\right)$, therefore there is the bounded inverse operator A_ε^{-1}. The existence and the boundedness of $A^{-1} = [I-(z+1)R_0(-1)]^{-1}$ would be proven analogously.

Then it can be written

$$R_\varepsilon^U(z) - R_0(z) = A_\varepsilon^{-1} R_\varepsilon^U(-1) - A^{-1} R_0(-1) = \left(A_\varepsilon^{-1} - A^{-1}\right) R_\varepsilon^U(-1) - A^{-1}\left(R_0(-1) - R_\varepsilon^U(-1)\right).$$

According to the triangle inequality, the following estimate is gained.

$$\left\|R_\varepsilon^U(z) - R_0(z)\right\|_{0\to 0} \le \left\|\left(A_\varepsilon^{-1} - A^{-1}\right) R_\varepsilon^U(-1) F\right\|_0 + \left\|A^{-1}\right\|_{0\to 0} \left\|\left(R_0(-1) - R_\varepsilon^U(-1)\right) F\right\|_0$$

Additionally, the equality mentioned below is satisfied.

$$A_\varepsilon^{-1} - A^{-1} = \frac{1}{z+1}\left[\frac{1}{z+1} - R_\varepsilon^U(-1)\right]^{-1} \left(R_\varepsilon^U(-1) - R_0(-1)\right)\left[\frac{1}{z+1} - R_0(-1)\right]^{-1}$$

Operators $B_\varepsilon = \left[\frac{1}{z+1} - R_\varepsilon^U(-1)\right]^{-1}$ and $B = \left[\frac{1}{z+1} - R_0(-1)\right]^{-1}$ are bounded, therefore $G = BR_\varepsilon^U(-1)F$ is function in $L^2(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)$. Consequently,

$$\left\|\left[R_\varepsilon^U(z) - R_0(z)\right] F\right\|_0 \le C(z)\left[\left\|\left(R_0(-1) - R_\varepsilon^U(-1)\right) G\right\|_0 + \left\|\left(R_0(-1) - R_\varepsilon^U(-1)\right) F\right\|_0\right] \underset{\varepsilon\mapsto 0}{\longrightarrow} 0$$

for any $F \in L^2(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)$ and z with $\operatorname{Im} z \neq 0$. □

2.2 The Norm-Resolvent Convergence

Lemma 2.2.1 (see [11], Lemma A.1)**.** Let $\{R_s\}_{s\in\mathbb{R}}$ be a family of bounded operators on a Hilbert space $\mathcal{H}$ and let R be a compact operator in $\mathcal{H}$. Suppose that

$$\left.\begin{array}{l} s_k \underset{k\mapsto\infty}{\longrightarrow} \infty \\ f_k \underset{k\mapsto\infty}{\longrightarrow} f \ \text{ in } \ \mathcal{H} \\ \forall k \in \mathbb{N},\ \|f_k\|_{\mathcal{H}} = 1 \end{array}\right\} \Longrightarrow R_{s_k} f_k \underset{k\mapsto\infty}{\longrightarrow} Rf \ \text{ in } \ \mathcal{H}, \tag{2.6}$$

for all $\{s_k\}_{k=1}^\infty \subset \mathbb{R}$ and $\{f_k\}_{k=1}^\infty \subset \mathcal{H}$. Then $\{R_s\}_{s\in\mathbb{R}}$ converges to R uniformly, i.e.,

$$\lim_{s\mapsto\infty} \|R_s - R\|_{\mathcal{H}\to\mathcal{H}} = 0. \tag{2.7}$$

Proof. Lemma will be proven by contradiction. Assume that (2.6) is violated and (2.7) holds. It means, there is $K > 0$, a sequence $\{s_k\}_{k=1}^\infty \subset \mathbb{R}$ such that $s_k \underset{k\mapsto\infty}{\longrightarrow} \infty$, a sequence $\{f_k\}_{k=1}^\infty \subset \mathcal{H}$ satisfying $\|f_k\|_{\mathcal{H}} = 1$ and

$$(\forall k \in \mathbb{N})\left(\left\|R_{s_k} f_k - Rf_k\right\|_{\mathcal{H}} \ge K > 0\right) \tag{2.8}$$

for all $n \in \mathbb{N}$. Since the sequence $\{f_k\}_{k=1}^\infty$ is bounded, there is a $\{f_{k_m}\}_{m=1}^\infty$ complying with $f_{k_m} \overset{w}{\underset{m\mapsto\infty}{\longrightarrow}} f$ in $\mathcal{H}$. The compactness of R implies

$$\lim_{m\mapsto\infty} \left\|Rf_{k_m} - Rf\right\|_{\mathcal{H}} = 0. \tag{2.9}$$

Furthermore, (2.8), (2.9) yield

$$0 < K \le \lim_{m\mapsto\infty} \left\|R_{s_{k_m}} f_{k_m} - Rf_{k_m}\right\|_{\mathcal{H}} = \lim_{m\mapsto\infty} \left\|R_{s_{k_m}} f_{k_m} - Rf\right\|_{\mathcal{H}} = 0.$$

It is a contradiction with the positivity of K. □

Theorem 2.2.2. Let the same assumptions as in Lemma 1.2.2 be supposed. Then H_ε converges to H_0 in the norm-resolvent sense.

Proof. Consider the sequence of the resolvent operators $\{R_\varepsilon\}_{\varepsilon>0}$. It means, the required boundedness is satisfied. The inequality in the sense of the associated quadratic forms

$$h_0 \geq -C\Delta^\Sigma,$$

where Δ^Σ denotes the Laplace-Beltrami operator on Σ, follows from the definition of h_0 and properties of $\overline{a}$. Since the spectrum of $-\Delta^\Sigma$ is purely discrete and its eingenvalues tend to infinity (see [8], Thm. 3.2.1), also the spectrum of h_0 is purely discrete. The definition of H_0 yields, the essential spectrum of H_0 is empty, then $R_0(z)$ is compact for all $z \in \rho(H_0)$ (see [2], Corol. 4.2.3).

Let $\{\varepsilon_k\}_{k=1}^\infty$ be an arbitrary sequence satysfying $\varepsilon_k \underset{k\mapsto\infty}{\longrightarrow} 0$ and $\varepsilon_k > 0$ for all $k \in \mathbb{N}$. From Theorem 2.1.2, it is known $\left\|\left[R_\varepsilon^U(z) - R_0(z)\right] F\right\|_0 \underset{\varepsilon\mapsto 0}{\longrightarrow} 0$ for any $F \in L^2(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)$ and $z \in \mathbb{C}$ with $\operatorname{Im} z \neq 0$. It implies

$$\left\|R_{\varepsilon_k}(z)F_{\varepsilon_k} - R_0(z)F\right\|_0 \leq \left\|R_{\varepsilon_k}(z)\right\|_{0\to 0}\left\|F_{\varepsilon_k} - F\right\|_0 + \left\|R_{\varepsilon_k}(z)F - R_0(z)F\right\|_0 \underset{k\mapsto\infty}{\longrightarrow} 0.$$

If $R_\varepsilon = R_{\frac{1}{s}} = \tilde{R}_s$ for all $s > 0$ and $R_{\varepsilon_k} = R_{\frac{1}{s_k}} = \tilde{R}_{s_k}$ for all $k \in \mathbb{N}$, the assumptions of Lemma 2.2.1 are satisfied and

$$\lim_{\varepsilon\mapsto 0} \|R_\varepsilon - R_0\|_{0\to 0} = 0.$$

□

2.2.1 The Rate of the Norm-Resolvent Convergence

Even the rate of the norm-resolvent convergence can be derived. Firstly, the alternative definition of an arbitrary norm is proven

$$\|h\| = \sup_{\varphi\neq 0} \frac{|(\varphi, h)|}{\|\varphi\|}. \tag{2.10}$$

Proof. The equality is proven by an upper estimate and a lower estimate. The upper estimate is shown by the Cauchy-Schwarz inequality

$$\sup_{\varphi\neq 0} \frac{|(\varphi, h)|}{\|\varphi\|} \leq \sup_{\varphi\neq 0} \frac{\|\varphi\|\,\|h\|}{\|\varphi\|} = \|h\|,$$

the lower estimate is carried out by taking $\varphi = h$

$$\sup_{\varphi\neq 0} \frac{|(\varphi, h)|}{\|\varphi\|} \geq \frac{|(h, h)|}{\|h\|} = \|h\|.$$

□

According to (2.10), the following relation is obtained.

$$\|R_\varepsilon - R_0\|_{0\to 0} = \sup_{F,G\in L^2(\Omega,\mathrm{d}\Sigma\wedge\mathrm{d}t)} \frac{|(F, (R_\varepsilon - R_0)\, G)_0|}{\|F\|_0\, \|G\|_0}$$

The sequence $\{\psi_\varepsilon\}_{\varepsilon>0}$ in $L^2(\Omega, f_\varepsilon(s,t)\, \mathrm{d}\Sigma \wedge \mathrm{d}t)$ and ψ in $L^2(\Omega, \mathrm{d}\Sigma \wedge \mathrm{d}t)$ are introduced by

$$\psi_\varepsilon = R_\varepsilon(z) U_\varepsilon^{-1} G, \tag{2.11}$$

$$\psi = R_0(z)P_0F \tag{2.12}$$

for any F, $G \in L^2(\Omega, d\Sigma \wedge dt)$ and $z \in \mathbb{C}$ with $\mathrm{Im}\, z \neq 0$ or $\mathrm{Re}\, z < 0$, where P_0 is defined by (1.39). Additionally, the equality (2.12) implies $\psi \in \mathcal{H}_0$. Equations (2.11), (2.12) are equivalent to

$$(H_\varepsilon - z)\,\psi_\varepsilon = U_\varepsilon^{-1}G, \tag{2.13}$$

$$(H_0 - z)\,\psi = P_0F. \tag{2.14}$$

Remark. In the following lemma, estimates are satisfied in the norm $\|\cdot\|_\varepsilon$ even in the norm $\|\cdot\|_0$, therefore the notation $\|\cdot\|$ is used.

Lemma 2.2.3. Let $\{\psi_\varepsilon\}_{\varepsilon>0}$ be defined by (2.11) for any $G \in L^2(\Omega, d\Sigma \wedge dt)$ and $z < 0$. Then the following estimates are valid.

$$\|\psi_\varepsilon\| \leq C\,|z|^{-1}\,\|G\| \tag{2.15}$$

$$\left\|\left|\nabla_g \psi_\varepsilon\right|_g\right\| \leq C\,|z|^{-1/2}\,\|G\| \tag{2.16}$$

$$\left\|\frac{\partial \psi_\varepsilon}{\partial t}\right\| \leq C\,|z|^{-1/2}\,\varepsilon\,\|G\| \tag{2.17}$$

$$\left\|P_0^\perp \psi_\varepsilon\right\| \leq C\,|z|^{-1/2}\,\varepsilon\,\|G\| \tag{2.18}$$

If ψ is defined by (2.12) for any $F \in L^2(\Omega, d\Sigma \wedge dt)$ and $z < 0$, then

$$\|\psi\| \leq C\,|z|^{-1}\,\|F\|, \tag{2.19}$$

$$\left\|\left|\nabla_g \psi\right|_g\right\| \leq C\,|z|^{-1/2}\,\|F\|. \tag{2.20}$$

Proof. Let Q_ε be the associated quadratic form of the operator H_ε, h_ε is the respective sesquilinear form. The equality (2.13) can be written in the form sense

$$(\psi_\varepsilon \in D(H_\varepsilon))\,(\forall \phi \in D(h_\varepsilon))\left(h_\varepsilon(\phi, \psi_\varepsilon) - (\phi, z\psi_\varepsilon)_\varepsilon = (\phi, U_\varepsilon^{-1}G)_\varepsilon\right),$$

for any $G \in L^2(\Omega, d\Sigma \wedge dt)$ and $z < 0$. Take $\phi = \psi_\varepsilon$, then the equation is explicitly

$$\int_\Omega a_\varepsilon \left[\frac{\partial \overline{\psi_\varepsilon}}{\partial s^\mu} G^{\mu\nu} \frac{\partial \psi_\varepsilon}{\partial s^\nu} + \frac{1}{\varepsilon^2}\left|\frac{\partial \psi_\varepsilon}{\partial t}\right|^2\right] f_\varepsilon\, d\Sigma \wedge dt - z\int_\Omega |\psi_\varepsilon|^2 f_\varepsilon\, d\Sigma \wedge dt = \int_\Omega \overline{\psi_\varepsilon} G f_\varepsilon^{1/2}\, d\Sigma \wedge dt. \tag{2.21}$$

Based on (1.6), the following estimate is gained

$$C\int_\Omega \left[\frac{\partial \overline{\psi_\varepsilon}}{\partial s^\mu} g^{\mu\nu} \frac{\partial \psi_\varepsilon}{\partial s^\nu} + \frac{1}{\varepsilon^2}\left|\frac{\partial \psi_\varepsilon}{\partial t}\right|^2\right] f_\varepsilon\, d\Sigma \wedge dt - z\int_\Omega |\psi_\varepsilon|^2 f_\varepsilon\, d\Sigma \wedge dt = \int_\Omega \overline{\psi_\varepsilon} G f_\varepsilon^{1/2}\, d\Sigma \wedge dt.$$

Using the Cauchy-Schwarz inequality, (1.5) and (1.4), it can be seen,

$$C\,|z|\,\|\psi_\varepsilon\|_0^2 \leq C\,|z|\,\|\psi_\varepsilon\|_\varepsilon^2 \leq C\,\|\psi_\varepsilon\|_0\,\|G\|_0$$

is valid. It implies (2.15). The estimate

$$C\left\|\left|\nabla_g \psi_\varepsilon\right|_g\right\|^2 \leq \|\psi_\varepsilon\|\,\|G\|$$

is obtained in the same way. The relation (2.16) is proven using (2.15). In addition, (2.17) can be shown analogously. It is known from Lemma 1.1.3,

$$\left\|P_0^{\perp}\psi_\varepsilon\right\|^2 \leq C\left\|\frac{\partial\psi_\varepsilon}{\partial t}\right\|^2 .$$

Then only use (2.17).

Since $\psi \in \mathcal{H}_0$, the weak formulation of the equation (2.14) can be written as

$$\int_\Omega \frac{\overline{a}}{2}\frac{\partial\overline{\psi}}{\partial s^\mu}g^{\mu\nu}\frac{\partial\psi}{\partial s^\nu}\,\mathrm{d}\Sigma\wedge\mathrm{d}t - z\int_\Omega |\psi|^2\,\mathrm{d}\Sigma\wedge\mathrm{d}t = \int_\Omega \overline{\psi}P_0F\,\mathrm{d}\Sigma\wedge\mathrm{d}t. \tag{2.22}$$

The estimate (2.19) is proven by the Cauchy-Schwarz inequality. The following relation is shown similarly.

$$C\left\|\left|\nabla_g\psi\right|_g\right\| \leq \|\psi\|\,\|P_0F\|$$

If (2.19) is applied, the proof is finished. □

Remark. The mentioned below relations can be gained in the same way as in Lemma 1.1.4.

$$\|P_0\psi_\varepsilon\| \leq C\,|z|^{-1}\,\|G\| \qquad \left\|\left|\nabla_g\left(P_0\psi_\varepsilon\right)\right|_g\right\| \leq C\,|z|^{-1/2}\,\|G\| \qquad \left\|\left|\nabla_g\left(P_0^{\perp}\psi_\varepsilon\right)\right|_g\right\| \leq C\,|z|^{-1/2}\,\|G\|$$

Theorem 2.2.4. Let the same assumptions as in Lemma 1.2.2 be supposed. R_ε and R_0 denote resolvent operators of H_ε and H_0. Then the rate of the norm-resolvent convergence is

$$\left\|R_\varepsilon^U(z) - R_0(z)\right\|_{0\to 0} \leq C(z)\max\left\{\varepsilon, d(\varepsilon)\right\} \tag{2.23}$$

for any $z \in \mathbb{C}$ with $\operatorname{Im} z \neq 0$. The function d is defined by

$$d(\varepsilon) = \operatorname*{ess\,sup}_{s\in\Sigma}\left|\int_{-1}^{1}\left(\frac{\overline{a}}{2} - a_\varepsilon f_\varepsilon\right)\mathrm{d}t\right|. \tag{2.24}$$

Proof. Firstly, put $z = -1$. Based on the properties of R_0 and relations (2.13), (2.14), it can be shown

$$\begin{aligned}
\left(F,\left(U_\varepsilon R_\varepsilon(-1)U_\varepsilon^{-1} - R_0(-1)\right)G\right)_0 &= \left(F, U_\varepsilon R_\varepsilon(-1)U_\varepsilon^{-1}G\right)_0 - \left(F, R_0(-1)G\right)_0\\
&= \left(\left(P_0 + P_0^{\perp}\right)F, U_\varepsilon\psi_\varepsilon\right)_0 - \left(R_0(-1)\left(P_0 + P_0^{\perp}\right)F, G\right)_0\\
&= \left(\left(H_0 + 1\right)\psi, U_\varepsilon\psi_\varepsilon\right)_0 + \left(P_0^{\perp}F, U_\varepsilon\psi_\varepsilon\right)_0 - \left(\psi, U_\varepsilon\left(H_\varepsilon + 1\right)\psi_\varepsilon\right)_0\\
&= \left(H_0\psi, U_\varepsilon\psi_\varepsilon\right)_0 + \left(P_0^{\perp}F, U_\varepsilon\psi_\varepsilon\right)_0 - \left(U_\varepsilon^{-1}\psi, H_\varepsilon\psi_\varepsilon\right)_\varepsilon\\
&= \hat{h}_0\left(\psi, U_\varepsilon\psi_\varepsilon\right) - h_\varepsilon\left(U_\varepsilon^{-1}\psi, \psi_\varepsilon\right) + \left(P_0^{\perp}F, U_\varepsilon\psi_\varepsilon\right)_0,
\end{aligned}$$

where $\hat{h}_0$ and h_ε are sesquilinear forms of H_0 and H_ε. It is explicitly equal to

$$\int_\Omega \frac{\overline{a}}{2}\frac{\partial\overline{\psi}}{\partial s^\mu}g^{\mu\nu}\frac{\partial P_0\left(f_\varepsilon^{1/2}\psi_\varepsilon\right)}{\partial s^\nu}\mathrm{d}\Sigma\wedge\mathrm{d}t - \int_\Omega a_\varepsilon\frac{\partial\left(f_\varepsilon^{-1/2}\overline{\psi}\right)}{\partial s^\mu}G^{\mu\nu}\frac{\partial\psi_\varepsilon}{\partial s^\nu}f_\varepsilon\,\mathrm{d}\Sigma\wedge\mathrm{d}t$$

$$-\frac{1}{\varepsilon^2}\int_\Omega a_\varepsilon\frac{\partial\left(f_\varepsilon^{-1/2}\overline{\psi}\right)}{\partial t}\frac{\partial\psi_\varepsilon}{\partial t}f_\varepsilon\,\mathrm{d}\Sigma\wedge\mathrm{d}t + \int_\Omega \overline{P_0^{\perp}F}\psi_\varepsilon f_\varepsilon^{1/2}\,\mathrm{d}\Sigma\wedge\mathrm{d}t$$

$$= \int_\Omega \left[\frac{\overline{a}}{2} \frac{\partial \overline{\psi}}{\partial s^\mu} g^{\mu\nu} \frac{\partial P_0 \left(f_\varepsilon^{1/2} \psi_\varepsilon \right)}{\partial s^\nu} - a_\varepsilon \frac{\partial \left(f_\varepsilon^{-1/2} \overline{\psi} \right)}{\partial s^\mu} G^{\mu\nu} \frac{\partial P_0 \psi_\varepsilon}{\partial s^\nu} f_\varepsilon \right] \mathrm{d}\Sigma \wedge \mathrm{d}t$$

$$- \int_\Omega a_\varepsilon \frac{\partial \left(f_\varepsilon^{-1/2} \overline{\psi} \right)}{\partial s^\mu} G^{\mu\nu} \frac{\partial P_0^\perp \psi_\varepsilon}{\partial s^\nu} f_\varepsilon \, \mathrm{d}\Sigma \wedge \mathrm{d}t - \frac{1}{\varepsilon^2} \int_\Omega a_\varepsilon \overline{\psi} \frac{\partial \left(f_\varepsilon^{-1/2} \right)}{\partial t} \frac{\partial \psi_\varepsilon}{\partial t} f_\varepsilon \, \mathrm{d}\Sigma \wedge \mathrm{d}t$$

$$+ \int_\Omega \overline{P_0^\perp F} \psi_\varepsilon f_\varepsilon^{1/2} \, \mathrm{d}\Sigma \wedge \mathrm{d}t.$$

Based on the properties of the orthogonal projection P_0, properties of f_ε, ψ_ε and the exchange of derivatives and the integral, the following relation can be shown.

$$\frac{\partial P_0 \left(f_\varepsilon^{1/2} \psi_\varepsilon \right)}{\partial s^\nu} = \frac{\partial}{\partial s^\nu} \left[\frac{1}{2} \int_{-1}^{1} \left(f_\varepsilon^{1/2} - 1 \right) \left(P_0 + P_0^\perp \right) \psi_\varepsilon \mathrm{d}t + P_0 \psi_\varepsilon \right]$$

$$= \frac{\partial P_0 \psi_\varepsilon}{\partial s^\nu} + \frac{1}{2} \frac{\partial P_0 \psi_\varepsilon}{\partial s^\nu} \int_{-1}^{1} \left(f_\varepsilon^{1/2} - 1 \right) \mathrm{d}t + \frac{1}{2} P_0 \psi_\varepsilon \int_{-1}^{1} \frac{\partial f_\varepsilon^{1/2}}{\partial s^\nu} \mathrm{d}t$$

$$+ \frac{1}{2} \int_{-1}^{1} \frac{\partial f_\varepsilon^{1/2}}{\partial s^\nu} P_0^\perp \psi_\varepsilon \mathrm{d}t + \frac{1}{2} \int_{-1}^{1} \left(f_\varepsilon^{1/2} - 1 \right) \frac{\partial P_0^\perp \psi_\varepsilon}{\partial s^\nu} \mathrm{d}t$$

Consequently, the mentioned above equality, the Cauchy-Schwarz inequality, Lemma 2.2.3 and properties of f_ε lead to

$$\left| \int_\Omega \frac{\overline{a}}{2} \frac{\partial \overline{\psi}}{\partial s^\mu} g^{\mu\nu} \left(\frac{\partial P_0 \left(f_\varepsilon^{1/2} \psi_\varepsilon \right)}{\partial s^\nu} - \frac{\partial P_0 \psi_\varepsilon}{\partial s^\nu} \right) \mathrm{d}\Sigma \wedge \mathrm{d}t \right| \le C \left\| \left| \nabla_g \psi \right|_g \right\|_0 \left[\left\| \left| \nabla_g f_\varepsilon^{1/2} \right|_g \right\|_{L^\infty(\Omega)} \left(\| P_0 \psi_\varepsilon \|_0 + \left\| P_0^\perp \psi_\varepsilon \right\|_0 \right) \right.$$

$$\left. + \left\| f_\varepsilon^{1/2} - 1 \right\|_{L^\infty(\Omega)} \left(\left\| \left| \nabla_g \left(P_0 \psi_\varepsilon \right) \right|_g \right\|_0 + \left\| \left| \nabla_g \left(P_0^\perp \psi_\varepsilon \right) \right|_g \right\|_0 \right) \right] \le \varepsilon \, \| F \|_0 \, \| G \|_0 \, .$$

For the integral

$$\int_\Omega a_\varepsilon \frac{\partial \left(f_\varepsilon^{-1/2} \overline{\psi} \right)}{\partial s^\mu} G^{\mu\nu} \frac{\partial \psi_\varepsilon}{\partial s^\nu} f_\varepsilon \, \mathrm{d}\Sigma \wedge \mathrm{d}t = \int_\Omega a_\varepsilon \left(\frac{\partial f_\varepsilon^{-1/2}}{\partial s^\mu} \overline{\psi} + f_\varepsilon^{-1/2} \frac{\partial \overline{\psi}}{\partial s^\mu} \right) G^{\mu\nu} \frac{\partial \psi_\varepsilon}{\partial s^\nu} f_\varepsilon \, \mathrm{d}\Sigma \wedge \mathrm{d}t$$

estimates

$$\left| \int_\Omega a_\varepsilon \frac{\partial f_\varepsilon^{-1/2}}{\partial s^\mu} \overline{\psi} G^{\mu\nu} \frac{\partial \psi_\varepsilon}{\partial s^\nu} f_\varepsilon \, \mathrm{d}\Sigma \wedge \mathrm{d}t \right| \le C \left\| \left| \nabla_g f_\varepsilon^{-1/2} \right|_g \right\|_{L^\infty(\Omega)} \left\| \left| \nabla_g \psi_\varepsilon \right|_g \right\|_\varepsilon \| \psi \|_\varepsilon \le \varepsilon C \, \| F \|_0 \, \| G \|_0 \, ,$$

$$\left| \int_\Omega a_\varepsilon \frac{\partial \overline{\psi}}{\partial s^\mu} \left(f_\varepsilon^{-1/2} G^{\mu\nu} - g^{\mu\nu} \right) \frac{\partial P_0 \psi_\varepsilon}{\partial s^\nu} \, f_\varepsilon \, \mathrm{d}\Sigma \wedge \mathrm{d}t \right| \le \left| \int_\Omega a_\varepsilon \frac{\partial \overline{\psi}}{\partial s^\mu} \left(f_\varepsilon^{-1/2} - 1 \right) G^{\mu\nu} \frac{\partial P_0 \psi_\varepsilon}{\partial s^\nu} \, f_\varepsilon \, \mathrm{d}\Sigma \wedge \mathrm{d}t \right|$$

$$+ \left| \int_\Omega a_\varepsilon \frac{\partial \overline{\psi}}{\partial s^\mu} \left(G^{\mu\nu} - C_\varepsilon^- g^{\mu\nu} \right) \frac{\partial P_0 \psi_\varepsilon}{\partial s^\nu} \, f_\varepsilon \, \mathrm{d}\Sigma \wedge \mathrm{d}t \right| + \left| \int_\Omega a_\varepsilon \frac{\partial \overline{\psi}}{\partial s^\mu} \left(C_\varepsilon^- - 1 \right) g^{\mu\nu} \frac{\partial P_0 \psi_\varepsilon}{\partial s^\nu} \, f_\varepsilon \, \mathrm{d}\Sigma \wedge \mathrm{d}t \right|$$

$$\le \varepsilon C \left\| \left| \nabla_g \psi \right|_g \right\|_\varepsilon \left\| \left| \nabla_g \left(P_0 \psi_\varepsilon \right) \right|_g \right\|_\varepsilon \le \varepsilon C \, \| F \|_0 \, \| G \|_0$$

are proved using the Cauchy-Schwarz inequality, Lemma 2.2.3, (1.6) and properties of f_ε. Furthermore, the inequalities

$$\left| \int_\Omega \left(\frac{\overline{a}}{2} - a_\varepsilon f_\varepsilon \right) \frac{\partial \overline{\psi}}{\partial s^\mu} g^{\mu\nu} \frac{\partial P_0 \psi_\varepsilon}{\partial s^\nu} \, \mathrm{d}\Sigma \wedge \mathrm{d}t \right| = \left| \int_\Sigma \int_{-1}^{1} \left(\frac{\overline{a}}{2} - a_\varepsilon f_\varepsilon \right) \mathrm{d}t \, \frac{\partial \overline{\psi}}{\partial s^\mu} g^{\mu\nu} \frac{\partial P_0 \psi_\varepsilon}{\partial s^\nu} \, \mathrm{d}\Sigma \right|$$

$$\leq d(\varepsilon)\, C \left\| \left|\nabla_g \psi\right|_g \right\|_0 \left\| \left|\nabla_g \left(P_0 \psi_\varepsilon\right)\right|_g \right\|_0 \leq d(\varepsilon)\, C \left\|F\right\|_0 \left\|G\right\|_0 ,$$

$$\left| \int_\Omega a_\varepsilon \frac{\partial \overline{\psi}}{\partial s^\mu} G^{\mu\nu} \frac{\partial P_0^\perp \psi_\varepsilon}{\partial s^\nu} f_\varepsilon^{1/2} \,\mathrm{d}\Sigma \wedge \mathrm{d}t \right| \leq \varepsilon C \left\|F\right\|_0 \left\|G\right\|_0$$

are satisfied, where the latter can be shown analogously as in Lemma 1.2.1. From the estimates mentioned above, it can be seen

$$\left| \int_\Omega \left[\frac{\overline{a}}{2} \frac{\partial \overline{\psi}}{\partial s^\mu} g^{\mu\nu} \frac{\partial P_0 \left(f_\varepsilon^{1/2} \psi_\varepsilon\right)}{\partial s^\nu} - a_\varepsilon \frac{\partial \left(f_\varepsilon^{-1/2} \overline{\psi}\right)}{\partial s^\mu} G^{\mu\nu} \frac{\partial \psi_\varepsilon}{\partial s^\nu} f_\varepsilon \right] \mathrm{d}\Sigma \wedge \mathrm{d}t \right| \leq \max\left\{d(\varepsilon), \varepsilon\right\} C \left\|F\right\|_0 \left\|G\right\|_0 .$$

Now rewrite the equation (2.13) in the form sense for the function $tK_1\psi$. It is explicitly

$$\int_\Omega a_\varepsilon t K_1 \frac{\partial \overline{\psi}}{\partial s^\mu} G^{\mu\nu} \frac{\partial \psi_\varepsilon}{\partial s^\nu} f_\varepsilon \,\mathrm{d}\Sigma \wedge \mathrm{d}t + \int_\Omega a_\varepsilon t \overline{\psi} \frac{\partial K_1}{\partial s^\mu} G^{\mu\nu} \frac{\partial \psi_\varepsilon}{\partial s^\nu} f_\varepsilon \,\mathrm{d}\Sigma \wedge \mathrm{d}t + \frac{1}{\varepsilon^2} \int_\Omega a_\varepsilon K_1 \overline{\psi} \frac{\partial \psi_\varepsilon}{\partial t} f_\varepsilon \,\mathrm{d}\Sigma \wedge \mathrm{d}t$$

$$- z \int_\Omega t K_1 \overline{\psi} \psi_\varepsilon f_\varepsilon \,\mathrm{d}\Sigma \wedge \mathrm{d}t = \int_\Omega t K_1 \overline{\psi} G f_\varepsilon^{1/2} \,\mathrm{d}\Sigma \wedge \mathrm{d}t.$$

It means

$$\left| \frac{1}{\varepsilon^2} \int_\Omega a_\varepsilon K_1 \overline{\psi} \frac{\partial \psi_\varepsilon}{\partial t} f_\varepsilon \,\mathrm{d}\Sigma \wedge \mathrm{d}t \right| \leq C \left\|F\right\|_0 \left\|G\right\|_0 ,$$

then the observation

$$\frac{\partial f^{-1/2}}{\partial t} = \frac{d-1}{2} \varepsilon K_1 + O(\varepsilon^2) \qquad \text{as } \varepsilon \mapsto 0$$

gives the following estimate for the sufficiently small ε.

$$\left| \frac{1}{\varepsilon^2} \int_\Omega a_\varepsilon \overline{\psi} \frac{\partial \left(f_\varepsilon^{-1/2}\right)}{\partial t} \frac{\partial \psi_\varepsilon}{\partial t} f_\varepsilon \,\mathrm{d}\Sigma \wedge \mathrm{d}t \right| \leq \varepsilon C \left\|F\right\|_0 \left\|G\right\|_0$$

The last part can be estimated using properties of projections and f_ε, the Cauchy-Schwarz inequality and Lemma 2.2.3.

$$\left| \int_\Omega \overline{P_0^\perp F} \psi_\varepsilon f_\varepsilon^{1/2} \,\mathrm{d}\Sigma \wedge \mathrm{d}t \right| = \left| \left(\left(P_0^\perp\right)^2 F, \psi_\varepsilon f_\varepsilon^{1/2} \right)_0 \right| = \left| \left(P_0^\perp F, P_0^\perp \left(\psi_\varepsilon f_\varepsilon^{1/2}\right) \right)_0 \right| \leq \left\|F\right\|_0 \left\| P_0^\perp \left(\psi_\varepsilon f_\varepsilon^{1/2}\right) \right\|_0$$

$$\leq \left\|F\right\|_0 \left(\left\| \psi_\varepsilon f_\varepsilon^{1/2} - \psi_\varepsilon \right\|_0 + \left\| P_0 \left(\psi_\varepsilon f_\varepsilon^{1/2} - \psi_\varepsilon \right) \right\|_0 + \left\| P_0^\perp \psi_\varepsilon \right\|_0 \right)$$

$$\leq C \left\|F\right\|_0 \left(\left\| f_\varepsilon^{1/2} - 1 \right\|_{L^\infty(\Omega)} \left\|\psi_\varepsilon\right\|_0 + \left\| P_0^\perp \psi_\varepsilon \right\|_0 \right) \leq \varepsilon C \left\|F\right\|_0 \left\|G\right\|_0$$

Hence, the total estimate is

$$\left| \left(F, \left(R_\varepsilon^U(-1) - R_0(-1) \right) G \right)_0 \right| \leq \max\left\{\varepsilon, d(\varepsilon)\right\} C \left\|F\right\|_0 \left\|G\right\|_0$$

and the norm satisfies

$$\left\| R_\varepsilon^U(-1) - R_0(-1) \right\|_{0\to 0} \leq C \max\left\{\varepsilon, d(\varepsilon)\right\}$$

Now the aim is to get this estimation for all z with $\operatorname{Im} z \neq 0$. The first resolvent identity (see [5], Thm. 5.13.) has to be applied as in Theorem 2.1.2, then the estimate

$$\left\| R_\varepsilon^U(z) - R_0(z) \right\|_{0\to 0} \leq \left[|z+1|^{-1} \left\|B_\varepsilon\right\|_{0\to 0} \left\|B\right\|_{0\to 0} \left\| R_\varepsilon^U(-1) \right\|_{0\to 0} + \left\| A^{-1} \right\|_{0\to 0} \right] \left\| \left(R_0(-1) - R_\varepsilon^U(-1) \right) \right\|_{0\to 0}$$

$$\leq C(z) \left\| \left(R_0(-1) - R_\varepsilon^U(-1) \right) \right\|_{0\to 0} \leq C(z) \max\left\{\varepsilon, d(\varepsilon)\right\}$$

is gained for all z with $\operatorname{Im} z \neq 0$. The notation is the same as in Theorem 2.1.2. □

Remark. In conclusion, the same choice of the function a_ε as in [17]

$$a_\varepsilon(s,t) = \begin{cases} a_+ & t \in (0,1) \\ a_- & t \in (-1,0) \end{cases}$$

gives $d(\varepsilon) = 0$. It means, the rate of the norm-resolvent convergence $\left\| R_\varepsilon^U(z) - R_0(z) \right\|_{0\to 0} \leq \varepsilon C(z)$ is satisfied for all z with $\operatorname{Im} z \neq 0$.

Conclusion

At the beginning of the study, the Neumann boundary spectral problem (3) is rigorously interpreted by the definition of the self-adjoint operator corresponding to this spectral problem. The self-adjoint operator is the Neumann Laplace operator with a non-homogeneous metric in thin domains defined by the associated quadratic form. The operator can be physically interpreted as the Hamiltonian of a (quasi)particle in a non-homogeneous nanostructure and the problem (3) as the respective stationary Schrödinger equation.

Furthermore, the behaviour of this operator is studied, while the width of the thin domain tends to zero. Then it is found out that the operator converges to the effective model in the spectral sense and in the resolvent sense. Both the strong resolvent convergence and the norm-resolvent convergence are proven. Finally, the rate of the norm-resolvent convergence is obtained. Conditions of the convergence are formulated and discussed.

It is worth mentioning, this study is the significant generalization of the T. Yachimura's article published in the journal Differential and Integral Equations in 2018, where only the piecewise constant metric was considered.

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