Czech Technical University in Prague
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Synchronization mechanisms in quantum Markov processes

Mechanismy synchronizace v kvantových markovovských procesech

Master's thesis

Diplomová práce

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Author's declaration:

I hereby declare to have written this work by myself and to have listed all sources used.

Prague, 18 August 2020 Daniel Štěrba

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Abstract: The work adresses the phenomenon of synchronization in qubit networks with Markovian evolution. It introduces the formalism of quantum Markov dynamical semigroups and Lindblad dynamics as a suitable model to study the evolution and asymptotics of open quantum systems. For a system of two qubits all possible synchronization- and phase-locking-enforcing evolution maps within the studied dynamics are found and classified, and their attractor spaces are described. It is shown how the discovered synchronization mechanisms can be applied to qubit networks, n-partite systems with bipartite interactions. It is further demonstrated that, depending on the particular synchronizing Lindblad operator, the evolution either completely destroys the reduced dynamics of individual qubits, or synchronizes an arbitrary network wherof interaction graph is weakly connected. Additionally, several other properties of the synchronizing and phase-locking maps are investigated.

Key words: synchronization, phase-locking, qubit networks, open quantum systems, quantum Markov processes, quantum dynamical semigroups, QMDS, Lindblad dynamics, asymptotic evolution, attractor space

Název práce:

Mechanismy synchronizace v kvantových markovovských procesech

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Abstrakt: Práce se zabývá otázkou synchronizace na qubitových sítích s markovskou evolucí. Seznamuje s formalismem kvantových markovských dynamických semigrup a představuje lindbladovskou dynamiku jako vhodný model pro studium vývoje a asymptotiky otevřených kvantových systémů. Pro systém dvou qubitů jsou nalezeny a klasifikovány veškeré možné evoluce v rámci studované dynamiky, které vynucují synchronizaci nebo phase-locking, a jsou popsány jejich atraktorové prostory. Je ukázáno, jak lze tyto objevené synchronizační mechanismy aplikovat na qubitové sítě, n-částicové systémy s bipartitními interakcemi. Dále je ukázáno, že v závislosti na konkrétním synchronizujícím lindbladovském operátoru evoluce buď zcela utlumí jednotlivé vlastní qubitové dynamiky, nebo synchronizuje libovolnou síť, jejíž interakční graf je alespoň slabě souvislý. Krom toho jsou studovány rozličné vlastnosti synchronizujících a phase-locking vynucujících evolučních zobrazení.

Klíčová slova: synchronizace, phase-locking, qubitové sítě, otevřené kvantové systémy, kvantové markovské procesy, kvantové dynamické semigrupy, QMDS, lindbladovská dynamika, asymptotický evoluce, atraktorový prostor

Contents

Introduction

The first to observe synchronization was reportedly Christiaan Huygens who noticed the tendency of two pendulum clocks to adjust to anti-phase oscillations when mounted on a common support bar, and described the discovery in his letters as early as in February 1665 [1]. Since then, synchronization has been thoroughly explored in a great variety of classical systems [2], yet it was not until very recently that the study of this ubiquitous phenomenon entered the quantum realm.
In both classical and quantum domain, synchronization is a very broad term. Various viewpoints and hence definitions and measures have been introduced [3], [4], [5], [6], [7], and systems investigated are numerous. The first works on the subject were typically concerned with the case of a forced synchronization induced by an external field, or entrainment, examples include a driven oscillator [8], an oscillator coupled to a qubit [9] or systems of van der Pol oscillators [10]. Another noteworthy field of research is represented by synchronization protocols, proposals of how to exploit system properties such as entanglement to achieve clock synchronization between two parties [11]. Finally, the main focus in current literature is on spontaneous synchronization, the situation when two or more individual subsystems tune their local dynamics to a common pace due to the presence of coupling. Prevalent is the study of so called transient synchronization, the emergence of synchronous behaviour in dissipative systems as a result of time-scale separation of decay rates of single modes [12]. In such a case the system goes through a long-lasting yet temporal phase of sychronized evolution, eventually approaching relaxation in the asymptotics. Among the examples of examined systems are oscillator networks [13], [14], spin systems [15], atomic lattices [16], qubits in bosonic environment [17], collision models [18] or simple few-body systems in dissipative environments [19]. It was nonetheless demonstrated that synchronization can arise temporarily as well as asymptotically, and that such an asymptotic behaviour can be associated with the presence of synchronous modes in the decoherence-free subspaces of the state space [15], [20]. Very recently, a different understanding of non-vanishing synchronized evolution was presented in the form of an analogue of the classical phase space limit cycles for a spin system of purely quantum nature [21]. The same authors also discuss the minimal quantum system which can exhibit this type of synchronization [7], providing a promising baseline for studying limit cycles synchronization in more complex spin networks. Synchronization can even occur as a concomitant of other phenomena. In one particular instance it was shown to be an accompanying effect of super- and subradiance [17].
There have been various atempts to establish a link between spontaneous synchronization and several possible local or global indicators such as entanglement [13], discord [14] or classical and quantum correlations in general [15]. To give an example, in [21] a syn-

chronization of two spins solely through their, conveniently chosen, mutual interaction was demonstrated to always come with a creation of entanglement, the converse not necessarily true. While the results might be promissing in some specific cases, so far no general connection between the emergence of spontaneous synchronization and any other phenomena has been found and the plausible mechanisms of quantum synchronization and as well as its very nature remain to great extend unknown.
An endeavour to better understand the phenomenon of synchronization on the quantum level motivates this work. The main idea is based on Huygens' original observation of two clocks. We investigate the emergence of spontaneous synchronization between two or more individual identical systems with their own inner dynamics that are coupled together, with the aim of understanding the underlying synchronizing mechanism. Identical systems have identical inner dynamics and natural frequencies, hence when it comes to synchronization we talk about phase synchronization. We are not concerned with temporal transient phases of synchronous behaviour preceeding dissipation and relaxation, as it is often the case in the current literature, see the brief overview above, rather we look into systems exhibiting sychronized dynamics in the asymptotics.
For the process of synchronization it is necessary to consider not only the possible mutual interaction of the individual constituents of the composite system in question but also the effects of the environment. Apart from the contact with the environment being practically inevitable, a closed system alone is not enough for the study of the phenomenon since unitary evolution in finite dimensions is always at least quasiperiodic [22]. A thrid party is essential for non-trivial occurence of synchronization. To account for possible environments and their interactions with the system it is best to view it as an open quantum system. One of the simplest and most convenient approaches used to describe the open dynamics and to study asymptotic behaviour is Markovian approximation. Within the approximation the framework of quantum Markov dynamical semigroups and Lindblad dynamics represents a suitable tool, and is employed in this work.

We begin with an introduction to the formalism and several key elements of the theory. Quantum Markov dynamical semigroups are described and their generators are cast into the well-known Lindblad form. The notion of an attractor space is introduced and a vital theorem is stated, revealing how the attractors of the evolution map and the structure of its generator are intertwined. Discussion of the concept of synchronization follows together with suitable definitions for our setup. Alongside synchronization we also propound the idea of phase-locking as its direct generalization to later show that the two can be explored simultaneously.
The main part of the work is contained in the following chapters. In chapter 3 we investigate in depth a system of two coupled non-interacting qubits, and explore and classify all synchronizing maps in the studied model. We make use of a theorem presented in the theoretical part in chapter 1 which links generators of the evolution map in the Lindblad form and the attractor space via commutation relations. Firstly, we assume all possible attractors corresponding to non-trivial synchronized asymptotic evolution and find generators of the evolution map which permit the existence of such attractors. From the resulting set of generators we then pick those that enforce synchronization on the entire attractor space and hence lead to synchronous asymptotic behaviour irrespective of initial conditions. We further study the obtained evolution maps and describe their attractor spaces. Chapter

4 is devoted to extending all the concepts and results from two qubits to qubit networks. We show how the synchronizing mechanisms identified in the case of two parties can be applied to systems of many, achieving the same effects for the entire qubit networks in some cases and destroying the single-qubit dynamics in other ones. Last but not least, we study some relevant properties of synchronizing Lindblad operators. The text is concluded with obtained results.

Supplementary material consisting of a derivation of normal matrix parameterization utilized in the work, numerical simulations illustrating the studied phenomenon and setup, and an overview of discovered synchronizing normal Lindblad operators and attractor spaces of their respective associated evolution maps can be found in appendices A, B and C in this order.

Chapter 1

Theoretical background

The evolution of an open quantum system is in general described by an irreversible linear completely positive trace non-increasing map acting on the space of linear operators on a Hilbert space. The open dynamics is often too complex for analytical solutions and certain additional simplifying assumptions need to be applied. A common approach is the use of Markovian approximation to describe dynamics of the system. Two main classes of quantum Markovian processes are commonly studied, namely discrete quantum Markov chains and continuous quantum Markov dynamical semigroups. With the latter being utilized throughout the rest of the work, this section gives a brief introduction to the necessary theory.

1.1 Preliminary

Assume a quantum system represented by a finite-dimensional Hilbert space $\mathscr{H}$, let $\mathcal{B}(\mathscr{H})$ be the associated space of all bounded linear operators on $\mathscr{H}$. For $A, B \in \mathcal{B}(\mathscr{H})$, $(A, B) = \mathrm{Tr}\left\{A^\dagger B\right\}$ stands for the corresponding scalar product and $\|A\|$ the induced norm thereof, with $A^\dagger$ being the adjoint operator of A defined via the scalar product $\langle\, ,\rangle$ on $\mathscr{H}$. A state of such a system is described by a density operator $\rho \in \mathcal{B}(\mathscr{H})$, a (generally not stricly) positive self-adjoint operator with a unit trace.

1.2 Quantum dynamical semigroups

Among all possible evolutions of a state of an open quantum system special attention is paid to the so called quantum Markovian dynamical semigroups. By the Markov property it is meant that the evolution depends only on the present state and is completely independent of its past. Further, we assume that the process is homogenous, that is the evolution from t_1 to t_2 depends solely on the time difference $\Delta t = t_2 - t_1$ and not on the actual points in time themselves. With these properties combined we arrive at the following definition.

Definiton 1.2.1. A one-parameter family of completely positive (CP) trace non-increasing maps $\mathcal{T}_t : \mathcal{B}(\mathscr{H}) \to \mathcal{B}(\mathscr{H})$, parameterized by $t \in \mathbb{R}_0^+$, satisfying

$$\mathcal{T}_t \mathcal{T}_s = \mathcal{T}_{t+s}, \tag{1.1}$$

$$\mathcal{T}_0 = I, \tag{1.2}$$

is called a quantum Markovian dynamical semigroup (QDMS).

1.3 Generators of QMDS and Lindblad operators

In this work we consider exclusively semigroups which are norm continuous. Such semigroups are known to be akin to exponential maps, namely they are of the form $\mathcal{T}_t = \exp(\mathcal{L}t)$ for some superoperator $\mathcal{L} \in \mathcal{B}(\mathcal{B}(\mathscr{H}))$. To describe continuous QMDS we make use of the results of [23], further discussed in [24], [25].

Theorem 1.3.1. *Let $\mathcal{T}_t$ be a continuous quantum dynamical semigroup (continuous in the parameter t above). Then the superoperator $\mathcal{T}_t \equiv \mathcal{T}$ is differentiable in t and is of the form*

$$\mathcal{T}_t = \exp(\mathcal{L}t), \tag{1.3}$$

where $\mathcal{L} : \mathcal{B}(\mathscr{H}) \to \mathcal{B}(\mathscr{H})$ is a linear map called the generator. The generator $\mathcal{L}$ can be split into

$$\mathcal{L}(\rho) = \phi(\rho) - K\rho - \rho K^\dagger, \tag{1.4}$$

where ϕ is completely positive and $K \in \mathcal{B}(\mathscr{H})$.

The master equation governing the evolution of an arbitrary state ρ in this model reads

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} = \mathcal{L}(\rho(t)). \tag{1.5}$$

A CP map ϕ admits a decomposition into Kraus operators [26], denoted here as $\{L_j\}$ and satisfying $\sum_j L_j^\dagger L_j \leq I$. Furthermore, the operator K can be split into its hermitian and antihermitian parts

$$K = \underbrace{\frac{1}{2}(K - K^\dagger)}_{iH} + \underbrace{\frac{1}{2}(K + K^\dagger)}_{\frac{1}{2}\phi^\dagger(I)+B}, \tag{1.6}$$

hereby defining a self-adjoint operator $H \in \mathcal{B}(\mathscr{H})$ and a positive operator $B \in \mathcal{B}(\mathscr{H})$, known as the optical potential. The introduction of B is motivated by the trace non-increasing property of $\mathcal{T}$. From

$$\operatorname{Tr} \mathcal{T}(\rho) = \operatorname{Tr}\{I\mathcal{T}(\rho)\} = \operatorname{Tr}\{\mathcal{T}^\dagger(I)\rho\} \leq \operatorname{Tr}\{I\rho\} = \operatorname{Tr}\rho, \tag{1.7}$$

for all states $\rho \in \mathcal{B}(\mathscr{H})$, it follows

$$\mathcal{T}^\dagger(I) \leq I, \tag{1.8}$$

which translates to the generator $\mathcal{L}^\dagger$ of $\mathcal{T}^\dagger$ as

$$\mathcal{L}^\dagger(I) = \phi^\dagger(I) - K^\dagger - K \leq 0. \tag{1.9}$$

Rewriting $\phi^\dagger(I) = \sum_j L_j^\dagger L_j$ using the Kraus operators, it can be seen that B is well defined and (not necessarily strictly) positive. Put together we arrive at the final expression of the generator of a continuous QMDS

$$\mathcal{L}(\rho) = -i[H,\rho] + \sum_j L_j \rho L_j^\dagger - \frac{1}{2}\Big\{L_j^\dagger L_j, \rho\Big\} - B\rho - \rho B^\dagger. \tag{1.10}$$

Here the operator H can be identified with the hamiltonian. Indeed, in the case of $L_j = B = 0$ equations (1.10) and (1.5) reduce to $\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i[H,\rho]$, standard expression for the unitary evolution of a closed system.
It is worth mentioning that an arbitrary choice of a self-adjoint operator H, positive operator B and operators $\{L_j\}$ satisfying $\sum_j L_j^\dagger L_j \leq I$ gives a valid generator $\mathcal{L}$ leading to a CP trace non-increasing map $\mathcal{T}_t$ at any time t and as such describes a physically admissible evolution of an open system.

The relation (1.10) simplifies in case of trace-preserving QMDS. The condition (1.8) reduces to the adjoint map being unital, $\mathcal{T}^\dagger(I) = I$, which in turn directly implies $0 = \mathcal{L}^\dagger(I) = \phi^\dagger(I) - K^\dagger - K$ as a special case of (1.9). Subsequently $B = 0$. The resulting equation is known as the Lindblad equation and reads

$$\mathcal{L}(\rho) = -i[H,\rho] + \sum_j L_j \rho L_j^\dagger - \frac{1}{2}\Big\{L_j^\dagger L_j, \rho\Big\}. \tag{1.11}$$

In this context, Kraus operators $\{L_j\}$ are usually reffered to as Lindblad operators. In the following chapters we only deal with trace-preserving quantum operations.

Note: So far we have only been working with time-evolving states in Schrödinger picture; the evolution of observables in Heisenberg picture can be described in a similar way [23], [24], [25]. Namely it is given by the semigroup $\mathcal{T}^\dagger$ of adjoint maps $\mathcal{T}_t^\dagger$, referred to as the adjoint semigroup, in the case of trace-preserving QMDS with a generator $\mathcal{L}^\dagger$ of the form

$$\mathcal{L}^\dagger(A) = i[H,A] + \sum_j L_j^\dagger A L_j - \frac{1}{2}\Big\{L_j^\dagger L_j, A\Big\}, \tag{1.12}$$

for an observable $A \in \mathcal{B}(\mathscr{H})$.

1.4 Attractor space and asymptotic dynamics

The dynamics of an open system is typically highly involved and thus complicated to analyze compared to that of a closed system as the non-unitary generator of the evolution may generally not be diagonalizable. However, should we only be concerned with the asymptotic dynamics of the system, there is an algebraic method for an anylytical treatment developed in [27], [28] at hand. For details the reader is advised to study the original papers, for the purpose of this work we only state the important related results.

The asymptotic spectrum $\sigma_{as}(\mathcal{L})$ of a generator $\mathcal{L}$ of a continuous QMDS (1.3) is the set of all purely imaginary points of its spectrum $\sigma(\mathcal{L})$ and eventually zero, i.e.

$$\sigma_{as}(\mathcal{L}) = \{\lambda \in \sigma(\mathcal{L}), \mathrm{Re}\lambda = 0\}. \tag{1.13}$$

The attractor space $\mathrm{Att}(\mathcal{T})$ of a QMDS $\mathcal{T} = \exp(\mathcal{L}t)$ is the subspace spanned by the eigenvectors of its generator corresponding to purely imaginary eigenvalues,

$$\mathrm{Att}(\mathcal{T}) = \bigoplus_{\lambda\in\sigma_{as}(\mathcal{L})} \mathrm{Ker}(\mathcal{L} - \lambda I). \tag{1.14}$$

We commonly refer to an element $X \in \mathrm{Att}(\mathcal{T})$ as attractor. An eigenvector X_λ of $\mathcal{L}$ associated with eigenvalue λ is also an eigenvector of $\mathcal{T}_t = \exp(\mathcal{L}t)$ associated with eigenvalue $\exp(\lambda t)$. It holds $|\exp(\lambda t)| = 1, \forall\lambda \in \sigma_{as}, \forall t \in \mathbb{R}_0^+$. It has been shown in [27] that it is always possible to diagonalize the part of generator $\mathcal{L}$ responsible for the asymptotic dynamics of quantum Markov process and therefore we can decompose (as a direct sum) the Hilbert space $\mathcal{B}(\mathscr{H})$, which represents a superset of the set of all possible states, into two subspaces $\mathrm{Att}(\mathcal{T})$ and Y. The former accounts for the asymptotic dynamics and the latter represents the part dying out during the evolution, i. e.

$$\mathcal{B}(\mathscr{H}) = \mathrm{Att}(\mathcal{T}) \oplus Y. \tag{1.15}$$

Assume $\{X_{\lambda,i}\}$ to be a basis of the attractor space $\mathrm{Att}(\mathcal{T})$, $\{X_k\}$ to be a basis of $\mathcal{B}(\mathscr{H})$ containing $\{X_{\lambda,i}\}$ and $\{X^k\}$ to be the basis of $\mathcal{B}^*(\mathscr{H})$ dual to $\{X_k\}$, i.e. $\mathrm{Tr}\left\{X_k^\dagger X^{k'}\right\} = \delta_{kk'}$. Denoting $X^{\lambda,i}$ the elements of $\{X^k\}$ which constitute the dual basis to $\{X_{\lambda,i}\}$, $\mathrm{Tr}\left\{X_{\lambda,i}^\dagger X^{\lambda',j}\right\} = \delta_{\lambda\lambda'}\delta_{ij}$, we can express the asymptotic dynamics of an initial state $\rho(0)$ as

$$\rho_{as}(t) = \sum_{\lambda\in\sigma_{as}(\mathcal{L}),i} \exp(\lambda t)\,\mathrm{Tr}\left\{\left(X^{\lambda,i}\right)^\dagger \rho(0)\right\} X_{\lambda,i} \tag{1.16}$$

and it holds

$$\lim_{t\to\infty} \|\rho(t) - \rho_{as}(t)\| = 0. \tag{1.17}$$

In general the construction of a suitable basis and its dual remains a challenging task. The subspaces $\mathrm{Ker}(\mathcal{L} - \lambda I), \lambda \in \sigma_{as}(\mathcal{L})$, corresponding to different eigenvalues need not be orthogonal, nor is the attractor space $\mathrm{Att}(\mathcal{T})$ necessarily orthogonal to the complementary subspace Y of vanishing states. Compared to the unitary evolution of a closed system, the existence of a simple relation between eigenvectors of the evolution map and its generator and those of their adjoint maps is not guaranteed. Provided the presence of a state preserved under evolution, a so-called faithful $\mathcal{T}$-state defined below, ones can easily be constructed from the others. If on the other hand no such state is found, there is no generally applicable approach to the problem. It is therefore complicated to link the asymptotic evolution of states and observables. For the purpose of this work, however, we only need to work in the Schrödinger picture.

To state the final structure theorem revealing a possible way of how to find the attractor space of a given continuous QMDS one more definition is needed.

Definiton 1.4.1. A state σ satisfying $\sigma > 0$ is called a faithful $\mathcal{T}$-state if

$$\mathcal{T}_t(\sigma) \le \sigma, \forall t > 0. \tag{1.18}$$

For a semigroup of trace-preserving superoperators of the form (1.3) the condition reduces to $\mathcal{T}_t(\sigma) = \sigma$, or $\mathcal{L}(\sigma) = 0$ respectively.
Finally, due to [28] the following holds.

Theorem 1.4.2. *Let $\mathcal{T}_t : \mathcal{B}(\mathscr{H}) \to \mathcal{B}(\mathscr{H})$ be a QMDS with generator $\mathcal{L}$ of the form* (1.10) *equipped with a faithful $\mathcal{T}$-state σ and let $X \in \mathcal{B}(\mathscr{H})$ be an attractor of $\mathcal{T}_t$ in Schrödinger picture associated with eigenvalue λ. Then the following set of equations holds*

$$\left[L_j, X\sigma^{-1}\right] = \left[L_j, \sigma^{-1}X\right] = \left[L_j^\dagger, X\sigma^{-1}\right] = \left[L_j^\dagger, \sigma^{-1}X\right] = 0, \tag{1.19}$$

$$\left[B, X\sigma^{-1}\right] = \left[B, \sigma^{-1}X\right] = 0, \tag{1.20}$$

$$\left[H, \sigma^{-1}X\right] = i\lambda\sigma^{-1}X, \quad \left[H, X\sigma^{-1}\right] = i\lambda X\sigma^{-1}. \tag{1.21}$$

If $\mathcal{T}_t$ is either trace-preserving or the faithful $\mathcal{T}$-state σ is stationary the reverse statement applies.

The theorem 1.4.2 reveals how the internal structure of the generator $\mathcal{L}$ determines the asymptotic spectrum and the corresponding attractor space of a given QMDS. It plays a pivotal role in our work.

1.5 Special case of normal Lindblad operators

In further application we use a simplifying assumption that all the Lindblad operators L_j in (1.11) are normal operators. The idea behind is that we want the identity, proportional to the maximally mixed state, to be preserved under evolution. The identity clearly satisfies $I > 0$ and as such is the faithful $\mathcal{T}$-state in the case $\mathcal{T}(I) = I$. The existence of a faithful state is essential for our work. It is a necessary condition for the structure theorem 1.4.2 to hold and thus something our approach to the topic is reliant on. Additionaly, it provides us with the ability to construct a dual basis to that of the attractor space, enabling us to make use of the expression for asymptotic evolution of states (1.16) or describe the evolution of observables in the Heisenberg picture if desired. The identity operator is merely the most convenient choice in regard to the persuit of simplicity[1], while also being quite a natural choice. The assumption is for an open system dynamics to preserve the maximally mixed state which could be reasonably expected from a great variety of systems.
In the absence of a faithful $\mathcal{T}$-state the analysis of asymptotic dynamics becomes noticeably more involved. A more general approach to the construction of attractors, not relying on the existence of a faithful state, is presented in [29]. However, it is considerably complicated and to the best of author's knowledge it has not been successfully applied to any problem of this scope yet.

Written in terms of the generator the requirement to preserve identity reads

[1] Technically, we could assume only that there exists a faithful $\mathcal{T}$-state without further specification, dropping the assumption of it being proprotional to identity. The theory would still apply the same, however, later in the work we would have to deal with significantly more intricate sets of commutation relations which would need to be solved not only for the Lindblad operators L_j or attractors X respectively, but also for the faithful $\mathcal{T}$-state σ concurrently. If we were to do so, we would mostly be unable to obtain general solutions, introducing unnecessary complexity to our method without obtaining any new results.

$$\mathcal{L}(I) = \sum_j L_j L_j^\dagger - L_j^\dagger L_j = 0, \tag{1.22}$$

which is satisfied for an arbitrary number and combination of Lindblad operators, in particular for a single Lindblad operator, if and only if $\left[L_j, L_j^\dagger\right] = 0$, i. e. if the operators L_j are normal.

In this particular case, the choice of a suitable basis and construction of its dual becomes a significantly simpler task. The eigenspaces $\mathrm{Ker}(\mathcal{L} - \lambda I), \lambda \in \sigma_{as}(\mathcal{L})$, forming the attractor space $\mathrm{Att}(\mathcal{T})$ are mutually orthogonal and the same holds for $\mathrm{Att}(\mathcal{T})$ and Y, see [27], [28].

$$\mathrm{Ker}(\mathcal{L} - \lambda_i I) \perp \mathrm{Ker}(\mathcal{L} - \lambda_j I) \quad \text{for} \quad \lambda_i, \lambda_j \in \sigma_{as}(\mathcal{L}),\, \lambda_i \neq \lambda_j, \tag{1.23}$$

$$\mathrm{Att}(\mathcal{T}) \perp Y. \tag{1.24}$$

Furthermore, the theorem 1.4.2 reduces to

Theorem 1.5.1. *Let $\mathcal{T}_t : \mathcal{B}(\mathscr{H}) \to \mathcal{B}(\mathscr{H})$ be a trace-preserving QMDS with generator $\mathcal{L}$ of the form* (1.11) *where all Lindblad operators $\{L_j\}$ are normal. An element $X \in \mathcal{B}(\mathscr{H})$ is an attractor of $\mathcal{T}_t$ associated with eigenvalue λ if and only if it holds*

$$[L_j, X] = \left[L_j^\dagger, X\right] = 0, \tag{1.25}$$

$$[H, X] = i\lambda X. \tag{1.26}$$

This theorem will be employed as a key tool to analyze attractors of QMDS and to design desired synchronization mechanisms.

Chapter 2

Synchronization and phase-locking

Before we start to investigate synchronization of particular quantum system we should first explain and clearly define what we mean by saying a quantum system is synchronized. In this chapter we first present our viewpoint of synchronization and definitions suitable for our work, an itroduction of the most common synchronization measures follows. Phase-locking is introduced as a straightforward generalization of synchronization.

2.1 Synchronisation and phase-locking

The notion of synchronization is used in various contexts. Therefore, before we move to the actual definitions of synchronization for the purpose of this work, we should motivate them briefly by discussing in layman's terms what it is we want to be synchronous and how to choose a suitable criterion, as the choice needs to be done accordingly to the investigated system. Let us begin with comparing the two main situations, namely the synchronization of frequencies and the synchronization of phases. Regarding the former, imagine a case of two detuned oscillators operating at two distinct frequencies. The intuitive understanding of their synchronization is an evolution towards oscillations at a single common frequency, possibly with a resulting constant phase shift bethween the two oscillators. However, this understanding brings at least two difficulties. First, the resulting common frequency will likely be set by and dependent on the outer synchronization mechanism. Second, once this mechanism is turned off the inner dynamics of the oscillators will tend to desynchronize their frequencies again. On the other hand, in the case of two identical pendulum clocks which from the very beginning oscillate with the same frequency, the natural is synchronization of their phases. When synchronized, they should move with an a priori determined phase difference, typically in-phase or anti-phase, irrespective of the initial shift.
The idea behind this works originates from the Huygens' clock experiment. As a result we are interested in the case of two or more identical systems, each with their own inner dynamics, which is the same for all of them. An example of such a system is a qubit network. In the beginning, local phases of the subsystems are random or unknown. The target is to achieve a predetermined known relation between them, in our case the synchronization of phases. This is an analogue to the phase synchronization of classical clocks. With identical inner dynamics, the systems will continue to evolve synchronously after the synchronization process even if the coupling and mutual interactions are interrupted,

resembling clocks taken apart. Such a synchronization mechanism will also necessarily need to be in accordance with the inner dynamics of the systems. Furthermore, we are primarily concerned with mechanisms that apply universally, independently of the given initial state. In other words, we are interested in a spontaneous phase synchronization. The reason for the last requirement is not unjustified. There is a key difference between the classical and quantum world to consider when it comes to synchronization. The state of a quantum system cannot be measured or otherwise determined without affecting it. The synchronization process should therefore not need to be adjusted to the state of the system since there are generally no means of extracting any information from it. Thus, a generaly applicable mechanism is desirable.
Now that we have established to examine spontaneous phase synchronization, we want to go more into datail about when two quantum systems are synchronized. That is not what it is to synchronize about the systems, but when the synchronization of that particular aspect occurs. Let us emphasize that quantum synchronization does not refer to any newly discovered phenomenon of quantum nature, unwitnessed in the classisal domain. It refers to the synchronization of quantum systems in the classical understanding, in any of but not limited to the meanings suggested so far. Various concepts and measures of synchronization applicable in the quantum realm appear in the current literature, see [3], [4], [5] for a brief overview. In the classical domain, the notion and measures of synchronization are typically built upon comparing systems trajectories in the phase space. In quantum systems, there are two different main approaches to consider. The first one is to look at the dynamics of local observables and their expectation values, the second is to directly compare the local density matrices or other representations of the quantum states. We choose the second approach based on the states themselves rather than observables, but will briefly comment on both to explain ourselves.
Last but not least, for the most part we are interested in the asymptotic dynamics and not in any transient effects. Consequently, we can make use of a more restrictive absolute understanding of synchronization rather than using a synchronization measure that would describe and quantify the process leading to a synchronized evolution. Hence our choice of setup and following definitions.

2.1.1 Synchronization of states

Assume a single quantum system with an associated Hilbert space $\mathscr{H}$. An n-component composite system of identical subsystems, copies of our original quantum system, is then associated with Hilbert space $\mathscr{H}^{\otimes n} = \mathscr{H}_1 \otimes \cdots \otimes \mathscr{H}_n$, $\mathscr{H}_i = \mathscr{H}$, $\forall i \in \{1, \ldots, n\}$. Let this composite system be in a state $\rho \in \mathcal{B}(\mathscr{H}^{\otimes n})$ and let us denote ρ_k the reduced state of the k^{th} component, obtained as a partial trace over all the remaining n-1 subsystems,

$$\rho_k = \mathrm{Tr}_{\otimes_{j \neq k} \mathscr{H}_j}\, \rho. \tag{2.1}$$

We then define synchronization and phase-locking as follows.

Definiton 2.1.1. Assume an n-component composite system in a state $\rho(t) \in \mathcal{B}(\mathscr{H}^{\otimes n})$ at time t. We say that the n individual systems in the reduced states $\rho_1(t), \ldots, \rho_n(t)$ are synchronized if for any pair of reduced states $\rho_j(t)$ and $\rho_k(t)$ there exists a stationary state

$\rho_{c_{jk}}$, $\frac{\partial \rho_{c_{jk}}}{\partial t} = 0$, such that

$$\rho_j(t) - \rho_k(t) = \rho_{c_{jk}}, \quad \forall t. \tag{2.2}$$

They achieve an asymptotic synchronization if they become synchronized in the limit $t \to \infty$, that is if any $\rho_j(t)$, $\rho_k(t)$ evolve such that

$$\lim_{t\to\infty} \left\| \rho_j(t) - \rho_k(t) - \rho_{c_{jk}} \right\| = 0, \tag{2.3}$$

We call a global state $\rho(t)$ with all reduces states ρ_j synchronized a synchronized state, and say that it achieves an asymptotic synchronization if all its subsystems do. We say that the quantum dynamical semigroup $\mathcal{T}$ and its generating Lindblad operators $\{L_j\}$ synchronize, enforce synchronization or lead to synchronization if the asymptotic state $\lim_{t\to\infty} \rho(t)$ of the evolution is synchronized for an arbitrary initial state $\rho(0)$. We call the operation $\mathcal{T}$ itself and its generating operators synchronizing or synchronization-enforcing.

According to this definition, the subsystems are synchronized if their non-stationary parts undergo the same evolution. We allow for a constant difference between the synchronized states, imposing constraints only on the dynamical part. To be able to distinguish we introduce a more restrictive second definition.

Definiton 2.1.2. We speak of complete synchronization if the reduced states $\rho_1(t), \ldots, \rho_n(t)$ of all subsystems in question are the same, i.e.

$$\rho_j(t) - \rho_k(t) = 0, \quad \forall j, k, t, \tag{2.4}$$

or respectively of asymptotic complete synchronization if

$$\lim_{t\to\infty} \|\rho_j(t) - \rho_k(t)\| = 0, \quad \forall j, k. \tag{2.5}$$

The rest of the terminology is defined as in the case of synchronization in definition 2.1.1.

In comparison with synchronization, the definition of complete synchronization additionally requires that all subsystems oscillate or otherwise evolve around the same stationary state.

Note: To satisfy our definition of synchronization 2.1.1 or that of complete synchronization 2.1.2 the quantum subsystems need not be evolving in time in general. For example, the reduced states of a maximally mixed state (proportional to the identity operator) cleary satisfy (2.4) and simultaneously the state of the entire system is stationary in the studied dynamics (1.5),(1.10). In this work, however, we focus on systems with non-trivial asymptotic evolution, i.e. on situations when the synchronization mechanism does not kill the inner dynamics.

Assuming only two subsystems we label them with letters A and B and, again denoting $\rho(t) \in \mathcal{B}(\mathscr{H}^{\otimes 2})$ the global state of the composite system, write the condition of complete synchronization as

$$\mathrm{Tr}_A\,\rho(t) = \mathrm{Tr}_B\,\rho(t). \tag{2.6}$$

Another important concept is antisynchronization, a process where subsystems reach mutually opposite phases. Both synchronization and antisynchronization are special cases of a so-called phase-locking which stands for a process of establishing a given constant phase shift. As this does not make sense for an arbitrary subset of subsystems and cannot be achieved for an arbitrary pair of subsystems in general, we introduce the definition exclusively for systems consisting of two subsystems. Analogously to the previous, we shall distinguish two situations. The first one characterized by the the systems' dynamical parts being the same and evolving with a constant phase difference, the second one additionally by the systems' stationary parts coinciding.

Definiton 2.1.3. Let A and B be two subsystems of a two-component system. At any time t the two systems A and B are in reduced states $\mathrm{Tr}_B\,\rho(t) = \rho_A(t)$ and $\mathrm{Tr}_A\,\rho(t) = \rho_B(t)$ of a global state $\rho(t)$. We denote the stationary part of the state of system $X \in \{A, B\}$ as $\rho_{X,st}$ and the dynamical (time-evolving) part as $\rho_{X,dyn}$,

$$\rho_A(t) = \rho_{A,st} + \rho_{A,dyn}(t), \tag{2.7}$$

$$\rho_B(t) = \rho_{B,st} + \rho_{B,dyn}(t), \tag{2.8}$$

$$\frac{\partial \rho_{A,st}}{\partial t} = \frac{\partial \rho_{B,st}}{\partial t} = 0. \tag{2.9}$$

We say that the subsystems A and B, in this order, are phase-locked with a phase shift $\varphi \in [0, 2\pi)$ if their dynamical parts satisfy

$$\rho_{A,dyn}(t) = e^{i\varphi}\rho_{B,dyn}(t), \tag{2.10}$$

i.e. if the dynamical parts of the subsystems differ only by a phase factor $e^{i\varphi}$. Asymptotic phase-locking is achieved if

$$\lim_{t\to\infty} \left\|\rho_{A,dyn}(t) - e^{i\varphi}\rho_{B,dyn}(t)\right\| = 0. \tag{2.11}$$

Phase-locking is a straightforward generalization of phase synchronization. Therefore, as an equivalent to the expression phase-locking we hereby introduce and hereafter parallelly use the term generalized synchronization.

We speak of phase-locking with a phase shift $\varphi \in [0, 2\pi)$ and simultaneous synchronization of the stationary parts, if in addition to (2.10) it holds

$$\rho_{A,st} = \rho_{B,st}. \tag{2.12}$$

We aternatively call such situation a generalized complete synchronization. The remaining terminology is defined in a similar fashion.

Note: i) For a trivial phase shift $\varphi = 0$ phase-locking reduces to synchronization, hence the term generalized synchronization.[1]
ii) In the case of a phase shift $\varphi = \pi$ we commonly speak of (complete) antisynchronization.
iii) For all but these two cases, synchronization and antisynchronization, the order of the two subsystems A and B does matter.

The concept of phase-locking naturally extends to more than just two subsystems. There are, however, some limitations as the structure of the system as a whole needs to be taken into account. More on the topic in chapter 4.

2.1.2 Synchronization of observables

A different approach is to characterize and define synchronization and phase-locking purely via the observables of the systems. The dynamics of two or more subsystems are characterized by the expectation values of chosen local observables and their time evolutions are compared by a classical criterion. The advantage is that this provides not only a definition of synchronization, but also a measure thereof. It is also well suited for the study of imperfect transient synchronization and for the study of synchronization of nonidentical subsystems.
On the other hand, the drawback of such an approach is that it only takes some of the observables into account, typically just a single one. This way it only makes use of partial information about the systems in question. Synchronization of one observable does not imply synchronization of other ones, nor does it guarantee that the systems are not in substantially different states.

The two concepts are, nonetheless, intertwined. Assume an n-component composite system of identical subsystems in a state $\rho \in \mathcal{B}(\mathscr{H}^{\otimes n})$. Synchronization with respect to a local observable $A \in \mathcal{B}(\mathscr{H})$ can be understood as a situation when the expectation value of A is the same on all of the individual subsystems

$$\mathrm{Tr}\left(A\rho_1\right) = \cdots = \mathrm{Tr}\left(A\rho_n\right), \tag{2.13}$$

or if we denote $A^{(l)} = I^{\otimes(l-1)} \otimes A \otimes I^{\otimes(n-l)}$ the local operator corresponding to the l$^{\text{th}}$ component

$$\left\langle A^{(1)} \right\rangle = \cdots = \left\langle A^{(n)} \right\rangle. \tag{2.14}$$

It is straightforward to show that complete synchronization in the sense of definition 2.1.2 implies synchronization of any local observable $A \in \mathcal{B}(\mathscr{H})$. The converse is not true, unless we extend the requirement on all possible observables $A \in \mathcal{B}(\mathscr{H})$ simultaneously. Formally, if a system in a state $\rho \in \mathcal{B}(\mathscr{H}^{\otimes n})$ has equal expectation values $\mathrm{Tr}\left(A^{(i)}\rho\right)$ on all its components i for all observables $A \in \mathcal{B}(\mathscr{H})$, then the reduced states ρ_i of individual component subsystems are the same. This follows immediatly from the fact that the trace is a scalar product on $\mathcal{B}(\mathscr{H})$.

[1]Another connection will be revealed in chapter 3 where it is shown that the (two-qubit) synchronization and the (two-qubit) phase-locking mechanisms with a given phase shift φ are in fact in a one-to-one correspondence and can smoothly be deformed ones into the others.

Therefore, when not restricting ourselves to some small subset of predetermined observables we actually require synchronization of states when requiring synchronization of observables, and vice versa.

By introducing a delay linked to the intrinsic frequency of the subsystems between the time-evolving expectation values of an appropriately chosen local observable the concept of synchronization naturally extends to phase-locking.

2.2 Synchronization measures

Synchronization measures provide a way of quantifying synchronization and describing the efficiency of a synchronization process. They can also account for possible errors in synchronization of the established states. Typically, one observable is chosen and a suitable criterion is applied to its expectation values on the subsystems.
Such a criterion is the Pearson's correlation coefficient defined for two real-valued time-dependent functions f, g via

$$C_{f,g}(t, \Delta t) = \frac{\overline{(f-\bar{f})(g-\bar{g})}}{\sqrt{\overline{(f-\bar{f})^2}\,\overline{(g-\bar{g})^2}}}, \tag{2.15}$$

where $t \in \mathbb{R}$ is used to denote time and $\bar{f} = \frac{1}{\Delta t}\int_t^{t+\Delta t} f(t')dt'$ is the mean value of f over a time window $(t, t+\Delta t)$, similarly for g, [4], [2]. The coefficient ranges from -1 to 1 with 1 corresponding to synchronization of f and g and -1 to antisynchronization of the two, irrespective of a possible constant difference between them. Incorporating a phase shift into the definitions of f, g one can easily modify the criterion to quantify the degree to which a desired phase-locking is achieved.
The criterion is commonly used in the current literature and it was successfully applied for example to position and momentum operator expectation values for two dissipating oscillators [14], in oscillator networks [13] or to spin operators in various spin systems [15], atomic lattices [16] or collision models [18].
Pearson's correlation coefficient is our synchronization measure of choice employed in the numerical simulations later in this work.

Another useful criterion is the so called synchronization error typically used for the study of chaotic systems [4]. It is defined for two systems as

$$S_c(t) = \left\langle (q_-^2(t) + p_-^2(t) \right\rangle^{-1}, \tag{2.16}$$

where $q_- = \frac{1}{\sqrt{2}}(q_1 - q_2)$ is the difference in position, the same for momentum p, trajectiories in the classical case and operators in the quantum realm. This measure is bounded in the quantum domain by the uncertainity relations. It was employed for example in [3] to quantify synchronization of a pair of coupled optomechanical oscillators.

Chapter 3

Two-qubit system

In this chapter we examine in detail a system of two identical two-level subsystems, a system of two identical qubits, with the evolution of its state ρ described by a quantum Markov dynamical semigroup whereof the generator takes the form (1.11) and all Lindblad operators $\{L_j\}$ are normal. In particular, we investigate all possible mechanisms of generalized synchronization realized by a QMDS with just a single normal Lindblad operator $L_1 \equiv L$, i.e.

$$\mathcal{L}(\rho) = -i[H, \rho] + L\rho L^\dagger - \frac{1}{2}\Big\{L^\dagger L, \rho\Big\}. \tag{3.1}$$

This assumption will be dropped later. Once we have a complete solution to the problem of synchronization mechanisms with a single Lindblad operator, the generalization to an arbitrary number of normal Lindblad operators is straightforward.
Our motivation for the choice involves several aspects. Firstly, despite the fact that two qubits constitute the simplest composite quantum system possible, two-qubit synchronization mechanisms are not well understood yet. Previous works on the subject are mostly restricted to both particular systems and specific mechanisms. Secondly, QMDS with normal Lindblad operators are significant because of a known faithful state, see chapter 1, section 1.5, and constitute a sufficiently broad family of quantum Markov processes for our work to be fitruitful. Thirdly, we are eventually concerned with synchronization mechanism capable of synchronizing a set of qubits connected in a qubit network. Such mechanisms should be aplicable to a network of an arbitrary size, hence also to the smallest one possible consisting of only two parties. Additionally, we hope that our analysis will contribute to the understanding of how an interplay between internal dynamics of quantum subsystems and their mediated mutual interaction result in synchronization. One of our aims is to uncover the mathematical structure of such synchronization mechanisms. In order to accomplish that we must keep the setup as simple as possible.

Let $\mathscr{H}_0$ be the Hilbert space corresponding to a single qubit and

$$H_0 = \begin{pmatrix} E_0 & 0 \\ 0 & E_1 \end{pmatrix}, \tag{3.2}$$

$E_0, E_0 \in \mathbb{R}$, be the Hamiltonian in the basis of its eigenvectors $|0\rangle$ and $|1\rangle$. For a closed system of a single free qubit the evolution of an initial state $|\psi\rangle \in \mathscr{H}_0$,

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle\,, \tag{3.3}$$

$a, b \in \mathbb{C}$, is generated by the Hamiltonian H_0 and given by

$$|\psi(t)\rangle = e^{-iE_1 t}\left(e^{-i(E_0-E_1)t} a\,|0\rangle + b\,|1\rangle\right). \tag{3.4}$$

The factor $e^{-iE_1 t}$ represents an overall phase prefactor and it is irrelevant as far as the qubit alone is concerned. The intrinsic frequency of the system dynamics is $\omega = E_0 - E_1$, given as the difference of eigenvalues. Indeed, the prefactor $e^{-iE_1 t}$ vanishes when expressed in the formalism of density matrices,

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)| = \begin{pmatrix} |a|^2 & e^{-i(E_0-E_1)t}\, a\bar{b} \\ e^{i(E_0-E_1)t}\, \bar{a}b & |b|^2 \end{pmatrix}, \tag{3.5}$$

revealing the frequency of the inner dynamics.

A system of two qubits is associated with the Hilbert space $\mathscr{H} = \mathscr{H}_0 \otimes \mathscr{H}_0$ and Hamiltonian $H = H_A + H_B \equiv H_1 \otimes I + I \otimes H_1$. Let $\mathscr{B} = (|00\rangle, |01\rangle, |10\rangle, |11\rangle)$ be a basis of $\mathscr{H}$, using the standard notation $|ij\rangle = |i\rangle \otimes |j\rangle$. In this basis the Hamiltonian H reads

$$H = \begin{pmatrix} 2E_0 & 0 & 0 & 0 \\ 0 & E_0 + E_1 & 0 & 0 \\ 0 & 0 & E_0 + E_1 & 0 \\ 0 & 0 & 0 & 2E_1 \end{pmatrix}. \tag{3.6}$$

We stick to $\mathscr{B}$ as our standard computational basis throughout the entire chapter.

3.1 Two-qubit synchronization and phase-locking

The goal is to find all possible two-qubit couplings that can be represented by a single normal operator L in the Lindblad equation (3.1) such that the time evolution described by the corresponding QMDS leads to synchronization, respectively phase-locking of the two qubits.
Let us begin with a seemingly reverse problem, a description of the attractor space of a given QMDS and discussion of its role in synchronization of states. For this purpose we apply theorem 1.5.1. To find all elements of the attractor space the commutation relations (1.26) can be used to separate the space $\mathcal{B}(\mathscr{H})$, superset of the space of all states, into five subspaces $X_{i\lambda}$ based on the corresponding associated eigenvalue λ,

$$X_0 = \mathrm{span}\{|00\rangle\langle 00|, |01\rangle\langle 01|, |01\rangle\langle 10|, |10\rangle\langle 01|, |10\rangle\langle 10|\, |11\rangle\langle 11|\}, \tag{3.7}$$

$$X_{2E_0-2E_1} = \mathrm{span}\{|00\rangle\langle 11|\}, \tag{3.8}$$

$$X_{2E_1-2E_0} = \mathrm{span}\{|11\rangle\langle 00|\}, \tag{3.9}$$

$$X_{E_0-E_1} = \mathrm{span}\{|00\rangle\langle 01|, |00\rangle\langle 10|, |01\rangle\langle 11|, |10\rangle\langle 11|\}, \tag{3.10}$$

$$X_{E_1-E_0} = \mathrm{span}\{|01\rangle\langle 00|\, |10\rangle\langle 00|, |11\rangle\langle 01|, |11\rangle\langle 10|\}. \tag{3.11}$$

The first one, X_0, corresponds to the stationary part of a possible asymptotic state that does not evolve in time and as such automatically satisfies our condition of synchronization (2.2). It is irrelevant for the synchronization of the dynamical part, however, it plays an important role in the question of complete synchronization discusssed in the next section.

The following two subspaces, namely $X_{2E_0-2E_1}$ and $X_{2E_1-2E_0}$, are trivial from the point of view of synchronization in the sense that any vectors $X_1 \in X_{2E_0-2E_1}$ and $X_2 \in X_{2E_1-2E_0}$ satisfy $\mathrm{Tr}_A X_1 = Tr_B X_1 = 0$ and $\mathrm{Tr}_A X_2 = Tr_B X_2 = 0$ respectively. As such they do not affect the reduced single-qubit states but contribute only to the asymptotic evolution of the composite system.

Finally, the last two subspaces $X_{E_1-E_0}$ and $X_{E_0-E_1}$ correspond to the non-trivial evolution of the reduced one-qubit states. The two subspaces are connected by the operation of complex conjugation. Consequently, solving the commutation relations (1.25) and (1.26) for one of the subspaces provides the solution for the other one.
Indeed, X is an eigenvector of a linear map ϕ with eigenvalue λ iff $X^\dagger$ is an eigenvector of ϕ with eigenvalue $\bar{\lambda}$,

$$\phi(X) = \lambda X \iff \phi(X^\dagger) = \bar{\lambda} X^\dagger. \tag{3.12}$$

It holds in general, and in this particular case it can also be seen from the fact that for the commutation relations it holds

$$[X, L] = \left[X, L^\dagger\right] = 0 \iff \left[X^\dagger, L\right] = \left[X^\dagger, L^\dagger\right] = 0, \tag{3.13}$$

$$[H, X] = i\lambda X \iff \left[H, X^\dagger\right] = i\bar{\lambda} X^\dagger, \tag{3.14}$$

for any matrices $X, L,\ H$, where H is self-adjoint, $\lambda \in \mathbb{C}$, and for which the expressions make sense.

Thus, we restrict ourselves to work only with the space $X_{E_0-E_1}$ and choose to parameterize an element $X \in X_{E_0-E_1}$ as

$$X = \alpha\,|00\rangle\langle 01| + \beta\,|00\rangle\langle 10| + \gamma\,|01\rangle\langle 11| + \delta\,|10\rangle\langle 11|\,, \tag{3.15}$$

or written in the matrix form

$$X = \begin{pmatrix} 0 & \alpha & \beta & 0 \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & \delta \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.16}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. Due to the restriction on $X_{E_0-E_1}$ the partial trace condition (2.2) of synchronization reduces to simple

$$\mathrm{Tr}_A\, X = \mathrm{Tr}_B\, X, \tag{3.17}$$

which in the chosen parameterization translates to

$$\alpha + \delta = \beta + \gamma. \tag{3.18}$$

Strictly speaking, in definition 2.1.1 we defined synchronization via the condition (2.2) only for states, not arbitrary elements of an attractor space. However, our definition extends naturally. Since the asymptotic evolution is given by (1.16), any state evolving in the asymptotics can be written as a sum of projections of an initial state onto the elements of the attractor space, each of the projections evolving with its own frequency determined by the corresponding eigenvalue. Hence, the condition of synchronization applied to a general asymptotic state immediately yields the same condition for attractors from the individual subspaces. The asymptotic state will be synchronized, irrespective of the initial state, if and only if the attractor space is formed by attractors satisfying the condition of synchronization on each of the subspaces. And it follows from the discussion above that this requirement is non-trivial only for the subspace $X_{E_0-E_1}$, or $X_{E_1-E_0}$ respectively.

In light of theorem 1.5.1, elements of the attractor space of a given QMDS are determined by the commutation relations (1.25), which in the case of a generator (3.1) read

$$[L, X] = \left[L^\dagger, X\right] = 0. \tag{3.19}$$

Therefore, our goal is to find all possible normal operators L such that the solution to the commutation relations (3.19) for $X \in X_{E_0-E_1}$ is non-trivial and satisfies the condition of synchronization (3.17).

To achieve that we will go through all such possible solutions X and find the Lindblad operators L that permit them, in order to subsequently pick out the ones that permit exclusively such solutions and no other. We will work in the parameterization given by (3.15) and discuss separatelly all possible attractors $X \in X_{E_1-E_2}$ satisfying the synchronization condition (3.18), sorted by the number of non-zero coefficients (denoted $\alpha, \beta, \gamma, \delta$) in the parametrization. The commutation relations (3.19) will give us a set of operators L for each possible attractor X and from these sets we will extract those operators that not only commute with the synchronized attractor X, but also enforce the synchronization condition on the entire associated attractor space.
Since by our definition the synchronization condition is also necessary for the complete synchronization, we will make use of the results in the next section where we further extract those operators L that even enforce the complete synchronization.

The same reasoning can be followed for phase-locking. In the above the definition of synchronization 2.1.1 is replaced by that of phase-locking 2.1.3, effectively changing (3.17) and (3.18) into

$$\operatorname{Tr}_A X = e^{i\varphi} \operatorname{Tr}_B X, \tag{3.20}$$

and

$$\alpha + \delta = e^{i\varphi} (\beta + \gamma), \tag{3.21}$$

where $\varphi \in [0, 2\pi)$ denotes the phase shift. As both cases are closely related and can be solved simultaneously, we will proceed with this generalized condition of synchronization

(3.20), (3.21). Synchronization in the sense of definition 2.1.1 can be obtained at any point simply by setting the phase shift $\varphi = 0$. This way we avoid repeating the same procedure twice and additionaly reveal the connection between synchronizing and phase-locking maps.

Lastly, note that any two Lindblad operators that differ only by an overall phase factor lead to the same evolution map due to the form of the generator (1.11), and any two Lindblad operators proportional one to the other result in the same asymptotic dynamics due to it being determined by the commutation relations (1.19). We will therefore sometimes omit the possible prefactors in the expressions below for simplicity.
Now for the actual analysis of possible attractors of synchronizing and phase-locking QMDS.

I. One non-zero coefficient:

This situation cannot occur as the generalized synchronization condition (3.21) requires at least two non-zero coefficients for non-trivial solutions.

II. Two non-zero coefficients:

a) $\alpha = -\delta \neq 0 \wedge \beta = \gamma = 0$ or $\beta = -\gamma \neq 0 \wedge \alpha = \delta = 0$

This corresponds to stationary asymptotic reduced states of individual qubits. Indeed, it can be seen from the parameterization (3.15) that both reduced operators of such attractor X are trivial. Consequently, it can only contribute to the evolution of mutual correlations.

b) $\delta = \beta = 0$ and $\alpha = e^{i\varphi}\gamma \neq 0$

The attractor X now reads

$$X = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & e^{-i\varphi}\alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \propto \begin{pmatrix} 0 & e^{i\varphi} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.22}$$

Denoting l_{ij} the matrix elements of L we can explicitly evaluate the commutation relations (3.19) in the form $XL = LX$ and $XL^\dagger = L^\dagger X$, resulting straightforwardly into a set of equations

$$l_{ij} = 0 \quad \text{for} \quad i \neq j, \tag{3.23}$$

$$l_{11} = l_{22} = l_{44}, \tag{3.24}$$

giving L of the form

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.25}$$

where $a \in \mathbb{C}$ and we factor out and drop an arbitrary complex prefactor.

We have found our first candidate for a Lindblad operator generating a synchronizing, respectively phase-locking QMDS, yet so far we have only shown that if the attractor X is suppposed to have a certain form satisfying the generalized synchronization condition (3.21) and the commutation relations (3.19) hold, then L has to have the form (3.25). However, for L to enforce synchronization or phase-locking, the existence of a synchronized or phase-locked attractor X is only a necessary condition, not a sufficient one. There might be other elements of the corresponding attractor space that do not satisfy the generalized synchronization condition. To show that an operator L generates a synchronizing, respectively phase-locking map, we need to prove the opposite relation, that is given an operator L, here by equation (3.25), and a general attractor $X' \in X_{E_0-E_1}$ satisfying commutation relations $[L, X'] = \left[L^\dagger, X'\right] = 0$, then X' necessarily satisfies the generalized synchronization condition $\mathrm{Tr}_A X' = e^{i\varphi} \mathrm{Tr}_B X'$.

Given the diagonal form of L close to identity, it is no surprise that this converse statement does not hold and that our candidate does not lead to a synchronizing, respectively phase-locking map. It can be seen from the fact that L does not depend on the set phase shift, hence it commutes with all attractors X of the form (3.22), irrespective of the value of φ, and cannot enforce synchronization or a particular phase shift.

c) $\alpha = \gamma = 0$ and $\delta = e^{i\varphi}\beta \neq 0$

Analogously to the previous case we arrive at

$$l_{ij} = 0 \quad \text{for} \quad i \neq j, \tag{3.26}$$

$$l_{11} = l_{33} = l_{44}, \tag{3.27}$$

giving L of the form

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.28}$$

for $b \in \mathbb{C}$, an arbitrary prefactor omitted.

Similarly to the previous case, this candidate for L does not lead to synchronization or phase-locking.

d) $\delta = \gamma = 0$ and $\alpha = e^{i\varphi}\beta \neq 0$

The attractor X reads

$$X = \begin{pmatrix} 0 & \alpha & e^{-i\varphi}\alpha & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \propto \begin{pmatrix} 0 & 1 & e^{-i\varphi} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.29}$$

Noticing that

$$X = \alpha \left|00\right\rangle \left(\left\langle 01\right| + e^{-i\varphi} \left\langle 10\right|\right), \tag{3.30}$$

we introduce a new orthonormal basis (e_1, e_2, e_3, e_4) where

$$e_1 = |00\rangle\,, \tag{3.31}$$

$$e_2 = \frac{1}{\sqrt{2}}(|01\rangle + e^{i\varphi}\,|10\rangle), \tag{3.32}$$

$$e_3 = \frac{1}{\sqrt{2}}(|10\rangle - e^{-i\varphi}\,|01\rangle), \tag{3.33}$$

$$e_4 = |11\rangle\,, \tag{3.34}$$

so that the transition matrix

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -e^{-i\varphi}\frac{1}{\sqrt{2}} & 0 \\ 0 & e^{i\varphi}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.35}$$

is unitary and the attractor X in the new basis reads

$$\tilde{X} = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \propto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.36}$$

Using the fact that the commutation relations (3.19) are invariant with respect to the change of basis we evaluate them directly to obtain $\tilde{L}$, the operator L in the new basis, which can then be transformed back into the original computational basis as $L = T\tilde{L}T^\dagger$. (3.19) implies

$$\tilde{l}_{11} = \tilde{l}_{22}, \tag{3.37}$$

$$\tilde{l}_{12} = \tilde{l}_{13} = \tilde{l}_{14} = \tilde{l}_{21} = \tilde{l}_{23} = \tilde{l}_{24} = \tilde{l}_{31} = \tilde{l}_{32} = \tilde{l}_{41} = \tilde{l}_{42} = 0, \tag{3.38}$$

leaving the lower right 2x2 submatrix arbitrary. That gives

$$\tilde{L} = \begin{pmatrix} c\, I_{2\times 2} & 0 \\ 0 & M \end{pmatrix}, \tag{3.39}$$

where $c \in \mathbb{C}$ and $M \in \mathbb{C}^{2x2}$. It is possible to factor out a phase factor and choose $c \in \mathbb{R}$ instead. The normality condition $\left[\tilde{L}, \tilde{L}^\dagger\right] = 0$, unaffected in form by the change of basis, can be written in blocks implying that L is normal if and only if the submatrix M is normal. We make use of the parameterization of a general 2x2 normal matrix (A.14), whereof derivation is available in the appendix A.

$$M = \begin{pmatrix} a & b \\ e^{i2k}b & a + me^{ik} \end{pmatrix} = aI + \begin{pmatrix} 0 & b \\ e^{i2k}b & me^{ik} \end{pmatrix}, \tag{3.40}$$

where $a, b \in \mathbb{C}$, $k, m \in \mathbb{R}$. Put together, we arrive at a candidate for synchronising, respectively phase-locking Lindblad operator L given by

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{e^{-i\varphi}}{\sqrt{2}} & 0 \\ 0 & \frac{e^{i\varphi}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & e^{i2k}\bar{b} & a + me^{ik} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{e^{-i\varphi}}{\sqrt{2}} & 0 \\ 0 & -\frac{e^{i\varphi}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.41}$$

where $a, b \in \mathbb{C}$, $c, k, m \in \mathbb{R}$ and $\varphi \in [0, 2\pi)$ is the desired phase shift.

To see if our candidate L leads to a synchronizing map, assume a general attractor $X' \in X_{E_0 - E_1}$ parameterized by $\alpha', \beta', \gamma', \delta' \in \mathbb{C}$ as follows

$$X' = \alpha' |00\rangle\langle 01| + \beta' |00\rangle\langle 10| + \gamma' |01\rangle\langle 11| + \delta' |10\rangle\langle 11| . \tag{3.42}$$

In our new basis it can be expressed as $\tilde{X'} = T^\dagger X' T$, resulting in

$$\tilde{X'} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \alpha' + e^{i\varphi}\beta' & \beta' - e^{-i\varphi}\alpha' & 0 \\ 0 & 0 & 0 & \gamma' + e^{-i\varphi}\delta' \\ 0 & 0 & 0 & \delta' - e^{i\varphi}\gamma' \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.43}$$

As $\tilde{X'}$ is assumed to be an attractor, $\left[\tilde{L}, \tilde{X'}\right] = \left[\tilde{L}^\dagger, \tilde{X'}\right] = 0$ holds. Written explicitly in the chosen parameterization and comparing matrix elements, this yields the following set of equations

$$\begin{aligned} (c - a)(\beta' - e^{-i\varphi}\alpha') &= 0, & (3.44) \\ b(\beta' - e^{-i\varphi}\alpha') &= 0, & (3.45) \\ e^{i2k}\bar{b}(\gamma' + e^{-i\varphi}\delta') &= 0, & (3.46) \\ [c - (a + me^{ik})](\gamma' + e^{-i\varphi}\delta') &= 0, & (3.47) \\ e^{i2k}\bar{b}(\delta' - e^{i\varphi}\gamma') &= 0, & (3.48) \\ me^{ik}(\delta' - e^{i\varphi}\gamma') &= 0. & (3.49) \\ & & (3.50) \end{aligned}$$

The first two constitute constraints on α' and β'. It follows from (3.44) and (3.45) that both $b \neq 0$ and $c \neq a$ implies $\alpha' = e^{i\varphi}\beta'$. And since the parameters α', β' do not appear in the remaining equations, the requirement

$$b \neq 0 \lor a \neq c \tag{3.51}$$

is necessary for L to be synchronizing in the generalized sense. However, of the two conditions in (3.51), only the former is also sufficient. Indeed, for $b \neq 0$ it follows from (3.46) and (3.48) that $\gamma' = \delta' = 0$. It is enough to multiply (3.46) by a factor $e^{i\varphi}$ and sum, respectively substract the two. Consequently, the generalized synchronization condition (3.21) holds.

On the other hand, for $b = 0$ the equations (3.46) and (3.48) vanish and the parameters γ', δ' are constrained solely by (3.47) and (3.49). In the case of $m = 0$, (3.49) is trivial and (3.47) implies $\delta' = -e^{i\varphi}\gamma'$, contradicting the generalized synchronization condition. Thus, $m \neq 0$ is needed, in which case the equation (3.49) yields $\delta' = e^{i\varphi}\gamma'$. Depending on the value of $c - a - me^{ik}$, (3.47) may additionaly compel $\gamma' = \delta' = 0$. In either case, the requirement $m \neq 0$ together with $a \neq c$ is sufficient to enforce synchronization, respectivelly phase-locking.

To sum up, an operator L given by (3.41) is synchronizing in the generalized sense if and only if at least one of the conditions

$$b \neq 0, \tag{3.52}$$

$$a \neq c \wedge m \neq 0, \tag{3.53}$$

is satisfied. Note that excluded from (3.41) are only operators which are diagonal in the new basis and such that $\tilde{L}_{11} = \tilde{L}_{22}$, $\tilde{L}_{33} = \tilde{L}_{44}$ or $\tilde{L}_{11} = \tilde{L}_{22} = \tilde{L}_{33}$, an insignificant set of measure zero.

We denote this first class of synchronization-, respectively phase-locking-enforcing Lindblad operators $\mathscr{L}_1$, an operator in it L_1 and write explicitly

$$L_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{e^{-i\varphi}}{\sqrt{2}} & 0 \\ 0 & \frac{e^{i\varphi}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & e^{i2k}\bar{b} & a + me^{ik} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{e^{-i\varphi}}{\sqrt{2}} & 0 \\ 0 & -\frac{e^{i\varphi}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.54}$$

where $a, b \in \mathbb{C}$, $c, k, m \in \mathbb{R}$, $b \neq 0$ or $(a \neq c \wedge m \neq 0)$ holds, and $\varphi \in [0, 2\pi)$ is the achieved phase shift.

e) $\alpha = \beta = 0$ and $\delta = e^{i\varphi}\gamma \neq 0$

This case can be solved similarly to the previous one and also the result bears a close resemblance to the above. Since the attractor can be written as

$$X = \delta \left(e^{-i\varphi} |01\rangle + |10\rangle\right) \langle 11| , \tag{3.55}$$

we choose a new basis (e_1, e_2, e_3, e_4) accordingly

$$e_1 = |00\rangle , \tag{3.56}$$

$$e_2 = \frac{1}{\sqrt{2}} \left(|01\rangle - e^{i\varphi} |10\rangle\right) , \tag{3.57}$$

$$e_3 = \frac{1}{\sqrt{2}} \left(e^{-i\varphi} |01\rangle + |10\rangle\right) , \tag{3.58}$$

$$e_4 = |11\rangle . \tag{3.59}$$

Again, the new basis is orthonormal and the transition matrix T is unitary. Following the same steps as in the previous case we arrive at a class of synchronizing, respectively phase-locking Lindblad operators, denoted $L_2 \in \mathscr{L}_2$. The result in a familiar form $L = T\tilde{L}T^\dagger$ reads

$$L_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{e^{-i\varphi}}{\sqrt{2}} & 0 \\ 0 & -\frac{e^{i\varphi}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & 0 & 0 \\ e^{i2k}\bar{b} & a + me^{ik} & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{e^{-i\varphi}}{\sqrt{2}} & 0 \\ 0 & \frac{e^{i\varphi}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.60}$$

where $a, b \in \mathbb{C}$, $c, k, m \in \mathbb{R}$, $b \neq 0$ or $(a + me^{ik} \neq c \wedge m \neq 0)$ holds, and $\varphi \in [0, 2\pi)$ is the achieved phase shift.

III. Three non-zero coefficients:

Remarkably, choosing any three of the coefficients non-zero and one equal to zero, the commutation relations (3.19) directly lead to

$$l_{ij} = 0 \quad \text{for} \quad i \neq j, \tag{3.61}$$

$$l_{11} = l_{22} = l_{33} = l_{44}, \tag{3.62}$$

so that the only operators L such that both L and $L^\dagger$ commute with X are multiples of identity,

$$L \propto I. \tag{3.63}$$

The explicit calculation is omitted here due to its length and simplicity. Since the identity operator commutes with any other operator, the commutation relation $[X', I] = 0$ trivially holds for any $X' \in X_{E_1 - E_2}$ and there are no constraints on X'. This case provides us with no synchronising operators L.

IV. Four non-zero coefficients:

The attractor X takes the form

$$X = \begin{pmatrix} 0 & \alpha & \beta & 0 \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & \delta \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.64}$$

with $\alpha, \beta, \gamma, \delta \neq 0$, satisfying $\alpha + \delta = e^{i\varphi}(\beta + \gamma)$. Looking for submatrices with nonzero determinant in the upper right corner of X we see immediately that $\operatorname{rank} X = 2$. Thus, to simplify evaluation of the commutation relations we introduce a new basis (e_1, e_2, e_3, e_4) such that $e_1, e_2 \in \operatorname{Ker} X$, spanning the two-dimensional kernel, and $e_3, e_4 \in (e_1, e_2)^\perp$. Let

$$e_1 = |11\rangle, \tag{3.65}$$

$$e_2 = \beta\,|12\rangle - \alpha\,|21\rangle\,, \tag{3.66}$$

$$e_3 = \bar{\alpha}\,|12\rangle + \bar{\beta}\,|21\rangle\,, \tag{3.67}$$

$$e_4 = |22\rangle\,, \tag{3.68}$$

and without loss of generality the parameters α, β are supposed to satisfy a normalization condition

$$|\alpha|^2 + |\beta|^2 = 1. \tag{3.69}$$

This only impacts rescalling of the attractor X and is thus irrelevant for the result. The reason behind is that at the same time the normalization ensures that the new basis is orthonormal, the transition matrix

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta & \bar{\alpha} & 0 \\ 0 & -\alpha & \bar{\beta} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.70}$$

is unitary and it holds $Xe_3 = e_1$, with no additional numerical prefactor, which further simplifies the form of X in the new basis. For the remaining basis element $e_4 = |22\rangle$, which comes from the original computational basis in order not to unnecessarily make the transition matrix T more complicated, we have $Xe_4 = \gamma\,|12\rangle + \delta\,|21\rangle$. Clearly $Xe_4 \in \mathrm{span}(e_2, e_3)$, a fact that can be used to define two new parameters $s, r \in \mathbb{C}$ via

$$Xe_4 = se_2 + re_3, \tag{3.71}$$

to take over the role of the parameters $\delta = -s\alpha + r\bar{\beta}$ and $\gamma = s\beta + r\bar{\alpha}$. This reparameterization merely helps structurize the disscusion below in simpler terms. The attractor X in the new basis reads

$$\tilde{X} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s \\ 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.72}$$

and the partial trace condition of generalized synchronisation (3.21) takes the form

$$(1-s)\alpha + r\bar{\beta} = e^{i\varphi}\left[(1+s)\beta + r\bar{\alpha}\right]. \tag{3.73}$$

This way we only need to examine the dependence on two parameters s and r while the other two, α and β, keep their role of defining a unitary change of basis (3.70). Again, the result will be of the form $L = T\tilde{L}T^{\dagger}$. In the folllowing we explore all possible situations one can meet.

Rewriting both matrices $\tilde{X}$ and $\tilde{L}$ in a block form

$$\tilde{X} = \begin{pmatrix} 0 & S \\ 0 & R \end{pmatrix}, \quad \tilde{L} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{3.74}$$

where

$$S = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}, \quad R = \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}, \tag{3.75}$$

introducing 2x2 matrices $A, B, C, D, S, R \in \mathbb{C}^{2x2}$, the commutation relations (3.19) imply, among other things, that

$$0 = SC = SB^\dagger, \tag{3.76}$$

$$0 = RC = RB^\dagger. \tag{3.77}$$

Hence, since at least one of the parameters r, s is nonzero it follows

$$B = C = 0, \tag{3.78}$$

and the now block-diagonal form of $\tilde{L}$ further simplifies the commutaion relations into

$$SD = AS, \tag{3.79}$$

$$RD = DR. \tag{3.80}$$

The same constraints hold for $A^\dagger, D^\dagger$ in place of A, D as well. Let us analyze the possible cases for the parameters s and r.

a) $s \neq 0$

Comparing matrix elements in (3.79), denoting $A = (a_{ij})$, $D = (d_{ij})$, we obtain

$$a_{11} = d_{11}, \tag{3.81}$$

$$a_{22} = d_{22}, \tag{3.82}$$

$$d_{12} = s a_{12}, \tag{3.83}$$

$$d_{21} = \frac{1}{s} a_{21}, \tag{3.84}$$

and by doing the same for $A^\dagger, D^\dagger$, taking complex conjugation and comparing with the above we arrive at

$$\bar{s} = \frac{1}{s} \implies |s| = 1. \tag{3.85}$$

If furthermore $r \neq 0$, the relation (3.80) implies

$$d_{12} = d_{21} = 0 \implies a_{12} = a_{21} = 0, \tag{3.86}$$

$$d_{11} = d_{22}, \tag{3.87}$$

and thus the only solutions for $\tilde{L}$ and consequently for L are multiplies of identity, which cannot enforce any form of synchronization. Therefore, we set $r = 0$. Since the matrices A and D are normal due to the block-diagonal shape of $\tilde{L}$ we parameterize A, and thus also D, using (A.14).

From the equations (3.73) and (3.85) it follows that

$$\frac{\bar{\alpha}}{\bar{\beta}} = -e^{-2i\varphi}\frac{\alpha}{\beta}, \tag{3.88}$$

respectively

$$\beta = \frac{1-s}{1+s}e^{-i\varphi}\alpha, \tag{3.89}$$

consequence of which is that regarding phase the difference between α and β can be expressed by a factor $\pm ie^{-i\varphi}$. For the value of the parameter s the relation (3.73) implies

$$s = \frac{\alpha - e^{i\varphi}\beta}{\alpha + e^{i\varphi}\beta}. \tag{3.90}$$

Note that the seemingly problematic cases $\alpha = e^{i\varphi}\beta$, implying $s = 0$, and $\alpha = -e^{i\varphi}\gamma$, for which neither the equation (3.90) is defined nor the generalized synchronization condition (3.73) is satisfied unless $s = 0$, are excluded as a consequence of (3.88). This reflects the fact that setting $s = r = 0$ is equivalent to $e_4 \in \operatorname{Ker} X$ and $\delta = \gamma = 0$, the situation discussed in II.d).

Together with the normalization condition $|\alpha|^2 + |\beta|^2 = 1$, (3.88) and (3.90) show that the choice of the parameter α determines two pairs (β, s), the two possibilities stemming from the two possible phase differences between α and γ. Importantly, they are non-equivalent in the sense that they correspond each to a different attractor X and Lindblad operator L.

There exist two families of distinct classes of operators L taking the form

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta & \bar{\alpha} & 0 \\ 0 & -\alpha & \bar{\beta} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & 0 & 0 \\ e^{i2k}\bar{b} & a + me^{ik} & 0 & 0 \\ 0 & 0 & a & sb \\ 0 & 0 & \bar{s}e^{i2k}\bar{b} & a + me^{ik} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \bar{\beta} & -\bar{\alpha} & 0 \\ 0 & \alpha & \beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.91}$$

where $a, b, \in \mathbb{C},\ k, m \in \mathbb{R}$, $\alpha \in \mathbb{C}$, $0 < |\alpha| < 1$,

$$\beta = \pm ie^{-i\varphi}\alpha\frac{\sqrt{1-|\alpha|^2}}{|\alpha|}, \tag{3.92}$$

$$s = \frac{1 \mp i\frac{\sqrt{1-|\alpha|^2}}{|\alpha|}}{1 \pm i\frac{\sqrt{1-|\alpha|^2}}{|\alpha|}}, \tag{3.93}$$

that commute each with the corresponding non-trivial attractor X of the form

$$X = \begin{pmatrix} 0 & \alpha & \beta & 0 \\ 0 & 0 & 0 & s\beta \\ 0 & 0 & 0 & -s\alpha \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.94}$$

Not apparent at first sight, the parameter s runs around the entire unit circle except for the points ± 1, the upper circle corresponding to the upper one of the possible signs in (3.92) and (3.93), the lower circle to the lower sign. Note also that s is independent of the phase shift φ, the information about the achieved phase difference between the two qubits is entirely encoded in the transition matrix T.

One can multiply the matrices in (3.91) to see that the phase of α can be included in the parameter b, whether $b \neq 0$ or not, without affecting the attractor X, as can be seen e. g. from the fact that the attractor X itself is proportional to the phase of α. We can therefore choose $\alpha \in \mathbb{R}$, $\alpha \in (0,1)$, removing a small redundancy in the description.

With α real we can further simplify the expressions (3.92), (3.93) for β and s by setting

$$\alpha = \cos t, \tag{3.95}$$

$t \in \left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right)$. Then

$$\beta = ie^{-i\varphi} \sin t, \tag{3.96}$$
$$s = e^{2it}, \tag{3.97}$$

the two families of classes of Lindblad operators represented by the $\pm$ sign accounted for by the two intervals for t.

In order to determine whether the operators L of the form (3.91) truly enforce synchronization or phase-locking we once again parameterize $X' \in X_{E_0-E_1}$ as in (3.42). In the new basis $\tilde{X}'$ reads

$$\tilde{X}' = \begin{pmatrix} 0 & \beta\alpha' - \alpha\beta' & \bar{\alpha}\alpha' + \bar{\beta}\beta' & 0 \\ 0 & 0 & 0 & \bar{\beta}\gamma' - \bar{\alpha}\delta' \\ 0 & 0 & 0 & \alpha\gamma' + \beta\delta' \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.98}$$

The commutation relations (3.19), which now take the form $\left[\tilde{L}, \tilde{X}'\right] = \left[\tilde{L}^\dagger, \tilde{X}'\right] = 0$, yield the following set of equations

$$\begin{aligned} 0 &= e^{i2k}\bar{b}(\beta\alpha' - \alpha\beta'), && (3.99)\\ 0 &= me^{ik}(\beta\alpha' - \alpha\beta'), && (3.100)\\ b(\bar{\beta}\gamma' - \bar{\alpha}\delta') &= sb(\bar{\alpha}\alpha' + \bar{\beta}\beta'), && (3.101)\\ e^{i2k}\bar{b}(\bar{\alpha}\alpha' + \bar{\beta}\beta') &= \bar{s}e^{i2k}\bar{b}(\bar{\beta}\gamma - \bar{\alpha}\delta'), && (3.102)\\ 0 &= \bar{s}e^{i2k}\bar{b}(\alpha\gamma' + \beta\delta'), && (3.103)\\ 0 &= me^{ik}(\alpha\gamma' + \beta\delta'), && (3.104) \end{aligned}$$

and analogously for $\tilde{L}^\dagger$, resulting in the same set of constraints. Let us first assume the case $b \neq 0$. It follows from (3.99) that

$$\beta' = \frac{\beta}{\alpha}\alpha' \tag{3.105}$$

and from (3.103) that

$$\gamma' = -\frac{\beta}{\alpha}\delta'. \tag{3.106}$$

Inserting these results into either (3.101) or (3.47), multiplying by α and utilizing the imposed normalization (3.69) yields

$$\delta' = -s\alpha'. \tag{3.107}$$

The fact that the generalized synchronization condition $\alpha' + \delta' = e^{i\varphi}(\beta' + \gamma')$ holds follows, using the relation (3.89),

$$\begin{aligned} e^{i\varphi}(\beta' + \gamma') &\overset{(3.105)(3.106)}{=} e^{i\varphi}\frac{\beta}{\alpha}\left(\alpha' - \delta'\right) \overset{(3.107)}{=} e^{i\varphi}\frac{\beta}{\alpha}\alpha'(1+s) \\ &\overset{(3.89)}{=} \alpha'(1-s) \overset{(3.107)}{=} \alpha' + \delta' \end{aligned} \tag{3.108}$$

This shows that

$$b \neq 0 \tag{3.109}$$

is a sufficient condition for L to be synchronising.

On the other hand, consider the case $b = 0$. The equations (3.99),(3.101),(3.102) and (3.103) become trivial. If additionally $m = 0$, the conditions (3.100) and (3.104) vanish as well, the operator L is a multiple of identity and as such does not enforce any synchronisation or phase-locking. Assume therefore $m \neq 0$. The equations (3.100) and (3.104) are the only non-trivial remaining constraints on the attractor X' stemming from the commutation relations and they retrieve the results (3.105) and (3.106). The relation (3.107) is not enforced in this case and the claim is that consequently the generalized synchronization condition does not hold. As a counterexample, let $\alpha' = \delta' = \frac{\alpha}{\beta}\beta' = -\frac{\alpha}{\beta}\gamma' \neq 0$. This is an attractor X' commuting with L and yet not satisfying the generalized synchronisation condition as

$$\alpha' + \delta' = 2\alpha' \neq 0 \tag{3.110}$$

does not equal

$$e^{i\varphi}(\beta' + \gamma') = e^{i\varphi}\frac{\beta}{\alpha}(\alpha' - \delta') = 0. \tag{3.111}$$

This proves that the condition $b \neq 0$ is also necessary.

Before we state the final result, note that the attractor X (3.94) associated with L is entirely determined by a single parameter α and a choice of $\pm$ in (3.92), (3.93). That is why we now select the parameter α and the binary choice of $\pm$ to parameterize two families of classes of Lindblad operators, as opposed to the rest of parameters which will again specify single operators within these classes. This way we always have a single class of operators corresponding to a particular attractor.

There are two families of classes of synchronizing, respectively phase-locking Lindblad operators $\mathscr{L}_{\alpha+}, \mathscr{L}_{\alpha-}$[1], with the class parameter $\alpha \in (0,1)$, consisting of operators $L_{\alpha+}, L_{\alpha-}$ (shortened to $L_{\alpha\pm}$) which take the form

$$L_{\alpha\pm} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta & \alpha & 0 \\ 0 & -\alpha & \bar{\beta} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & 0 & 0 \\ e^{i2k}\bar{b} & a + me^{ik} & 0 & 0 \\ 0 & 0 & a & sb \\ 0 & 0 & \bar{s}e^{i2k}\bar{b} & a + me^{ik} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \bar{\beta} & -\alpha & 0 \\ 0 & \alpha & \beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.112}$$

where $a, b \in \mathbb{C},\ k, m, \alpha \in \mathbb{R},\ 0 < \alpha < 1,\ b \neq 0,$

$$\beta = \pm i e^{-i\varphi}\sqrt{1-\alpha^2}, \tag{3.113}$$

$$s = \frac{\alpha \mp i\sqrt{1-\alpha^2}}{\alpha \pm i\sqrt{1-\alpha^2}}, \tag{3.114}$$

and $\varphi \in [0, 2\pi)$ is the achieved phase shift. Respectively,

$$L_{\cos t\,\mathrm{sgn}t} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & ie^{-i\varphi}\sin t & \cos t & 0 \\ 0 & -\cos t & -ie^{i\varphi}\sin t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & 0 & 0 \\ e^{i2k}\bar{b} & a+me^{ik} & 0 & 0 \\ 0 & 0 & a & e^{2it}b \\ 0 & 0 & e^{-2it}e^{i2k}\bar{b} & a+me^{ik} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -ie^{i\varphi}\sin t & -\cos t & 0 \\ 0 & \cos t & ie^{-i\varphi}\sin t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.115}$$

where $a, b \in \mathbb{C},\ k, m, \alpha \in \mathbb{R},\ b \neq 0,\ t \in \left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right),\ \cos t = \alpha$, and $\varphi \in [0, 2\pi)$ is the achieved phase shift.

b) $r \neq 0$

We have already demonstrated that if both s and r are non-zero, the only operators commuting with such attractors X are multiples of identity. Therefore, we assume $s = 0$ further on. The relations $SD = AS$ and $RD = DR$ imply

$$\alpha_{11} = d_{11} = d_{22}, \tag{3.116}$$

$$a_{12} = a_{21} = d_{12} = d_{21} = 0. \tag{3.117}$$

Hence, L simplifies into

[1] We use the same symbols to denote both the individual classes and the encompassing families in the belief that this slight abuse of notation will not be an obstacle to the reader's understanding and that a strict distinction would on the contrary only cause confusion.

$$L=\begin{pmatrix}1&0&0&0\\0&\beta&\bar{\alpha}&0\\0&-\alpha&\bar{\beta}&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}a&0&0&0\\0&b&0&0\\0&0&a&0\\0&0&0&a\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&\bar{\beta}&-\bar{\alpha}&0\\0&\alpha&\beta&0\\0&0&0&1\end{pmatrix}\tag{3.118}$$

where $a,b,\alpha,\beta\in\mathbb{C}$, $|\alpha|^2+|\beta|^2=1$. The parameter r is given

$$r=\frac{\alpha-e^{i\varphi}\beta}{e^{i\varphi}\bar{\alpha}-\bar{\beta}},\tag{3.119}$$

for $\alpha\neq e^{i\varphi}\beta$, and r can be arbitrary for $\alpha=e^{i\varphi}\beta$. Note that $\delta=r\bar{\beta}$ and $\gamma=r\bar{\alpha}$ due to (3.71), so that

$$X=\begin{pmatrix}0&\alpha&\beta&0\\0&0&0&r\bar{\alpha}\\0&0&0&r\bar{\beta}\\0&0&0&0\end{pmatrix}.\tag{3.120}$$

In the case $\alpha=e^{i\varphi}\beta$, r arbitrary, the parameter r does not affect the operator L nor the transition matrix T. It merely parameterizes attractors X (3.120) which are given by the parameters α and r. We use the plural here as different values of parameter r correspond to different attractors X. In the case $\alpha\neq e^{i\varphi}\beta$, the assumed attractor (3.120) is fully determined by the parameters α,β.

The operator L of the form (3.118), however, does not lead to a synchronizing or phase-locking map for $\alpha\neq e^{i\varphi}\beta$ as it is not constraining the phase shift φ in any way, and it reduces to a special case of (3.60) for $\alpha=e^{i\varphi}\beta$.

c) $s=r=0$

By the definition of s and r, see equation (3.71), this is equivalent to $e_4\in\operatorname{Ker}X$ and $\delta=\gamma=0$. The case was already discussed, see II. d).

3.2 Two-qubit complete synchronization

In this part we investigate continuous quantum Markovian dynamical semigroups with generator of the form (3.1) which enforce asymptotic generalized complete synchronization in the sense of definition 2.1.3 for an arbitrary initial state of two qubits.
As the requirement of complete synchronization is stronger than that of synchronization, it is sufficient to inspect the Lindblad operators L found in section 3.1 to pick out those that additionaly enforce identical stationary parts of the asymptotic reduced individual-qubit states. To proceed we assume a general stationary attractor $X_{st}\in X_0$ given by

$$X_{st}=A\left|00\right\rangle\!\left\langle00\right|+B\left|01\right\rangle\!\left\langle01\right|+C\left|01\right\rangle\!\left\langle10\right|+D\left|10\right\rangle\!\left\langle01\right|+E\left|10\right\rangle\!\left\langle10\right|+F\left|11\right\rangle\!\left\langle11\right|,\tag{3.121}$$

parameterized by six variables $A,B,C,D,E,F\in\mathbb{C}$. In terms of these parameters the condition of complete synchronization (2.12) reduces to

$$B = E. \tag{3.122}$$

We now require that for a given operator L the commutation relations (3.19) impose the condition of complete synchronization (3.122) on X_{st}. As we only consider operators synchronizing the dynamical parts of asymptotic reduced qubit evolutions, the result will be a complete synchronization of the entire system. In the following we discuss the two classes and the two families of classes of synchronizing Lindblad operators from section 3.1.

The analysis of operators L_1, L_2 representing the first two classes $\mathscr{L}_1, \mathscr{L}_2$ can be performed in a similar fashion and therefore only the case of the former is presented in detail. To begin with, consider an operator L_1 of the form (3.54). Using the same change of basis as introduced in the respective part of section 3.1, given by the transition matrix (3.35), the attractor X_{st} in the new basis reads

$$\tilde{X}_{st} = \frac{1}{2}\begin{pmatrix} 2A & 0 & 0 & 0 \\ 0 & B + e^{i\varphi}C + e^{-i\varphi}D + E & -e^{-i\varphi}B + C - e^{-2i\varphi}D + e^{-i\varphi}E & 0 \\ 0 & -e^{i\varphi}B - e^{2i\varphi}C + D + e^{i\varphi}E & B - e^{i\varphi}C - e^{-i\varphi}D + E & 0 \\ 0 & 0 & 0 & 2F \end{pmatrix}. \tag{3.123}$$

The commutation relations (3.19) written explicitly in the new basis give rise to a set of equations

$$e^{-i\varphi}(c-a)(-B + e^{i\varphi}C - e^{-i\varphi}D + E) = 0, \tag{3.124}$$
$$e^{-i\varphi}b\,(-B + e^{i\varphi}C - e^{-i\varphi}D + E) = 0, \tag{3.125}$$
$$e^{i\varphi}(c-a)(-B - e^{i\varphi}C + e^{-i\varphi}D + E) = 0, \tag{3.126}$$
$$b\,(B - e^{i\varphi}C - e^{-i\varphi}D + E) = b\,2F, \tag{3.127}$$
$$e^{i\varphi}e^{i2k}\bar{b}\,(-B - e^{i\varphi}C + e^{-i\varphi}D + E) = 0, \tag{3.128}$$
$$e^{i2k}\bar{b}\,(B - e^{i\varphi}C - e^{-i\varphi}D + E) = e^{i2k}\bar{b}\,2F. \tag{3.129}$$

Since at least one of the conditions $b \neq 0$ and $c \neq a$ holds, equations (3.124),(3.125) and (3.126), (3.128) simplify into

$$B - e^{i\varphi}C + e^{-i\varphi}D - E = 0, \tag{3.130}$$
$$B + e^{i\varphi}C - e^{-i\varphi}D - E = 0, \tag{3.131}$$

which combined together gives $B - E = 0$, so that the condition (3.122) is always satisfied. We have found out that the whole class $\mathscr{L}_1$ of operators L_1, originally designed to synchronize the two subsystems, actually enforces complete synchronization.

The same holds for the operators L_2 given by (3.60). The proof is analogous to the one above and as such is not presented here.

To discuss the remaining two continua of classes $\mathscr{L}_{\alpha+}, \mathscr{L}_{\alpha-}$ we consider operators $L_{\alpha+}, L_{\alpha-}$ of the form (3.112). The attractor X_{st} in the respective basis reads

$$\tilde{X}_{st} = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & |\beta|^2 B + \alpha^2 E - \alpha\bar{\beta}C - \alpha\beta D & \alpha\bar{\beta}B - \alpha\bar{\beta}E + \bar{\beta}^2 C - \alpha^2 D & 0 \\ 0 & \alpha\beta B - \alpha\beta E - \alpha^2 C + \beta^2 D & \alpha^2 B + |\beta|^2 E + \alpha\bar{\beta}C + \alpha\beta D & 0 \\ 0 & 0 & 0 & F \end{pmatrix}, \tag{3.132}$$

and, using the fact that $b \neq 0$ in (3.112), the commutation relations (3.19) yield the following set of equations

$$|\beta|^2 B + \alpha^2 E - \alpha\bar{\beta}C - \alpha\beta D = A, \tag{3.133}$$

$$\alpha\bar{\beta}B - \alpha\bar{\beta}E + \bar{\beta}^2 C - \alpha^2 D = 0, \tag{3.134}$$

$$\alpha\beta B - \alpha\beta E - \alpha^2 C + \beta^2 D = 0, \tag{3.135}$$

$$\alpha^2 B + |\beta|^2 E + \alpha\bar{\beta}C + \alpha\beta D = F. \tag{3.136}$$

Dividing (3.134) and (3.135) by $\bar{\alpha}\bar{\beta}$ and $\alpha\beta$ respectively, we can express the difference $B - E$ to obtain

$$B - E = \frac{\bar{\beta}}{\bar{\alpha}}C - \frac{\bar{\alpha}}{\bar{\beta}}D, \tag{3.137}$$

$$B - E = \frac{\beta}{\alpha}D - \frac{\alpha}{\beta}C. \tag{3.138}$$

These expressions admit various non-trivial solutions (two linear equations for four variables). Nonetheless, summed together and plugged in (3.113) for β,

$$\begin{aligned} B - E &= \frac{1}{2}\left[\left(\frac{\alpha}{\beta} - \frac{\bar{\beta}}{\alpha}\right)C + \left(\frac{\alpha}{\bar{\beta}} - \frac{\beta}{\alpha}\right)F\right] \\ &= \pm i\frac{1-2\alpha^2}{2\alpha\sqrt{1-\alpha^2}}\left(e^{i\varphi}C - e^{-i\varphi}D\right), \end{aligned} \tag{3.139}$$

they provide us with the desired constraint. Consequently, for

$$\alpha = \frac{1}{\sqrt{2}} \tag{3.140}$$

implying

$$\beta = \frac{\pm i e^{-i\varphi}}{\sqrt{2}}, \tag{3.141}$$

$$s = \mp i, \tag{3.142}$$

the condition (3.122) is satisfied and the Lindblad operators $L_{\alpha+}, L_{\alpha-}$ enforce generalized complete synchronization.

To show that this condition (3.140) is not only sufficient but also necessary, a simple counterexample can be given. Let $\alpha \neq \frac{1}{\sqrt{2}}$ and $D = -e^{2i\varphi}C$. The equations (3.137), (3.138) merge into

$$B - E = \pm i e^{i\varphi} \frac{1 - 2\alpha^2}{\alpha\sqrt{1-\alpha^2}} C, \tag{3.143}$$

a non-zero expression for $C \neq 0$. Moreover, (3.133) and (3.136) can be viewed as only introducing new variables A and F respectively. A nontrivial attractor X_{st} violating the condition (3.122) exists.

The operators $L_{\alpha\pm}$ lead to complete synchronization in the generalized sense if and only if $\alpha = \frac{1}{\sqrt{2}}$, in which case

$$L_{\frac{1}{\sqrt{2}}\pm} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\pm i e^{-i\varphi}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{\mp i e^{i\varphi}}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & 0 & 0 \\ e^{i2k}\bar{b} & a + m e^{ik} & 0 & 0 \\ 0 & 0 & a & \mp i b \\ 0 & 0 & \pm i e^{i2k}\bar{b} & a + m e^{ik} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\mp e^{i\varphi}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{\pm e^{-i\varphi}}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.144}$$

where $a, b \in \mathbb{C}$, $k, m \in \mathbb{R}$ and $b \neq 0$.

Note: Reparameterizing $(\alpha, \beta) \to (\pm i e^{i\varphi}\alpha, \pm i e^{i\varphi}\beta)$, which is equivalent to $b \to \pm i e^{i\varphi} b$, we are able to cast the operators $L_{\frac{1}{\sqrt{2}}+}, L_{\frac{1}{\sqrt{2}}-}$ into an even more familiar shape

$$L_{\frac{1}{\sqrt{2}}\pm} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{\mp i e^{-i\varphi}}{\sqrt{2}} & 0 \\ 0 & \frac{\mp i e^{i\varphi}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & 0 & 0 \\ e^{i2k}\bar{b} & a + m e^{ik} & 0 & 0 \\ 0 & 0 & a & \mp i b \\ 0 & 0 & \pm i e^{i2k}\bar{b} & a + m e^{ik} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{\pm e^{-i\varphi}}{\sqrt{2}} & 0 \\ 0 & \frac{\pm e^{i\varphi}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.145}$$

where $a, b \in \mathbb{C}$, $k, m \in \mathbb{R}$ and $b \neq 0$.

3.3 Attractor spaces of synchronizing maps on two qubits

3.3.1 Attractor spaces of synchronizing maps

Previously, we established two classes and two families of classes of normal Lindblad operators L that induce (complete) synchronization or phase-locking of two qubits. We did, however, only partially discussed the corresponding attractors, whereof role is essential in determining the asymptotic dynamics of the system in question. Having a basis of the attractor space of an associated QMDS we can write the asymptotic evolution of an arbitrary initial state via (1.16). Revealing the entire structure of the attractor spaces of the respective maps is the subject of this section.

We will go through all the synchronizing and phase-locking maps we found and will describe all their possible attractors, consecutively considering elements of the subspaces $X_{E_0-E_1}$, $X_{E_1-E_0}$, X_0, $X_{2E_0-2E_1}$ and $X_{2E_1-2E_0}$. Since attractors corresponding to different eigenvalues are mutually orthogonal, see (1.23), these subspaces are to be dealt with separately.

Let us begin with the operators L_1, given by (3.54). There are already some partial results found in the section 3.1. The attractor corresponding to the first dynamical part of reduced qubit dynamics $X_{d1} \in X_{E_0-E_1}$, formerly denoted simply X or X' which we now reserve for a general attractor $X \in \mathrm{Att}(\mathcal{T})$, is subject to the constraints (3.44) to (3.49)

$$(c-a)(\beta - e^{-i\varphi}\alpha) = 0, \tag{3.44}$$
$$b(\beta - e^{-i\varphi}\alpha) = 0, \tag{3.45}$$
$$e^{i2k}\bar{b}(\gamma + e^{-i\varphi}\delta) = 0, \tag{3.46}$$
$$[c-(a+me^{ik})](\gamma + e^{-i\varphi}\delta) = 0, \tag{3.47}$$
$$e^{i2k}\bar{b}(\delta - e^{i\varphi}\gamma) = 0, \tag{3.48}$$
$$me^{ik}(\delta - e^{i\varphi}\gamma) = 0, \tag{3.49}$$

whereof solution splits into several cases depending on the value of parameters a, b, c, k, m. For $b \neq 0 \vee c - a - me^{ik} \neq 0$,[2] the attractor X_{d1} reads

$$X_{d1} = \begin{pmatrix} 0 & \alpha & e^{-i\varphi}\alpha & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.146}$$

$\alpha \in \mathbb{C}$. Consequently, the attractor $X_{d2} \in X_{E_1-E_0}$ representing the second dynamical part, related to the previous one by the operation of complex conjugation, has the form

$$X_{d2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \zeta & 0 & 0 & 0 \\ e^{i\varphi}\zeta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.147}$$

$\zeta \in \mathbb{C}$. The parameters α and ζ are uncorrelated, the complex conjugation is merely a connection between the two subspaces $X_{E_0-E_1}$ and $X_{E_1-E_0}$.

Deliberately, we postpone the discussion of the excluded special case $b = 0$, $m = e^{-ik}(c - a) \neq 0$ until later. Instead, let us proceed with the stationary part. The corresponding attractor $X_{st} \in X_0$ was already shown to satisfy (3.124) to (3.131). In particular, recall

$$B - e^{i\varphi}C + e^{-i\varphi}D - E = 0, \tag{3.130}$$
$$B + e^{i\varphi}C - e^{-i\varphi}D - E = 0, \tag{3.131}$$

and

$$b\,(B - e^{i\varphi}C - e^{-i\varphi}D + E) = b\,2F, \tag{3.127}$$
$$e^{i2k}\bar{b}\,(B - e^{i\varphi}C - e^{-i\varphi}D + E) = e^{i2k}\bar{b}\,2F. \tag{3.129}$$

[2] Recall that in the case $b = 0$ it holds $c \neq a \wedge m \neq 0$. The requirement $c - a - me^{ik} \neq 0$ is additional to that of enforcing synchronization.

By summing (3.130) and (3.131) we obtained

$$E = B, \tag{3.148}$$

and by substracting them we further get

$$D = e^{2i\varphi}C. \tag{3.149}$$

If $b = 0$, there are no additional constraints and it holds

$$X_{st} = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & e^{-i\varphi}C & 0 \\ 0 & e^{i\varphi}C & B & 0 \\ 0 & 0 & 0 & F \end{pmatrix}, \tag{3.150}$$

where $A, B, C, F \in \mathbb{C}$. In the case $b \neq 0$, F is no longer a free parameter as both (3.127) and (3.129) imply

$$F = B - e^{i\varphi}C, \tag{3.151}$$

so that

$$X_{st} = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & e^{-i\varphi}C & 0 \\ 0 & e^{i\varphi}C & B & 0 \\ 0 & 0 & 0 & B - e^{i\varphi}C \end{pmatrix}, \tag{3.152}$$

where $A, B, C \in \mathbb{C}$.

Lastly, assume possible attractors $X_{c1} \in X_{2E_0-2E_1}$ and $X_{c2} \in X_{2E_1-2E_0}$, associated purely with correlations within the system, and parameterize the former

$$X_{c1} = \sigma \left|11\right\rangle \left\langle 22\right|, \tag{3.153}$$

$\sigma \in \mathbb{C}$. The commutation relations yield

$$[c - (a + me^{ik})]\,\sigma = 0, \tag{3.154}$$

$$e^{i2k}\bar{b}\,\sigma = 0, \tag{3.155}$$

which results in

$$\sigma = 0, \tag{3.156}$$

for all values of the parameters a, b, c, m, k but the excluded case $b = 0$, $m = e^{-ik}(c-a) \neq 0$. It follows that both X_{c1} and X_{c2} are trivial.

Put together, we obtain parameterization of a general attractor associated with an operator L_1 given by (3.54), excluding the case $b = 0$, $m = e^{-ik}(c - a) \neq 0$. Denoting a corresponding QMDS $\mathcal{T}_{L_1}$ the attractor space $\mathrm{Att}(\mathcal{T}_{L_1})$ thereof reads

$$\mathrm{Att}(\mathcal{T}_{L_1}) = \mathrm{span}\left\{\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&e^{-i\varphi}&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\1&0&0&0\\e^{i\varphi}&0&0&0\\0&0&0&0\end{pmatrix},\right.$$
$$\left.\frac{1}{\sqrt{3}}\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix}0&0&0&0\\0&0&e^{-i\varphi}&0\\0&e^{i\varphi}&0&0\\0&0&0&-e^{i\varphi}\end{pmatrix}\right\}, \tag{3.157}$$

in the case $b \neq 0$, and for $b = 0$, $m \neq e^{-ik}(c-a)$ then

$$\mathrm{Att}(\mathcal{T}_{L_1}) = \mathrm{span}\left\{\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&e^{-i\varphi}&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\1&0&0&0\\e^{i\varphi}&0&0&0\\0&0&0&0\end{pmatrix},\right.$$
$$\left.\frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\0&0&e^{-i\varphi}&0\\0&e^{i\varphi}&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}\right\}, \tag{3.158}$$

where $\varphi \in [0, 2\pi)$ is the enforced phase shift.

Note that an attractor $X \in \mathrm{Att}(\mathcal{T}_{L_1})$ is not necessarily a state and that its different parts evolve with different frequencies. For convenience, we write the attractor space $\mathrm{Att}(\mathcal{T}_{L_1})$ as a linear span of orthonormal generators associated each with a single eigenvalue of $\mathcal{T}_{L_1}$. The same for QMDS generated by other classes of Lindblad operators.

The attractor space associated with the second class $\mathscr{L}_2$ of synchronizing Lindblad operators L_2 would be discussed analogously, we only state the results. Namely, the attractor space $\mathrm{Att}(\mathcal{T}_{L_2})$ of a QMDS $\mathcal{T}_{L_2}$ generated by L_2 given by (3.60), excluding the case $b = 0$, $a = c$, reads

$$\mathrm{Att}(\mathcal{T}_{L_2}) = \mathrm{span}\left\{\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix}-e^{i\varphi}&0&0&0\\0&0&e^{-i\varphi}&0\\0&e^{i\varphi}&0&0\\0&0&0&0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&e^{i\varphi}\\0&0&0&0\end{pmatrix},\right.$$
$$\left.\frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&e^{-i\varphi}&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}\right\}, \tag{3.159}$$

in the case $b \neq 0$, and for $b = 0$, $a \neq c$ then

$$\mathrm{Att}(\mathcal{T}_{L_2}) = \mathrm{span}\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-i\varphi} & 0 \\ 0 & e^{i\varphi} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \right.$$
$$\left. \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & e^{i\varphi} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & e^{-i\varphi} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}. \tag{3.160}$$

where $\varphi \in [0, 2\pi)$ is the enforced phase shift.

To address the aforementioned special case, assume an operator L_1 given by (3.54) where $b = 0$, $m = e^{-ik}(c - a)$ and note that it in fact coincides with an operator L_2 given by (3.60) where $b = 0$, $a = c$,[3] as well as with the operator L given by (3.118) where $\alpha = e^{i\varphi}\beta$, the case excluded from otherwise not synchronising operators (3.118) and noted to reduce to L_2.
This can be best seen multiplying the matrices $T\tilde{L}T^{\dagger}$ in the expressions mentioned. Such an operator L, from here on denoted L_s, reads

$$L_s = \frac{1}{2} \begin{pmatrix} 2c & 0 & 0 & 0 \\ 0 & c + a & e^{-i\varphi}(c - a) & 0 \\ 0 & e^{i\varphi}(c - a) & c + a & 0 \\ 0 & 0 & 0 & 2c \end{pmatrix}, \tag{3.161}$$

where $c \in \mathbb{R}, a \in \mathbb{C}$, $a \neq c$ and $\varphi \in [0, 2\pi)$ is the desired phase shift. It marks the overlap of the two otherwise distinct classes $\mathscr{L}_1$ and $\mathscr{L}_2$ of synchronizing Lindblad operators L_1 and L_2. It is worth mentioning that for $c = -a = 1$ the operator L_s given by (3.161) reduces to

$$SWAP_{\varphi} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & e^{-i\varphi} & 0 \\ 0 & e^{i\varphi} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.162}$$

which is a generalization of the well-known and studied SWAP operator, often appearing in the context of quantum synchronization [6]. Traditionally, the SWAP operator acts simply as an exchange of states on two qubits, i. e. $|i\rangle \otimes |j\rangle \xleftrightarrow{SWAP} |j\rangle \otimes |i\rangle$, corresponding here to setting $\varphi = 0$. The generalization lies in adding a phase factor $e^{i\varphi}$ which additionaly introduces a phase shift between the swapped parties.
In fact, to reveal and point out the connection to the $SWAP_{\varphi}$ operator we can rewrite L_s as

[3] In the case of L_1 we are left with parameters $c \in \mathbb{R}$ and $a \in \mathbb{C}$, in the case of L_2 the remaining parameters are $c, m, k \in \mathbb{R}$. Reparameterizing the latter by introducing a parameter $a = me^{ik} - c \in \mathbb{C}$ to substitute for m and k, we obtain the former.

$$L_s = x\,I + y\,SWAP_\varphi, \tag{3.163}$$

where we introduced two new parameters $x, y \in \mathbb{C}$ such that $x = c + a, y = c - a \neq 0$. One of them can be chosen real by factoring out a global phase prefactor. Special cases of this operator L_s, without the added phase shift φ, are sometimes referred to as partial swap operators in the literature [18].
From the point of view of asymptotic dynamics, the operators L_s given by (3.161) or (3.163) constitues a certain generalization of the SWAP, respectively $SWAP_\varphi$ operator as they represent the maximal set of normal Lindblad operators that result in the same asymptotic behaviour as the SWAP, respectively $SWAP_\varphi$ operator in the studied type of evolution. Thus the subscript s for swap in L_s.
Note that while neither adding a multiple of identity to the Lindblad operator nor multiplying it by a prefactor changes the solution to the commutation relations (3.19), both actions result in a change in the generator (3.1) and thus in a different evolution map. Yet, although the transient dynamics of such modified QMDS may differ, both their asymptotic spectra and their attractor spaces remain unchanged. These two possibilities of modifying the Lindblad operators are included in the parameterizations of the other classes of synchronizing and phase-locking operators L_1, L_2 and $L_{\alpha\pm}$ as well. The ability to alter the Lindblad operators in this way is a trivial consequence of the theorem 1.4.2. Bearing that in mind, the class of operators L_s given by (3.161) or (3.163) respectively can effectively be reduced to the $SWAP_\varphi$ operator (3.162). The result thus translates to following. Apart from the trivial modifications, there is no other normal Lindblad operator that would exhibit the same behaviour with respect to the asymptotic dynamics, namely that it would be associated with the same attractor space, as the SWAP operator, respectively its generalization the $SWAP_\varphi$ operator.

Regarding the corresponding attractor space, alone the fact that the two operator classes do not form disjoint sets already points towards the existence of richer, more complex structures of the attractor spaces of maps generated by the Lindblad operators from the overlap. That is due to our procedure where we first assumed a particular attractor for which we subsequently found the whole class of commuting and especially synchronization-enforcing Lindblad operators. As the two classes $\mathscr{L}_1$ and $\mathscr{L}_2$ originate from different assumed attractors, the attractor space of a QMDS generated by an operator L_s (3.161) from their overlap contains at least the two attractor spaces of maps generated by L_1 and L_2 as its subspaces, with the possibility of their union being only a proper subset of the encompassing attractor space of the map generated by L_s. This holds since by excluding the operators from the overlap when discussing the attractor spaces corresponding to L_1 and L_2 we kept some of the constraints stemming from the commutation relations which now disappear. Hence, those solutions must apply here as well.
This statement is confirmed by evaluating the constraints again for this case. Vanishing of (3.46) and (3.47) which allows the parameters γ, δ to be nonzero corresponds to including the attractor associated with the eigenvalue $E_0 - E_1$ of the second class. The attractors accounting for the stationary part of asymptotic evolution match as (3.127) and (3.129) are also trivial. And when it comes to X_{c1}, X_{c2}, the constraints (3.154) and (3.155), again, vanish.

Thus, the attractor space $\mathrm{Att}(\mathcal{T}_{L_s})$ of a QMDS $\mathcal{T}_{L_s}$ generated by operator L_s given by (3.161) reads

$$\begin{aligned}\mathrm{Att}(\mathcal{T}_{L_s}) = \mathrm{span}\Bigg\{ &\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\0&0&e^{-i\varphi}&0\\0&e^{i\varphi}&0&0\\0&0&0&0\end{pmatrix},\\ &\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&e^{-i\varphi}&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&e^{i\varphi}\\0&0&0&0\end{pmatrix},\\ &\frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\1&0&0&0\\e^{i\varphi}&0&0&0\\0&0&0&0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&e^{-i\varphi}&0\end{pmatrix}, \begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\\ &\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}\Bigg\},\end{aligned} \tag{3.164}$$

where $\varphi \in [0, 2\pi)$ is the enforced phase shift.

Once we finish the discussion below regarding the attractor spaces of QMDS generated by $L_{\alpha+}$ and $L_{\alpha-}$ it will have been demonstrated that among all synchronizing and phase-locking normal Lindblad operators the operators L_s of the form (3.161) generate maps equipped with the richest attractor structure, namely with an attractor space of the highest and exclusively achieved dimension. Moreover, they turn out to be the only ones additionally preserving the subspaces $X_{2E_0-2E_1}$ and $X_{2E_1-2E_0}$. In a sense the Lindblad operators L_s generate evolution maps which preserve the greatest piece of information about the initial state. Recall that in terms of attractors of a given QMDS the resulting asymptotic state is given by (1.16). Furthermore, to each linearly independent attractor one can associate one linearly independent constant of motion, whereof expectation value remains unchanged along all trajectories.

Last but not least, we take a look at the remaining two families of classes $\mathscr{L}_{\alpha+}$, $\mathscr{L}_{\alpha-}$ of synchronizing, respectively phase-locking Lindblad operators and examinate in detail the attractor spaces of QMDS generated by operators $L_{\alpha+}$ and $L_{\alpha-}$ given by (3.112).
The attractor X_{d1} standing for the dynamical part of the reduced single-qubit asymptotic evolution was already found and is given by (3.94), (3.113) and (3.114). The attractor X_{st} representing the stationary part, assuming the parameterisation (3.121), is subject to constraints (3.133) to (3.136)

$$(1-\alpha^2)B+\alpha^2E \pm ie^{i\varphi}\alpha\sqrt{1-\alpha^2}C \mp ie^{-i\varphi}\alpha\sqrt{1-\alpha^2}D = A, \tag{3.133}$$

$$B-E \mp ie^{i\varphi}\frac{\sqrt{1-\alpha^2}}{\alpha}C \mp ie^{-i\varphi}\frac{\alpha}{\sqrt{1-\alpha^2}}D = 0, \tag{3.134}$$

$$B-E \pm ie^{i\varphi}\frac{\alpha}{\sqrt{1-\alpha^2}}C \pm ie^{-i\varphi}\frac{\sqrt{1-\alpha^2}}{\alpha}D = 0, \tag{3.135}$$

$$\alpha^2B+(1-a^2)E \mp ie^{i\varphi}\alpha\sqrt{1-\alpha^2}C \pm ie^{-i\varphi}\alpha\sqrt{1-\alpha^2}D = F, \tag{3.136}$$

where we already divided (3.134) and (3.135) by $\alpha\bar{\beta}$ and $\alpha\beta$ respectively and plugged in for β from (3.113). Substracting (3.134) and (3.135) yields

$$D = -e^{2i\varphi}C, \tag{3.165}$$

and inserting this result back into either of the same equations gives

$$E = B \pm ie^{i\varphi}\frac{2\alpha^2-1}{\alpha\sqrt{1-\alpha^2}}C. \tag{3.166}$$

Consequently, (3.133) and (3.136) simplify to

$$A = B \pm ie^{i\varphi}\frac{2\alpha^4-(2\alpha^2-1)^2}{\alpha\sqrt{1-\alpha^2}}C, \tag{3.167}$$

$$F = B \mp ie^{i\varphi}\frac{2\alpha^4-(2\alpha^2-1)^2}{\alpha\sqrt{1-\alpha^2}}C. \tag{3.168}$$

For $X_{c1} \in X_{2E_0-2E_1}$ in the parameterization (3.153) the commutation relations (3.19) impose the following constraints

$$\bar{s}e^{i2k}\bar{b}\,\sigma = 0, \tag{3.169}$$

$$me^{ik}\,\sigma = 0, \tag{3.170}$$

$$e^{i2k}b\,\sigma = 0, \tag{3.171}$$

resulting in simple

$$\sigma = 0, \tag{3.172}$$

since $b \neq 0$ was required for $L_{\alpha+}$, $L_{\alpha-}$ to enforce synchronization or phase-locking.

Put together, the attractor space $\mathrm{Att}(\mathcal{T}_{L_{\alpha\pm}})$ of a QMDS $\mathcal{T}_{L_{\alpha\pm}}$ generated by $L_{\alpha\pm}$ of the form (3.112) reads

$$
\begin{aligned}
\mathrm{Att}(\mathcal{T}_{L_{\alpha\pm}}) = \mathrm{span}\Bigg\{ & \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \frac{1}{\sqrt{(2\alpha^2-1)^2+\frac{2}{\alpha^2}}}\begin{pmatrix} 0 & 1 & \pm i e^{-i\varphi}\frac{\sqrt{1-\alpha^2}}{\alpha} & 0 \\ 0 & 0 & 0 & e^{-i\varphi}2(1-\alpha^2)\pm i e^{-i\varphi}\frac{\sqrt{1-\alpha^2}}{\alpha}(2\alpha^2-1) \\ 0 & 0 & 0 & 1-2\alpha^2\pm i2\alpha\sqrt{1-\alpha^2} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
& \frac{1}{\sqrt{(2\alpha^2-1)^2+\frac{2}{\alpha^2}}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \mp i e^{i\varphi}\frac{\sqrt{1-\alpha^2}}{\alpha} & 0 & 0 & 0 \\ 0 & e^{i\varphi}2(1-\alpha^2)\mp i e^{i\varphi}\frac{\sqrt{1-\alpha^2}}{\alpha}(2\alpha^2-1) & 1-2\alpha^2\mp i2\alpha\sqrt{1-\alpha^2} & 0 \end{pmatrix}, \\
& \frac{\alpha\sqrt{1-\alpha^2}}{\sqrt{8\alpha^8-32\alpha^6+42\alpha^4-18\alpha^2+3}}\begin{pmatrix} \pm i\frac{2\alpha^4-(2\alpha^2-1)^2}{\alpha\sqrt{1-\alpha^2}} & 0 & 0 & 0 \\ 0 & 0 & e^{-i\varphi} & 0 \\ 0 & -e^{i\varphi} & \pm i\frac{2\alpha^2-1}{\alpha\sqrt{1-\alpha^2}} & 0 \\ 0 & 0 & 0 & \mp i\frac{2\alpha^4-(2\alpha^2-1)^2}{\alpha\sqrt{1-\alpha^2}} \end{pmatrix}\Bigg\}.
\end{aligned}
\tag{3.173}
$$

In the special case of Lindblad operators $L_{\frac{1}{\sqrt{2}}\pm}$ given by (3.144) which enforce even generalized complete synchronization, obtainable by setting $\alpha = \frac{1}{\sqrt{2}}$, the attractor space $\mathrm{Att}(\mathcal{T}_{L_{\frac{1}{\sqrt{2}}\pm}})$ of a QMDS $\mathcal{T}_{L_{\frac{1}{\sqrt{2}}\pm}}$ reduces to a simpler form

$$
\begin{aligned}
\mathrm{Att}(\mathcal{T}_{L_{\frac{1}{\sqrt{2}}\pm}}) = \mathrm{span}\Bigg\{ & \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & 1 & \pm i e^{-i\varphi} & 0 \\ 0 & 0 & 0 & e^{-i\varphi} \\ 0 & 0 & 0 & \pm i \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
& \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \mp i e^{i\varphi} & 0 & 0 & 0 \\ 0 & e^{i\varphi} & \mp i & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} \pm i & 0 & 0 & 0 \\ 0 & 0 & e^{-i\varphi} & 0 \\ 0 & -e^{i\varphi} & 0 & 0 \\ 0 & 0 & 0 & \mp i \end{pmatrix}\Bigg\}.
\end{aligned}
\tag{3.174}
$$

To sum up, we observe that the attractor spaces of synchronization- or phase-locking-enforcing maps generated by normal Lindblad operators L given by (3.54), (3.60), (3.161) or (3.112) are in general four- to ten-dimensional, subject to several additional conditions, with these conditions and precise structures of the attractor spaces discussed and explicitly presented above.

A comprehensive overview of the results is found in appendix C. It consists of a list of all two-qubit generalized (complete) synchronization mechanisms and attractor spaces of their respective associated QMDS.

3.3.2 Combining several Lindblad operators

In the entire chapter so far we have restricted ourselves to work with just a single Lindblad operator in the generator of the dynamics (3.1). Having found all synchronizing and phase-locking operators in this setup and having fully described the attractor spaces of the

corresponding evolution maps, we can now discuss the situation of an arbitrary number of normal Lindblad operators combined in the generator.
The structure theorem 1.5.1 states that an attractor X of a QMDS must satisfy the commutation relations (1.25) with each of the Lindblad operators L_j present in the generator $\mathcal{L}$ separately, i. e. it must commute with all of them. It also shows that the attractor space is completely determined by the solution to the commutation relations. Hence, in the case of several operators L_j simultaneously appearing in the generator the attractor space of such evolution map is given by the intersection of the attractor spaces of maps with generators containing the individual operators L_j one at a time.
We discovered several classes of normal Linbdlad operators leading to synchronization, respectively phase-locking, and described the corresponding attractor spaces. We found that the attractor spaces and consequently the asymptotic dynamics is preserved within each class, up to minor changes subject to additional conditions in the part associated with stationary states in the case of classes $\mathscr{L}_1, \mathscr{L}_2$ of operators L_1 and L_2, and up to the overlap of these two classes consisting of the L_s operators. We furthermore showed that the attractor space, in particular the part responsible for the non-trivial asymptotic evolution of the reduced states, is unique to each class. From these facts two conclusions can be immediately drawn.
Firstly, if any number of synchronization-, respectively phase-locking-enforcing normal Lindblad operators from two or more distinct classes[4] are combined in the generator then the evolution given by the resulting QMDS will always result in a stationary state, irrespective of initial conditions. This follows from the observation that in intersection with the subspace $X_{E_0-E_1} \subset \mathcal{B}(\mathscr{H})$ the intersection of the attractor spaces corresponding to Lindblad operators from different classes is trivial.
Secondly, the Lindblad operators from one class can be combined arbitrarily withnout affecting the asymptotic evolution, again up to possible minor changes in the stationary part in the case of classes $\mathscr{L}_1, \mathscr{L}_2$ of operators L_1, L_2, mentioned above and described in detail in the relevant subsection 3.3.1 of this section and chapter. In the construction we always began with the attractor and found all commuting normal operators to subsequently pick out the ones that enforce synchronization or phase-locking. That implies that with each class we can even include any number of those operators of the same type discovered and later discarded during the process as they still commute with the relevant attractors, even though they do not have the ability to enforce synchronization or phase-locking on their own. The resulting generator will still lead to a synchronization- or phase-locking-enforcing evolution map. In other words, the operators L_1 given by (3.54) can be combined with operators L given by (3.41) and so on. From the construction, these are the only such normal Linblad operators. It is worth noting that the synchronization- or phase-locking-non-enforcing normal operators form a set of meassure zero within the set of all normal operators commuting with the relevant attractors. Therefore, interestingly, almost all normal Lindblad operators compatible with synchronization or phase-locking also enforce it.
Everything stated so far applies to combining Lindblad operators with a set identical phase shift $\varphi \in [0, 2\pi)$. Naturally, operators inducing two or more different phase shifts appearing concurrently in the generator will result in a stationary asymptotic state. The reasoning

[4]With an "exception" of the operators from the overlap, which technicallly both are from the same class and are from two distinct classes simultaneously. Those are no equivalence classes, just a slack analogy.

behind is the same as above in the case of Lindblad operators from different classes. The relevant attractor is unique to each class and every phase shift.

Chapter 4

Qubit networks

In this chapter we address the synchronization and phase-locking phenomena in so called qubit networks, n-qubit systems with bipartite interactions. Having thoroughly analyzed the smallest qubit network consisting of two qubits, we now show how the previously found synchronization mechanisms represented by normal Lindblad operators can by applied to a general system of n qubits. Our main aim is to answer the question which of the two-qubit synchronization mechanisms are able to synchronize an arbitrary quantum network.
The chapter is divided into several parts. In the beginning, we present the necessary terminology and notation together with a detailed construction of the network Lindbladian. Next, we meticulously discuss a particular synchronization mechanism as an example of qubit network synchronization, addressing especially two points. First, that the synchronizing Lindblad operators originally designed for two-qubit synchronization truly pairwise synchronize the qubits they are applied to also in a network, as it is to be expected by their design. Second, that the reduced single-qubit dynamics are not destroyed by the particular mechanism when applied to a network where the qubits mutually interact each with more than one other in general. Subsequently, we perform a similar analysis for a general two-qubit synchronization mechanism and consecutively address these two aspects for all classes of synchronizing Lindblad operators from chapter 3.
Throughout the discussion of synchronization in qubit networks we purposedly avoid its generalization to phase-locking. The investigation into phase-locking in qubit networks represents a complex problem, closely related to the study of network topologies and graph theory, and is beyond the scope of this work.

4.1 Preliminary

This section it devoted mainly to introducing the terminology and notation, and to the construction of Lindbladians for qubit networks.
Let us start with the associated Hilbert space and Hamiltonian. A single qubit is associated with Hilbert space $\mathscr{H}_0$ and Hamiltonian H_0 given by (3.2). The Hilbert space of a n-qubit system is then $\mathscr{H} = \mathscr{H}_0^{\otimes n}$ and its free Hamiltonian H equals

$$H = \sum_{j=1}^{n} I^{\otimes(j-1)} \otimes H_0 \otimes I^{\otimes(n-j)}. \tag{4.1}$$

Let $\mathscr{B} = (|0\dots0\rangle, |0\dots01\rangle, |0\dots010\rangle, |0\dots011\rangle, \dots, |1\dots1\rangle)$ be a basis of its eigenvectors, using again the standard notation $|i_1 \dots i_n\rangle = |i_1\rangle \otimes \dots \otimes |i_n\rangle$. We will use this basis throughout the entire chapter unless explicitly stated otherwise. The Hamiltonian H reads

$$H = \begin{pmatrix} nE_0 & 0 & \cdots & \cdots & 0 \\ 0 & (n-1)E_0 + E_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & nE_1 \end{pmatrix}. \tag{4.2}$$

A local observable $A \in \mathcal{B}(\mathscr{H}_0)$ on the i^{th} compoment of the composite system corresponds to a global operator $A^{(i)} \in \mathcal{B}(\mathscr{H})$ defined by

$$A^{(i)} = I^{\otimes(i-1)} \otimes A \otimes I^{\otimes(n-i)}. \tag{4.3}$$

For a two-qubit operation $L \in \mathcal{B}(\mathscr{H}_0^{\otimes 2})$, such as one of the two-qubit synchronizing Lindblad operators from chapter 3, we introduce a similar concept and notation. First, let $\Pi_{kl} \in \mathcal{B}(\mathscr{H})$ be a permutation operator swapping the k^{th} and l^{th} qubit, i.e. $\Pi_{kl} : |i_1, \dots, i_k, \dots, i_l, \dots, i_n\rangle \mapsto |i_1, \dots, i_l, \dots, i_k, \dots, i_n\rangle$. In the formalism of density operators $\rho \in \mathcal{B}(\mathscr{H})$, the qubit-swapping operator Π_{kl} naturally acts by conjugation,

$$\begin{aligned} &\Pi_{kl} |i_1, \dots, i_k, \dots, i_l, \dots, i_n\rangle \langle j_1, \dots, j_k, \dots, j_l \dots, j_n| \Pi_{kl} = \\ &= |i_1, \dots, i_l, \dots, i_k, \dots, i_n\rangle \langle j_1, \dots, j_l, \dots, j_k \dots, j_n|, \end{aligned} \tag{4.4}$$

and in a basis decomposition it can be expressed as

$$\Pi_{kl} = \sum_{\substack{i_a \in \{0,1\}, \\ a \in \{1,\dots,n\}}} |i_1, \dots, i_l, \dots, i_k, \dots, i_n\rangle\langle i_1, \dots, i_k, \dots, i_l, \dots, i_n|. \tag{4.5}$$

It holds $\Pi_{kl} = \Pi_{lk} = \Pi_{kl}^\dagger = \Pi_{kl}^{-1}$ and $\Pi_{kk} = I$. For an operator $L \in \mathcal{B}(\mathscr{H}_0^{\otimes 2})$ we can now introduce a corresponding global operator $L^{(kl)} \in \mathcal{B}(\mathscr{H})$ representing the action of the two-qubit operator L on qubits k and l in this order. Since two-qubit operations are in general asymmetric in the sense that they are not invariant with respect to swapping of the two qubits, which is true in particular for the vast majority of phase-locking two-qubit Lindblad operators and, as it turns out, also for many of the purely synchronizing ones, see chapter 5, section 5.3 for details, we need to distinguish between two situations. If $1 \leq k < l \leq n$ we define the global operator $L^{(kl)} \in \mathcal{B}(\mathscr{H})$ to be

$$L^{(kl)} = \Pi_{k+1,l}\,(I^{\otimes(k-1)} \otimes L \otimes I^{\otimes(n-k-1)})\,\Pi_{k+1,l}. \tag{4.6}$$

The operator $L^{(kl)}$ first orders the qubits k and l one after the other, then locally applies the operator L, and finally puts the qubits back into the original order. By the same reasoning, if $1 \le l < k \le n$ we set

$$L^{(kl)} = \Pi_{kl}\,\underbrace{\Pi_{l+1,k}\,(I^{\otimes(l-1)} \otimes L \otimes I^{\otimes(n-l-1)})\,\Pi_{l+1,k}}_{L^{(lk)},\text{ where } l<k}\,\Pi_{kl}, \tag{4.7}$$

that is we simply swap the qubits k and l and apply the previous definition, effectively substituting $\Pi_{kl}L^{(lk)}\Pi_{kl}$ for $L^{(kl)}$. Note that the construction of a global operator $L^{(kl)}$ via (4.6) and (4.7) preserves properties, in particular normality of the operator L.

This enables us to construct the Lindbladian (1.11) using the two-qubit generalized synchronization mechanisms. For example, for a single Lindblad operator $L^{(kl)}$, i.e. for an operator $L \in \mathcal{B}(\mathscr{H}_0^{\otimes 2})$ acting on the k^{th} and the l^{th} qubit in this order, $k < l$ for simplicity, the generator (1.11) reads

$$\begin{aligned}
\mathcal{L}(\rho) &= -i[H,\rho] + L^{(kl)}\rho(L^{(kl)})^\dagger - \frac{1}{2}\Big\{(L^{(kl)})^\dagger L^{(kl)}, \rho\Big\} \\
&= -\,i[H,\rho] + \Pi_{k+1,l}(I^{\otimes(k-1)} \otimes L \otimes I^{\otimes(n-k-1)})\Pi_{k+1,l}\,\rho\,\Pi_{k+1,l}(I^{\otimes(k-1)} \otimes L^\dagger \otimes I^{\otimes(n-k-1)})\Pi_{k+1,l} \\
&\quad - \frac{1}{2}\Big\{\Pi_{k+1,l}(I^{\otimes(k-1)} \otimes L^\dagger \otimes I^{\otimes(n-k-1)})\Pi_{k+1,l}\,\Pi_{k+1,l}(I^{\otimes(k-1)} \otimes L \otimes I^{\otimes(n-k-1)})\Pi_{k+1,l},\, \rho\Big\} \\
&= -\,i[H,\rho] + \Pi_{k+1,l}(I^{\otimes(k-1)} \otimes L \otimes I^{\otimes(n-k-1)})(\Pi_{k+1,l}\rho\Pi_{k+1,l})(I^{\otimes(k-1)} \otimes L^\dagger \otimes I^{\otimes(n-k-1)})\Pi_{k+1,l} \\
&\quad - \frac{1}{2}\Pi_{k+1,l}\Big\{I^{\otimes(k-1)} \otimes L^\dagger L \otimes I^{\otimes(n-k-1)}, \Pi_{k+1,l}\rho\Pi_{k+1,l}\Big\}\Pi_{k+1,l}.
\end{aligned} \tag{4.8}$$

Indeed, the action of qubit-swapping, localy applying the operator L, and swapping back can be clearly seen also in the generator form.
In the beginning, we restrict ourselves to work with a single synchronization mechanism given by a single two-qubit Lindblad operator L. Using the above construction of operators $L^{(kl)}$, we will apply this mechanism onto individual pairs of qubits. We do not consider any direct interactions of more than two qubits.

To describe which pairs of qubits of an n-party system are connected and interacting we employ the notion of a directed graph $\mathcal{G} = \{\mathcal{N}, \mathcal{E}\}$, where $\mathcal{N}$ is the set of vertices or nodes, representing the n qubits and labeled $1, \ldots, n$, and $\mathcal{E}$ is the set of directed edges which stand for the mutual two-qubit interactions. Namely, the Lindblad operator L is applied to each pair of adjacent qubits, taking the orientation of the connecting edge into account. We call such a graph $\mathcal{G}$ an interaction graph and denote its adjacency matrix G. The entire system is referred to as a qubit network.
The evolution of a qubit network is given by a QMDS $\mathcal{T}$ whereof generator $\mathcal{L}$ of the form (1.11) can be written as

$$\mathcal{L}(\rho) = -i[H,\rho] + \sum_{\substack{i,j\in\{1,\ldots,n\},\\ G_{ij}\neq 0}} L^{(ij)}\rho(L^{(ij)})^\dagger - \frac{1}{2}\Big\{(L^{(ij)})^\dagger L^{(ij)},\rho\Big\}, \tag{4.9}$$

where H is the free n-qubit Hamiltonian (4.1), $G \in \mathbb{R}^{n\times n}$ is the adjacency matrix of an interaction graph, and $L^{(ij)}$ are the Lindblad operators given by (4.6),(4.7) constructed from a fixed single two-qubit normal operator L.

Having a well-defined concept of a qubit network and a suitable construction of its Lindbladian, we can proceed with the main task which is the analysis of its asymptotic dynamics. By the theorem 1.5.1, an attractor $X \in \mathrm{Att}(\mathcal{T})$ associated with the eigenvalue λ is given by the solution to the commutation relations (1.25), (1.26), i.e.

$$\Big[L^{(ij)},X\Big] = \Big[(L^{(ij)})^\dagger,X\Big] = 0, \quad \forall i,j\in\{1,\ldots,n\}\,\text{such that}\,G_{ij}\neq 0, \tag{4.10}$$

$$[H,X] = i\lambda X. \tag{4.11}$$

Note: Using a weighted graph and/or a set of graphs, this description can naturally be extended to account for weighted interactions and/or any number of distinct two-qubit operators L acting on individual qubit pairs. The weights do not have any influence on the asymptotics of the system, they merely scale the Lindblad operators and by that alter the transient dynamics and convergence rate towards an asymptotic state.

In order for us to be able to analyze the asymptotic dynamics of a qubit network via solving commutation relations in the light of theorem 1.5.1, a suitable parameterization of attractors is needed. Let $X \in \mathcal{B}(\mathscr{H})$. It can be written as

$$X = \sum_{\substack{i_a,j_a\in\{0,1\},\\ a\in\{1,\ldots,n\}}} X^{i_1\ldots i_n}_{j_1\ldots j_n}\,|i_1,\ldots,i_n\rangle\langle j_1,\ldots,j_n|\,, \tag{4.12}$$

where $X^{i_1\ldots i_n}_{j_1\ldots j_n}\in\mathbb{C}$ are the hereby introduced basis coefficients. To shorten the notation, let us denote $\vec{i},\vec{j}\in\{0,1\}^n$ the multiindices $i_1\ldots i_n$ and $j_1\ldots j_n$. The equation (4.12) reads

$$X = \sum_{\vec{i},\vec{j}\in\{0,1\}^n} X^{\vec{i}}_{\vec{j}}\left|\vec{i}\right\rangle\!\left\langle\vec{j}\right|. \tag{4.13}$$

Two multiindices $\vec{i}\in\{0,1\}^k$ and $\vec{j}\in\{0,1\}^l$ written one following the other, unseparated by a comma, should be understood as a multiindex $\vec{m}\in\{0,1\}^{k+l}$, $\vec{m} = \vec{i}\,\vec{j} = i_1\ldots i_k\,j_1\ldots j_l$. Additionally, for a multiindex $\vec{j}\in\{0,1\}^n$ we call the $n\in\mathbb{N}$ its length and define $\left|\vec{j}\right|$ to be

$$\left|\vec{j}\right| = \sum_{a=1}^{n} j_a. \tag{4.14}$$

We write $\vec{i} > \vec{j}$ if $\left|\vec{i}\right| > \left|\vec{j}\right|$ for two multiindices of the same length and for a multiindex $\vec{j}\in\{0,1\}^n$ and $k\in\mathbb{Z}$ denote $\vec{j}+k\in\{0,1\}^n$ an unspecified multiindex such that $\left|\vec{j}+k\right| = \left|\vec{j}\right| + k$.

As was the case with a two qubit system, we can separate the space of operators $\mathcal{B}(\mathscr{H})$ into several subspaces using the commutation relation (1.26) with the Hamiltonian H (4.2). Denoted again $X_{i\lambda}$, where λ is the associated eigenvalue, those $2n+1$ subspaces read

$$X_0 = \operatorname{span}\left\{ |i_1 \dots i_n\rangle\langle j_1 \dots j_n| \, , \sum_{a=1}^{n} i_a = \sum_{a=1}^{n} j_a \right\}, \tag{4.15}$$

$$X_{E_1-E_0} = \operatorname{span}\left\{ |i_1 \dots i_n\rangle\langle j_1 \dots j_n| \, , \sum_{a=1}^{n} i_a = \sum_{a=1}^{n} j_a + 1 \right\}, \tag{4.16}$$

$$X_{E_0-E_1} = \operatorname{span}\left\{ |i_1 \dots i_n\rangle\langle j_1 \dots j_n| \, , \sum_{a=1}^{n} i_a = \sum_{a=1}^{n} j_a - 1 \right\}, \tag{4.17}$$

$$X_{2E_1-2E_0} = \operatorname{span}\left\{ |i_1 \dots i_n\rangle\langle j_1 \dots j_n| \, , \sum_{a=1}^{n} i_a = \sum_{a=1}^{n} j_a + 2 \right\}, \tag{4.18}$$

$$\vdots$$

$$X_{nE_1-nE_0} = \operatorname{span}\left\{ |1 \dots 1\rangle\langle 0 \dots 0| \right\}, \tag{4.19}$$

$$X_{nE_0-nE_1} = \operatorname{span}\left\{ |0 \dots 0\rangle\langle 1 \dots 1| \right\}, \tag{4.20}$$

or shortly

$$X_{k(E_1-E_0)} = \operatorname{span}\left\{ \left| \vec{i} \right\rangle\!\left\langle \vec{j} \right| , \left| \vec{i} \right| - \left| \vec{j} \right| = k \right\}, \tag{4.21}$$

where $\vec{i}, \vec{j} \in \{0,1\}^n$, $k \in \{-n, \dots, n\}$.

For the reduced one-qubit dynamics, only three of them are relevant, namely the subspaces $X_0, X_{E_1-E_0}$ and $X_{E_0-E_1}$. Two more contribute to the two-qubit reduced dynamics and so on. Only elements of the subspaces $X_{k(E_1-E_0)}$ in (4.21) with the absolute value of k less than or equal to the number of concerned qubits survive when tracing out the remaining subsystems. The other ones contribute exclusively to higher order correlations.
Except for X_0, the subspaces are pairwise related by the operation of complex conjugation, which takes $X_{k(E_1-E_0)}$ to $X_{-k(E_1-E_0)}$ and vice versa. It is therefore enough to consider only one of the pair when solving the commutation relations (1.25).

4.2 Two-qubit synchronization mechanisms in qubit networks

4.2.1 A simple synchronization mechanism - an illustrating example

Let us begin with one of the simplest synchronizing normal Lindblad operators, namely with the operator L_2 given by (3.60) where $a = c = k = m = \varphi = 0$, throughout this subsection denoted simply L, i.e.

$$L = \begin{pmatrix} 0 & b & -b & 0 \\ \bar{b} & 0 & 0 & 0 \\ -\bar{b} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.22}$$

where $b \in \mathbb{C}$, $b \neq 0$. The parameter b is kept unspecified to better illustrate the action of the operator L in the upcoming analysis.
Next, we apply it to a single pair of qubits, without loss of generality to the qubits 1 and 2 in this order[1], via an operator $L^{(12)}$ constructed using (4.6). That is we assume the generator $\mathcal{L}$ of a QMDS $\mathcal{T}$ given by (4.9) where L is given by (4.22) and the adjacency matrix G of the interaction graph satisfies $G_{12} = 1$. The operarator $L^{(12)}$ acts on the elements of the computational basis from the left as

$$|00\, i_3 \dots i_n\rangle\langle j_1 \dots j_n| \mapsto \bar{b}\, |01\, i_3 \dots i_n\rangle\langle j_1 \dots j_n| - \bar{b}\, |10\, i_3 \dots i_n\rangle\langle j_1 \dots j_n|\,, \tag{4.23}$$
$$|01\, i_3 \dots i_n\rangle\langle j_1 \dots j_n| \mapsto b\, |00\, i_3 \dots i_n\rangle\langle j_1 \dots j_n|\,, \tag{4.24}$$
$$|10\, i_3 \dots i_n\rangle\langle j_1 \dots j_n| \mapsto -b\, |00\, i_3 \dots i_n\rangle\langle j_1 \dots j_n|\,, \tag{4.25}$$
$$|11\, i_3 \dots i_n\rangle\langle j_1 \dots j_n| \mapsto 0, \tag{4.26}$$

where $\vec{i} \in \{0,1\}^{n-2}$, labeled from 3 to n to match the qubit labeling, $\vec{j} \in \{0,1\}^n$; and from the right as

$$|k_1 \dots k_n\rangle\langle 00\, l_3 \dots l_n| \mapsto b\, |k_1 \dots k_n\rangle\langle 01\, l_3 \dots l_n| - b\, |k_1 \dots k_n\rangle\langle 10\, l_3 \dots l_n|\,, \tag{4.27}$$
$$|k_1 \dots k_n\rangle\langle 01\, l_3 \dots l_n| \mapsto \bar{b}\, |k_1 \dots k_n\rangle\langle 00\, l_3 \dots l_n|\,, \tag{4.28}$$
$$|k_1 \dots k_n\rangle\langle 10\, l_3 \dots l_n| \mapsto -\bar{b}\, |k_1 \dots k_n\rangle\langle 00\, l_3 \dots l_n|\,, \tag{4.29}$$
$$|k_1 \dots k_n\rangle\langle 11\, l_3 \dots l_n| \mapsto 0, \tag{4.30}$$

where $\vec{k} \in \{0,1\}^n$, $\vec{l} \in \{0,1\}^{n-2}$, labeled again from 3 to n.

Let $X \in X_{E_1 - E_0}$ be an attractor of the QMDS from one of the two subspaces relevant for the asymptotic time evolution of reduced single-qubit states. Using the parameterization (4.12), the commutation relations (4.10), i.e.

$$\left[L^{(12)}, X\right] = \left[(L^{(12)})^\dagger, X\right] = 0, \tag{4.31}$$

yield the following sets of equations

$$bX^{01\,\vec{i}}_{00\,\vec{i}} - bX^{10\,\vec{i}}_{00\,\vec{i}} = 0, \tag{4.32}$$
$$-\bar{b}X^{11\,\vec{i}}_{01\,\vec{i}} + \bar{b}X^{11\,\vec{i}}_{10\,\vec{i}} = 0, \tag{4.33}$$
$$-bX^{01\,\vec{i}}_{00\,\vec{i}} = 0, \tag{4.34}$$
$$bX^{10\,\vec{i}}_{00\,\vec{i}} = 0, \tag{4.35}$$

$\forall \vec{i} \in \{0,1\}^{n-2}$,

[1] Alhough, in this particular case the order is not important as the operator L (4.22) acts symmetrically on the two qubits. Changing $L^{(12)}$ to $L^{(21)}$ is here equivalent to changing L to $-L$, which leaves the generator and consequently the QMDS unchanged.

$$\bar{b}X^{00\,\vec{j+1}}_{00\,\vec{j}} - \bar{b}X^{01\,\vec{j+1}}_{01\,\vec{j}} + \bar{b}X^{01\,\vec{j+1}}_{10\,\vec{j}} = 0, \tag{4.36}$$

$$-\bar{b}X^{00\,\vec{j+1}}_{00\,\vec{j}} - \bar{b}X^{10\,\vec{j+1}}_{01\,\vec{j}} + \bar{b}X^{10\,\vec{j+1}}_{10\,\vec{j}} = 0, \tag{4.37}$$

$$bX^{01\,\vec{j+1}}_{01\,\vec{j}} - bX^{10\,\vec{j+1}}_{01\,\vec{j}} - bX^{00\,\vec{j+1}}_{00\,\vec{j}} = 0, \tag{4.38}$$

$$bX^{01\,\vec{j+1}}_{10\,\vec{j}} - bX^{10\,\vec{j+1}}_{10\,\vec{j}} + bX^{00\,\vec{j+1}}_{00\,\vec{j}} = 0, \tag{4.39}$$

$\forall \vec{j}, \vec{j+1} \in \{0,1\}^{n-2}$,

$$-\bar{b}X^{00\,\vec{k+2}}_{01\,\vec{k}} + \bar{b}X^{00\,\vec{k+2}}_{10\,\vec{k}} = 0, \tag{4.40}$$

$$\bar{b}X^{00\,\vec{k+2}}_{01\,\vec{k}} = 0, \tag{4.41}$$

$$\bar{b}X^{00\,\vec{k+2}}_{10\,\vec{k}} = 0, \tag{4.42}$$

$$bX^{01\,\vec{k+2}}_{11\,\vec{k}} - bX^{10\,\vec{k+2}}_{11\,\vec{k}} = 0, \tag{4.43}$$

$\forall \vec{k}, \vec{k+2} \in \{0,1\}^{n-2}$,

$$\bar{b}X^{00\,\vec{l+3}}_{11\,\vec{l}} = 0, \tag{4.44}$$

$\forall \vec{l}, \vec{l+3} \in \{0,1\}^{n-2}$,

$$bX^{11\,\vec{m}-1}_{00\,\vec{m}} = 0, \tag{4.45}$$

$\forall \vec{m}, \vec{m}-1 \in \{0,1\}^{n-2}$.

The equations are purposedly sorted by the difference in the number of zeros and ones in the unspecified multitindices in the lower and upper indices of the coefficients $X^{\vec{i}}_{\vec{j}}$.
For the reduced dynamics of qubits 1 and 2 only the first set, comprising equations (4.32) to (4.35), is relevant. This can be best seen if we calculate the partial trace of the attractor X over the remaining $n-2$ subsystems. Denoting $\mathrm{Tr}_{i_1\ldots i_k}$ the trace $\mathrm{Tr}_{\mathscr{H}_0^{\otimes k}}$ over the k Hilbert spaces standing for qubits $i_1, \ldots, i_k$, the reduced attractor reads

$$\begin{aligned} \mathrm{Tr}_{3\ldots n} X = \mathrm{Tr}_{3\ldots n} \left\{ \sum_{\substack{\vec{i},\vec{j}\in\{0,1\}^n,\\ |\vec{i}|=|\vec{j}|+1}} X^{i_1\ldots i_n}_{j_1\ldots j_n} \left| i_1, \ldots, i_n \right\rangle\!\left\langle j_1, \ldots, j_n \right| \right\} = \\ = \sum_{\vec{i}\in\{0,1\}^{n-2}} X^{01\,\vec{i}}_{00\,\vec{i}} \left|01\right\rangle\!\left\langle 00\right| + X^{10\,\vec{i}}_{00\,\vec{i}} \left|10\right\rangle\!\left\langle 00\right| + \left|11\right\rangle\!\left\langle 01\right| X^{11\,\vec{i}}_{01\,\vec{i}} + X^{11\,\vec{i}}_{10\,\vec{i}} \left|11\right\rangle\!\left\langle 10\right|. \end{aligned} \tag{4.46}$$

Therefore, only elements with matching indices in the positions 3 to n in the decomposition (4.12) of X affect the reduced attractor. Due to the restriction onto $X_{E_1-E_0}$ the synchronization condition (2.2) for the reduced bipartite system of qubits 1 and 2 reduces to $\mathrm{Tr}_{13\ldots n} X = \mathrm{Tr}_{2\ldots n} X$ which gives

$$\sum_{\vec{i}\in\{0,1\}^{n-2}} X^{01\,\vec{i}}_{00\,\vec{i}} + X^{11\,\vec{i}}_{10\,\vec{i}} = \sum_{\vec{i}\in\{0,1\}^{n-2}} X^{10\,\vec{i}}_{00\,\vec{i}} + X^{11\,\vec{i}}_{01\,\vec{i}}. \tag{4.47}$$

The equations (4.34) and (4.35) imply

$$X^{01\,\vec{i}}_{00\,\vec{i}} = X^{10\,\vec{i}}_{00\,\vec{i}} = 0, \tag{4.48}$$

$\forall\vec{i}\in\{0,1\}^{n-2}$. Consequently, (4.32) is always satisfied. Lastly, (4.33) implies

$$X^{11\,\vec{i}}_{10\,\vec{i}} = X^{11\,\vec{i}}_{01\,\vec{i}}, \tag{4.49}$$

$\forall\vec{i}\in\{0,1\}^{n-2}$. These two results (4.48) and (4.49) together show that the synchronization condition (4.47) for qubits 1 and 2 is satisfied since it is satisfied for every element of the sum.

The second set of equations, (4.36) to (4.39), comprises constraints on a part of the attractor X which does not affect the reduced dynamics of qubits 1 and 2, yet can still be relevant for other qubit pairs. Because in the case of multiindices $\vec{j},\vec{j}+1$ that are matching up to a single position, say the k^{th} one, the reduced attractor $\mathrm{Tr}_{1\ldots(k-1)(k+1)\ldots n}\,X$ contains elements with coefficients appearing in (4.36) to (4.39). That makes this set of equations especially important as it represents a type of constraints on X non-existing in the case of only two qubits which could, theoretically, together with similar constraints produced by other Lindblad operators $L^{(ij)}$ enforce trivial asymptotic dynamics of some or all individual qubits in the network.
To continue, let us consider the quadruples of equations (4.36) to (4.39) with the same multiindices $\vec{j},\vec{j}+1$ in all four equations. Divided by b or $\bar{b}$ respectively, (4.36) substracted from (4.39) implies

$$X^{01\,\vec{j}+1}_{01\,\vec{j}} = X^{10\,\vec{j}+1}_{10\,\vec{j}}. \tag{4.50}$$

That makes (4.39) a multiple of (4.36) and (4.38) a multiple of (4.37). The summation of (4.36) and (4.37) further yields

$$X^{10\,\vec{j}+1}_{01\,\vec{j}} = X^{01\,\vec{j}+1}_{10\,\vec{j}}, \tag{4.51}$$

The whole set thus reduces to (4.50), (4.51) and

$$X^{00\,\vec{j}+1}_{00\,\vec{j}} - X^{01\,\vec{j}+1}_{01\,\vec{j}} + X^{01\,\vec{j}+1}_{10\,\vec{j}} = 0, \tag{4.52}$$

$\forall\vec{j},\vec{j}+1\in\{0,1\}^{n-2}$. Naturally, these constraints alone are not enough for the QMDS to enforce synchronization on any additional qubit pair. They do, nevertheless, show to what extend and how the synchronizing operator L (4.22) applied to qubits 1 and 2 affects the asymptotic time evolution of the remaining parties, and constitute a part of its effect on mutual correlations.

Finally, the remaining two sets, i.e. equations (4.40) to (4.45), concern exclusively higher order correlations and no reduced single-qubit dynamics.

Analogous analysis can be performed for a stationary attractor $X_{st} \in X_0$. The commutation relations (4.31) yield

$$X_{st}{}^{00\,\vec{i}}_{00\,\vec{i}} - X_{st}{}^{01\,\vec{i}}_{01\,\vec{i}} + X_{st}{}^{01\,\vec{i}}_{10\,\vec{i}} = 0, \tag{4.53}$$
$$-X_{st}{}^{00\,\vec{i}}_{00\,\vec{i}} - X_{st}{}^{10\,\vec{i}}_{01\,\vec{i}} + X_{st}{}^{10\,\vec{i}}_{10\,\vec{i}} = 0, \tag{4.54}$$
$$X_{st}{}^{01\,\vec{i}}_{01\,\vec{i}} - X_{st}{}^{10\,\vec{i}}_{01\,\vec{i}} - X_{st}{}^{00\,\vec{i}}_{00\,\vec{i}} = 0, \tag{4.55}$$
$$X_{st}{}^{01\,\vec{i}}_{10\,\vec{i}} - X_{st}{}^{10\,\vec{i}}_{10\,\vec{i}} + X_{st}{}^{00\,\vec{i}}_{00\,\vec{i}} = 0, \tag{4.56}$$

$\forall \vec{i} \in \{0,1\}^n$, i.e. the already solved set of equations (4.36) to (4.39) for X with a substitute $\vec{i}$ for both $\vec{j}$ and $\vec{j}+1$, and four more sets of equations which are irrelevant for the reduced attractor $\mathrm{Tr}_{3\ldots n} X_{st}$. Hence

$$X_{st}{}^{01\,\vec{i}}_{01\,\vec{i}} = X_{st}{}^{10\,\vec{i}}_{10\,\vec{i}}, \tag{4.57}$$
$$X_{st}{}^{10\,\vec{i}}_{01\,\vec{i}} = X_{st}{}^{01\,\vec{i}}_{10\,\vec{i}}, \tag{4.58}$$
$$X_{st}{}^{00\,\vec{i}}_{00\,\vec{i}} = X_{st}{}^{01\,\vec{i}}_{01\,\vec{i}} + X_{st}{}^{01\,\vec{i}}_{10\,\vec{i}}, \tag{4.59}$$

$\forall \vec{i} \in \{0,1\}^n$, the consequence of which is that the condition of complete synchronization (2.4) for qubits 1 and 2, reduced here to $\mathrm{Tr}_{2\ldots n} X_{st} = \mathrm{Tr}_{13\ldots n} X_{st}$ or explicitly

$$\sum_{\vec{i} \in \{0,1\}^{n-2}} X_{st}{}^{01\,\vec{i}}_{01\,\vec{i}} = \sum_{\vec{i} \in \{0,1\}^{n-2}} X_{st}{}^{10\,\vec{i}}_{10\,\vec{i}}, \tag{4.60}$$

is satisfied because it is satisfied for every element of the sum due to (4.57). The QMDS completely synchronizes qubits 1 and 2, as was to be expected.

So far we have described the constraints on attractors relevant for the asymptotic reduced states of qubits 1 and 2 induced by the Lindblad operator $L^{(12)}$, showing that their complete synchronization is enforced also in the case of an otherwise arbitrary qubit network. Furthermore, we obtained some additional information about how the parts of the attractor space relevant in regard to the asymptotics of the remaining qubits and to some of the system correlations are affected.
Let us complete the picture by discussing the reduced two-qubit attractors for qubits 1 and 2. Attractors lying in the subspaces X_0 and $X_{E_1-E_0}$ were already analyzed, so only the subspace $X_{2E_1-2E_0}$ is left. Assume $X_c \in X_{2E_1-2E_0}$. From the constraints given by the commutation relations (4.31), a single one is non-trivial for $\mathrm{Tr}_{3\ldots n} X_c$, namely

$$X_c{}^{11\,\vec{i}}_{00\,\vec{i}} = 0. \tag{4.61}$$

Summarizing briefly, while still having only partial information about the attractor space $\mathrm{Att}(\mathcal{T})$, we have uncovered the form of a general reduced two-qubit attractor of qubits 1 and 2, i.e. $\mathrm{Tr}_{3\ldots n} X$, $X \in \mathrm{Att}(\mathcal{T})$, and showed that it corresponds to an element of the attractor space of a QMDS on two qubits generated by the same Lindblad operator L. Indeed, with a slight abuse of notation, if $X \in \mathrm{Att}(\mathcal{T})$ then $\mathrm{Tr}_{3\ldots n} X \in \mathrm{Att}(\mathcal{T}_{L_2})$ given by

(C.7).

The effect of $L^{(12)}$ discussed, we proceed with applying the synchronizing operator L (4.22) to all pairs of nodes in the qubit network. Formally speaking, we assume the interaction graph to be the complete directed graph. That is the evolution is described by a QMDS whereof generator takes the form

$$\mathcal{L}(\rho) = -i[H,\rho] + \sum_{i,j\in\{1,\dots,n\}} L^{(ij)}\rho(L^{(ij)})^\dagger - \frac{1}{2}\Big\{(L^{(ij)})^\dagger L^{(ij)},\rho\Big\}, \tag{4.62}$$

where H is the free Hamiltonian (4.1) and $L^{(ij)}$ is given by (4.6) and (4.7) respectively with L given by (4.22). Later we will refine the assumption to a general weakly connected[2] interaction graph.

In the above the qubits 1 and 2 together with the Lindblad operator $L^{(12)}$ can easily be replaced by an arbitrary pair $p, q \in \{1,\dots,n\}$, implying immediatelly that complete synchronization is enforced for any pair of adjacent qubits, in this case for all qubits in the network. This can also be demonstrated directly. Let $X \in X_{E_1-E_0}$. The attractor is determined by the commutation relations (4.10), i.e. for each pair of qubits p and q it holds

$$\Big[L^{(pq)},X\Big] = \Big[(L^{(pq)})^\dagger,X\Big] = 0. \tag{4.63}$$

The commutation relations (4.63) thus recreate the equations (4.32) to (4.45) and further yield their analogues for every qubit pair. For the reduced single-qubit dynamics we are primarily concerned with a subset of these equations, namely with the constraints

$$-X_{\vec{i}\,0\,\vec{j}\,1\,\vec{k}}^{\vec{i}\,1\,\vec{j}\,1\,\vec{k}} + X_{\vec{i}\,1\,\vec{j}\,0\,\vec{k}}^{\vec{i}\,1\,\vec{j}\,1\,\vec{k}} = 0, \tag{4.64}$$
$$X_{\vec{i}\,0\,\vec{j}\,0\,\vec{k}}^{\vec{i}\,0\,\vec{j}\,1\,\vec{k}} = 0, \tag{4.65}$$
$$X_{\vec{i}\,0\,\vec{j}\,0\,\vec{k}}^{\vec{i}\,1\,\vec{j}\,0\,\vec{k}} = 0, \tag{4.66}$$

$\forall \vec{i} \in \{0,1\}^{p-1}, \vec{j} \in \{0,1\}^{q-p-1}, \vec{k} \in \{0,1\}^{n-q}$, all for every pair of distinct indices $p, q \in \{1,\dots,n\}$. These are the equations (4.48) and (4.49) for all individual qubit pairss. We can rewrite them using simplified notation

$$-X_{\dots0\dots1\dots}^{\dots1\dots1\dots} + X_{\dots1\dots0\dots}^{\dots1\dots1\dots} = 0, \tag{4.67}$$
$$X_{\dots0\dots0\dots}^{\dots0\dots1\dots} = 0, \tag{4.68}$$
$$X_{\dots0\dots0\dots}^{\dots1\dots0\dots} = 0, \tag{4.69}$$

bearing in mind that the dots stand for respective appropriate multiindices. The synchronization condition (2.2) for an arbitrary pair of qubits p and q reads

[2] A graph is weakly connected if replacing all its edges with undirected edges results in a connected graph. A weakly connected interaction graph means that every qubit in the network interacts with at least one other qubit, one way or another.

$$\mathrm{Tr}_{1\ldots(q-1)(q+1)\ldots n} X = \mathrm{Tr}_{1\ldots(p-1)(p+1)\ldots n} X, \tag{4.70}$$

or explicitly

$$\sum_{\substack{\vec{i}\in\{0,1\}^{p-1},\\ \vec{j}\in\{0,1\}^{q-p-1},\\ \vec{k}\in\{0,1\}^{n-q}}} X^{\vec{i}\,0\,\vec{j}\,1\,\vec{k}}_{\vec{i}\,0\,\vec{j}\,0\,\vec{k}} + X^{\vec{i}\,1\,\vec{j}\,1\,\vec{k}}_{\vec{i}\,1\,\vec{j}\,0\,\vec{k}} = \sum_{\substack{\vec{i}\in\{0,1\}^{p-1},\\ \vec{j}\in\{0,1\}^{q-p-1},\\ \vec{k}\in\{0,1\}^{n-q}}} X^{\vec{i}\,1\,\vec{j}\,0\,\vec{k}}_{\vec{i}\,0\,\vec{j}\,0\,\vec{k}} + X^{\vec{i}\,1\,\vec{j}\,1\,\vec{k}}_{\vec{i}\,0\,\vec{j}\,1\,\vec{k}}, \tag{4.71}$$

and it is always satisfied since it is satisfied for every element of the sum due to (4.64), (4.65) and (4.66). This can be best seen rewriting (4.71) as

$$\sum \underbrace{X^{\ldots0\ldots1\ldots}_{\ldots0\ldots0\ldots}}_{=0} - \underbrace{X^{\ldots1\ldots0\ldots}_{\ldots0\ldots0\ldots}}_{=0} + \underbrace{X^{\ldots1\ldots1\ldots}_{\ldots1\ldots0\ldots} - X^{\ldots1\ldots1\ldots}_{\ldots0\ldots1\ldots}}_{=0} = 0. \tag{4.72}$$

Similarly, for $X_{st} \in X_0$ the commutation relations (4.10) yield, among other things, equations (4.53) to (4.56) and their other-qubit-pair analogues, which give after manipulation

$$X_{st}{}^{\vec{i}\,0\,\vec{j}\,1\,\vec{k}}_{\vec{i}\,0\,\vec{j}\,1\,\vec{k}} = X_{st}{}^{\vec{i}\,1\,\vec{j}\,0\,\vec{k}}_{\vec{i}\,1\,\vec{j}\,0\,\vec{k}}, \tag{4.73}$$

$$X_{st}{}^{\vec{i}\,1\,\vec{j}\,0\,\vec{k}}_{\vec{i}\,0\,\vec{j}\,1\,\vec{k}} = X_{st}{}^{\vec{i}\,0\,\vec{j}\,1\,\vec{k}}_{\vec{i}\,1\,\vec{j}\,0\,\vec{k}}, \tag{4.74}$$

$$X_{st}{}^{\vec{i}\,0\,\vec{j}\,0\,\vec{k}}_{\vec{i}\,0\,\vec{j}\,0\,\vec{k}} = X_{st}{}^{\vec{i}\,0\,\vec{j}\,1\,\vec{k}}_{\vec{i}\,0\,\vec{j}\,1\,\vec{k}} + X_{st}{}^{\vec{i}\,0\,\vec{j}\,1\,\vec{k}}_{\vec{i}\,1\,\vec{j}\,0\,\vec{k}}, \tag{4.75}$$

$\forall \vec{i} \in \{0,1\}^{p-1}, \vec{j} \in \{0,1\}^{q-p-1}, \vec{k} \in \{0,1\}^{n-q}$, all for every pair of distinct indices $p, q \in \{1,\ldots,n\}$. In the simplified notation then

$$X_{st}{}^{\ldots0\ldots1\ldots}_{\ldots0\ldots1\ldots} = X_{st}{}^{\ldots1\ldots0\ldots}_{\ldots1\ldots0\ldots}, \tag{4.76}$$

$$X_{st}{}^{\ldots1\ldots0\ldots}_{\ldots0\ldots1\ldots} = X_{st}{}^{\ldots0\ldots1\ldots}_{\ldots1\ldots0\ldots}, \tag{4.77}$$

$$X_{st}{}^{\ldots0\ldots0\ldots}_{\ldots0\ldots0\ldots} = X_{st}{}^{\ldots0\ldots1\ldots}_{\ldots0\ldots1\ldots} + X_{st}{}^{\ldots0\ldots1\ldots}_{\ldots1\ldots0\ldots}. \tag{4.78}$$

Consequently, the condition of complete synchronization (2.4), reduced here to

$$\mathrm{Tr}_{1\ldots(p-1)(p+1)\ldots n} X_{st} = \mathrm{Tr}_{1\ldots(q-1)(q+1)\ldots n} X_{st}, \tag{4.79}$$

is satisfied for an arbitrary pair of qubits p and q since

$$\sum \underbrace{X_{st}{}^{\ldots0\ldots1\ldots}_{\ldots0\ldots1\ldots} - X_{st}{}^{\ldots1\ldots0\ldots}_{\ldots1\ldots0\ldots}}_{=0} = 0, \tag{4.80}$$

due to (4.73).

To sum up, a QMDS with generator $\mathcal{L}$ given by (4.62), i.e. a QMDS generated by a two-qubit Lindblad operator L (4.22) and a complete interaction graph, corresponding to pairwise application of the operator L to all nodes, enforces complete synchronization of all qubits within the system.

What we have not shown so far is that the evolution given by such a QMDS does not kill the reduced single-qubit dynamics entirely. After all, we ignored a significant portion of all the constraints on the attractor space stemming from the commutation relations (4.10). In particular, we did not consider the analogues of the equations (4.36) to (4.39), respectively (4.50) to (4.52) after manipulation, which describe the effects of Lindblad operators $L^{(ij)}$ on the reduced dynamics of the other-than-applied-to qubits. Those read

$$X^{...0...1...}_{...0...1...} = X^{...1...0...}_{...1...0...}, \tag{4.81}$$

$$X^{...1...0...}_{...0...1...} = X^{...0...1...}_{...1...0...}, \tag{4.82}$$

$$X^{...0...0...}_{...0...0...} = X^{...0...1...}_{...0...1...} + X^{...0...1...}_{...1...0...}, \tag{4.83}$$

for every pair of positions of the dislayed indices and all possible multiindices in places of dots such that the coefficients $X^{\vec{i}}_{\vec{j}}$ satisfy $|\vec{i}| = |\vec{j}| + 1$. We will not solve these equations. Instead, to prove that the QMDS preserves the reduced single-qubit dynamics, except for a very specific choice of initial conditions, we construct an attractor from $X_{E_1-E_0}$ whose contribution to the partial trace onto each single-qubit subsystem is, in general, nonzero. Consider $X \in X_{E_1-E_0}$ of the form

$$X = |11\dots1\rangle\langle01\dots1| + \dots + |1\dots11\rangle\langle1\dots10|\,, \tag{4.84}$$

a sum of all basis elements with just a single zero appearing in them. As an attractor, X is constrained exclusively by the equations (4.64), which it satisfies, and hence truly $X \in \mathrm{Att}(\mathcal{T})$. The fact that such an attractor exists guarantees that the asymptotic evolution of individual qubits given by our QMDS is non-trival for almost all initial conditions.

Let us finish this part with pointing out that the assumption of the interaction graph being a complete directed graph is unnecessarily strong for the complete synchronization of the entire network to be achieved. Instead, it is enough to assume any weakly connected graph. Indeed, what is needed is that no two qubits are dynamically separated, i.e. for any pair of qubits there exists a chain of two-qubit interactions connecting them which effectively results in a non-direct interaction between them. Consequently, the enforced constraints (4.64) to (4.66) and (4.73) cover and link every single pair of qubits in the network. Since they imply simple equalities, the transitivity does the rest of the work. This requirement stated formally is that replacing all edges with undirected ones there exists a path between each pair of vertices, which means precisely that the interaction graph is weakly connected. The non-triviality of the asymtotic single-qubit dynamics certainly remains ensured as the step from the complete to a weakly connected graph results at the most in some of the constraints on attractors being removed.

4.2.2 Two qubit synchronization mechanisms in qubit networks

In the previous subsection it was demonstrated how a simple synchronizing two-qubit Lindblad operator can be applied to a qubit network, and that it is capable of enforcing synchronization of any pair of adjacent qubits, in particular of all qubits in the entire network in the case of a weakly connected interaction graph. Moreover, the particular investigated mechanism does so without destroying the reduced single-qubit dynamics. In the following we generalize these results. On the one hand, we show that all two-qubit synchronization-enforcing Lindblad operators from chapter 3 enforce the corresponding type of synchronization also on qubit networks, on the other hand we demonstrate that some of them simultaneously render the single-qubit dynamics trivial.

Let $L \in \mathcal{B}(\mathscr{H}_0^{\otimes 2})$ be a two-qubit synchronizing Lindblad operator, $G \in \mathbb{R}^{n\times n}$ be the adjacency matrix of an interaction graph and $\mathcal{T}$ be the QMDS with generator $\mathcal{L}$ given by (4.9). Let us denote L^{ab}_{cd}, where $a,b,c,d \in \{0,1\}$, the matrix elements of L given by the decomposition

$$L = \sum_{a,b,c,d\in\{0,1\}} L^{ab}_{cd}\,|ab\rangle\langle cd|\,, \tag{4.85}$$

similarly to the ceofficients $X^{\vec{i}}_{\vec{j}}$. Then, for an attractor $X \in \mathrm{Att}(\mathcal{T})$ the commutation relations (4.10) yield for every pair of indices $p,q \in \{1,\ldots,n\}, p<q$, such that $G_{pq} \neq 0$ the following set of equations

$$\sum_{r,s\in\{0,1\}} L^{ab}_{rs}\,X^{\vec{i}\,r\,\vec{j}\,s\,\vec{k}}_{\vec{l}\,c\,\vec{m}\,d\,\vec{n}} - L^{rs}_{cd}\,X^{\vec{i}\,a\,\vec{j}\,b\,\vec{k}}_{\vec{l}\,r\,\vec{m}\,s\,\vec{n}} = 0, \tag{4.86}$$

$\forall \vec{i},\vec{l} \in \{0,1\}^{p-1}, \forall \vec{j},\vec{m} \in \{0,1\}^{q-p-1}, \forall \vec{k},\vec{n} \in \{0,1\}^{n-q}$, and $\forall a,b,c,d \in \{0,1\}$. Analogously for $p>q$. Using again the simplified notation for improved intelligibility, (4.86) reads

$$\sum_{r,s\in\{0,1\}} L^{ab}_{rs}\,X^{\ldots r \ldots s \ldots}_{\ldots c \ldots d \ldots} - L^{rs}_{cd}\,X^{\ldots a \ldots b \ldots}_{\ldots r \ldots s \ldots} = 0, \tag{4.87}$$

$\forall a,b,c,d \in \{0,1\}$. Here the indices a and b, r and s, and c and d are on the p^{th} and q^{th} position respectively.
From these constraints we are particularly interested in those where $\vec{i}=\vec{l}, \vec{j}=\vec{m}, \vec{k}=\vec{n}$ so that we can analyze the parts of the attractor X which do not vanish when tracing out the n-2 or n-1 subsystems and thus are relevant for the reduced attractors $\mathrm{Tr}_{1\ldots(p-1)(p+1)\ldots(q-1)(q+1)\ldots n}\,X$, $\mathrm{Tr}_{1\ldots(p-1)(p+1)\ldots n}\,X$ and $\mathrm{Tr}_{1\ldots(q-1)(q+1)\ldots n}\,X$. Furthermore, we are primarily concerned with the restrictions onto some of the subspaces $X_{k(E_1-E_0)}$, namely $X_{E_1-E_0}$ to discuss synchronization, X_0 to further explore complete synchronization and optionally $X_{2E_1-2E_0}$ to get the full reduced attractor space of adjacent qubit pairs.

The main reason for the general form (4.86) and discussion of the commutation relations (4.10) is to demonostrate that their most relevant subsets, described in the previous paragraph, were already thoroughly analyzed and solved in chapter 3.
Indeed, since the Lindblad operator $L^{(pq)}$ acts locally on qubits p and q, $n-2$ elements of each multiindex appearing in the terms $X^{\vec{i}}_{\vec{j}}|\vec{i}\rangle\langle\vec{j}|$ of the attractor decomposition (4.12) are unaffected by its action. The commutation relations thus yield up to $2^{2(n-2)}$ sets of equations of the form (4.86) identical in shape except for the change in multiindices $\vec{i}$ to $\vec{n}$. Not only can all those sets of equations be solved simultaneously, by crossing out the multiindices $\vec{i}$ to $\vec{n}$ we obtain the previously investigated commutation relations for a system of two qubits, as leaving the multiindices out is equivalent to assuming $n=2$. Restricting ourselves to a subset of the attractor space such that the partial trace $\mathrm{Tr}_{1\ldots(p-1)(p+1)\ldots(q-1)(q+1)\ldots n}\,X$ is non-trivial for every element X from this subset,

the number of the sets of equations stemming from the commutation relations for each operator $L^{(pq)}$ further reduces to $2^{(n-2)}$. They also simplify slightly to

$$\sum_{r,s\in\{0,1\}} L^{ab}_{rs} X^{\vec{i}\,r\,\vec{j}\,s\,\vec{k}}_{\vec{i}\,c\,\vec{j}\,d\,\vec{k}} - L^{rs}_{cd} X^{\vec{i}\,a\,\vec{j}\,b\,\vec{k}}_{\vec{i}\,r\,\vec{j}\,s\,\vec{k}} = 0, \tag{4.88}$$

$\forall\vec{i} \in \{0,1\}^{p-1}, \forall\vec{j} \in \{0,1\}^{q-p-1}, \forall\vec{k}, \in \{0,1\}^{n-q}$, and $\forall a, b, c, d \in \{0,1\}$.

More importantly, an attractor $X \in X_{k(E_1-E_0)}$ is associated with the same eigenvalue $k(E_1-E_0)$ as its two-qubit counterpart or reduced attractor $\mathrm{Tr}_{1\ldots(p-1)(p+1)\ldots(q-1)(q+1)\ldots n} X$, and this restriction ensures that no additional terms appear in the equations (4.88) compared to the two-qubit case when assuming X from a particular subset $X_{k(E_1-E_0)}$. This is because the associated eigenvalue $k(E_1-E_0)$ is fully determined by the two indices on the p^{th} and the two indices on q^{th} positions in the attractor deceompositon (4.12).
Consequently, we already have partial solutions for $X \in X_0$, $X \in X_{E_1-E_0}$ and $X \in X_{2E_1-2E_0}$ of the just described subset (4.88) of constraints (4.86) for every two-qubit synchronizing normal Lindblad operator L. We write partial for we have complete solutions for every fixed pair of indices p and q, i.e. for every Lindblad operator $L^{(pq)}$, separately, but not the intersection of these solutions which together with other constraints determine the attractor space.

It is worth emphasizing that the restriction to (4.88) does not cover all constraints imposed on an attractor $X \in X_0$, $X \in X_{E_1-E_0}$ or $X \in X_{2E_1-2E_0}$ by the commutation relations (4.86). It is a valuable tool to straightforwardly demontrate that all QMDS constructed using any two-qubit synchronizing Lindblad operator L and an arbitrary weakly connected interaction graph enforce synchronization of the entire network, yet it does not address the fact that some of those QMDS actually kill the reduced single-qubit dynamics completely. Therefore, we will go through several classes of synchronizing normal Lindblad operators L from chapter 3 and individually discuss their properties with respect to qubit networks. Let $X \in X_{E_1-E_0}$ and $X_{st} \in X_0$, both in the parameterization (4.12), and assume the interaction graph to be weakly connected.

First, consider the case $L = L_1$ given by (3.54). The commutation relations (4.88) follow the pattern of equations (3.44) to (3.49) for X and the pattern of (3.124) to (3.129) for X_{st}, implying particularly

$$X^{\ldots 0 \ldots 1 \ldots}_{\ldots 0 \ldots 0 \ldots} = X^{\ldots 1 \ldots 0 \ldots}_{\ldots 0 \ldots 0 \ldots}, \tag{4.89}$$

$$X^{\ldots 1 \ldots 1 \ldots}_{\ldots 0 \ldots 1 \ldots} = X^{\ldots 1 \ldots 1 \ldots}_{\ldots 1 \ldots 0 \ldots}, \tag{4.90}$$

$$X_{st}{}^{\ldots 0 \ldots 1 \ldots}_{\ldots 0 \ldots 1 \ldots} = X_{st}{}^{\ldots 1 \ldots 0 \ldots}_{\ldots 1 \ldots 0 \ldots}, \tag{4.91}$$

for all adjacent pairs of qubits. Due to the weak connectivity of the interaction graph and transitivity of equality, (4.89), (4.90) and (4.91) hold for every pair of qubits in the network, i.e. for every pair of displayed indices. The condition of complete synchronization (2.4), here

$$\mathrm{Tr}_{2\ldots n} X = \cdots = \mathrm{Tr}_{1\ldots(n-1)} X, \tag{4.92}$$

$$\operatorname{Tr}_{2\ldots n} X_{st} = \cdots = \operatorname{Tr}_{1\ldots(n-1)} X_{st}, \tag{4.93}$$

written equivalently for an arbitrary pair of qubits reads after rearanging

$$\sum \underbrace{X^{\ldots 0 \ldots 1 \ldots}_{\ldots 0 \ldots 0 \ldots} - X^{\ldots 1 \ldots 0 \ldots}_{\ldots 0 \ldots 0 \ldots}}_{=0} + \underbrace{X^{\ldots 1 \ldots 1 \ldots}_{\ldots 1 \ldots 0 \ldots} - X^{\ldots 1 \ldots 1 \ldots}_{\ldots 0 \ldots 1 \ldots}}_{=0} = 0, \tag{4.94}$$

$$\sum \underbrace{X_{st}{}^{\ldots 0 \ldots 1 \ldots}_{\ldots 0 \ldots 1 \ldots} - X_{st}{}^{\ldots 1 \ldots 0 \ldots}_{\ldots 1 \ldots 0 \ldots}}_{=0} = 0, \tag{4.95}$$

and is satisfied due to (4.89), (4.90) and (4.91). The corresponding QMDS enforces complete synchronization. To show that it does not concurrently kill the reduced single-qubit dynamics except for a very specific choice of initial conditions, we present an example of an attractor $X \in X_{E_1-E_0} \cap \operatorname{Att}(\mathcal{T})$ subject exclusively to constraints (4.89) which it satisfies, namely

$$X = |10\ldots0\rangle\langle 00\ldots0| + \cdots + |0\ldots01\rangle\langle 0\ldots00|\,, \tag{4.96}$$

that has a non-trivial partial trace onto each single-qubit subsystem. Similarly to the illustrating example discussed in subsection 4.2.1.
Such an attractor can, in fact, be found systematically. The constraints (4.86) for $X \in X_{E_1-E_0}$ can be sorted by the value of $|\vec{i}| + |\vec{j}| + |\vec{k}| - |\vec{l}| - |\vec{m}| - |\vec{n}|$, as was done for (4.32) to (4.45), to demonstrate which of the equations (4.86) not contained in (4.88) can affect the part of attractor X with non-trivial partial trace onto at least one of the single-qubit subsystems. These are the equations with $|\vec{i}|+|\vec{j}|+|\vec{k}|-|\vec{l}|-|\vec{m}|-|\vec{n}| = 1$, which necessarily follow the pattern of (4.88) for $X_{st} \in X_0$, with one of the upper multiindices replaced by itself $+1$. That is because the elements $X^{i_1\ldots i_n}_{j_1\ldots j_n} |i_1 \ldots i_n\rangle\langle j_1 \ldots j_n|$ constituting such a part of the attractor X need to have all but a single pair of indices i_a, j_a pairwise matching, including especially the indices on the positions affected by the action of the Lindblad operator.

Unsurprisingly, the same can be shown also in the case $L = L_2$ given by (3.60). A QMDS given by (4.9) where $L = L_2$ enforces complete synchronization of all qubits in the network for every weakly connected interaction graph and almost all initial conditions. Since the reasoning behind is practically identical to the case $L = L_1$, we skip the proof.

Next, we discuss the remaining cases $L = L_{\alpha\pm}$. We present in detail only the instance $L = L_{\frac{1}{\sqrt{2}}\pm}$, both derivation and results are analogous for a general $\alpha \in (0, 1)$. In the same way as for L_1 and L_2, the commutation relations (4.88) for $X \in X_{E_1-E_0}$ immediately imply pairwise synchronization of reduced single-qubit dynamics, hence also the synchronization of all qubits in the network for every weakly connected interaction graph. The key question is whether the resulting single-qubit evolution will be non-trivial.
The operator $L_{\frac{1}{\sqrt{2}}\pm}$ differs from L_1, L_2 in one key aspect. The attractor space of its associated QMDS is not permutation invariant, neither by itself nor in the intersection with $X_{E_1-E_0}$, see 5.2 for details. This fact limits the possible interaction graphs which may result in non-trivial asymptotic evolution. For any two adjacent qubits p and q connected by directed edges in both directions the attractor X needs to satisfy the commutation relations for both $L^{(pq)}$ and $L^{(qp)}$ and, consequently, the reduced attractor $\operatorname{Tr}_{1\ldots(p-1)(p+1)\ldots(q-1)(q+1)\ldots n} X$

necessarily lies in some permutation invariant subspace of the attractor space of the corresponding two-qubit QMDS (3.174). Since such a subspace is trivial in the intersection with $X_{E_1-E_0}$ due to the form of (3.174), the reduced attractor is always stationary. Because of the enforced synchronization with other qubits, the asymptotic dynamics of all reduced single-qubit states throughout the entire network are trivial.
Interestingly, the same can be shown for any weakly connected interaction graph containing at least one vertex of a degree bigger than or equal to two, i.e. for any network where there exists a qubit interacting with at least two other qubits (or a pair of qubits interacting in a way just described). And that is bassicaly any non-trivial network of more than two qubits. Indeed, assume a weakly connected interaction graph such that its adjacency matrix G satisfies $G_{pq} = G_{qr} = 1$, i.e. the qubit p is adjacent to q and q is adjacent to r, without loss of generality $p < q < r$, and the synchronizing operator L is applied in this order. In other words, the Lindbladian (4.9) contains $L^{(pq)}$ and $L^{(qr)}$. The commutation relations (4.86) for $X \in X_{E_1-E_0}$ yield several sets of equations out of which we are particularly interested in

$$X^{...0...1...}_{...0...0...} = X^{...1...1...}_{...0...1...}, \tag{4.97}$$

$$X^{...1...0...}_{...0...0...} = X^{...1...1...}_{...1...0...}, \tag{4.98}$$

$$X^{...0...1...}_{...0...1...} = X^{...1...0...}_{...1...0...}, \tag{4.99}$$

$$X^{...1...0...}_{...0...0...} = \mp i X^{...0...1...}_{...0...0...}, \tag{4.100}$$

$$X^{...0...0...}_{...0...0...} = X^{...0...1...}_{...0...1...} \pm i X^{...0...1...}_{...1...0...}, \tag{4.101}$$

$$X^{...1...1...}_{...1...1...} = X^{...0...1...}_{...0...1...} \mp i X^{...0...1...}_{...0...0...}, \tag{4.102}$$

where the dots stand for every possible combination of multiindices such that the coefficients $X^{\vec{i}}_{\vec{j}}$ are coefficients of the basis decomposition (4.12) of the attractor X, and the explicitly displayed indices are on the positions of adjacent qubit pairs, especially on the positions p,q and r. The asymptotic dynamics of the qubit q is determined by the reduced attractor $\mathrm{Tr}_{1...(q-1)(q+1)...n}\, X$ which, written with emphasis on the interaction with qubits p and r, reads

$$\mathrm{Tr}_{1...(q-1)(q+1)...n}\, X = \left(\sum X^{...0...1...0...}_{...0...0...0...} + X^{...0...1...1...}_{...0...0...1...} + X^{...1...1...0...}_{...1...0...0...} + X^{...1...1...1...}_{...1...0...1...}\right) |1\rangle\langle 0| \,. \tag{4.103}$$

For the coefficients it follows from (4.97),(4.98) and (4.99) that

$$X^{...0...1...0...}_{...0...0...0...} = X^{...1...1...1...}_{...1...0...1...}, \tag{4.104}$$

from (4.97), (4.98), (4.101) and (4.102) that

$$X^{...0...1...1...}_{...0...0...1...} + X^{...1...1...0...}_{...1...0...0...} = 2X^{...1...1...1...}_{...1...0...1...}, \tag{4.105}$$

and from (4.97), (4.98) and (4.100) that

$$X^{...0...1...1...}_{...0...0...1...} + X^{...1...1...0...}_{...1...0...0...} = 0. \tag{4.106}$$

Put together, (4.104),(4.105),(4.106) imply

$$\sum X^{\ldots 0\ldots 1\ldots 0\ldots}_{\ldots 0\ldots 0\ldots 0\ldots} + X^{\ldots 0\ldots 1\ldots 1\ldots}_{\ldots 0\ldots 0\ldots 1\ldots} + X^{\ldots 1\ldots 1\ldots 0\ldots}_{\ldots 1\ldots 0\ldots 0\ldots} + X^{\ldots 1\ldots 1\ldots 1\ldots}_{\ldots 1\ldots 0\ldots 1\ldots} = 0, \tag{4.107}$$

which in turn gives

$$\mathrm{Tr}_{1\ldots(q-1)(q+1)\ldots n}\, X = 0. \tag{4.108}$$

The reduced dynamics of the qubit q necessarily vanishes in the asymptotics. Consequently, so does the reduced dynamics of every qubit in the network due to mutual synchronization. The same result is achieved when assuming different orientations of the edges between p and q, and q and r, realizable by substituting $L^{(pq)}$, $L^{(rq)}$ or $L^{(qp)}$, $L^{(qr)}$ for $L^{(pq)}$, $L^{(qr)}$, that is for both two incoming and two outcoming edges instead of one incoming to and one outcoming from the vertex q. Changing orientation merely flips the sign of $\pm$ and $\mp$ in (4.100), (4.101), (4.102) for the particular interaction, the derivation of (4.108) in these cases is analogous.

To sum up, the Lindblad operators $L_{\frac{1}{\sqrt{2}}\pm}$ (3.145), and similarly $L_{\alpha\pm}$ (3.112), are unsuitable for the synchronization of qubit networks since they kill the asymptotic reduced dynamics of all single qubits for any non-trivial network of more than two parties. Conversely, the operators L_1 (3.54) and L_2 (3.60), including the operators L_s (3.161), enforce non-trivial complete synchronization of all qubits in any network with an interaction graph that is at least weakly connected.

Chapter 5

Properties of two-qubit synchronizing and phase-locking maps

In chapter 3 we found and described the generators of all possible QMDS with normal Lindblad operators that enforce synchronization or phase-locking of two qubits. We further found their respective attractor spaces and provided their full parameterization. In chapter 4 we showed how the two-qubit generalized synchronization mechanisms can be applied to qubit networks. This chapter is devoted to the study of their various other properties. We once again restric ourselves to the case of two qubits, as our aim si to better understand the very nature of generalized synchronization mechanisms on the basic level.

5.1 Visibility of asymptotic reduced states

Synchronization and phase-locking are mediated by the interaction of subsystems with their common environment. Such irreversible processes are typically accompanied by decoherence and dephasing, i. e. by information leak into the environment. In open quantum systems, synchronization processes are inevitably accompanied with gradual loss of information about the initial state and possible destruction of internal dynamics of the subsystems. While in the transient synchronization such processes lead to complete relaxation of the whole system, in the asymptotic synchronization and phase-locking the internal dynamics at least partly survive. Thus, in this part we address the following question. Once an initial state of two qubits is synchronized or phase-locked by one of the mechanisms described in chapter 3, how visible and detectable will the resulting non-trivial time evolution of the individual qubit states be? To what extend does the internal dynamics of qubits survive the process of generalized synchronization?

For a global two-qubit state $\rho(t) \equiv (\rho_{ij})(t) \in \mathcal{B}(\mathscr{H}_1^{\otimes 2})$ the reduced states $\rho_A(t), \rho_B(t) \in \mathcal{B}(\mathscr{H}_1)$ read

$$\rho_A = \begin{pmatrix} \rho_{11}(t) + \rho_{22}(t) & \rho_{13}(t) + \rho_{24}(t) \\ \rho_{31}(t) + \rho_{42}(t) & \rho_{33}(t) + \rho_{44}(t) \end{pmatrix}, \tag{5.1}$$

$$\rho_B = \begin{pmatrix} \rho_{11}(t) + \rho_{33}(t) & \rho_{12}(t) + \rho_{34}(t) \\ \rho_{21}(t) + \rho_{43}(t) & \rho_{22}(t) + \rho_{44}(t) \end{pmatrix}. \tag{5.2}$$

A general density matrix $\rho(t) \in \mathcal{B}(\mathscr{H}_1)$ describing a state of a qubit has the form

$$\rho(t) = \begin{pmatrix} x & ye^{iEt} \\ \bar{y}e^{-iEt} & 1-x \end{pmatrix}, \tag{5.3}$$

where $y \in \mathbb{C}$, $x, E \in \mathbb{R}$, and from the positivity of $\rho(t)$ it holds $|y| \leq \sqrt{x - x^2}$. In our case $E = E_0 - E_1$ as given by the Hamiltonian (3.2). In the asymptotics, the evolution is driven by (1.16) and hence the coefficents x, y are determined by the elements of dual basis of the attractor space acting on the initial state.

Remark: The asymptotic state is strongly dependent on initial conditions. Seemingly, for a given synchronizing map we could choose such an initial state that when projected onto the attractor space, the part responsible for non-trivial time evolution in the asymptotics vanishes. In such a case the synchronization mechanism kills the internal dynamics in spite of the presence of an attractor associated with a non-zero eigenvalue in the attractor space. This is inevitable since quantum evolution always has a fixed point. However, this would require a very specific choice of the initial state as it would have to lie in the orthogonal complement of the mentioned attractor in the space of all operators on the system Hilbert space. This orthogonal complement is a set of codimension at least one (there might exist more independent such attractors) and will consequently constitute a set of measure zero in the space of all states. We can conclude that a synchronizing, respectively phase-locking map enforces generaliezd synchronization with non-trivial asymptotic evolution for almost every initial condition.

The question remains how perceptible this asymptotic evolution will be. To extract the information about the time evolution we can calculate the expectation value $\langle \sigma_1 \rangle (t) = \mathrm{Tr}\left\{\rho^\dagger(t)\sigma\right\}$ of the observable

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.4}$$

For a general qubit state (5.3) this results, after manipulation, into

$$\langle \sigma \rangle (t) = |y| \cos(Et + \phi), \tag{5.5}$$

where $\phi \in \mathbb{R}$ accounts for the phase of y. Equivalently, we could express the probabilities $p_1 = \mathrm{Tr}\{\rho(t)M_1\}$, $p_2 = \mathrm{Tr}\{\rho(t)M_2\}$ of the corresponding projective measurements $M_1 = \frac{0}{1}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|)$ and $M_2 = \frac{0}{1}(|0\rangle - |1\rangle)(\langle 0| - \langle 1|)$. It holds

$$p_1(t) = \frac{1}{2}\left\{1 + 2|y| \cos(Et + \phi)\right\}, \tag{5.6}$$

$$p_2(t) = \frac{1}{2}\left\{1 - 2|y| \cos(Et + \phi)\right\}, \tag{5.7}$$

$$(p_1 - p_2)(t) = 2|y| \cos(Et + \phi). \tag{5.8}$$

Therefore, the visibility of the time evolution in the asymptotics scales with $|y|$, the absolute value of the off-diagonal parameter y. The greater the $|y|$, the bigger the amplitude in (5.5) and the easier it is to observe the non-trivial evolution of a qubit in the state (5.3).

We will now compare the attractor spaces of synchronizing and phase-locking maps described in section 3.3, chapter 3, as they limit the possible asymptotic states, to see if the choice of the generalized synchronization mechanisms in general affects how well the internal dynamics is preserved. The aim is to study visibility independently of initial conditions, despite the fact that they play a crucial role in determining the asymptotic state and hence a very few conclusions may be drawn without specifying them. It will be shown that in some cases the evolution map may actually limit or suppress the visibility of the time evolution of the resulting synchronised asymptotic states, irrespective of the initial conditions.

Any asymptotic state of a QMDS lies in its attractor space. Therefore, assume an element X of the attractor space of a QMDS, which is also a state. Using parameterizations (3.15) and (3.121) for its dynamical and stationary parts, the reduced one-qubit states X_A, X_B read

$$X_A = \begin{pmatrix} A+B & \beta+\gamma \\ \bar{\beta}+\bar{\gamma} & E+F \end{pmatrix}, \tag{5.9}$$

$$X_B = \begin{pmatrix} A+E & \alpha+\delta \\ \bar{\alpha}+\bar{\delta} & B+F \end{pmatrix}. \tag{5.10}$$

In the parameterization (5.3) of a general qubit state, the time t set to zero, these two reduced states correspond to

$$x = A+B, \tag{5.11}$$

$$y = \beta+\gamma, \tag{5.12}$$

for qubit A and

$$x = A+E, \tag{5.13}$$

$$y = \alpha+\delta, \tag{5.14}$$

for qubit B. Let the assumed QMDS be one of the synchronizing, respectively phase-locking maps discovered in chapter 3. The two qubits being synchronized in the generalized sense, (3.21) holds and thus the deciding factor $|y|$ is the same for both of them. It is also independent of the phase shift φ. Depending on the synchronization mechanism, additional constraints on $A, B, E, \alpha, \beta, \gamma, \delta$ may apply.
Let us first address the stationary part. Regardless of the particular synchronizing map, the corresponding attractor space yields effectively no limitations on the possible values of parameter x for either of the qubits. It varies with the initial conditions over its entire range for each generalized synchronization mechanism. Thus, it is only the part of the attractor space associated with the non-trivial time evolution that can pose initial-condition-independent restrictions on the visibility of asymptotic sates. We are concerned solely with the value of $|y|$, which in turn is given by (5.12). For all completely synchronizing maps, described in section 3.2, chapter 3, the parameters β and γ determining y are either independent, as it is the case for Lindblad operators L_1 and L_2 given by (3.54), (3.60), or linearly dependent for the operators $L_{\frac{1}{\sqrt{2}}\pm}$ given by (3.145). If so, $\gamma = \mp i\beta$.

In either case, the visibility parameter $|y|$ is in general not limited by the particular synchronizing mechanism and can reach any value between 0 and $\frac{1}{2}$, depending on the initial conditions.
We conclude that all completely synchronizing maps exhibit the same visibility of the asymptotic evolution.

On the other hand, for the synchronizing maps generated by Lindblad operators $L_{\alpha\pm}$ given by (3.112), the form of a general attractor (3.173) results in the value of y depending not only on the initial conditions, but also on a parameter α of $L_{\alpha\pm}$ itself. In particular, the attractor X given by (3.94), which is a generator of the attractor space corresponding to the non-trivial asymptotic dynamics, reads after normalisation

$$X = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \alpha & \pm ie^{-i\varphi}\sqrt{1-\alpha^2} & 0 \\ 0 & 0 & 0 & \pm ie^{-i\varphi}s\sqrt{1-\alpha^2} \\ 0 & 0 & 0 & -s\alpha \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.15}$$

where s is given by (3.114). It is the projection of an initial state onto this attractor X that determines the value of the visibility parameter $|y|$. It holds

$$y \propto \alpha\sqrt{1-\alpha^2}. \tag{5.16}$$

Varying α accounts for the choice of a particular class of synchronizing Lindblad operators and does not change the norm of attractor X in (5.15), making it a suitable tool for comparing the synchronizing mechanisms with respect to the visibility of asymptotic states.

We can plot the factor above as a function of α.

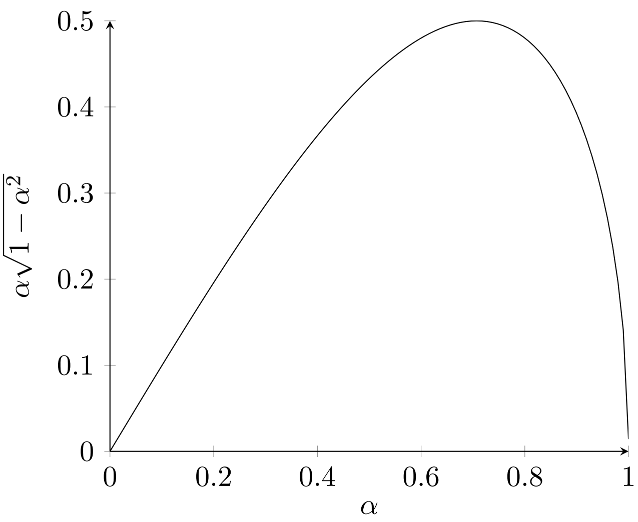

Figure 5.1: Plot of the proportional factor $\alpha\sqrt{1-\alpha^2}$ of the visibility parameter $|y|$ for synchronizing maps generated by Lindblad operators $L_{\alpha\pm}$ given by (3.112).

We can see that in the case of Lindblad operators $L_{\alpha\pm}$ with the value of α close to 0 or 1 the asymptotic dynamics is strongly suppressed. The asymptotic reduced states will resemble stationary ones, independently of the initial conditions. Interestingly, the plotted function reaches its maximum for $\alpha = \frac{1}{\sqrt{2}}$ when the class of Lindblad operators $L_{\alpha\pm}$ becomes completely synchronizing. From this point of view the generalized complete synchronization results in, perhaps counterintuitively, better visibility than the less restrictive synchronization or phase-locking.

Remark: Note that the enforced phase shift φ between the two qubits plays no role whatsoever in respect of visibility.

5.2 Global symmetry of synchronized states

By their very nature, the complete synchronization mechanisms make two qubits locally indistinguishable. This section addresses a question whether the two qubits also become indistinguishable from a global point of view. It turns out to depend on the particular synchronizing mechanism.

Indistinguishability of a bipartite quantum state means that at any time of its evolution it is described by a permutationally invariant state. By permutation invariance we mean that the global state is invariant with respect to the exchange of the two qubits. Consequently, no measurement can discern the order of the subsystems.
For a map to enforce asymptotic permutation invariance for arbitrary initial conditions, any state lying in its attractor space need be permutation invariant. Formally, an attractor $X \in \mathcal{B}(\mathscr{H}_1^{\otimes 2})$ is permutation invariant if it is invariant with respect to conjugation by the SWAP operator, denoted here as $\Pi = |00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 11|$, a special case of Π_{kl}, namely Π_{12}, given by (4.4), (4.5). It holds $\Pi = \Pi^\dagger = \Pi^{-1}$, and the condition can be written as

$$X = \Pi\, X\, \Pi^{-1}. \tag{5.17}$$

Denoting X_{ij} the matrix elements of X, this condition simplifies into

$$X_{12} = X_{21}, \tag{5.18}$$
$$X_{34} = X_{24}, \tag{5.19}$$
$$X_{22} = X_{33}, \tag{5.20}$$
$$X_{23} = X_{32}, \tag{5.21}$$
$$\tag{5.22}$$

or equivalently, in the parameterization (3.15) and (3.121) previously used, into

$$\alpha = \beta, \tag{5.23}$$
$$\gamma = \delta, \tag{5.24}$$
$$B = E, \tag{5.25}$$
$$C = D. \tag{5.26}$$

We do not explicitly state the condition of permutation invariance (5.17) for the projection of the attractor X onto the subspace $X_{E_1-E_0}$, related to the subspace $X_{E_0-E_1}$ by complex conjugation, as it is satisfied if and only if the conditions (5.18), (5.19), or equivalently (5.23), (5.24), are.
Having fully described the attractor spaces of all synchronizing QMDS with normal Lindblad operators in section 3.3, chapter 3, we can directly conjugate a general element X of the attractor space by Π for each of them to see whether (5.17) holds.

Firstly, let us state the obvious. Only purely synchronizing and no phase-locking maps with a phase shift $\varphi \neq 0$ qualify. Clearly, in the case of a phase-locked non-sycnhronized asymptotic state the exchange of qubits would affect the state and result in a violation of (5.17). Written nearly unduly formally, the equations (5.23), (5.24) together yield the synchronization condition (3.18) which contradicts the generalized synchronization condition (3.21), unless the phase shift φ is zero or the reduced states of the attractor X in question are stationary. Consequently, no phase-locking map but a synchronizing one is compatible with permutation invariance of the asymptotic state, as is to be expected.
Among the purely synchronizing maps, naturally, only the completely synchronizing ones need to be taken into account. Indeed, the condition (5.25) is the condition of complete synchronization (3.122).

And now for the actual relevant cases. Consider Lindblad operators L_1 and L_2 given by (3.54), (3.60), $\varphi = 0$ including the overlap of this two classes represented by the partial swap operator (3.161). An arbitrary element X of the attractor space of a corresponding QMDS has the form (3.158), (3.160) or (3.164) respectively, with $\varphi = 0$, which always satisfies (5.17). Consequently, the permutation invariance of the asymptotic state is enforced for all initial conditions.
On the other hand, consider QMDS with Lindblad operators $L_{\frac{1}{\sqrt{2}}\pm}$ given by (3.144), $\varphi = 0$. An element X of the attractor space of such QMDS has the form (3.174) which violates (5.17), specifically it violates the conditions (5.23), (5.24) and (5.26), unless all coefficients involved are zero. That is a case of X stationary with additional restrictions and it would require very specific initial conditions. In general, any non-stationary asymptotic state of a QMDS with Lindblad operators $L_{\frac{1}{\sqrt{2}}\pm}$ of the form (3.144) is guaranteed not to be permutation invariant.

We can sumarize these observations. A non-stationary asymptotic two-qubit state of a QMDS with normal Lindblad operators is permutation invariant if and only if the QMDS is generated by the completely synchronizing operators L_1 or L_2 of the form (3.54), (3.60), $\varphi = 0$, including the case of partial swap operators (3.163). The Lindblad operators $L_{\frac{1}{\sqrt{2}}\pm}$, on the other hand, simultaneously enforce complete synchronization and prevent permutation invariance. Reduced states of a permutation invariant state are completely synchronized, nonetheless, a global state of completely synchronized one-qubit reduced states does not need to be permutation invariant.

5.3 Symmetry of synchronizing mechanisms

In the previous section we have shown that not all completely synchronizing mechanisms, which make two qubits localy indistinguishable, also make them indistinguishable globally. In the following text we explore the symmetry with respect to the exchange of qubits for the synchronizing mechanisms themselves.

Permutation invariance of a two-qubit state $X \in \mathcal{B}(\mathscr{H}_1^{\otimes 2})$ is formally expressed as an invariance with respect to conjugation by the SWAP operator (5.17). A particular synchronizing mechanism represented by a Lindblad operator L acts on a state X via the generator (3.1) of a QMDS governing the evolution. Since the Hamiltonian is permutation invariant, the action of the Lindbladian on the state X with the two qubits exchanged can be written using the conjugation of the state X by the SWAP operator both before and after applying $\mathcal{L}$.

$$\begin{aligned}\Pi\mathcal{L}(\Pi X\Pi)\Pi &= -i[H,X] + \Pi(L(\Pi X\Pi)L^\dagger)\Pi - \frac{1}{2}\Pi\Big\{L^\dagger L, \Pi X\Pi\Big\}\Pi = \\ &= -i[H,X] + (\Pi L\Pi)X(\Pi L\Pi)^\dagger - \frac{1}{2}\Big\{(\Pi L\Pi)^\dagger(\Pi L\Pi), X\Big\}.\end{aligned} \tag{5.27}$$

Therefore, to have a Lindblad operator L act on the two qubits in the opposite order it needs to be replaced by $\Pi L\Pi$. This is a special case of operators $L^{(kl)}$, here $L^{(21)}$, introduced via (4.7) in chapter 4 on networks. A synchronizing mechanism given by a Lindblad operator L acts symmetrically on two qubits, namely it is invariant with respect to the exchange of qubits, if

$$L = \Pi L\Pi \tag{5.28}$$

holds. This is not entirely precise. Due to the form of generator (1.11) of a QMDS, two Lindblad operator that differ only by a phase prefactor constitute identical Lindbladians and generate the same evolution map. And we should account for this ambiguity. Hence, we hone the statement above as follows. A synchronizing mechanism given by a Lindblad operator L acts symmetrically on two qubits, namely it is invariant with respect to the exchange of qubits, if there exists $\psi \in \mathbb{R}$ such that

$$L = e^{i\psi}\Pi L\Pi. \tag{5.29}$$

Multiplying the matrices in defining expression (3.54),(3.60) and (3.112) of the synchronizing Lindblad operators L_1, L_2 and $L_{\alpha\pm}$, see appendix C, equations (C.1), (C.6) and (C.9) for the resulting single-matrix forms, and applying the conjugation by the SWAP operator, we can directly see which of the mechanisms act symmetrically and which do not.
Clearly, we can immediatelly exclude all phase-locking-enforcing operators L with a phase shift $\varphi \notin \{0, \pi\}$ and incompletely synchronizing permutation-noninvariant-stationary-part-

enforcing operators $L_{\alpha\pm}$ for $\alpha \neq \frac{1}{\sqrt{2}}$. Hence, only the completely synchronizing and anti-synchronizing operators L_1, L_2 and $L_{\frac{1}{\sqrt{2}}\pm}$ need to be discussed.
To begin with $L_{\frac{1}{\sqrt{2}}\pm}$, these operators are permutation invariant for neither $\varphi = 0$ nor $\varphi = \pi$. Interestingly, for both values of φ the conjugation by the SWAP operator changes the operator class from one to the other, i.e. it takes an operator $L_{\frac{1}{\sqrt{2}}+}$ to an operator $L_{\frac{1}{\sqrt{2}}-}$ and vice versa. Moving to L_1, L_2 and starting with the case of synchronization, $\varphi = 0$, it can be easily verified that both operators are permutation invariant if and only if $b = 0$ in the parameterization (3.54)/(C.1) and (3.60)/(C.6) respectively. This includes but is not limited to the overlapping class of operators L_s of the form (3.163)/(C.4), containing the SWAP operator. Nevertheless, in the case $b \neq 0$ a synchronizing Lindblad operator L_1 or L_2 subject to the symmetry transformation of conjugation by the SWAP operator remains to be a synchronizing Lindblad operator of the same class. The transformation merely changes b to $-b$. This gives us another set of symmetrically acting synchronization mechanisms, namely the operators L_1 and L_2 where $b \neq 0$ and $a = c = m = 0$. Such operators are not permutation invariant since $\Pi L_{1(2)} \Pi = -L_{1(2)}$, yet for the same reason the generated QMDS remain unchanged by this transformation. It is the case of (5.29) where $\psi = \pi$.
Finally, the antisynchronizing operators L_1, L_2, $\varphi = \pi$, are permutation invariant.

To sum up, only a small part of the synchronizing mechanisms uncovered in chapter 3 is invariant with respect to the exchange of qubits. A vast majority of them does not treat the qubits equally. There even exist Lindblad operators whereof corresponding QMDS result in a permutation invariant asymptotic state for all initial conditions in spite of the Lindblad operators not being permutation invariant themselves. However, for each such an operator a permutation invariant Lindblad operator can be found within the same class. That is the case of the synchronizing operators L_1, L_2. On the other hand, there are Lindblad operators, namely the antisynchronizing operators L_1, L_2, which are permutation invariant yet their corresponding QMDS enforce permutation non-invariant asymptotic states.

5.4 Mutual relation of synchronizing and phase-locking mechanisms

When defining different degrees of synchronization and phase-locking for the purpose of this work we introduced the definitions of generalized synchronization and generalized complete synchronization 2.1.3 to concurrently account for both synchronization and phase-locking with a non-zero phase shift. Using these terms throughtout chapter 3 we managed to simultaneously describe various two-qubit synchronization and phase-locking mechanisms in the form of normal Lindblad operators and their respective attractor spaces. Notably, we did so without ever specifying or otherwise limiting the possible phase shift between the concerned parties. Indeed, notice that in the parameterizations 3.54, 3.60, 3.112 of synchronizing, respectively phase-locking Lindblad operators $L_1, L_2, L_{\alpha\pm}$ the phase shift φ appears as a free parameter which affects nothing but the achieved phase difference between the two qubits. The relevant attractor is also the only part of the attractor space of an associated QMDS that is subject to change when varying the parameter φ.

From the particular expressions 3.54, 3.60, 3.112 for $L_1, L_2, L_{\alpha\pm}$, covering all synchronizing and phase-locking normal Lindblad operators on two qubits, the following is evident. The set of all phase-locking operators with an arbitrary but fixed phase shift φ_1 and the set of all phase-locking operators with an arbitrary but fixed phase shift φ_2 can be smoothly deformed one into the other, and a simple mapping from the former to the latter changing $\varphi_1 \mapsto \varphi_2$ is a homeomorphism in the subspace topology, induced by the standard topology on $\mathbb{R}^{4\times 4}$. This holds also for $\varphi_1 = 0$, i.e. for synchronizing Lindblad operators. Restriction of such a map to any of the individual classes $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_{\alpha\pm}$ or to any other subset retains this property.
From this point of view, the two-qubit synchronizing normal Lindblad operators do not take up any distinctive place among phase-locking mechanisms, they merely represent their particular subset. Morover, the smooth dependence of the achieved phase shift on the chosen Lindblad operator is a very desirable property. It is a good starting point, perhaps even a necessity for a possible future experimental realization of the studied system.

5.5 Entanglement generation and destruction

Synchronization and phase-locking are certain forms of correlation between the involved parties. It is natural to ask whether during the evolution towards a synchronized or phase-locked state any other form of correlation, such as entanglement, arises. In the following we briefly address the relation between two-qubit generalized synchronization mechanisms discovered in chapter 3 and entanglement of the asymptotic states.
The connection between synchronization and entanglement formation has been tackled in the literature without much success so far. It is an open question and, unfortunately, it will remain so throughout this section as well. Equipped with a broad class of synchronizing and phase-locking QMDS, we can, theoretically, judge whether some of them inevitably enforce generation, destruction or preservation of entanglement between the qubits and when. Practically, it turns out to be a real hassle. If anything, we obtained evidence suggesting that synchronization and entanglement formation or destruction are not directly correlated.

For a quantification of entanglement of two-qubit states we use concurrence, an explicitly calculable entanglement monotone, monotonously related to the entanglement of formation [30], [31]. The two concepts are defined as follows.

Given a pure state $\rho = |\psi\rangle\langle\psi|$, the entropy of entanglement E is defined as the entropy of either subsystem

$$E(\rho) = -\operatorname{Tr}(\rho_A \log \rho_A) = -\operatorname{Tr}(\rho_B \log \rho_B). \tag{5.30}$$

For a mixed state ρ the entanglement of formation is defined to be the average entropy of entanglement of the pure states in a pure state decomposition, minimized over all possible decompositions

$$E(\rho) = \inf \sum_i p_i E(\rho_i), \tag{5.31}$$

where $\rho = \sum_i p_i \rho_i = \sum_i p_i |\psi_i\rangle \langle\psi_i|$ stands for a pure state decomposition. It turns out that for two-qubit states this quantity can be explicitly expressed and calculated.

For a pure state $|\psi\rangle$ the concurrence C is defined as

$$C(|\psi\rangle) = \left| \langle\psi| |\tilde{\psi}\rangle \right|, \tag{5.32}$$

where $|\tilde{\psi}\rangle$ stands for the result of applying a spin-flip operation onto $|\psi\rangle$, i. e. $|\tilde{\psi}\rangle = (\sigma_y \otimes \sigma_y) |\bar{\psi}\rangle$ with $|\bar{\psi}\rangle$ being the complex conjugation of $|\psi\rangle$ in the standard basis and

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{5.33}$$

The concurrence of a mixed state ρ is defined to be the average concurrence of the pure states in a pure state decomposition, minimized over all possible decompositions

$$C(\rho) = \inf \sum_i p_i C(\rho_i), \tag{5.34}$$

where $\rho = \sum_i p_i \rho_i = \sum_i p_i |\psi_i\rangle \langle\psi_i|$ is a pure state decomposition of ρ. Remarkably, there exists an explicit formula for the concurrence C [30], [31]. Denote $\tilde{\rho}$ the spin-fliped operator ρ

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\bar{\rho}(\sigma_y \otimes \sigma_y), \tag{5.35}$$

where $\bar{\rho}$ stands for the complex conjugation of ρ in the standard basis. Then

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{5.36}$$

where λ_i are the square roots of eigenvalues of $\rho\tilde{\rho}$ in descending order[1] .

Concurrence is related to the entanglement of formation via

$$E(C) = f\left(\frac{1 + \sqrt{1 - C^2}}{2}\right), \tag{5.37}$$

$$f(x) = -x \log x - (1 - x) log(1 - x). \tag{5.38}$$

The entanglement of formation E as a function of concurrence C is monotonous. Hence, concurrence is a suitable measure of entanglement and quantifying of entanglement of two qubits reduces to the problem of finding the eigenvalues of $\rho\tilde{\rho}$. We used this tool to study the mechanisms of generalized synchronization.

[1] All the eigenvalues are real and non-negative as $\rho\tilde{\rho}$ is a product of two positive-semidefinite matrices.

Basically, one of the following options can occur during time evolution

a) An initially separable state remains separable throughout the evolution and in the asymptotics.
b) A separable state temporarily becomes entangled during the evolution yet results in a separable state again.
c) A separable state becomes entangled.
d) An initially entangled state evolves towards a separable asymptotic state.
e) An entangled state remains entangled with the entanglement remaining the same according to the chosen measure.
f) An entangled state remains entangled and the entanglement increases or decreases, possibly nonmonotonously, according to the chosen measure.

Analytically, we were unable to obtain any conclusive results regarding for what kind of initial conditions does any of the synchronizing or phase-locking mechanisms enforce one of the cases described above. The concurrence of the asymptotic state is strongly dependent on initial conditions, in a manner too complex to draw any conclusions. In numerical simulations, we witnessed all above listed possible scenarios for each of the classes of synchronizing and phase-locking Lindblad operators, based on the initial conditions. See appendix B for examples.

Every mechanism of generalized synchronization described in this work is capable of creating, destroying, preserving and both increasing and decreasing entanglement of a pair of qubits. No studied form of synchronization is in a simple relation to entanglement and neither is entanglement a suitable indicator of synchronization. The generality of our approach to the search of synchronization mechanisms within QMDS is an advantage here as this basically shows that entanglement formation or destruction is not directly connected to synchronization for a large class of possible evolutions of a two-qubit system.

Conclusion

This thesis addressed the phenomenon of synchronization in open quantum systems with Markovian evolution. An introduction to the formalism and theory of quantum Markov dynamical semigroups, with emphasis on the asymptotics, was given. The concept of synchronization of quantum systems in the current literature was discussed and suitable definitions of two different degrees of synchronization for composite systems of identical subsystems with internal dynamics were provided, together with the generalization of synchronization to phase-locking. Within the framework of Lindbladian dynamics with normal Lindblad operators we then investigated systems of n qubits.

We begun with an in-depth study of a system of two. Using a theorem that links the Lindblad operators, i.e. the structure of the generator of the evolution map, and the attractor space of the generated map via commutation relations, we found all normal Lindblad operators that enforce synchronization or phase-locking of the reduced states of a pair of qubits. From the synchronizing and phase-locking mechanisms we subsequently picked those which not only synchronize or phase-lock the dynamical time-evolving parts of the asymptotic single-qubit states, but also enforce synchronization of their stationary parts, resulting in the case of synchronization in identical reduced states of the individual qubits. The synchronization- and phase-locking-enforcing normal Lindblad operators were separated into several classes based on a part of the attractor space of the corresponding generated QMDS relevant for the asymptotic time evolution of the reduced qubit states. We further found the entire attractor spaces and showed them to be preserved within each class or its particular subset, further classifying the synchronization, respectively phase-locking mechanisms. With these results for a single normal Lindblad operator in the generator we generalized the studied dynamics to an arbitrary number and any combination of normal Lindblad operators.

Next, we applied the synchronization mechanisms to qubit networks, n-partite systems with bipartite interactions. It was presented how a corresponding Lindbladian can be constructed from the two-qubit synchronizing normal Lindblad operators and how the relevant parts of the attractor spaces of generated QMDS can be analyzed. We found that while all discovered synchronizing mechanisms enforce pairwise synchronization of adjacent qubits also in an arbitrary network, only two of the classes of synchronizing operators are actually applicable to a general network of more than two qubits. For those we were able to demonstrate that they synchronize all qubits in every network whereof interaction graph is weakly connected. The remaining synchronization mechanisms, on the contrary, were shown to destroy the asymptotic dynamics of individual qubits for every non-trivial network of more that two parties.

Last but not least, we studied some properties of the two-qubit synchronizing mechanisms and asymptotic states they lead to. Regarding visibility of the asymptotic time evolution of the reduced single-qubit states, it turned out that it depends mostly merely on the initial conditions, with an exception of two families of classes of synchronizing, respectively phase-locking Lindblad operators for which it it suppressed by a factor depending solely on the operator class. We also investigated the invariance of the synchronization mechanism and of the attractor spaces of corresponding QMDS with respect to the exchange of qubits. We showed that not all of the synchronization processes enforce a permutation invariant global two-qubit state, in fact, many do the exact opposite. Furthermore, we found that for the majority of synchronizing Lindblad operators, whether associated with a permutation invariant attractor space or not, the action of qubit-swapping changes the Lindbladian and, hence, their action on the two qubits is asymmetrical. Besides, we tackled the question of a possible connection between synchronization and entanglement, using concurrence as an entanglement measure of choice in two-qubit systems and numerical simulations as a supplementary tool, albeit without any worthy conclusive results. Additionally, an observation was made that the synchronizing and the phase-locking two-qubit normal Lindblad operators with an arbitrary, but fixed achieved phase shift are in one-to-one correspondence, share various properties including their classification into classes, and can smoothly be deformed ones into the others.

This thesis represents an extensive study of synchronization and phase-locking processes on qubit networks, realizable by quantum Markov dynamical semigroups with normal Lindblad operators. The greater part of the work comprises original and previously unpublished results. Besides, it opens various possibilities for future investigation, such as into phase-locking on qubit networks with regard to their topology, generalization to non-normal Lindblad operators or even beyond continuous Markovian evolution, consideration of systems more complex than qubits and networks made thereof or even combination of different subsystems. The author believes that this thesis builds a solid foundation for further research on the topic and will contribute to the endeavour to understand the very nature of the phenomenon of synchronization on the quantum level.

Appendix A

Parameterization of normal matrices

Here we derive a possible parameterization for 2x2 normal matrices, which is used in the main body of the work. From the general case suitable parametrizations of unitary, hermitian and antihermitian matrices are obtained. The parameterization of normal matrices is essential to the work, the added part serves as a tool to extraxt unitary, hermitian or antihermitian generalized-synchronization-enforcing Lindblad operators from the more general case of normal ones when needed.

A.1 Normal matrices

Assume a matrix $M \in \mathbb{C}^{2x2}$ parameterized by $a, b, c, d \in \mathbb{C}$,

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{A.1}$$

satisfying

$$\left[M, M^{\dagger}\right] = 0. \tag{A.2}$$

Evaluating this commutation relation element-wise yields the following set of equations

$$|b|^2 = |c|^2, \tag{A.3}$$

$$a\bar{c} + b\bar{d} = \bar{a}b + \bar{c}d. \tag{A.4}$$

It follows from (A.3) that we can rewrite c as

$$c = e^{ik}b, \tag{A.5}$$

where $k \in \mathbb{R}$. Inserting that into (A.4) and rearranging the terms gives

$$\underbrace{(d-a)}_{me^{il}} e^{-ik}\bar{b} = \underbrace{(\bar{d}-\bar{a})}_{me^{-il}} b, \tag{A.6}$$

denoting $d - a = me^{il}$ with $m, l \in \mathbb{R}$. This equation is trivially satisfied for $b = 0$, implying $c = 0$ and leaving a, d arbitrary, or for $a = d$, leaving b, k arbitrary. In other cases we can divide (A.6) by its right-hand side. The result reads

$$e^{i2l}e^{-ik}e^{-i2\arg(b)} = 1, \tag{A.7}$$

implying $l = \frac{k}{2} + \arg(b)$. We can drop the other solutions $l + n\pi$, $n \in \mathbb{Z}$, which only account for the change of sign of m and introduce unnecessary redundance. Hence

$$d = a + me^{i\left(\frac{k}{2}+\arg(b)\right)}. \tag{A.8}$$

Put together, an arbitrary normal matrix $M \in \mathbb{C}^{2\times 2}$ can be parameterized as

$$M = \begin{pmatrix} a & b \\ e^{ik}b & a + me^{i\left(\frac{k}{2}+\arg(b)\right)} \end{pmatrix}, \tag{A.9}$$

where $a, b \in \mathbb{C}$, $k, m \in \mathbb{R}$, and we additionaly set $\arg(0) = 0$ so that the parameterization (A.14) covers also the previously discussed case $b = 0$.

An alternative parameterization can be obtained by the transformation

$$k \to 2k - 2\arg(b), \tag{A.10}$$

which corresponds to choosing

$$c = e^{i2k}\bar{b}, \tag{A.11}$$

instead of (A.5). The equation (A.7) simplifies into

$$e^{i2l}e^{-i2k} = 1, \tag{A.12}$$

implying $l = k$, again dropping the solutions $l = k + n\pi$, $n \in \mathbb{Z}$. Then, instead of the relation (A.8), we get

$$d = a + me^{ik}. \tag{A.13}$$

Finally, the parameterization of an arbitrary normal matrix $M \in \mathbb{C}^{2\times 2}$ reads

$$M = \begin{pmatrix} a & b \\ e^{i2k}\bar{b} & a + me^{ik} \end{pmatrix}, \tag{A.14}$$

where $a, b \in \mathbb{C}$, $k, m \in \mathbb{R}$.

A.2 Unitary matrices

Assume a unitary matrix $U \in \mathbb{C}^{2\times 2}$. Using the parameterization (A.14) to evaluate the unitarity condition $UU^\dagger = U^\dagger U = I$ yields

$$|a|^2 + |b|^2 = 1 \tag{A.15}$$

$$|a|^2 + |b|^2 + ame^{-ik} + \bar{a}me^{ik} + m^2 = 1, \tag{A.16}$$

$$abe^{-i2k} + \bar{a}b + bme^{-ik} = 0. \tag{A.17}$$

Inserting (A.15) into (A.16) the latter can be rewritten as

$$m\left[m + 2|a|\cos\left(k - \arg(a)\right)\right] = 0. \tag{A.18}$$

Assuming $m = 0$, (A.17) implies $k = \frac{\pi}{2} + \arg(a)$ and we can write U as

$$U = \begin{pmatrix} a & b \\ -\bar{b}e^{i2\arg(a)} & a \end{pmatrix}, \tag{A.19}$$

$a, b \in \mathbb{C}$, $|a|^2 + |b|^2 = 1$.

For $m = -2|a|\cos\left(k - \arg(a)\right)$, the equation (A.17) is satisfied for arbitrary $k \in \mathbb{R}$. Decomposing the cosine into two exponentials again, we get

$$d = a + me^{ik} = a - |a|e^{i\arg(a)} - |a|e^{i(2k - \arg(a))} = \bar{a}e^{i(\pi+2k)}. \tag{A.20}$$

Let us denote $\omega = \pi + 2k$, replacing the parameter k. We arrive at a parameterization of U of the form

$$U = \begin{pmatrix} a & b \\ -\bar{b}e^{i\omega} & \bar{a}e^{i\omega} \end{pmatrix}, \tag{A.21}$$

where $a, b \in \mathbb{C}$, $|a|^2 + |b|^2 = 1$, $\omega \in \mathbb{R}$.

Note that the previous solutions (A.19) are contained here as well, obtainable by simply setting $\omega = 2\arg(a)$ for a given a. Consequently, (A.21) is a parameterization describing all possible 2x2 unitary matrices and the one used in the work. Since it holds $\det U = e^{i\omega}$, a parameterization of all special unitary matrices is obtained by fixing $\omega = 0$.

A.3 Hermitian matrices

Assume a hermitian matrix $H \in \mathbb{C}^{2\times 2}$ in the parameterization (A.14). It follows immediately from the hermicity, $H^\dagger = H$, that $a \in \mathbb{R}$ and

$$k = 0. \tag{A.22}$$

Subsequently, reintroducing $d = a + m$ to replace the parameter m, all hermitian 2x2 matrices can be parameterized as

$$H = \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix}, \tag{A.23}$$

where $a, d \in \mathbb{R}$, $b \in \mathbb{C}$.

A.4 Skew-hermitian matrices

Assume a skew-hermitian matrix $A \in \mathbb{C}^{2\times 2}$ in the parameterization (A.14). Similarly to the previous case, the condition $A^\dagger = -A$ implies $a \in i\mathbb{R}$ and

$$k = \frac{\pi}{2}. \tag{A.24}$$

It follows that $e^{ik} = i$, $e^{i2k} = -1$ and we can reintroduce $d = a + im \in i\mathbb{R}$. Together, all skew-hermitian 2x2 matrices can be parameterized as

$$A = \begin{pmatrix} a & b \\ -\bar{b} & d \end{pmatrix}, \tag{A.25}$$

where $a, d \in i\mathbb{R}$, $b \in \mathbb{C}$, or in a slightly different form

$$A = \begin{pmatrix} ia & b \\ -\bar{b} & id \end{pmatrix}, \tag{A.26}$$

where $a, d \in \mathbb{R}$, $b \in \mathbb{C}$.

Appendix B

Numerical simulation

The vast majority of this work does not rely on any numerical simulation to obtain the outcome. However, in order to get a better idea about the evolution towards asymptotics, to test the analytical results and to illustrate the studied systems and setups, a simple MATLAB script was written. Some examples are provided on the following pages, the script itself is available from the author upon request[1].

The QMDS governing the evolution is generated by the Lindbladian (1.11). The script reads the user-provided adjacency matrix of an interaction graph of a qubit network, generalized synchronization mechanism class and required phase shift. Within the classes of synchronizing, respectively phase-locking normal Lindblad operators the particular operators are chosen randomly, unless otherwise specified. The initial conditions are randomly generated as well, either from separable states or from entangled ones, as indicated.
In each evolution step the reduced states $\rho_1(t), \rho_2(t), ..., \rho_n(t)$ are calculated from the global state $\rho(t)$ and the distances of these reduced states from a fixed randomly generated qubit test state ρ_{test}, i.e. $\|\rho_1(t) - \rho_{test}\|, \|\rho_2(t) - \rho_{test}\|, ..., \|\rho_n(t) - \rho_{test}\|$, are plotted. For a two-qubit state the norm of their difference $\|\rho_1(t) - \rho_2(t)\|$ is plotted as well. The second plot displays the expectation values $\left\langle \sigma^{(1)}(t) \right\rangle, \left\langle \sigma^{(2)}(t) \right\rangle, ..., \left\langle \sigma^{(n)}(t) \right\rangle$ of a local observable σ (5.4) for the individual reduced states. The third one shows the values of Pearson's correlation coefficient (2.15) $C_{\left\langle \sigma^{(1)} \right\rangle, \left\langle \sigma^{(j)} \right\rangle}(t, \Delta t)$ for the expectation values of σ, one for each qubit being compared with qubit 1, taken over a time window Δt of 30 time units and modified by the asymptotic phase shift between them. For a two-qubit state, additionally, the concurrence C (5.36) is displayed. The time axes are aligned and equally scaled and the interaction graph is depicted or otherwise described.

[1]Availability is only guaranteed until the day of this thesis defence.

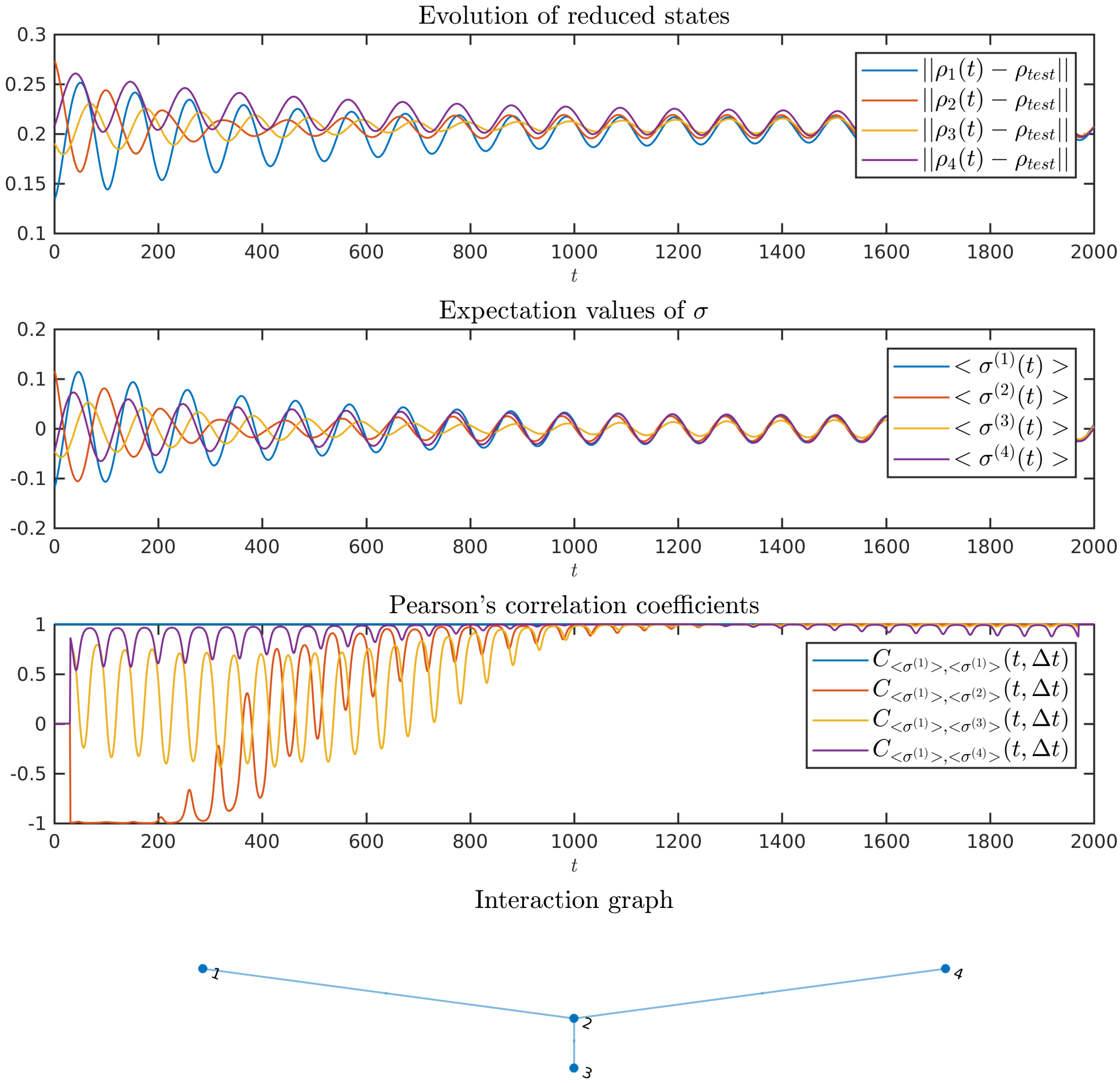

Figure B.1: Complete synchronization of a four-qubit network with an interaction graph whereof adjacency matrix reads $G_{12} = G_{23} = G_{42} = 1$ and $G_{ij} = 0$ otherwise, enforced by Lindblad operator L_2 given by (3.60). Both the particular operator L_2 and initial conditions were generated randomly. The first plot shows the time evolution of the reduced single-qubit states $\rho_i(t)$ compaired against a stationary test state ρ_{test}. The second one displays the expectation values $\left\langle \sigma^{(i)} \right\rangle (t)$ of the observable σ (5.4) for individual qubits. In the third one we can see the values of Pearson's correlation coefficients $C_{\left\langle \sigma^{(1)} \right\rangle, \left\langle \sigma^{(j)} \right\rangle}(t, \Delta t)$ (2.15) of the expectation values of σ for individual qubit pairs. The interaction graph is depicted at the bottom.

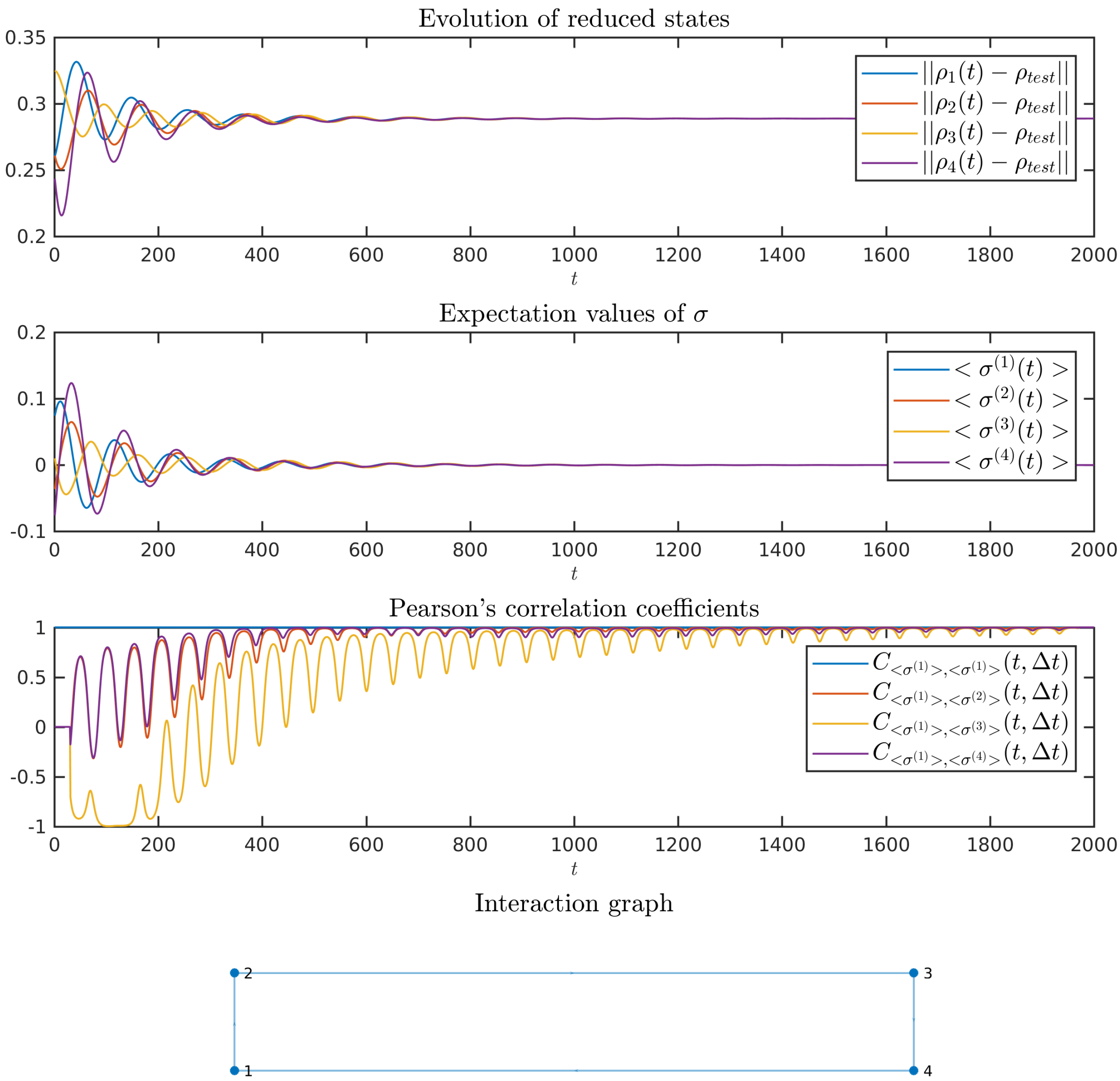

Figure B.2: The destruction of single-qubit dynamics in a four-qubit network with an interaction graph whereof adjacency matrix reads $G_{12} = G_{23} = G_{34} = G_{41} = 1$ and $G_{ij} = 0$ otherwise, enforced by Lindblad operator $L_{\alpha+}$ given by (3.112). Both the particular operator $L_{\alpha+}$ and initial conditions were generated randomly. The first plot shows the time evolution of the reduced single-qubit states $\rho_i(t)$ compaired against a stationary test state ρ_{test}. The second one dipslays the expectation values $\langle \sigma^{(i)} \rangle (t)$ of the observable σ (5.4) for individual qubits. In the third one we can see the values of Pearson's correlation coefficients $C_{\langle \sigma^{(1)} \rangle, \langle \sigma^{(j)} \rangle}(t, \Delta t)$ (2.15) of the expectation values of σ for individual qubit pairs. The interaction graph is depicted at the bottom.

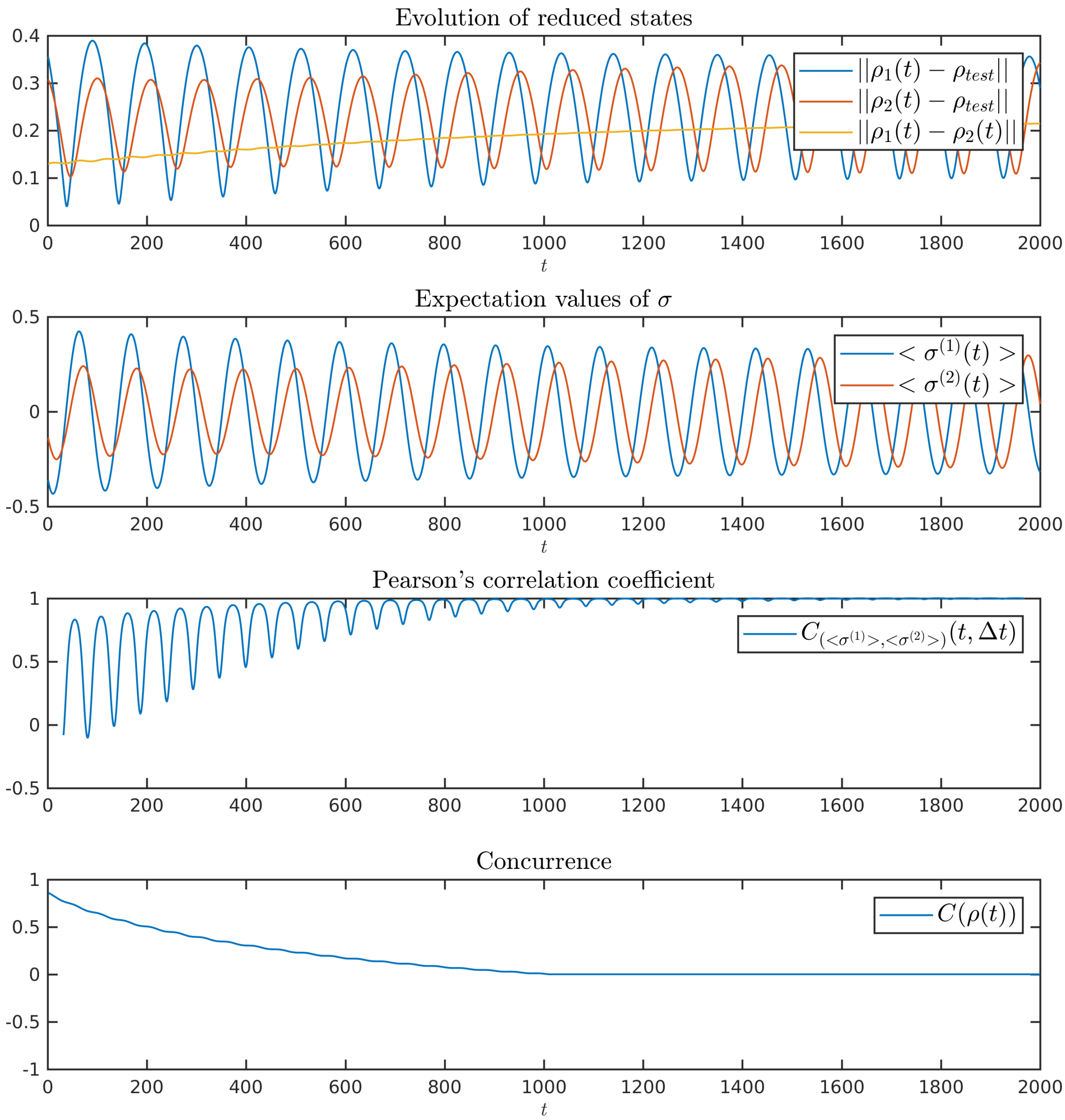

Figure B.3: Phase-locking with a phase shift $\varphi = \frac{\pi}{2}$ enforced by a Lindblad operator $L_{\alpha-}$(3.112) on two qubits. Destruction of entanglement during the process. Both the particular operator $L_{\alpha-}$ and initial conditions were generated randomly. The first plot shows the time evolution of the reduced single-qubit states $\rho_i(t)$ compaired against a stationary test state ρ_{test}, and the norm of their difference. The second one displays the expectation values $\left\langle \sigma^{(i)} \right\rangle (t)$ of the observable σ (5.4) for individual qubits. In the third one we can see the value of Pearson's correlation coefficients$C_{\left\langle \sigma^{(1)} \right\rangle, \left\langle \sigma^{(j)} \right\rangle}(t, \Delta t)$ (2.15) of the expectation values of σ for individual qubit pairs, modified by the phase shift φ. The last plot shows the time evolution of concurrence.

Appendix C

Overview of synchronizing and phase-locking normal Lindblad operators

In this appendix the reader finds an overview of all two-qubit generalized (complete) synchronization mechanisms given by normal Lindblad operators and the attractor spaces of their respective corresponding QMDS with generators (3.1), written as linear spans of orthonormal bases of generators associated each with a single eigenvalue for convenience. The division into two classes and two separate families of classes from the work is kept, with special attention devoted to several particular cases.

$$
\begin{aligned}
L_1 = &\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{e^{-i\varphi}}{\sqrt{2}} & 0 \\ 0 & \frac{e^{i\varphi}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & e^{i2k}\bar{b} & a + me^{ik} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{e^{-i\varphi}}{\sqrt{2}} & 0 \\ 0 & -\frac{e^{i\varphi}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \\
&= \frac{1}{2} \begin{pmatrix} 2c & 0 & 0 & 0 \\ 0 & c+a & e^{-i\varphi}(c-a) & -\sqrt{2}e^{-i\varphi}b \\ 0 & e^{i\varphi}(c-a) & c+a & \sqrt{2}b \\ 0 & -\sqrt{2}e^{i\varphi}e^{2ik}\bar{b} & \sqrt{2}e^{2ik}\bar{b} & 2(a+me^{ik}) \end{pmatrix},
\end{aligned}
\tag{C.1}
$$

where $a, b \in \mathbb{C}$, $c, k, m \in \mathbb{R}$, $b \neq 0$ or $a \neq c \wedge m \neq 0$ holds, and $\varphi \in [0, 2\pi)$ is the achieved phase shift. Denoting a corresponding QMDS $\mathcal{T}_{L_1}$ the attractor space $\mathrm{Att}(\mathcal{T}_{L_1})$ thereof reads

$$
\begin{aligned}
\mathrm{Att}(\mathcal{T}_{L_1}) = \mathrm{span}\Bigg\{ &\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&e^{-i\varphi}&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\1&0&0&0\\e^{i\varphi}&0&0&0\\0&0&0&0\end{pmatrix}, \\
&\frac{1}{\sqrt{3}}\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix}0&0&0&0\\0&0&e^{-i\varphi}&0\\0&e^{i\varphi}&0&0\\0&0&0&-e^{i\varphi}\end{pmatrix}\Bigg\},
\end{aligned}
\tag{C.2}
$$

in the case $b \neq 0$. For $b = 0$, $m \neq e^{-ik}(c-a)$ then

$$
\begin{aligned}
\mathrm{Att}(\mathcal{T}_{L_1}) = \mathrm{span}\Bigg\{ &\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&e^{-i\varphi}&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\1&0&0&0\\e^{i\varphi}&0&0&0\\0&0&0&0\end{pmatrix}, \\
&\frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\0&0&e^{-i\varphi}&0\\0&e^{i\varphi}&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}\Bigg\}.
\end{aligned}
\tag{C.3}
$$

In the remaining case, i.e. $b = 0$, $m = e^{-ik}(c-a)$, the operator L_1 reduces to

$$
\begin{aligned}
L_s &= \frac{1}{2}\begin{pmatrix}2c&0&0&0\\0&c+a&e^{-i\varphi}(c-a)&0\\0&e^{i\varphi}(c-a)&c+a&0\\0&0&0&2c\end{pmatrix} = \\
&= (c+a)\underbrace{\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}}_{I_{4\times 4}} + (c-a)\underbrace{\begin{pmatrix}1&0&0&0\\0&0&e^{-i\varphi}&0\\0&e^{i\varphi}&0&0\\0&0&0&1\end{pmatrix}}_{SWAP_{\varphi}},
\end{aligned}
\tag{C.4}
$$

where $a, c \in \mathbb{C}$, $c \neq a$, $\varphi \in [0, 2\pi)$ is the phase shift, and the attractor space $\mathrm{Att}(\mathcal{T}_{L_s})$ of a corresponding QMDS $\mathcal{T}_{L_s}$ reads

$$\begin{aligned}\mathrm{Att}(\mathcal{T}_{L_s}) = \mathrm{span}\Bigg\{&\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\0&0&e^{-i\varphi}&0\\0&e^{i\varphi}&0&0\\0&0&0&0\end{pmatrix},\\ &\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&e^{-i\varphi}&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&e^{i\varphi}\\0&0&0&0\end{pmatrix},\\ &\frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\1&0&0&0\\e^{i\varphi}&0&0&0\\0&0&0&0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&e^{-i\varphi}&0\end{pmatrix}, \begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\\ &\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}\Bigg\}.\end{aligned} \tag{C.5}$$

$$\begin{aligned}L_2 &= \begin{pmatrix}1&0&0&0\\0&\frac{1}{\sqrt{2}}&\frac{e^{-i\varphi}}{\sqrt{2}}&0\\0&-\frac{e^{i\varphi}}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}a&b&0&0\\e^{i2k}\bar{b}&a+me^{ik}&0&0\\0&0&c&0\\0&0&0&c\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&\frac{1}{\sqrt{2}}&-\frac{e^{-i\varphi}}{\sqrt{2}}&0\\0&\frac{e^{i\varphi}}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\0&0&0&1\end{pmatrix} =\\ &= \frac{1}{2}\begin{pmatrix}2a&\sqrt{2}b&-\sqrt{2}e^{-i\varphi}b&0\\\sqrt{2}e^{2ik}\bar{b}&c+a+me^{ik}&e^{-i\varphi}(c-a-me^{ik})&0\\-\sqrt{2}e^{i\varphi}e^{2ik}\bar{b}&e^{i\varphi}(c-a-me^{ik})&c+a+me^{ik}&0\\0&0&0&2c\end{pmatrix},\end{aligned} \tag{C.6}$$

where $a, b \in \mathbb{C}$, $c, k, m \in \mathbb{R}$, $b \neq 0$ or $a + me^{ik} \neq c \wedge m \neq 0$ holds, and $\varphi \in [0, 2\pi)$ is the achieved phase shift. The attractor space $\mathrm{Att}(\mathcal{T}_{L_2})$ of a corresponding QMDS $\mathcal{T}_{L_2}$ reads

$$\begin{aligned}\mathrm{Att}(\mathcal{T}_{L_2}) = \mathrm{span}\Bigg\{&\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix}-e^{i\varphi}&0&0&0\\0&0&e^{-i\varphi}&0\\0&e^{i\varphi}&0&0\\0&0&0&0\end{pmatrix},\\ &\frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&e^{i\varphi}\\0&0&0&0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&e^{-i\varphi}&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}\Bigg\},\end{aligned} \tag{C.7}$$

in the case $b \neq 0$. For $b = 0$, $a \neq c$ then

$$\mathrm{Att}(\mathcal{T}_{L_2}) = \mathrm{span}\left\{\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\0&0&e^{-i\varphi}&0\\0&e^{i\varphi}&0&0\\0&0&0&0\end{pmatrix},\right.$$
$$\left.\frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&e^{i\varphi}\\0&0&0&0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&e^{-i\varphi}&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}\right\}. \tag{C.8}$$

In the remaining case, i.e. $b = 0$, $c = a$, the operator L_2 reduces to L_s (C.4) and the attractor space of a corresponding QMDS is given by (C.5).

$$L_{\alpha\pm} = \begin{pmatrix}1&0&0&0\\0&\beta&\alpha&0\\0&-\alpha&\bar{\beta}&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}a&b&0&0\\e^{i2k}\bar{b}&a+me^{ik}&0&0\\0&0&a&sb\\0&0&\bar{s}e^{i2k}\bar{b}&a+me^{ik}\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&\bar{\beta}&-\alpha&0\\0&\alpha&\beta&0\\0&0&0&1\end{pmatrix} =$$
$$= \begin{pmatrix}a & \mp ie^{i\varphi}\sqrt{1-\alpha^2}b & -\alpha b & 0\\ \pm ie^{-i\varphi}e^{2ik}\bar{b} & a+(1-a^2)me^{ik} & \mp ie^{-i\varphi}\alpha\sqrt{1-\alpha^2}me^{ik} & \alpha sb\\ -\alpha e^{2ik}\bar{b} & \pm ie^{i\varphi}\alpha\sqrt{1-\alpha^2}me^{ik} & a+\alpha^2 me^{ik} & \mp ie^{i\varphi}\sqrt{1-\alpha^2}sb\\ 0 & \alpha\bar{s}e^{2ik}\bar{b} & \pm ie^{-i\varphi}\sqrt{1-\alpha^2}\bar{s}e^{2ik}\bar{b} & a+me^{ik}\end{pmatrix}, \tag{C.9}$$

where $a, b \in \mathbb{C},\ k, m, \alpha \in \mathbb{R}$, $0 < \alpha < 1$, $b \neq 0$,

$$s = \frac{\alpha \mp i\sqrt{1-\alpha^2}}{\alpha \pm i\sqrt{1-\alpha^2}}, \tag{C.10}$$

$$\beta = \pm ie^{-i\varphi}\sqrt{1-\alpha^2}, \tag{C.11}$$

and $\varphi \in [0, 2\pi)$ is the achieved phase shift. The attractor space $\mathrm{Att}(\mathcal{T}_{L_{\alpha\pm}})$ of a corresponding QMDS $\mathcal{T}_{L_{\alpha\pm}}$ reads

$$
\begin{aligned}
\mathrm{Att}(\mathcal{T}_{L_{\alpha\pm}}) = \mathrm{span}\Bigg\{ & \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \frac{1}{\sqrt{(2\alpha^2-1)^2+\frac{2}{\alpha^2}}}\begin{pmatrix} 0 & 1 & \pm ie^{-i\varphi}\frac{\sqrt{1-\alpha^2}}{\alpha} & 0 \\ 0 & 0 & 0 & e^{-i\varphi}2(1-\alpha^2)\pm ie^{-i\varphi}\frac{\sqrt{1-\alpha^2}}{\alpha}(2\alpha^2-1) \\ 0 & 0 & 0 & 1-2\alpha^2\pm i2\alpha\sqrt{1-\alpha^2} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
& \frac{1}{\sqrt{(2\alpha^2-1)^2+\frac{2}{\alpha^2}}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \mp ie^{i\varphi}\frac{\sqrt{1-\alpha^2}}{\alpha} & 0 & 0 & 0 \\ 0 & e^{i\varphi}2(1-\alpha^2)\mp ie^{i\varphi}\frac{\sqrt{1-\alpha^2}}{\alpha}(2\alpha^2-1) & 1-2\alpha^2\mp i2\alpha\sqrt{1-\alpha^2} & 0 \end{pmatrix}, \\
& \frac{\alpha\sqrt{1-\alpha^2}}{\sqrt{8\alpha^8-32\alpha^6+42\alpha^4-18\alpha^2+3}}\begin{pmatrix} \pm i\frac{2\alpha^4-(2\alpha^2-1)^2}{\alpha\sqrt{1-\alpha^2}} & 0 & 0 & 0 \\ 0 & 0 & e^{-i\varphi} & 0 \\ 0 & -e^{i\varphi} & \pm i\frac{2\alpha^2-1}{\alpha\sqrt{1-\alpha^2}} & 0 \\ 0 & 0 & 0 & \mp i\frac{2\alpha^4-(2\alpha^2-1)^2}{\alpha\sqrt{1-\alpha^2}} \end{pmatrix}\Bigg\}.
\end{aligned}
\tag{C.12}
$$

For $\alpha = \frac{1}{\sqrt{2}}$ the operators $L_{\alpha\pm}$ become the completely synchronizing operators $L_{\frac{1}{\sqrt{2}}\pm}$,

$$
\begin{aligned}
L_{\frac{1}{\sqrt{2}}\pm} = & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{\mp ie^{-i\varphi}}{\sqrt{2}} & 0 \\ 0 & \frac{\mp ie^{i\varphi}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & 0 & 0 \\ e^{2ik}\bar{b} & a+me^{ik} & 0 & 0 \\ 0 & 0 & a & \mp ib \\ 0 & 0 & \pm ie^{2ik}\bar{b} & a+me^{ik} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{\pm ie^{-i\varphi}}{\sqrt{2}} & 0 \\ 0 & \frac{\pm ie^{i\varphi}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \\
= & \frac{1}{2}\begin{pmatrix} 2a & \mp\sqrt{2}ie^{i\varphi}b & -\sqrt{2}b & 0 \\ \pm\sqrt{2}ie^{-i\varphi}e^{2ik}\bar{b} & 2a+me^{ik} & \mp ie^{-i\varphi}me^{ik} & \mp i\sqrt{2}b \\ -\sqrt{2}e^{2ik}\bar{b} & \pm ie^{i\varphi}me^{ik} & 2a+me^{ik} & -\sqrt{2}e^{i\varphi}b \\ 0 & \pm\sqrt{2}ie^{2ik}\bar{b} & -\sqrt{2}e^{-i\varphi}e^{2ik}\bar{b} & a+me^{ik} \end{pmatrix},
\end{aligned}
\tag{C.13}
$$

wherof corresponding QMDS $\mathcal{T}_{L_{\frac{1}{\sqrt{2}}\pm}}$ have the attractor spaces $\mathrm{Att}(\mathcal{T}_{L_{\frac{1}{\sqrt{2}}\pm}})$,

$$
\begin{aligned}
\mathrm{Att}(\mathcal{T}_{L_{\frac{1}{\sqrt{2}}\pm}}) = \mathrm{span}\Bigg\{ & \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & 1 & \pm ie^{-i\varphi} & 0 \\ 0 & 0 & 0 & e^{-i\varphi} \\ 0 & 0 & 0 & \pm i \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
& \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \mp ie^{i\varphi} & 0 & 0 & 0 \\ 0 & e^{i\varphi} & \mp i & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} \pm i & 0 & 0 & 0 \\ 0 & 0 & e^{-i\varphi} & 0 \\ 0 & -e^{i\varphi} & 0 & 0 \\ 0 & 0 & 0 & \mp i \end{pmatrix}\Bigg\}.
\end{aligned}
\tag{C.14}
$$

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