Czech Technical University in Prague
Faculty of Electrical Engineering

Vietoris-Rips complex

Bachelor thesis

Daria Dunina
Study Programme: Open Informatics
Thesis Supervisor: Ing. Matěj Dostál, PhD.

Prague, 2021

BACHELOR'S THESIS ASSIGNMENT

I. Personal and study details

Student's name: **Dunina Daria** Personal ID number: **483747**

Faculty / Institute: **Faculty of Electrical Engineering**

Department / Institute: **Department of Computer Science**

Study program: **Open Informatics**

Specialisation: **Software**

II. Bachelor's thesis details

Bachelor's thesis title in English:

Vietoris-Rips complex

Bachelor's thesis title in Czech:

Vietorisův-Ripsův komplex

Guidelines:

Vietoris-Rips (V-R) complexes are simplicial complexes arising from metric spaces. They play an important role in the study of (co)homology of metric spaces [2]. Practical applications include coverage in sensor networks [4].
The aim of the thesis is to describe basic constructions of V-R complexes in the language of category theory [3].
The student will define a category of metric spaces suitable for the study of V-R complexes. They will describe the "simplicial category" Δ such that presheaves on Δ are precisely simplicial complexes. They will study the (singular functor/realisation functor) adjunction and find a simplex functor such that its corresponding singular functor describes precisely the construction of a V-R complex. Geometric realisation of V-R complexes will be computed in this setting.
Advanced topics can be optionally covered: parametrisation of the above construction w.r.t. ε, density of the simplex functor.
The style and presentation of the thesis will be theoretical.

Bibliography / sources:

[1] R. Engelking, General topology, Heldermann-Verlag, 1989.
[2] J.-C. Hausmann, On the Vietoris–Rips complexes and a cohomology theory for metric spaces, in: Prospects in Topology: Proceedings of a conference in honour of William Browder , Annals of Mathematics Studies 138, Princeton University Press 1995, 175–188.
[3] S. Mac Lane, Categories for the working mathematician (2nd ed.), Springer, 1998.
[4] V. de Silva, R. Ghrist

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Date of bachelor's thesis assignment: **10.02.2021** Deadline for bachelor thesis submission: **21.05.2021**

Assignment valid until: **30.09.2022**

Ing. Matěj Dostál, Ph.D.
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Head of department's signature

prof. Mgr. Petr Páta, Ph.D.
Dean's signature

III. Assignment receipt

The student acknowledges that the bachelor’s thesis is an individual work. The student must produce her thesis without the assistance of others, with the exception of provided consultations. Within the bachelor’s thesis, the author must state the names of consultants and include a list of references.

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Date of assignment receipt

Student’s signature

Declaration

I hereby declare I have written this bachelor thesis independently and quoted all the sources of information used in accordance with methodological instructions on ethical principles for writing an academic thesis. Moreover, I state that this thesis has neither been submitted nor accepted for any other degree.

In Prague, 2021

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Daria Dunina

Acknowledgements

Foremost, I would like to express sincere gratitude to my thesis supervisor Matěj Dostál. Without his guidance, patience and persistent help, this thesis would not have been possible.

I would also like to express my deepest appreciation to Jiří Velebil for the continuous support and inspiring me to explore mathematics in the first place.

And I am extremely grateful to my mother for her love, caring and sacrifices for educating me and preparing for my future.

Abstract

[EN] This thesis is devoted to the study of Vietoris-Rips complexes and their geometric realization. We first introduce the necessary basics of category theory and the theory of metric spaces. The notions of a simplicial set and Vietoris-Rips complex are defined in the usual manner and then via categorical means. We present an abstract account of geometric realization and show its instantiation in the special case of the realization of Vietoris-Rips complexes.

Keywords: simplex, simplicial set, Vietoris-Rips complex, geometric realization, category theory.

[CZE] Tato práce je věnována studiu Vietorisových-Ripsových komplexů a jejich geometrické realizaci. Nejprve zavedeme nezbytné základy teorie kategorií a teorie metrických prostorů. Pojmy simpliciální množiny a Vietorisova-Ripsova komplexu jsou definovány klasickým způsobem a poté kategoriálními prostředky. Obecně zavedeme teorii geometrické realizace a jako speciální příklad prostudujeme realizaci Vietorisových-Ripsových komplexů.

Klíčová slova: simplex, simpliciální množina, Vietorisův-Ripsův komplex, geometrická realizace, teorie kategorií.

Preface

In many areas of applied mathematics (for example feature extraction, geometry of sensor networks, etc.) the following ideas are used:

(1) If one wants to extract certain properties of a complicated space, it is often useful to encode these properties into a combinatorial structure that arises from a clever "triangulation" of the space.

(2) Given a "triangular recipe" for a space (i.e., a specification of the "triangulation"), one wants to produce a space that fits given specification the best.

Intuitively, one can expect some loss of information which is not considered as significant. Thus the processes are not supposed to be inverse to each other. However, they should be tied together "as much as possible" in the sense that one process determines the other.

In this thesis we study a special instance of the above. Namely:

(1) Out of a *metric* space we produce a *simplicial set* that is called a *Vietoris-Rips complex* of the space.

    We explain how the complex is supposed to capture a *sampling* of the space.

(2) Out of a *general* simplicial set we produce the corresponding metric space that "fits the best" the combinatorial structure hidden in the simplicial set.

We show that these two processes can be viewed as an instance of a much more general machinery from category theory. More in detail, we show that they are *adjoint* to each other in a precise sense. In fact, the adjunction arises in a certain canonical way.

We also demonstrate that our approach is equivalent to the "classical" concept of a Vietoris-Rips concept known from the literature. We believe, however, that the approach of category theory has a potential of further development.

This thesis has the following structure (we emphasize the main results):

- **Chapter 1** is devoted to a brief introduction of the fundamentals of category theory needed in the thesis. All of the theory is standard and well-known.

- **Chapter 2** describes various categories of metric spaces and their cocompleteness properties.

  Main result: We show that the category of extended metric spaces is cocomplete.

- **Chapter 3** introduces the notions of a simplicial set and a Vietoris-Rips complex in a classical way. Then we describe Vietoris-Rips complexes and their geometric realization as categorical concepts.

  Main result: We show that Vietoris-Rips complex can be described as a presheaf. Also we describe the geometric realization of simplicial sets as a left adjoint of a special functor.

Contents

Chapter 1

Introduction to Category Theory

> Category theory takes a bird's eye view of mathematics. From high in the sky, details become invisible, but we can spot patterns that were impossible to detect from ground level.
>
> Tom Leinster

In this chapter we will briefly go through the fundamentals of category theory that will be used throughout the thesis.

In Section 1.1.1 we will introduce the concept and formal definition of a category, define what a functor and a natural transformation is, followed by some examples. Section 1.1.2 is dedicated to the concept of colimits. Section 1.1.3 describes the notion of an adjunction.

An interested reader could find a more involved treatment in materials listed the References section, for example [5], [9].

1.1 Categories, Functors and Natural Transformations

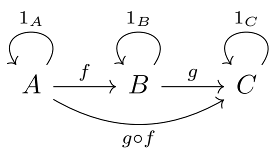
1.1.1 Categories

A category consists of two types of "things": a collection of objects and morphisms (arrows) between them. Morphisms can be composed. The composition is associative and there are identity morphisms for each object.

Before we give a formal definition of a category let us see an example to get a feeling of what we will be dealing with.

1.1.1 Example Imagine some kind of collection of all sets together with a collection of all set functions. Together those two collections construct a category *Set*.

Notice that for each set A there is the identity function $1_A : A \longrightarrow A$, which sends all elements of A to themselves. The other observation is that we can compose those functions: whenever there is $f : A \longrightarrow B$ and $g : B \longrightarrow C$, we know that there is $g \circ f : A \longrightarrow C$.



Now we abstract the example above.

1.1.2 Definition A *category* $\mathscr{C}$ consists of a collection of objects that will be denoted by A,B,C, etc., and morphisms (arrows) between them. Each morphism is associated with two operations:

Domain, which assigns to each morphism f an object $A = dom(f)$;
Codomain, which assigns to each morphism f an object $B = cod(f)$;
These operations are best understood if f is represented by an actual arrow starting at its domain and pointing to its codomain:

$$A \xrightarrow{f} B \quad \text{or} \quad f : A \longrightarrow B$$

We also write

$$hom_{\mathscr{C}}(A, B) \text{ or } \mathscr{C}(A, B) = \{f \mid f \text{ in } \mathscr{C}, dom(f) = A, cod(f) = B\}$$

for the set of morphisms from A to B called “hom-set”.
The morphisms must obey the following laws:

(1) There is a map $comp_{A,B,C} : \mathscr{C}(A, B) \times \mathscr{C}(B, C) \longrightarrow \mathscr{C}(A, C)$ such that for each $f \in \mathscr{C}(A, B), g \in \mathscr{C}(B, C)$, the morphism $comp(f, g)$ is denoted by $g \circ f$.

This is also named as the *composition law*: whenever the codomain of one morphism matches the domain of another, there is a morphism that is their composition, i.e. given $f : A \longrightarrow B$ and $g : B \longrightarrow C$ there is a morphism $h = g \circ f : A \longrightarrow C$;

(2) The composition is *associative*: $(h \circ g) \circ f = h \circ (g \circ f)$ whenever

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \;;$$

(3) Each object X has an *identity morphism* 1_X: $X \longrightarrow X$ which satisfies $1_Y \circ f = f \circ 1_X = f$ for any $X \xrightarrow{f} Y$.

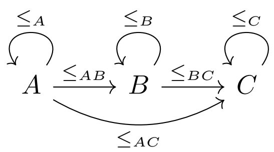
1.1.3 Examples The following are examples of categories.

(1) *Set* The category of all small sets and all functions between them as morphisms.

(2) Vct_K The category of all vector spaces over field K and linear transformations as morphisms.

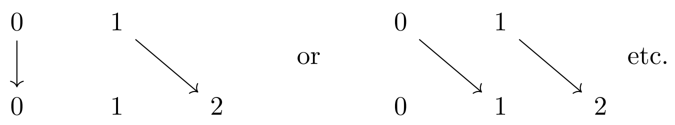
(3) A poset P could be something to construct a category from. The resulting category $\mathscr{P}$ would have elements of P as objects and there would be a morphism between objects A and B whenever $A \leq B$.

Notice that we can compose those morphisms: whenever $A \leq B$ and $B \leq C$ $(A, B, C \in \mathscr{P})$, we know that $A \leq C$. There also is an identity morphism $A \leq A$.



(4) Δ*(Simplicial category)* A category with objects $[n] := \{0, 1, ..., n-1\}, n \in \mathbb{N}$ and monotone maps as morphisms.

A monotone map between [1] and [2] could be illustrated as follows:



We say that a category is *small* if both $ob(\mathscr{C})$ (the collection of objects of $\mathscr{C}$) and $hom_{\mathscr{C}}$ (the collection of morphisms of $\mathscr{C}$) are actually sets and not proper classes, and *large* otherwise.
A category is *discrete* if it has only identity morphisms.

1.1.4 Definition A morphism $f : A \longrightarrow B$ in $\mathscr{C}$ is called *invertible* whenever there exists a morphism $g : B \longrightarrow A$ in $\mathscr{C}$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$.

1.1.5 Definition Every category $\mathscr{C}$ has an opposite category denoted as $\mathscr{C}^{op}$ which has the same objects as $\mathscr{C}$. There is a morphism $f : B \longrightarrow A$ in $\mathscr{C}^{op}$ whenever there is a morphism $f : A \longrightarrow B$ in $\mathscr{C}$. A composite of morphisms $g \circ f$ in $\mathscr{C}^{op}$ is defined to be the composite $f \circ g$ in $\mathscr{C}$.

1.1.2 Functors

A *functor* is a morphism of categories. As a category consists of objects and morphisms which have completely different "behaviour", functor consists of two parts as well: one of them "takes care" of objects, the other one — of morphisms.

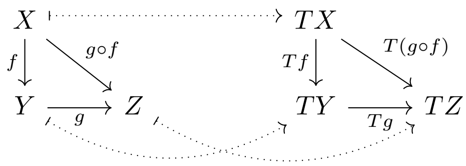
1.1.6 Definition For categories $\mathscr{C}$ and $\mathscr{B}$ a functor $T : \mathscr{C} \longrightarrow \mathscr{B}$ consists of two suitably related assignments: The object assignment T, which assigns to each object C of $\mathscr{C}$ an object TC of $\mathscr{B}$ and the morphism assignment (also as written T) which assigns to each morphism $f : X \longrightarrow X'$ a morphism $Tf : TX \longrightarrow TX'$ of $\mathscr{B}$, in such a way that the equalities:

- $T(1_C) = 1_{TC}$;
- $T(g \circ f) = Tg \circ Tf$ (whenever the composite $g \circ f$ is defined in $\mathscr{C}$);

hold.

So a functor preserves identity morphisms:

and preserves composition:



1.1.7 Examples The following are examples of functors:

(1) The underlying functor $U : Vct_K \longrightarrow Set$. For every vector space V over the field K the object UV is the set of vectors in V, and each linear transformation f is sent to corresponding set map Uf.

(2) Vice versa, we can define the *"free vector space"* functor $F : Set \longrightarrow Vct_K$, which would assign to each set $S \in ob(Set)$ the free vector space $F(S)$ with basis S. Every set map $f : S \longrightarrow R$ is mapped to the corresponding linear transformation defined by the assignment of the elements of the bases f.

(3) Let $\mathscr{X}$ and $\mathscr{Y}$ be posets considered as categories. Functors correspond to monotone maps.

(4) A functor of type $F : \mathscr{A}^{op} \longrightarrow Set$ is called a *presheaf* on the category $\mathscr{A}$.

An extremely important presheaf for this thesis are presheaves of the form $\Delta^{op} \longrightarrow Set$. The idea is that, given $F : \Delta^{op} \longrightarrow Set$, the set $F([n])$ is the set of "n-dimensional simplices", for each $[n]$ in Δ. We will talk more about it later in Chapter 3.

(5) For every category $\mathscr{A}$ and every object A_0 of $\mathscr{A}$, we define the *representable* functor $\mathscr{A}(A_0,—) : \mathscr{A} \longrightarrow Set$ as follows:

(a) An object A is sent to the set $\mathscr{A}(A_0, A)$ of all morphisms from A_0 to A.

(b) Given $f : A \longrightarrow A'$, the mapping $\mathscr{A}(A_0, f) : \mathscr{A}(A_0, A) \longrightarrow \mathscr{A}(A_0, A')$ sends $h : A_0 \longrightarrow A$ to $f \circ h : A_0 \longrightarrow A'$.

We will slightly generalise the notion of a representable functor in Definition 1.1.12.

1.1.3 Natural Transformations

1.1.8 Definition Given two functors $T, S : \mathscr{C} \longrightarrow \mathscr{B}$ a *natural transformation* $\tau : S \longrightarrow T$ assigns to each object X of $\mathscr{C}$ a morphism $\tau_X : SX \longrightarrow TX$ of $\mathscr{B}$ in such a way that every morphism $f : X \longrightarrow X'$ in $\mathscr{C}$ yields a diagram

$$\begin{array}{c} X \\ \downarrow{\scriptstyle f} \\ X' \end{array} \qquad\qquad \begin{array}{ccc} SX & \xrightarrow{\tau_X} & TX \\ {\scriptstyle Sf}\downarrow & & \downarrow{\scriptstyle Tf} \\ SX' & \xrightarrow{\tau_{X'}} & TX' \end{array}$$

such that the square on the right commutes.

Notice that the natural transformation τ is the totality of *all* the morphisms τ_X, where each τ_X is referred to as a *component* of τ.

1.1.9 Definition A natural transformation τ with every component τ_X invertible in $\mathscr{B}$ is called *natural equivalence* or *natural isomorphism*, in symbols $\tau : S \cong T$. We also call this type on natural transformation *invertible*.

1.1.10 Remark The inverse natural transformation τ^{-1} of a natural transformation τ consists of inverses τ_X^{-1} of components τ_X of τ for each $X \in \mathscr{C}$.

1.1.11 Definition *Natural bijection* is a natural isomorphism between two *Set*-valued functors.

1.1.12 Definition A functor $H : \mathscr{A} \longrightarrow Set$ such that H is naturally isomorphic to $\mathscr{A}(A_0, —) : \mathscr{A} \longrightarrow Set$ is called a representable functor. The object A_0 is called the representing object.

Having defined the notion of categories, functors and natural transformations, we may introduce another specific type of categories.

1.1.13 Definition Given a small categories $\mathscr{B}$ and a category $\mathscr{C}$, we construct the *functor category* $\mathscr{C}^{\mathscr{B}}$ also written as $[\mathscr{B}, \mathscr{C}]$ with functors $T : \mathscr{B} \longrightarrow \mathscr{C}$ as objects and natural transformations between two such functors as morphisms.

1.2 Colimits in a Category

Colimits are a categorical generalisation of well-known constructions from set theory: *disjoint union* and *quotient set* construction. These correspond to colimit notions of a *coproduct* and a *coequalizer*.

Let us proceed to a formal definition.

1.2.1 Definition Suppose $\mathscr{D}$ is a small category. A functor $D : \mathscr{D} \longrightarrow \mathscr{X}$ is called a *diagram* of the *scheme* $\mathscr{D}$ in $\mathscr{X}$.

Informally speaking, the diagram *"imprints"* the pattern of the scheme in the category we are working with.

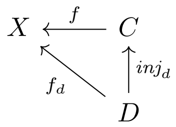
1.2.2 Definition A *cocone* for $D : \mathscr{D} \longrightarrow \mathscr{X}$ is a tuple (X, f_d), where $f_d : Dd \longrightarrow X$ is a collection indexed by objects of $\mathscr{D}$, such that the triangle

$$\begin{array}{ccc} X & \xleftarrow{f_d} & Dd \\ & {\scriptstyle f_{d'}}\nwarrow & \downarrow{\scriptstyle D\delta} \\ & & Dd' \end{array}$$

commutes, for every morphism $\delta : d \longrightarrow d'$ in $\mathscr{D}$.

1.2.3 Definition A cocone (C, inj_d) for D is called a *colimit* of D, provided it has the following universal property:

For every cocone $(X, (f_d)_{d \in \mathscr{D}})$ for D there is a unique $f : C \longrightarrow X$ such that the triangle



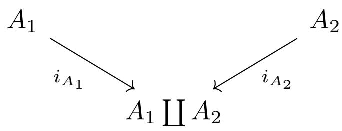
commutes, for every d in $\mathscr{D}$.

1.2.4 Definition A category X is called *cocomplete*, if it has colimits of all diagrams.

There are various types of special colimits, we will take a closer look at two of them: *coproducts* and *coequalizers*.

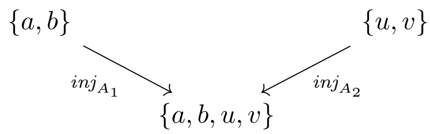
1.2.5 Definition If $\mathscr{I}$ is a discrete index category with two objects, a functor $D : \mathscr{I} \longrightarrow \mathscr{X}$ is a tuple $\langle A_1, A_2 \rangle$ of objects of $\mathscr{X}$. The colimit of D is called a *binary coproduct* of A_1 and A_2, and it is written $A_1 + A_2$.

The colimit diagram consists of the colimit object $A_1 + A_2$ and morphisms i_{A_1} and i_{A_2} called the *injections* in the coproduct (though they are not required to be injective in any sense of the word).



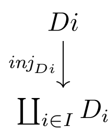
1.2.6 Example Binary coproducts in *Set* are disjoint unions of two sets.

If $A_1 = \{a, b\}, A_2 = \{u, v\}$ are sets, then their binary coproduct is comprised of the disjoint union $A_1 + A_2$ and injection maps $inj_{A_1} : A_1 \longrightarrow A_1 + A_2$, $inj_{A_2} : A_2 \longrightarrow A_1 + A_2$.



1.2.7 Definition If $\mathscr{I}$ is a discrete index category, a functor $D : \mathscr{I} \longrightarrow \mathscr{X}$ can be thought of as a collection $\{Di \mid i \in ob(\mathscr{I})\}$. The colimit of D is called a *coproduct* of $\{Di \mid i \in ob(\mathscr{I})\}$, and is written $\coprod_{i \in I} Di$.

The colimit diagram consists of the colimit object $\coprod_{i \in I} Di$ and morphisms $inj_{Di} : Di \longrightarrow \coprod_{i \in I} Di$ called the *injections* into the coproduct.



1.2.8 Examples The coproduct of any two objects exists in many of the familiar categories, where it has a variety of names as indicated in the following list:

(1) Coproducts in *Set* are disjoint unions of those sets.

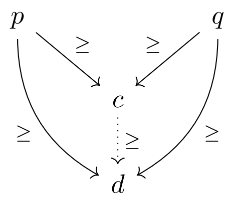
The coproduct is constructed analogously to the binary coproduct in *Set*.

Notice that every element $y \in \coprod_{i \in I} Di$ can be written as $inj_{D_j}(x)$ for some $j \in I$ and $x \in X_j$.

(2) Coproducts in a *poset*.

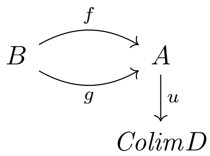
In a poset P, the coproduct of a set of elements $\{p_1, p_2, ...\}$ is their least upper bound.

For example, the colimit of a diagram of two elements p, q is an element c such that $c \geq p$ and $c \geq q$, and if there is any element d with $d \geq p$ and $d \geq q$, then $d \geq c$. That is, c is the least upper bound (supremum) of p and q.



1.2.9 Definition If $\mathscr{J}$ is the scheme $\cdot \rightrightarrows \cdot$, a functor $D : \mathscr{J} \longrightarrow \mathscr{X}$ is a pair $f, g : B \longrightarrow A$ of parallel morphisms of $\mathscr{X}$. A colimit object $ColimD$, when it exists, is called a *coequalizer* of f and g.

The colimit diagram is



, $\quad u \circ f = u \circ g.$

1.2.10 Remark Notice how in the diagram above there is no morphism $k : B \longrightarrow ColimD$, which seemingly contradicts the definition of a cocone (Definition 1.2.3).

The morphism $k : B \longrightarrow ColimD$ is in fact thought of, but is not needed to be illustrated. To fulfill the cocone properties, we need the following equalities to hold:

$$k = u \circ f;$$

$$k = u \circ g.$$

Which can be simplified to one equality:

$$u \circ f = u \circ g,$$

which implies the existence of the morphism $k : B \longrightarrow ColimD$ but not necessarily mentions it.

1.2.11 Example In *Set*, the coequalizer of two functions $f, g : X \longrightarrow Y$ is the projection $p : Y \longrightarrow {}^{Y}\!/_{E}$ on the quotient set of Y by the least equivalence relation $E \subseteq Y \times Y$ which contains all pairs $\langle fx, gx \rangle$ for $x \in X$.

As E contains all the pairs $\langle fx, gx \rangle$, it is obvious that $p \circ f = p \circ g$.

Having any other set Z together with a morphism $z : Y \longrightarrow Z$ such that $z \circ f = z \circ g$, we can construct a unique morphism $m : {}^{Y}\!/_{E} \longrightarrow Z$ as follows.

Define $m : {}^{Y}\!/_{E} \longrightarrow Z$ by the assignment $m : py = [y] \mapsto zy$. This definition uses the fact that p is obviously surjective.

The verification of the fact that m does not depend on the choice of the representative is left as a small exercise to the reader.

The notions of a coproduct and a coequalizer are "representative" examples of colimits. In fact, the existence of all coproducts and coequalizers implies the existence of colimits.

1.2.12 Theorem (Maranda's Theorem) *For a category $\mathscr{X}$, the following are equivalent:*

(1) *$\mathscr{X}$ has all colimits.*

(2) *$\mathscr{X}$ has all coproducts and all coequalizers.*

1.3 Adjunction

The notion of adjunction captures the fact that sometimes two functors are related to each other in some particularly "good" way. They are not necessary inverse to each other but they still let us "go back and forth". Let us consider categories Vct_K and Set. Instead of looking at every object of Vct_K as a complicated structure, we could look only at the set of vectors in that space.

Let us take a closer look at the forgetful functor $U : Vct_K \longrightarrow Set$ and a functor $F : Set \longrightarrow Vct_K$ as described in Examples 1.1.7.

Each function $g : S \longrightarrow U(V)$ (function from some set to a used-to-be vector space) extends to a unique linear transformation $f : F(S) \longrightarrow V$ (a transformation between a used-to-be set and a vector space). This correspondence $\psi : g \mapsto f$ has an inverse $\varphi : f \mapsto f\mid_S$, the restriction of f to S, hence there is a bijection

$$\Theta_{S,V} : Vct_K(F(S), V) \cong Set(S, U(V))$$

of hom-sets.

The bijection $\Theta_{S,V}$ is defined "in the same way" for every set S and every vector space V, which means that the $\Theta_{S,V}$ are the components of a natural transformation Θ when both sides above are regarded as functors of S and V. It suffices to verify naturality in S and V separately.

Naturality in S means that for each morphism $h : S' \longrightarrow S$ the diagram

$$\begin{array}{ccc} Vct_K(F(S), V) & \xrightarrow{\Theta} & Set(S, U(V)) \\ {\scriptstyle Vct_K(Fh,V)}\big\downarrow & & \big\downarrow{\scriptstyle Set(H,U(V))} \\ Vct_K(F(S'), V) & \xrightarrow[\Theta]{} & Set(S', U(V)) \end{array}$$

commutes. Naturality in V is expressed analogously .

1.3.1 Definition Let $\mathscr{A}$ and $\mathscr{X}$ be categories. An *adjunction* between $\mathscr{X}$ and $\mathscr{A}$ is a triple $\langle F, U, \varphi\rangle$, where F and U are functors

$$\begin{array}{c} \mathscr{A} \\ {\scriptstyle F}\big\uparrow \; \big\downarrow{\scriptstyle U} \\ \mathscr{X} \end{array}$$

and φ is a function which assigns to each pair $X \in \mathscr{X}$, $A \in \mathscr{A}$ of objects a bijection of sets

$$\varphi = \varphi_{X,A} : \mathscr{A}(FX, A) \cong \mathscr{X}(X, UA)$$

which is natural in X and A.

Given such an adjunction, the functor F is said to be a *left adjoint* to U, while U is called a *right adjoint* to F, written $F \dashv U$.

1.3.2 Theorem *Suppose $F : \mathscr{A} \longrightarrow \mathscr{X}$ has a right adjoint. Then F preserves any colimit existing in $\mathscr{X}$.*

Two categories being connected by a pair of adjoint functors is an extremely useful property that has a lot of powerful implications (such as the one above). Therefore there are various ways to conclude that there is an adjunction between two categories. One of them will become handy later Chapter 2.

1.3.3 Definition Suppose $U : \mathscr{A} \longrightarrow \mathscr{X}$ is given. We say that an object F_0X, together with a morphism $\eta_X : X \longrightarrow UF_0X$, is a *free object* on X (w.r.t. U), provided that the following property is satisfied:

For every $f : X \longrightarrow UA$ there is a *unique* $f^{\#} : F_0X \longrightarrow A$ such that the triangle

$$\begin{array}{ccc} UF_0X & \xrightarrow{Uf^{\#}} & UA \\ {\scriptstyle \eta_X}\big\uparrow & \nearrow_{f} & \\ X & & \end{array}$$

commutes.

1.3.4 Theorem *Each adjunction $\langle F, U, \varphi \rangle$ between categories $\mathscr{X}$ and $\mathscr{A}$ is completely determined by the functor $U : \mathscr{A} \longrightarrow \mathscr{X}$ and an object $F_0X \in \mathscr{A}$ for each $X \in \mathscr{X}$ together with a morphism $\eta_X : X \longrightarrow UF_0X$ such that F_0X together with η_X is free on X.*

Then the functor F (a left adjoint) has object function F_0 and is defined on morphisms $h : X \longrightarrow X'$ by $UFh \circ \eta_X = \eta_{X'} \circ h$.

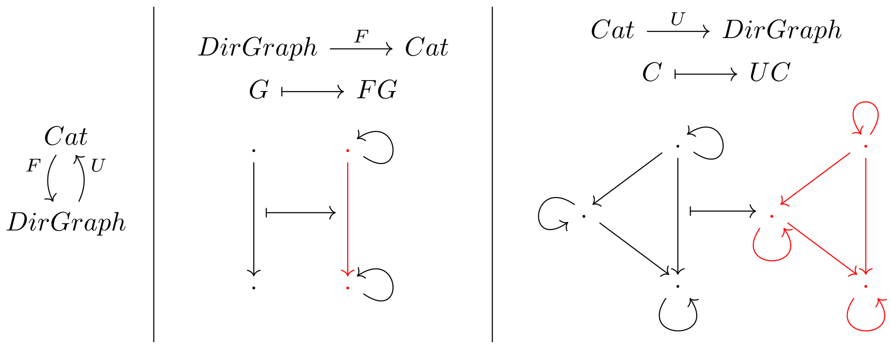
1.3.5 Remark There are more criteria that completely determine an adjunction except for Theorem 1.3.4, the interested reader can read more about them in [5].

1.3.6 Example We now proceed with showing another example of an adjunction. Those types of adjunctions are called free-forgetful adjunctions.

Whenever a functor $U : \mathscr{X} \longrightarrow \mathscr{A}$ ignores some data or structure in $\mathscr{X}$ and has a left adjoint $F : \mathscr{A} \longrightarrow \mathscr{X}$, the left adjoint constructs "free objects".

In the beginning of the present section such an adjunction was presented between Vct_K and Set. Another illustration lies in the connection between directed graphs and categories. Both involve vertices/objects and edges/morphisms. Every directed graph gives rise to a category, and every category is a directed graph (with extra data).

More formally, there is an adjunction involving the category *DirGraph* of directed graphs and the category *Cat* of categories.



In the picture above, the functor F turns a graph G into a category FG by viewing vertices as objects and edges as morphisms. It also inserts identity morphisms at each vertex, and declares the set of morphisms between two vertices to be the set of all finite paths between them. Composition is then concatenation of those paths.

On the other hand, the functor U assigns to a category C its underlying graph UC. It just forgets the identity and composition axioms, which are not needed to specify a graph.

Chapter 2

Categories of Metric Spaces

This chapter is devoted to the exploration of a "category of metric spaces". We use quotes here for the reason that the notion of a *metric space* allows a few variations, from which a category can be constructed. The main goal is to define a cocomplete category, which will be crucial for proceeding in the study of Vietoris-Rips complexes further in the thesis.

This chapter will be like looking for a light switch in a dark room, where we go step by step and bump into an obstacle at some point. This will prompt us to come up with new more abstract notions to proceed in the search.

2.1 Metric Spaces

One of the most natural operations in school geometry is to measure distance between points. This very fact lays at the heart of the development of geometry, which was initially the science concerned with making measurements.

In such circumstances, it is natural to try to abstract the properties of distance and allow for a more general notion of a space where distances could be measured.

This idea is the basis for the concept of metric spaces, we spell out the definition[1].

2.1.1 Definition By a *metric space* we mean an arbitrary set X together with a function $d : X \times X \longrightarrow \mathbb{R}$ which associates to every pair x, y of elements of X a non-negative real number $d(x, y)$ in such way that the following axioms are obeyed:

(1) $d(x, y) = 0$ if and only if $x = y$,

(2) $d(x, y) = d(y, x)$ for every $x, y \in X$ (symmetry),

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for every $x, y, z \in X$ (triangle inequality).

The members of the set X are conventionally called *points*, the function d is know as the *metric* and the value $d(x, y)$ of the metric corresponding to the points $x, y \in X$ is said to be the *distance* between these points.

We can refer to a metric space as (X, d_X) or just X, not mentioning the metric explicitly, but implying that X has an inner structure of an underlying set with a metric function.

2.1.2 Examples These are examples of metric spaces:

(1) *The discrete metric space.* This space consists of an arbitrary set X and a metric d defined by the formula:

$$d = \begin{cases} 0, & \text{for } x = y, \\ 1, & \text{for } x \neq y. \end{cases} \tag{2.1}$$

(2) *The real line* $\mathbb{R}$. This is the space consisting of the set $\mathbb{R}$ of the real numbers with metric defined by the formula $d(x, y) = |x - y|$.

[1] For a thorough treatment of the theory of metric spaces we refer the reader to the book [2].

(3) *The space of maps.* Suppose X is a non-empty set and Y is a metric space with the property $\sup\{d(y,y') \mid y,y' \in Y\} < \infty$. Consider the set P of all maps $f : X \longrightarrow Y$. Define the distance between two points f and g in P by the formula

$$d_P(f,g) = \sup\{d(f(x),g(x)) \mid x \in X\}.$$

It is easy to verify that all of the functions mentioned in above examples are indeed metrics.

2.1.3 Definition Suppose that $(X,d_X),(Y,d_Y)$ are metric spaces. We say that a map $f : X \longrightarrow Y$ is *non-expanding* when $d_Y(f(x),f(x')) \leq d_X(x,x')$ for every $x,x' \in X$.

2.1.4 Definition Let (A,d) be a metric space, let B be a subset of A and let $d_B : B \times B \longrightarrow \mathbb{R}$ be the restriction of d to B. Then d_B is the *subspace metric* of d with respect to B. The metric space (B,d_B) is called a *metric subspace* of (A,d).

We define *Met* to be the category of all metric spaces and non-expanding maps as morphisms.

2.1.5 Remark Indeed *Met* forms a category.

- The identity map $1_X : X \longrightarrow X$ is non-expanding and thus it is the identity morphism of (X,d) in *Met*.
- Having a pair $f : A \longrightarrow B, g : B \longrightarrow C$ of non-expanding maps between metric spaces (where A,B,C are metric spaces), their composition $g \circ f$ is a non-expanding map as well. Additionally, composition of non-expanding maps is associative.

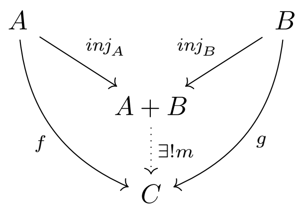
Met has a lot of interesting properties as a category. But it does not have the property we would want it to have: *Met* is not cocomplete.

2.1.6 Proposition *Met does not have all coproducts.*

Proof. We will prove this statement using a proof by contradiction.

Let us assume that *Met* has all binary coproducts.

Thus, for every two metric spaces A, B there exists a metric space $A+B$ together with two non-expanding maps $inj_A : A \longrightarrow A+B$ and $inj_B : B \longrightarrow A+B$ such that, for any other other metric space C together with two non-expanding maps $f : A \longrightarrow C, g : B \longrightarrow C$, there exists a unique $m : A+B \longrightarrow C$, such that the following diagram

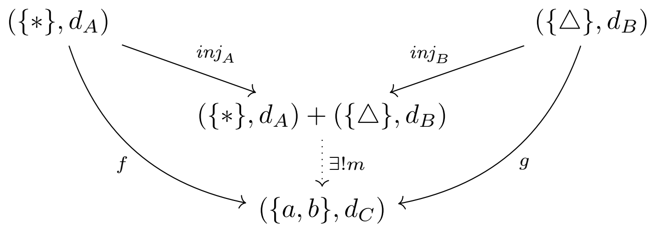


commutes.

Let A and B be one-element metric spaces, $A = (\{*\},d_A)$ and $B = (\{\triangle\},d_B)$. The coproduct then is a metric space $((\{*\},d_A)+(\{\triangle\},d_B),d_+)$ and $d_+(inj_A(*),inj_B(\triangle)) = r$, where r is some real number.

Let C be the two-element metric space $C = (\{a,b\},d_C)$ with $d_C(a,b) > r$.

The last part of our set up will be in defining f and g. Let $f : A \longrightarrow C$ be a non-expanding map defined by the assignment like $f : * \mapsto a$; $g : B \longrightarrow C$ defined by the assignment $g : \triangle \mapsto b$.



What would m look like? As a non-expanding map m should abide $d_+(inj_A(*), inj_B(\triangle)) \geq d_C(a, b)$, where d_+ is the coproduct metric.

However, the only way to construct a non-expanding map from the coproduct is to send each element from the coproduct to the same element in C, which makes the diagram non-commutative. Thus, it is impossible to define any mediating morphism $m : A + B \longrightarrow C$ in this case.

This particular coproduct $A + B$ does not exist in *Met*, therefore *Met* does not have all coproducts and therefore is not cocomplete. ■

For this reason we will define a slightly different category of *"extended"* metric spaces: spaces where the distance between points is allowed to be infinite. In the following section we properly define this category and try our luck in proving its cocompleteness.

2.2 Extended Metric Spaces

2.2.1 Definition By an *extended metric space* we mean an arbitrary set X together with a function

$$d : X \times X \longrightarrow \mathbb{R} \cup \{\infty\}$$

which associates to every pair x, y of elements of X a non-negative extended real number $d(x, y)$ in such way that the following axioms are obeyed:

(1) $d(x, y) = 0$ if and only if $x = y$,

(2) $d(x, y) = d(y, x)$ for every $x, y \in X$,

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for every $x, y, z \in X$.

Observe that every metric space is thus also an extended metric space.

2.2.2 Remark Analogously to Remark 2.1.5 we can define category Met_e of *extended* metric spaces as objects and non-expanding maps as morphisms.

2.2.3 Proposition *Category Met_e has all coproducts.*

Proof. Let us propose that the coproduct $\coprod_{i \in I} Di$ of extended metric spaces $\{Di \mid i \in I\}$ is a disjoint union the underlying sets of those metric spaces $\{Di \mid i \in I\}$ and an extended metric d_∞.

Recall from Examples 1.2.8 that every element $y \in \coprod_{i \in I} Di$ can be written as $inj_{D_j}(x)$ for some $j \in I$ and $x \in X_j$.

d_∞ is defined as

$$d_\infty(inj_{Di}(x), inj_{Dj}(y)) = \begin{cases} d_{Di}(x, y), & \text{if } i = j; \\ \infty, & \text{otherwise.} \end{cases} \tag{2.2}$$

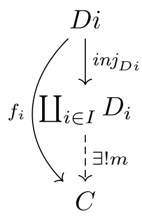
Observe that d_∞ satisfies all requirements for an extended metric:

(1) For each pair $x, y \in \coprod_{i \in I} Di$ the inequality $d_\infty(x, y) \geq 0$ holds.

(2) For symmetry $d_\infty(inj_{Di}(x), inj_{Dj}(y)) = \begin{cases} d_{Di}(x, y) = d_{Di}(y, x) = d_\infty(inj_{Dj}(y), inj_{Di}(x)), & \text{if } i = j; \\ \infty = d_\infty(inj_{Dj}(y), inj_{Di}(x)), & \text{otherwise.} \end{cases}$

(3) For every $x, y, z \in \coprod_{i \in I} Di$ the inequality $d_\infty(x, z) \leq d_\infty(x, y) + d_\infty(y, z)$ holds. This follows easily by case analysis.

To prove that the object above is indeed a coproduct we have to verify the universal property: given any $\{f_i : Di \longrightarrow C \mid i \in I\}$, we will show that there is a unique $m : \coprod_{i \in I} D_i \longrightarrow C$ for which all the triangles



commute.

In order for the diagram above to commute a potential morphism m must be a non expanding map defined by the assignment $m : inj_{Di}(x) \mapsto f_i(x)$ for $x \in Di$.

Since we defined $\coprod_{i \in I} Di$ as a disjoint union of $\{Di \mid i \in I\}$, there can exist only one such map. The last part of the proof is showing that m is indeed a non-expanding map.

For each $inj_{Di}(x), inj_{Dj}(y) \in \coprod_{i \in I} Di$ the inequality $d_C(m(inj_{Di}(x)), m(inj_{Dj}(y))) \leq d_\infty(inj_{Di}(x), inj_{Dj}(y))$ must hold.

We will divide the proof in two parts:

(1) Suppose $i = j$.
Thus $d_\infty(inj_{Di}(x), inj_{Dj}(y)) = d_{Di}(x, y)$ by definition.

$$\begin{aligned} d_C(m(inj_{Di}(x)), m(inj_{Dj}(y))) &= d_C(f_i(x), f_i(y)) \\ &\leq d_{Di}(x, y) \\ &= d_\infty(inj_{Di}(x), inj_{Dj}(y)). \end{aligned} \tag{2.3}$$

(2) Suppose $i \neq j$. Thus $d_\infty(inj_{Di}(x), inj_{Dj}(y)) = \infty$.

$$\begin{aligned} d_C(m(inj_{Di}(x)), m(inj_{Dj}(y))) &\leq \infty \\ &= d_\infty(inj_{Di}(x), inj_{Dj}(y)). \end{aligned} \tag{2.4}$$

This way we have shown that the coproduct in Met_e can be computed as their disjoint sum. ∎

Thus, to prove cocompleteness of Met_e, all we have left is to construct coequalizers in Met_e (by Theorem 1.2.12).

2.2.4 Proposition *Category Met_e has all coequalizers.*

Construction of coequalizers in Met_e is *possible* but rather tricky. We will divide it into several steps in the following section using the notion of an extended *pseudometric* space.

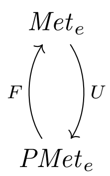
2.3 Extended Pseudometric Spaces

In this section we will show the construction of coequalizers in Met_e step by step.

(1) First we will describe a new category $PMet_e$ (of extended pseudometric spaces).

(2) The category $PMet_e$ has coequalizers. We will present their construction.

(3) We will show that there is an adjunction



(4) Using the left adjoint $F : PMet_e \longrightarrow Met_e$ we will show that the category Met_e also has coequalizers.

2.3.1 Definition By an *extended pseudometric space* we mean an arbitrary set X together with a function

$$d : X \times X \longrightarrow \mathbb{R} \cup \{\infty\}$$

which associates to every pair x, y of elements of X a non-negative extended real number $d(x, y)$ in such way that the following axioms are obeyed:

(1) $d(x, x) = 0$, for every $x \in X$,

(2) $d(x, y) = d(y, x)$ for every $x, y \in X$,

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for every $x, y, z \in X$.

Observe that every extended metric space is thus also an extended pseudometric space.

2.3.2 Remark Analogously to Remark 2.1.5 we can define category the $PMet_e$ of *extended* metric spaces as objects and non-expanding maps as morphisms.

2.3.3 Proposition *The category $PMet_e$ has all coequalizers.*

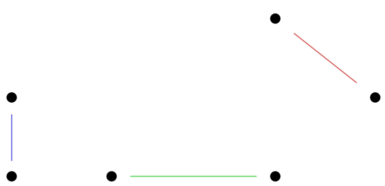
2.3.4 Remark Before presenting the proof itself, let us bring a little intuition behind it, so that formal definitions are not too confusing.

Let A, B be extended pseudometric spaces, $f, g : A \longrightarrow B$ morphisms.

$$\begin{array}{c} A \\ f \downarrow \; \downarrow g \\ B \\ c \downarrow \\ C \end{array}$$

Imagine that f and g are telling us, which points in B are going to have a *"teleport"* between them. This way those points are still different, but there is no distance between them. We group all the points connected by the same "teleport" with an equivalence relation E, and call them $[-]_E$.

Then we construct a metric which "iterates" through all the possible ways of going trough points and their "teleports".



Points with a "teleport" are connected with coloured lines between them. We can see that the path from the "blue" group to the "red" group is shorter, if we go though the "green" group.

This way, our metric is nothing more than the shortest possible way to "jump" between connected points, using their "teleports". Let us proceed with a formal construction of coequalizers in $PMet_e$.

Proof. Let A, B be extended pseudometric spaces, $f, g : A \longrightarrow B$ morphisms.

$$\begin{array}{c} A \\ f \downarrow \; \downarrow g \\ B \\ c \downarrow \\ C \end{array}$$

Let us propose that the coequalizer C is an extended metric space $({}^{B}\!/_{E}, d_C)$, where ${}^{B}\!/_{E}$ is a quotient set by the least equivalence relation $E \subseteq Y \times Y$, which contains all pairs $\langle fa, ga\rangle$, for each $a \in A$.

Morphism c is the projection of B to ${}^{B}\!/_{E}$: $c : b \mapsto [b]_E$, for each $b \in B$.

To define d_C we need to introduce two notions:

(1) Define function $d^* : [-]_E \times [-]_E \longrightarrow \mathbb{R}$ as

$$d^*([x]_E, [y]_E) = \inf\{d_B(\overline{x}, \overline{y}) \mid \overline{x}, \overline{y} \in B; \overline{x} \sim x; \overline{y} \sim y\}.$$

(2) Define an *admissible i-track* $A^i_{x,y}$ as the following set of numbers for each $i \in \mathbb{N}$:

- $A^0_{x,y} = \{d^*([x]_E, [y]_E)\}$;
- $A^1_{x,y} = \{d^*([x]_E, [x_0]_E) + d^*([x_0]_E, [y]_E) \mid x_0 \in B\}$;
- $A^2_{x,y} = \{d^*([x]_E, [x_0]_E) + d^*([x_0]_E, [x_1]_E) + d^*([x_1]_E, [y]_E) \mid x_0, x_1 \in B\}$;
- etc.

Define the extended pseudometric d_C as

$$d_C([x]_E, [y]_E) = \inf \bigcup_{i=0}^{\infty} A^i_{x,y}.$$

To prove that the construction above is indeed an extended pseudometric we have to verify two properties:

(1) The first property to be proved is symmetry, we check whether the equation $d_C([x]_E, [y]_E) = d_C([y]_E, [x]_E)$ holds. This is equivalent to proving that $A^i_{x,y} = A^i_{y,x}$ for each $i \in \mathbb{N}$

- $A^0_{x,y} = \{d^*([x]_E, [y]_E)\} = \{d^*([y]_E, [x]_E)\} = A^0_{y,x}$;
- $$\begin{aligned} A^1_{x,y} &= \{d^*([x]_E, [x_0]_E) + d^*([x_0]_E, [y]_E) \mid x_0 \in B\} \\ &= \{d^*([x_0]_E, [x]_E) + d^*([y]_E, [x_0]_E) \mid x_0 \in B\} = A^1_{y,x}; \end{aligned}$$
- $$\begin{aligned} A^2_{x,y} &= \{d^*([x]_E, [x_0]_E) + d^*([x_0]_E, [x_1]_E) + d^*([x_1]_E, [y]_E) \mid x_0, x_1 \in B\} \\ &= \{d^*([x_0]_E, [x]_E) + d^*([x_1]_E, [x_0]_E) + d^*([y]_E, [x_1]_E) \mid x_0, x_1 \in B\} = A^2_{y,x}; \end{aligned}$$
- etc.

The statements above use the obvious observation that d^* is symmetric.

(2) The second property is the triangle inequality, we check whether $d_C([x]_E, [y]_E) \leq d_C([x]_E, [z]_E) + d_C([z]_E, [y]_E)$, for each $[z]_E \in C$.

$$\begin{aligned} d_C([x]_E, [z]_E) + d_C([z]_E, [y]_E) &= \inf \bigcup_{i=0}^{\infty} A^i_{x,z} + \inf \bigcup_{i=0}^{\infty} A^i_{z,y} \\ &= \inf\{\bigcup_{i=0}^{\infty} A^i_{x,z} + \bigcup_{i=0}^{\infty} A^i_{z,y}\} \\ &\geq d_C([x]_E, [y]_E). \end{aligned} \tag{2.5}$$

Last inequality follows from the inclusion $\bigcup_{i=0}^{\infty} A^i_{x,z} + \bigcup_{i=0}^{\infty} A^i_{z,y} \subseteq \bigcup_{i=0}^{\infty} A^i_{x,y}$. The + sign means sum in the statement above, not coproduct.

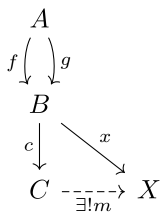
Next step to the definition of C as a coequalizer is to verify that $c : B \longrightarrow C$ is a non-expanding map in order for this projection to exist in $PMet_e$.

For each $b_1, b_2 \in B$ the following holds

$$\begin{aligned} d_C(c(b_1), c(b_2)) &= d_C([b_1], [b_2]) \\ &= \inf \bigcup_{i=0}^{\infty} A^i_{x,y} \\ &\leq A^0_{x,y} = d^*([b_1], [b_2]) \\ &\leq d_B(b_1, b_2). \end{aligned} \tag{2.6}$$

Observe that $c \circ f = c \circ g$ by the definition of c and E. Thus c coequalizes f and g.

In order for $C = (B/_E, d_C)$ to be a coequalizer we have to verify the universal property.



Let X be an extended pseudometric space, let $x : B \longrightarrow X$ coequalize f and g. There should exist a unique mediating morphism $m : C \longrightarrow X$ such that the triangle in the diagram above commutes.

In order for the diagram to commute, the morphism m must by defined by the assignment

$$m : cb \mapsto xb$$

for each $b \in B$. The above works, since c is surjective.

In addition, the morphism m must be a non-expanding map. To prove this we will show that any way to *"jump"* between *"connected points"* in C (refer to Remark 2.3.4 to recall what the phrases in quotes stand for) is shorter than an imprint of this path in the extended pseudometric space B. On the other hand the image under x of this imprint on B is shorter in X. Then we will use the infimum definition of metric in C to prove that m in non-expanding.

For each $c(b_1), c(b_2)$ the inequality $d_C(c(b_1), c(b_2)) \geq d_X(m \circ c(b_1), m \circ c(b_2))$ must hold.

$$\begin{array}{rcll} d_C(c(b_1), c(b_2)) & = & d_C([b_1], [b_2]) & \\ & = & \inf \bigcup_{i=0}^{\infty} A^i_{x,y} & \\ & \leq & t, & (\text{ for all } t \in A^k_{x,y}, \text{ for all } k \in \mathbb{N}) \\ & = & d^*([b_1], [p_0]) + ... + d^*([p_k], [b_2]) & \\ & = & \inf\{d_B(\overline{b_1}, \overline{p_0}) \mid \overline{b_1} \sim b_1, \overline{p_0} \sim p_0\} + ... & \\ & & ... + \inf\{d_B(\overline{p_k}, \overline{b_2}) \mid \overline{p_k} \sim p_k, \overline{b_2} \sim b_2\} & \\ & \leq & d_B(b'_1, p'_0) + ... + d_B(p'_k, b'_2), & (b'_i \sim b_i, \forall p'_i \sim p_i). \end{array}$$

On the other hand, for each b'_i, p'_i the inequality holds:

$$\begin{array}{rcll} d_B(b'_1, p'_0) + ... + d_B(p'_k, b'_2) & \geq & d_X(x(b'_1), x(p'_0)) + ... + d_X(x(p'_k), x(b'_2)), & (\text{ by } x \text{ being non-expanding}) \\ & \geq & d_X(x(b'_1), x(b'_2)) & (\text{ by triangle inequality}) \\ & = & d_X(m \circ c(b'_1), m \circ c(b'_2)) & \\ & = & d_X(m \circ c(b_1), m \circ c(b_2)). & \end{array}$$

Hence $d_X(m \circ c(b_1), m \circ c(b_2))$ is a lower bound for the set $\{\{d_B(b_{1i}, k_{0i}) \mid b_{1i} \sim b_1, p_{0i} \sim p_0\} + ... + \{d_B(p_{ki}, b_{2i}) \mid p_{ki} \sim p_k, b_{2i} \sim b_2\}\}$.

This implies $d_X(m \circ c(b_1), m \circ c(b_2)) \leq t, \forall t \in A^k_{x,y}, \forall k \in \mathbb{N}$.

Thus $d_X(m \circ c(b_1), m \circ c(b_2)) \leq \inf \bigcup_{i=0}^{\infty} A^i_{x,y} = d_C(c(b_1), c(b_2))$

■

To show that Met_e has coequlizers, we have to show that there is a connection between objects in Met_e and $PMet_e$. And indeed there is: we can construct an extended metric space from an extended pseudometric space in a universal way: we simply glue all the elements with distance 0 together.

2.3.5 Proposition *There is an adjunction between Met_e and $PMet_e$.*

PROOF. To construct an adjunction between $PMet_e$ and Met_e we will define a functor $U : Met_e \longrightarrow PMet_e$ and a special object function F_0. This function F_0 would assign to each object (P, d_P) of $PMet_e$ an object $F_0(P, d_P)$ in Met_e together with a morphism $\eta_{(P,d_P)} : (P, d_P) \longrightarrow UF_0(P, d_P)$ such that $F_0(P, d_P)$ with $\eta_{(P,d_P)}$ form a free object on (P, d_P) by the Definition 1.3.3. Then due to Theorem 1.3.4 there is an adjunction between $PMet_e$ and Met_e.

Define the functor $U : Met_e \longrightarrow PMet_e$ as a "classical" forgetful functor.

(1) Every extended metric space (X, d_X) is mapped by U to an exact copy of itself in category $PMet_e$.

(2) Every morphism $f : (X, d_X) \longrightarrow (Y, d_Y)$ in Met_e is also mapped to its exact copy in $PMet_e$.

$$Met_e \xrightarrow{\quad U \quad} PMet_e$$

$$\begin{array}{ccc} (X, d_X) & \longmapsto & U(X, d_X) = (X, d_X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Uf=f} \\ (Y, d_Y) & \longmapsto & U(Y, d_Y) = (Y, d_Y) \end{array}$$

Define an object function F_0 as follows. Every extended pseudometric space (P, d_P) is mapped to:

(1) An extended metric space $F_0(P, d_P) = (P', d_{P'})$, where:

(a) P' is a quotient set by the least equivalence relation $E' \subseteq P \times P$, which contains all pairs $\langle p_1, p_2 \rangle$, for each $p_1, p_2 \in P$ such that $d_P(p_1, p_2) = 0$.

(b) Extended metric $d_{P'}$ is defined by the rule

$$d_{P'}([p], [q]) = d_P(p, q).$$

It is obvious that $d_{P'}$ obeys all the axioms for an extended metric from Definition 2.2.1. As we work with classes of equivalence, we have to verify that $d_{P'}$ does not depend on the choice of a representative.

Let $p_1, p_2 \in P$ and $d_P(p_1, p_2) = 0$. We want to prove that for each $q \in P$ the equality $d_{P'}([p_1], [q]) = d_{P'}([p_2], [q])$ holds.

$$d_{P'}([p_1], [q]) \leq d_{P'}([p_1], [p_2]) + d_{P'}([p_2], [q]) = d_{P'}([p_2], [q]);$$
$$d_{P'}([p_2], [q]) \leq d_{P'}([p_2], [p_1]) + d_{P'}([p_1], [q]) = d_{P'}([p_1], [q]).$$

After reducing we get two inequalities:

$$d_{P'}([p_1], [q]) \leq d_{P'}([p_2], [q]);$$

$$d_{P'}([p_1], [q]) \geq d_{P'}([p_2], [q]).$$

Thus

$$d_{P'}([p_1], [q]) = d_{P'}([p_2], [q]).$$

The independence on the choice of the representative in the right equivalence class in $d_{P'}$ follows from the symmetry of $d_{P'}$.

(2) A morphism $\eta_{(P,d_P)} : (P, d_P) \longrightarrow UF_0(P, d_P)$ is defined by the following assignment:

$$\eta_{(P,d_P)} : p \mapsto [p]_{E'},$$

for each $p \in P$.

In order for $F_0(P, d_P)$ together with $\eta_{(P,d_P)} : (P, d_P) \longrightarrow UF_0(P, d_P)$ to be a free object on (P, d_P), the universal property must be satisfied. This means that for every $f : (P, d_P) \longrightarrow U(A, d_A)$, where $(A, d_A) \in ob(Met_e)$, there is a unique $f^{\#} : F_0(P, d_P) \longrightarrow (A, d_A)$ such that the triangle below

$$F(P, d_P) \xrightarrow{f^{\#}} (A, d_A)$$

$$\begin{array}{ccc} UF_0(P, d_P) & \xrightarrow{Uf^{\#}} & U(A, d_A) \\ {\scriptstyle \eta_{(P,d_P)}}\uparrow & \nearrow{\scriptstyle f} & \\ (P, d_P) & & \end{array}$$

commutes.

By definition of U, the morphism $Uf^{\#}$ is equal to $f^{\#}$. Thus, we only need to prove that there exists a unique $Uf^{\#}$ with the properties needed.

In order for the triangle above to commute, $Uf^{\#}$ must be defined by the assignment:

$$Uf^{\#} : [p]_{E'} \mapsto fp.$$

To verify that $Uf^{\#}$ is defined correctly we have to prove two properties:

(1) The value of the morphism $Uf^{\#}$ at a given equivalence class does not depend on the choice of the representative.

Let $p, p' \in (P, d_P)$ and $d_P(p, p') = 0$:

$$d_P(p, p') = 0 \text{ implies } d_{UA}(p, p') = 0 \text{ implies } fp = fp' \text{ by the definition of } U.$$

(2) The morphism $Uf^{\#}$ is non-expanding. Let $[p], [p'] \in UF_0(P, d_P)$:

$$d_{U(A,d_A)}(Uf^{\#}([p]), Uf^{\#}([p']))) = d_{U(A,d_A)}(fp, fp') \leq d_P(p, p') = d_{P'}([p], [p']).$$

Thus, $F_0(P, d_P)$ together with $\eta_{(P,d_P)}$ is a free object on (P, d_P) for each $(P, d_P) \in PMet_e$ and by Theorem 1.3.4 functor F is a left adjoint. ■

2.3.6 Proposition *The category Met_e has all coequalizers.*[2]

Proof. Let A, B be extended metric spaces and let $f, g : A \longrightarrow B$ be morphisms in Met_e.

We have successfully proved that there exists a coequalizer of Uf and Ug in $PMet_e$.

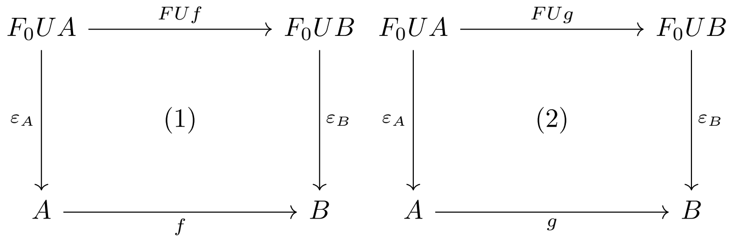
$$UA \underset{Ug}{\overset{Uf}{\rightrightarrows}} UB \xrightarrow{c} C$$

By Theorem 1.3.2, the functor F as defined in Proposition 2.3.5 preserves all colimits. Thus F_0C is a coequalizer of FUf and FUg.

$$F_0UA \underset{FUg}{\overset{FUf}{\rightrightarrows}} F_0UB \xrightarrow{Fc} F_0C$$

[2]This fact is an instance of a more abstract statement about reflective categories. You can read more about it in [7].

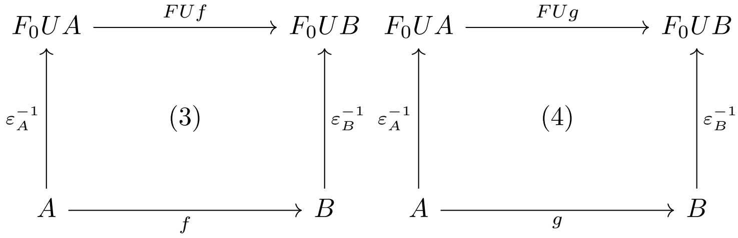
Notice that we can define a natural transformation $\varepsilon : FU \longrightarrow 1_{Met_e}$. Define ε by components $\varepsilon_A : F_0UA \longrightarrow A$. The morphism ε_A maps an equivalence class $[x]_{E'}$ to its representative x. Definitions of U and F imply that every $[x]_{E'}$ consists of only one point. It is easy to see that ε is indeed a natural transformation, thus the squares $(1), (2)$



(2.7)

commute.

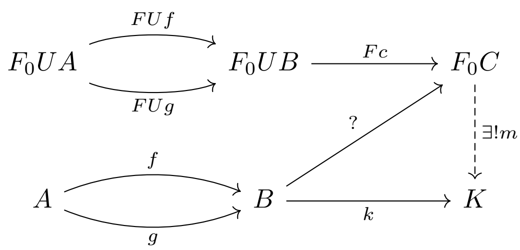
Since each ε_A is invertible, we obtain an inverse natural transformation ε^{-1}, thus the squares $(3), (4)$



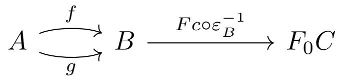
(2.8)

commute as well.

We want to prove that F_0C is not only a coequalizer of FUf and FUg, but also of f and g. To achieve this goal, we have to define a morphism $? : B \longrightarrow F_0C$, which would coequalize f and g such that the universal property is satisfied.



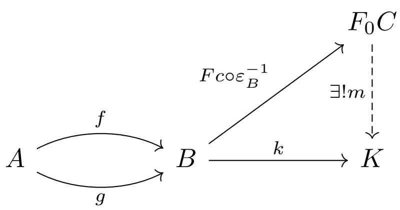
(1) To prove that F_0C is a coequalizer for f and g, we will first show that $Fc \circ \varepsilon_B$ coequalizes f and g.



We will show that $Fc \circ \varepsilon_B^{-1} \circ f = Fc \circ \varepsilon_B^{-1} \circ g$:

$$\begin{aligned} Fc \circ \varepsilon_B^{-1} \circ f &= Fc \circ FUf \circ \varepsilon_A^{-1} \text{ by (2.8)} \\ &= Fc \circ FUg \circ \varepsilon_A^{-1} \text{ by the properties of coequalizer } Fc \\ &= Fc \circ \varepsilon_B^{-1} \circ g \text{ by (2.8).} \end{aligned} \tag{2.9}$$

(2) Next step is to verify the universal property. Let K be an extended metric space and $k : B \longrightarrow K$ such that $k \circ f = k \circ g$. We want to prove that there exists a unique morphism $m : F_0C \longrightarrow K$ such that the triangle in the diagram below



commutes.

We will show that $k \circ \varepsilon_B$ coequlizes FUf and FUg:

$$
\begin{aligned}
k \circ f = k \circ g \text{ iff} \\
k \circ \varepsilon_B \circ FUf \circ \varepsilon_A^{-1} = k \circ \varepsilon_B \circ FUg \circ \varepsilon_A^{-1} \text{ by (2.7) and (2.8) iff} \\
k \circ \varepsilon_B \circ FUf = k \circ \varepsilon_B \circ FUg.
\end{aligned}
\tag{2.10}
$$

Thus by the universal property of Fc, there exists a unique $m : F_0C \longrightarrow K$ such that

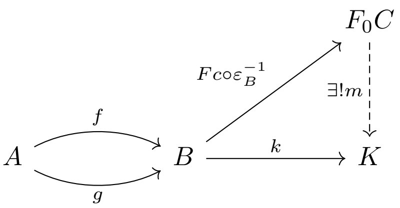
$$m \circ Fc = k \circ \varepsilon_B.$$

Which is equivalent to

$$m \circ Fc \circ \varepsilon_B^{-1} = k.$$

Thus m is the unique mediating morphism $F_0C \longrightarrow K$ for F_0C together with $Fc \circ \varepsilon_B^{-1}$.

Hence F_0C is a coequalizer for f and g and whole diagram below



commutes. ■

The following fact will be crucial in Chapter 3.

2.3.7 Theorem *Met_e is cocomplete.*

PROOF. We have proved that Met_e has all coproducts in Proposition 2.2.3, so by Proposition 2.3.6 and Theorem 1.2.12 we conclude that Met_e is cocomplete. ■

Chapter 3

Vietoris-Rips Complexes and Geometric Realization

In this chapter we will introduce the reader to the notion of a simplicial set and continue with defining Vietoris-Rips complexes with a classical metric spaces approach. See [6] and [3] for a more thorough treatment.

Then we continue with more advanced category theory notions, which allow us to define Vietoris-Rips complexes as a functor and define geometric realization as a categorical concept.

3.1 Brief Introduction to Simplicial Sets

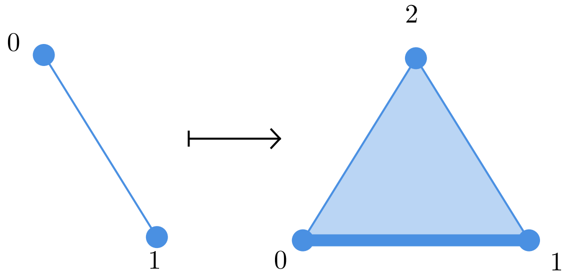
We have already mentioned the concept of *simplices* in previous chapters. Now we will properly define them.

3.1.1 Definition Let Δ be the category whose objects are finite, non-empty, linearly ordered sets $[n] = \{0, 1, ..., n\}$ with the usual ordering, we will refer to them as *simplices*. Morphisms in Δ are order-preserving functions (between those sets).

In Examples 1.1.3 we illustrated simplices and order preserving functions as plain sequences of numbers and arrows between them. Now we will give them a geometrical interpretation.

A simplex (of dimension n) could be imagined as a set of $(n+1)$ points generating a n-dimensional polyhedron. So [1] could be imagined as two sides of a line segment, [2] — vertices of a triangle, [3] — vertices of a tetrahedron, etc.

A map $[1] \longrightarrow [2]$ defined by the assignment $0 \mapsto 0; 1 \mapsto 1;$ could be illustrated as a map from a line segment to a triangle.



The simplex category Δ is generated by two particularly important families of morphisms (maps) called *coface* maps d and *codegeneracy* maps s.

3.1.2 Definition Coface and codegeneracy maps are defined respectively as

$$d^i : [n-1] \longrightarrow [n] \qquad 0 \leq i \leq n; \qquad s^i : [n+1] \longrightarrow [n] \qquad 0 \leq i \leq n;$$

$$d^i(k) = \begin{cases} k, & k < i, \\ k+1, & k \geq i. \end{cases} \qquad s^i(k) = \begin{cases} k, & k \leq i, \\ k-1, & k \geq i. \end{cases} \tag{3.1}$$

Thus d^i is the only (order-preserving) injection $[n-1] \longrightarrow [n]$, that "misses" i; and s^i is the only (order-preserving) surjection $[n+1] \longrightarrow [n]$ that "hits" i twice.

These morphisms satisfy several obvious relations:

$$\begin{aligned} d^j d^i &= d^i d^{j-1}, && i < j, \\ s^j s^i &= s^i s^{j+1}, && i \leq j, \\ s^j d^i &= \begin{cases} 1, & i = j, j+1, \\ d^i s^{j-1}, & i < j, \\ d^{i-1} s^j, & i > j+1. \end{cases} \end{aligned}$$

It is not difficult to verify that every morphism of Δ can be expressed as a composite of coface and codegeneracy maps.

3.1.3 Remark Any simplex obtained as an image of a codegeneracy map $s^i : [n+1] \longrightarrow [n]$ is called a *degenerate* $[n+1]$-simplex.

3.1.4 Definition A *simplicial set* is a presheaf $\Delta^{op} \longrightarrow Set$. More generally, for any category $\mathscr{C}$, a *simplicial object* in $\mathscr{C}$ is a functor $X : \Delta^{op} \longrightarrow \mathscr{C}$.

Let $X : \Delta^{op} \longrightarrow \mathscr{C}$ be a simplicial set. It is standard to write X_n for the set $X[n]$ and call its elements $[n]$-*simplices*. We visualize an $[n]$-simplex $x \in X_n$ as an n-dimensional polyhedron, whose $(n+1)$ vertices are ordered $0, ..., n$.

If X is a simplicial set, the maps

$$\begin{aligned} d_i &= Xd^i : X_n \longrightarrow X_{n-1} && 0 \leq i \leq n; \\ s_i &= Xs^i : X_n \longrightarrow X_{n+1} && 0 \leq i \leq n \end{aligned}$$

are called *face* and *degeneracy* maps, respectively: each face map assigns, to each $x \in X_n$, an $[n-1]-$simplex $d_i(x) \in X_{n-1}$. Notice, that there are $(n+1)$ such maps. By convention, the face $d_i(x)$ is the one not containing the i-th vertex of x.

To each $x \in X_n$, the degeneracy maps associate an $[n+1]$-simplex $s_i(x) \in X_{n+1}$. The $[n+1]$-simplex $s_i(x)$ has x as its i-th and $(i+1)$-faces. The intuition is that the projection that collapses the edge from the i-th to the $(i+1)$-th vertex to a point returns the n-simplex x.

The morphisms d_i and s_i will then satisfy relations dual to the equations for coface and codegeneracy maps in Definition 3.1.2.

We write *sSet* for the category of simplicial sets, which is simply the functor category $[\Delta^{op}, Set]$ (recall Definition 1.1.13), thus morphisms $f : X \longrightarrow Y$ are natural transformations.

In fact, the data of a simplicial set are completely specified by the sets X_n and the maps d_i, s_i (with some relations to be satisfied) in the sense of the following alternative definition.

3.1.5 Definition (Alternative) A *simplicial set* X is a collection of sets X_n for each integer $n \geq 0$ together with functions $d_i : X_n \longrightarrow X_{n-1}$ and $s_i : X_n \longrightarrow X_{n+1}$ for all $0 \leq i \leq n$ and for each n satisfying the following relations:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i, && i < j, \\ s_i s_j &= s_{j+1} s_i, && i \leq j, \\ d_i s_j &= \begin{cases} 1, & i = j, j+1, \\ s_{j-1} d_i, & i < j, \\ s_j d_{i-1}, & i > j+1. \end{cases} \end{aligned}$$

3.2 The Classical Approach to Vietoris-Rips Complexes

3.2.1 Definition ([4]) Given a metric space (X, d_X) and a real number $\varepsilon \geq 0$, its *Vietoris-Rips complex* is a simplicial set with the set $V_\varepsilon(X, d_X)_n$ of $[n]$-simplices defined by

$$V_\varepsilon(X, d_X)_n = \{(x_0, ..., x_n) \in X^{n+1} \mid \max_{i,j}\{d_X(x_i, x_j)\} \leq \varepsilon\}.$$

Morphisms d_i are defined as follows:

$$d_i : V_\varepsilon(X, d_X)_n \longrightarrow V_\varepsilon(X, d_X)_{n-1}, \qquad 0 \leq i \leq n,$$
$$(x_0, ..., x_n) \mapsto (x_{d^i(0)}, ..., x_{d^i(n-1)}).$$

Morphisms s_i are defined as follows:

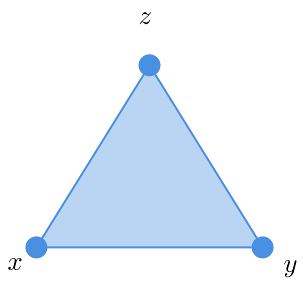
$$s_i : V_\varepsilon(X, d_X)_n \longrightarrow V_\varepsilon(X, d_X)_{n+1}, \qquad 0 \leq i \leq n,$$
$$(x_0, ..., x_n) \mapsto (x_{s^i(0)}, ..., x_{s^i(n+1)}).$$

Thus, an $(n + 1)$-tuple $(x_0, ..., x_n)$ is declared to be an $[n]$-simplex, whenever the set $\{x_0, ..., x_n\}$ has a diameter at most ε in the metric space (X, d_X).

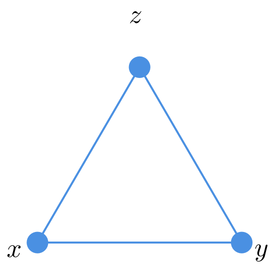
3.2.2 Examples The following are examples of Vietoris-Rips complexes:

(1) An abstract example. Having a metric space (X, d_X) consisting of three points x, y, z and $d_X(x, y) = 3, d_X(y, z) = 2, d_X(x, z) = 5$ and varying the parameter ε, its Vietoris-Rips complex is the following simplicial set:

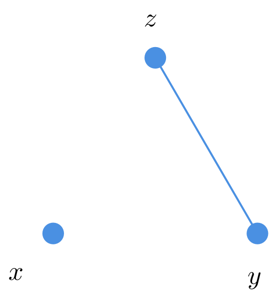
(a) $\varepsilon \geq 5$



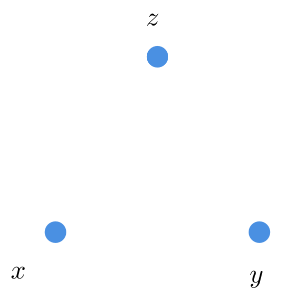
(b) $3 \leq \varepsilon < 5$



(c) $2 \leq \varepsilon < 3$



(d) $0 \leq \varepsilon < 2$



(2) An applied example [8]. Vietoris-Rips Complexes can be used for a modelling purposes in wireless networks.

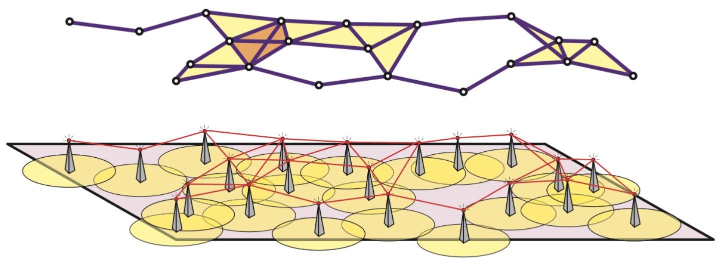


Figure 3.1: Source [8]

A collection of sensor nodes generates a cover in the workspace as seen in the bottom part of the figure. The Vietoris-Rips complex of the network is an abstract simplicial complex which has no localization or coordinate data as seen in top part of the figure. In the example illustrated, the Vietoris-Rips complex encodes the communication network as one [3]-simplex, eleven [2]-simplices, and seven [1]-simplices connected as shown. The 'holes' in this Vietoris-Rips complex reflect the holes in the sensor cover.

3.3 Categorical Approach to Vietoris-Rips Complexes

In this section we describe Vietoris-Rips complexes with the means of category theory. We start with more advanced category theoretical notions which will allow us to make several important conclusions about Vietoris-Rips complexes.

3.3.1 Presheaf Categories and Adjoints

3.3.1 Definition ([1]) Given any functor $D : \mathscr{D} \longrightarrow \mathscr{K}$ define the functor

$$\widetilde{D} : \mathscr{K} \longrightarrow [\mathscr{D}^{op}, Set]$$

as follows:

(1) $\widetilde{D}$ assigns to each object K in $\mathscr{K}$ a presheaf $\widetilde{D}K = \mathscr{K}(D\_\_, K) : \mathscr{D}^{op} \longrightarrow Set$.

  (a) Given an object d in $\mathscr{D}$, we define $\widetilde{D}Kd$ to be the hom-set $\mathscr{K}(Dd, K)$.

  (b) Given $\delta : d \longrightarrow d'$ in $\mathscr{D}$, the map $\widetilde{D}K\delta$ is the map $\mathscr{K}(D\delta, K) : \mathscr{K}(Dd', K) \longrightarrow \mathscr{K}(Dd, K)$ defined by pre-composition with $D\delta$.

(2) The action of $\widetilde{D}$ on morphisms of $\mathscr{K}$ is defined as follows:

Given $\omega : k \longrightarrow k'$ the natural transformation

$$\widetilde{D}\omega = \mathscr{K}(D\_\_, K) \longrightarrow \mathscr{K}(D\_\_, K')$$

is defined by post-composition with ω.

3.3.2 Remark It is easy to verify that the construction of $\widetilde{D}$ in Definition 3.3.1 indeed yields a functor.

The functor $\widetilde{D} : \mathscr{K} \longrightarrow [\mathscr{D}^{op}, Set]$ will be the right adjoint of a functor $\_\_ * D : [\mathscr{D}^{op}, Set] \longrightarrow \mathscr{K}$. Explicitly, the value of $\_\_ * D$ at an object X of $[\mathscr{D}^{op}, Set]$ is a particular type of a colimit called a *coend*. We will not describe coends in their full generality but apply them in our special case.

3.3.3 Definition For any set S and object C of $\mathscr{C}$, the *copower* $S \cdot C$ of C by S is the coproduct $\coprod_S C$ of copies of C indexed by S.

3.3.4 Lemma *For any $C, Y \in \mathscr{C}$ and $S \in Set$ we have the following bijection*

$$\mathscr{C}(S \cdot C, Y) \cong Set(S, \mathscr{C}(C, Y))$$

natural in Y.

Proof. Immediate from the properties of coproducts. ■

In particular, we may form copowers $Xd' \cdot Dd$, for any $d, d' \in \mathscr{D}$ and any $X : \mathscr{D}^{op} \longrightarrow Set$. A morphism $f : d \longrightarrow d'$ of $\mathscr{D}$ induces a map

$$Xd' \cdot Df : Xd' \cdot Dd \longrightarrow Xd' \cdot Dd',$$

defined by requiring that the following diagram:

$$\begin{array}{ccc} \coprod_{x'\in Xd'} Dd & \xrightarrow{Xd'\cdot Df} & \coprod_{x'\in Xd'} Dd' \\ {\scriptstyle inj_{x'}}\uparrow & & \nearrow{\scriptstyle inj_{x'}} \\ Dd & \xrightarrow[Df]{} & Dd' \end{array} \tag{3.2}$$

to commute for every $x' \in Xd'$.

Intuitively $Xd' \cdot Df$ applies Df to the copy of Dd in the component corresponding to $x' \in Xd'$ and includes it in the component corresponding to x' in $Xd' \cdot Dd'$.

The morphism $f : d \longrightarrow d'$ also induces a map

$$Xf \cdot Dd : Xd' \cdot Dd \longrightarrow Xd \cdot Dd,$$

defined by the universal property of coproducts:

$$\begin{array}{ccc} \coprod_{x'\in Xd'} Dd & \xrightarrow{Xf\cdot Dd} & \coprod_{x\in Xd} Dd \\ {\scriptstyle inj_{x'}}\uparrow & \nearrow{\scriptstyle inj_{Xf(x')}} & \\ Dd & & \end{array} \tag{3.3}$$

which maps the component corresponding to $x \in Xd'$ to the component corresponding to $Xf(x) \in Xd$.

3.3.5 Definition Consider the diagram whose objects are copowers $Xd' \cdot Dd$ for $d, d' \in \mathscr{D}$ and $X : \mathscr{D}^{op} \longrightarrow Set$; and whose arrows consist of morphisms $Xd' \cdot Df : Xd' \cdot Dd \longrightarrow Xd' \cdot Dd'$ and $Xf \cdot Dd : Xd' \cdot Dd \longrightarrow Xd \cdot Dd$ for each $f : d \longrightarrow d' \in \mathscr{D}$. A *wedge* under this diagram is an object e of $\mathscr{K}$ together with morphisms $\gamma_d : Xd \cdot Dd \longrightarrow e$ such that the squares

$$\begin{array}{ccc} Xd' \cdot Dd & \xrightarrow{Xd'\cdot Df} & Xd' \cdot Dd' \\ {\scriptstyle Xf\cdot Dd}\downarrow & & \downarrow{\scriptstyle \gamma_{d'}} \\ Xd \cdot Dd & \xrightarrow[\gamma_d]{} & e \end{array}$$

commute for each f.

The *coend*, denoted by $\int^d Xd \cdot Dd$, is defined to be a universal wedge: having any other wedge e together with morphisms γ_d, there exists a unique morphism $m : \int^d Xd \cdot Dd \longrightarrow e$ such that the following diagram

$$\begin{array}{ccc} Xd \cdot Dd & \xrightarrow{\alpha_d} & \int^d Xd \cdot Dd \\ & \searrow^{\gamma_d} & \downarrow {\scriptstyle \exists ! m} \\ & & e \end{array}$$

commutes for all $d \in \mathscr{D}$.

3.3.6 Remark ([6]) Equivalently, $\int^d Xd \cdot Dd$ is a coequalizer of the diagram

$$\coprod_{f:d\longrightarrow d'} Xd' \cdot Dd \overset{\omega}{\underset{\theta}{\rightrightarrows}} \coprod_d Xd \cdot Dd \dashrightarrow \int^d Xd \cdot Dd,$$

where ω and θ are defined by the universal property of coproducts:

$$\begin{array}{ccc} Xd' \cdot Dd & \xrightarrow{Xd' \cdot Df} & Xd' \cdot Dd' \\ {\scriptstyle inj_f}\downarrow & & \downarrow{\scriptstyle inj_{d'}} \\ \coprod_{f:d\longrightarrow d'} Xd \cdot Dd & \xrightarrow[\omega]{} & \coprod_d Xd \cdot Dd \end{array} \qquad \begin{array}{ccc} Xd' \cdot Dd & \xrightarrow{Xf \cdot Dd} & Xd \cdot Dd \\ {\scriptstyle inj_f}\downarrow & & \downarrow{\scriptstyle inj_d} \\ \coprod_{f:d\longrightarrow d'} Xd \cdot Dd & \xrightarrow[\omega]{} & \coprod_d Xd \cdot Dd \end{array}$$

where ω and θ are defined by the universal property of coproducts.

Any cocomplete category $\mathscr{K}$ thus has all coends in the sense of Definition 3.3.5.

Now we have defined everything we need to give a proper definition of $\_ * D$.

3.3.7 Definition Define the functor $\_ * D : [\mathscr{D}^{op}, Set] \longrightarrow \mathscr{K}$ for any cocomplete $\mathscr{K}$ as follows:

(1) Action on objects is defined by the assignment:

$$X * D = \int^d Xd \cdot Dd.$$

(2) If $\alpha : X \longrightarrow Y$ is a map of $X : \mathscr{D}^{op} \longrightarrow Set$ and $Y : \mathscr{D}^{op} \longrightarrow Set$, then $\alpha * D : X * D \longrightarrow Y * D$ is defined by the universal property of coends.

3.3.8 Remark Since the diagram below commutes (as we will show now), the morphism $\alpha : X \longrightarrow Y$ indeed induces a wedge $\int^d Yd \cdot Dd$ together with morphisms $\beta_d \circ (\alpha_d \cdot Dd)$ from the diagram

$$\begin{array}{ccccc} Xd' \cdot Dd & & \xrightarrow{Xd' \cdot Df} & & Xd' \cdot Dd' \\ & \searrow^{\alpha_{d'} \cdot Dd} & & & \searrow^{\alpha_{d'} \cdot Dd'} \\ {\scriptstyle Xf \cdot Dd}\downarrow & & Yd' \cdot Dd & \xrightarrow{Yd' \cdot Df} & Yd' \cdot Dd' \\ Xd \cdot Dd & & {\scriptstyle Yf \cdot Dd}\downarrow & & \downarrow{\scriptstyle \beta_{d'}} \\ & \searrow^{\alpha_d \cdot Dd} & & & \\ & & Yd \cdot Dd & \xrightarrow[\beta_d]{} & \int^d Yd \cdot Dd \end{array} \tag{3.4}$$

for X to the object $Y * D$, where each $\alpha_{d'} \cdot Dd$ is defined by the universal property of coproducts:

$$\begin{array}{ccc} Dd & \xrightarrow{inj_{\alpha_{d'}(x')}} & \\ {\scriptstyle inj_{x'}}\downarrow & \searrow & \\ \coprod_{x' \in Xd'} Dd & \xrightarrow{\alpha_{d'} \cdot Dd} & \coprod_{y' \in Yd'} Dd \end{array} \tag{3.5}$$

By the definition of (3.3), (3.5) the following diagram:

$$\begin{tikzcd}
 & Xd' \cdot Dd \arrow[r, "\alpha_{d'} \cdot Dd"] \arrow[d, "Xf \cdot Dd"'] & Yd' \cdot Dd \arrow[d, "Yf \cdot Dd"] \\
Dd \arrow[ur, "inj_{x'}"] \arrow[dr, "inj_{Xf(x')}"'] \arrow[urr, bend left, "inj_{\alpha_{d'}(x')}"] \arrow[drr, bend right, "inj_{\alpha_{d'}(Xf(x'))}"'] & & \\
 & Xd \cdot Dd \arrow[r, "\alpha_d \cdot Dd"] & Yd \cdot Dd
\end{tikzcd} \tag{3.6}$$

is commutative.

So is the following diagram

$$\begin{tikzcd}
Dd \arrow[r, "Df"] \arrow[d, dashed, "inj_{x'}"] \arrow[dd, bend right, "inj_{\alpha_{d'}(x')}"'] & Dd' \arrow[d, dashed, "inj_{x'}"] \arrow[dd, dashed, bend left, "inj_{\alpha_{d'}(x')}"] \\
Xd' \cdot Dd \arrow[r, "Xd' \cdot Df"] \arrow[d, "\alpha_{d'} \cdot Dd"] & Xd' \cdot Dd' \arrow[d, "\alpha_{d'} \cdot D'"] \\
Yd' \cdot Dd \arrow[r, "Yd' \cdot Df"] & Yd' \cdot Dd'
\end{tikzcd} \tag{3.7}$$

by definition of (3.2), (3.5).

Hence the universal property of the coend defines a map $\alpha * D : X * D \longrightarrow Y * D$.

3.3.9 Theorem *The functor* $\_ * D$ *is a left adjoint to* $D : \mathscr{K} \longrightarrow [\mathscr{D}^{op}, Set]$ *for any cocomplete* $\mathscr{K}$*.*

Proof. We will prove the adjointness of two functors by showing the bijection of sets

$$\mathscr{K}(X * D, K) \cong [\mathscr{D}^{op}, Set](X, \widetilde{D}K)$$

for each $X \in [\mathscr{D}^{op}, Set]$ and for each $K \in \mathscr{K}$.

Each $\gamma : X \longrightarrow \widetilde{D}K \in [\mathscr{D}^{op}, Set](X, \widetilde{D}K)$ is a natural transformation and thus consists of components $\gamma_d : Xd \longrightarrow \widetilde{D}Kd$. From the properties of natural transformation, we know that the following diagram

$$\begin{tikzcd}
Xd' \arrow[r, "\gamma_{d'}"] \arrow[d, "Xf"'] & \mathscr{K}(Dd', K) \arrow[d, "\mathscr{K}(Df,K)"] \\
Xd \arrow[r, "\gamma_d"] & \mathscr{K}(Dd, K)
\end{tikzcd} \tag{3.8}$$

commutes for all $f : d \longrightarrow d'$.

Inspecting the elements we obtain the identity

$$\begin{tikzcd}
x' \arrow[r, mapsto, "\gamma_{d'}"] \arrow[d, mapsto, "Xf"'] & & & \gamma_{d'}(x') \arrow[d, mapsto, "\mathscr{K}(Df,K)"] \\
Xf(x') \arrow[r, mapsto, "\gamma_d"'] & \gamma_{d'}(Xf(x')) \;=\; \gamma_d(x) & = & \gamma_{d'}(x') \circ Df
\end{tikzcd}$$

for all $x \in Xd'$ for all $d' \in \mathscr{D}$.

From the property of copowers from Lemma 3.3.4, we know that the following sets are naturally isomorphic:

$$Set(Xd, (\widetilde{D}K)d) \cong \mathscr{K}(Xd \cdot Dd, K).$$

Thus there exists a bijection of sets defined by the assignment:

$$\gamma_d : Xd \longrightarrow (\widetilde{D}K)d \longmapsto \gamma'_d : Xd \cdot Dd \longrightarrow K,$$

where γ'_d is defined by the universal property of copower as follows:

$$\begin{array}{ccc} \coprod_{x \in Xd} Dd & \xrightarrow{\gamma'_d} & K \\ \uparrow{\scriptstyle inj_x} & \nearrow{\scriptstyle \gamma_d(x)} & \\ Dd & & \end{array} \tag{3.9}$$

With this definition of γ' the square

$$\begin{array}{ccc} Xd' \cdot Dd & \xrightarrow{Xd' \cdot Df} & Xd' \cdot Dd' \\ \downarrow{\scriptstyle Xf \cdot Dd} & & \downarrow{\scriptstyle \gamma'_{d'}} \\ Xd \cdot Dd & \xrightarrow[\gamma'_d]{} & K \end{array} \tag{3.10}$$

commutes for all $d, d' \in \mathscr{D}$ as we show below. The object K together with the morphisms $\gamma'_d : Xd \cdot Dd \longrightarrow K$ is thus a wedge.

To show that (3.10) commutes, we will show that

$$\begin{array}{cccc} Dd & & & \\ & \searrow{\scriptstyle inj_{x'}} & & \\ & Xd' \cdot Dd & \xrightarrow{Xd' \cdot Dd} & Xd' \cdot Dd' \\ & \downarrow{\scriptstyle Xf \cdot Dd} & & \downarrow{\scriptstyle \gamma'_{d'}} \\ & Xd \cdot Dd & \xrightarrow[\gamma'_d]{} & K \end{array}$$

commutes for all $f : d \longrightarrow d'$ and $x' \in Xd'$.

Observe that the diagram

$$\begin{array}{ccc} Dd & \xrightarrow{inj_{x'}} & Xd' \cdot Dd \\ & \searrow{\scriptstyle inj_{Xf(x')}} & \downarrow{\scriptstyle Xf \cdot Dd} \\ & & Xd \cdot Dd \\ & \searrow{\scriptstyle \gamma_d(Xf(x'))} & \downarrow{\scriptstyle \gamma'_d} \\ & & K \end{array}$$

commutes: upper triangle from the definition of $Xd \cdot Dd$ (see (3.3)), lower triangle from the definition of γ'_d (see (3.9)).

And that

$$\begin{array}{ccc} Dd & \xrightarrow{Df} & Dd' \\ \downarrow{\scriptstyle inj_{x'}} & & \downarrow{\scriptstyle inj_{x'}} \\ Xd' \cdot Dd & \xrightarrow{Xd' \cdot Df} & Xd' \cdot Dd' \\ & & \downarrow{\scriptstyle \gamma'_{d'}} \\ & & K \end{array} \quad (Dd' \xrightarrow{\gamma_{d'}(x')} K)$$

commutes: the top square by the definition of $Xd' \cdot Df$ (see (3.2)), the right triangle by $\gamma'_{d'}$ (3.9).

Since the following diagram

$$\begin{array}{ccc} Dd & \xrightarrow{Df} & Dd' \\ & \searrow^{\gamma_d(x)} & \downarrow{\scriptstyle \gamma_{d'}(x')} \\ & & K \end{array}$$

commutes from the definition of γ_d (see (3.8)), we have proved that the diagram (3.10) commutes.

An object K together with morphisms $\gamma_d, d \in \mathscr{D}$ is thus a wegde and the universal property defines us a unique morphism $\gamma' : \int^d Xd \cdot Dd \longrightarrow K$.

Hence there is a bijection

$$[\mathscr{D}^{op}, Set](X, \widetilde{D}K) \cong \mathscr{K}(X * D, K)$$

$$\gamma : X \longrightarrow \widetilde{D}Kd \longmapsto \gamma' : \int^d Xd \cdot Dd \longrightarrow K.$$

It is straightforward to show that this bijection is natural in K. ■

3.3.2 Vietoris-Rips Complex and its Geometric Realization

In this section we are already equipped with all the category theory we need. Here we instantiate the general theory introduced in Section 3.3.1.

Our first goal is to give a categorical definition of a Vietoris-Rips complex. Intuitively speaking, Vietoris-Rips complex is a simplicial set generated from a metric space in a special way: the intuition from Definition 3.2.1 is that we have a certain sieve, which only allows simplices with a certain diameter through it and those simplices then construct a simplicial set.

Now we will approach this from another point of view. Refer to objects of Δ as "abstract simplices". We will interpret each abstract simplex as a certain metric space. In order to do that we will define a functor $J_\varepsilon : \Delta \longrightarrow Met_e$.

Since J_ε will be a functor, it will not only assign to each abstract simplex $[n]$ a metric space $J_\varepsilon[n]$ (the "discrete simplex metric space"), but the abstract structural relations between simplices of various dimensions will be preserved.

This "interpretation" functor J_ε will be then used to describe the construction of a Vietoris-Rips complex of a metric space.

3.3.10 Definition Given $\varepsilon \geq 0$ we define the *discrete simplex functor* $J_\varepsilon : \Delta \longrightarrow Met_e$ as follows:

The object $[n]$ is mapped to the metric subspace $(\{\varepsilon \cdot e_0, ..., \varepsilon \cdot e_n\}, d)$ of $\mathbb{R}^{n+1}$ equipped with the Manhattan distance, where $(e_0, ..., e_n)$ denotes the canonical basis of $\mathbb{R}^{n+1}$.

The action of J_ε on morphisms is defined by the assignment on the coface and codegeneracy morphisms in Δ:

(1) the morphism $d^i : [n-1] \longrightarrow [n]$ is mapped to:

$$J_\varepsilon d^i : J_\varepsilon[n-1] \longrightarrow J_\varepsilon[n]$$

$$(\{\varepsilon \cdot e_0, ..., \varepsilon \cdot e_{n-1}\}, d) \longrightarrow (\{\varepsilon \cdot e_0, ..., \varepsilon \cdot e_n\}, d), \text{ where}$$

$$\varepsilon \cdot e_j \mapsto \varepsilon \cdot e_{d^i(j)};$$

(2) the morphism $s^i : [n+1] \longrightarrow [n]$ is mapped to:

$$J_\varepsilon s^i : J_\varepsilon[n+1] \longrightarrow J_\varepsilon[n]$$

$$(\{\varepsilon \cdot e_0, ..., \varepsilon \cdot e_{n+1}\}, d) \longrightarrow (\{\varepsilon \cdot e_0, ..., \varepsilon \cdot e_n\}, d), \text{ where}$$

$$\varepsilon \cdot e_j \mapsto \varepsilon \cdot e_{s^i(j)}.$$

We will refer to this functor as to the interpretation of a simplicial category.

3.3.11 Definition Given an extended metric space (X, d) and $\varepsilon > 0$, we define the *Vietoris-Rips complex* of (X, d) to be the presheaf $\widetilde{J_\varepsilon}(X, d) : \Delta^{op} \longrightarrow Set$.

3.3.12 Remark The beauty of category theory lays in such short and powerful definitions like this one. Let us compare this definition to Definition 3.2.1 from Section 3.2.

The functor $\widetilde{J_\varepsilon}(X, d) : \Delta^{op} \longrightarrow Set$ maps an object $[n] \in \Delta$ to a hom-set $Met_e(J_\varepsilon[n], (X, d))$ (refer to the Definition 3.3.1 for a full description of $\widetilde{J_\varepsilon}$). Thus we get a set of maps from a metric space of n points, where the distance of each two separate points is exactly ε.

To have a map $f : J_\varepsilon[n] \longrightarrow (X, d) \in Met_e(J_\varepsilon[n], (X, d))$ means to get a sequence of $(n+1)$ points of (X, d), with each pair having distance at most ε (a "discrete ε-simplex" in (X, d)). To have all such maps is equal to having all such sequences. Definition 3.3.11 therefore yields the same data as Definition 3.2.1.

The functor J_ε interprets abstract simplices as metric spaces. These simplices can be thought of as "building blocks" of a simplicial set. A natural question arises: if we can interpret separate simplices as metric spaces, can we extend this interpretation from simplices to simplicial sets? That is, can we use the discrete simplex functor J_ε to give a "geometric realization" of any simplicial set X?

The answer is positive: we just need to "glue together" the interpretations of the simplices of X based on the relations between the simplices in X.

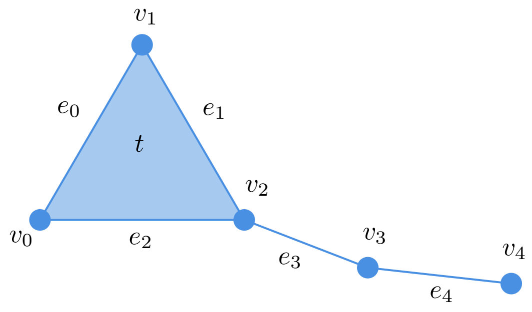
Moreover, no further work is needed — we use the results of Section 3.3.1 to define the "geometric realization of X", namely we use the existence of a left adjoint to the "Vietoris-Rips functor" $\widetilde{J_\varepsilon}$.

3.3.13 Theorem *The functor $\widetilde{J_\varepsilon} : Met_e \longrightarrow sSet$ has a left adjoint $\_ * J_\varepsilon : sSet \longrightarrow Met_e$. We call $\_ * J_\varepsilon$ the $\varepsilon-$geometric realization functor.*

Proof. The category Met_e is cocomplete, $sSet$ is a presheaf category $[\Delta^{op}, Set]$, and J_ε is a functor of the form $\Delta \longrightarrow Met_e$.

Therefore Theorem 3.3.13 is an instance of Theorem 3.3.9. ■

3.3.14 Example Let X be a simplicial set



generated by the non-degenerate simplices $t \in X_2; e_0, e_1, e_2, e_3, e_4 \in X_1; v_0, v_1, v_2, v_3, v_4 \in X_0$, satisfying the obvious face conditions (e.g. $d_0(e_2) = v_2$).

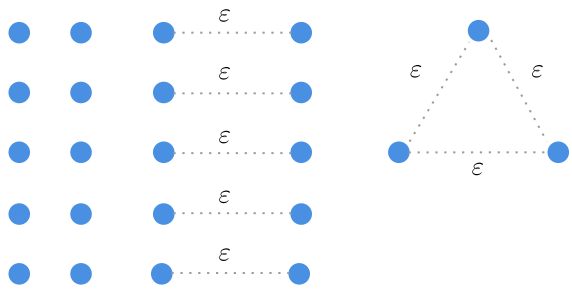
We will now describe what the coend construction from Definition 3.3.7 amounts to for the simplicial set X.

Simplicial sets can be imagined as a collection of simplices together with the information about face and degeneracy relations between the simplices. The first step in the geometric realization of X is to construct a collection of *interpretations* of abstract simplices in X. We obtain the copower

$$X_n \cdot J_\varepsilon[n],$$

(i.e., the extended metric space obtained as the X_n-fold coproduct of $J_\varepsilon[n]$), for each $[n] \in \Delta$.

The non-degenerate simplices in X are thus interpreted as ten interpretations of [0]-simplices, five interpretations of [1]-simplices and one interpretation of a [2]-simplex as shown below:



Since Met_e has all coproducts, we can form the coproduct $\coprod_{[n]\in\Delta} X_n \cdot J_\varepsilon[n]$, the extended metric space formed by the disjoint union of $X_n \cdot J_\varepsilon[n]$ indexed by all objects $[n] \in \Delta$. Recall from Remark 3.3.6 that this is the first step in construction of the coend $\int^{[n]} X_n \cdot J_\varepsilon[n]$ (the geometric realization of X).

In the next step of geometric realization the face and degeneracy relations present in X are used to "glue adjacent simplices together": more formally, we form a quotient of the extended metric space $\coprod_{[n]\in\Delta} X_n \cdot J_\varepsilon[n]$.

Of course, the resulting quotient space is precisely the coend

$$X * J_\varepsilon = \int^{[n]} X_n \cdot J_\varepsilon[n].$$

The informal notion of gluing simplices together is captured formally by describing an equivalence relation on the set of points of $\coprod_{[n]\in\Delta} X_n \cdot J_\varepsilon[n]$. Recall from Section 2.3 that forming a quotient of an extended metric space is a two-step process.

We will now describe the equivalence relation that gives rise to the quotient extended pseudometric space, using the fact that the second step (forming an extended *metric* space) consists only of an identification of points with mutual distance 0).

Since $X * J_\varepsilon$ is a coend, the diagram

$$\begin{array}{ccc} X_m \cdot J_\varepsilon[n] & \xrightarrow{X_m \cdot J_\varepsilon f} & X_m \cdot J_\varepsilon[m] \\ {\scriptstyle Xf \cdot J_\varepsilon[n]}\downarrow & & \downarrow{\scriptstyle \gamma'_m} \\ X_n \cdot J_\varepsilon[n] & \xrightarrow[\gamma'_n]{} & \int^{[n]} X_n \cdot J_\varepsilon[n] \end{array} \tag{3.11}$$

has to commute for each $f : [n] \longrightarrow [m] \in \Delta$, where γ' is the universal wedge of the coend $\int^{[n]} X_n \cdot J_\varepsilon[n]$.

Element-wise, the images of any point in $X_m \cdot J_\varepsilon[n]$ along both paths in diagram (3.11) have to be equal.

Recall from Remark 3.3.6 that the coend $\int^{[n]} X_n \cdot J_\varepsilon[n]$ can be computed as a quotient of the coproduct $\coprod_{[n]\in\Delta} X_n \cdot J_\varepsilon[n]$. Informally this means that certain points of the metric space $\coprod_{[n]\in\Delta} X_n \cdot J_\varepsilon[n]$ are "glued together".

The equivalence relation giving rise to the quotient metric space $\int^{[n]} X_n \cdot J_\varepsilon[n]$ is therefore defined using the pairs

$$\begin{array}{ccc} X_m \cdot J_\varepsilon[n] & \xrightarrow{X_m \cdot J_\varepsilon f} & X_m \cdot J_\varepsilon[m] \\ {\scriptstyle Xf \cdot J_\varepsilon[n]}\downarrow & & \\ X_n \cdot J_\varepsilon[n] & & \end{array}$$

of morphisms for each $f : [n] \longrightarrow [m]$ in the following way.

For every point $(x, \varepsilon \cdot e_i) \in X_m \cdot J_\varepsilon[n]$ we obtain a pair $(Xf(x), \varepsilon \cdot e_i)$ and $(x, J_\varepsilon f(\varepsilon \cdot e_i)) = (x, \varepsilon \cdot e_{fi})$ of points in $\coprod_{[n]\in\Delta} X_n \cdot J_\varepsilon[n]$ which are to be equal in $\int^{[n]} X_n \cdot J_\varepsilon[n]$. The set of all such pairs generates the desired equivalence relation. Compare this description of the equivalence relation to the one in [3].

For example, given any $x \in X_1$ and choosing $f = d^0 : [0] \longrightarrow [1]$, the following two points will be equal in $\int^{[n]} X_n \cdot J_\varepsilon[n]$:

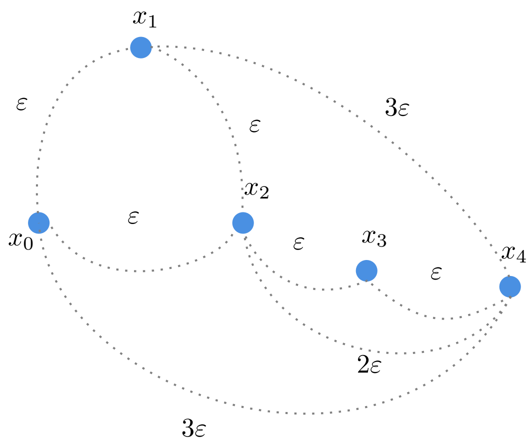
$$(d_0(x), \varepsilon \cdot e_0) \sim (x, \varepsilon \cdot e_{d^0(0)}). \tag{3.12}$$

The element $d_0(x) \in X_0$ is the endpoint of the [1]-simplex x in X. Being a [0]-simplex, $d_0(x)$ is interpreted as a point in $X_0 \cdot J_\varepsilon[0]$. However, the [1]-simplex x is interpreted as a pair $(x, \varepsilon \cdot e_0), (x, \varepsilon \cdot e_1)$ of points with distance ε in $X_1 \cdot J_\varepsilon[1]$.

Obviously, the interpretations of $d_0(x)$ and of the endpoint $(x, \varepsilon \cdot e_1)$ (of the interpretation of x) should coincide. This is the geometric meaning of the equivalence (3.12).

The family of diagrams 3.11 then represents all the identifications of points in $\int^{[n]} X_n \cdot J_\varepsilon[n]$ given by all morphisms in Δ.

Thus the $\varepsilon-$geometric realization of X depicted above is the following metric space:



carried by points $\{x_0, x_1, x_2, x_3, x_5\}$ with the metric denoted in the figure.

Summary and Future Work

In this thesis we have laid down the basics of the categorical approach to Vietoris-Rips complex.

The foundational observation was that one can work within the category Met_e of *extended metric spaces* and *non-expanding maps*. We showed that the categorical properties of Met_e ensure the basic correspondence between spaces and simplicial sets. Namely, we gave the *"singular construction"* that yields a Vietoris-Rips complex for every metric space, and the *geometric realisation* that produces a metric space out of a simplicial set.

Moreover, the above two processes are adjoint to each other in a precise sense of Category Theory.

In fact, we assume that one can build up homological theories starting with our setup. However, such a general theory would require mastering much deeper techniques of homology theory and category theory than those presented in the thesis. We therefore postpone this topic to future work.

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