Bachelor Project

Czech Technical University in Prague

F3

Faculty of Electrical Engineering
Department of Radio Engineering

Differential Forms and Electrodynamics

Josef Gajdůšek

Supervisor: doc. Martin Bohata
Study program: Open Electronic Systems
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ZADÁNÍ BAKALÁŘSKÉ PRÁCE

I. OSOBNÍ A STUDIJNÍ ÚDAJE

Příjmení: **Gajdůšek** Jméno: **Josef** Osobní číslo: **459275**

Fakulta/ústav: **Fakulta elektrotechnická**

Zadávající katedra/ústav: **Katedra radioelektroniky**

Studijní program: **Otevřené elektronické systémy**

II. ÚDAJE K BAKALÁŘSKÉ PRÁCI

Název bakalářské práce:

Diferenciální formy a elektrodynamika

Název bakalářské práce anglicky:

Differential Forms and Electrodynamics

Pokyny pro vypracování:

Student will study a formulation of classical electrodynamics in the framework of differential forms. Besides an introduction to fundamental notions of the theory of differential forms, the work should include an explicit discussion of differential forms on Minkowski spacetime together with their applications to electrodynamics. In particular, Maxwell's equations and their direct consequences should be reformulated by means of differential forms. The work should also contain simple illustrative examples.

Seznam doporučené literatury:

[1] H. Flanders: Differential Forms with Applications to the Physical Sciences, Dover Publications, New York, 1989.
[2] T. Frankel: The Geometry of Physics, Cambridge University Press, Cambridge, 2012.

Jméno a pracoviště vedoucí(ho) bakalářské práce:

doc. RNDr. Martin Bohata, Ph.D., katedra matematiky FEL

Jméno a pracoviště druhé(ho) vedoucí(ho) nebo konzultanta(ky) bakalářské práce:

Datum zadání bakalářské práce: **07.02.2020** Termín odevzdání bakalářské práce: ______________

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doc. RNDr. Martin Bohata, Ph.D. podpis vedoucí(ho) práce	doc. Ing. Josef Dobeš, CSc. podpis vedoucí(ho) ústavu/katedry	prof. Mgr. Petr Páta, Ph.D. podpis děkana(ky)

III. PŘEVZETÍ ZADÁNÍ

Student bere na vědomí, že je povinen vypracovat bakalářskou práci samostatně, bez cizí pomoci, s výjimkou poskytnutých konzultací.
Seznam použité literatury, jiných pramenů a jmen konzultantů je třeba uvést v bakalářské práci.

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Datum převzetí zadání — Podpis studenta

Acknowledgements

I would like to thank my supervisor, doc. Martin Bohata, whose guidance and expertise were invaluable for writing this thesis.

Declaration

Prohlašuji, že jsem předloženou práci vypracoval samostatně a že jsem uvedl veškeré použité informační zdroje v souladu s Metodickým pokynem o dodržování etických principů při přípravě vysokoškolských závěrečných prací.

V Praze, 14. August 2020

Abstract

This thesis deal with the classical theory of electromagnetic field via the framework of differential forms. The first portion contains a short introduction to the theoretical background, while in the second we present the electromagnetic field 2-form, state Maxwell's equations and discuss the electromagnetic potential and the Lorenz gauge. A special attention is given to the invariance of the laws of electrodynamics under isometries of the Minkowski space. The whole theory is illustrated by simple examples.

Keywords: differential forms, electrodynamics, Maxwell's equations, theoretical physics

Supervisor: doc. Martin Bohata
Jugoslavských Partyzánů 1580,
Praha 6

Abstrakt

Tato bakalářská práce se zabývá klasickou teorií elektromagnetického pole vyjádřenou v jazyce diferenciálních forem. První část obsahuje krátký úvod do teoretického pozadí, zatímco ve druhé části uvedeme elektromagnetickou 2-formu, formulujeme Maxwellovy rovnice a pojednáme o elektromagnetickém potenciálu a Lorenzově kalibrační podmínce. Zvláštní pozornost je věnována invarianci zákonů elektrodynamiky pod isometriemi Minkowského prostoru. Celá teorie je ilustrována jednoduchými příklady.

Klíčová slova: diferenciální formy, elektrodynamika, Maxwellovy rovnice, teoretická fyzika

Překlad názvu: Diferenciální formy a elektrodynamika

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Chapter 1

Introduction

The theory of electrodynamics is an important foundation of electrical engineering. First complete formulation is attributed to Maxwell in the late 19th century, however his formulation bears little resemblence to the way we treat electrodynamics today. Bowing to the limited mathematical tools available at the time, Maxwell expressed the laws of electrodynamics in cartesian coordinates as a system of 20 partial differential equations.

Some time later, Hamilton introduced quaternions into physics, expressing Maxwell's equations in his new formalism. Building on his work, Heaviside and Gibbs separated the "curl" and "divergence" from the original quaternionic ∇ operator, bringing Maxwell's equations to the form we know today.

The main purpose of this text is to explore a modern approach to the theory of electromagnetic fields based on tools employed in differential geometry, primarily differential forms. This formulation gained great importance after the introduction of general relativity into physics and it provides a deeper insight into the theoretical structure underlying the laws of electrodynamics.

Moreover, it allows us to perform coordinate transformations in a clear and rigorous way. As we then set the time and space coordinates on equal footings, allowing us to discuss electrodynamics in arbitrary (inertial as well as noninertial) coordinate systems in the Minkowski spacetime.

The whole machinery of differential forms also works on curved spacetime and so the theory of electromagnetic fields can then simply be extended to include interactions with general relativity [MTWK17]. It is also worth noting that differential forms are widely used in many areas of theoretical physics. Besides electrodynamics, they appear, for example, in thermodynamics or general relativity [Sze12].

In Chapter 2, we introduce multiple purely algebraic concepts, important for later the chapters. We construct the vector space of multivectors, including the exterior algebra, with multiplication called the wedge product. Additionally, we investigate the Hodge star map on multivectors.

Chapter 3 deals with tangent and cotangent spaces on open subsets of $\mathbf{R}^n$. We additionally introduce differential forms, the primary objects of interest for the rest of the text. On them, we define the exterior derivative, the codifferential and Laplace-de Rham operators. We also discuss the pullback of differential forms and inner product fields, and how it acts as a method of performing coordinate transformations. The last section presents several concepts of traditional multivariable calculus rephrased in the symbolism of differential forms.

Chapter 4 is devoted to electrodynamics expressed in the language of differential forms. The first section defines the Minkowski space and formulates the Maxwell's equations. Additionally, we discuss the electromagnetic potential, including its uniqueness and the Lorenz gauge condition. We then derive some of the basic consequences, such as the wave and continuity equations. In the next section, we discuss isometries of the Minkowski space and show how they preserve the laws of electromagnetism in different inertial coordinate systems. Several specific families of isometries are also derived from basic principles. The final section is dedicated to electromagnetic fields in vacuum, including several examples such as the relativistic Doppler effect and the electromagnetic field of a moving charged particle.

Chapter 2

Algebraic Concepts

2.1 Dual Spaces

First we are going to introduce the concept of the dual space of a vector space. Dual spaces are critical concepts for the theory of differential forms and are going to accompany us for the entirety of this text. A standard reference is [Axl17] or any other text on advanced linear algebra.

In this text, a *vector space* is taken as a finite dimensional vector space over the real numbers, unless implied otherwise.

Definition 2.1 (Linear form)**.** Let V be a vector space. A *linear form* is then a linear map from V to $\mathbf{R}$.

Definition 2.2 (Dual space)**.** A dual space of a vector space V, which we shall denote by V^*, is the vector space of all linear forms.

Intuitively, linear forms work as "measuring sticks" for elements of our vector space.

Definition 2.3 (Dual basis)**.** Let V be a vector space with a basis denoted $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$. Then we define the dual basis of V^* to be the tuple of linear forms $(\mathbf{f}^1, \ldots, \mathbf{f}^n)$ where each $\mathbf{f}^i$ acts on the basis of V as follows:

$$\mathbf{f}^i(\mathbf{e}_k) = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k. \end{cases}$$

In other words, taking arbitrary $\mathbf{v} = \sum_k \alpha^k \mathbf{e}_k \in V$, we have

$$\mathbf{f}^i(\mathbf{v}) = \mathbf{f}^i\left(\sum_k \alpha^k \mathbf{e}_k\right) = \alpha^i.$$

Proposition 2.4. *Dual basis is a basis of the dual space.*

Proof. First we show that the dual basis generates V^*. Take a $\mathbf{f} \in V^*$ and any $\mathbf{v} \in V$. We then compute the coordinates with respect to the dual basis

$$\mathbf{f}(\mathbf{v}) = \mathbf{f}\left(\sum_i \mathbf{f}^i(\mathbf{v})\, \mathbf{e}_i\right) = \sum_i \mathbf{f}^i(\mathbf{v})\, \mathbf{f}(\mathbf{e}_i) = \left(\sum_i \mathbf{f}(\mathbf{e}_i)\, \mathbf{f}^i\right)(\mathbf{v}).$$

Now we prove that the dual basis is linearly independent. Assume by contradiction that there is a nontrivial linear combination that sums to the zero form $\mathbf{f}_0$, that is

$$\sum_i \sigma_i \mathbf{f}^i = \mathbf{f}_0.$$

Evaluating this sum on any element $\mathbf{e}_k$ of the original basis yields

$$\sigma_k = \left(\sum_i \sigma_i \mathbf{f}^i\right)(\mathbf{e}_k) = \mathbf{f}_0(\mathbf{e}_k) = 0.$$

□

Corollary 2.5. $\dim V = \dim V^*$.

The concept of a dual basis suggests a possible way of identifying a vector space with its dual space. This is not basis independent as Example 2.6 shows. We are going to revisit this concept later in Proposition 2.20.

Example 2.6. Take $\mathbf{R}^2$ with bases $(\mathbf{x}_1, \mathbf{x}_2)$ and $(\mathbf{y}_1, \mathbf{y}_2) = (\mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_2)$. Let $(\mathbf{f}^1, \mathbf{f}^2)$ and $(\mathbf{g}^1, \mathbf{g}^2)$ be the corresponding dual bases.
Denote $\mathbf{v} = \mathbf{x}_1 = \mathbf{y}_1 - \mathbf{y}_2$. Now we compute $\mathbf{f}^2(\mathbf{v}) = 0$ and $\mathbf{g}^2(\mathbf{v}) = -1$. Even though $\mathbf{x}_2 = \mathbf{y}_2$ holds the forms $\mathbf{f}^2$ and $\mathbf{g}^2$ are not equal.

Definition 2.7 (Transpose of a linear map)**.** Let $F : V \to W$ be a linear map. We then define the *transpose* $F^T : W^* \to V^*$ as $F^T : \mathbf{g} \mapsto \mathbf{g} \circ F$. In other words, taking a form $\mathbf{g} \in W^*$ and a vector $\mathbf{v} \in V$ we have $(F^T\mathbf{g})(\mathbf{v}) = \mathbf{g}(F\mathbf{v})$

It can be shown that if we express F as a matrix, then expressing F^T with respect to the dual basis coincides with the usual notion of matrix transposition. This is left to specialized linear algebra texts such as [Axl17].

Definition 2.8 (Multilinear map)**.** Let V be a vector space and let $f : V^k \to \mathbf{R}$ be a map. We call f an *k-linear map*[1] if it is linear in each of its arguments separately. In other words, given any vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}$, any $\alpha \in \mathbf{R}$ and any index $1 \le p \le k$ we have

$$f(\mathbf{v}_1, \ldots, \mathbf{v}_p + \alpha\mathbf{v}, \ldots, \mathbf{v}_k) = f(\mathbf{v}_1, \ldots, \mathbf{v}_p, \ldots, \mathbf{v}_k) + \alpha f(\mathbf{v}_1, \ldots, \mathbf{v}, \ldots, \mathbf{v}_k).$$

Observe that just as it is sufficient to know the value of a linear map on all basis vectors of its domain to fully determine its value on the entire vector space, it is also sufficient to know the value of a k-linear map on every k-tuple of basis vectors.

Definition 2.9 (Alternating multilinear map)**.** Given a k-linear map on V, we call it *alternating* if swapping two adjacent arguments changes the sign of the image. More concretely, given any $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V$ we have

$$f(\mathbf{v}_1, \ldots, \mathbf{v}_p, \mathbf{v}_{p+1}, \ldots, \mathbf{v}_k) = -f(\mathbf{v}_1, \ldots, \mathbf{v}_{p+1}, \mathbf{v}_p, \ldots, \mathbf{v}_k).$$

An important example of an alternating multilinear map is the determinant, which, given an n-dimensional vector space V is an n-linear map on V.

[1] Multilinear maps such as these are also often called *covariant tensors* in the physics literature.

2.2 Exterior Algebra

In this section we shall introduce the concept of exterior algebra. Exterior algebra allows us to take conceptually introduce "orientation" and "length" to subpsaces of a vector space.

Definition 2.10 (Space of multivectors)**.** Let V be a vector space with a basis of $(\mathbf{e}_1, \dots, \mathbf{e}_n)$. We define the *space of multivectors*, denoted by $\Lambda^*(V)$ as the vector space of formal sums of symbols with the form $\mathbf{e}_I$ where $I \subseteq \{1, \dots, n\}$[2]. We are also going to freely interchange our indexing set I with a k-tuple containing the elements of I in ascending order. We also define the *space of k-vectors*, denoted $\Lambda^k(V)$, as the subspace of $\Lambda^*(V)$ generated by considering elements $\mathbf{e}_I$ with $|I| = k$.

To make our life easier, we additionaly define an isomorphism between $\Lambda^1(V)$ and V given by $\mathbf{e}_i \mapsto \mathbf{e}_{\{i\}}$. Therefore from now on, we are not going to make any distinction between the underlying vector space and the space of 1-vectors.

Example 2.11 (2-vectors over $\mathbf{R}^3$)**.** We take $\mathbf{R}^3$ with basis denoted as $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. Then the basis of $\Lambda^2(\mathbf{R}^3)$ is $(\mathbf{e}_{12}, \mathbf{e}_{23}, \mathbf{e}_{13})$ and the basis of $\Lambda^3(\mathbf{R}^3)$ is $(\mathbf{e}_{123})$.

Definition 2.12 (Wedge product)**.** Given a vector space V, we define the *wedge product* as the bilinear map $\wedge : \Lambda^*(V) \times \Lambda^*(V) \to \Lambda^*(V)$ defined by

$$\mathbf{e}_I \wedge \mathbf{e}_J = \begin{cases} \operatorname{sgn} \binom{IJ}{I \cup J} \mathbf{e}_{I \cup J} & \text{if } I \cap J = \emptyset, \\ 0 & \text{if } I \cap J \neq \emptyset, \end{cases}$$

Where IJ denotes concatenation of I and J when expressed as ordered tuples in ascending order and sgn denotes the permutation sign of IJ.

The usual way of visualizing the wedge product of several 1-vectors is as an oriented parallelotope with edges formed by the individual vectors, as illustrated by Figure 2.1. An interactive demonstration of this concept can be found in [Bos].

[2]We are not going to do any arithmetic with the indices, meaning they just serve as symbols. We are thus free to start indexing from 0 later.

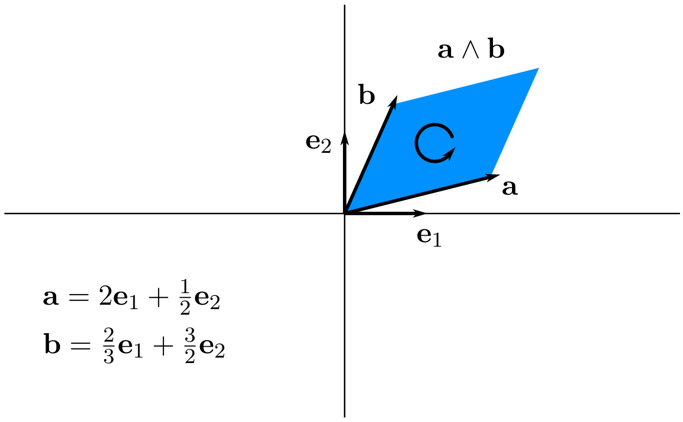

Figure 2.1: Wedge product of two vectors on $\mathbf{R}^2$

Proposition 2.13. *For any $\mathbf{e}_I \in \Lambda^k(V)$ we have*

$$(\alpha \mathbf{e}_\emptyset) \wedge \mathbf{e}_I = \alpha \mathbf{e}_I.$$

In other words, the space of 0-vectors together with the wedge product can be used to perform scalar multiplication of k-vectors.

Proof. By the definition of the wedge product we have

$$(\alpha \mathbf{e}_\emptyset) \wedge \mathbf{e}_I = \alpha \left(\mathbf{e}_\emptyset \wedge \mathbf{e}_I\right) = \alpha \left(\operatorname{sgn} \begin{pmatrix} \emptyset I \\ \emptyset \cup I \end{pmatrix} \mathbf{e}_{\emptyset \cup I} \right) = \alpha \mathbf{e}_I.$$

□

Proposition 2.14 (Properties of the exterior algebra)**.** *Let V be an n-dimensional vector space. Then the following properties hold:*

- $\dim \Lambda^k(V) = \binom{n}{k} = \binom{n}{n-k} = \dim \Lambda^{n-k}(V)$.
- $\wedge$ *is associative.*
- *For any basis k-vector $\mathbf{e}_I \in \Lambda^k(V)$ where $I = (i_1, \dots, i_k)$ we have*

  $$\mathbf{e}_I = \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}.$$

- *For any $\boldsymbol{\omega} \in \Lambda^k(V)$ and $\boldsymbol{\tau} \in \Lambda^l(V)$ we have $\boldsymbol{\omega} \wedge \boldsymbol{\tau} = (-1)^{kl} \boldsymbol{\tau} \wedge \boldsymbol{\omega}$.*
- *1-vectors $\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_k \in \Lambda^1(V)$ are linearly dependent if and only if*

  $$\boldsymbol{\omega}_1 \wedge \cdots \wedge \boldsymbol{\omega}_k = \mathbf{0}.$$

The proofs are computationally intensive and left to a specialized text such as [KST02] or [Fra17].

Example 2.15 (Wedge product on 1-vectors in $\mathbf{R}^3$)**.** Given arbitrary two vectors $\mathbf{v} = \alpha^1\mathbf{e}_1 + \alpha^2\mathbf{e}_2 + \alpha^3\mathbf{e}_3$ and $\mathbf{w} = \beta^1\mathbf{e}_1 + \beta^2\mathbf{e}_2 + \beta^3\mathbf{e}_3$ in $\mathbf{R}^3$, we compute their wedge product.

$$\begin{aligned}\mathbf{v} \wedge \mathbf{w} &= (\alpha^1\mathbf{e}_1 + \alpha^2\mathbf{e}_2 + \alpha^3\mathbf{e}_3) \wedge (\beta^1\mathbf{e}_1 + \beta^2\mathbf{e}_2 + \beta^3\mathbf{e}_3) \\ &= (\alpha^1\beta^2 - \alpha^2\beta^2)\mathbf{e}_1 \wedge \mathbf{e}_2 + (\alpha^2\beta^3 - \alpha^3\beta^2)\mathbf{e}_2 \wedge \mathbf{e}_3 + (\alpha^3\beta^1 - \alpha^1\beta^3)\mathbf{e}_3 \wedge \mathbf{e}_1.\end{aligned}$$

Notice that this is very similar to the standard cross product formula, except our result is a 2-vector. This illustrates the important concept of axial vectors found in many areas of physics.

As the vector spaces of 2-vectors and 1-vectors are of equal dimension in $\mathbf{R}^3$, we could attempt to identify them and keep working with 1-vectors only. However, as will be illustrated in Example 4.1, this might often be misleading.

2.3 Inner Product Spaces

Definition 2.16 (Inner product)**.** Let $\langle -|-\rangle : V \times V \to \mathbf{R}$. Then we call $\langle -|-\rangle$ an *inner product* if it satisties the following three properties.

- (Bilinearity)
  For any $\mathbf{v}, \mathbf{w}, \mathbf{u} \in V$ and $\alpha \in \mathbf{R}$ we have $\langle \alpha\mathbf{v} + \mathbf{w}|\mathbf{u}\rangle = \alpha\langle \mathbf{v}|\mathbf{u}\rangle + \langle \mathbf{w}|\mathbf{u}\rangle$
- (Symmetry)
  For any $\mathbf{v}, \mathbf{w} \in V$ we have $\langle \mathbf{v}|\mathbf{w}\rangle = \langle \mathbf{w}|\mathbf{v}\rangle$
- (Nondegeneracy)
  Given a nonzero $\mathbf{v} \in V$, there always exists $\mathbf{w} \in V$ such that $\langle \mathbf{v}|\mathbf{w}\rangle \neq 0$

  We call a vector space equipped with an inner product an *inner product space.*

This definition of the inner product is somewhat weaker than usually used in literature (which assumes positive definiteness). Some of the differences are highlighted in Example 2.17.

Example 2.17 (Minkowski inner product on $\mathbf{R}^4$)**.** We define the *Minkowski inner product* on $\mathbf{R}^4$ as

$$\langle (\alpha^0, \alpha^1, \alpha^2, \alpha^3)^T | (\beta^0, \beta^1, \beta^2, \beta^3)^T \rangle = -\alpha^0\beta^0 + \alpha^1\beta^1 + \alpha^2\beta^2 + \alpha^3\beta^3.$$

The Minkowski inner product is not positive definite. This yields several interesting observations. If we use this inner product to define the concept of "length", we have zero-length vectors other than the zero vector. Likewise if we define orthogonality using this inner product, we have nonzero vectors which are orthogonal to themselves.

Definition 2.18 (Orthonormal basis)**.** Let V be an inner product space. We call a basis $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ of V *orthonormal* if the following holds for any $\mathbf{e}_i, \mathbf{e}_k$.

$$\langle \mathbf{e}_i | \mathbf{e}_k \rangle = \begin{cases} \pm 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k. \end{cases}$$

Proposition 2.19. *Any inner product space has an orthonormal basis.*

This can be proved using the standard theorem on diagonalization of symmetric matrices. Note that the orthonormal basis is not determined uniquely.

Proposition 2.20 (Musical isomorphisms)**.** *Given an inner product space V, we can define a linear map*[3] $(-)^\flat : V \to V^*$ *by* $\mathbf{v}^\flat = \langle \mathbf{v} | - \rangle$*. This map is an isomorphism and we are going to denote its inverse by* $(-)^\sharp$*.*
In other words, given a linear form $\mathbf{f} \in V^*$*, we can always find a unique vector* $\mathbf{v} \in V$*, such that for all* $\mathbf{w} \in V$*, we have* $\mathbf{f}(\mathbf{w}) = \langle \mathbf{v} | \mathbf{w} \rangle$*.*

Proof. Linearity of $(-)^\flat$ follows from bilinearity of the inner product. By nondegeneracy of the inner product, $(-)^\flat$ is also necessarily injective. Combining this with Corollary 2.5 and the Rank-Nullity theorem also means that $(-)^\flat$ is surjective. □

Definition 2.21 (Linear isometry)**.** Let V and W be two inner product spaces and let $\langle - | - \rangle_V$ and $\langle - | - \rangle_W$ denote their respective inner products. We call an isomorphism $T : V \to W$ a *linear isometry* if, for any two vectors $\mathbf{x}, \mathbf{y} \in V$ we have $\langle \mathbf{x} | \mathbf{y} \rangle_V = \langle T\mathbf{x} | T\mathbf{y} \rangle_W$.

Definition 2.22 (Contravariant inner product)**.** Let V be an inner product space. We then define the *contravariant inner product on* V^*, temporarily denoted $\langle - | - \rangle^* : V^* \times V^* \to \mathbf{R}$, for every $\mathbf{f}, \mathbf{g} \in V^*$, as

$$\langle \mathbf{f} | \mathbf{g} \rangle^* = \langle \mathbf{f}^\sharp | \mathbf{g}^\sharp \rangle.$$

It is easy to verify that this indeed defines a valid inner product on V^*. Additionally, this definition automatically makes $(-)^\sharp$ a linear isometry between V and V^*.

From now on, we shall not make an explicit distinction between the covariant and contravariant inner products. The type of the arguments shall determine which one is to be used.

Remark 2.23. This construction is much more useful in matrix form. First we pick a basis of an inner product space V and then represent elements of V as column vectors with components given by their coordinates. We also represent elements of V^* as row vectors with respect to the dual basis.

Afterwards, we can express the inner product of any $\mathbf{x}, \mathbf{y} \in V$ as $\langle \mathbf{x} | \mathbf{y} \rangle = \mathbf{x}^T \mathbf{G} \mathbf{y}$, where $\mathbf{G}$ is a symmetric square matrix. We then have $\mathbf{x}^\flat = \mathbf{x}^T \mathbf{G}$ and $\mathbf{f}^\sharp = \mathbf{G}^{-1} \mathbf{f}^T$ for a linear form $\mathbf{f} \in V^*$. This means we have

$$\langle \mathbf{f} | \mathbf{h} \rangle^* = \left(\mathbf{G}^{-1} \mathbf{f}^T \right)^T \mathbf{G} \left(\mathbf{G}^{-1} \mathbf{h}^T \right) = \mathbf{f} \mathbf{G}^{-1} \mathbf{G} \left(\mathbf{h} \mathbf{G}^{-1} \right)^T = \mathbf{f} \mathbf{G}^{-1} \mathbf{h}^T.$$

[3] We read $\mathbf{v}^\flat$ as "v-flat" and $\mathbf{f}^\sharp$ as "f-sharp". These symbols come from music notation where $\flat$ means "lower in pitch" and $\sharp$ means "higher in pitch".

Therefore to compute the matrix of the contravariant inner product, we just need to invert the matrix of the original inner product.

2.4 Hodge Star

In this section we shall introduce the concept of the *Hodge star* map. This is a linear map which identifies k-vectors with $(n-k)$-vectors in a way that is consistent with a given inner product.

Definition 2.24 (Inner product on k-vectors)**.** Let V be an inner product space with a basis denoted as $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$. We then define an inner product on $\Lambda^k(V)$ as

$$\langle \mathbf{e}_I | \mathbf{e}_J \rangle = \langle \mathbf{e}_{i_1 \ldots i_k} | \mathbf{e}_{j_1 \ldots j_k} \rangle = \det \begin{pmatrix} \langle \mathbf{e}_{i_1} | \mathbf{e}_{j_1} \rangle & \langle \mathbf{e}_{i_1} | \mathbf{e}_{j_2} \rangle & \ldots & \langle \mathbf{e}_{i_1} | \mathbf{e}_{j_k} \rangle \\ \langle \mathbf{e}_{i_2} | \mathbf{e}_{j_1} \rangle & \langle \mathbf{e}_{i_2} | \mathbf{e}_{j_2} \rangle & \ldots & \langle \mathbf{e}_{i_2} | \mathbf{e}_{j_k} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{e}_{i_k} | \mathbf{e}_{j_1} \rangle & \langle \mathbf{e}_{i_k} | \mathbf{e}_{j_2} \rangle & \ldots & \langle \mathbf{e}_{i_k} | \mathbf{e}_{j_k} \rangle \end{pmatrix},$$

extending by bilinearity to the entire vector space. The matrix of inner products is usually called the *Gram matrix*. To make our definition work for 0-vectors, we consider the determinant of the empty matrix to be 1.

The fact that this inner product is a valid inner product can be easily verified by picking an orthonormal basis and using basic properties of the determinant. It can also be shown that this construction is, in fact, independent of the basis chosen.

Remark 2.25. We are often going to deal with an orthonormal basis. This means that the Gramian matrix is going to have non-zero entries only on the main diagonal. Therefore our definition of the inner product reduces to

$$\langle \mathbf{e}_I | \mathbf{e}_J \rangle = \langle \mathbf{e}_{i_1 \ldots i_k} | \mathbf{e}_{j_1 \ldots j_k} \rangle = \prod_{p=1}^{k} \langle \mathbf{e}_{i_p} | \mathbf{e}_{j_p} \rangle.$$

This also means that if I and J are distinct, our inner product is automatically zero.

Definition 2.26 (Volume form)**.** Given an n-dimensional vector space V, we call any element $\boldsymbol{\sigma} \in \Lambda^n(V)$ a *volume form* if it is an orthonormal basis of $\Lambda^n(V)$. Additionally, we define the *metric sign*[4] as $s = \langle \boldsymbol{\sigma} | \boldsymbol{\sigma} \rangle = \pm 1$.

The volume form is not unique, however we do not have too many choices, as the following proposition shows.

Proposition 2.27 (A volume form is determined up to a sign)**.** *Given two volume forms $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$, we have $\langle \boldsymbol{\sigma}_1 | \boldsymbol{\sigma}_1 \rangle = \langle \boldsymbol{\sigma}_2 | \boldsymbol{\sigma}_2 \rangle$ and also $\boldsymbol{\sigma}_1 = \pm \boldsymbol{\sigma}_2$.*

Proof. As volume forms are bases by assumption, we have

$$\boldsymbol{\sigma}_1 = \alpha \boldsymbol{\sigma}_2$$

[4] Note that some authors omit this term. Including it makes our sign convention agree with several popular Computer Algebra Systems as outlined in Appendix A.

for some real nonzero α. We then have

$$\langle \boldsymbol{\sigma}_1 | \boldsymbol{\sigma}_1 \rangle = \alpha^2 \langle \boldsymbol{\sigma}_2 | \boldsymbol{\sigma}_2 \rangle.$$

Volume forms are orthonormal, so we either have $\langle \boldsymbol{\sigma}_1 | \boldsymbol{\sigma}_1 \rangle = \langle \boldsymbol{\sigma}_2 | \boldsymbol{\sigma}_2 \rangle$ or $\langle \boldsymbol{\sigma}_1 | \boldsymbol{\sigma}_1 \rangle = -\langle \boldsymbol{\sigma}_2 | \boldsymbol{\sigma}_2 \rangle$. The second choice yields $1 = -\alpha^2$, which has no real solutions and thus violates the first assumption, proving that $\langle \boldsymbol{\sigma}_1 | \boldsymbol{\sigma}_1 \rangle = \langle \boldsymbol{\sigma}_2 | \boldsymbol{\sigma}_2 \rangle$. This then results in $1 = \alpha^2$, constraining α to either 1 or -1. □

Now we consider a volume form $\boldsymbol{\sigma}$ and take a fixed $\boldsymbol{\lambda} \in \Lambda^k(V)$. We define the linear map

$$\begin{aligned} \varphi_{\boldsymbol{\lambda}} : \Lambda^{n-k}(V) &\to \Lambda^n(V), \\ \boldsymbol{\mu} &\mapsto \boldsymbol{\lambda} \wedge \boldsymbol{\mu}. \end{aligned}$$

As $\varphi_{\boldsymbol{\lambda}}$ is a linear map and the space of n-vectors is one-dimensional, we can define a linear form $\mathbf{f}_{\boldsymbol{\lambda}} \in \Lambda^{n-k}(V)^*$ such that

$$\boldsymbol{\lambda} \wedge \boldsymbol{\mu} = s\mathbf{f}_{\boldsymbol{\lambda}}(\boldsymbol{\mu})\,\boldsymbol{\sigma}$$

holds. Now we use Proposition 2.20 to find a unique vector $\star\boldsymbol{\lambda} \in \Lambda^{n-k}(V)$ such that $\star\boldsymbol{\lambda} = \mathbf{f}_{\boldsymbol{\lambda}}{}^\sharp$ Ultimately, for any $\boldsymbol{\mu}$, we have

$$\boldsymbol{\lambda} \wedge \boldsymbol{\mu} = s\langle \star\boldsymbol{\lambda} | \boldsymbol{\mu} \rangle \boldsymbol{\sigma}.$$

This discussion leads to the following definition.

Definition 2.28 (Hodge star)**.** The map $\star : \Lambda^k(V) \to \Lambda^{n-k}(V)$ is called a *Hodge star* if

$$\boldsymbol{\lambda} \wedge \boldsymbol{\mu} = s\langle \star\boldsymbol{\lambda} | \boldsymbol{\mu} \rangle \boldsymbol{\sigma}$$

for all $\boldsymbol{\lambda} \in \Lambda^k(V)$ and all $\boldsymbol{\mu} \in \Lambda^{n-k}(V)$.

Proposition 2.29. *The Hodge star is a linear map.*

Proof. We take any $\boldsymbol{\lambda}, \boldsymbol{\omega} \in \Lambda^k(V)$ and $\alpha \in \mathbf{R}$. We now have, for any $\boldsymbol{\mu} \in \Lambda^{n-k}(V)$, the following:

$$\begin{aligned} s\langle \star(\alpha\boldsymbol{\lambda} + \boldsymbol{\omega}) | \boldsymbol{\mu} \rangle \boldsymbol{\sigma} &= (\alpha\boldsymbol{\lambda} + \boldsymbol{\omega}) \wedge \boldsymbol{\mu} \\ &= \alpha\,(\boldsymbol{\lambda} \wedge \boldsymbol{\mu}) + \boldsymbol{\omega} \wedge \boldsymbol{\mu} \\ &= (\alpha s\langle \star\boldsymbol{\lambda} | \boldsymbol{\mu} \rangle + s\langle \star\boldsymbol{\omega} | \boldsymbol{\mu} \rangle)\,\boldsymbol{\sigma}. \end{aligned}$$

As this holds for any $\boldsymbol{\mu}$, linearity is established by non-degeneracy of the inner product. □

Example 2.30 (Hodge star on $\mathbf{R}^2$ with the standard inner product)**.** We take an orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2)$ of $\mathbf{R}^2$ with the standard inner product given by $\langle \alpha^1\mathbf{e}_1 + \alpha^2\mathbf{e}_2 | \beta^1\mathbf{e}_1 + \beta^2\mathbf{e}_2 \rangle = \alpha^1\beta^1 + \alpha^1\beta^2$.
First let us compute $\star\mathbf{e}_1$. For this, we consider the equations

$$\begin{aligned} \mathbf{e}_1 \wedge \mathbf{e}_1 = 0\boldsymbol{\sigma} = \langle \star\mathbf{e}_1 | \mathbf{e}_1 \rangle \boldsymbol{\sigma}, \\ \mathbf{e}_1 \wedge \mathbf{e}_2 = 1\boldsymbol{\sigma} = \langle \star\mathbf{e}_1 | \mathbf{e}_2 \rangle \boldsymbol{\sigma}. \end{aligned}$$

This obviously requires that $\star\mathbf{e}_1 = \mathbf{e}_2$. Similarly for $\star\mathbf{e}_2$, we have

$$\begin{aligned}\mathbf{e}_2 \wedge \mathbf{e}_1 &= -1\boldsymbol{\sigma} = \langle \star\mathbf{e}_2 | \mathbf{e}_1 \rangle \boldsymbol{\sigma}, \\ \mathbf{e}_2 \wedge \mathbf{e}_2 &= 0\boldsymbol{\sigma} = \langle \star\mathbf{e}_2 | \mathbf{e}_2 \rangle \boldsymbol{\sigma}.\end{aligned}$$

Which forces $\star\mathbf{e}_2 = -\mathbf{e}_1$. In this case, the Hodge star corresponds to a quarter turn counterclockwise rotation.

Proposition 2.31 (Additional properties of the Hodge star). *Let V be an inner product space, let $\boldsymbol{\lambda} \in \Lambda^k(V)$, let $\boldsymbol{\sigma} \in \Lambda^n(V)$ be the volume form and let $s = \langle \boldsymbol{\sigma} | \boldsymbol{\sigma} \rangle$. Then the following identities hold.*

- *(Involution up to a sign)*
  $\star\star\boldsymbol{\lambda} = s(-1)^{k(n-k)}\boldsymbol{\lambda}$.
- *(Inverse map)*
  $\star^{-1} = s(-1)^{k(n-k)}\star$.
- *(Hodge star of the basis 0-vector)*
  $\star\mathbf{e}_\emptyset = \boldsymbol{\sigma}$.
- *(Hodge star of the volume form)*
  $\star\boldsymbol{\sigma} = s\mathbf{e}_\emptyset$.

The relationship of the Hodge star with the wedge product is summarized in the following proposition:

Proposition 2.32. *For any two k-vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, we have*

$$\boldsymbol{\alpha} \wedge \star\boldsymbol{\beta} = \langle \boldsymbol{\alpha} | \boldsymbol{\beta} \rangle \boldsymbol{\sigma}.$$

Proof. We compute using Definition 2.26 and Proposition 2.31:

$$\begin{aligned}\boldsymbol{\alpha} \wedge \star\boldsymbol{\beta} &= (-1)^{k(n-k)} \star\boldsymbol{\beta} \wedge \boldsymbol{\alpha} \\ &= (-1)^{k(n-k)} s\langle \star\star\boldsymbol{\beta} | \boldsymbol{\alpha} \rangle \boldsymbol{\sigma} \\ &= (-1)^{2k(n-k)} s^2 \langle \boldsymbol{\beta} | \boldsymbol{\alpha} \rangle \boldsymbol{\sigma} \\ &= \langle \boldsymbol{\alpha} | \boldsymbol{\beta} \rangle \boldsymbol{\sigma}.\end{aligned}$$

□

This grants us a convenient formula for both computing the Hodge star and for computing the inner product. The example below illustrates this process.

Example 2.33. We consider $\mathbf{R}^3$ with the standard inner product and an orthonormal basis $(\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_2)$. To compute $\star\mathbf{e}_2$, we have

$$\mathbf{e}_2 \wedge \star\mathbf{e}_2 = \langle \mathbf{e}_2 | \mathbf{e}_2 \rangle \boldsymbol{\sigma} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3.$$

This requires that we "complement" $\mathbf{e}_2$ with the other basis vectors to get the volume form and thus we can easily see $\star\mathbf{e}_2 = \mathbf{e}_3 \wedge \mathbf{e}_1$.

Chapter 3

Differential Forms

In this chapter, we shall introduce differential forms, which are the main tool we are going to use in the rest of this text. The main idea here is to take the constructs of the previous chapter and attach them to every point of an open subset of $\mathbf{R}^n$, in a sufficiently smooth fashion as to avoid the pitfalls of real analysis.

Our treatment will be limited in scope, avoiding discussion of differentiable manifolds entirely. This is a fascinating topic in and of itself, with elementary introductions available in [Tu11] or [Fra17].

3.1 Tangent and Cotangent Spaces

Definition 3.1 (Domains)**.** We are going to call and open subset Ω of $\mathbf{R}^n$ an *n-domain* or just *domain* if the dimension is not important. The vector space of smooth real-valued functions on Ω shall be denoted as $C^\infty(\Omega)$. In coordinates, this means that for any function $f : \Omega \to \mathbf{R}$ in $C^\infty(\Omega)$, all its partial derivatives of every degree exist and are continuous.

We start off by defining the concept of the *tangent space*. In general there are two equivalent ways of constructing tangent spaces, both of which are equally important. For brevity, we are going to use a construction based on point derivations of smooth functions, the alternative being a construction based on tangent vectors to smooth curves on a domain. An interested reader may consult [Lee09] or [Fra17] for further details.

Definition 3.2 (Derivation at a point and the tangent space)**.** Let Ω be a domain and let $p \in \Omega$. We then call any linear map $\mathbf{v}_p : C^\infty(\Omega) \to \mathbf{R}$ a *derivation at point p* if, for any $f, g \in C^\infty(\Omega)$, it satisties the product rule

$$\mathbf{v}_p(fg) = \mathbf{v}_p(f)g(p) + f(p)\mathbf{v}_p(g).$$

It can be shown that the set of all derivations at a point is a vector space with the usual definitions of scalar multiplication and vector addition.
We call this vector space the *tangent space at p* and denote it as Ω_p.

Definition 3.3 (Basis of the tangent space). Let Ω be an n-domain and let $p = (x^1, \ldots, x^n) \in \Omega$. We then define

$$\boldsymbol{\partial}_{x^i}|_p(f) = \frac{\partial f}{\partial x^i}(p).$$

It can be shown that $\boldsymbol{\partial}_{x^i}|_p$ is a derivation and additionally that the n-tuple

$$(\boldsymbol{\partial}_{x^1}|_p, \ldots, \boldsymbol{\partial}_{x^n}|_p)$$

then forms a basis of Ω_p.
A more detailed discussion can be found in [Lee09].

One may look at the tangent space as an additional n-dimensional vector space which is attached at every point of Ω. It is important to note however, that there might not be any "natural" way of identifiying vectors in a tangent space with coordinate vectors of Ω (in fact, the coordinate "vectors" do not usually even form a vector space, we only care about the topological structure of Ω) or vectors in tangent spaces at two different points, even though they have the same dimension. Visually, this is illustrated by Figure 3.1.

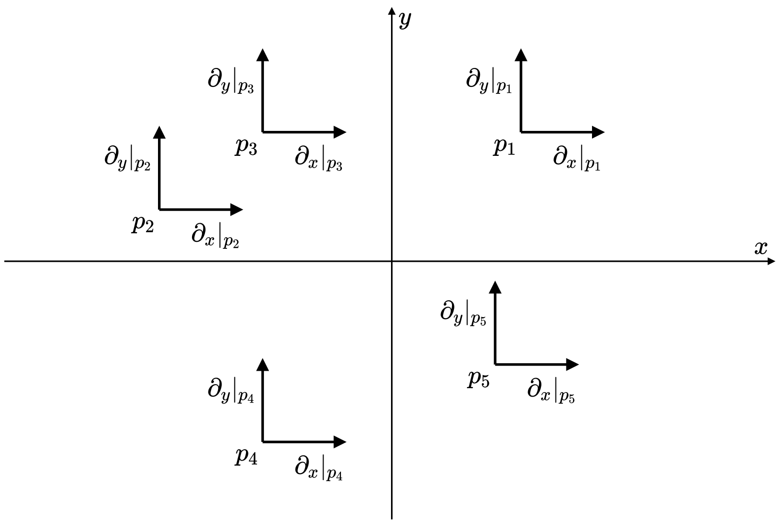

Figure 3.1: Visualization of the tangent spaces of $\mathbf{R}^2$

Visually, it is a good idea to think of objects in Ω as "points in space" and objects in Ω_p as "arrows originating at p".

Definition 3.4 (Cotangent space). We define the *cotangent space at* p, denoted as Ω_p^* to be the dual space of a tangent space Ω_p. Given a basis

$$(\boldsymbol{\partial}_{x^1}|_p, \ldots, \boldsymbol{\partial}_{x^n}|_p)$$

of the tangent space, we are going to denote the dual basis by

$$(\mathrm{d}x^1|_p, \ldots, \mathrm{d}x^n|_p).$$

To make our life easier, we shall often omit the $|_p$ symbolism if the base point is obvious from context. Additionally, if our coordinates are numbered and we are dealing only with one set of such coordinates, we are going to denote the basis tangent vectors as $\boldsymbol{\partial}_{x^i} = \boldsymbol{\partial}_i$.

As cotangent vectors are linear forms on the tangent space, we would like to extend this concept to cotangent k-vectors. The following definition provides such a concept, constructing an isomorphism between cotangent k-vectors and the vector space of alternating k-linear forms.

Definition 3.5. Given an n-domain Ω, a point $p \in \Omega$, a basis cotangent k-vector $\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \in \Lambda^k(\Omega_p^*)$ we define its value on any k-tuple of tangent basis vectors $\boldsymbol{\partial}_{j_1}, \ldots, \boldsymbol{\partial}_{j_k}$ at p as

$$(\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k})(\boldsymbol{\partial}_{j_1}, \ldots, \boldsymbol{\partial}_{j_k}) = \begin{cases} \operatorname{sgn} \binom{i_1 \ldots i_k}{j_1 \ldots j_k} & \text{if } \{i_1, \ldots, i_k\} = \{j_1, \ldots, j_p\}, \\ 0 & \text{otherwise.} \end{cases}$$

Further we extend this definition such that $\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p}$ is a multilinear map on Ω_p. This map is also alternating. Note that alternating maps are closed under addition and as such any cotangent k-vector can be thought of as an an alternating multilinear map.

One may construct k-vectors, including the wedge product, as alternating linear maps on the tangent space directly, which sacrifices some of the visual intuition for a more robust foundation. See [Tu11] for further details.

3.2 Differential Forms

This section serves to introduce *differential forms*. These will be the primary objects of our interest in the rest of this text.

Definition 3.6 (Vector field)**.** Let Ω be an n-domain. We then define a *vector field on* Ω to be a smooth map defined on Ω which assigns every point $p \in \Omega$ an element of Ω_p.
In coordinates, for a vector field $\mathbf{v}$, we may write

$$\mathbf{v}(x^1, \ldots, x^n) = \sum_{i=1}^{n} v^i(x^1, \ldots, x^n)\boldsymbol{\partial}_i$$

where the components v^i are elements of $C^\infty(\Omega)$. We are going to denote the vector space of all vector fields on Ω as $\mathfrak{X}(\Omega)$.

Definition 3.7 (Differential form)**.** Let Ω be an n-domain. We then define a *differential k-form on* Ω as a smooth map defined on Ω which assigns every point $p \in \Omega$ an element of $\Lambda^k(\Omega_p^*)$.
In coordinates, for a differential form $\boldsymbol{\omega}$, we may write

$$\boldsymbol{\omega}(x^1, \ldots, x^n) = \sum_{|I|=k} \omega_I(x^1, \ldots, x^n)\mathrm{d}x^I.$$

Where the functions ω_I are smooth for every I. For further brevity, we denote the vector space of all k-forms defined on Ω as $\mathcal{E}^k(\Omega)$. As suggested by Proposition 2.13, we are going to identify $C^\infty(\Omega)$ with $\mathcal{E}^0(\Omega)$ by $f \mapsto f\mathrm{d}x^\emptyset$.

Definition 3.8 (Frames and coframes)**.** The tuple of vector fields $(\boldsymbol{\partial}_{x^1}, \ldots, \boldsymbol{\partial}_{x^n})$ is called a *frame* and the tuple of differential 1-forms $(\mathrm{d}x^1, \ldots, \mathrm{d}x^n)$ is called a *coframe*.

Definition 3.9 (Exterior derivative)**.** Let Ω be an n-domain. We then define the *exterior derivative* as a map[1] $\mathrm{d} : \mathcal{E}^k(\Omega) \to \mathcal{E}^{k+1}(\Omega)$ given by

$$\mathrm{d}\boldsymbol{\omega}(x) = \sum_{|I|=k} \sum_{i=1}^{n} \frac{\partial \omega_I}{\partial x^i}(x) \mathrm{d}x^i \wedge \mathrm{d}x^I.$$

For the sake of expediency, we have chosen a rather inelegant definition of the exterior derivative. The reader may consult [Fra17] for a more axiomatic definition.

Proposition 3.10 (The differential)**.** *For a* 0*-form* $\boldsymbol{\omega} = \omega_\emptyset \, \mathrm{d}x^\emptyset \in \mathcal{E}^0(\Omega)$*, we have*

$$\mathrm{d}\boldsymbol{\omega}\,(x) = \sum_{i=1}^{n} \frac{\partial \omega_\emptyset}{\partial x^i} \, \mathrm{d}x^i \, .$$

Proof. Directly from the definition, we get

$$\begin{aligned} \mathrm{d}\boldsymbol{\omega}(x) &= \sum_{|I|=0} \sum_{i=1}^{n} \frac{\partial \omega_I}{\partial x^i}(x) \mathrm{d}x^i \wedge \mathrm{d}x^I, \\ &= \sum_{i=1}^{n} \frac{\partial \omega_\emptyset}{\partial x^i}(x) \mathrm{d}x^i \wedge \mathrm{d}x^\emptyset = \sum_{i=1}^{n} \frac{\partial \omega_\emptyset}{\partial x^i}(x) \mathrm{d}x^i. \end{aligned}$$

□

This coincides with the usual notion of the "differential" as used in many branches of physics. Additionally, from now on, we shall not make an explicit distinction between 0-forms and smooth functions, identifying them by $f \mapsto f \, \mathrm{d}x^\emptyset$.

Proposition 3.11 (Differential of the coordinate function)**.** *Let* Ω *be an* n*-domain and let* $1 \leq i \leq n$*. Consider a* 0*-form* $f : \Omega \to \mathbf{R}$ *defined as*

$$f(x^1, \ldots, x^n) = x^i.$$

Then we have $\mathrm{d}f = \mathrm{d}x^i$*.*

This proposition simply asserts that there is no reason to distinguish between the statement $\mathrm{d}(x^i)$ as the exterior derivative applied to the coordinate function and $\mathrm{d}x^i$ as a basis vector of the cotangent space.

Proposition 3.12 (Properties of the exterior derivative)**.** *For any* $\boldsymbol{\omega}, \boldsymbol{\tau} \in \mathcal{E}^p(\Omega)$ *and* $\boldsymbol{\mu} \in \mathcal{E}^q(\Omega)$ *the following holds:*

1. *(Additivity)*
   $\mathrm{d}(\boldsymbol{\omega} + \boldsymbol{\tau}) = \mathrm{d}\boldsymbol{\omega} + \mathrm{d}\boldsymbol{\tau}$

[1] We have in fact defined an entire family of linear maps, one for every k. We are not going to distinguish these notationally however.

2. *(Derivation with respect to the wedge product)*
   $\mathrm{d}(\boldsymbol{\omega} \wedge \boldsymbol{\mu}) = \mathrm{d}\boldsymbol{\omega} \wedge \boldsymbol{\mu} + (-1)^p \boldsymbol{\omega} \wedge \mathrm{d}\boldsymbol{\mu}$

3. *(Nilpotency)*
   $\mathrm{d}(\mathrm{d}\boldsymbol{\omega}) = \mathbf{0}$

The proofs are left to specialized texts such as [KST02] or [Fra17].

Transforming differential forms

At this point, we are ready to develop mechanisms for performing coordinate transformations of differential forms.

For the rest of this subsection, Ω is going to denote an n-domain and Ψ denotes an m-domain. Coordinates in Ω and Ψ will be denoted $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^m)$ respectively. We also take φ to be a smooth map $\Omega \to \Psi$.

Definition 3.13 (Pushforward of a tangent vector)**.** We define the *pushforward* of a tangent vector at point $p \in \Omega$ as a map $\varphi_* : \Omega_p \to \Psi_{\varphi(p)}$, which, for any function $g \in C^\infty(\Psi)$ and any tangent vector $\mathbf{v} \in \Omega_p$, is given by

$$(\varphi_* \mathbf{v})(g) = \mathbf{v}(g \circ \varphi).$$

Proposition 3.14. *The pushforward is a linear map between tangent spaces.*

Proof. First we need to verify that, for any tangent vector $\mathbf{v} \in \Omega_p$, its pushforward $\varphi_* \mathbf{v}$ is actually a tangent vector, that is, it is a derivation at $\varphi(p)$. We consider two functions $f, g \in C^\infty(\Psi)$ and get

$$\begin{aligned}
(\varphi_* \mathbf{v})(fg) &= \mathbf{v}((fg) \circ \varphi) \\
&= \mathbf{v}((f \circ \varphi)(g \circ \varphi)) \\
&= \mathbf{v}(f \circ \varphi) g(\varphi(p)) + f(\varphi(p)) \mathbf{v}(g \circ \varphi).
\end{aligned}$$

To show linearity of φ_*, we have, for any two tangent vectors $\mathbf{v}, \mathbf{w} \in \Omega_p$ and $\alpha \in \mathbf{R}$, the following

$$\begin{aligned}
(\varphi_*(\mathbf{v} + \alpha \mathbf{w}))(g) &= (\mathbf{v} + \alpha \mathbf{w})(g \circ \varphi) \\
&= \mathbf{v}(g \circ \varphi) + \alpha \mathbf{w}(g \circ \varphi) \\
&= (\varphi^* \mathbf{v})(g) + \alpha(\varphi^* \mathbf{w})(g).
\end{aligned}$$

$\square$

An illustration of the pushforward is depicted by Figure 3.2.

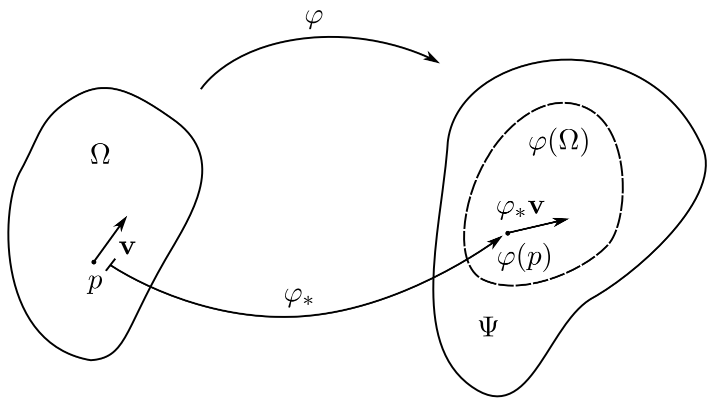

Figure 3.2: Illustration of the pushforward

Using the chain rule from multivariable calculus, we can express the pushforward in coordinates.

Proposition 3.15 (Pushforward of tangent basis). *If we write φ in components, that is, as an m-tuple of functions:*

$$\left(\varphi^1(x^1,\dots,x^n),\dots,\varphi^m(x^1,\dots,x^n)\right).$$

For the tangent basis vectors, we then have

$$\varphi_*\left(\boldsymbol{\partial}_{x^i}\right)=\sum_{j=1}^{m}\frac{\partial\varphi^j}{\partial x^i}\boldsymbol{\partial}_{y^j}.$$

The matrix of the pushforward expressed with respect to the tangent basis is usually called the Jacobian matrix.

Definition 3.16 (Pullback of a contangent vector). We define the *pullback* of a cotangent vector at point $\varphi(p)$ as a linear map $\varphi^*:\Psi^*_{\varphi(p)}\to\Omega^*_p$ which, for any tangent vector $\mathbf{v}\in\Omega_p$ and any cotangent vector $\boldsymbol{\omega}\in\Psi^*_{\varphi(p)}$, satistifes

$$(\varphi^*\boldsymbol{\omega})(\mathbf{v})=\boldsymbol{\omega}(\varphi_*\mathbf{v}).$$

In other words, we have that

$$\begin{tikzcd}
\Psi_{\varphi(p)} \arrow[r, "\boldsymbol{\omega}"] & \mathbf{R} \\
\Omega_p \arrow[u, "\varphi_*"] \arrow[ur, "\varphi^*\boldsymbol{\omega}"'] &
\end{tikzcd}$$

commutes. This is just a special version of Definition 2.7. Expressed in coordinates, we have, for the pullback of the basis vector $\mathrm{d}y^i$ acting on a tangent vector $\boldsymbol{\partial}_{x^j}$, the following:

$$\begin{aligned}\left(\varphi^*(\mathrm{d}y^i)\right)(\boldsymbol{\partial}_{x^j}) &= \mathrm{d}y^i\left(\varphi_*\left(\boldsymbol{\partial}_{x^j}\right)\right)\\ &= \mathrm{d}y^i\left(\sum_{q=1}^{r}\frac{\partial\varphi^q}{\partial x^j}\boldsymbol{\partial}_{y^q}\right)\\ &= \frac{\partial\varphi^i}{\partial x^j}\mathrm{d}x^j\left(\boldsymbol{\partial}_{x^j}\right).\end{aligned}$$

In other words, we have

$$\varphi^*(\mathrm{d}y^i) = \sum_{j=1}^{n}\frac{\partial\varphi^i}{\partial x^j}\mathrm{d}x^j = \mathrm{d}\varphi^i.$$

That is, pulling back a cotangent basis vector is identical to computing the exterior derivative of the 0-form given by φ^i (which is just the differential) and evaluating it at p.

The pullback is illustrated by Figure 3.3.

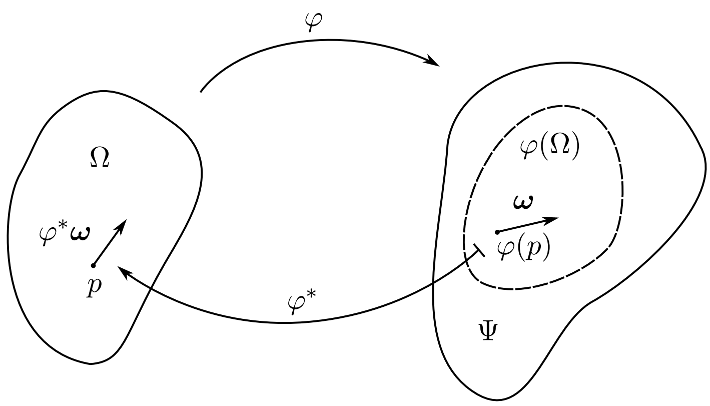

Figure 3.3: Illustration of the pullback

Remark 3.17. We are often going to use an convenient abuse of notation — by naming both our coordinate and the components of our smooth map identically (e.g. $y^i = \varphi^i$) we get

$$\varphi^*(\mathrm{d}y^i) = \mathrm{d}y^i = \sum_{j=1}^{n}\frac{\partial y^i}{\partial x^j}\mathrm{d}x^j.$$

Altough aesthetically pleasing, we need to always remember that the cotangent basis vectors $\mathrm{d}y^i$ and $\mathrm{d}x^j$ live in different tangent spaces and as such one cannot really be equal to a linear combination of the others.

As usual, we are going to extend this pointwise operation on a cotangent vector to act on an entire differential form at all points of its domain. This preserves smoothness, but proof is left to specialized texts such as [Tu11].

Definition 3.18 (Pullback of a differential form)**.** We define the *pullback* $\varphi^* : \mathcal{E}^k(\Psi) \to \mathcal{E}^k(\Omega)$ of a differential k-form as a linear map which for any vector fields $\mathbf{v}_1, \ldots, \mathbf{v}_k$ on Ω

$$(\varphi^* \boldsymbol{\omega})(\mathbf{v}_1, \ldots, \mathbf{v}_k) = (\boldsymbol{\omega} \circ \varphi)(\varphi_* \mathbf{v}_1, \ldots, \varphi_* \mathbf{v}_k)$$

at every point of Ω.

An important reason for dealing with differential forms instead of vector fields is the fact that it is always possible to pullback a differential form using a smooth map. This is however not true if we want to pushforward an entire vector field! If our smooth map is not surjective, we do not have a way of determining which tangent vectors to assign outside of its range and if our smooth map is not injective, we might have too many vector to choose from.

Proposition 3.19 (Pullback in coordinates)**.** *In coordinates, the pullback of a differential k-form on Ψ results on in*

$$\varphi^* \boldsymbol{\omega} = \sum_{|I|=p} (\omega_I \circ \varphi) \, \mathrm{d}\varphi^{i_1} \wedge \cdots \wedge \mathrm{d}\varphi^{i_p}.$$

The pullback interacts very nicely with the exterior derivative and the wedge product. Some of the identities are summarized below.

Proposition 3.20 (Properties of the pullback)**.** *Let $\boldsymbol{\omega} \in \mathcal{E}^p(\Psi)$ and $\boldsymbol{\sigma} \in \mathcal{E}^q(\Psi)$. Then the following equalities hold:*

1. *(Additivity)*
   If $p = q$, then $\varphi^*(\boldsymbol{\omega} + \boldsymbol{\sigma}) = \varphi^*(\boldsymbol{\omega}) + \varphi^*(\boldsymbol{\sigma})$
2. *(Algebra homomorphism)*
   $\varphi^*(\boldsymbol{\omega} \wedge \boldsymbol{\sigma}) = \varphi^*(\boldsymbol{\omega}) \wedge \varphi^*(\boldsymbol{\sigma})$
3. *(Commutes with the exterior derivative)*
   $\varphi^*(\mathrm{d}\boldsymbol{\omega}) = \mathrm{d}(\varphi^*(\boldsymbol{\omega}))$

Remark 3.21*.* We are often going to use the pullback to perform coordinate transforms. This is particularly useful if the smooth map involved is injective. In that case, we call it a *chart*.[2] Intuitively, we want to think of charts as "engraving" the coordinate grid of the domain onto the codomain.

As a specific example, let us consider polar coordinates on the unit disk. We take a rectangle $\Omega = \{(r, \theta) \in \mathbf{R}^2 \mid 0 < r < 1 \ \ \& \ \ 0 < \theta < 2\pi\}$ and the disk $D = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 < 1\}$. Then we consider the map $\varphi : \Omega \to D$ given by

$$x = r\cos(\theta) \quad \text{and} \quad y = r\sin(\theta).$$

This is illustrated by Figure 3.4. Notice that the range of φ is not the entire disk, but we have a "cut" along the positive x-axis. This could be a potential

[2]Technically, charts are usually smooth injective maps *from* an open subset of the object we want to introduce coordinates on *to* a subset of $\mathbf{R}^n$. We do not want to run into topological difficulties and thus are using this definition instead.

problem as a smooth function on Ω might not admit a smooth extension to D. Formally, this can be solved with the introduction of an *atlas*, which is a collection of charts with ranges that collectively cover the entirety of D.

We are not going to worry about this issue too much however, as all coordinate transformations we are going to use are sufficiently nicely behaved. For instance, we can take $\Omega' = \{(r, \theta) \mid |0 < r < 1 \quad \& \quad -\pi < \theta < \pi\}$ and use an identical prescription to φ to define another chart $\varphi' : \Omega' \to D$ which covers the positive x-axis, missing the negative x-axis instead (both of them still miss the origin). We can thus stay ambiguous as to what domain we are using and in effect use all of them at once.

Additionally, if using a chart, we are sometimes going to leave out the explicit pullback symbolism, understanding that an expression involving the chart coordinates is just a way of representing a particular object on a subset of our primary domain of interest.

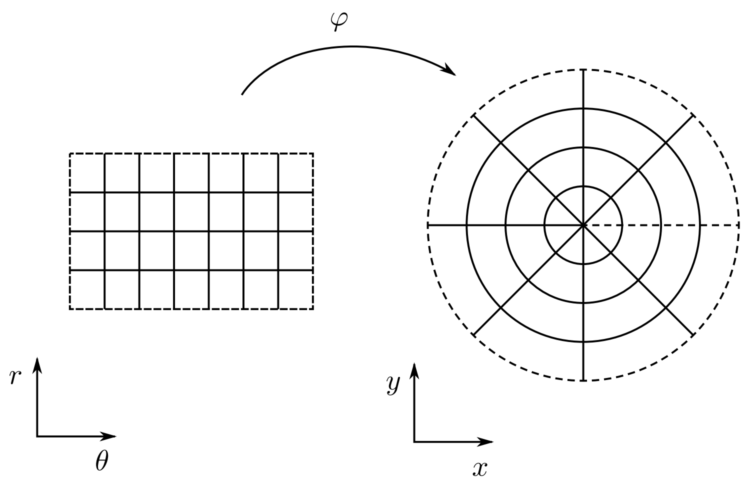

Figure 3.4: Polar coordinates on the unit disk

Inner product fields

Additional structure we need to introduce on our tangent spaces is an inner product. This concept allows us to measure "length" of tangent and cotangent vectors.

Altough it may seem like we can always introduce an inner product field, we have to be aware that we are enforcing additional physical meaning. For instance, if our domain was describing state of a an electric motor, with coordinates measuring quantities such as winding current or angular velocity, there might not be a physically acceptable way to introduce an inner product field.

Definition 3.22 (Inner product field)**.** Let Ω be an n-domain. We then define an *inner product field* on Ω as a smooth map which assigns to every point

$p = (x^1, \ldots, x^n) \in \Omega$ an inner product on Ω_p. In coordinates we have

$$\langle \boldsymbol{\partial}_{x^i} | \boldsymbol{\partial}_{x^j} \rangle|_p = g_{ij}\,(p)\,,$$

where every $g_{ij} \in C^\infty(\Omega)$.

We also extend Proposition 2.20 from a pointwise operation to the entire Ω and thus get an isomorphism between $\mathfrak{X}(\Omega)$ and $\mathcal{E}^0(\Omega)$.

Definition 3.23 (Pullback of an inner product field)**.** Let Ω and Ψ be n-domains and let $\varphi : \Omega \to \Psi$ be a smooth map. Furthermore, we require[3] the pushforward φ_* to be an isomorphism between tangent spaces at every point in Ω. We then define the pullback of an inner product field on Ψ to an inner product field on Ω as

$$\varphi^*\left(\langle - | - \rangle\right)(\mathbf{v}, \mathbf{w}) = \langle \varphi_* \mathbf{v} | \varphi_* \mathbf{w} \rangle.$$

Of course an inner product on the tangent space is not that useful for our purposes — however we can simply use Definition 2.22 to transform it to an inner product on the cotangent space. We then extend the inner product to differential k-forms as per Definition 2.24, additionally constructing the Hodge star in the process.

Definition 3.24 (Euclidean space)**.** We define the *Euclidean space*, denoted by $\mathbf{E}^n$, as $\mathbf{R}^n$ with an inner product field given by

$$\langle \alpha^1 \boldsymbol{\partial}_1 + \cdots + \alpha^n \boldsymbol{\partial}_n | \beta^1 \boldsymbol{\partial}_1 + \cdots + \beta^n \boldsymbol{\partial}_n \rangle = \alpha^1 \beta^1 + \cdots + \alpha^n \beta^n.$$

Coordinates of $\mathbf{E}^n$ shall be denoted by $(x^1, \ldots, x^n)$ unless stated otherwise. Our volume form is given by $\boldsymbol{\sigma} = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$. We are going to work with vector fields on $\mathbf{E}^n$ later, so we note the musical isomorphisms take the form $(\mathrm{d}x^i)^\sharp = \boldsymbol{\partial}_i$.

Example 3.25 (Polar coordinates)**.** Consider the plane $\mathbf{E}^2$, with coordinates denoted by x and y. We wish to introduce polar coordinates. To achieve this, we take the domain $\Omega = \{(r, \theta) \in \mathbf{R}^2 \mid r > 0 \ \ \& \ \ 0 < \theta < 2\pi\}$ and the map $\varphi : \Omega \to \mathbf{E}^2$ given as

$$\begin{aligned} x(r, \theta) &= r\cos(\theta), \\ y(r, \theta) &= r\sin(\theta). \end{aligned}$$

First we compute the pushforward of tangent vectors in Ω to tangent vectors in $\mathbf{E}^2$ as

$$\begin{aligned} \varphi_* \boldsymbol{\partial}_r &= \cos(\theta)\, \boldsymbol{\partial}_x + \sin(\theta)\, \boldsymbol{\partial}_y, \\ \varphi_* \boldsymbol{\partial}_\theta &= -r\sin(\theta)\, \boldsymbol{\partial}_x + r\cos(\theta)\, \boldsymbol{\partial}_y. \end{aligned}$$

We would like to transform the inner product field on $\mathbf{E}^2$ to an inner product

[3] This is required in order to preserve the nondegeneracy of our new inner product.

field on Ω by using Definition 3.23. To achieve this, we evaluate

$$\begin{aligned}
\varphi^*(\langle -|-\rangle)(\boldsymbol{\partial}_r, \boldsymbol{\partial}_r) &= \langle \varphi_* \boldsymbol{\partial}_r | \varphi_* \boldsymbol{\partial}_r \rangle \\
&= \langle \cos(\theta)\, \boldsymbol{\partial}_x + \sin(\theta)\, \boldsymbol{\partial}_y | \cos(\theta)\, \boldsymbol{\partial}_x + \sin(\theta)\, \boldsymbol{\partial}_y \rangle \\
&= 1, \\
\varphi^*(\langle -|-\rangle)(\boldsymbol{\partial}_\theta, \boldsymbol{\partial}_\theta) &= \langle -r\sin(\theta)\, \boldsymbol{\partial}_x + r\cos(\theta)\, \boldsymbol{\partial}_y | -r\sin(\theta)\, \boldsymbol{\partial}_x + r\cos(\theta)\, \boldsymbol{\partial}_y \rangle \\
&= r^2, \\
\varphi^*(\langle -|-\rangle)(\boldsymbol{\partial}_r, \boldsymbol{\partial}_\theta) &= \langle \cos(\theta)\, \boldsymbol{\partial}_x + \sin(\theta)\, \boldsymbol{\partial}_y | -r\sin(\theta)\, \boldsymbol{\partial}_x + r\cos(\theta)\, \boldsymbol{\partial}_y \rangle \\
&= 0.
\end{aligned}$$

Therefore, using Remark 2.23, we can convert this inner product to the contagent space, as follows:

$$\langle \mathrm{d}r|\mathrm{d}r\rangle = 1, \quad \langle \mathrm{d}\theta|\mathrm{d}\theta\rangle = \frac{1}{r^2}, \quad \langle \mathrm{d}r|\mathrm{d}\theta\rangle = 0.$$

Now we wish to compute the Hodge star in polar coordinates. We already have an inner product field, but we have yet to figure out how our volume form transforms under this coordinate change.

For this we compute the pullback of the coframe of $\mathbf{E}^2$ to differential forms in Ω. We can do this simply by calculating the differentials of our coordinate maps as follows

$$\begin{aligned}
\mathrm{d}x &= \cos(\theta)\mathrm{d}r - r\sin(\theta)\mathrm{d}\theta, \\
\mathrm{d}y &= \sin(\theta)\mathrm{d}r + r\cos(\theta)\mathrm{d}\theta.
\end{aligned}$$

And now

$$\begin{aligned}
\boldsymbol{\sigma} &= \mathrm{d}x \wedge \mathrm{d}y = (\cos(\theta)\mathrm{d}r - r\sin(\theta)\mathrm{d}\theta) \wedge (\sin(\theta)\mathrm{d}r + r\cos(\theta)\mathrm{d}\theta) \\
&= (\cos(\theta)\mathrm{d}r - \sin(\theta)(r\mathrm{d}\theta)) \wedge (\sin(\theta)\mathrm{d}r + \cos(\theta)(r\mathrm{d}\theta)) \\
&= \mathrm{d}r \wedge r\mathrm{d}\theta.
\end{aligned}$$

We can easily verify that the tuple $(\mathrm{d}r, r\mathrm{d}\theta)$ forms an orthonormal basis with respect to our inner product field on differential forms at every point of Ω.

Now we proceed using the usual approach. To compute $\star\mathrm{d}r$ we take

$$\begin{aligned}
\mathrm{d}r \wedge \mathrm{d}r &= 0\boldsymbol{\sigma} = \langle \star\mathrm{d}r|\mathrm{d}r\rangle\boldsymbol{\sigma}, \\
\mathrm{d}r \wedge \mathrm{d}\theta &= \frac{1}{r}\boldsymbol{\sigma} = \langle \star\mathrm{d}r|\mathrm{d}\theta\rangle\boldsymbol{\sigma}.
\end{aligned}$$

And thus $\star\mathrm{d}r = r\mathrm{d}\theta$. Similarly for $\star\mathrm{d}\theta$, we have

$$\begin{aligned}
\mathrm{d}\theta \wedge \mathrm{d}r &= -\frac{1}{r}\boldsymbol{\sigma} = \langle \star\mathrm{d}\theta|\mathrm{d}r\rangle\boldsymbol{\sigma}, \\
\mathrm{d}\theta \wedge \mathrm{d}\theta &= 0\boldsymbol{\sigma} = \langle \star\mathrm{d}\theta|\mathrm{d}\theta\rangle\boldsymbol{\sigma}.
\end{aligned}$$

And thus $\star\mathrm{d}\theta = -\frac{1}{r}\mathrm{d}r$. Notice that in multivariable calculus, we often find ourselves manipulating unit tangent vectors. This means we would, for example, have

$$\boldsymbol{\theta}_0 = \frac{1}{r}\boldsymbol{\partial}_\theta.$$

We shall not use this notation here, but it is worth keeping in mind.

As we can see from the above example, computing coordinate transformations manually is quite tedious. Luckily, several computer algebra software packages support perfoming automated differential-geometric computations. One option is further expanded upon in Appendix A (this example is contained specifically in Notebook A.1). This is extremely convenient as it allows us to perform even very complicated coordinate transformations in automated fashion.

Definition 3.26 (Isometry)**.** Let Ω and Ψ connected be n-domains, let $\langle -|-\rangle_\Omega$ and $\langle -|-\rangle_\Psi$ be inner product fields on Ω and Ψ respectively and let $\varphi : \Omega \to \Psi$ be a smooth bijection with a smooth inverse. We call φ an *isometry* if φ_* is a linear isometry between $\Psi_{\varphi(p)}$ and Ω_p at every point of $p \in \Omega$. In other words, for every pair of tangent vectors $\mathbf{v}, \mathbf{w} \in \Omega_p$ we have

$$\langle \mathbf{v}|\mathbf{w}\rangle_\Omega|_p = \langle \varphi_*\mathbf{v}|\varphi_*\mathbf{w}\rangle_\Psi|_{\varphi(p)}.$$

In the rest of this section, we denote Ω, Ψ, their respective inner product fields and an isometry φ as above. An isometry constructs linear isometries between tangent spaces. However, we also want to work with cotangent vectors and thus we need to show that the pullback is also a linear isometry between cotangent spaces using the respective contravariant inner products.

Proposition 3.27. *For every pair cotangent vectors $\boldsymbol{\mu}, \boldsymbol{\omega} \in \Psi^*_{\varphi(p)}$ we have*

$$\langle \boldsymbol{\mu}|\boldsymbol{\omega}\rangle_\Psi|_{\varphi(p)} = \langle \varphi^*\boldsymbol{\mu}|\varphi^*\boldsymbol{\omega}\rangle_\Omega|_p.$$

Proof. Let $p \in \Omega$ and let $\boldsymbol{\mu}, \boldsymbol{\omega} \in \Psi^*_{\varphi(p)}$. First we show that we have $\varphi_*((\varphi^*\boldsymbol{\mu})^\sharp) = \boldsymbol{\mu}^\sharp$ for any $\boldsymbol{\mu}$. In other words, we want to show that

$$\begin{tikzcd}
\Psi^*_{\varphi(p)} \arrow[r, "\varphi^*"] \arrow[d, "\sharp"'] & \Omega^*_p \arrow[d, "\sharp"] \\
\Psi_{\varphi(p)} & \Omega_p \arrow[l, "\varphi_*"]
\end{tikzcd}$$

commutes. Take any vector $\mathbf{v} \in \Omega_p$ and compute

$$\begin{aligned}
\langle \varphi_*((\varphi^*\boldsymbol{\mu})^\sharp)|\varphi_*\mathbf{v}\rangle_\Psi &= \langle (\varphi^*\boldsymbol{\mu})^\sharp|\mathbf{v}\rangle_\Omega \\
&= (\varphi^*\boldsymbol{\mu})(\mathbf{v}) \\
&= \boldsymbol{\mu}(\varphi_*\mathbf{v}) \\
&= \langle \boldsymbol{\mu}^\sharp|\varphi_*\mathbf{v}\rangle_\Psi.
\end{aligned}$$

Now as φ_* is an isomorphism by assumption and the inner product is nondegenerate, we have the desired equality.

With this fact in hand, we can now show

$$\begin{aligned}
\langle \boldsymbol{\mu}|\boldsymbol{\omega}\rangle_\Psi &= \langle \boldsymbol{\mu}^\sharp|\boldsymbol{\omega}^\sharp\rangle_\Psi \\
&= \langle \varphi_*((\varphi^*\boldsymbol{\mu})^\sharp)|\varphi_*((\varphi^*\boldsymbol{\omega})^\sharp)\rangle_\Psi \\
&= \langle (\varphi^*\boldsymbol{\mu})^\sharp|(\varphi^*\boldsymbol{\omega})^\sharp\rangle_\Omega \\
&= \langle \varphi^*\boldsymbol{\mu}|\varphi^*\boldsymbol{\omega}\rangle_\Omega.
\end{aligned}$$

□

Proposition 3.28. *For every pair of differential k-forms $\boldsymbol{\mu}, \boldsymbol{\omega} \in \mathcal{E}^k(\Psi)$, we have*

$$\varphi^* \langle \boldsymbol{\mu} | \boldsymbol{\omega} \rangle_\Psi = \langle \varphi^* \boldsymbol{\mu} | \varphi^* \boldsymbol{\omega} \rangle_\Omega .$$

Proof. For 1-forms, this is simply a consequence of the previous proposition. Picking a point $p \in \Omega$, we have

$$(\varphi^* \langle \boldsymbol{\mu} | \boldsymbol{\omega} \rangle_\Psi)(p) = (\langle \boldsymbol{\mu} | \boldsymbol{\omega} \rangle_\Psi \circ \varphi)(p) = \langle \boldsymbol{\mu} | \boldsymbol{\omega} \rangle_\Psi |_{\varphi(p)} = \langle \varphi^* \boldsymbol{\mu} | \varphi^* \boldsymbol{\omega} \rangle_\Omega |_p .$$

For k-forms, this can be extended simply by taking the definition of the inner product based on the Gramian matrix and considering that the pullback distributes over the wedge product. □

We would like to study the interaction between the Hodge star and an isometric pullback. First we note that orthonormality is preserved if we convert an orthonormal tangent basis to a cotangent basis using Definition 2.22. Now we can prove the following important proposition.

Proposition 3.29 (Isometry pullback and the Hodge star)**.** *We have, for any k-form $\boldsymbol{\omega} \in \mathcal{E}^k(\Psi)$, that*

$$\varphi^* \star \boldsymbol{\omega} = \star \varphi^* \boldsymbol{\omega} \quad \text{or} \quad \varphi^* \star \boldsymbol{\omega} = - \star \varphi^* \boldsymbol{\omega} .$$

We call φ orientation-preserving in the first case and orientation-reversing in the second case.

Proof. Label the volume form on Ω as $\boldsymbol{\sigma}_\Omega$ and the volume form on Ψ as $\boldsymbol{\sigma}_\Psi$. First we note that as φ^* is a linear isometry, it maps orthonormal volume forms to volume forms and thus we get, by Proposition 2.27, that $\varphi^* \boldsymbol{\sigma}_\Psi = \pm \boldsymbol{\sigma}_\Omega$. As the domains are connected by assumption and the process is continuous, the sign is identical at every point. The metric sign is then also necessarily preserved. Now we consider an $(n-k)$-form $\boldsymbol{\mu}$ on Ψ and a (fixed) k-form $\boldsymbol{\omega}$. We have, from Proposition 2.32, that

$$s(-1)^{k(n-k)} \boldsymbol{\mu} \wedge \boldsymbol{\omega} = \boldsymbol{\mu} \wedge \star\star \boldsymbol{\omega} = \langle \boldsymbol{\mu} | \star \boldsymbol{\omega} \rangle_\Psi \boldsymbol{\sigma}_\Psi .$$

Now we apply φ^* to the equality and get

$$\begin{aligned}
\langle \varphi^* \boldsymbol{\mu} | \star \varphi^* \boldsymbol{\omega} \rangle_\Omega \boldsymbol{\sigma}_\Omega &= (\varphi^* \boldsymbol{\mu}) \wedge \star \star (\varphi^* \boldsymbol{\omega}) \\
&= s(-1)^{n(n-k)} (\varphi^* \boldsymbol{\mu}) \wedge (\varphi^* \boldsymbol{\omega}) \\
&= \varphi^* (\langle \boldsymbol{\mu} | \star \boldsymbol{\omega} \rangle_\Psi \boldsymbol{\sigma}_\Psi) \\
&= \varphi^* (\langle \boldsymbol{\mu} | \star \boldsymbol{\omega} \rangle_\Psi)(\varphi^* \boldsymbol{\sigma}_\Psi) \\
&= \pm \langle \varphi^* \boldsymbol{\mu} | \varphi^* (\star \boldsymbol{\omega}) \rangle_\Omega \boldsymbol{\sigma}_\Omega .
\end{aligned}$$

Now as φ is a bijection by assumption and φ^* is an isomorphism, we can show the equality at every point of Ω by picking a suitable $\boldsymbol{\mu}$. □

Remark 3.30. If we pullback an inner product field, as defined in Definition 3.23, we automatically get an isometry. This is particularly useful when using pullbacks to change coordinates, as we can transform the inner product field first, and then commute the pullback over every instance of the Hodge star, writing any differential form involved in the new coordinates directly.

3.3 Additional Tools

We can compose the tools we have built so far to create several additional constructs which will be useful in their own right.

Definition 3.31 (The codifferential and Laplace-de Rham)**.** Given a inner product field on an n-domain Ω with metric sign s, we can define the *codifferential* δ as a linear map $\mathcal{E}^k(\Omega) \to \mathcal{E}^{k-1}(\Omega)$ given by

$$\delta = (-1)^k \star^{-1} \mathrm{d} \star = (-1)^{n(k-1)+1} s \star \mathrm{d} \star.$$

We also define the *Laplace-de Rham* operator as $\Delta : \mathcal{E}^k(\Omega) \to \mathcal{E}^k(\Omega)$ given by

$$\Delta = \mathrm{d}\delta + \delta\mathrm{d}.$$

It helps to visualize the sequence of mappings $\star\mathrm{d}\star$, which constitute the codifferential as

$$\mathcal{E}^k(\Omega) \xrightarrow{\star} \mathcal{E}^{n-k}(\Omega) \xrightarrow{\mathrm{d}} \mathcal{E}^{n-k+1}(\Omega) \xrightarrow{\star} \mathcal{E}^{k-1}(\Omega).$$

From the nilpotency of d, we can easily see that:

Proposition 3.32 (The codifferential is nilpotent)**.** *We have* $\delta\delta = 0$*.*

From Proposition 3.29, we also have:

Proposition 3.33 (The codifferential commutes with isometric pullback)**.** *For an isometry* φ*, we have* $\varphi^*\delta = \delta\varphi^*$*.*

Definition 3.34 (Closed and exact forms)**.** A k-form $\boldsymbol{\omega} \in \mathcal{E}^k(\Omega)$ is called *closed* if $\mathrm{d}\boldsymbol{\omega} = \mathbf{0}$ and *exact* if there exists a differential form $\boldsymbol{\lambda} \in \mathcal{E}^{k-1}(\Omega)$ such that $\mathrm{d}\boldsymbol{\lambda} = \boldsymbol{\omega}$. Additionally, $\boldsymbol{\omega}$ is said to be *co-closed* if $\delta\boldsymbol{\omega} = \mathbf{0}$ and *co-exact* if there exists a differential form $\boldsymbol{\gamma} \in \mathcal{E}^{k+1}(\Omega)$ such that $\delta\boldsymbol{\gamma} = \boldsymbol{\omega}$. Furthermore, we call $\boldsymbol{\omega}$ *harmonic* if $\Delta\boldsymbol{\omega} = \mathbf{0}$.

Additionally note that the set of all closed differential forms is a subspace of all differential k-forms, that is, the kernel of the exterior derivative. The set of all exact differential forms is then the range of the exterior derivative. Similarly for co-closed and co-exact forms and the codifferential.

By Proposition 3.12, we can easily see that every exact form is automatically closed. The converse however is not always true.

Proposition 3.35 (Poincaré lemma)**.** *On* $\mathbf{R}^n$ *every closed form is exact.*[4] *In other words, if we have a differential form* $\boldsymbol{\omega} \in \mathcal{E}^k(\mathbf{R}^n)$ *such that* $\mathrm{d}\boldsymbol{\omega} = \mathbf{0}$*, then there exists a differential form* $\boldsymbol{\lambda} \in \mathcal{E}^{k-1}(\mathbf{R}^n)$ *such that* $\mathrm{d}\boldsymbol{\lambda} = \boldsymbol{\omega}$*.*

This proposition is highly nontrivial. An interested reader may consult [Tu11] or [Fra17] for more details. Note that it is a purely topological concept and does not require the introduction of any inner product field.

[4] This theorem can be generalized to some special open subsets of $\mathbf{R}^n$. Intuitively, we need such subset to contain no "holes".

3.4 Connections to Multivariable Calculus

In this section, we are going to work with the Euclidean space to form a bridge between the formalism of differential forms and multivariable calculus.

First important thing to note is that in multivariable calculus, we traditionally do not distinguish between tangent vectors and the elements of $\mathbf{E}^n$ itself. This is unfortunate, as they are fundamentally different objects, which is especially apparent when changing coordinate systems.

Consider a scalar function $f \in \mathcal{E}^0(\mathbf{E}^n)$. We know that the exterior derivative is computed as

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i\,.$$

This looks almost like the gradient, but it does not behave as expected under pullbacks. For instance, the exterior derivative of a scalar function on $\mathbf{E}^3$ expressed in spherical coordinates is given by

$$\mathrm{d}f = \frac{\partial f}{\partial r}\,\mathrm{d}r + \frac{\partial f}{\partial \theta}\,\mathrm{d}\theta + \frac{\partial f}{\partial \varphi}\,\mathrm{d}\varphi\,.$$

We remind ourselves that the gradient is usually thought of as a vector field and thus define:

Definition 3.36 (Gradient)**.** The gradient is a linear map $\nabla : \mathcal{E}^0(\mathbf{E}^n) \to \mathfrak{X}(\mathbf{E}^n)$ given by $\nabla f = (\mathrm{d}f)^\sharp$. In other words, the gradient is a vector field on $\mathbf{E}^n$ such that for every vector field $\mathbf{v}$ on $\mathbf{E}^n$ we have

$$\langle \nabla f | \mathbf{v} \rangle = \mathrm{d}f\,(\mathbf{v}).$$

Expressed in coordinates in $\mathbf{E}^n$, this means

$$\nabla f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\boldsymbol{\partial}_{x^i}.$$

Example 3.37 (Gradient in polar coordinates)**.** We are going to expand upon Example 3.25 by computing the gradient in polar coordinates. We remind ourselves that we considered the set $\Omega = \{(r,\theta) \in \mathbf{R}^2 \mid r > 0 \ \ \& \ \ 0 < \theta < 2\pi\}$ and consider a 0-form f on Ω. Its exterior derivative is then given as

$$\mathrm{d}f = \frac{\partial f}{\partial r}\,\mathrm{d}r + \frac{\partial f}{\partial \theta}\,\mathrm{d}\theta\,.$$

Now we have to use the euclidean inner product we have transformed to Ω to compute $(\mathrm{d}f)^\sharp$. We have

$$(\boldsymbol{\partial}_r)^\flat = \mathrm{d}r \quad \text{and} \quad (\boldsymbol{\partial}_\theta)^\flat = r^2\,\mathrm{d}\theta$$

and thus

$$(\mathrm{d}r)^\sharp = \boldsymbol{\partial}_r \quad \text{and} \quad (\mathrm{d}\theta)^\sharp = \frac{1}{r^2}\boldsymbol{\partial}_\theta.$$

Which yields

$$\nabla f = \frac{\partial f}{\partial r}\boldsymbol{\partial}_r + \frac{\partial f}{\partial \theta}\frac{1}{r^2}\boldsymbol{\partial}_\theta = \frac{\partial f}{\partial r}\mathbf{r}_0 + \frac{\partial f}{\partial \theta}\frac{1}{r}\boldsymbol{\theta}_0,$$

where

$$\mathbf{r}_0 = \boldsymbol{\partial}_r \quad \text{and} \quad \boldsymbol{\theta}_0 = \frac{1}{r}\boldsymbol{\partial}_\theta.$$

Definition 3.38 (Divergence and curl)**.** We define the divergence $\boldsymbol{\nabla}\boldsymbol{\cdot} : \mathfrak{X}(\mathbf{E}^3) \to \mathcal{E}^0(\mathbf{E}^3)$ of a vector field $\mathbf{v}$ on $\mathbf{E}^3$ as

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{v} = \star\mathrm{d}\star\mathbf{v}^\flat.$$

We also define the curl $\boldsymbol{\nabla}\times : \mathfrak{X}(\mathbf{E}^3) \to \mathfrak{X}(\mathbf{E}^3)$ as

$$\boldsymbol{\nabla}\times\mathbf{v} = (\star\mathrm{d}\mathbf{v}^\flat)^\sharp.$$

Example 3.39 (Divergence and curl in cartesian coordinates)**.** For a vector field

$$\mathbf{v} = v^1\boldsymbol{\partial}_1 + v^2\boldsymbol{\partial}_2 + v^3\boldsymbol{\partial}_3$$

on $\mathbf{E}^3$, we compute the divergence as

$$\begin{aligned}
\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{v} &= \star\mathrm{d}\star\mathbf{v}^\flat \\
&= \star\mathrm{d}\star\Big(v^1\,\mathrm{d}x^1 + v^2\,\mathrm{d}x^2 + v^3\,\mathrm{d}x^3\Big) \\
&= \star\mathrm{d}\Big(v^1\,\mathrm{d}x^2\wedge\mathrm{d}x^3 + v^2\,\mathrm{d}x^3\wedge\mathrm{d}x^1 + v^3\,\mathrm{d}x^1\wedge\mathrm{d}x^2\Big) \\
&= \star\left(\frac{\partial v^1}{\partial x^1} + \frac{\partial v^2}{\partial x^2} + \frac{\partial v^3}{\partial x^3}\right)\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 \\
&= \frac{\partial v^1}{\partial x^1} + \frac{\partial v^2}{\partial x^2} + \frac{\partial v^3}{\partial x^3}.
\end{aligned}$$

For the curl, we get

$$\begin{aligned}
\mathrm{d}\mathbf{v}^\flat = &\left(\frac{\partial v^2}{\partial x^1} - \frac{\partial v^1}{\partial x^2}\right)\mathrm{d}x^1\wedge\mathrm{d}x^2 + \\
&\left(\frac{\partial v^1}{\partial x^3} - \frac{\partial v^3}{\partial x^1}\right)\mathrm{d}x^3\wedge\mathrm{d}x^1 + \\
&\left(\frac{\partial v^3}{\partial x^2} - \frac{\partial v^2}{\partial x^3}\right)\mathrm{d}x^2\wedge\mathrm{d}x^3
\end{aligned}$$

and thus

$$\boldsymbol{\nabla}\times\mathbf{v} = \left(\frac{\partial v^2}{\partial x^1} - \frac{\partial v^1}{\partial x^2}\right)\boldsymbol{\partial}_3 + \left(\frac{\partial v^1}{\partial x^3} - \frac{\partial v^3}{\partial x^1}\right)\boldsymbol{\partial}_2 + \left(\frac{\partial v^3}{\partial x^2} - \frac{\partial v^2}{\partial x^3}\right)\boldsymbol{\partial}_1.$$

To summarize, we have the following diagram for the gradient, curl and divergence we defined on $\mathbf{E}^3$.

$$\begin{array}{ccccccc}
\mathcal{E}^0(\mathbf{E}^3) & \xrightarrow{\mathrm{d}} & \mathcal{E}^1(\mathbf{E}^3) & \xrightarrow{\mathrm{d}} & \mathcal{E}^2(\mathbf{E}^3) & \xrightarrow{\mathrm{d}} & \mathcal{E}^3(\mathbf{E}^3) \\
\big\downarrow & & \sharp\big\downarrow & & \sharp\circ\star\big\downarrow & & \star\big\downarrow \\
C^\infty(\Omega) & \xrightarrow{\nabla} & \mathfrak{X}(\Omega) & \xrightarrow{\boldsymbol{\nabla}\times} & \mathfrak{X}(\Omega) & \xrightarrow{\boldsymbol{\nabla}\boldsymbol{\cdot}} & C^\infty(\Omega)
\end{array}$$

This diagram illustrates several important points about multivariable calculus. As it works with vector fields only, all the complexity has to be present in the various operators, relying heavily on being the ability to identify 2-forms and 1-forms present only in $\mathbf{E}^3$ and the musical isomorphisms.

If we leverage differential forms, we can work in arbitrary dimension and with arbitrary inner product fields, or even without any inner product field at all. We can then think of the divergence as "codifferential of a 1-form" and the curl as "exterior derivative of a 1-form". The gradient can be replaced with exterior derivative of a 0-form and converted to a vector field only when needed.

We can also use what we know about the exterior derivative to derive the identities we know from multivariable calculus. This is best left to specialized textbooks, but we are going to present an example.

Proposition 3.40. *For any vector field* $\mathbf{v}$ *on* $\mathbf{E}^n$ *we have* $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \mathbf{v} = \mathbf{0}$.

Proof. We compute using the definitions and nilpotency of the exterior derivative

$$\begin{aligned}\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \mathbf{v} &= \star \mathrm{d} \star \star \mathrm{d} \mathbf{v}^\flat \\ &= \star \mathrm{d} \star \star \mathrm{d} \mathbf{v}^\flat \\ &= \star \mathrm{d}\mathrm{d} \mathbf{v}^\flat \\ &= \mathbf{0}.\end{aligned}$$

□

Chapter 4

Electrodynamics

At this point, we have the tools necessary to study electrodynamics in the formalism of differential forms. We are going to consider a conservative approach which precludes the introduction of material dependence. However note that materials are a macroscopic abstraction and thus the laws of electromagnetism describe the behavior of the electromagnetic field even inside materials.

We are going to start with a motivating example:

Example 4.1. It is well known that the magnetic vector field generated by an infinitely long wire oriented along the x^3 axis with constant unit current can be described by

$$\left(\frac{-x^2}{(x^1)^2+(x^2)^2}\right)\boldsymbol{\partial}_{x^1}+\left(\frac{x^1}{(x^1)^2+(x^2)^2}\right)\boldsymbol{\partial}_{x^2}.$$

We would like to transform this vector field by a reflection along the plane spanned by the x^2 and x^3 axis. We thus setup a coordinate transform as

$$y^1=-x^1,\quad y^2=x^2,\quad y^3=x^3.$$

Now if we push forward our vector field (the coordinate transform is bijective, so we can push forward entire vector fields), we get

$$\left(\frac{y^2}{(y^1)^2+(y^2)^2}\right)\boldsymbol{\partial}_{y^1}+\left(\frac{-y^1}{(y^1)^2+(y^2)^2}\right)\boldsymbol{\partial}_{y^2}.$$

Physically, this does not make much sense. Our transformation preserved both the wire and the direction of the current imposed on it. Yet we have a different result in our new coordinate system!

However, if instead of a vector field we consider the magnetic field to be a 2-form given by

$$\boldsymbol{\mathcal{B}}=\left(\frac{-x^2}{(x^1)^2+(x^2)^2}\right)\mathrm{d}x^2\wedge\mathrm{d}x^3+\left(\frac{x^1}{(x^1)^2+(x^2)^2}\right)\mathrm{d}x^1\wedge\mathrm{d}x^3\,.$$

and its pullback is then

$$\varphi^*\boldsymbol{\mathcal{B}}=\left(\frac{-y^2}{(y^1)^2+(y^2)^2}\right)\mathrm{d}y^2\wedge\mathrm{d}y^3+\left(\frac{y^1}{(y^1)^2+(y^2)^2}\right)\mathrm{d}y^1\wedge\mathrm{d}y^3\,.$$

The sign changes cancel out and we get identical result in the new coordinate system.

Classically, we would call the magnetic field a *pseudovector* and just remember that we have to apply different rules while transforming its coordinates. However, in the theory of differential forms, we can represent pseudovectors as a distinct object type and thus the coordinate transformations are implicitly handled correctly.

4.1 Basic Definitions

Definition 4.2 (Minkowski space)**.** We define the *Minkowski space*, denoted by $\mathbf{M}$, as the domain $\mathbf{R}^4$ with an inner product field given by

$$\begin{aligned}&\langle \alpha^0\boldsymbol{\partial}_0 + \alpha^1\boldsymbol{\partial}_1 + \alpha^2\boldsymbol{\partial}_2 + \alpha^3\boldsymbol{\partial}_3 | \beta^0\boldsymbol{\partial}_0 + \beta^1\boldsymbol{\partial}_1 + \beta^2\boldsymbol{\partial}_2 + \beta^3\boldsymbol{\partial}_3\rangle \\ &= -\alpha^0\beta^0 + \alpha^1\beta^1 + \alpha^2\beta^2 + \alpha^3\beta^3\end{aligned}$$

at every point. We are going to denote coordinates in $\mathbf{M}$ by (x^0, x^1, x^2, x^3). Our volume form is given as $\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$.

We want to think of the x^0 coordinate as the "time" coordinate and the remaining coordinates as "spatial" coordinates. Additionally, we are using "geometric units", that is, units where the time coordinate and spatial coordinates have the same dimensions.

To simplify some further computations, we also define:

Definition 4.3 (Position 1-form)**.** A *position 1-form* on the Minkowski space is defined as

$$\boldsymbol{\mathcal{R}} = -x^0\,\mathrm{d}x^0 + x^1\,\mathrm{d}x^1 + x^2\,\mathrm{d}x^2 + x^3\,\mathrm{d}x^3\,.$$

And now we are ready to start constructing the theory of electromagnetism.

Definition 4.4 (Electromagnetic field)**.** Given a *source* 1*-form* $\boldsymbol{\mathcal{J}}$ on $\mathbf{M}$, we call a 2-form $\boldsymbol{\mathcal{F}}$ on $\mathbf{M}$ *electromagnetic field associated with* $\boldsymbol{\mathcal{J}}$ if it satisfies the *homogeneous Maxwell's equation*, given by

$$\mathrm{d}\boldsymbol{\mathcal{F}} = \mathbf{0}$$

and the *inhomogeneous Maxwell's equation*, given by

$$\delta\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{J}}.$$

To connect the electromagnetic field to the traditional theory of electromagnetism, we are going to name its components in the following definition.

Definition 4.5 (Components of the electromagnetic field)**.** Given an electromagnetic field $\boldsymbol{\mathcal{F}}$ associated with $\boldsymbol{\mathcal{J}}$, we define the *electric* 1*-form* $\boldsymbol{\mathcal{E}}$ and *magnetic* 2*-form* $\boldsymbol{\mathcal{B}}$, with components named by

$$\begin{aligned}\boldsymbol{\mathcal{E}} &= E_1\,\mathrm{d}x^1 + E_2\,\mathrm{d}x^2 + E_3\,\mathrm{d}x^3\,,\\ \boldsymbol{\mathcal{B}} &= B_1\,\mathrm{d}x^2\wedge\mathrm{d}x^3 + B_2\,\mathrm{d}x^3\wedge\mathrm{d}x^1 + B_3\,\mathrm{d}x^1\wedge\mathrm{d}x^2\,.\end{aligned}$$

Then we set

$$\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{E}} \wedge \mathrm{d}x^0 + \boldsymbol{\mathcal{B}}.$$

Note that we can control every coframe 2-form independently and thus we have not restricted our collection of electromagnetic fields in any way.

We also name the components of $\boldsymbol{\mathcal{J}}$ as

$$\boldsymbol{\mathcal{J}} = -\rho \,\mathrm{d}x^0 + J_1 \,\mathrm{d}x^1 + J_2 \,\mathrm{d}x^2 + J_3 \,\mathrm{d}x^3 \,.$$

Proposition 4.6 (Components of any electromagnetic field satisfy Maxwell's equations)**.** *We shall compare our expression with the traditional vector calculus form in cartesian coordinates.*

Proof. As per Definition 4.5, we construct vector fields on $\mathbf{E}^3$ from the components of $\boldsymbol{\mathcal{F}}$ as

$$\begin{aligned}
\mathbf{E} &= E_1\boldsymbol{\partial}_1 + E_2\boldsymbol{\partial}_2 + E_3\boldsymbol{\partial}_3, \\
\mathbf{B} &= B_1\boldsymbol{\partial}_1 + B_2\boldsymbol{\partial}_2 + B_3\boldsymbol{\partial}_3, \\
\mathbf{J} &= J_1\boldsymbol{\partial}_1 + J_2\boldsymbol{\partial}_2 + J_3\boldsymbol{\partial}_3.
\end{aligned}$$

Note that the components still depend on time, so we have in fact constructed an entire family of vector fields on $\mathbf{E}^3$. This is necessary as the traditional formulation from vector calculus does not consider the time coordinate to be a proper coordinate.

First we compute $\mathrm{d}\boldsymbol{\mathcal{F}} = \mathbf{0}$ as

$$\begin{aligned}
\mathrm{d}\boldsymbol{\mathcal{F}} &= \mathrm{d}(\boldsymbol{\mathcal{E}} \wedge \mathrm{d}x^0 + \boldsymbol{\mathcal{B}}) \\
&= \mathrm{d}\boldsymbol{\mathcal{E}} \wedge \mathrm{d}x^0 + \mathrm{d}\boldsymbol{\mathcal{B}} \\
&= \left(\frac{\partial E_2}{\partial x^1} - \frac{\partial E_1}{\partial x^2}\right) \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \\
&\quad \left(\frac{\partial E_1}{\partial x^3} - \frac{\partial E_3}{\partial x^1}\right) \mathrm{d}x^0 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^1 + \\
&\quad \left(\frac{\partial E_3}{\partial x^2} - \frac{\partial E_2}{\partial x^3}\right) \mathrm{d}x^0 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \\
&\quad \frac{\partial B_3}{\partial x^0} \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \\
&\quad \frac{\partial B_2}{\partial x^0} \mathrm{d}x^0 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^1 + \\
&\quad \frac{\partial B_1}{\partial x^0} \mathrm{d}x^0 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \\
&\quad \left(\frac{\partial B_1}{\partial x^1} + \frac{\partial B_2}{\partial x^2} + \frac{\partial B_3}{\partial x^3}\right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \,.
\end{aligned}$$

Setting all the components to zero shows that this is equivalent to

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial x^0} \quad \text{and} \quad \boldsymbol{\nabla} \cdot \mathbf{B} = \mathbf{0}.$$

Similarly, by computing $\delta\boldsymbol{\mathcal{F}}$ we get

$$\boldsymbol{\nabla} \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial x^0} \quad \text{and} \quad \boldsymbol{\nabla} \cdot \mathbf{E} = \rho.$$

See Notebook A.3 for the details of this computation. $\square$

From nilpotency of δ we can easily get

Proposition 4.7 (Continuity equation)**.** *We have* $\delta\boldsymbol{\mathcal{J}} = \mathbf{0}$.

Proof. Evaluate $\mathbf{0} = \delta\delta\boldsymbol{\mathcal{F}} = \delta\boldsymbol{\mathcal{J}}$. $\square$

4.2 The Electromagnetic Potential

As the homogeneous Maxwell's equation just states that $\boldsymbol{\mathcal{F}}$ is a closed differential form, we can use Proposition 3.35 to get a potential for any electromagnetic field.

Definition 4.8 (Electromagnetic potential)**.** Given an electromagnetic field $\boldsymbol{\mathcal{F}}$, we call any 1-form $\boldsymbol{\mathcal{A}}$ such that $\mathrm{d}\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{F}}$ an *electromagnetic potential* of $\boldsymbol{\mathcal{F}}$.

With electromagnetic potentials, the Maxwell's equation simplify even further. The homogeneous equation is satisfied automatically by nilpotency of the exterior derivative and thus we are only left with

$$\delta\,\mathrm{d}\boldsymbol{\mathcal{A}} = \star\mathrm{d}\star\mathrm{d}\boldsymbol{\mathcal{A}} = \delta\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{J}}.$$

Any 1-form generates a valid electromagnetic field for some source 1-form. However the exterior derivative is not injective and thus the electromagnetic potential is not determined uniquely. We can characterize this more rigorously by the following proposition.

Proposition 4.9 (Gauge freedom)**.** *Let* $\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{A}}' \in \mathcal{E}^1(\mathbf{M})$ *and let* $\boldsymbol{\mathcal{A}}$ *be an electromagnetic potential for* $\boldsymbol{\mathcal{F}}$*. Then* $\boldsymbol{\mathcal{A}}'$ *is also an electromagnetic potential for* $\boldsymbol{\mathcal{F}}$ *if and only if* $\boldsymbol{\mathcal{A}} - \boldsymbol{\mathcal{A}}'$ *is exact.*

Proof. First we assume $\mathrm{d}\boldsymbol{\mathcal{A}}' = \boldsymbol{\mathcal{F}}$. Then we have

$$\mathrm{d}(\boldsymbol{\mathcal{A}} - \boldsymbol{\mathcal{A}}') = \mathrm{d}\boldsymbol{\mathcal{A}} - \mathrm{d}\boldsymbol{\mathcal{A}}' = \boldsymbol{\mathcal{F}} - \boldsymbol{\mathcal{F}} = \mathbf{0}.$$

and thus $\boldsymbol{\mathcal{A}} - \boldsymbol{\mathcal{A}}'$ is closed and by Theorem 3.35 it is also exact.

Next we assume exactness, therefore $\mathrm{d}\boldsymbol{\lambda} = \boldsymbol{\mathcal{A}} - \boldsymbol{\mathcal{A}}'$ for some $\boldsymbol{\lambda} \in \mathcal{E}^0(\mathbf{M})$. Then we have

$$\mathbf{0} = \mathrm{d}\,\mathrm{d}\boldsymbol{\lambda} = \mathrm{d}(\boldsymbol{\mathcal{A}} - \boldsymbol{\mathcal{A}}') = \boldsymbol{\mathcal{F}} - \mathrm{d}\boldsymbol{\mathcal{A}}'$$

and thus $\boldsymbol{\mathcal{A}}'$ is an electromagnetic potential of $\boldsymbol{\mathcal{F}}$. $\square$

As the electromagnetic potential is not determined uniquely, we are not going to name its components, referring to them by number of the corresponding coframe form only.

A natural way to constrain the set of valid electromagnetic potentials is by introducing additional constraints, called *gauge conditions*. A convenient gauge condition for our setting can be stated as follows.

Definition 4.10 (Lorenz gauge condition)**.** An electromagnetic potential $\boldsymbol{\mathcal{A}}$ is said to be in the *Lorenz gauge*[1] if it satisfies $\delta\boldsymbol{\mathcal{A}} = \mathbf{0}$. In other words, we demand that $\boldsymbol{\mathcal{A}}$ is co-closed.[2]

In coordinates, this yields

$$\frac{\partial A_0}{\partial x^0} - \frac{\partial A_1}{\partial x^1} - \frac{\partial A_2}{\partial x^2} - \frac{\partial A_3}{\partial x^3} = 0.$$

With this calibration condition, the equation for the electromagnetic potential is reduced to the well-known wave equation.

Proposition 4.11 (Wave equation)**.** *Given an electromagnetic potential $\boldsymbol{\mathcal{A}}$ in the Lorenz gauge, we have $\Delta\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{J}}$.*

Proof. Compute

$$\Delta\boldsymbol{\mathcal{A}} = (\mathrm{d}\delta + \delta\mathrm{d})\,\boldsymbol{\mathcal{A}} = \delta\,\mathrm{d}\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{J}}.$$

□

In coordinates, the wave equation for the electromagnetic potential is

$$\frac{\partial^2 A_i}{\partial {x^0}^2} - \frac{\partial^2 A_i}{\partial {x^1}^2} - \frac{\partial^2 A_i}{\partial {x^2}^2} - \frac{\partial^2 A_i}{\partial {x^3}^2} = J_i.$$

Notice that we never had to specify some special "wave operator" acting on vectors. The information necessary to get to the hyperbolic partial differential equation is included in the Minkowski inner product field.

The Lorenz gauge is still not sufficient to constrain the electromagnetic potential fully, however it gives us some space to work with. More specifically, we have the following proposition:

Proposition 4.12. *Let $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{A}}'$ be electromagnetic potentials of $\boldsymbol{\mathcal{F}}$ in the Lorenz gauge. Then $\boldsymbol{\mathcal{A}} - \boldsymbol{\mathcal{A}}'$ is harmonic.*

Proof. We consider

$$\Delta(\boldsymbol{\mathcal{A}} - \boldsymbol{\mathcal{A}}') = \boldsymbol{\mathcal{J}} - \boldsymbol{\mathcal{J}} = \mathbf{0}.$$

□

Beware that if a 1-form satisfies the wave equation, it is not necessarily co-closed. We thus need to still make sure that the electromagnetic potential is in the Lorenz gauge.

We can also formulate the wave equation for the electromagnetic field itself in a very similar fashion.

Proposition 4.13 (Wave equation for the electromagnetic field)**.** *Given an electromagnetic field $\boldsymbol{\mathcal{F}}$ associated to $\boldsymbol{\mathcal{J}}$, we have*

$$\Delta\boldsymbol{\mathcal{F}} = \mathrm{d}\boldsymbol{\mathcal{J}}\,.$$

[1] Named after Ludvig Lorenz, not Hendrik Lorentz after whom some other concepts in this thesis are named.

[2] We are not going to prove that any electromagnetic field actually has an electromagnetic potential in the Lorenz gauge. This may in fact require some additional constraints on the electromagnetic field itself.

4.3 Isometries of the Minkowski Space

To study coordinate transformations of the Maxwell's equations, we would like to find out which smooth maps of the Minkowski space are isometries. In our case, isometries formalize the idea of transforming between "inertial coordinate frames of reference" which are coordinate systems where laws of special relativity hold in unchanged form.

Proposition 4.14 (Maxwell's equations and isometries). *Maxwell's equations are invariant under isometries. That is, given an isometry φ of $\mathbf{M}$ and an electromagnetic field $\boldsymbol{\mathcal{F}}$ associated with $\boldsymbol{\mathcal{J}}$, $\varphi^*\boldsymbol{\mathcal{F}}$ is an electromagnetic field associated with $\varphi^*\boldsymbol{\mathcal{J}}$.*

Additionally, if $\boldsymbol{\mathcal{A}}$ is an electromagnetic potential of $\boldsymbol{\mathcal{F}}$, then $\varphi^*\boldsymbol{\mathcal{A}}$ is an electromagnetic potential of $\varphi^*\boldsymbol{\mathcal{F}}$ and if $\boldsymbol{\mathcal{A}}$ is in the Lorenz gauge then $\varphi^*\boldsymbol{\mathcal{A}}$ is also in the Lorenz gauge.

Proof. Call our isometry φ. Applying φ^* to the homogeneous equation yields

$$\varphi^*(\mathrm{d}\boldsymbol{\mathcal{F}}) = \mathrm{d}(\varphi^*\boldsymbol{\mathcal{F}}) = \mathbf{0}$$

and pulling back the inhomogeneous equation results in

$$\varphi^*(\delta\boldsymbol{\mathcal{F}}) = \varphi^*(\star\mathrm{d}\star\boldsymbol{\mathcal{F}}) = \delta(\varphi^*\boldsymbol{\mathcal{F}}) = \varphi^*\boldsymbol{\mathcal{J}}.$$

If $\mathrm{d}\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{F}}$, then $\varphi^*\mathrm{d}\boldsymbol{\mathcal{A}} = \mathrm{d}\varphi^*\boldsymbol{\mathcal{A}} = \varphi^*\boldsymbol{\mathcal{F}}$ and if $\delta\boldsymbol{\mathcal{A}} = \mathbf{0}$ then $\varphi^*\delta\boldsymbol{\mathcal{A}} = \delta\varphi^*\boldsymbol{\mathcal{A}} = \mathbf{0}$. □

In the following examples, we are going to parametrize several important families of isometries on the Minkowski space. By applying the definition, we can easily see that:

Example 4.15 (Translational isometries). Smooth maps given by

$$y^i = x^i + x_0^i,$$

where x_0^i are constant are isometries.

As composition of isometries is again an isometry, we can now restrict ourselves to mappings which leave the origin fixed.

Example 4.16 (Parity transformations). Smooth maps given by

$$y^i = \pm x^i.$$

are also isometries.

Example 4.17 (Lorentz boosts). Now we wish to find out which smooth maps involving the time coordinate and a single spatial coordinate are isometries. These transformations are usually called *Lorentz boosts* in the literature.

We start with defining our coordinate transformation $\varphi : \mathbf{M} \to \mathbf{M}$ as

$$\begin{aligned} y^0 &= f(x^0, x^1), & y^1 &= g(x^0, x^1), \\ y^2 &= x^2, & y^3 &= x^3, \end{aligned}$$

where f and g are some unknown smooth functions such that $f(0,0) = g(0,0) = 0$.

For brevity, we set $\frac{\partial f}{\partial x^i}(x^0, x^1) = f_i(x^0, x^1)$ and compute our cotangent basis transformations as

$$\begin{aligned}
\varphi^*(\mathrm{d}y^0) &= f_0(x^0, x^1)\,\mathrm{d}x^0 + f_1(x^0, x^1)\,\mathrm{d}x^1\,, \\
\varphi^*(\mathrm{d}y^1) &= g_0(x^0, x^1)\,\mathrm{d}x^0 + g_1(x^0, x^1)\,\mathrm{d}x^1\,, \\
\varphi^*(\mathrm{d}y^2) &= \mathrm{d}x^2\,, \\
\varphi^*(\mathrm{d}y^3) &= \mathrm{d}x^3\,.
\end{aligned}$$

We wish to have an isometry, in other words we wish that any pair of cotangent vectors $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbf{M}_p^*$ satisfy $\varphi^*\langle\boldsymbol{\alpha}|\boldsymbol{\beta}\rangle = \langle\varphi^*\boldsymbol{\alpha}|\varphi^*\boldsymbol{\beta}\rangle$. Obviously the third and fourth equations do not add any information about our solution, so we are left with

$$\begin{aligned}
\langle\varphi^*(\mathrm{d}y^0)|\varphi^*(\mathrm{d}y^0)\rangle &= -(f_0)^2 + (f_1)^2 = -1 = \varphi^*\langle\mathrm{d}y^0|\,\mathrm{d}y^0\rangle, \\
\langle\varphi^*(\mathrm{d}y^1)|\varphi^*(\mathrm{d}y^1)\rangle &= -(g_0)^2 + (g_1)^2 = 1 = \varphi^*\langle\mathrm{d}y^1|\,\mathrm{d}y^1\rangle, \\
\langle\varphi^*(\mathrm{d}y^1)|\varphi^*(\mathrm{d}y^0)\rangle &= -f_0 g_0 + f_1 g_1 = 0 = \varphi^*\langle\mathrm{d}y^1|\,\mathrm{d}y^0\rangle.
\end{aligned}$$

These are tabulated partial differential equations (see [Pol] for details). From the first two equations we have solutions of the form

$$\begin{aligned}
f(x^0, x^1) &= A_0 x^0 - A_1 x^1 && \text{where } A_0^2 - A_1^2 = 1, \\
g(x^0, x^1) &= -B_0 x^0 + B_1 x^1 && \text{where } B_1^2 - B_0^2 = 1.
\end{aligned}$$

Substituting the solutions into the third equation additionally yields

$$A_0 B_0 = A_1 B_1. \tag{4.1}$$

To get our solution into a more traditional form, we notice that the coefficients A_0 and A_1 are constrained to lie on a hyperbola. If we pick a single branch of said hyperbola[3], we can get a hyperbolic angle θ (illustrated by Figure 4.1) such that

$$A_0 = \cosh(\theta), \quad A_1 = \sinh(\theta).$$

For the second pair of coefficient we temporarily pick a hyperbolic angle ψ and set

$$B_1 = \cosh(\psi), \quad B_0 = \sinh(\psi).$$

Substituting these equalities to equation 4.1 yields

$$\cosh(\theta)\sinh(\psi) = \sinh(\theta)\cosh(\psi)$$

and thus $\psi = \theta$ as tanh is bijective.

Parametrized using the parameter θ, our coordinate transformation is given

[3]Note that this removes some valid solutions. Namely we are keeping only those Lorentz boosts which are orientation preserving.

by

$$
\begin{aligned}
y^0 &= \cosh(\theta)x^0 - \sinh(\theta)x^1, \\
y^1 &= -\sinh(\theta)x^0 + \cosh(\theta)x^1, \\
y^2 &= x^2, \\
y^3 &= x^3.
\end{aligned}
$$

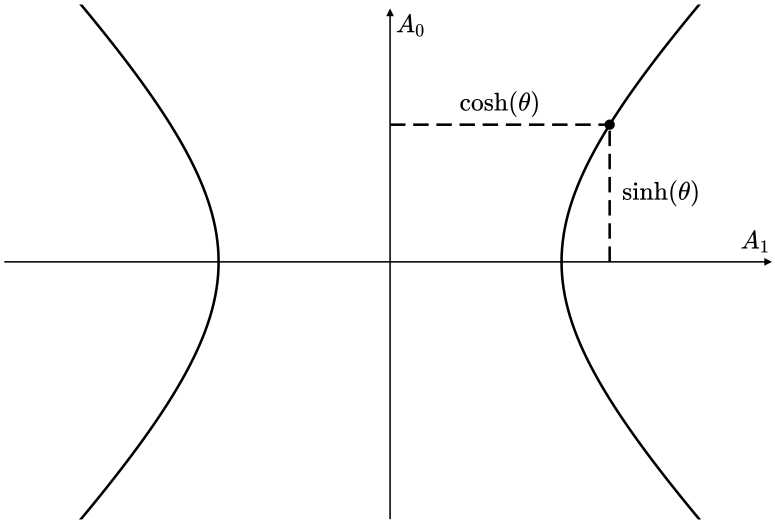

Figure 4.1: Parameters of the Lorentz boost

Another parametrization of the Lorentz boost which is quite popular in the literature can be obtained by substituting

$$\beta = \tanh(\theta) \quad \text{and} \quad \gamma = \cosh(\theta),$$

yielding

$$y^0 = \gamma(x^0 - \beta x^1) \quad \text{and} \quad y^1 = \gamma(x^1 - \beta x^0).$$

Beware that this can be somewhat misleading as the parameters γ and β cannot be chosen independently. The parameter β is usually intrepreted as the *relative velocity* of the two coordinate systems and the parameter γ is then called the *Lorentz factor*.

Computing Example 4.17 using a computer algebra system is additionally shown in Notebook A.4.

Example 4.18 (Spatial rotations)**.** What is left are isometries involving two spatial coordinates. Our coordinate transform is then given as

$$
\begin{aligned}
y^0 &= x^0, \\
y^1 &= f(x^1, x^2), \\
y^2 &= g(x^1, x^2), \\
y^3 &= x^3.
\end{aligned}
$$

Applying the same approach as in Example 4.17 yields

$$\begin{aligned}(f_1)^2 + (f_2)^2 &= 1,\\ (g_1)^2 + (g_2)^2 &= 1,\\ f_1 g_1 + f_2 g_2 &= 0.\end{aligned}$$

Solutions for these partial differential equations are

$$\begin{aligned}f(x^1, x^2) &= A_1 x^1 + A_2 x^2 \qquad && \text{where } A_1^2 + A_2^2 = 1,\\ g(x^1, x^2) &= B_1 x^1 + B_2 x^2 \qquad && \text{where } B_1^2 + B_2^2 = 1.\end{aligned}$$

The third equation gives an additional constraint in the form of

$$A_1 B_1 + A_2 B_2 = 0. \tag{4.2}$$

Note that the parameters A_1 and A_2 are constrained to the unit circle. This means we can find an angle θ such that

$$A_1 = \cos(\theta), \quad A_2 = -\sin(\theta).$$

equation (4.2) gives us two ways of picking the coefficients B_1 and B_2. To get a positively oriented isometry we have to pick

$$B_1 = \sin(\theta), \quad B_2 = \cos(\theta).$$

Our parametrized isometry is thus given by

$$\begin{aligned}y^0 &= x^0,\\ y^1 &= \cos(\theta)x^1 - \sin(\theta)x^2,\\ y^2 &= \sin(\theta)x^1 + \cos(\theta)x^2,\\ y^3 &= x^3.\end{aligned}$$

Remark 4.19. It is interesting to note that our isometries turned out to be affine in the global coordinate vectors. This is often silently assumed, but we had no reason to believe that would be the case, as we generally only care about the topological structure of $\mathbf{M}$. It can however be shown that every isometry of the Minkowski space can be expressed as such. A proof of this fact can be found in [Wei72].

An important consequence is that the position 1-form "looks the same" after changing coordinates using an isometry which leaves the origin fixed.

Of great importance are scalars which are invariant under change of inertial coordinate system. We are going to compute two of these in the next proposition.

Proposition 4.20 (Invariants of the electromagnetic field). *Given an electromagnetic field $\boldsymbol{\mathcal{F}}$, the following two scalar functions are invariant under orientation-preserving isometries*

$$\star(\boldsymbol{\mathcal{F}} \wedge \star\boldsymbol{\mathcal{F}}) \quad \text{and} \quad \star(\boldsymbol{\mathcal{F}} \wedge \boldsymbol{\mathcal{F}})\,.$$

Proof. Naming our isometry as φ and using Proposition 3.29, we compute

$$\varphi^*(\star(\mathcal{F} \wedge \star\mathcal{F})) = \star((\varphi^*\mathcal{F}) \wedge \star(\varphi^*\mathcal{F}))$$

and

$$\varphi^*(\star(\mathcal{F} \wedge \mathcal{F})) = \star((\varphi^*\mathcal{F}) \wedge (\varphi^*\mathcal{F}))\,.$$

□

Expanding $\mathcal{F}$ in components, we use Proposition 2.32 to observe that the first invariant is just the magnitude of $\mathcal{F}$ and thus we have

$$\begin{aligned}\star(\mathcal{F} \wedge \star\mathcal{F}) &= -\langle\mathcal{F}|\mathcal{F}\rangle \\ &= -\langle\mathcal{E} \wedge \mathrm{d}x^0 + \mathcal{B}|\mathcal{E} \wedge \mathrm{d}x^0 + \mathcal{B}\rangle \\ &= \langle\mathcal{E}|\mathcal{E}\rangle - \langle\mathcal{B}|\mathcal{B}\rangle.\end{aligned}$$

For the second invariant we have

$$\begin{aligned}\star(\mathcal{F} \wedge \mathcal{F}) &= \star\Big((\mathcal{E} \wedge \mathrm{d}x^0 + \mathcal{B}) \wedge (\mathcal{E} \wedge \mathrm{d}x^0 + \mathcal{B})\Big) \\ &= \star\Big(\mathcal{B} \wedge \mathcal{E} \wedge \mathrm{d}x^0 + \mathcal{E} \wedge \mathrm{d}x^0 \wedge \mathcal{B}\Big) \\ &= 2\star\Big(\mathcal{B} \wedge \mathcal{E} \wedge \mathrm{d}x^0\Big)\,.\end{aligned}$$

4.4 Vacuum Fields

Definition 4.21 (Vacuum electromagnetic field)**.** An electromagnetic field $\mathcal{F}$ is called a *vacuum electromagnetic field* if its source 1-form is the zero form. In other words, we have

$$\begin{aligned}\mathrm{d}\mathcal{F} &= \mathbf{0}, \\ \delta\mathcal{F} &= \mathbf{0}.\end{aligned}$$

If $\mathcal{A}$ is an electromagnetic potential of $\mathcal{F}$, we have

$$\delta\,\mathrm{d}\mathcal{A} = \mathbf{0}.$$

As vacuum electromagnetic fields are both closed and co-closed, they exhibit an interesting kind of symmetry.

Proposition 4.22. *Let $\mathcal{F}$ be a vacuum electromagnetic field. Then $\star\mathcal{F}$ is also a vacuum electromagnetic field.*

In essence, this means that we can exchange the components of the magnetic 2-form and the electric 1-form (in a way that is compatible with the Hodge star) and still get another valid vacuum electromagnetic field.

Example 4.23 (Plane wave)**.** Take an arbitrary[4] smooth function $f : \mathbf{R} \to \mathbf{R}$ and any *angular frequency*[5] $\omega \in \mathbf{R}$. We want to consider an electromagnetic

[4]In practice, it is often convenient to work with complex-valued differential forms. These can be introduced without much issue, see [DK16] for an example.

[5]Note that f is not necessarily periodic, meaning our name for ω is just a variable name.

potential given by

$$\boldsymbol{\mathcal{A}} = f(\omega(x^1 - x^0))\,\mathrm{d}x^2\,.$$

Before we proceed, we are going to introduce alternative coordinates on $\mathbf{M}$, given by

$$x^0 = y^+ - y^-, \quad x^1 = y^+ + y^-, \quad x^2 = y^2, \quad x^3 = y^3,$$

valid on $\mathbf{R}^4$. The coordinates y^+ and y^- are usually called *null coordinates*, as we have $\langle \mathrm{d}y^+ | \,\mathrm{d}y^+\rangle = \langle \mathrm{d}y^- | \,\mathrm{d}y^-\rangle = 0$. Note that our coframe is no longer orthonormal, which makes computing the Hodge star somewhat more involved.

In these coordinates, we have

$$\boldsymbol{\mathcal{A}} = f(2\omega y^-)\,\mathrm{d}y^2\,.$$

Now we compute the electromagnetic field as

$$\boldsymbol{\mathcal{F}} = \mathrm{d}\boldsymbol{\mathcal{A}} = 2\omega f'(2\omega y^-)\,\mathrm{d}y^- \wedge \mathrm{d}y^2\,.$$

Next we want to show that $\boldsymbol{\mathcal{F}}$ is a vacuum electromagnetic field. It suffices to show $\mathrm{d}{\star}\boldsymbol{\mathcal{F}} = \mathbf{0}$, by

$$\mathrm{d}{\star}\boldsymbol{\mathcal{F}} = \mathrm{d}\Big(2\omega f'(2\omega y^-)\,\mathrm{d}y^3 \wedge \mathrm{d}y^-\Big) = \mathbf{0},$$

as $\star\big(\mathrm{d}y^- \wedge \mathrm{d}y^2\big) = \mathrm{d}y^3 \wedge \mathrm{d}y^-$.

If we want to express $\boldsymbol{\mathcal{F}}$ in cartesian coordinates, we can use the inverse coordinate transformation, given by

$$y^+ = \frac{1}{2}x^1 + \frac{1}{2}x^0, \quad y^- = \frac{1}{2}x^1 - \frac{1}{2}x^0, \quad y^2 = x^2, \quad y^3 = x^3.$$

Which then results in

$$\boldsymbol{\mathcal{F}} = \omega f'(\omega(x^1 - x^0))(\mathrm{d}x^1 - \mathrm{d}x^0) \wedge \mathrm{d}x^2\,.$$

Now that we have an example of a vacuum electromagnetic field, we are going to solve a classical problem, first explored in the seminal work on special relativity [Ein05].

Example 4.24 (Relativistic longtitudal Doppler effect)**.** We want to explore how the plane wave changes after undergoing a Lorentz boost to a different inertial reference frame moving in the direction of propagation.[6] We setup our coordinate transformation φ as

$$\begin{aligned} x^0 &= \cosh(\theta)y^0 - \sinh(\theta)y^1, \\ x^1 &= -\sinh(\theta)y^0 + \cosh(\theta)y^1 \end{aligned}$$

leaving the rest of coordinates unchanged. The pullback of a plane wave

[6] We have not defined "direction of propagation" as this requires the stress-energy tensor.

solution is then

$$\begin{aligned}\varphi^*\boldsymbol{\mathcal{A}} &= \varphi^*(f(\omega(x^1 - x^0))\,\mathrm{d}x^2)\\ &= f(\omega(-\sinh(\theta)y^0 + \cosh(\theta)y^1 - \cosh(\theta)y^0 + \sinh(\theta)y^1))\,\mathrm{d}y^2\\ &= f(\underbrace{\omega(\cosh(\theta) + \sinh(\theta))}_{\omega'}(x^1 - x^0))\,\mathrm{d}y^2\,.\end{aligned}$$

Thus our angular frequency changed by a factor of

$$\frac{\omega'}{\omega} = \cosh(\theta) + \sinh(\theta) = \sqrt{\frac{1+\beta}{1-\beta}}.$$

We have two classical approximations for the Doppler shift, given by

$$\frac{\omega'}{\omega} \approx 1 + \beta, \quad \frac{\omega'}{\omega} \approx \frac{1}{1-\beta},$$

for a moving receiver and a moving transmitter respectively. The first expression is the first order Taylor expansion of ω'/ω and the second expression is the Taylor expansion of ω/ω'. These approximations are depicted by Figure 4.2.

What is left is to compute the amplitude of the plane wave in the new coordinate system. We can just argue by symmetry of the transformed potential and get

$$\varphi^*\boldsymbol{\mathcal{F}} = \omega' f(\omega'(y^1 - y^0))(\mathrm{d}x^1 - \mathrm{d}x^0) \wedge \mathrm{d}x^2\,.$$

That is, the relative amplitude ratio is again given by ω'/ω.

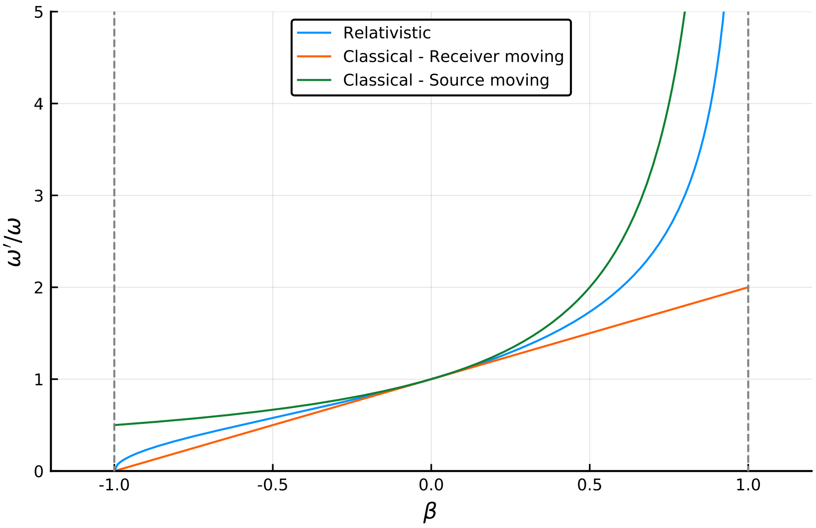

Figure 4.2: Comparison the the relativistic Doppler shift and its classical approximations

The discussion in Example 4.23 also shows that there are no longtitudal plane waves in vacuum.

Proposition 4.25. *Let $f : \mathbf{R} \to \mathbf{R}$ be an arbitrary smooth function and consider a vacuum electromagnetic field $\boldsymbol{\mathcal{F}}$ with a potential given by $\boldsymbol{\mathcal{A}} = f(x^1 - x^0)\,\mathrm{d}x^1$. Then $\boldsymbol{\mathcal{F}}$ is invariant under translational isometries.*

Proof. We first compute $\boldsymbol{\mathcal{F}}$ as

$$\mathrm{d}\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{F}} = f'(x^1 - x^0)\,\mathrm{d}x^1 \wedge \mathrm{d}x^0$$

and then the inhomogeneous Maxwell's equation by

$$\begin{aligned}
\mathrm{d}\star\boldsymbol{\mathcal{F}} &= \mathrm{d}\Big(f'(x^1 - x^0)\,\mathrm{d}x^2 \wedge \mathrm{d}x^3\Big) \\
&= -f''(x^1 - x^0)\,\mathrm{d}x^0 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + f''(x^1 - x^0)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\
&= \mathbf{0}.
\end{aligned}$$

Therefore f is a linear function, that is, $f(x^1 - x^0) = A(x^1 - x^0) + B$ for some real constants A and B. But, the electromagnetic field is then given by

$$\boldsymbol{\mathcal{F}} = A\,\mathrm{d}x^1 \wedge \mathrm{d}x^0\,,$$

which is just an electrostatic field. □

Example 4.26 (Field generated by a point charge)**.** We want to compute the electrical field generated by a stationary charge at the origin. First we note that the charge density is not going to be defined at the origin and thus we restrict ourselves to

$$\mathbf{M}_0 = \{(x^0, x^1, x^2, x^3) \in \mathbf{M} \mid x^1 \neq 0 \;\;\&\;\; x^2 \neq 0 \;\;\&\;\; x^3 \neq 0\},$$

that is, the Minkowski space with a line removed. We wish to argue by symmetry, so we introduce spherical coordinates on the domain

$$S = \{(t, r, \theta, \varphi) \in \mathbf{R}^4 \mid r \in (0, \infty) \;\;\&\;\; \theta \in (0, \pi) \;\;\&\;\; \varphi \in (0, 2\pi)\}$$

and a coordinate change map $\psi : S \to \mathbf{M}_0$ given by

$$\begin{aligned}
x^0 &= t, & x^1 &= r\cos(\varphi)\sin(\theta), \\
x^2 &= r\sin(\varphi)\sin(\theta), & x^3 &= r\cos(\theta).
\end{aligned}$$

As the setup is spherically symmetrical and invariant in time, we are going to assume that the components of the electric 1-form depend only on the radial coordinate.

We thus parametrize our electric 1-form as

$$\boldsymbol{\mathcal{E}} = E_r(r)\,\mathrm{d}r + E_\theta(r)\,\mathrm{d}\theta + E_\varphi(r)\,\mathrm{d}\varphi\,.$$

We want to argue by symmetry further, assuming that the result is symmetric under reflections. We setup a pair of coordinate transformations given by

$$\begin{aligned}
\psi_1 : S \to S \quad (t, r, \theta, \varphi) &\mapsto (t, r, -\theta, \varphi), \\
\psi_2 : S \to S \quad (t, r, \theta, \varphi) &\mapsto (t, r, \theta, -\varphi).
\end{aligned}$$

Note that the composition $\psi_1 \circ \psi$ is just ψ composed with

$$(x^0, x^1, x^2, x^3) \mapsto (x^0, -x^1, -x^2, x^3)$$

and similarly for $\psi_2 \circ \psi$ we have ψ composed with

$$(x^0, x^1, x^2, x^3) \mapsto (x^0, x^1, -x^2, x^3)$$

We would surely expect our resulting field to be invariant with respect to these transformations, meaning we want

$$\psi_1{}^* \boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}$$
$$E_r(r)\,\mathrm{d}r - E_\theta(r)\,\mathrm{d}\theta + E_\varphi(r)\,\mathrm{d}\varphi = E_r(r)\,\mathrm{d}r + E_\theta(r)\,\mathrm{d}\theta + E_\varphi(r)\,\mathrm{d}\varphi$$

and also

$$\psi_2{}^* \boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}$$
$$E_r(r)\,\mathrm{d}r + E_\theta(r)\,\mathrm{d}\theta - E_\varphi(r)\,\mathrm{d}\varphi = E_r(r)\,\mathrm{d}r + E_\theta(r)\,\mathrm{d}\theta + E_\varphi(r)\,\mathrm{d}\varphi$$

which forces $E_\theta(r) = E_\varphi(r) = 0$. Now we need to actually enforce that the electromagnetic field, given by $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{E}} \wedge \mathrm{d}t$ is closed and co-closed.

To show that $\boldsymbol{\mathcal{F}}$ is closed, we just need to show that $\boldsymbol{\mathcal{E}}$ is closed and thus compute

$$\mathrm{d}\boldsymbol{\mathcal{E}} = \mathrm{d}(E_r(r)\,\mathrm{d}r) = 0,$$

adding no additional restrictions on E_r.

To get $\boldsymbol{\mathcal{F}}$ to be co-closed, we just have to ensure that $\mathrm{d}{\star}\boldsymbol{\mathcal{F}} = \mathbf{0}$. We have

$$\begin{aligned}
\mathrm{d}{\star}\boldsymbol{\mathcal{F}} &= \mathrm{d}\,{\star}(E_r(r)\,\mathrm{d}r \wedge \mathrm{d}t) \\
&= \mathrm{d}\Big(E_r(r) r^2 \sin(\theta)\,\mathrm{d}\theta \wedge \mathrm{d}\varphi\Big) \\
&= \left(2rE_r(r) + r^2 \frac{\partial E_r}{\partial r}\right) \sin(\theta)\,\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\varphi \\
&= \mathbf{0}.
\end{aligned}$$

As both r and $\sin(\theta)$ are positive on our coordinate range, we can divide and get an ordinary differential equation given by

$$2E_r(r) + rE_r'(r) = 0.$$

This differential equation then has solutions of the form

$$E_r(r) = \frac{A}{r^2}$$

where A is an arbitrary real constant.

If we want to express the electromagnetic field in cartesian coordinates, we can use the position 1-form and get

$$\boldsymbol{\mathcal{F}} = A \left(\frac{\boldsymbol{\mathcal{R}}}{((x^1)^2 + (x^2)^2 + (x^3)^2)^{\frac{3}{2}}} \right) \wedge \mathrm{d}x^0$$

as exterior multiplication annihilates the $\mathrm{d}x^0$ term. Note that we cannot really consider a multiple of $\boldsymbol{\mathcal{R}}$ an electric 1-form as per Definition 4.5 because it contains a non-zero $\mathrm{d}x^0$ term. However this expression will be useful in the next example.

Example 4.27 (Moving point charge)**.** We want to transform the previously acquired expression for the electromagnetic field of a stationary charge to a moving inertial coordinate frame in the direction of the x^3 axis. We set $r_x = ((x^1)^2 + (x^2)^2 + (x^3)^2)^{\frac{1}{2}}$ and normalize by setting $A = 1$, getting

$$\boldsymbol{\mathcal{F}} = \frac{\boldsymbol{\mathcal{R}}}{r_x^3} \wedge \mathrm{d}x^0 .$$

We setup our coordinate transformation φ as

$$x^0 = \gamma(y^0 - \beta y^3) \quad \text{and} \quad x^3 = \gamma(y^3 - \beta y^0)$$

leaving the rest of coordinates unchanged. We also set $r_y = ((y^1)^2 + (y^2)^2 + (y^3)^2)^{\frac{1}{2}}$ As per the definition of the position 1-form and Remark 4.19, we have

$$\begin{aligned} r_x^2 &= \langle \boldsymbol{\mathcal{R}} | \boldsymbol{\mathcal{R}} \rangle + (x^0)^2, \\ r_y^2 &= \langle \boldsymbol{\mathcal{R}} | \boldsymbol{\mathcal{R}} \rangle + (y^0)^2 \end{aligned}$$

and thus

$$r_x^2 = r_y^2 + \underbrace{(x^0)^2 - (y^0)^2}_{s^2} = r_y^2 + s^2.$$

Transforming the electromagnetic field then results in

$$\boldsymbol{\mathcal{F}} = \frac{\boldsymbol{\mathcal{R}}}{(r_y^2 + s^2)^{\frac{3}{2}}} \wedge \gamma(\mathrm{d}y^0 - \beta\,\mathrm{d}y^3).$$

This expression is still not very satisfying, so we transform it to a cylindrical coordinate system by

$$y^0 = t, \quad y^1 = \rho\cos(\theta), \quad y^2 = \rho\sin(\theta), \quad y^3 = z.$$

Now we have

$$\boldsymbol{\mathcal{R}} = -t\,\mathrm{d}t + \rho\,\mathrm{d}\rho + z\,\mathrm{d}z\,, \quad \mathrm{d}y^0 = \mathrm{d}t\,, \quad \mathrm{d}y^3 = \mathrm{d}z\,.$$

which results in

$$\begin{aligned} \boldsymbol{\mathcal{F}} &= \frac{-t\,\mathrm{d}t + \rho\,\mathrm{d}\rho + z\,\mathrm{d}z}{(r_y^2 + s^2)^{\frac{3}{2}}} \wedge \gamma(\mathrm{d}t - \beta\,\mathrm{d}z) \\ &= \underbrace{\frac{\gamma}{(r_y^2 + s^2)^{\frac{3}{2}}}}_{K} \left((\rho\,\mathrm{d}\rho + z\,\mathrm{d}z) \wedge \mathrm{d}t + \beta\,(t\,\mathrm{d}t - \rho\,\mathrm{d}\rho) \wedge \mathrm{d}z\right) \\ &= K\left((\rho\,\mathrm{d}\rho + z\,\mathrm{d}z) \wedge \mathrm{d}t + \beta\,(t\,\mathrm{d}t - \rho\,\mathrm{d}\rho) \wedge \mathrm{d}z\right) \\ &= K\left((\rho\,\mathrm{d}\rho + z\,\mathrm{d}z) \wedge \mathrm{d}t - \beta t\,\mathrm{d}z \wedge \mathrm{d}t + \beta\rho\,\mathrm{d}z \wedge \mathrm{d}\rho\right) \\ &= K\left((\rho\,\mathrm{d}\rho + (z - \beta t)\,\mathrm{d}z) \wedge \mathrm{d}t + \beta\rho\,\mathrm{d}z \wedge \mathrm{d}\rho\right) \end{aligned}$$

Now we can properly separate the electric and magnetic components

$$\boldsymbol{\mathcal{E}} = K\left(\rho\,\mathrm{d}\rho + (z - \beta t)\,\mathrm{d}z\right) \quad \text{and} \quad \boldsymbol{\mathcal{B}} = K\beta\rho\,\mathrm{d}z \wedge \mathrm{d}\rho\,.$$

Example 4.28 (Homopolar generator)**.** A homopolar generator is a device which generates electricity by rotating a conductive disk in a homogenous magnetic field. We are going to compute the electromagnetic field as seen from the rotating coordinate frame of the disk.

The magnetic 2-form is homogeneous and oriented along the plane spanned by the x^1 and x^2 axis, that is

$$\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{B}} = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \,.$$

Classically, we would say that the magnetic field points in the direction of the x^3 axis, however this does not translate well to our visualization of 2-forms as parallelograms.

First, we transform $\boldsymbol{\mathcal{F}}$ to cylindrical coordinates on the domain

$$P = \{(t, r, \theta, z) \in \mathbf{R}^4 \mid r > 0 \ \ \& \ \ 0 < \theta < 2\pi\}$$

where our smooth map $\varphi : P \to \mathbf{M}$ is given as

$$x^0 = t, \quad x^1 = r\cos(\theta), \quad x^2 = r\sin(\theta), \quad x^3 = z$$

and thus

$$\varphi^* \mathrm{d}x^1 = \cos(\theta)\,\mathrm{d}r - r\sin(\theta)\,\mathrm{d}\theta\,, \quad \varphi^* \mathrm{d}x^2 = \sin(\theta)\,\mathrm{d}r + r\cos(\theta)\,\mathrm{d}\theta\,.$$

Which results in

$$\varphi^* \boldsymbol{\mathcal{F}} = r\,\mathrm{d}r \wedge \mathrm{d}\theta\,.$$

Now we want to rotate this coordinate system at constant angular velocity ω. We take the set

$$P' = \{(t', r', \theta', z') \in \mathbf{R}^4 \mid r > 0 \ \ \& \ \ 0 < \theta' - \omega t' < 2\pi\}$$

and define a smooth map $\psi : P' \to P$ given by

$$t = t', \quad r = r', \quad \theta = \theta' - \omega t', \quad z = z',$$

This means we have

$$\psi^* \mathrm{d}r = \mathrm{d}r'\,, \quad \psi^* \mathrm{d}\theta = \mathrm{d}\theta' - \omega\,\mathrm{d}t'$$

and our electromagnetic field then looks like

$$\begin{aligned} \phi^* \varphi^* \boldsymbol{\mathcal{F}} &= r'\,\mathrm{d}r' \wedge (\mathrm{d}\theta' - \omega\,\mathrm{d}t') \\ &= r'\,\mathrm{d}r' \wedge \mathrm{d}\theta' - r'\omega\,\mathrm{d}r' \wedge \mathrm{d}t'\,. \end{aligned}$$

Thus in our rotating coordinate frame we have obtained a non-zero electric field in the radial direction. However beware, we have defined the components of the electromagnetic field on $\mathbf{M}$ and verified that Maxwell's equations hold in unchanged form only if our transformations are isometric. Therefore interpreting the components stands on somewhat shaky ground.

If we also pullback the inner product field, we notice that our frame and coframe are no longer orthonormal. Notably, we have

$$\langle \boldsymbol{\partial}_{t'} | \boldsymbol{\partial}_{t'} \rangle = \omega^2 r'^2 - 1 \quad \text{and} \quad \langle \boldsymbol{\partial}_{t'} | \boldsymbol{\partial}_{\theta'} \rangle = -\omega r'^2 .$$

Now we see that the magnitude of $\boldsymbol{\partial}_{t'}$ is not constant and even swaps signs as the tangential velocity approaches 1 (that is, the speed of light), thus interpreting the t' coordinate as the "time" may be somewhat ill-conceived.

To further illustrate, we are going to pretend that we have obtained $\phi^* \varphi^* \boldsymbol{\mathcal{F}}$ just by transforming an electromagnetic field to polar coordinates. That is, we take

$$\boldsymbol{\mathcal{F}}' = r \, \mathrm{d}r \wedge \mathrm{d}\theta - r\omega \, \mathrm{d}r \wedge \mathrm{d}t \, .$$

We know that the homogeneous Maxwell's equation has to be satisfied as pullbacks commute with exterior differentiation, however the inhomogeneous equation might not be. So we compute $\delta \boldsymbol{\mathcal{F}}'$

$$\begin{aligned} \delta \boldsymbol{\mathcal{F}}' &= \star \mathrm{d} \star \boldsymbol{\mathcal{F}}' \\ &= \star \mathrm{d} \Big(\mathrm{d}t \wedge \mathrm{d}z - r^2 \omega \, \mathrm{d}\theta \wedge \mathrm{d}z \Big) \\ &= \star (-2 r\omega \, \mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}z) \\ &= 2\omega \, \mathrm{d}t \, . \end{aligned}$$

And now unsurprisingly, our source 1-form is not zero. Of course this is only possible because we purposefully ignored applying ψ^* to our original electromagnetic field. If we pullback the Minkowski inner product field, the Hodge star acts in a compatible way and the codifferential produces zero.

This example is also presented in Notebook A.8.

Chapter 5

Conclusion

In this thesis, we first gave an overview of the tools necessary for formulating the theory of electrodynamics in the language of differential forms. Afterwards, we discussed the laws of electromagnetism on the Minkowski spacetime both in their general form and on concrete examples.

This text only scratches the surface of the tools available in differential geometry. There are many ways in which we could continue our study, some of which are summarized in the text below.

The most evident omission is the lack of integration on differential forms. This is a crucial concept, allowing us to generalize the well known divergence, Green's and Kelvin-Stokes theorems into arbitrary dimensions. Any text on differential geometry will be a sufficient resource for this topic.

Another missing topic is constructing differential forms on smooth manifolds as opposed to the restriction on open subsets of $\mathbf{R}^n$ we have employed here. This may seem overly general, but, as an example, we often deal in practice with electromagnetic fields which are periodic in time or space. The circle manifold is then a natural environment for such fields.

Altough not critical for the laws of physics, introducing material dependencies would be convenient from a practical standpoint. See [Fra17], [Fla89] or [con20] for hints as to how to implement these.

The Fourier transform can also be extended to work on certain smooth manifolds, being interpreted as decomposition of differential forms into a sum of of eigenvectors of the Laplace-de Rham operator, as explained in [Won]. This is a concept frequently employed in the study of waveguides.

From the perspective of numerical computations, discrete differential geometry is an active area of research, serving as an analogy of differential geometry performed on discrete meshes. A standard reference can be found in [Cra]. An implementation of discrete exterior calculus for the Julia programming language can be found in [Sch], including an example computing the cavity resonator modes of a rectangular box.

Appendix A

Computer Algebra Systems

There are several computer algebra packages which have support for performing various differential-geometric computations. We are going to focus on SageMath, with its comprehensive built-in SageManifolds library [sag20]. Other options are available, such as SymPy [MSP⁺17] or Maple with its DifferentialGeometry package.

For a complete explanation of the various functionality implemented in SageManifolds, the reader is encouraged to visit the official tutorials and examples.

The rest of this appendix is provided as a set of companion files for various examples in this thesis.

Notebook A.1 (Polar). *This notebook shows how to compute transformations to polar coordinates as shown in Example 3.25.*

Notebook A.2 (PolarCharts). *This notebook is a version of Notebook A.1, using the concept of charts available in SageManifolds as indicated in Remark 3.21.*

Notebook A.3 (Electrodynamics). *This notebook shows basic computations involving the electromagnetic field and the electromagnetic potential.*

Notebook A.4 (LorentzBoosts). *This notebook shows how to compute Lorentz Boosts as per Example 4.17.*

Notebook A.5 (Minkowski). *This notebook computes various useful identities for working in the Minkowski space.*

Notebook A.6 (PlaneWave). *This notebook computes the plane wave solution, including the Doppler effect.*

Notebook A.7 (PointParticle). *This notebook contains computations involving the field generated by a moving charged particle.*

Notebook A.8 (HomopolarGenerator). *This notebook contains computations involving the homopolar generator.*

Appendix B

Identities

This appendix serves as a collection of useful identities. The necessary context for these identities is contained in the main text. See [Wik20] for a more comprehensive list.

Differential Forms

$$
\begin{aligned}
&\boldsymbol{\omega} \wedge (\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2) = \boldsymbol{\omega} \wedge \boldsymbol{\tau}_1 + \boldsymbol{\omega} \wedge \boldsymbol{\tau}_2 \\
&(\boldsymbol{\omega} \wedge \boldsymbol{\tau}) \wedge \boldsymbol{\lambda} = \boldsymbol{\omega} \wedge (\boldsymbol{\tau} \wedge \boldsymbol{\lambda}) \\
&\boldsymbol{\omega} \wedge \boldsymbol{\tau} = (-1)^{kl} \boldsymbol{\tau} \wedge \boldsymbol{\omega} && \text{where } \boldsymbol{\omega} \in \mathcal{E}^k(\Omega),\ \boldsymbol{\tau} \in \mathcal{E}^l(\Omega) \\
&f(\boldsymbol{\tau} \wedge \boldsymbol{\omega}) = (f\boldsymbol{\tau}) \wedge \boldsymbol{\omega} && \text{where } f \in \mathcal{E}^0(\Omega) \\
&\boldsymbol{\omega} \wedge \boldsymbol{\omega} = \mathbf{0} && \text{where } \boldsymbol{\omega} \in \mathcal{E}^k(\Omega) \text{ and } k \text{ is odd}
\end{aligned}
$$

$$
\begin{aligned}
&\varphi^*(\boldsymbol{\omega} \wedge \boldsymbol{\tau}) = \varphi^*\boldsymbol{\omega} \wedge \varphi^*\boldsymbol{\tau} \\
&\varphi^* \mathrm{d}\boldsymbol{\omega} = \mathrm{d}\varphi^*\boldsymbol{\omega}
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{d}\boldsymbol{\omega} + \boldsymbol{\tau} = \mathrm{d}\boldsymbol{\omega} + \mathrm{d}\boldsymbol{\tau} \\
&\mathrm{d}(\boldsymbol{\omega} \wedge \boldsymbol{\tau}) = \mathrm{d}\boldsymbol{\omega} \wedge \mathrm{d}\boldsymbol{\tau} + (-1)^k \boldsymbol{\omega} \wedge \mathrm{d}\boldsymbol{\tau} && \text{where } \boldsymbol{\omega} \in \mathcal{E}^k(\Omega) \\
&\mathrm{d}(\mathrm{d}\boldsymbol{\omega}) = \mathbf{0}
\end{aligned}
$$

Hodge Star

$$
\begin{aligned}
\star\star &= s(-1)^{k(n-k)}\,\mathrm{id} & \star^{-1} &= s(-1)^{k(n-k)}\star \\
\star\,\mathrm{d}x^{\emptyset} &= \boldsymbol{\sigma} & \star\boldsymbol{\sigma} &= s\,\mathrm{d}x^{\emptyset} \\
\boldsymbol{\alpha} \wedge \star\boldsymbol{\beta} &= \langle \boldsymbol{\alpha} | \boldsymbol{\beta} \rangle \boldsymbol{\sigma}
\end{aligned}
$$

Minkowski Space

Hodge Star

$$\star dx^0 = -dx^1 \wedge dx^2 \wedge dx^3 \qquad \star dx^1 = -dx^0 \wedge dx^2 \wedge dx^3$$
$$\star dx^2 = dx^0 \wedge dx^1 \wedge x^3 \qquad \star dx^3 = -dx^0 \wedge dx^1 \wedge dx^2$$
$$\star(dx^0 \wedge dx^1) = -dx^2 \wedge dx^3 \qquad \star(dx^0 \wedge dx^2) = dx^1 \wedge dx^3$$
$$\star(dx^0 \wedge dx^3) = -dx^1 \wedge dx^2 \qquad \star(dx^1 \wedge dx^2) = dx^0 \wedge dx^3$$
$$\star(dx^1 \wedge dx^3) = -dx^0 \wedge dx^2 \qquad \star(dx^2 \wedge dx^3) = dx^0 \wedge dx^1$$
$$\star(dx^0 \wedge dx^1 \wedge dx^2) = -dx^3 \qquad \star(dx^0 \wedge dx^1 \wedge dx^3) = dx^3$$
$$\star(dx^0 \wedge dx^2 \wedge dx^3) = -dx^1 \qquad \star(dx^1 \wedge dx^2 \wedge dx^3) = -dx^0$$

0-forms	$\star\star = -\mathrm{id}$	
1-forms	$\star\star = \mathrm{id}$	$\delta = \star d \star$
2-forms	$\star\star = -\mathrm{id}$	$\delta = \star d \star$
3-forms	$\star\star = \mathrm{id}$	$\delta = \star d \star$
4-forms	$\star\star = -\mathrm{id}$	$\delta = \star d \star$

Appendix C

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