Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Antipalindromic numbers

Antipalindromická čísla

Bachelor's Degree Project

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ZADÁNÍ BAKALÁŘSKÉ PRÁCE

Student:	Stanislav Kruml
Studijní program:	Aplikace přírodních věd
Obor:	Aplikovaná informatika
Název práce (česky):	Antipalindromická čísla
Název práce (anglicky):	Antipalindromic numbers

Pokyny pro vypracování:

1. Seznámení se s definicí palindromického a antipalindromického čísla v přirozené bázi.
2. Nastudování vlastností palindromických čísel (dělitelnost, prvočísla, mezery, mocniny, pořadí).
3. Zkoumání analogických vlastností pro antipalindromická čísla.
4. Formulace a řešení nových otázek pro antipalindromická čísla.
5. Program pro testování zkoumaných vlastností antipalindromických čísel.

Doporučená literatura:

1. X. Droubay, J. Justin, G. Pirillo, Episturmian words and some constructions of de Luca and Rauzy. Theoret. Comput. Sci. 255, 2001, 539–553.

2. Ľ. Dvořáková, D. Ryzák, Antipalindromy. Rozhledy matematicko-fyzikální 1, 2019, 2-10.

3. E. Pelantová, Š. Starosta, Constructions of words rich in palindromes and pseudopalindromes. DMTCS 18, 2016, # 16.

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V Praze dne 23. října 2019

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Fakulta jaderná a fyzikálně inženýrská
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Author's declaration:

I declare that this Bachelor's Degree Project is entirely my own work and I have listed all the used sources in the bibliography.

Prague, July 22, 2020 Stanislav Kruml

Název práce:
Antipalindromická čísla

Autor: Stanislav Kruml

Obor: Aplikovaná informatika

Druh práce: Bakalářská práce

Vedoucí práce: doc. Ing. Ľubomíra Dvořáková, Ph.D., FJFI ČVUT v Praze, Katedra matematiky

Abstrakt: Každý jistě slyšel o palindromech, slovech, která zůstanou stejná, když je čteme zezadu dopředu. Například kajak, radar nebo rotor. V matematice se studují palindromická čísla, což jsou přirozená čísla, jejichž zápis v nějaké přirozené bázi je palindrom. Zkoumají se palindromická prvočísla, palindromické čtverce a vyšší mocniny, čísla palindromická ve více bázích atd. V této práci studujeme antipalindromická čísla, což jsou přirozená čísla, jejichž zápis v nějaké přirozené bázi je antipalindrom. Jde o novou strukturu (definovanou v roce 2018), pro kterou je celá řada vlastností neprostudovaná. My jsme získali nové výsledky týkající se minimálního a maximálního počtu palindromických čísel mezi antipalindromickými a naopak, mezer mezi (anti)palindromickými čísly, antipalindromických čtverců a vyšších mocnin a čísel, která jsou antipalindromická ve více bázích. Ke všem studovaným otázkám jsme vytvořili uživatelsky přátelskou aplikaci.

Klíčová slova: zápis v bázi, palindromy, antipalindromy, palindromická čísla, antipalindromická čísla

Title:
Antipalindromic numbers

Author: Stanislav Kruml

Abstract: Everybody has certainly heard about palindromes: words that stay the same when read backwards. For instance, kayak, radar, or rotor. Mathematicians are interested in palindromic numbers: positive integers whose expansion in a certain integer base is a palindrome. The following problems are studied: palindromic primes, palindromic squares and higher powers, multi-base palindromic numbers etc. In this project, antipalindromic numbers are studied: positive integers whose expansion in a certain integer base is an antipalindrome. It is a new structure (defined in 2018) for which a lot of problems are open. New results were obtained, concerning the minimum and maximum number of antipalindromic numbers between palindromic numbers and vice versa, the gaps between (anti)palindromic numbers, antipalindromic squares and higher powers, and multi-base antipalindromic numbers. A user-friendly application was created for all the questions studied.

Key words: expansion in a base, palindromes, antipalindromes, palindromic numbers, antipalindromic numbers

Contents

Introduction

Palindromes, words that stay the same when read backwards, are commonly known. It is not surprising that in natural languages, it is impossible to find extremely long palindromes. The longest palindrome in English is “tattarrattat”. However, its victory is doubtful since tattarrattat is a neologism created by James Joyce in his novel Ulysses [6]: it expresses loud knocking at the door:

“I was just beginning to yawn with nerves thinking he was trying to make a fool of me when I knew his tattarrattat at the door.”

Palindromic phrases are even more interesting. They provide palindromes if punctuation, capitalization, and spaces are ignored. Some popular palindromic phrases in English are:

“Do geese see God?”
“A man, a plan, a canal: Panama. ”
“Madam, in Eden, I’m Adam.”

Mathematicians are interested in palindromic numbers: positive integers whose expansion in a certain integer base is a palindrome. Let us make a list of the problems studied:

1. *Palindromic squares, cubes, and higher powers in base* 10: The first nine terms of the sequence $1^2, 11^2, 111^2, 1111^2, \ldots$ are palindromic numbers $1, 121, 12321, 1234321, \ldots$ (sequence A002477 in the OEIS [9]). The only known non-palindromic number whose cube is a palindromic number is 2201, and Simmons [8] conjectured that the fourth roots of all palindromic fourth powers are palindromic numbers of the form $10^n + 1$. Simmons [7] also conjectured there are no palindromic numbers of the form n^k for $k > 4$ and $n > 1$.

2. *Palindromic primes*: The first 20 decimal palindromic primes are (sequence A002385 in the OEIS [9]):

   $$2, 3, 5, 7, 11, 101, 131, 151, 181, 191, 313, 353, 373, 383, 727, 757, 787, 797, 919, 929, \ldots$$

   Except for 11, all palindromic primes have an odd number of digits because the divisibility test for 11 indicates that every palindromic number with an even number of digits is divisible by 11. On the one hand, it is not known if there are infinitely many palindromic primes in base 10. The largest known decimal palindromic prime has 474,501 digits (found in 2014):

   $$10^{474500} + 999 \cdot 10^{237249} + 1.$$

   On the other hand, it is known that, for any base, almost all palindromic numbers are composite [1]. It means the ratio of palindromic composites and all palindromic numbers less than n tends to 1.

   Binary palindromic primes include the Mersenne primes and the Fermat primes [1]. All binary palindromic primes except the number 3 (having the expansion 11 in base 2) have an odd number of digits; palindromic numbers with an even number of digits are divisible by 3. Let us write down the sequence of binary expansions of the first binary palindromic primes (sequence A117697 in the OEIS [9]):

   $$11, 101, 111, 10001, 11111, 1001001, 1101011, 1111111, 100000001, 100111001, 110111011, \ldots$$

[1]A Mersenne prime is a prime of the form $2^p - 1$, where p is a prime. A Fermat prime is a prime of the form $2^{2^n} + 1$.

3. *Multi-base palindromic numbers*: Any positive integer n is palindromic in all bases b with $b \geq n+1$ because n is then a single-digit number, and also in base $n-1$ because the expansion of n in base $n-1$ equals 11. But, it is more interesting to consider bases smaller than the number itself. For instance, the number 105 is palindromic in bases $4, 8, 14, 20, 34, 104$; the expansions of 105 in these bases are:

   $$(105)_4 = 1221, \quad (105)_8 = 151, \quad (105)_{14} = 77, \quad (105)_{20} = 55, \quad (105)_{34} = 33, \quad (105)_{104} = 11.$$

   A palindromic number in base b whose expansion is made up of palindromic sequences of length ℓ arranged in a palindromic order is palindromic in base b^ℓ. For example, the number 24253 has the expansion in base 2 equal to $(24253)_2 = 101\ 111\ 010\ 111\ 101$, i.e., it is made up of palindromes of length 3, and its expansion in base $2^3 = 8$ is equal to $(24253)_8 = 57275$.

4. *Sum of palindromes*: Every positive integer can be written as the sum of three palindromic numbers in every number system with base 5 or greater [4].

In this project, we are dealing with antipalindromic numbers in various integer bases. We examine and compare properties of palindromic numbers and antipalindromic numbers, bringing a number of new results. These are structured as follows. Chapter 1 contains the theoretical part about antipalindromic numbers. In Section 1.1, we show the formal definition of an antipalindromic number in an integer base and its basic properties following from the definition. In the following section, Section 1.2, previously discovered properties of antipalindromic numbers are described. Section 1.3 informs about the number of (anti)palindromic numbers of a certain length and their mutual positions. Results concerning the length of gaps between antipalindromic numbers are demonstrated in Section 1.4. In Section 1.5, antipalindromic squares and higher powers are examined. Section 1.6 contains information about numbers that are antipalindromic in two or more bases at the same time. Chapter 2 offers a description of the application that is a part of this project. In Conclusion, we summarize our results and provide a list of conjectures and open problems.

Chapter 1

(Anti)palindromic numbers

1.1 Definition and basic properties

Let us start with a formal definition of palindromic and antipalindromic numbers and their basic properties.

Definition 1. *Let $b \in \mathbb{N}, b \geq 2$. Consider a natural number m whose expansion in base b is of the following form*

$$m = a_n b^n + \cdots + a_1 b + a_0,$$

where $a_0, a_1, \ldots, a_n \in \{0, 1, \ldots, b-1\}, a_n \neq 0$. We usually write $(m)_b = a_n \ldots a_1 a_0$. Then m is called

1. a palindromic number *in base b if its digits in base b satisfy the condition:*

$$a_j = a_{n-j} \quad \text{for all } j \in \{0, 1, \ldots, n\}, \tag{1.1}$$

2. an antipalindromic number *in base b if its digits in base b satisfy the condition:*

$$a_j = b - 1 - a_{n-j} \quad \text{for all } j \in \{0, 1, \ldots, n\}. \tag{1.2}$$

The length of the expansion of the number m is usually denoted $|m|$.

Example 1. *Consider distinct bases b and have a look at antipalindromic numbers in these bases:*

- 395406 *is an antipalindromic number in base $b = 10$.*
- $(1581)_3 = 2011120$ *is an antipalindromic number in base $b = 3$.*
- $(52)_2 = 110100$ *is an antipalindromic number in base $b = 2$.*

Theorem 1. *An antipalindromic number can have an odd number of digits only in an odd base b. The digit in the center is then equal to $\frac{b-1}{2}$.*

Proof. Let us denote the digits of the considered antipalindromic number $a_0, a_1, a_2, \ldots, a_{2n}$. Pair the digits and add $a_0 + a_{2n}, a_1 + a_{2n-1}, \ldots, a_{n-1} + a_{n+1}$. From the definition, each pair has a total of $b - 1$. That leaves us with the digit a_n that must be paired with itself: $2a_n = b - 1$. Therefore, the digit a_n is an integer only for $b = 2k + 1$, where $k \in \mathbb{N}$, i.e., for an odd b. Furthermore, $a_n = \frac{b-1}{2}$. □

Theorem 2. *An antipalindromic number can be palindromic at the same time only if b is an odd number and all the digits are equal to $\frac{b-1}{2}$.*

Proof. Consider an antipalindromic number with digits $a_0, a_1, \ldots, a_n$. For this number to be palindromic, $a_j = a_{n-j}$ must be true for each $j \in \{0, 1, \ldots, n\}$. From the definition of an antipalindromic number, it follows that $a_j + a_{n-j} = b - 1$. For each $j \in \{0, 1, \ldots, n\}$ we obtain $a_j = \frac{b-1}{2}$, i.e., all the digits are equal to $\frac{b-1}{2}$ and the base b must therefore be odd. □

1.2 Previously discovered properties

Let us summarize some already known properties of antipalindromic numbers, see [1] for more details.

Lemma 3. *Let m be a natural number and its expansion in base b be equal to $a_n b^n + a_{n-1} b^{n-1} + \ldots + a_1 b + a_0$. Then m is divisible by $b-1$ if and only if the sum of its digits is divisible by $b-1$, i.e., $a_n + a_{n-1} + \ldots + a_1 + a_0 \equiv 0 \bmod b-1$.*

Theorem 4. *Any antipalindromic number with an even number of digits in base b is divisible by $b-1$.*

Proof. Consider an antipalindromic number

$$m = a_n b^n + a_{n-1} b^{n-1} + \ldots + a_1 b + a_0$$

for an odd n. From the definition, it is true that $a_j + a_{n-j} = b-1$ for each $j \in \{0, 1, \ldots, n\}$. The number of digits is even, hence

$$a_n + a_{n-1} + \ldots + a_1 + a_0 = (b-1)\frac{n+1}{2} \equiv 0 \pmod{b-1}.$$

Using Lemma 3, the antipalindromic number m is divisible by the number $b-1$. □

Theorem 5. *An antipalindromic number with an odd number of digits in base b is divisible by $\frac{b-1}{2}$.*

Proof. Consider the antipalindromic number

$$m = a_{2n} b^{2n} + a_{2n-1} b^{2n-1} + \ldots + a_1 b + a_0.$$

The digit sum of the number $m - a_n b^n$ is divisible by $b-1$. From Lemma 3, we also know that the number $m - a_n b^n$ itself is divisible by $b-1$ and, therefore, by $\frac{b-1}{2}$. From the definition, $a_n = \frac{b-1}{2}$. The number m is a sum of two numbers divisible by $\frac{b-1}{2}$. □

The following properties are linked to prime numbers.

Theorem 6. *Let base $b > 3$. Then there exists at most one antipalindromic prime number p in base b: $p = \frac{b-1}{2}$.*

Proof. Theorems 4 and 5 show that every antipalindromic number is divisible either by $\frac{b-1}{2}$ or $b-1$. Although $b-1$ may be a prime number, it is never antipalindromic. □

Theorem 7. *Let base $b = 2$. Then there exists only one antipalindromic number $p = 2$, $(p)_2 = 10$.*

Proof. Every antipalindromic number in base $b = 2$ is even. 2 is the only even prime number. □

Theorem 8. *Let base $b = 3$. Every antipalindromic prime in this base has an odd number of digits $n \geq 3$.*

Proof. From Theorem 4, antipalindromic numbers with an even number of digits in base $b = 3$ are even. The only antipalindromic number in this base with one digit is 1. □

Lemma 9. *Antipalindromic numbers in base $b = 3$ beginning with a digit 2 are divisible by 3.*

Proof. Consider an antipalindromic number $m = a_n 3^n + a_{n-1} 3^{n-1} + \ldots + a_1 3 + a_0$, where $a_n = 2$. The sum of a_n and a_0 need to be equal to 2, therefore $a_0 = 0$. All the summands are divisible by three. □

Theorem 10. *All antipalindromic primes in base $b = 3$ can be expressed as $6k + 1$, where $k \in \mathbb{N}$.*

Proof. Consider an antipalindromic prime $m = a_{2n}3^{2n} + a_{2n-1}3^{2n-1} + \ldots + a_1 3 + a_0$. (The number of digits must be odd.) From Lemma 9, a_0 is equal to 1. Let us pair the members of the antipalindromic number m (except for a_{2n}, a_n and a_0): $a_{2n-j}3^{2n-j} + a_j 3^j$, $j \in \{1, \ldots, n-1\}$. Let us prove that for each $j \in \{1, \ldots, n-1\}$ there exists $s \in \mathbb{N}$ satisfying

$$3^j(a_{2n-j}3^{2n-2j} + a_j) = 6s.$$

We can only consider three possibilities: $a_{2n-j} = 2, a_j = 0$, or $a_{2n-j} = a_j = 1$, or $a_{2n-j} = 0, a_j = 2$. In either case, the equation holds because there is an even number inside the bracket. We then get

$$\begin{aligned} m &= a_{2n}3^{2n} + a_n 3^n + a_0 + 6l \\ &= 3^{2n} + 3^n + 1 + 6l \\ &= 3^n(3^n + 1) + 1 + 6l \end{aligned}$$

for some $l \in \mathbb{N}_0$. The first summand is also divisible by 6, therefore m can indeed be expressed as $6k + 1$ for some $k \in \mathbb{N}$.

□

The application that is a part of this project can be used for searching antipalindromic primes in base 3. During an extended search, the first 637807 antipalindromic primes have been found. The first 100 antipalindromic primes in base 3 are listed in the appendix of this project. Let us now list at least the first 10 of them, along with their expansions in base 3:

13	111
97	10121
853	1011121
1021	1101211
1093	1111111
7873	101210121
8161	102012021
8377	102111021
9337	110210211
12241	121210101

1.3 Occurences of palindromic numbers between neighboring antipalindromic numbers and vice versa

While it is obvious that there are more palindromic numbers than antipalindromic ones in any given base, the way palindromic and antipalindromic numbers are ordered might not be so trivial. One of the modes in the application that is a part of this project is dedicated to the calculation of both palindromic and antipalindromic sequences to show the manner in which these numbers occur in relation to one another, see Figure 1.1.

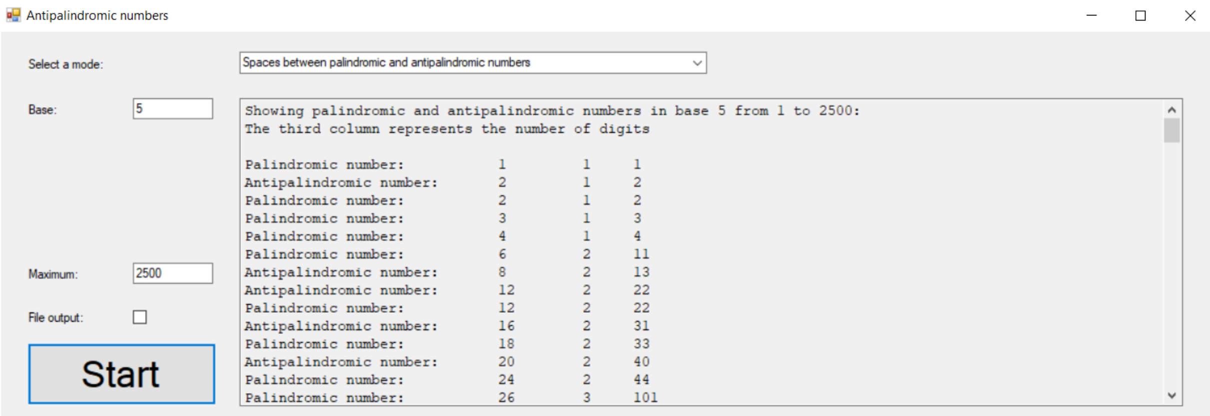

Figure 1.1: Application mode: Spaces between palindromic and antipalindromic numbers

Let us answer the following questions:

Question 1. *What is the* **minimum** *number of palindromic numbers that may occur between two neighboring antipalindromic numbers in a given base b?*

Question 2. *What is the* **minimum** *number of antipalindromic numbers that may occur between two neighboring palindromic numbers in a given base b?*

Question 3. *What is the* **maximum** *number of palindromic numbers that may occur between two neighboring antipalindromic numbers in a given base b?*

Question 4. *What is the* **maximum** *number of antipalindromic numbers that may occur between two neighboring palindromic numbers in a given base b?*

Definition 2. *Consider a string $u = u_0 u_1 \dots u_n$, where $u_i \in \{0, 1, \dots, b-1\}$. The* palindromic complement *of u in base b is $P(u) = u_n u_{n-1} \dots u_0$ and the* antipalindromic complement *of u in base b is $A(u) = (b-1-u_n)(b-1-u_{n-1}) \dots (b-1-u_0)$.*

Definition 3. *Two antipalindromic numbers m, n in base b, where $m < n$, are called* neighboring *if there are no antipalindromic numbers in base b larger than m and smaller than n at the same time.*

Example 2. *The numbers $m = 81$ and $n = 90$ are evidently neighboring antipalindromic numbers in base 10.*

Let us describe the form of neighboring antipalindromic numbers m and n in base b, $m < n$, depending on the length of their expansion in base b.

1. If $(m)_b$ and $(n)_b$ have the same and even length, then we have

$$(m)_b = wA(w) \quad \text{and} \quad (n)_b = vA(v),$$

where $w = (x)_b$ and $v = (x+1)_b$ for some $x \in \mathbb{N}$.

2. If $(m)_b$ and $(n)_b$ have the same and odd length, then b is an odd base and we have

$$(m)_b = w\tfrac{b-1}{2}A(w) \quad \text{and} \quad (n)_b = v\tfrac{b-1}{2}A(v),$$

where $w = (x)_b$ and $v = (x+1)_b$ for some $x \in \mathbb{N}$.

3. If $(m)_b$ is of odd length $2k+1$ and $(n)_b$ is of even length $2k+2$, then b is an odd base and we have

$$(m)_b = \underbrace{(b-1)\dots(b-1)}_{k-\text{times}}\tfrac{b-1}{2}\underbrace{0\dots0}_{k-\text{times}} \quad \text{and} \quad (n)_b = 1\underbrace{0\dots0}_{k-\text{times}}\underbrace{(b-1)\dots(b-1)}_{k-\text{times}}(b-2).$$

4. If $(m)_b$ is of even length $2k$ and $(n)_b$ is of odd length $2k+1$, then b is an odd base and we have

$$(m)_b = \underbrace{(b-1)\dots(b-1)}_{k-\text{times}}\underbrace{0\dots0}_{k-\text{times}} \quad \text{and} \quad (n)_b = 1\underbrace{0\dots0}_{(k-1)-\text{times}}\tfrac{b-1}{2}\underbrace{(b-1)\dots(b-1)}_{(k-1)-\text{times}}(b-2).$$

5. If $(m)_b$ is of even length $2k$ and $(n)_b$ is of even length $2k+2$, then b is an even base and we have

$$(m)_b = \underbrace{(b-1)\dots(b-1)}_{k-\text{times}}\underbrace{0\dots0}_{k-\text{times}} \quad \text{and} \quad (n)_b = 1\underbrace{0\dots0}_{k-\text{times}}\underbrace{(b-1)\dots(b-1)}_{k-\text{times}}(b-2).$$

Example 3. *Consider the base $b = 10$. The antipalindromic numbers 5814 and 5904 illustrate case 1., the antipalindromic numbers 9900 and 100998 illustrate case 5. Let us underline that in base 10, if $w_2 - w_1 = 1$, then $w_1A(w_1)$ and $w_2A(w_2)$ are neighboring.*

Example 4. *Consider the base $b = 3$. The antipalindromic numbers $m = 460$ and $n = 510$ with expansions $(m)_3 = 122001$ and $(n)_3 = 200220$ illustrate case 1., the antipalindromic numbers $m = 145$ and $n = 177$ with expansions $(m)_3 = 12101$ and $(n)_3 = 20120$ illustrate case 2., the antipalindromic numbers $m = 225$ and $n = 268$ with expansions $(m)_3 = 22100$ and $(n)_3 = 100221$ illustrate case 3. and the antipalindromic numbers $m = 72$ and $n = 97$ with expansions $(m)_3 = 2200$ and $(n)_3 = 10121$ illustrate case 4.*

Definition 4. *Two palindromic numbers m, n in base b, where $m < n$, are called* neighboring *if there are no palindromic numbers in base b larger than m and smaller than n at the same time.*

Example 5. *The numbers $m = 88$ and $n = 99$ are evidently neighboring palindromic numbers in base 10.*

Let us describe the form of neighboring palindromic numbers m and n in base b, $m < n$, depending on the length of their expansion in base b.

1. If $(m)_b$ and $(n)_b$ have the same and even length, then we have

$$(m)_b = wP(w) \quad \text{and} \quad (n)_b = vP(v),$$

where $w = (x)_b$ and $v = (x+1)_b$ for some $x \in \mathbb{N}$.

2. If $(m)_b$ and $(n)_b$ have the same and odd length, then we have either

$$(m)_b = waP(w) \quad \text{and} \quad (n)_b = w(a+1)P(w),$$

where $w = (x)_b$ for some $x \in \mathbb{N}$ and $a \in \{0, \dots, b-2\}$, or

$$(m)_b = w(b-1)P(w) \quad \text{and} \quad (n)_b = v0P(v),$$

where $w = (x)_b$ and $v = (x+1)_b$ for some $x \in \mathbb{N}$.

3. If $(m)_b$ is of length k and $(n)_b$ is of length $k+1$, then we have

$$(m)_b = \underbrace{(b-1)\dots(b-1)}_{k-\text{times}} \quad \text{and} \quad (n)_b = 1 \underbrace{0\dots0}_{(k-1)-\text{times}} 1.$$

Example 6. *Consider the base $b = 10$. The palindromic numbers 789987 and 790097 illustrate case 1., the palindromic numbers 7892987 and 7893987 illustrate case 2.a), the palindromic numbers 7899987 and 7900097 illustrate case 2.b) and the palindromic numbers 999 and 1001 illustrate case 3.*

First, let us answer questions 1. and 2. concerning the minimum number.

Theorem 11. *Let $b \in \mathbb{N}, b \geq 2$. Then there exist two neighboring antipalindromic numbers m, n in base b such that there is no palindromic number p in base b satisfying $m \leq p \leq n$. Similarly, there exist two neighboring palindromic numbers m, n in base b such that there is no antipalindromic number a in base b satisfying $m \leq a \leq n$.*

Proof.

- If b is an even base, $b > 2$, then we may set $(m)_b = (\frac{b}{2}-1)(\frac{b}{2})$ and $(n)_b = (\frac{b}{2})(\frac{b}{2}-1)$. If b is an odd base, $b > 3$, then we may set $(m)_b = 1(\frac{b-1}{2})(\frac{b-1}{2})(b-2)$ and $(n)_b = 1(\frac{b+1}{2})(\frac{b-3}{2})(b-2)$. If $b = 2$, then we may set $(m)_2 = 1010$ and $(n)_2 = 1100$. If $b = 3$, then we may set $(m)_3 = 2020$ and $(n)_3 = 2110$. Then m and n are neighboring antipalindromic numbers in base b such that there is no palindromic number in base b between them.
- We may set $(m)_b = \underbrace{(b-1)\dots(b-1)}_{k-\text{times}}$ and $(n)_b = 1 \underbrace{0\dots0}_{(k-1)-\text{times}} 1$ for some $k > 1$. Then m and n are neighboring palindromic numbers in base b such that there is no antipalindromic number in base b between them.

$\square$

Example 7. *Consider the base $b = 10$. There is no palindromic number between 45 and 54 (the nearest palindromic numbers are 44 and 55). There is no antipalindromic number between 99 and 101 (the nearest antipalindromic numbers are 90 and 1098).*

Example 8. *Consider the base $b = 3$. There is no palindromic number between $m = 60$ and $n = 66$, i.e., $(m)_3 = 2020$ and $(n)_3 = 2110$. There is no antipalindromic number between $m = 8$ and $n = 10$, i.e., $(m)_3 = 22$ and $(n)_3 = 101$.*

It remains to answer questions 3. and 4. concerning the maximum number. We will state the results depending on parity of base b.

Odd base

Theorem 12. *1) Let m, n be two neighboring antipalindromic numbers in base b. Then the maximum number of palindromic numbers p satisfying $m \leq p \leq n$ equals $b+1$.*
2) Let m, n be two neighboring palindromic numbers in base b. Then the maximum number of antipalindromic numbers a satisfying $m \leq a \leq n$ equals two.

Proof. 1. Consider situations for neighboring antipalindromic numbers.
In case 1., there are at most two palindromic numbers between m and n – if such palindromic number exists, its expansion in base b must be equal to $wP(w)$ or $vP(v)$.
In case 2., there are at most $b+1$ palindromic numbers between m and n – their expansions are equal to
$w(\frac{b-1}{2})P(w), \dots, w(b-1)P(w), v0P(v), \dots, v(\frac{b-1}{2})P(v)$. The maximum number of palindromic numbers can be reached, for example, by choosing $w = \frac{(b-1)}{2}(b-1)(b-1)$. The first palindromic number greater than w is then equal to $\frac{b-1}{2}(b-1)(b-1)\frac{b-1}{2}(b-1)(b-1)\frac{b-1}{2}$ and the last one smaller than v is equal to $\frac{b+1}{2}00\frac{b-1}{2}00\frac{b+1}{2}$.

In case 3., there are $\frac{b+1}{2}+1$ palindromic numbers between m and n – their expansions are

$$\underbrace{(b-1)\dots(b-1)}_{k\text{–times}}\,\tfrac{b-1}{2}\,\underbrace{(b-1)\dots(b-1)}_{k\text{–times}},$$

$$\underbrace{(b-1)\dots(b-1)}_{k\text{–times}}(\tfrac{b-1}{2}+1)\,\underbrace{(b-1)\dots(b-1)}_{k\text{–times}},$$

$$\dots\dots,$$

$$\underbrace{(b-1)\dots(b-1)}_{(2k+1)\text{–times}},$$

$$1\,\underbrace{0\dots0}_{2k\text{–times}}\,1.$$

In case 4., there are $\frac{b+1}{2}+1$ palindromic numbers between m and n – their expansions are equal to

$$\underbrace{(b-1)\dots(b-1)}_{2k\text{–times}},$$

$$1\quad\underbrace{0\dots0}_{(2k-1)\text{–times}}\quad 1,$$

$$1\quad\underbrace{0\dots0}_{(k-1)\text{–times}}\quad 1\quad\underbrace{0\dots0}_{(k-1)\text{–times}}\quad 1,$$

$$\dots\dots,$$

$$1\quad\underbrace{0\dots0}_{(k-1)\text{–times}}\quad\tfrac{b-1}{2}\quad\underbrace{0\dots0}_{(k-1)\text{–times}}\quad 1.$$

It follows from the above analysis that the maximum number is attained in case 2, and it equals $b+1$.

2. Consider situations for neighboring palindromic numbers. In case 1., if there are antipalindromic numbers between m and n, then they have expansions in base b equal to $wA(w)$ or $vA(v)$.
   In case 2., there is at most one antipalindromic number between m and n – its expansion is equal to $w\frac{b-1}{2}A(w)$.
   In case 3., there is no antipalindromic number between m and n.

   It follows that the maximum number of antipalindromic numbers between palindromic numbers is two. Consider, for instance, palindromic numbers m and n with expansions $(m)_b = \frac{b-1}{2}\,\frac{b-1}{2}$ and $(n)_b = \frac{b+1}{2}\,\frac{b+1}{2}$. Then the antipalindromic numbers between them are m and s with $(s)_b = \frac{b+1}{2}\,\frac{b-3}{2}$. □

Example 9. *Consider the base $b = 3$. Let $m = 145$ and $n = 177$. These numbers are antipalindromic with expansions $(m)_3 = 12101$ and $(n)_3 = 20120$. There are 4 palindromic numbers between m and n, these numbers are $151, 160, 164, 173$ with expansions $12121, 12221, 20002, 20102$. For palindromic numbers $m = 4$ and $n = 8$ with expansions 11 and 22, there are two antipalindromic numbers between m and n – these numbers are m and $s = 6$, i.e., $(s)_3 = 20$.*

Even base

Theorem 13. *Let b be an even base. The maximum number of palindromic numbers between two neighboring antipalindromic numbers in base b is not bounded. In particular, between the largest antipalindromic number with $2k$ digits and the smallest antipalindromic number with $2k + 2$ digits, there are exactly $(b-1) \cdot b^k + 2$ palindromic numbers. Let m, n be two neighboring palindromic numbers in base b. Then the maximum number of antipalindromic numbers a satisfying $m \le a \le n$ equals two.*

Proof.

- Consider the largest antipalindromic number with $2k$ digits and expansion $\underbrace{(b-1)\ldots(b-1)}_{k\text{-times}}\underbrace{0\ldots0}_{k\text{-times}}$. The only larger palindromic number with $2k$ digits is the largest one possible – with all digits equal to $b-1$. The smallest antipalindromic number with $2k + 2$ digits has the expansion $1\underbrace{0\ldots0}_{k\text{-times}}\underbrace{(b-1)\ldots(b-1)}_{k\text{-times}}(b-2)$ and there is only one smaller palindromic number with $2k + 2$ digits – its expansion equals to 100...001. It remains to prove that there are exactly $(b-1) \cdot b^k$ palindromic numbers of length $2k + 1$.

  Denote the considered palindromic number $(u_{2k})\underbrace{(u_{2k-1})\ldots(u_{k+1})}_{k-1 \text{ digits}}(u_k)\underbrace{(u_{k+1})\ldots(u_{2k-1})}_{k-1 \text{ digits}}(u_{2k})$. All digits from u_{2k-1} to u_k (their number is equal to k) can be chosen arbitrarily from the set $0, 1, \ldots, b-1\}$, u_{2k} cannot be zero. Therefore, the number of all palindromic numbers with $2k + 1$ digits is $(b-1) \cdot b^k$.

- Consider situations for neighboring palindromic numbers. In case 1., if there are antipalindromic numbers between m and n, then they have expansions in base b equal to $wA(w)$ or $vA(v)$.
  In case 2., there is no antipalindromic number between m and n.

  It follows that the maximum number of antipalindromic numbers between palindromic numbers is two. Let $b > 2$ and consider, for instance, palindromic numbers m and n with expansions $(m)_b = \frac{b-2}{2}\frac{b-2}{2}$ and $(n)_b = \frac{b}{2}\frac{b}{2}$. Then the antipalindromic numbers between them are r and s with $(r)_b = \frac{b-2}{2}\frac{b}{2}$ and $(s)_b = \frac{b}{2}\frac{b-2}{2}$. Let $b = 2$ and consider $(m)_2 = 1001$ and $(n)_2 = 1111$. Then the antipalindromic numbers between them are r and s, where $(r)_2 = 1010$ and $(s)_2 = 1100$.

$\square$

Example 10. *Consider the base $b = 4$. Let $m = 12$ and $n = 78$. These numbers are antipalindromic with expansions $(m)_4 = 30$ and $(n)_4 = 1032$ and also neighboring. There are 14 palindromic numbers between m and n, these numbers are $15, 17, 21, 25, 29, 34, 38, 42, 46, 51, 55, 59, 63, 65$ with expansions $33, 101, 111, 121, 131, 202, 212, 222, 232, 303, 313, 323, 333, 1001$.*

Consider again the base $b = 4$. Let $m = 5$ and $n = 10$. These numbers are palindromic with expansions 11 and 22. There are 2 antipalindromic numbers between m and n, these numbers are 6 and 9 with expansions 12 and 21.

1.4 Gaps between neighboring antipalindromic numbers

To produce a list of antipalindromic numbers using an algorithm, it is useful to employ a formula that calculates the gap between the current antipalindromic number and the following one. In this section, we present a set of such formulae, some of which have been put to use in the source code of the application that is a part of this project. The application was also used to verify our results.

Let us answer the following question:

Question 5. *How big are the gaps between neighboring antipalindromic numbers?*

Example 11. *Consider the base $b = 10$. When listing the antipalindromic numbers of length 4 from the smallest to the largest, notice which digits are changing:*

$$1098, 1188, 1278, 1368, 1458, 1548, 1638, 1728, 1818, 1908, 2097, 2187, ... \tag{1.3}$$

Two types of gap lengths occur in this case: Either the second digit is increased by one and the third digit is decreased by one, and then the gap is of length 90. Or the first digit is increased by one and the fourth digit is decreased by one, the second digit (equal to 9) and the third digit (equal to 0) are exchanged, then the gap is of length 189. The n-th number in the list (1.3) is followed by the gap of length 90 if and only if n is not divisible by 10. There are exactly 10 antipalindromic numbers with the same first digit.

Example 12. *Consider the base $b = 3$. When listing the antipalindromic numbers of length 6 from the smallest to the largest, notice which digits are changing:*

$$100221, 101121, 102021, 110211, 111111, 112011, 120201, 121101, 122001, 200220, ... \tag{1.4}$$

Three types of gap lengths occur in this case:

- *The third digit is increased by one and the fourth digit is decreased by one, then the gap length equals 18.*
- *The second digit is increased by one and the fifth digit is decreased by one, the third digit (equal to 2) and the fourth digit (equal to 0) are exchanged. Then the gap length equals 42.*
- *The first digit is increased by one and the last digit is decreased by one, the second digit (equal to 2) and the fifth digit (equal to 0) are exchanged, the third digit (equal to 2) and the fourth digit (equal to 0) are exchanged. Then the gap length equals 50.*

The n-th number in the list (1.4) is followed by the gap

1. *of length 18 if and only if n is not divisible by 3;*
2. *of length 42 if and only if n is divisible by 3 but not by 9;*
3. *of length 50 if and only if n is divisible by 9.*

There are exactly 3 antipalindromic numbers with the same first two digits. There are exactly 9 antipalindromic numbers with the same first digit (1 or 2). Since there are 18 antipalindromic numbers of this length in total, only one gap of length 50 occurs - between the numbers having expansions 122001 and 200220.

Definition 5. *Let us define a mapping M_b which assigns to an integer i the maximum non-negative number k such that $i \equiv 0 \pmod{b^k}$. Furthermore, let $A_b(n)$ denote a sequence of all n-digit antipalindromic numbers, ordered from the smallest to the largest.*
Remark: It is easy to count the number of members of the sequence $A_b(n)$ depending on parity of n:

- *If n is even, then the number of members is equal to* $(b-1)\cdot b^{\frac{n-2}{2}}$.
- *If n is odd and b is odd, then the number of members is equal to* $(b-1)\cdot b^{\frac{n-3}{2}}$. *(The middle digit of all members of* $A_b(n)$ *is* $\frac{b-1}{2}$.*)*
- *If n is odd and b is even, then the sequence is empty.*

Theorem 14. *Consider an odd b and an odd* $n>2$. *Then for all* $i\in\{1,\ldots,(b-1)\cdot b^{\frac{n-3}{2}}-1\}$, *it is true:*

$$A_b(n)_{i+1}-A_b(n)_i=(b^2-1)\cdot\sum_{k=0}^{M_b(i)} b^{\frac{n-3}{2}-k}.$$

Proof. If $M_b(i)=0$, we only need to change two of the digits: the ones on either side of the middle digit. On the left side, we add 1, and on the right side, we subtract 1. This gives us the gap $(b^2-1)b^{(n-3)/2}$. If $M_b(i)=1$, the digit on the left from the central digit of $A_b(n)_i$ is equal to $b-1$ and the right one is 0. The number n must be equal to or greater than 5 because $M_b(i)=1$ means we are changing four digits. In this case, we are changing the left digit to 0 and the right one to $b-1$. The digit corresponding to $b^{\frac{n-3}{2}}$ is increased by 1, and, symmetrically, the digit corresponding to $b^{\frac{n+5}{2}}$ is decreased by 1, giving us the gap

$$\Big(b^4-(b-1)\cdot b^3+(b-1)\cdot b-1\Big)\cdot b^{\frac{n-5}{2}},$$

which is equal to

$$\begin{aligned}(b^3+b^2-b-1)\cdot b^{\frac{n-5}{2}} &= (b+1)\cdot(b^2-1)\cdot b^{\frac{n-5}{2}}\\ &= \left(1+\frac{1}{b}\right)\cdot(b^2-1)\cdot b^{\frac{n-3}{2}}.\end{aligned}$$

By continuing to raise the value of $M_b(i)=m$, it can be seen that the difference is always

$$\Big(b^{2m+2}-(b-1)\cdot(b^{2m+1}+b^{2m}+\ldots+b^{m+2}\Big)+(b-1)\cdot\Big(b^m+b^{m-1}+\ldots+b)-1\Big)\cdot b^{\frac{n-2m-3}{2}},$$

which can be reduced to

$$\begin{aligned}\Big(b^{m+2}+b^{m+1}-b-1\Big)\cdot b^{\frac{n-2m-3}{2}} &= (b+1)(b^{m+1}-1)\cdot b^{\frac{n-2m-3}{2}}\\ &= (b^m+b^{m-1}+\ldots+1)\cdot(b^2-1)\cdot b^{\frac{n-2m-3}{2}}\\ &= \left(1+\frac{1}{b}+\ldots+\frac{1}{b^m}\right)\cdot(b^2-1)\cdot b^{\frac{n-3}{2}}.\end{aligned}$$

□

Theorem 15. *Consider an even* $n>3$. *Then for all* $i\in\{1,\ldots,(b-1)\cdot b^{\frac{n-2}{2}}-1\}$, *it is true:*

a) $M_b(i)=0$:

$$A_b(n)_{i+1}-A_b(n)_i=(b-1)\cdot b^{\frac{n-2}{2}},$$

b) $M_b(i)>0$:

$$A_b(n)_{i+1}-A_b(n)_i=(b-1)\cdot b^{\frac{n-2}{2}}+(b^2-1)\cdot\left(\sum_{k=1}^{M_b(i)} b^{\frac{n-2}{2}-k}\right).$$

Proof. Similar to the proof of Theorem 14. □

However, the neighboring antipalindromic numbers do not necessarily have the same length. There are three more possible gap lengths to be considered.

Example 13. *Consider the base $b = 3$. Let $(m)_3 = 22100$ and $(n)_3 = 100221$. These two numbers are neighboring because even though their length is different, there is no antipalindromic number between them.*

Example 14. *Consider the base $b = 3$. Let $(m)_3 = 222000$ and $(n)_3 = 1001221$. These two numbers are neighboring because even though their length is different, there is no antipalindromic number between them.*

Example 15. *Consider the base $b = 10$. Let $m = 9900$ and $n = 100998$. These two numbers are neighboring. 9900 is the largest antipalindromic number of length 4 and 100998 is the smallest antipalindromic number of length 6. Furthermore, since the base is even, there are no antipalindromic numbers of an odd length.*

Theorem 16. *For an odd b and an odd n, the gap between the largest antipalindromic number of length n in base b and the smallest antipalindromic number of length $n+1$ in base b is equal to*

$$\tfrac{3b+1}{2} \cdot b^{\frac{n-1}{2}} - 2.$$

Proof. The smaller of the numbers has the expansion $\underbrace{(b-1)\dots(b-1)}_{\frac{n-1}{2}\text{–times}}\left(\frac{b-1}{2}\right)\underbrace{0\dots0}_{\frac{n-1}{2}\text{–times}}$ and the greater one $1\underbrace{0\dots0}_{\frac{n-1}{2}\text{–times}}\underbrace{(b-1)\dots(b-1)}_{\frac{n-1}{2}\text{–times}}(b-2)$. By subtracting these two numbers, we get the difference

$$\tfrac{3b+1}{2} \cdot b^{\frac{n-1}{2}} - 2.$$

□

Theorem 17. *For an odd b and an even n, the gap between the largest antipalindromic number of length n in base b and the smallest antipalindromic number of length $n+1$ in base b is equal to*

$$\tfrac{b+3}{2} \cdot b^{\frac{n}{2}} - 2.$$

Proof. The smaller of the numbers has the expansion $\underbrace{(b-1)\dots(b-1)}_{\frac{n}{2}\text{–times}}\underbrace{0\dots0}_{\frac{n}{2}\text{–times}}$ and the greater one $1\underbrace{0\dots0}_{\frac{n-2}{2}\text{–times}}\left(\frac{b-1}{2}\right)\underbrace{(b-1)\dots(b-1)}_{\frac{n-2}{2}\text{–times}}(b-2)$. By subtracting these two numbers, we get the difference

$$\tfrac{b+3}{2} \cdot b^{\frac{n}{2}} - 2.$$

□

Theorem 18. *For an even b, the gap between the largest antipalindromic number of an even length n in base b and the smallest antipalindromic number of length $n+2$ in base b is equal to*

$$(b-1) \cdot b^n + (b+1) \cdot b^{\frac{n}{2}} - 2.$$

Proof. As we know, for an even base b, there are no antipalindromic numbers of an odd length. The smaller of the numbers has the expansion $\underbrace{(b-1)\dots(b-1)}_{\frac{n}{2}\text{–times}}\underbrace{0\dots0}_{\frac{n}{2}\text{–times}}$ and the greater one $1\underbrace{0\dots0}_{\frac{n}{2}\text{–times}}\underbrace{(b-1)\dots(b-1)}_{\frac{n}{2}\text{–times}}(b-2)$. By subtracting these two numbers, we get the difference

$$(b-1) \cdot b^n + (b+1) \cdot b^{\frac{n}{2}} - 2.$$

□

1.5 Squares and other powers as antipalindromes

For palindromic numbers, squares and higher powers were considered in [7, 8] by G. J. Simmons more than thirty years ago. Simmons proved that there were infinitely many palindromic squares, cubes and biquadrates. However, his conjecture was that for $k > 4, k \in \mathbb{N}$, no integer n exists, such that n^k is a palindromic number (in the decimal base). This conjecture is still open. That is definitely not the case for antipalindromic numbers as $3^7 = 2187$ is antipalindromic in base 10.

Let us answer the following question:

Question 6. *Are there any antipalindromic integer squares?*

The initial observation suggested that bases $b = n^2 + 1, n \in \mathbb{N}$, have the most antipalindromic squares and the computer application provided additional insight needed to prove this observation not only for squares but for other powers as well. Table 1.1 expresses the number of antipalindromic squares smaller than 10^{12} in some bases to underline the differences between the bases of the form $n^2 + 1$ and the others.

base	n=20	21	22	23	24	25	26	27	28	29	30
n^2	3	13	3	14	9	11	35	9	6	17	1
n^2+1	47	44	48	53	55	57	68	59	61	71	66
n^2+2	2	2	2	2	2	1	1	2	2	1	1

Table 1.1: Number of antipalindromic squares smaller than 10^{12} in particular bases

As the exponent is raised, the differences become more significant but the numbers rise faster, see Table 1.2:

base	n=4	5	6	7	8	9	10	11	12
n^4	0	1	0	1	0	1	0	0	0
n^4+1	6	6	8	10	13	13	13	13	13
n^4+2	0	0	0	0	0	0	0	0	0

Table 1.2: Number of antipalindromic biquadrates smaller than 10^{15} in particular bases

Example 16. *Consider the base $b = 10$. Any antipalindromic number in this base must be divisible by 9. Every double-digit number divisible by 9 (except 99) is antipalindromic:*

$$18, 27, 36, 45, 54, 63, 72, 81, 90.$$

The number 9 is a square, so if a square is divided by 9, it still is a square.

$$36 = 4 \cdot 9 = 2^2 \cdot 3^2 = 6^2, \quad 81 = 9 \cdot 9 = 9^2.$$

Lemma 19. *For $b = n^2 + 1$ and $m \in \{2, 3, \dots, n\}$, the number $(m \cdot n)^2$ is antipalindromic.*

Proof. Since $b = n^2 + 1$, we can modify the expression as follows: $(m \cdot n)^2 = m^2 \cdot (b - 1)$. This number has the expansion $(m^2 - 1)\,(b - m^2)$. □

Question 7. *Are there any higher integer powers that are also antipalindromic numbers?*

Example 17. *Consider the base $b = 28$. Any antipalindromic number in this base with an even number of digits must be divisible by 27. Every double-digit number divisible by 27 (except the one with expansion 27 27) is antipalindromic. See the expansions of such numbers:*

$$(1)(26), (2)(25), (3)(24), \dots, (24)(3), (25)(2), (26)(1), (27)(0).$$

The number 27 is a third power of 3, so if a third power of any number is divided by 27, it still is a third power of an integer.

$$(7)(20) = (216)_{28} = 8 \cdot 27 = 2^3 \cdot 3^3 = 6^3, \quad (26)(1) = (729)_{28} = 27 \cdot 27 = 3^3 \cdot 3^3 = 9^3.$$

Theorem 20. *For $b = n^k + 1$ and $m \in \{2, 3, \ldots, n\}$, the number $(m \cdot n)^k$ is antipalindromic.*

Proof. Since $b = n^k + 1$, we can modify the expression as follows: $(m \cdot n)^k = m^k \cdot (b - 1)$. This number has the expansion $(m^k - 1)(b - m^k)$. $\square$

For odd powers and high enough bases, other patterns exist.

Lemma 21. *For each $m > 1$, there exists a number c such that in every base $b > c$, the following number is antipalindromic:*

$$[m \cdot (b - 1)]^3.$$

It suffices to put $c = 3 \cdot m^3 - 1$.

Proof. Let us write down the expansion of $[m \cdot (b - 1)]^3$:

$$([m \cdot (b - 1)]^3)_b = (m^3 - 1) \quad (b - 3 \cdot m^3) \quad (3 \cdot m^3 - 1) \quad (b - m^3).$$

$\square$

Theorem 22. *For each $m > 1$ and odd $k > 1$, there exists a number c such that in every base $b > c$, the following number is antipalindromic:*

$$[m \cdot (b - 1)]^k.$$

It suffices to put $c = \binom{k}{\frac{k-1}{2}} \cdot m^k - 1$.

Proof. From the binomial theorem:

$$[m \cdot (b - 1)]^k = m^k \cdot \sum_{i=0}^{k} (-1)^i \cdot \binom{k}{i} \cdot b^{k-i}.$$

Since $\binom{k}{\frac{k-1}{2}}$ is the maximum number among $\binom{k}{i}$ for $i \in \{0, 1, ..., k\}$, the expansion in base b equals:

$$\left([m \cdot (b - 1)]^k\right)_b = \left(m^k \cdot \binom{k}{0} - 1\right)\left(b - m^k \cdot \binom{k}{1}\right) \ldots \left(m^k \cdot \binom{k}{k-1} - 1\right)\left(b - m^k \cdot \binom{k}{k}\right).$$

$\square$

1.6 Multi-base antipalindromic numbers

Let us study the question whether there are numbers that are antipalindromic simultaneously in more bases. In his 2010 paper, Bašić showed [2] that for any list of bases, there exists a number with palindromic expansions in each of the bases. This does not necessarily apply to antipalindromic numbers as there exist sets of bases (e.g., 6 and 8) for which our application mode for searching multi-base antipalindromic numbers was unable to find any simultaneous antipalindromic numbers.

In 2014, Bérczes and Ziegler discussed [3] multi-base palindromic numbers and proposed a list of the first 53 numbers palindromic in bases 2 and 10 simultaneously. The simple application that is a part of this bachelor project has only been able to find one number with an antipalindromic expansion in these bases. This number, 3276, is also antipalindromic in other 19 distinct bases, see Table 1.3. The next greater number that is antipalindromic both in base 2 and 10 must be greater than 10^{10} and divisible by 18.

It is not uncommon for a number to be antipalindromic in multiple bases. In this section, we provide a proof that if a number is antipalindromic in a unique base, then the number must be prime or equal to 1, see Theorem 23.

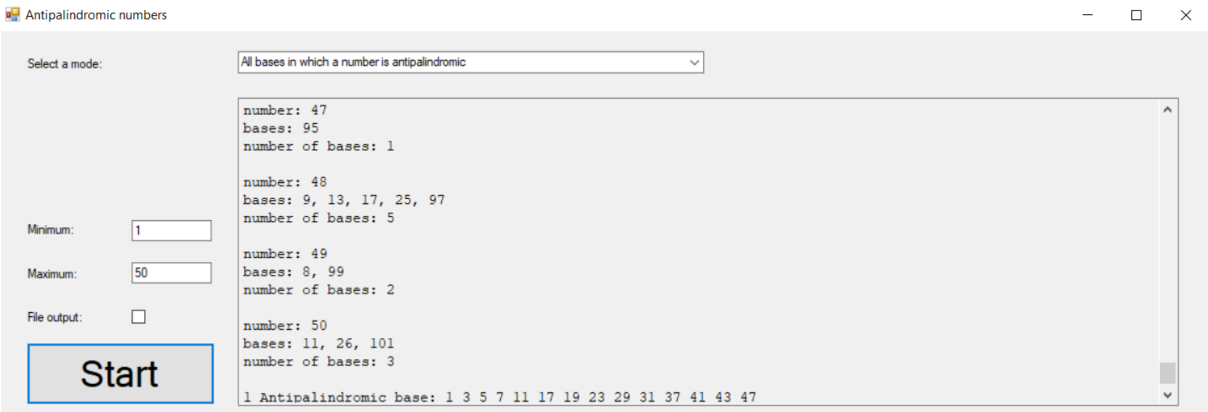

Figure 1.2: Application mode: All bases in which a number is antipalindromic

Definition 6. *An antipalindromic number is called* multi-base *if it is antipalindromic in at least two different bases.*

Observation 1. *Every number $m \in \mathbb{N}$ is antipalindromic in base $2m+1$.*

Example 18. *The number 6192 is a multi-base antipalindromic number because* $(6192)_9 = 8440$ *and* $(6192)_3 = 22111100$.

Theorem 23. *For any composite number $a \in \mathbb{N}$, we can find at least two bases b, c such that this number has an antipalindromic expansion in both of them.*

Proof. Assume that

$$\begin{aligned} a &= m \cdot n, \ m, n \in \mathbb{N}, \ m \geq n, \\ b &= \frac{a}{n} + 1, \\ c &= 2a + 1. \end{aligned}$$

Expansions of a in bases b, c are equal to:

$$\begin{aligned} (a)_{\frac{a}{n}+1} &= (n-1)(\tfrac{a}{n} - n + 1), \\ (a)_{2a+1} &= a. \end{aligned}$$

$\square$

base	expansion
2	110011001100
4	303030
10	3276
64	53 10
79	41 37
85	38 46
92	35 56
118	27 90
127	25 101
157	20 136
183	17 165
235	13 221
253	12 240
274	11 262
365	8 356
469	6 462
547	5 541
820	3 816
1093	2 1090
1639	1 1637
6553	3276

Table 1.3: Antipalindromic expansions of the number 3276 in 21 bases

Theorem 24. *For every $n \in \mathbb{N}$, there exist infinitely many numbers that are antipalindromic in at least n bases.*

Proof. Consider a number a such that $a = (2n)!$. Theorem 23 indicates that the number a is antipalindromic in bases $\frac{a}{2} + 1, \frac{a}{3} + 1, \dots, \frac{a}{n} + 1$ and also $2a + 1$. □

Theorem 25. *Let $b \in \mathbb{N}, b \geq 2$. Then there exists $m \in \mathbb{N}$ such that m is antipalindromic in base b and in at least one more base less than m.*

Proof.

$b = 2$	$m := 12$	$(12)_2 = 1100$	$(12)_4 = 30$
$b = 3$	$m := 72$	$(72)_3 = 2200$	$(72)_9 = 80$
$b \geq 4$	$m := 4 \cdot (b-1)$	$(m)_b = 3\ (b-4)$	$(m)_{2b-1} = 1\ (2b-3)$

□

Theorem 26. *Let $p, q \in \mathbb{N}$ such that $\gcd(p,q) = d$, $p = p' \cdot d$, $q = q' \cdot d$ and $p \geq q' > 1$, $q \geq p' > 1$. Then the number $m = p' \cdot q' \cdot d = p \cdot q' = q \cdot p'$ is antipalindromic in bases $p+1$ and $q+1$.*

Proof. We have

$$(m)_{p+1} = (q'-1)(p+1-q'),$$
$$(m)_{q+1} = (p'-1)(q+1-p').$$

□

Example 19. *Let $p = 4$, $q = 6$, then $\gcd(4,6) = 2$.*
The number $m = 12$ is antipalindromic in bases 5 and 7: $(12)_5 = 22$, $(12)_7 = 15$.

Theorem 27. *Let $b \in \mathbb{N}, b \geq 2$. An antipalindromic number m in base b^n, where $(m)_{b^n} = u_k \dots u_1 u_0$ and $u_k \geq b^{n-1}$, is simultaneously antipalindromic in base b if and only if the expansion of u_j in base b of length n (i.e., completed with zeroes if necessary) is a palindrome for all $j \in \{0, 1, \dots, k\}$.*

Proof. The digits of m in base b^n satisfy $0 \leq u_j \leq b^n - 1$. Let us denote the expansion of u_j in base b by $(u_j)_b = v_{j,n-1} \dots v_{j,1} v_{j,0}$ (where the expansion of u_j in base b is completed with zeroes in order to have the length n if necessary). The antipalindromic complement $A(u_j)$ of u_j in base b^n equals $b^n - 1 - u_j$ and its expansion in base b equals $(A(u_j))_b = (b - 1 - v_{j,n-1}) \dots (b - 1 - v_{j,1})(b - 1 - v_{j,0})$. Since m is antipalindromic in base b^n, we have $u_{k-j} = A(u_j) = b^n - 1 - u_j$ for all $j \in \{0, 1, \dots, k\}$.
Let us now consider the expansion of m in base b: it is obtained by concatenation of the expansions of u_j in base b for $j \in \{0, 1, \dots, k\}$, i.e.,

$$(m)_b = (u_k)_b \dots (u_1)_b (u_0)_b.$$

Following the assertion that $u_k \geq b^{n-1}$, the expansion $(u_k)_b$ starts in a non-zero. Thus, the length of the expansion $(m)_b$ equals $n \cdot |(m)_{b^n}|$.

The number m is antipalindromic in base b if and only if $(u_{k-j})_b = A((u_j)_b)$ for all $j \in \{0, 1, \dots, k\}$, i.e.,

$$\begin{aligned} (b - 1 - v_{j,n-1}) \dots (b - 1 - v_{j,1})(b - 1 - v_{j,0}) &= A(v_{j,n-1} \dots v_{j,1} v_{j,0}) \\ &= (b - 1 - v_{j,0})(b - 1 - v_{j,1}) \dots (b - 1 - v_{j,n-1}). \end{aligned}$$

Consequently, m is antipalindromic in base b if and only if $(u_j)_b = (v_{j,n-1} \dots v_{j,1} v_{j,0})$ is a palindrome for all $j \in \{0, 1, \dots, k\}$. □

Example 20. *Consider $m = 73652$. Then $(m)_{27} = 3\ 20\ 6\ 23 = u_3 u_2 u_1 u_0$, thus m is antipalindromic in base $27 = 3^3$. However, $(m)_3 = 10202020212$, thus m is not antipalindromic in base 3. If we cut $(m)_3$ into blocks of length 3, then all of them are palindromic. However, the first one equals 010 and it starts in zero, hence the assumption $u_3 \geq 9$ of Theorem 27 is not met.*

Example 21. *Consider $b = 10$. The number 6633442277556633 is an antipalindromic number both in base 10 and 100.*

Chapter 2

Description of the application

Both the Visual Studio solution and the standalone application can be found at

`https://github.com/Kruml3/Antipalindromic-numbers`

2.1 Source code

The application that is a part of this project was created using Microsoft Visual Studio 2019 with the source code written in C#. The main source code file consists of a major function *antipalindromy*, where most of the business logic is described, and a few auxiliary functions simplifying calculations and writing the result to the output. More precisely, there are auxiliary functions fulfilling the following tasks:

- Determining whether a number is palindromic or antipalindromic in a given base.
- Calculating powers.
- Passing a text to a textbox or to a file.

The function *antipalindromy* has a special mode for each of the nine selectable tasks. The first mode is only used to show palindromic and antipalindromic numbers in a selected base using the previously mentioned auxiliary functions. That can also be applied to the second mode showing palindromic primes in a given base. The mode for antipalindromic primes in base 3 is the most complex one as the algorithm is swift and can be easily used for numbers exceeding 10^{15}. For each possible expansion length, the function first creates an array of gap lengths between each two antipalindromic numbers using a formula that was derived in Theorem 14. Then, the antipalindromic numbers themselves are calculated. Finally, these numbers are tested for primality and, possibly, passed on to the output. The two modes for multi-base antipalindromic numbers, again, use a combination of the auxiliary functions. Sums of three antipalindromic numbers are computed in the following two modes. An internal resource is used here, containing the information about gap lengths, so as not to calculate them each time. The mode for antipalindromic powers only uses the auxiliary functions. The last mode only recursively calls this function in the previous mode while altering the parameters with each call.

The application also offers the user different sets of parameters depending on the selected mode and checks if all inputs are valid. When one or more of the inputs are invalid or too high, the application notifies the user with a simple error message.

2.2 Application from user perspective

The application first needs to be installed using the executable file *setup.exe*. After the installation and a succesful launch, it presents its only window to the user, see Figure 2.1. The combo box lets the

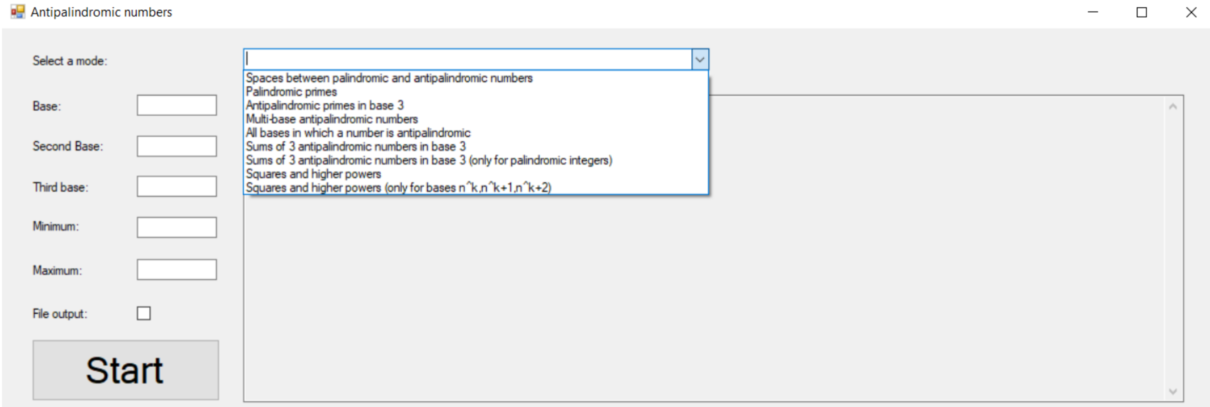

Figure 2.1: The application upon opening

user select one of the nine modes and lists a set of parameters needed to perform the computation. All calculations are launched by pressing the *Start* button.

In the *Spaces between palindromic and antipalindromic numbers* mode, a desired base and a maximum value must be assigned. After that, all palindromic and antipalindromic numbers from the selected range are shown, along with their respective expansions, expansion lengths and, in the end, the maximum number of palindromic numbers between two antipalindromic numbers and vice-versa in the selected range. (See Section 1.3 for details.) The *Palindromic primes* mode is used to calculate all palindromic primes in the desired base and range and their expansions, and shows their number. The mode *Antipalindromic primes in base 3* only needs to be given a maximum value and shows all antipalindromic numbers in base 3 less than the selected value, their expansions, and their number. (There is never more than one antipalindromic prime in any other base, as proven in Theorems 6 and 7.) The *Multi-base antipalindromic numbers* mode lets the user choose two bases and a maximum and presents all numbers from 1 to the maximum that are antipalindromic in both of these bases, and their expansion in each base. After ticking the check box *Three bases*, an additional base can be selected and the application shows only the numbers that are antipalindromic in all three bases at once, see Figure 2.2. Computer experiments using this mode helped us obtain results in Section 1.6.

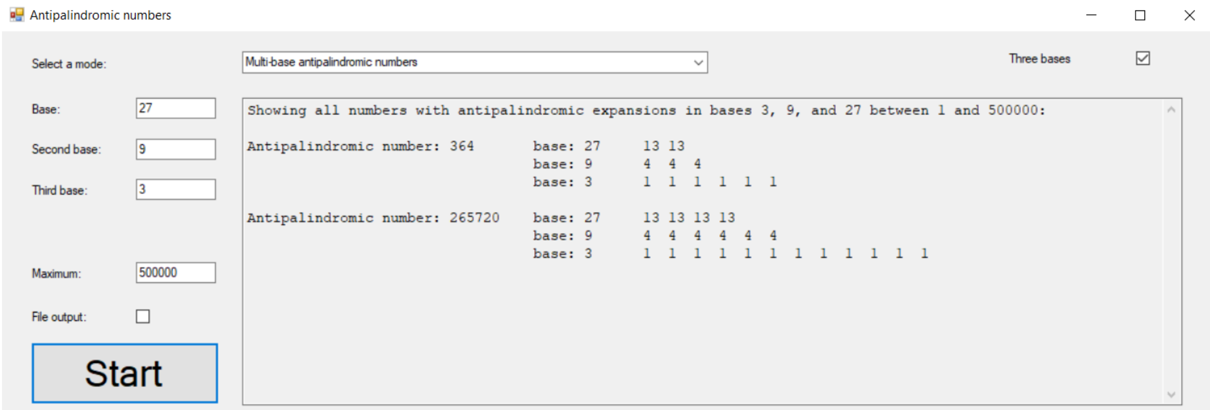

Figure 2.2: Application mode: Multi-base antipalindromic numbers

In the mode *All bases in which a number is antipalindromic*, the user inputs the desired number range and after the launch, the application lists all bases in which each number from the desired range has an antipalindromic expansion and, in the end, lists apart all numbers from the desired range that only have one such base. The following mode, *Sums of 3 antipalindromic numbers in base 3*, again offers the user the choice of a number range and shows one of the possible ways each number can be expressed as a sum of three antipalindromic numbers in base 3. If the number cannot be expressed as such, the application only shows the message 'CANNOT BE A SUM OF THREE ANTIPALINDROMIC NUMBERS IN BASE 3' for this number. The mode *Sums of 3 antipalindromic numbers in base 3 (only for palindromic integers)* is very similar. However, it only lists palindromic numbers from the selected range. The eighth mode, *Squares and higher powers*, needs to be given a maximum base, an exponent, and a maximum value. For each base, from 2 up to the selected one, the application lists all numbers from the selected range that are antipalindromic and have an integer root for the required exponent, along with their respective expansions and their number. Computer experiments using this mode helped us obtain results in Section 1.5. The last mode, *Squares and higher powers (only for bases n^k, n^k+1, n^k+2)*, is used to show the difference between "n^k + 1" bases and the rest, in terms of the number of antipalindromic k-th powers. These bases are likely to have many antipalindromic k-th powers, as stated in Theorem 20. The user only inputs a desired exponent and a maximum value of the number. The optimal maximum base is, in contrast with the previous mode, calculated by the application.

The user is allowed to tick the box *File output*, which opens up a dialogue window, letting the user create an output file in a desired location. The output is then both presented on the screen and written to the text file.

Conclusion

This project carries out a thorough study of antipalindromic numbers and describes analogies and differences between palindromic and antipalindromic numbers. It brings a number of new results and open problems. Let us list them here:

- We determined the number of (anti)palindromic numbers of a certain length and the maximum and minimum number of antipalindromic numbers between palindromic numbers and vice versa.
- We provided an explicit formula for the length of gaps between neighboring antipalindromic numbers.
- We found several classes of antipalindromic squares and higher powers.
- We described pairs of bases such that there is a number antipalindromic in both of these bases. Moreover, we obtained the following interesting results concerning multi-base antipalindromic numbers:
  - For any composite number, there exist at least two bases such that this number is antipalindromic in both of them.
  - For every $n \in \mathbb{N}$, there exist infinitely many numbers that are antipalindromic in at least n bases.
  - Let $b \in \mathbb{N}, b \geq 2$. Then there exists $m \in \mathbb{N}$ such that m is antipalindromic in base b and in at least one more base less than m.

We created a user-friendly application for all the questions studied. Based on computer experiments, we state the following conjectures and open problems:

1. Are there infinitely many antipalindromic primes in base 3? (We know there is never more than one antipalindromic prime in any other base except for 3.) During an extended search, the first 637807 antipalindromic primes have been found.
2. We conjecture it is possible to express any integer number (except for 24, 37, 49, 117, and 421) as the sum of three antipalindromic numbers in base 3. Our computer program shows that the answer is positive up to $5 \cdot 10^6$.
3. We conjecture it is possible to express any palindromic number in base 3 as the sum of three antipalindromic numbers in base 3. This conjecture follows evidently from the previous one, and we verified it even for larger numbers, up to 10^8.
4. Is there a pair of bases such that it is impossible to find any number that has an antipalindromic expansion in both of them? According to our computer experiments, suitable candidates seem to be the bases 6 and 8. It is to be studied in the future.

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Appendices

Lists of antipalindromic numbers

In the first appendix, the first 31 antipalindromic numbers in base $b = 2$ (i.e., all antipalindromic numbers with 10 or fewer digits), the first 53 antipalindromic numbers in base $b = 3$ (i.e., all antipalindromic numbers with 7 or fewer digits), and the first 99 antipalindromic numbers in base $b = 10$ can be found.

List of antipalindromic numbers in base $b = 2$

- Antipalindromic numbers with 2 digits: 2
  $(2)_2 = 10$.

- Antipalindromic numbers with 4 digits: 10, 12
  $(10)_2 = 1010$, $(12)_2 = 1100$.

- Antipalindromic numbers with 6 digits: 38, 42, 52, 56
  $(38)_2 = 100110$, $(42)_2 = 101010$, $(52)_2 = 110100$, $(56)_2 = 111000$.

- Antipalindromic numbers with 8 digits: 142, 150, 170, 178, 204, 212, 232, 240
  $(142)_2 = 10001110$, $(150)_2 = 10010110$, $(170)_2 = 10101010$, $(178)_2 = 10110010$,
  $(204)_2 = 11001100$, $(212)_2 = 11010100$, $(232)_2 = 11101000$, $(240)_2 = 11110000$.

- Antipalindromic numbers with 10 digits: 542, 558, 598, 614, 666, 682, 722, 738, 796, 812, 852, 868, 920, 936, 976, 992
  $(542)_2 = 1000011110$, $(558)_2 = 1000101110$, $(598)_2 = 1001010110$, $(614)_2 = 1001100110$,
  $(666)_2 = 1010011010$, $(682)_2 = 1010101010$, $(722)_2 = 1011010010$, $(738)_2 = 1011100010$,
  $(796)_2 = 1100011100$, $(812)_2 = 1100101100$, $(852)_2 = 1101010100$, $(868)_2 = 1101100100$,
  $(920)_2 = 1110011000$, $(936)_2 = 1110101000$, $(976)_2 = 1111010000$, $(992)_2 = 1111100000$.

List of antipalindromic numbers in base $b = 3$

- Antipalindromic numbers with 1 digit: 1
  $(1)_3 = 1$.

- Antipalindromic numbers with 2 digits: 4, 6
  $(4)_3 = 11$, $(6)_3 = 20$.

- Antipalindromic numbers with 3 digits: 13, 21
  $(13)_3 = 111$, $(21)_3 = 210$.

- Antipalindromic numbers with 4 digits: 34, 40, 46, 60, 66, 72
  $(34)_3 = 1021$, $(40)_3 = 1111$, $(46)_3 = 1201$, $(60)_3 = 2020$, $(66)_3 = 2110$, $(72)_3 = 2200$.

- Antipalindromic numbers with 5 digits: 97, 121, 145, 177, 201, 225
  $(97)_3 = 10121$, $(121)_3 = 11111$, $(145)_3 = 12101$,
  $(177)_3 = 20120$, $(201)_3 = 21110$, $(225)_3 = 22100$.

- Antipalindromic numbers with 6 digits: 268, 286, 304, 346, 364, 382, 424, 442, 460, 510, 528, 546, 588, 606, 624, 666, 684, 702
  $(268)_3 = 100221$, $(286)_3 = 101121$, $(304)_3 = 102021$,
  $(346)_3 = 110211$, $(364)_3 = 111111$, $(382)_3 = 112011$,
  $(424)_3 = 120201$, $(442)_3 = 121101$, $(460)_3 = 122001$,
  $(510)_3 = 200220$, $(528)_3 = 201120$, $(546)_3 = 202020$,
  $(588)_3 = 210210$, $(606)_3 = 211110$, $(624)_3 = 212010$,
  $(666)_3 = 220200$, $(684)_3 = 221100$, $(702)_3 = 222000$.

- Antipalindromic numbers with 7 digits: 781, 853, 925, 1021, 1093, 1165, 1261, 1333, 1405, 1509, 1581, 1653, 1749, 1821, 1893, 1989, 2061, 2133
  $(781)_3 = 1001221$, $(853)_3 = 1011121$, $(925)_3 = 1021021$,
  $(1021)_3 = 1101211$, $(1093)_3 = 1111111$, $(1165)_3 = 1121011$,
  $(1261)_3 = 1201201$, $(1333)_3 = 1211101$, $(1405)_3 = 1221001$,
  $(1509)_3 = 2001220$, $(1581)_3 = 2011120$, $(1653)_3 = 2021020$,
  $(1749)_3 = 2101210$, $(1821)_3 = 2111110$, $(1893)_3 = 2121010$,
  $(1989)_3 = 2201200$, $(2061)_3 = 2211100$, $(2133)_3 = 2221000$.

List of antipalindromic numbers in base $b = 10$

- Antipalindromic numbers with 2 digits: 18, 27, 36, 45, 54, 63, 72, 81, 90

- Antipalindromic numbers with 4 digits: 1098, 1188, 1278, 1368, 1458, 1548, 1638, 1728, 1818, 1908, 2097, 2187, 2277, 2367, 2457, 2547, 2637, 2727, 2817, 2907, 3096, 3186, 3276, 3366, 3456, 3546, 3636, 3726, 3816, 3906, 4095, 4185, 4275, 4365, 4455, 4635, 4725, 4815, 4905, 5094, 5184, 5274, 5364, 5454, 5634, 5724, 5814, 5904, 6093, 6183, 6273, 6363, 6453, 6633, 6723, 6813, 6903, 7092, 7182, 7272, 7362, 7452, 7632, 7722, 7812, 7902, 8091, 8181, 8271, 8361, 8451, 8631, 8721, 8811, 8901, 9090, 9180, 9270, 9360, 9450, 9540, 9630, 9720, 9810, 9900

Lists of multi-base antipalindromic numbers

In the second appendix, the first 17 numbers that are antipalindromic in bases $b = 3$ and $c = 9$ and the first 7 numbers that are antipalindromic in bases $b = 2$, $c = 4$ and $d = 16$ can be found.

List of numbers antipalindromic in bases $b = 3$ and $c = 9$

- Antipalindromic numbers with 2/1 digits: 4
  $((4)_3 = 11, (4)_9 = 4)$

- Antipalindromic numbers with 4/2 digits: 40, 72
  $((40)_3 = 1111, (40)_9 = 44)$, $((72)_3 = 2200, (72)_9 = 80)$

- Antipalindromic numbers with 6/3 digits: 364, 684
  $((364)_3 = 111111, (364)_9 = 444)$, $((684)_3 = 221100, (684)_9 = 840)$

- Antipalindromic numbers with 8/4 digits: 2992, 3280, 3586, 5904, 6192, 6480
  $((2992)_3 = 11002211, (2992)_9 = 4084)$, $((3280)_3 = 11111111, (3280)_9 = 4444)$,
  $((3586)_3 = 11220011, (3586)_9 = 4804)$, $((5904)_3 = 22002200, (5904)_9 = 8080)$,
  $((6192)_3 = 22111100, (6192)_9 = 8440)$, $((6480)_3 = 22220000, (6480)_9 = 8800)$.

- Antipalindromic numbers with 10/5 digits: 26644, 29524, 32404, 5284, 55764, 58644
  $((26644)_3 = 1100112211, (26644)_9 = 40484)$,
  $((29524)_3 = 1111111111, (29524)_9 = 44444)$,
  $((32404)_3 = 1122110011, (32404)_9 = 48404)$,
  $((52884)_3 = 2200112200, (52884)_9 = 80480)$,
  $((55764)_3 = 2211111100, (55764)_9 = 84440)$,
  $((58644)_3 = 2222110000, (58644)_9 = 88400)$.

List of numbers antipalindromic in bases $b = 2$, $c = 4$, and $d = 16$

- Antipalindromic numbers with 8/4/2 digits: 240
  $((240)_2 = 11110000, (240)_4 = 3300, (240)_{16} = 15\ 0)$

- Antipalindromic numbers with 16/8/4 digits: 61680, 65280
  $((61680)_2 = 1111000011110000, (65280)_4 = 33003300, (61680)_{16} = 15\ 0\ 15\ 0)$,
  $((65280)_2 = 1111111100000000, (65280)_4 = 33330000, (65280)_{16} = 15\ 15\ 0\ 0)$,

- Antipalindromic numbers with 24/12/6 digits: 15732720, 15790320, 16715520, 16773120
  $((15732720)_2 = 111100000000111111110000, (15732720)_4 = 330000333300$,
  $(15732720)_{16} = 15\ 0\ 0\ 15\ 15\ 0)$,
  $((15790320)_2 = 111100001111000011110000, (15790320)_4 = 330033003300$,
  $(15790320)_{16} = 15\ 0\ 15\ 0\ 15\ 0)$,
  $((16715520)_2 = 111111110000111100000000, (16715520)_4 = 333300330000$,
  $(16715520)_{16} = 15\ 15\ 0\ 15\ 0\ 0)$,
  $((16773120)_2 = 111111111111000000000000, (16773120)_4 = 333333000000$,
  $(16773120)_{16} = 15\ 15\ 15\ 0\ 0\ 0)$.

Lists of antipalindromic squares

In the third appendix, the first 3 numbers that are squares and antipalindromic in base $b = 2$, the first 10 numbers that are squares and antipalindromic in base $b = 3$, and the first 10 numbers that are squares and antipalindromic in base $b = 10$ can be found. The following square in base $b = 2$ must be greater than 10^{17}.

List of antipalindromic squares in base $b = 2$

- Antipalindromic squares in base $b = 2$: 13924, 56644, 16160400

  $(13924)_2 = 11011001100100$, $(56644)_2 = 1101110101000100$,
  $(16160400)_2 = 111101101001011010010000$.

List of antipalindromic squares in base $b = 3$

- Antipalindromic squares in base $b = 3$: 1, 4, 121, 225, 2500, 302500, 606841, 73427761, 5993701561, 6453390889

  $(1)_3 = 1$, $(4)_3 = 11$, $(121)_3 = 11111$, $(225)_3 = 22100$, $(2500)_3 = 10102121$,
  $(302500)_3 = 120100221201$, $(606841)_3 = 1010211102121$, $(73427761)_3 = 12010011111221201$,
  $(5993701561)_3 = 120110201012120211201$, $(6453390889)_3 = 121122202012020001101$

List of antipalindromic squares in base $b = 10$

- Antipalindromic squares in base $b = 10$: 36, 81, 5184, 367236, 3636813636, 8911548801, 388186318116, 479355446025, 531720972864, 811604593881

Lists of antipalindromic primes

In the fourth appendix, the first 100 antipalindromic primes in base $b = 3$ can be found. For their expansions in base $b = 3$, see Table 2.1.

List of antipalindromic primes in base $b = 3$

- Antipalindromic primes in base $b = 3$: 13, 97, 853, 1021, 1093, 7873, 8161, 8377, 9337, 12241, 62989, 63853, 66733, 74797, 79861, 81373, 82021, 84181, 86413, 91381, 92317, 64477, 95773, 98893, 100189, 101701, 111997, 114157, 534841, 552553, 556441, 560977, 578689, 580633, 591937, 600361, 631249, 637729, 648097, 652921, 663937, 677113, 677113, 681001, 685537, 687481, 698713, 703249, 738121, 742657, 751297, 769081, 795217, 797161, 812281, 814873, 816817, 825241, 84101, 849721, 854257, 873913, 897553, 902089, 913321, 917209, 948169, 954649, 993961, 998497, 1009153, 1022113, 1024921, 1028809, 4832056, 4857973, 4936381, 4944157, 4964461, 5029261, 5225821, 5259517, 5310061, 5397181, 5259517, 5310061, 5397181, 5422453, 5436709, 5456149, 5487253, 5515333, 5534773, 5625493, 5633269, 5673013, 5731981, 5757253, 5790949, 5816221, 5909173, 5968141, 5973973

Prime number	Expansion	Prime number	Expansion
13	111	751297	1102011120211
97	10121	769081	1110001222111
853	1011121	795217	1111101211111
1021	1101211	797161	1111111111111
1093	1111111	812281	1112021020111
7873	101210121	814873	1112101210111
8161	102012021	816817	1112111110111
8377	102111021	825241	1112221000111
9337	110210211	841081	1120201202011
12241	121210101	849721	1121011121011
62989	10012101221	854257	1121101211011
63853	10020120221	873913	1122101210011
66733	10101112121	897553	1200121012201
74797	10210121021	902089	1200211102201
79861	11001112211	913321	1201101211201
81373	11010121211	917209	1201121011201
82021	11011111211	948169	1210011122101
84181	11021110211	954649	1210111112101
86413	11101112111	993961	1212111110101
91381	11122100111	998497	1212201200101
92317	11200122011	1009153	1220021022001
94477	11210121011	1022113	1220221002001
95773	11212101011	1024921	1221001221001
98893	12000122201	1028809	1221021021001
100189	12002102201	4832053	100002111022221
101701	12011111201	4857973	100010210212221
111997	12200122001	4936381	100021210102221
114157	12210121001	4944157	100022012002221
534841	1000011122221	4964461	100100012221221
552553	1001001221221	5029261	100110111211221
556441	1001021021221	5225821	100211111110221
560977	1001111111221	5259517	100220012200221
578689	1002101210221	5310061	100222210000221
580633	1002111110221	5397181	101011012112121
591937	1010001222121	5422453	101012111012121
600361	1010111112121	5436709	101020012202121
631249	1012001220121	5456149	101021012102121
637729	1012101210121	5487253	101022210002121
648097	1012221000121	5515333	101101012121121
652921	1020011122021	5534773	101102012021121
663937	1020201202021	5625493	101120210201121
672577	1021011121021	5633269	101121012101121
677113	1021101211021	5673013	101200012220121
681001	1021121011021	5731981	101210012210121
685537	1021211101021	5757253	101211111110121
687481	1021221001021	5790949	101220012200121
698713	1022111110021	5816221	101221111100121
703249	1022201200021	5909173	102010012212021
738121	1101111111211	5968141	102020012202021
742657	1101201201211	5973973	102020111202021

Table 2.1: Prime antipalindromic numbers and their expansions in base $b = 3$