Czech Technical University in Prague
Faculty of Electrical Engineering
Department of Computer Science

Master`s Thesis

HPC TECHNOLOGIES FOR SIMULATION OF DIFFRACTION OF ELECTROMAGNETIC WAVES BY SPHERICAL OBSTACLES

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Study Program: Open Informatics
Field of Study: Software Engineering
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MASTER'S THESIS ASSIGNMENT

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Bibliography / sources:

1. Pleshchinskii, N. B., & Tumakov, D. N. (2015). A New Approach to Investigation of Maxwell Equations. 36(1), 15–27.
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Springer International Publishing, 2019.

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Declaration

I hereby declare that I have completed this thesis independently and that I have listed all the literature and publications used.

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Abstract

PUZANKOVA, Anastasiia: HPC technologies for simulation of diffraction of electromagnetic waves by spherical obstacles. [Master's Thesis] – Czech Technical University in Prague. Faculty of Electrical Engineering, Department of Computer Science. Supervisor: Ing. Miroslav Bureš, Ph.D.

Various real objects are well approximated by spherical bodies. Maxwell's equations, in turn, are also one of the fundamental in physics, as they are used in a huge range of applied problems The problem of diffraction of electromagnetic waves by a sphere plays an essential role in the theory of wave propagation along the Earth's surface. The problem of diffraction of an electromagnetic wave by a sphere is a key problem in electrodynamics. A system of equations that describes the diffraction of an incident plane electromagnetic wave on a spherical surface of a given radius was obtained. In this work general solution to the Maxwell equations in spherical coordinates is implemented. Several different versions of the numerical solution were implemented using parallel technologies. Experiments were carried out, results obtained and analyzed.

Keywords: diffraction problems, parallel algorithms, high-performance computing, algorithms optimization

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Chapter 1: Introduction

1.1. Problem description

Various real objects are well approximated by spherical bodies. There is a large number of works devoted to the study of wave diffraction by spherical bodies. In particular, electromagnetic waves are considered.

Maxwell's equations, in turn, are also one of the fundamental in physics, as they are used in a huge range of applied problems, such as plasma acceleration and photonic crystals.

In this regard, boundary value problems for Maxwell's equations in spherical coordinates arise in various fields of physics [1]. The classical problem is the diffraction of a plane electromagnetic wave by a uniform sphere of arbitrary size. The theory of Mie, developed by G. Mie back in 1908, is devoted to it. This theory is widely used today.

The problem of diffraction of electromagnetic waves by a sphere plays an essential role in the theory of wave propagation along the Earth's surface [2]. At present, there is great interest in the problems of the synthesis of spherical antennas and in the problems of scattering of the electromagnetic field by biological objects [3].

The problem of diffraction of an electromagnetic wave by a sphere is a key problem in electrodynamics [4]. An analytical solution to this problem in the form of a series of spherical harmonics can be found in the book [5]. Specific solutions of Maxwell's equations can be obtained by the method of separation of variables after passing to the equations for Debye potentials or Hertz vectors [4].

Parallel algorithms and technologies are widely used in solving various problems of electrodynamics. Similar algorithms can be used to speed up the computation of various parts of the problem solution. The algorithm can be used to calculate the matrix elements for the numerical solution of the diffraction problem on flat screens by the Rao-Wilton-Glisson method [6]. When solving the diffraction problem, parallel calculations can be applied to calculate auxiliary integrals over a spherical screen [7]. The problem of diffraction on a conducting screen can be reduced to solving a pseudodifferential equation, the numerical method of solving which is implemented using parallel computations [8, 9].

In this work, the general representation of the electromagnetic field in spherical coordinates is built directly on the basis of the analysis of the Maxwell system of equations. And parallel computations are used to calculate the expansion coefficients of the components of the electromagnetic field.

The aim of this work is to develop a set of programs that simulate the problem of diffraction of a plane electromagnetic wave by a spherical surface using various parallel programming technologies and select the most efficient program.

To achieve this goal, this work solves the following tasks:

– study of existing approaches to solving the problem of diffraction of electromagnetic waves;

– study of various technologies of parallel computing, as well as the possibility of their application for solving diffraction problems

– obtaining a system of equations describing the diffraction of an incident plane electromagnetic wave onto a spherical surface of a given radius;

– transformation of the resulting system and obtaining expressions for calculating the expansion coefficients of the components of the electromagnetic field;

– implementation of a numerical solution, calculation of decomposition components;

– implementation of several different versions of a numerical solution using parallel computing technologies;

– computational experiments.

1.2. Thesis organization

The Master's Thesis consists of an introduction, problem statement, theoretical, algorithmic, and practical chapters and a conclusion.

The first chapter gives a brief overview of existing methods and approaches to the considering problem.

The second chapter provides detailed statement of the problem of electromagnetic wave diffraction by spherical surface. This chapter provides all the necessary theoretical information regarding the considered problem.

The third chapter describes as complete as possible analytical solution of considered problem representing the considered system of equations.

The fourth chapter contains a description of the development process together with the optimizations and decisions made and the description of developed software.

The fifth chapter contains a description of the results obtained.

The sixth chapter contains conclusions, conclusions and a description of possible future work.

Chapter 2: Problem and tasks statement

This work considers the spherical surface and the plane electromagnetic wave incident on this surface. This work addresses the problem of electromagnetic wave diffraction by spherical surface in the spherical coordinates (r, θ, α).

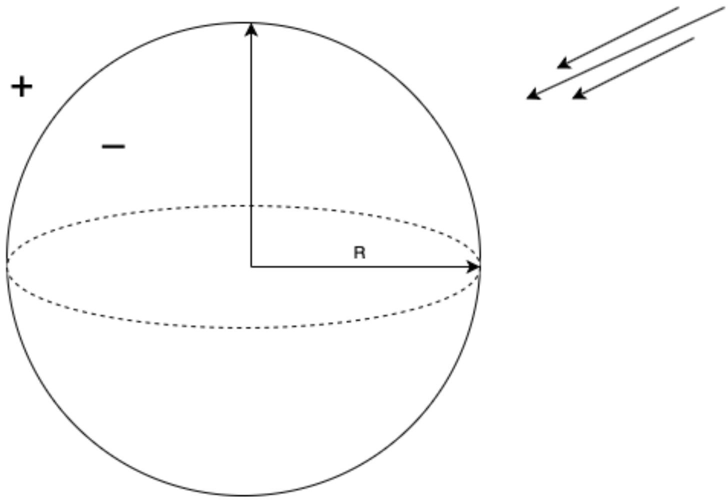

Figure 1 – Model of the spherical surface with incident wave

The spherical surface with the radius $r = R$ splits the space into two parts (Figure 1): the interior of the sphere ($r < R$) and the exterior of the sphere ($r > R$).

For the description of waves uses a second-order linear partial differential equation – the wave equation (1), which has the following form:

$$\frac{\partial^2 U}{\partial t^2} = v^2 \Delta U, \qquad (1)$$

where Δ – Laplace operator, so

$$\Delta U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}.$$

The solution to this equation is in the form $U = e^{-iwt}V(x, y, z)$. And substituting it into equation (1) it turns out for an equation of the form:

$$\Delta V + k^2 V = 0,$$

where $k^2 = \dfrac{\omega^2}{a^2}$. This equation is called the Helmholtz equation. The fundamental solutions of this equation are the Bessel functions in the one-dimensional case and the Hankel functions in the two-dimensional case, respectively [10].

2.1. Plane wave

A plane wave is a wave whose front is flat (plane). The surface of the constant phase is called the wave surface, or the wave front, in this case, the wave by the shape of the wave front is called plane (there are also spherical, cylindrical, and other waves). The plane wave front is unlimited in size, the phase velocity vector is perpendicular to the front. So a plane wave is a special case of wave: a physical quantity whose value, at any moment, is constant over any plane that is perpendicular to a fixed direction in space (Figure 2).

The equation of any wave is a solution to a differential equation called the wave equation (1). A plane wave is a particular solution of the wave equation and a convenient theoretical model [14].

A plane wave is described by the following equation:

$$A(z) = A_0 e^{ikz},$$

where A_0 – wave amplitude.

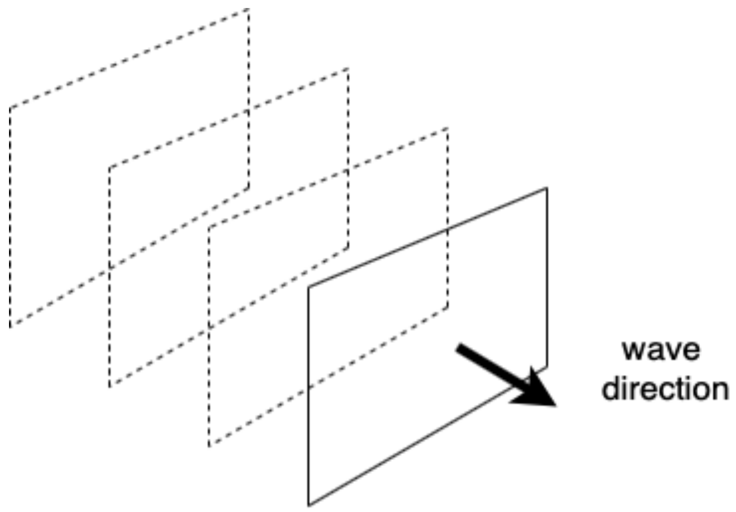

Figure 2 – Model of a plane wave

The book [11] contains a derivation of the expansion of a plane wave in terms of Legendre polynomials. This expansion has the following form

$$e^{ikz} = e^{ikr\cos\theta} = \sum_{n=0}^{\infty} i^n (2n+1) j_n(kr) P_n(\cos\theta), \tag{2}$$

where $j_n(\cdot)$ – Bessel functions of the first kind, $P_n(\cos\theta)$ – Legendre polynomials.

Thus, the expansion of a plane wave in Legendre polynomials does not contain the associated Legendre polynomials [12] A plane electromagnetic wave irradiating a sphere can be thought of as a superposition of spherical waves emerging from the center of the sphere.

2.2. Electromagnetic field

Maxwell equations set is a system of equations in differential or integral form describing the electromagnetic field. This system has different forms of representation. This work considers two Maxwell equations (3) for the complex amplitudes of the harmonic electromagnetic field in the:

$$\begin{aligned} \operatorname{rot} \mathbf{H} &= i\omega\varepsilon_0\varepsilon\mathbf{E}, \\ \operatorname{rot} \mathbf{E} &= -i\omega\mu_0\mu\mathbf{H}. \end{aligned} \tag{3}$$

in spherical coordinates (the time dependence has the form $e^{i\omega t}$).

The $\operatorname{rot} \mathbf{F}$ is the vector operator also known as $\operatorname{curl} \mathbf{F}$, where $\mathbf{F}$ is a vector field. The $\operatorname{rot} \mathbf{F}$ could be defined as following: $\operatorname{rot} \mathbf{F} \equiv \nabla \times \mathbf{F}$, where ∇ is a vector differential operator, such as $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$.

Considering that this work operates with the spherical coordinates, the set of equations (3) could be written in the coordinate form in the following way:

$$\begin{gathered}
\frac{1}{r\sin\theta}\frac{\partial(\sin\theta\cdot H_\alpha)}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial H_\theta}{\partial\alpha} = i\omega\varepsilon_0\varepsilon E_r, \\
\frac{1}{r\sin\theta}\frac{\partial H_r}{\partial\alpha} - \frac{1}{r}\frac{\partial(rH_\alpha)}{\partial r} = i\omega\varepsilon_0\varepsilon E_\theta, \\
\frac{1}{r}\frac{\partial(rH_\theta)}{\partial r} - \frac{1}{r}\frac{\partial H_r}{\partial\theta} = i\omega\varepsilon_0\varepsilon E_\alpha, \\
\frac{1}{r\sin\theta}\frac{\partial(\sin\theta\cdot E_\alpha)}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial E_\theta}{\partial\alpha} = -i\omega\mu_0\mu H_r, \\
\frac{1}{r\sin\theta}\frac{\partial E_r}{\partial\alpha} - \frac{1}{r}\frac{\partial(rE_\alpha)}{\partial r} = -i\omega\mu_0\mu H_\theta, \\
\frac{1}{r}\frac{\partial(rE_\theta)}{\partial r} - \frac{1}{r}\frac{\partial E_r}{\partial\theta} = -i\omega\mu_0\mu H_\alpha.
\end{gathered}$$

The spherical functions $S_{n,m}(\theta,\alpha) = P_n^{(m)}(\cos\theta)e^{im\alpha}$ form a complete orthogonal system of functions on the sphere, where $m = 0, \pm 1, \ldots$, $n = |m|, |m|+1, \ldots$, $P_n^{(m)}(\cdot)$ – associated Legendre polynomials.

Denote $\varphi_n(\theta) = P_n^{(m)}(\cos\theta)$, $n = |m|, |m|+1, \ldots$, besides, here and below, for the briefness of notation, the value m is reduced.

Fourier coefficients of the functions $E_r, E_\theta, E_\alpha, H_r, H_\theta, H_\alpha$ in the series of the following form: $A(r,\theta,\alpha) = \sum_{m=-\infty}^{+\infty} A_m(r,\theta)e^{im\alpha}$ are solutions of the set of equations:

$$\frac{\partial(\sin\theta\cdot H_\alpha)}{\partial\theta} - imH_\theta = i\omega\varepsilon_0\varepsilon r\sin\theta\cdot E_r,$$

$$
\begin{aligned}
&imH_r - \sin\theta\frac{\partial(rH_\alpha)}{\partial r} = i\omega\varepsilon_0\varepsilon r\sin\theta\cdot E_\theta,\\
&\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial\theta} = i\omega\varepsilon_0\varepsilon r\cdot E_\alpha,\\
&\frac{\partial(\sin\theta\cdot E_\alpha)}{\partial\theta} - imE_\theta = -i\omega\mu_0\mu r\sin\theta\cdot H_r,\\
&imE_r - \sin\theta\frac{\partial(rE_\alpha)}{\partial r} = -i\omega\mu_0\mu r\sin\theta\cdot H_\theta,\\
&\frac{\partial(rE_\theta)}{\partial r} - \frac{\partial E_r}{\partial\theta} = -i\omega\mu_0\mu rH_\alpha.
\end{aligned} \tag{4}
$$

The general solution of the set of equations (4) in the case when $m \neq 0$ has the following form (The more detailed solution could be found in [1]):

$$
\begin{aligned}
&E_r(r,\theta) = \frac{1}{im}\sum_{n=|m|}^{+\infty}\frac{n(n+1)}{r}\left[c_n\zeta_n^{(1)}(kr) + d_n\zeta_n^{(2)}(kr)\right]\varphi_n(\theta),\\
&E_\theta(r,\theta) = -\frac{i\omega\mu_0\mu}{\sin\theta}\sum_{n=|m|}^{+\infty}\left[a_n\zeta_n^{(1)}(kr) + b_n\zeta_n^{(2)}(kr)\right]\varphi_n(\theta) +\\
&+\frac{1}{im}\sum_{n=|m|}^{+\infty}\left[c_n\frac{1}{r}\frac{d}{dr}(r\zeta_n^{(1)}(kr)) + d_n\frac{1}{r}\frac{d}{dr}(r\zeta_n^{(2)}(kr))\right]\varphi_n'(\theta),\\
&E_\alpha(r,\theta) = \frac{\omega\mu_0\mu}{m}\sum_{n=|m|}^{+\infty}[a_n\zeta_n^{(1)}(kr) + b_n\zeta_n^{(2)}(kr)]\varphi_n'(\theta) +\\
&+\frac{1}{\sin\theta}\sum_{n=|m|}^{+\infty}\left[c_n\frac{1}{r}\frac{d}{dr}(r\zeta_n^{(1)}(kr)) + d_n\frac{1}{r}\frac{d}{dr}(r\zeta_n^{(2)}(kr))\right]\varphi_n(\theta),\\
&H_r(r,\theta) = \frac{1}{im}\sum_{n=|m|}^{+\infty}\frac{n(n+1)}{r}[a_n\zeta_n^{(1)}(kr) + b_n\zeta_n^{(2)}(kr)]\varphi_n(\theta),\\
&H_\theta(r,\theta) = -\frac{i\omega\varepsilon_0\varepsilon}{\sin\theta}\sum_{n=|m|}^{+\infty}[c_n\zeta_n^{(1)}(kr) + d_n\zeta_n^{(2)}(kr)]\varphi_n(\theta) +\\
&+\frac{1}{im}\sum_{n=|m|}^{+\infty}\left[a_n\frac{1}{r}\frac{d}{dr}(r\zeta_n^{(1)}(kr)) + b_n\frac{1}{r}\frac{d}{dr}(r\zeta_n^{(2)}(kr))\right]\varphi_n'(\theta),
\end{aligned} \tag{5}
$$

$$H_\alpha(r,\theta) = -\frac{\omega\varepsilon_0\varepsilon}{m}\sum_{n=|m|}^{+\infty}[c_n\zeta_n^{(1)}(kr) + d_n\zeta_n^{(2)}(kr)]\varphi_n'(\theta) +$$

$$+\frac{1}{\sin\theta}\sum_{n=|m|}^{+\infty}\left[a_n\frac{1}{r}\frac{d}{dr}(r\zeta_n^{(1)}(kr)) + b_n\frac{1}{r}\frac{d}{dr}(r\zeta_n^{(2)}(kr))\right]\varphi_n(\theta),$$

where $\zeta_n^{(j)}(z) = \sqrt{\frac{\pi}{2z}}H^{(j)}_{n+\frac{1}{2}}(z), j = 1,2$ – spherical Hankel functions of the first and second kind, respectively (spherical Bessel functions of the third kind), $H_n^{(j)}$ – Hankel functions, and a_n, b_n, c_n, d_n –are arbitrary constants.

In the particular case when $m = 0$, the system (4) splits into 2 independent systems of equations and the general solution is the following (the derivation of the solution is also considered in more detail in [1]):

$$E_r(r,\theta) = \sum_{n=1}^{+\infty}\frac{n(n+1)}{r}\left[c_n\zeta_n^{(1)}(kr) + d_n\zeta_n^{(2)}(kr)\right]\varphi(\theta),$$

$$E_\theta(r,\theta) = \sum_{n=1}^{+\infty}\left[c_n\frac{1}{r}\frac{d}{dr}\left(r\zeta_n^{(1)}(kr)\right) + d_n\frac{1}{r}\frac{d}{dr}\left(r\zeta_n^{(2)}(kr)\right)\right]\varphi'(\theta),$$

$$E_\alpha(r,\theta) = i\omega\mu_0\mu\sum_{n=1}^{+\infty}[a_n\zeta_n^{(1)}(kr) + b_n\zeta_n^{(2)}(kr)]\varphi'(\theta),$$

$$H_r(r,\theta) = \sum_{n=1}^{+\infty}\frac{n(n+1)}{r}\left[a_n\zeta_n^{(1)}(kr) + b_n\zeta_n^{(2)}(kr)\right]\varphi(\theta),$$

$$H_\theta(r,\theta) = \sum_{n=1}^{+\infty}\left[a_n\frac{1}{r}\frac{d}{dr}\left(r\zeta_n^{(1)}(kr)\right) + b_n\frac{1}{r}\frac{d}{dr}\left(r\zeta_n^{(2)}(kr)\right)\right]\varphi'(\theta),$$

$$H_\alpha(r,\theta) = -i\omega\varepsilon_0\varepsilon\sum_{n=1}^{+\infty}\left[c_n\zeta_n^{(1)}(kr) + d_n\zeta_n^{(2)}(kr)\right]\varphi'(\theta),$$

where $\zeta_n^{(j)}(z) = \sqrt{\frac{\pi}{2z}}H^{(j)}_{n+\frac{1}{2}}(z), j = 1,2$ – spherical Hankel functions of the first and second kind, respectively (spherical Bessel functions of the third kind), $H_n^{(j)}$ – Hankel functions, and a_n, b_n, c_n, d_n –are arbitrary constants.

The general solution of the Maxwell set of equations (4) in the spherical coordinates has the following form:

$$\mathbf{E}(r,\theta,\alpha) = \sum_{m=-\infty}^{+\infty}\mathbf{E}_m(r,\theta)e^{im\alpha},$$
$$\mathbf{H}(r,\theta,\alpha) = \sum_{m=-\infty}^{+\infty}\mathbf{H}_m(r,\theta)e^{im\alpha}. \tag{6}$$

where the components of vector-functions $\mathbf{E}_m = (E_{m,r}, E_{m,\theta}, E_{m,\alpha})$ and $\mathbf{H}_m = (H_{m,r}, H_{m,\theta}, H_{m,\alpha})$ are defined by formulas (5).

2.3. Boundary value problem

Boundary conditions for an electromagnetic field are conditions that connect the values of the intensities and inductions of magnetic and electric fields on opposite sides of surfaces characterized by a certain surface density of electric charge and electric current.

When a solution in a finite volume is considered, it is necessary to take into account the conditions at the boundaries of the body with the surrounding infinite space. The boundary conditions are obtained from Maxwell's equations by passing to the limit. In cases where there is a boundary between two regions of space, it is necessary to take into account that the tangential components of the vectors of the electric and magnetic fields **E** and **H** must be continuous.

In this work, the latitude θ and the longitude α are considered as the tangential variables.

The solution of the Maxwell set equations can be determined uniquely when considered the tangential components of the vectors **E** and **H** on the sphere.

A solution to the Maxwell set equations is called positively oriented if the representation (6) consists of summands with Hankel functions of the second kind only ($a_n = 0$, $c_n = 0\ \forall n$). A solution is called negatively oriented if the representation (6) consists of summands with Hankel functions of the first kind only ($b_n = 0$, $d_n = 0\ \forall n$). The sum of two oppositely oriented solutions is a non-oriented solution.

The boundary value problem for the system of Maxwell equations in coordinate form is divided into an infinite set of boundary value problems for the system of equations (4).

2.3.1. Exterior boundary value problem

The tangential components of vectors **E** and **H**, such as $E_\theta(r,\theta)$, $E_\alpha(r,\theta)$, $H_\theta(r,\theta)$, $H_\alpha(r,\theta)$ in the case of a positively oriented solution and $m = 0$ have the following form:

$$E_\theta(r,\theta) = \sum_{n=1}^{+\infty} d_n \frac{1}{r}\frac{d}{dr}\left(r\zeta_n^{(2)}(kr)\right)\varphi_n'(\theta),$$

$$E_\alpha(r,\theta) = i\omega\mu_0\mu\sum_{n=1}^{+\infty} b_n\zeta_n^{(2)}(kr)\varphi_n'(\theta),$$

$$H_\theta(r,\theta) = \sum_{n=1}^{+\infty} b_n \frac{1}{r}\frac{d}{dr}\left(r\zeta_n^{(2)}(kr)\right)\varphi_n'(\theta),$$

$$H_\alpha(r,\theta) = -i\omega\varepsilon_0\varepsilon\sum_{n=1}^{+\infty} d_n\zeta_n^{(2)}(kr)\varphi_n'(\theta).$$

In the case when $m \neq 0$ the tangential components of vectors **E** and **H** of a positively oriented solution have the following form:

$$E_\theta(r,\theta) = -\frac{i\omega\mu_0\mu}{\sin\theta}\sum_{n=|m|}^{+\infty} b_n\zeta_n^{(2)}(kr)\varphi_n(\theta) + \frac{1}{im}\sum_{n=|m|}^{+\infty} d_n\frac{1}{r}\frac{d}{dr}(r\zeta_n^{(2)}(kr))\varphi_n'(\theta),$$

$$E_{\alpha}(r,\theta) = \frac{\omega\mu_0\mu}{m}\sum_{n=|m|}^{+\infty} b_n\zeta_n^{(2)}(kr)\varphi_n'(\theta) + \frac{1}{\sin\theta}\sum_{n=|m|}^{+\infty} d_n\frac{1}{r}\frac{d}{dr}(r\zeta_n^{(2)}(kr))\varphi_n(\theta), \tag{7}$$

$$H_{\theta}(r,\theta) = \frac{1}{im}\sum_{n=|m|}^{+\infty} b_n\frac{1}{r}\frac{d}{dr}\left(r\zeta_n^{(2)}(kr)\right)\varphi_n'(\theta) + \frac{i\omega\varepsilon_0\varepsilon}{\sin\theta}\sum_{n=|m|}^{+\infty} d_n\zeta_n^{(2)}(kr)\varphi_n(\theta),$$

$$H_{\alpha}(r,\theta) = \frac{1}{\sin\theta}\sum_{n=|m|}^{+\infty} b_n\frac{1}{r}\frac{d}{dr}\left(r\zeta_n^{(2)}(kr)\right)\varphi_n(\theta) - \frac{\omega\varepsilon_0\varepsilon}{m}\sum_{n=|m|}^{+\infty} d_n\zeta_n^{(2)}(kr)\varphi_n'(\theta).$$

where $\zeta_n^{(j)}(z) = \sqrt{\frac{\pi}{2z}}H^{(j)}_{n+\frac{1}{2}}(z), j = 1{,}2$ – spherical Hankel functions of the first and second kind, respectively (spherical Bessel functions of the third kind), $H_n^{(j)}$ – Hankel functions, and a_n, b_n, c_n, d_n –are arbitrary constants. This solution in the detail could be found in [1].

2.3.2. Interior boundary value problem

If there are no sources of the electromagnetic field inside the sphere, then the class of solutions must be changed. The coefficients in front of the Hankel functions of various kinds should be the same, since if the term oriented in one way brings energy to the sphere, then the term oriented in a different way takes this energy from the sphere. In this case, only Bessel functions are used in the general solution and in each particular solution.

The tangential components $E_{\theta}(r,\theta)$, $E_{\alpha}(r,\theta)$, $H_{\theta}(r,\theta)$, $H_{\alpha}(r,\theta)$ of vectors **E** and **H** in the center of the sphere in the case when $m = 0$ have the following form:

$$E_{\theta}(r,\theta) = \sum_{n=1}^{+\infty} c_n\frac{1}{r}\frac{d}{dr}\left(rj_n(kr)\right)\varphi'(\theta),$$

$$E_{\alpha}(r,\theta) = i\omega\mu_0\mu\sum_{n=1}^{+\infty} a_nj_n(kr)\varphi'(\theta),$$

$$H_{\theta}(r,\theta) = \sum_{n=1}^{+\infty} a_n\frac{1}{r}\frac{d}{dr}\left(rj_n(kr)\right)\varphi'(\theta),$$

$$H_{\alpha}(r,\theta) = -i\omega\varepsilon_0\varepsilon\sum_{n=1}^{+\infty} c_nj_n(kr)\varphi'(\theta),$$

where $j_n(z) = \sqrt{\frac{\pi}{2z}}J_{n+\frac{1}{2}}(z)$ – spherical Bessel function of the first kind and J_n – Bessel function of the first kind respectively.

In the case when $m \neq 0$ he tangential components $E_{\theta}(r,\theta)$, $E_{\alpha}(r,\theta)$, $H_{\theta}(r,\theta)$, $H_{\alpha}(r,\theta)$ of vectors **E** and **H** o have the following form:

$$E_{\theta}(r,\theta) = -\frac{i\omega\mu_0\mu}{\sin\theta}\sum_{n=|m|}^{+\infty} a_nj_n(kr)\varphi_n(\theta) + \frac{1}{im}\sum_{n=|m|}^{+\infty} c_n\frac{1}{r}\frac{d}{dr}(r\zeta_n^{(1)}(kr))\varphi_n'(\theta),$$

$$
\begin{aligned}
E_\alpha(r,\theta) &= \frac{\omega\mu_0\mu}{m}\sum_{n=|m|}^{+\infty} a_n j_n(kr)\varphi_n'(\theta) + \frac{1}{\sin\theta}\sum_{n=|m|}^{+\infty} c_n \frac{1}{r}\frac{d}{dr}(rj_n(kr))\varphi_n(\theta), \\
H_\theta(r,\theta) &= \frac{1}{im}\sum_{n=|m|}^{+\infty} a_n \frac{1}{r}\frac{d}{dr}(rj_n(kr))\varphi_n'(\theta) + \frac{i\omega\varepsilon_0\varepsilon}{\sin\theta}\sum_{n=|m|}^{+\infty} c_n j_n(kr)\varphi_n(\theta), \\
H_\alpha(r,\theta) &= \frac{1}{\sin\theta}\sum_{n=|m|}^{+\infty} a_n \frac{1}{r}\frac{d}{dr}(rj_n(kr))\varphi_n(\theta) - \frac{\omega\varepsilon_0\varepsilon}{m}\sum_{n=|m|}^{+\infty} c_n j_n(kr)\varphi_n'(\theta),
\end{aligned}
\tag{8}
$$

where $j_n(z) = \sqrt{\dfrac{\pi}{2z}} J_{n+\frac{1}{2}}(z)$ – spherical Bessel function of the first kind and J_n – Bessel function of the first kind respectively.

2.4. Incident wave

In the considered formulation of the problem, an electromagnetic wave falls on the surface of the sphere. It is necessary to find the electromagnetic field generated as a result of the diffraction of this wave, namely, a negatively oriented wave inside the surface of the sphere and positively oriented outside the sphere.

As the notation for the dielectric constant and the wavenumber inside and outside the sphere used $\varepsilon_+, k_+, \varepsilon_-, k_-$ respectively.

This work considers the tangential components of the vectors **E** and **H** of the incident wave in the case when $m = 0$:

$$
\begin{aligned}
E_\theta^0(r,\theta) &= \sum_{n=1}^{+\infty} c_n^0 \frac{1}{r}\frac{d}{dr}\left(r\zeta_n^{(1)}(k_+r)\right)\varphi'(\theta), \\
E_\alpha^0(r,\theta) &= i\omega\mu_0\mu\sum_{n=1}^{+\infty} a_n^0\zeta_n^{(1)}(k_+r)\varphi'(\theta), \\
H_\theta^0(r,\theta) &= \sum_{n=1}^{+\infty} a_n^0 \frac{1}{r}\frac{d}{dr}\left(r\zeta_n^{(1)}(k_+r)\right)\varphi'(\theta), \\
H_\alpha^0(r,\theta) &= -i\omega\varepsilon_0\varepsilon_+\sum_{n=1}^{+\infty} c_n^0\zeta_n^{(1)}(k_+r)\varphi'(\theta).
\end{aligned}
\tag{9}
$$

And tangential components of the vectors **E** and **H** of the incident wave in the case when $m \neq 0$, respectively:

$$
\begin{aligned}
E_\theta^0(r,\theta) &= -\frac{i\omega\mu_0\mu}{\sin\theta}\sum_{n=|m|}^{+\infty} a_n^0\zeta_n^{(1)}(k_+r)\varphi_n(\theta) + \frac{1}{im}\sum_{n=|m|}^{+\infty} c_n^0\frac{1}{r}\frac{d}{dr}(r\zeta_n^{(1)}(k_+r))\varphi_n'(\theta), \\
E_\alpha^0(r,\theta) &= \frac{\omega\mu_0\mu}{m}\sum_{n=|m|}^{+\infty} a_n^0\zeta_n^{(1)}(k_+r)\varphi_n'(\theta) + \frac{1}{\sin\theta}\sum_{n=|m|}^{+\infty} c_n^0\frac{1}{r}\frac{d}{dr}(r\zeta_n^{(1)}(k_+r))\varphi_n(\theta)
\end{aligned}
\tag{10}
$$

$$H^0_\theta(r,\theta) = \frac{1}{im}\sum_{n=|m|}^{+\infty} a^0_n \frac{1}{r}\frac{d}{dr}(r\zeta^{(1)}_n(k_+r))\varphi'_n(\theta) + \frac{i\omega\varepsilon_0\varepsilon_+}{\sin\theta}\sum_{n=|m|}^{+\infty} c^0_n\zeta^{(1)}_n(k_+r)\varphi_n(\theta)$$
$$H^0_\alpha(r,\theta) = \frac{1}{\sin\theta}\sum_{n=|m|}^{+\infty} a^0_n \frac{1}{r}\frac{d}{dr}(r\zeta^{(1)}_n(k_+r))\varphi_n(\theta) - \frac{\omega\varepsilon_0\varepsilon_+}{m}\sum_{n=|m|}^{+\infty} c^0_n\zeta^{(1)}_n(k_+r)\varphi'_n(\theta).$$

This work considers the plane electromagnetic wave incident on the spherical surface with the radius $r = R$. Regarding this and the fact that in the expansion (2) does not contain the associated Legendre polynomials, the tangential components (9,10) of the vectors **E** and **H** of the incident wave will contain only the summands with $m = 1$ and $m = -1$.

2.5. Bessel functions

Bessel functions are a family of functions that are canonical solutions of the Bessel differential equation:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)y = 0,$$

where α – an arbitrary number called the order[15].

Bessel functions of the first kind, denoted by $J_\alpha(z)$, are solutions that are finite at the point $z = 0$. It is possible to define these functions using the Taylor series expansion [15]:

$$J_\alpha(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)}\left(\frac{z}{2}\right)^{2m+\alpha},$$

where $\Gamma(\cdot)$ – Euler's Gamma function.

Bessel functions of the second kind, also called Neumann functions, are solutions $Y_\alpha(z)$ of the Bessel equation, infinite at the point $x = 0$, they are also related by a relation with Bessel functions of the first kind [15]:

$$Y_\alpha(z) = \frac{J_\alpha(z)\cos(\alpha\pi) - J_{-\alpha}(z)}{\sin(\alpha\pi)}.$$

2.5.1. Spherical Bessel functions

The spherical Bessel functions of the first and second kind ($j_n(z)$ and $y_n(z)$) are related to the ordinary Bessel $J_n(z)$ and Neumann $Y_n(z)$ functions by the following relations, according to 10.1.1 from [13]:

$$j_n(z) = \frac{\pi}{2z}J_{n+\frac{1}{2}}(z),$$
$$y_n(z) = \frac{\pi}{2z}Y_{n+\frac{1}{2}}(z).$$

2.5.2. Calculation methods of spherical Bessel functions

There are several ways to calculate spherical Bessel functions, in addition to using definition, in particular, there are generating functions, as well as the recurrence relation described in [14]:

$$f_{n-1}(z) + f_{n+1}(z) = \frac{(2n+1)}{z} f_n(z),$$

where $f_n(z)$ is a function that can take values of $j_n(z)$, $y_n(z)$, $h_n^{(1)}(z)$ and $h_n^{(2)}(z)$.

Thus, this recurrence relation is valid for spherical Bessel functions of the first and second kind, as well as for spherical Hankel functions of the first and second kind.

Let's perform some transformation on the recurrence relation given above:

$$f_n(z) = \frac{(2n-1)}{z} f_{n-1}(z) - f_{n-2}(z).$$

Thus, a recurrent formula for calculating spherical Bessel functions of any order in terms of its two previous values is obtained. In order to use it, it is necessary to calculate two initial values for these functions.

There is an expression (in more details described in [14]) for the spherical Bessel functions in terms of elementary functions. In particular, these expressions exist for functions with orders zero and one:

$$j_0(z) = \frac{\sin z}{z},$$
$$j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z},$$

$$y_0(z) = -\frac{\cos z}{z},$$
$$y_1(z) = -\frac{\cos z}{z^2} - \frac{\sin z}{z}.$$

Thus, any spherical Bessel function of the first, second, and third kind of any integer order can be numerically calculated using relations and initial values provided above.

Chapter 3: Analytical solution

This work considers two different cases of the spherical surface. The dielectric surface and metal surface. Depending on it, there are two cases of electromagnetic wave diffraction by the spherical surface. This work considers plane incident electromagnetic wave in the form (10). This work seeks the reflected wave in the form (7) and the wave in the region inside the sphere in the form (8), when it is relevant.

3.1. The dielectric surface of the sphere

First let's consider a case with dielectric spherical surface and plane electromagnetic wave incident to it.

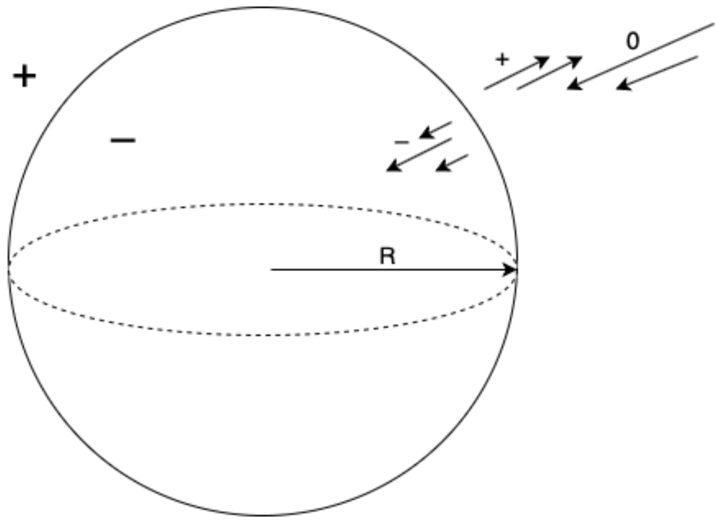

Figure 3 - The model of diffraction in the case of dielectric spherical surface

In the case of dielectric spherical surface the incident electromagnetic wave (electric and magnet components) should the sum of reflected wave and wave in the region inside the sphere (Figure 3, please notice that the figure provides only schematic notation).

$$E^0 = E^- - E^+,$$
$$H^0 = H^- - H^+. \tag{11}$$

The tangential components of electromagnetic wave have to be continuous on the spherical surface, regarding this let's equate the tangential components of the vectors **E** and **H** on the sphere ($r = R$) and will consider the fact that the incident wave will contain only the summands with $m = 1$ and $m = -1$.

3.1.1. Derivation the equation set

First let's consider equations for one of the components of vector **E**, for this equate corresponding components of (7), (8) and (10):

$$E^0_\theta = E^-_\theta - E^+_\theta,$$

$$-\frac{i\omega\mu_0\mu}{\sin\theta}\sum_{n=1}^{+\infty} a_n^0\zeta_n^{(1)}(k_+R)\varphi_n(\theta) + \frac{1}{im}\sum_{n=1}^{+\infty} c_n^0\frac{1}{R}\frac{d}{dr}(r\zeta_n^{(1)}(k_+r))\varphi_n'(\theta) =$$

$$= -\frac{i\omega\mu_0\mu}{\sin\theta}\sum_{n=1}^{+\infty} a_n j_n(k\_R)\varphi_n(\theta) + \frac{1}{im}\sum_{n=1}^{+\infty} c_n\frac{1}{R}\frac{d}{dr}(r\zeta_n^{(1)}(k\_r))\varphi_n'(\theta) +$$

$$+ \frac{i\omega\mu_0\mu}{\sin\theta}\sum_{n=1}^{+\infty} b_n\zeta_n^{(2)}(k_+R)\varphi_n(\theta) - \frac{1}{im}\sum_{n=1}^{+\infty} d_n\frac{1}{R}\frac{d}{dr}(r\zeta_n^{(2)}(k_+r))\varphi_n'(\theta).$$

In this expansion in terms of the associated Legendre polynomials (denoted as $\varphi_n(\theta) = P_n^{(m)}(\cos\theta)$), the coefficients of the corresponding function or its derivative must be equal, respectively, so this system of n equations can be divided into two:

$$-\frac{i\omega\mu_0\mu}{\sin\theta}\sum_{n=1}^{+\infty} a_n^0\zeta_n^{(1)}(k_+R)\varphi_n(\theta) = -\frac{i\omega\mu_0\mu}{\sin\theta}\sum_{n=1}^{+\infty} a_n j_n(k\_R)\varphi_n(\theta) +$$

$$+ \frac{i\omega\mu_0\mu}{\sin\theta}\sum_{n=1}^{+\infty} b_n\zeta_n^{(2)}(k_+R)\varphi_n(\theta),$$

$$\frac{1}{im}\sum_{n=1}^{+\infty} c_n^0\frac{1}{R}\frac{d}{dr}(r\zeta_n^{(1)}(k_+r))\varphi_n'(\theta) = \frac{1}{im}\sum_{n=1}^{+\infty} c_n\frac{1}{R}\frac{d}{dr}(r\zeta_n^{(1)}(k\_r))\varphi_n'(\theta) -$$

$$-\frac{1}{im}\sum_{n=1}^{+\infty} d_n\frac{1}{R}\frac{d}{dr}(r\zeta_n^{(2)}(k_+r))\varphi_n'(\theta).$$

Reduce the common coefficients and the functions $\varphi_n(\theta)$ itself and their derivatives:

$$a_n^0\zeta_n^{(1)}(k_+R) = a_n j_n(k\_R) - b_n\zeta_n^{(2)}(k_+R),$$

$$c_n^0\frac{1}{R}\frac{d}{dr}(r\zeta_n^{(1)}(k_+r)) = c_n\frac{1}{R}\frac{d}{dr}(r\zeta_n^{(1)}(k\_r)) - d_n\frac{1}{R}\frac{d}{dr}(r\zeta_n^{(2)}(k_+r)).$$

Now let's consider another component of vector **E**, for this equate corresponding components of (7), (8) and (10), respectively:

$$E_\alpha^0 = E_\alpha^- - E_\alpha^+,$$

$$\frac{\omega\mu_0\mu}{m}\sum_{n=|m|}^{+\infty} a_n^0\zeta_n^{(1)}(k_+R)\varphi_n'(\theta) + \frac{1}{\sin\theta}\sum_{n=|m|}^{+\infty} c_n^0\frac{1}{R}\frac{d}{dr}(r\zeta_n^{(1)}(k_+r))\varphi_n(\theta) =$$

$$= \frac{\omega\mu_0\mu}{m}\sum_{n=|m|}^{+\infty} a_n j_n(k\_R)\varphi_n'(\theta) + \frac{1}{\sin\theta}\sum_{n=|m|}^{+\infty} c_n\frac{1}{R}\frac{d}{dr}(r j_n(k\_r))\varphi_n(\theta) -$$

$$-\frac{\omega\mu_0\mu}{m}\sum_{n=|m|}^{+\infty} b_n\zeta_n^{(2)}(k_+R)\varphi_n'(\theta) - \frac{1}{\sin\theta}\sum_{n=|m|}^{+\infty} d_n\frac{1}{R}\frac{d}{dr}(r\zeta_n^{(2)}(k_+r))\varphi_n(\theta).$$

Perform similar transformations to split this system into two and reduce the common coefficients and the functions $\varphi_n(\theta)$ itself and their derivatives:

$$
\begin{aligned}
a_n^0 \zeta_n^{(1)}(k_+R) &= a_n j_n(k\_R) - b_n \zeta_n^{(2)}(k_+R), \\
c_n^0 \frac{1}{R}\frac{d}{dr}(r\zeta_n^{(1)}(k_+r)) \ &= \ c_n \frac{1}{R}\frac{d}{dr}(rj_n(k\_r)) - d_n \frac{1}{R}\frac{d}{dr}(r\zeta_n^{(2)}(k_+r)).
\end{aligned} \tag{12}
$$

Thus, by consideration of component E_α of the vector $\mathbf{E}$, two similar duplicating equations were obtained.

Now let's consider the component H_θ of the vector $\mathbf{H}$:

$$
H_\theta^0 = H_\theta^- - H_\theta^+,
$$

$$
\frac{1}{im}\sum_{n=1}^{+\infty} a_n^0 \frac{1}{R}\frac{d}{dr}(r\zeta_n^{(1)}(k_+r))\varphi_n'(\theta) \ + \ \frac{i\omega\varepsilon_0\varepsilon_+}{\sin\theta}\sum_{n=1}^{+\infty} c_n^0 \zeta_n^{(1)}(k_+R)\varphi_n(\theta) \ =
$$

$$
= \frac{1}{im}\sum_{n=1}^{+\infty} a_n \frac{1}{R}\frac{d}{dr}(rj_n(k\_r))\varphi_n'(\theta) + \frac{i\omega\varepsilon_0\varepsilon_-}{\sin\theta}\sum_{n=1}^{+\infty} c_n j_n(k\_R)\varphi_n(\theta) -
$$

$$
-\frac{1}{im}\sum_{n=1}^{+\infty} b_n \frac{1}{R}\frac{d}{dr}\left(r\zeta_n^{(2)}(k_+r)\right)\varphi_n'(\theta) - \frac{i\omega\varepsilon_0\varepsilon_+}{\sin\theta}\sum_{n=1}^{+\infty} d_n \zeta_n^{(2)}(k_+R)\varphi_n(\theta)\,.
$$

Similarly to the previous case with the vector $\mathbf{E}$, in this expansion in terms of the associated Legendre polynomials the coefficients of the corresponding function or its derivative must be equal, respectively, so this system of n equations can be divided into two:

$$
\frac{i\omega\varepsilon_0\varepsilon_+}{\sin\theta}\sum_{n=1}^{+\infty} c_n^0 \zeta_n^{(1)}(k_+R)\varphi_n(\theta) \ = \ \frac{i\omega\varepsilon_0\varepsilon_-}{\sin\theta}\sum_{n=1}^{+\infty} c_n j_n(k\_R)\varphi_n(\theta) -
$$

$$
- \frac{i\omega\varepsilon_0\varepsilon_+}{\sin\theta}\sum_{n=1}^{+\infty} d_n \zeta_n^{(2)}(k_+R)\varphi_n(\theta),
$$

$$
\frac{1}{im}\sum_{n=1}^{+\infty} a_n^0 \frac{1}{R}\frac{d}{dr}(r\zeta_n^{(1)}(k_+r))\varphi_n'(\theta) \ = \ \frac{1}{im}\sum_{n=1}^{+\infty} a_n \frac{1}{R}\frac{d}{dr}(rj_n(k\_r))\varphi_n'(\theta) -
$$

$$
-\frac{1}{im}\sum_{n=1}^{+\infty} b_n \frac{1}{R}\frac{d}{dr}\left(r\zeta_n^{(2)}(k_+r)\right)\varphi_n'(\theta).
$$

Let's reduce the common coefficients and the functions $\varphi_n(\theta)$ itself and their derivatives and write down the result.

$$
\begin{aligned}
\varepsilon_+ c_n^0 \zeta_n^{(1)}(k_+R) \ &= \ \varepsilon_- c_n j_n(k\_R) \ - \ \varepsilon_+ d_n \zeta_n^{(2)}(k_+R), \\
a_n^0 \frac{1}{r}\frac{d}{dr}(r\zeta_n^{(1)}(k_+R)) \ &= \ a_n \frac{1}{r}\frac{d}{dr}(rj_n(k\_R)) - b_n \frac{1}{r}\frac{d}{dr}\left(r\zeta_n^{(2)}(k_+R)\right).
\end{aligned} \tag{13}
$$

Consideration of H_α will get the duplicating equation set by this reason it is not provided in this work.

The outcome of performance under system (11) is two resulting systems (12) and (13) which is form an equation set for future consideration.

Denote the following:

$$\{f_n\}(k,R) = \frac{1}{R}\frac{d}{dr}\left(rf_n(kr)\right)\Bigg|_{r=R}, \tag{14}$$

where f_n is a function that can take values of j_n and $\zeta_n^{(i)},\ i = 1{,}2$, i.e. Bessel functions of the first and third kind (Hankel functions).

The result of consideration of continuity tangential components of the vectors **E** and **H** on the sphere ($r = R$) the following set of equations was get:

$$\begin{gathered} a_n j_n(k_-R) - b_n\zeta_n^{(2)}(k_+R) = a_n^0\zeta_n^{(1)}(k_+R), \\ a_n\{j_n\}(k_-,R) - b_n\{\zeta_n^{(2)}\}(k_+,R) = a_n^0\{\zeta_n^{(1)}\}(k_+,R), \\ \varepsilon_- c_n j_n(k_-R) - \varepsilon_+ d_n\zeta_n^{(2)}(k_+R) = \varepsilon_+ c_n^0\zeta_n^{(1)}(k_+R), \\ c_n\{j_n\}(k_-,R) - d_n\{\zeta_n^{(2)}\}(k_+,R) = c_n^0\{\zeta_n^{(1)}\}(k_+,R), \end{gathered} \tag{15}$$

where $\zeta_n^{(i)},\ i = 1{,}2$ – spherical Hankel functions of the first and second kind, respectively.

3.1.2. Analytical solution of the equation set in case dielectric surface

This part of the work considers analytical solution of the equation set (15).

FIrst lest change the notation for the spherical Hankel functions (Bessel functions of the third kind) to more frequently used in these latter days:

$$\zeta_n^{(i)}(z) = h_n^{(i)}(z),\ i = 1{,}2$$

Applying this notation to the equation set (15), the result is the following:

$$\begin{gathered} a_n j_n(k_-R) - b_n h_n^{(2)}(k_+R) = a_n^0 h_n^{(1)}(k_+R), \\ a_n\{j_n\}(k_-,R) - b_n\{h_n^{(2)}\}(k_+,R) = a_n^0\{h_n^{(1)}\}(k_+,R), \\ \varepsilon_- c_n j_n(k_-R) - \varepsilon_+ d_n h_n^{(2)}(k_+R) = \varepsilon_+ c_n^0 h_n^{(1)}(k_+R), \\ c_n\{j_n\}(k_-,R) - d_n\{h_n^{(2)}\}(k_+,R) = c_n^0\{h_n^{(1)}\}(k_+,R). \end{gathered} \tag{16}$$

It is necessary to take into consideration this system with respect to unknown arbitrary constants a_n, b_n, c_n, d_n.

It is easy to see that in the system under consideration the equations are pairwise independent, i.e. each of unknown arbitrary constants are only in two of given equations, therefore let's consider them as 2 independent systems of 2 equations:

$$a_n j_n(k\_R) - b_n h_n^{(2)}(k_+R) = a_n^0 h_n^{(1)}(k_+R),$$
$$a_n\{j_n\}(k_-, R) - b_n\{h_n^{(2)}\}(k_+, R) = a_n^0\{h_n^{(1)}\}(k_+, R).$$

$$\varepsilon\_c_n j_n(k\_R) - \varepsilon_+ d_n h_n^{(2)}(k_+R) = \varepsilon_+ c_n^0 h_n^{(1)}(k_+R),$$
$$c_n\{j_n\}(k_-, R) - d_n\{h_n^{(2)}\}(k_+, R) = c_n^0\{h_n^{(1)}\}(k_+, R).$$

Let’s consider the first system firstly:

$$a_n j_n(k\_R) - b_n h_n^{(2)}(k_+R) = a_n^0 h_n^{(1)}(k_+R),$$
$$a_n\{j_n\}(k_-, R) - b_n\{h_n^{(2)}\}(k_+, R) = a_n^0\{h_n^{(1)}\}(k_+, R).$$

The denotation (14) used here to short the system, but it makes it not perfectly clear. Let's reduce it and go back to the derivatives. To do so, let’s replace notation (14) with the complete form of it and provide the whole system here:

$$a_n j_n(k\_R) - b_n h_n^{(2)}(k_+R) = a_n^0 h_n^{(1)}(k_+R),$$
$$a_n \frac{1}{R}\frac{d}{dr}\left(r j_n(k\_r)\right)\Bigg|_{r=R} - b_n \frac{1}{R}\frac{d}{dr}\left(r h_n^{(2)}(k_+r)\right)\Bigg|_{r=R} = a_n^0 \frac{1}{R}\frac{d}{dr}\left(r h_n^{(1)}(k_+r)\right)\Bigg|_{r=R}.$$

Now lets differentiate the second equation according to given formulas and after this let’s reduce the multiplicand $\frac{1}{R}$:

$$a_n j_n(k\_R) - b_n h_n^{(2)}(k_+R) = a_n^0 h_n^{(1)}(k_+R),$$
$$a_n \left(j_n(k\_R) + R\frac{d}{dr} j_n(k\_r)\Bigg|_{r=R} \right) - b_n \left(h_n^{(2)}(k_+R) + R\frac{d}{dr} h_n^{(2)}(k_+r)\Bigg|_{r=R} \right) = \tag{17}$$
$$= a_n^0 \left(h_n^{(1)}(k_+R) + R\frac{d}{dr} h_n^{(1)}(k_+r)\Bigg|_{r=R} \right).$$

In the following step the formula 10.1.21 from the [13] considered and several transformations made of it:

$$\frac{n+1}{z} f_n(z) + \frac{d}{dz} f_n(z) = f_{n-1}(z),$$
$$\frac{d}{dz} f_n(z) = f_{n-1}(z) - \frac{n+1}{z} f_n(z),$$

$$\frac{d}{dz}f_n(kz)\bigg|_{z=R} = \frac{kd}{kdz}f_n(kz)\bigg|_{z=R} = \left[kz = t; kdz = dt\right] = \frac{kd}{dt}f_n(t)\bigg|_{t=kR} =$$

$$\left(kf_{n-1}(t) - k\frac{n+1}{t}f_n(t)\right)\bigg|_{t=kR} = kf_{n-1}(kR) - \frac{n+1}{R}f_n(kR)$$

Thus, in the transformations above, a formula for differentiating the Bessel functions at the point is obtained:

$$\frac{d}{dr}f_n(kr)\bigg|_{r=R} = kf_{n-1}(kR) - \frac{n+1}{R}f_n(kR). \tag{18}$$

Let's use this formula (18) to perform transformations under the second equation in the equation set (17) and represent it as a equation without derivatives in it:

$$\begin{gathered}
a_n j_n(k_R) - b_n h_n^{(2)}(k_+R) = a_n^0 h_n^{(1)}(k_+R), \\
a_n\left(j_n(k_R) \ + \ R(k_j_{n-1}(k_R) - \frac{n+1}{R}j_n(k_R))\right) \ - \\
- \ b_n\left(h_n^{(2)}(k_+R) \ + \ R(k_+h_{n-1}^{(2)}(k_+R) - \frac{n+1}{R}h_n^{(2)}(k_+R))\right) \ = \\
= \ a_n^0\left(h_n^{(1)}(k_+R) \ + \ R(k_+h_{n-1}^{(1)}(k_+R) - \frac{n+1}{R}h_n^{(1)}(k_+R))\right).
\end{gathered}$$

Subtract the first equation from the second one:

$$\begin{gathered}
a_n j_n(k_R) - b_n h_n^{(2)}(k_+R) = a_n^0 h_n^{(1)}(k_+R), \\
a_n\left(R(k_j_{n-1}(k_R) - \frac{n+1}{R}j_n(k_R))\right) \ - \\
- \ b_n\left(R(k_+h_{n-1}^{(2)}(k_+R) - \frac{n+1}{R}h_n^{(2)}(k_+R))\right) \ = \\
= \ a_n^0\left(R(k_+h_{n-1}^{(1)}(k_+R) - \frac{n+1}{R}h_n^{(1)}(k_+R))\right).
\end{gathered} \tag{19}$$

The result is the system of $2n$ linear equations with $2n$ unknown constant coefficients. Denote short notation for the functions:

$$\begin{gathered}
X_n = j_n(k_R), \\
Y_n = -\,h_n^{(2)}(k_+R),
\end{gathered}$$

$$
\begin{aligned}
Z_n &= R\left(k_- j_{n-1}(k_- R) - \frac{n+1}{R} j_n(k_- R)\right),\\
T_n &= -\left(R(k_+ h_{n-1}^{(2)}(k_+ R) - \frac{n+1}{R} h_n^{(2)}(k_+ R))\right),\\
V_n &= a_n^0 h_n^{(1)}(k_+ R),\\
W_n &= a_n^0\left(R(k_+ h_{n-1}^{(1)}(k_+ R) - \frac{n+1}{R} h_n^{(1)}(k_+ R))\right).
\end{aligned}
\tag{20}
$$

The introduced notation (20) could be used to simplify the notation of system (19). Thus, further system under consideration is the following:

$$
\begin{aligned}
a_n X_n + b_n Y_n &= V_n,\\
a_n Z_n + b_n T_n &= W_n.
\end{aligned}
\tag{21}
$$

The resulting system is a simple system of $2n$ linear equations, which can be considered as n systems of 2 linear equations, because the coefficients are independent for each pair of equations. Let's find the general solutions for this system.

$$
\begin{bmatrix} X_n & Y_n & V_n \\ Z_n & T_n & W_n \end{bmatrix} = \begin{bmatrix} 1 & \frac{Y_n}{X_n} & \frac{V_n}{X_n} \\ Z_n & T_n & W_n \end{bmatrix} = \begin{bmatrix} 1 & \frac{Y_n}{X_n} & \frac{V_n}{X_n} \\ 0 & T_n - Z_n\frac{Y_n}{X_n} & W_n - Z_n\frac{V_n}{X_n} \end{bmatrix} =
$$

$$
= \begin{bmatrix} 1 & \frac{Y_n}{X_n} & \frac{V_n}{X_n} \\ 0 & \frac{T_n X_n - Z_n Y_n}{X_n} & \frac{W_n X_n - Z_n V_n}{X_n} \end{bmatrix} = \begin{bmatrix} 1 & \frac{Y_n}{X_n} & \frac{V_n}{X_n} \\ 0 & 1 & \frac{W_n X_n - Z_n V_n}{T_n X_n - Z_n Y_n} \end{bmatrix} =
$$

$$
= \begin{bmatrix} 1 & 0 & \frac{1}{X_n}\left(V_n - Y_n\frac{W_n X_n - Z_n V_n}{T_n X_n - Z_n Y_n}\right) \\ 0 & 1 & \frac{W_n X_n - Z_n V_n}{T_n X_n - Z_n Y_n} \end{bmatrix}
$$

Thus, the solution of system (21) was obtained in an explicit form, which means the explicit expressions can be written for the coefficients of system (19). The resulting solution for the system (19), which is first part of the equation set (16) is the following:

$$
\begin{aligned}
a_n &= \frac{1}{X_n}\left(V_n - Y_n\frac{W_n X_n - Z_n V_n}{T_n X_n - Z_n Y_n}\right),\\
b_n &= \frac{W_n X_n - Z_n V_n}{T_n X_n - Z_n Y_n},
\end{aligned}
\tag{22}
$$

where all used coefficients defined in (20).

Now let's consider the second part of the system (16), which is formed by a system of $2n$ equations and can de describe as following:

$$\varepsilon_- c_n j_n(k_- R) - \varepsilon_+ d_n h_n^{(2)}(k_+ R) = \varepsilon_+ c_n^0 h_n^{(1)}(k_+ R),$$
$$c_n \{j_n\}(k_-, R) - d_n \{h_n^{(2)}\}(k_+, R) = c_n^0 \{h_n^{(1)}\}(k_+, R).$$

Firstly let's divide the first equation on ε_+ and denote $\varepsilon = \dfrac{\varepsilon_-}{\varepsilon_+}$, in this way the system can be reduced to the previous one.

$$\begin{aligned} \varepsilon c_n j_n(k_- R) - d_n h_n^{(2)}(k_+ R) &= c_n^0 h_n^{(1)}(k_+ R), \\ c_n \{j_n\}(k_-, R) - d_n \{h_n^{(2)}\}(k_+, R) &= c_n^0 \{h_n^{(1)}\}(k_+, R). \end{aligned} \tag{23}$$

The only difference between this system of equations and the one already considered before, which is represents first part of the system (16), is the coefficient ε in front of unknown arbitrary constant c_n in the first equation.

To get the explicit solution the same steps needs to be applied. First let's differentiate the second equation in the system (23) and reduce the multiplicand $\dfrac{1}{R}$, then apply the formula (18) to perform transformations under the second equation in the system (23) and represent it as equation without derivatives in it. And then subtract one equation from another. The result is the system of $2n$ linear equations with $2n$ unknown constant coefficients, which can be represented in a general view as a system (21).

Thus the explicit solution for the system (23) for the unknown arbitrary constant c_n, d_n is the following:

$$\begin{aligned} c_n &= \frac{1}{\hat{X}_n}\left(\hat{V}_n - \hat{Y}_n \frac{\hat{W}_n \hat{X}_n - \hat{Z}_n \hat{V}_n}{\hat{T}_n \hat{X}_n - \hat{Z}_n \hat{Y}_n}\right), \\ d_n &= \frac{\hat{W}_n \hat{X}_n - \hat{Z}_n \hat{V}_n}{\hat{T}_n \hat{X}_n - \hat{Z}_n \hat{Y}_n}, \end{aligned} \tag{24}$$

where all used coefficients defined as following:

$$\begin{aligned} \hat{X}_n &= \varepsilon j_n(k_- R), \\ \hat{Y}_n &= -h_n^{(2)}(k_+ R), \\ \hat{Z}_n &= R\left(k_- j_{n-1}(k_- R) - \frac{n+1}{R} j_n(k_- R)\right), \\ \hat{T}_n &= -\left(R(k_+ h_{n-1}^{(2)}(k_+ R) - \frac{n+1}{R} h_n^{(2)}(k_+ R))\right), \\ \hat{V}_n &= a_n^0 h_n^{(1)}(k_+ R), \end{aligned} \tag{25}$$

$$\hat{W}_n = a_n^0 \left(R(k_+ h_{n-1}^{(1)}(k_+R) - \frac{n+1}{R} h_n^{(1)}(k_+R)) \right).$$

Thus, an explicit solution was obtained for the system (16).

$$\begin{gathered} a_n = \frac{1}{X_n}\left(V_n - Y_n \frac{W_n X_n - Z_n V_n}{T_n X_n - Z_n Y_n}\right), \\ b_n = \frac{W_n X_n - Z_n V_n}{T_n X_n - Z_n Y_n}, \\ c_n = \frac{1}{\hat{X}_n}\left(\hat{V}_n - \hat{Y}_n \frac{\hat{W}_n \hat{X}_n - \hat{Z}_n \hat{V}_n}{\hat{T}_n \hat{X}_n - \hat{Z}_n \hat{Y}_n}\right), \\ d_n = \frac{\hat{W}_n \hat{X}_n - \hat{Z}_n \hat{V}_n}{\hat{T}_n \hat{X}_n - \hat{Z}_n \hat{Y}_n}, \end{gathered} \tag{26}$$

where all used coefficients are defined in (20) and (25).

3.2. The metal surface of the sphere

Now let's consider another case with metal spherical surface and plane electromagnetic wave incident to it.

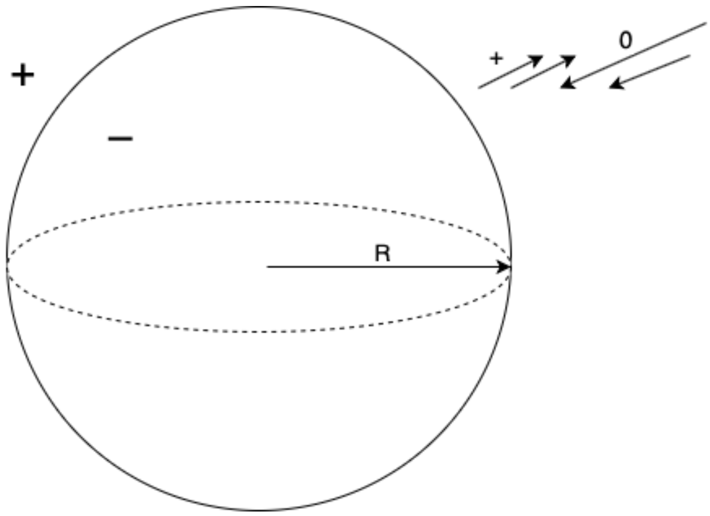

Figure 4 – The model of diffraction in the case of metal spherical surface

In the case of metal spherical surface the incident electromagnetic wave (electric and magnet components) should be equal to the reflected wave (Figure 4, please notice that the figure provides only schematic notation).

$$E^0 = -E^+,$$

$$H^0 = -H^+.$$

The tangential components of electromagnetic wave have to be continuous on the spherical surface, regarding this let's equate the tangential components of the vectors **E** and **H** on the sphere ($r = R$) and will consider the fact that the incident wave will contain only the summands with $m = 1$ and $m = -1$.

Exterior boundary value problem is formulating as following:

$$
\begin{aligned}
E_\theta(r,\theta) &= -\frac{i\omega\mu_0\mu}{\sin\theta}\sum_{n=|m|}^{+\infty} b_n\zeta_n^{(2)}(kr)\varphi_n(\theta) + \frac{1}{im}\sum_{n=|m|}^{+\infty} d_n\frac{1}{r}\frac{d}{dr}(r\zeta_n^{(2)}(kr))\varphi_n'(\theta),\\
E_\alpha(r,\theta) &= \frac{\omega\mu_0\mu}{m}\sum_{n=|m|}^{+\infty} b_n\zeta_n^{(2)}(kr)\varphi_n'(\theta) + \frac{1}{\sin\theta}\sum_{n=|m|}^{+\infty} d_n\frac{1}{r}\frac{d}{dr}(r\zeta_n^{(2)}(kr))\varphi_n(\theta),\\
H_\theta(r,\theta) &= \frac{1}{im}\sum_{n=|m|}^{+\infty} b_n\frac{1}{r}\frac{d}{dr}\left(r\zeta_n^{(2)}(kr)\right)\varphi_n'(\theta) + \frac{i\omega\varepsilon_0\varepsilon}{\sin\theta}\sum_{n=|m|}^{+\infty} d_n\zeta_n^{(2)}(kr)\varphi_n(\theta),\\
H_\alpha(r,\theta) &= \frac{1}{\sin\theta}\sum_{n=|m|}^{+\infty} b_n\frac{1}{r}\frac{d}{dr}\left(r\zeta_n^{(2)}(kr)\right)\varphi_n(\theta) - \frac{\omega\varepsilon_0\varepsilon}{m}\sum_{n=|m|}^{+\infty} d_n\zeta_n^{(2)}(kr)\varphi_n'(\theta).
\end{aligned}
\tag{27}
$$

The tangential components of incident electromagnetic wave in the same time could be represented in the following way:

$$
\begin{aligned}
E_\theta^0(r,\theta) &= -\frac{i\omega\mu_0\mu}{\sin\theta}\sum_{n=|m|}^{+\infty} a_n^0\zeta_n^{(1)}(k_+r)\varphi_n(\theta) + \frac{1}{im}\sum_{n=|m|}^{+\infty} c_n^0\frac{1}{r}\frac{d}{dr}(r\zeta_n^{(1)}(k_+r))\varphi_n'(\theta)\\
E_\alpha^0(r,\theta) &= \frac{\omega\mu_0\mu}{m}\sum_{n=|m|}^{+\infty} a_n^0\zeta_n^{(1)}(k_+r)\varphi_n'(\theta) + \frac{1}{\sin\theta}\sum_{n=|m|}^{+\infty} c_n^0\frac{1}{r}\frac{d}{dr}(r\zeta_n^{(1)}(k_+r))\varphi_n(\theta)\\
H_\theta^0(r,\theta) &= \frac{1}{im}\sum_{n=|m|}^{+\infty} a_n^0\frac{1}{r}\frac{d}{dr}(r\zeta_n^{(1)}(k_+r))\varphi_n'(\theta) + \frac{i\omega\varepsilon_0\varepsilon_+}{\sin\theta}\sum_{n=|m|}^{+\infty} c_n^0\zeta_n^{(1)}(k_+r)\varphi_n(\theta)\\
H_\alpha^0(r,\theta) &= \frac{1}{\sin\theta}\sum_{n=|m|}^{+\infty} a_n^0\frac{1}{r}\frac{d}{dr}(r\zeta_n^{(1)}(k_+r))\varphi_n(\theta) - \frac{\omega\varepsilon_0\varepsilon_+}{m}\sum_{n=|m|}^{+\infty} c_n^0\zeta_n^{(1)}(k_+r)\varphi_n'(\theta).
\end{aligned}
\tag{28}
$$

3.2.1. Derivation the equation set

Let's consider equations for the components of vector **E**, regarding the fact that wave have to be continuous on the spherical surface, the following equations should hold:

$$E_\theta^0(r,\theta) = -E_\theta^+(r,\theta), \tag{29.1}$$

$$E_\alpha^0(r,\theta) = -E_\alpha^+(r,\theta). \tag{29.2}$$

First let's consider equation (29.1), for this let's equate corresponding components of (27) and (28) and write down the result:

$$-\frac{i\omega\mu_0\mu}{\sin\theta}\sum_{n=|m|}^{+\infty} a_n^0 h_n^{(1)}(k_+r)\varphi_n(\theta) + \frac{1}{im}\sum_{n=|m|}^{+\infty} c_n^0\frac{1}{r}\frac{d}{dr}(rh_n^{(1)}(k_+r))\varphi_n'(\theta) =$$

$$= \frac{i\omega\mu_0\mu}{\sin\theta} \sum_{n=|m|}^{+\infty} b_n h_n^{(2)}(k_+ r)\varphi_n(\theta) - \frac{1}{im} \sum_{n=|m|}^{+\infty} d_n \frac{1}{r} \frac{d}{dr}(r h_n^{(2)}(k_+ r))\varphi_n'(\theta)$$

In the following step the formula 8.5.4 from the [13] considered:

$$(z^2 - 1)\frac{d}{dz} P_n^{(m)}(z) = n z P_n^{(m)}(z) - (n + m) P_{n-1}^{(m)}(z). \tag{30}$$

As it was said above, the notation $\varphi_n(\theta) = P_n^{(m)}(\cos\theta)$ used in the expressions. Regarding this notation the following could be considered:

$$\varphi_n'(\theta) = \sin\theta \cdot \frac{d}{dz} P_n^{(m)}(z),$$

where $z = \cos\theta$.

Considering all of the above, the following expression could be used to simplify the equations set (29) and corresponding to it:

$$-\sin\theta \cdot \varphi_n'(\theta) = n\cos\theta\varphi_n(\theta) - (n + m)\varphi_{n-1}(\theta).$$

Let's continue consideration of the equation (29.1):

$$-\frac{i\omega\mu_0\mu}{\sin\theta} \sum_{n=|m|}^{+\infty} a_n^0 h_n^{(1)}(k_+ r)\varphi_n(\theta) \; -$$

$$-\frac{1}{im\sin\theta} \sum_{n=|m|}^{+\infty} c_n^0 \frac{1}{r}\frac{d}{dr}(r h_n^{(1)}(k_+ r))(n\cos\theta\varphi_n(\theta) - (n + m)\varphi_{n-1}(\theta)) \; =$$

$$= \frac{i\omega\mu_0\mu}{\sin\theta} \sum_{n=|m|}^{+\infty} b_n h_n^{(2)}(k_+ r)\varphi_n(\theta) +$$

$$+ \frac{1}{im\sin\theta} \sum_{n=|m|}^{+\infty} d_n \frac{1}{r}\frac{d}{dr}(r h_n^{(2)}(k_+ r))(n\cos\theta\varphi_n(\theta) - (n + m)\varphi_{n-1}(\theta)),$$

$$-i\omega\mu_0\mu\sum_{n=|m|}^{+\infty}a_n^0h_n^{(1)}(k_+r)\varphi_n(\theta)-$$

$$-\frac{1}{im}\sum_{n=|m|}^{+\infty}c_n^0\frac{1}{r}\frac{d}{dr}(rh_n^{(1)}(k_+r))(n\cos\theta\varphi_n(\theta)-(n+m)\varphi_{n-1}(\theta)) =$$

$$=i\omega\mu_0\mu\sum_{n=|m|}^{+\infty}b_nh_n^{(2)}(k_+r)\varphi_n(\theta)+$$

$$+\frac{1}{im}\sum_{n=|m|}^{+\infty}d_n\frac{1}{r}\frac{d}{dr}(rh_n^{(2)}(k_+r))(n\cos\theta\varphi_n(\theta)-(n+m)\varphi_{n-1}(\theta))$$

To further simplification let's combine summands with the corresponding index of he associated Legendre polynomials:

$$-i\omega\mu_0\mu\sum_{n=|m|}^{+\infty}(a_n^0h_n^{(1)}(k_+r)+b_nh_n^{(2)}(k_+r))\varphi_n(\theta) -$$

$$-\frac{1}{im}\sum_{n=|m|}^{+\infty}c_n^0\frac{1}{r}\frac{d}{dr}(rh_n^{(1)}(k_+r))(n\cos\theta)\varphi_n(\theta)+$$

$$+\frac{1}{im}\sum_{n=|m|}^{+\infty}c_n^0\frac{1}{r}\frac{d}{dr}(rh_n^{(1)}(k_+r))(n+m)\varphi_{n-1}(\theta)) =$$

$$=\frac{1}{im}\sum_{n=|m|}^{+\infty}d_n\frac{1}{r}\frac{d}{dr}(rh_n^{(2)}(k_+r))(n\cos\theta)\varphi_n(\theta)-$$

$$-\frac{1}{im}\sum_{n=|m|}^{+\infty}d_n\frac{1}{r}\frac{d}{dr}(rh_n^{(2)}(k_+r))(n+m)\varphi_{n-1}(\theta)),$$

$$i\omega\mu_0\mu\sum_{n=|m|}^{+\infty}(a_n^0h_n^{(1)}(k_+r)+b_nh_n^{(2)}(k_+r))\varphi_n(\theta) +$$

$$+\frac{1}{im}\sum_{n=|m|}^{+\infty}(n\cos\theta)\left(c_n^0\frac{1}{r}\frac{d}{dr}(rh_n^{(1)}(k_+r))+d_n\frac{1}{r}\frac{d}{dr}(rh_n^{(2)}(k_+r))\right)\varphi_n(\theta)=$$

$$=\frac{1}{im}\sum_{n=|m|}^{+\infty}(n+m)\left(d_n\frac{1}{r}\frac{d}{dr}(rh_n^{(2)}(k_+r))+c_n^0\frac{1}{r}\frac{d}{dr}(rh_n^{(1)}(k_+r))\right)\varphi_{n-1}(\theta)$$

Now let’s consider equation (29.2) and perform on it similar actions to simplify and lead to a similar form:

$$-\frac{\omega\mu_0\mu}{m\sin\theta}\sum_{n=|m|}^{+\infty}a_n^0h_n^{(1)}(k_+r)(n\cos\theta\varphi_n(\theta)-(n+m)\varphi_{n-1}(\theta)) +$$

$$+\frac{1}{\sin\theta}\sum_{n=|m|}^{+\infty} c_n^0 \frac{1}{r}\frac{d}{dr}(rh_n^{(1)}(k_+r))\varphi_n(\theta) =$$

$$= -\frac{\omega\mu_0\mu}{m\sin\theta}\sum_{n=|m|}^{+\infty} b_n h_n^{(2)}(k_+r)(n\cos\theta\varphi_n(\theta) - (n+m)\varphi_{n-1}(\theta)) +$$

$$+\frac{1}{\sin\theta}\sum_{n=|m|}^{+\infty} d_n \frac{1}{r}\frac{d}{dr}(rh_n^{(2)}(k_+r))\varphi_n(\theta),$$

$$-\frac{\omega\mu_0\mu}{m}\sum_{n=|m|}^{+\infty} a_n^0 h_n^{(1)}(k_+r)(n\cos\theta\varphi_n(\theta) - (n+m)\varphi_{n-1}(\theta)) +$$

$$+\sum_{n=|m|}^{+\infty} c_n^0 \frac{1}{r}\frac{d}{dr}(rh_n^{(1)}(k_+r))\varphi_n(\theta) =$$

$$= -\frac{\omega\mu_0\mu}{m}\sum_{n=|m|}^{+\infty} b_n h_n^{(2)}(k_+r)(n\cos\theta\varphi_n(\theta) - (n+m)\varphi_{n-1}(\theta)) +$$

$$+\sum_{n=|m|}^{+\infty} d_n \frac{1}{r}\frac{d}{dr}(rh_n^{(2)}(k_+r))\varphi_n(\theta)$$

$$\sum_{n=|m|}^{+\infty} c_n^0 \frac{1}{r}\frac{d}{dr}(rh_n^{(1)}(k_+r))\varphi_n(\theta) \;-\; \sum_{n=|m|}^{+\infty} d_n \frac{1}{r}\frac{d}{dr}(rh_n^{(2)}(k_+r))\varphi_n(\theta) \;=$$

$$= \frac{\omega\mu_0\mu}{m}\sum_{n=|m|}^{+\infty} a_n^0 h_n^{(1)}(k_+r)(n\cos\theta\varphi_n(\theta) - (n+m)\varphi_{n-1}(\theta)) -$$

$$-\frac{\omega\mu_0\mu}{m}\sum_{n=|m|}^{+\infty} b_n h_n^{(2)}(k_+r)(n\cos\theta\varphi_n(\theta) - (n+m)\varphi_{n-1}(\theta))$$

And finally to further simplification let's combine summands with the corresponding index of the associated Legendre polynomials:

$$\sum_{n=|m|}^{+\infty}\left(c_n^0 \frac{1}{r}\frac{d}{dr}(rh_n^{(1)}(k_+r)) \;-\; d_n \frac{1}{r}\frac{d}{dr}(rh_n^{(2)}(k_+r))\right)\varphi_n(\theta) \;=$$

$$= \frac{\omega\mu_0\mu}{m}\sum_{n=|m|}^{+\infty} a_n^0 h_n^{(1)}(k_+r) n\cos\theta\varphi_n(\theta) \;-\; \frac{\omega\mu_0\mu}{m}\sum_{n=|m|}^{+\infty} a_n^0 h_n^{(1)}(k_+r)(n+m)\varphi_{n-1}(\theta) -$$

$$-\frac{\omega\mu_0\mu}{m}\sum_{n=|m|}^{+\infty} b_n h_n^{(2)}(k_+r) n\cos\theta\varphi_n(\theta) \;+\; \frac{\omega\mu_0\mu}{m}\sum_{n=|m|}^{+\infty} b_n h_n^{(2)}(k_+r)(n+m)\varphi_{n-1}(\theta),$$

$$\sum_{n=|m|}^{+\infty}\left(c_n^0\frac{1}{r}\frac{d}{dr}(rh_n^{(1)}(k_+r)) \;-\; d_n\frac{1}{r}\frac{d}{dr}(rh_n^{(2)}(k_+r))\right)\varphi_n(\theta) +$$

$$+\frac{\omega\mu_0\mu}{m}\sum_{n=|m|}^{+\infty} n\cos\theta\left(b_nh_n^{(2)}(k_+r) \;-\; a_n^0h_n^{(1)}(k_+r)\right)\varphi_n(\theta) \;=$$

$$=\frac{\omega\mu_0\mu}{m}\sum_{n=|m|}^{+\infty}(n+m)\left(b_nh_n^{(2)}(k_+r) - a_n^0h_n^{(1)}(k_+r)\right)\varphi_{n-1}(\theta).$$

The result of a consideration of equations (29) for the tangential components of the vector **E** is the following equation set:

$$\begin{gathered}
i\omega\mu_0\mu\sum_{n=|m|}^{+\infty}(a_n^0h_n^{(1)}(k_+r)+b_nh_n^{(2)}(k_+r))\varphi_n(\theta) \;+ \\
+\frac{1}{im}\sum_{n=|m|}^{+\infty}(n\cos\theta)\left(c_n^0\frac{1}{r}\frac{d}{dr}(rh_n^{(1)}(k_+r))+d_n\frac{1}{r}\frac{d}{dr}(rh_n^{(2)}(k_+r))\right)\varphi_n(\theta)= \\
=\frac{1}{im}\sum_{n=|m|}^{+\infty}(n+m)\left(d_n\frac{1}{r}\frac{d}{dr}(rh_n^{(2)}(k_+r))+c_n^0\frac{1}{r}\frac{d}{dr}(rh_n^{(1)}(k_+r))\right)\varphi_{n-1}(\theta), \\
\sum_{n=|m|}^{+\infty}\left(c_n^0\frac{1}{r}\frac{d}{dr}(rh_n^{(1)}(k_+r)) \;-\; d_n\frac{1}{r}\frac{d}{dr}(rh_n^{(2)}(k_+r))\right)\varphi_n(\theta) + \\
+\frac{\omega\mu_0\mu}{m}\sum_{n=|m|}^{+\infty} n\cos\theta\left(b_nh_n^{(2)}(k_+r) \;-\; a_n^0h_n^{(1)}(k_+r)\right)\varphi_n(\theta) \;= \\
\frac{\omega\mu_0\mu}{m}\sum_{n=|m|}^{+\infty}(n+m)\left(b_nh_n^{(2)}(k_+r) - a_n^0h_n^{(1)}(k_+r)\right)\varphi_{n-1}(\theta),
\end{gathered} \tag{31}$$

where $\varphi_n(\theta) = P_n^{(m)}(\cos\theta)$.

3.2.2. Analytical solution of the equation set in case of metal surface

First let's consider first equation from equation set (31):

$$\begin{gathered}
i\omega\mu_0\mu\sum_{n=|m|}^{+\infty}(a_n^0h_n^{(1)}(k_+r)+b_nh_n^{(2)}(k_+r))\varphi_n(\theta) \;+ \\
+\frac{1}{im}\sum_{n=|m|}^{+\infty}(n\cos\theta)\left(c_n^0\frac{1}{r}\frac{d}{dr}(rh_n^{(1)}(k_+r))+d_n\frac{1}{r}\frac{d}{dr}(rh_n^{(2)}(k_+r))\right)\varphi_n(\theta)= \\
=\frac{1}{im}\sum_{n=|m|}^{+\infty}(n+m)\left(d_n\frac{1}{r}\frac{d}{dr}(rh_n^{(2)}(k_+r))+c_n^0\frac{1}{r}\frac{d}{dr}(rh_n^{(1)}(k_+r))\right)\varphi_{n-1}(\theta)
\end{gathered}$$

And change the variables in the right part of it, to make the indexes of the associated Legendre polynomials are equal:

$$n - 1 = \bar{n},$$
$$n = \bar{n} + 1.$$

The result is the following:

$$i\omega\mu_0\mu \sum_{n=|m|}^{+\infty} (a_n^0 h_n^{(1)}(k_+r) + b_n h_n^{(2)}(k_+r))\varphi_n(\theta) \ +$$

$$+\frac{1}{im}\sum_{n=|m|}^{+\infty}(n\cos\theta)\left(c_n^0\frac{1}{r}\frac{d}{dr}(rh_n^{(1)}(k_+r)) + d_n\frac{1}{r}\frac{d}{dr}(rh_n^{(2)}(k_+r))\right)\varphi_n(\theta) =$$

$$=\frac{1}{im}\sum_{\bar{n}=|m|-1}^{+\infty}(\bar{n}+m+1)\left(d_{\bar{n}+1}\frac{1}{r}\frac{d}{dr}(rh_{\bar{n}+1}^{(2)}(k_+r)) + c_{\bar{n}+1}^0\frac{1}{r}\frac{d}{dr}(rh_{\bar{n}+1}^{(1)}(k_+r))\right)\varphi_{\bar{n}}(\theta)$$

Next let's rewrite it and reduce the sums into one:

$$i\omega\mu_0\mu \sum_{n=|m|}^{+\infty} (a_n^0 h_n^{(1)}(k_+r) + b_n h_n^{(2)}(k_+r))\varphi_n(\theta) \ +$$

$$+\frac{1}{im}\sum_{n=|m|}^{+\infty}(n\cos\theta)\left(c_n^0\frac{1}{r}\frac{d}{dr}(rh_n^{(1)}(k_+r)) + d_n\frac{1}{r}\frac{d}{dr}(rh_n^{(2)}(k_+r))\right)\varphi_n(\theta) =$$

$$=\frac{1}{im}\sum_{\bar{n}=|m|}^{+\infty}(\bar{n}+m+1)\left(d_{\bar{n}+1}\frac{1}{r}\frac{d}{dr}(rh_{\bar{n}+1}^{(2)}(k_+r)) + c_{\bar{n}+1}^0\frac{1}{r}\frac{d}{dr}(rh_{\bar{n}+1}^{(1)}(k_+r))\right)\varphi_{\bar{n}}(\theta) \ +$$

$$+\frac{1}{im}(|m|-1+m+1)(d_{|m|-1+1}\frac{1}{r}\frac{d}{dr}(rh_{|m|-1+1}^{(2)}(k_+r)) +$$
$$+\ c_{|m|-1+1}^0\frac{1}{r}\frac{d}{dr}(rh_{|m|-1+1}^{(1)}(k_+r)))\varphi_{|m|-1}(\theta),$$

$$i\omega\mu_0\mu \sum_{n=|m|}^{+\infty} (a_n^0 h_n^{(1)}(k_+r) + b_n h_n^{(2)}(k_+r))\varphi_n(\theta) \ +$$

$$+\frac{1}{im}\sum_{n=|m|}^{+\infty}(n\cos\theta)\left(c_n^0\frac{1}{r}\frac{d}{dr}(rh_n^{(1)}(k_+r)) + d_n\frac{1}{r}\frac{d}{dr}(rh_n^{(2)}(k_+r))\right)\varphi_n(\theta) -$$

$$-\frac{1}{im}\sum_{\bar{n}=|m|}^{+\infty}(\bar{n}+m+1)\left(d_{\bar{n}+1}\frac{1}{r}\frac{d}{dr}(rh_{\bar{n}+1}^{(2)}(k_+r)) + c_{\bar{n}+1}^0\frac{1}{r}\frac{d}{dr}(rh_{\bar{n}+1}^{(1)}(k_+r))\right)\varphi_{\bar{n}}(\theta) \ =$$

$$=\frac{1}{im}(|m|+m)\left(d_{|m|}\frac{1}{r}\frac{d}{dr}(rh_{|m|}^{(2)}(k_+r)) + c_{|m|}^0\frac{1}{r}\frac{d}{dr}(rh_{|m|}^{(1)}(k_+r))\right)\varphi_{|m|-1}(\theta)$$

$$-\omega\mu_0\mu m \sum_{n=|m|}^{+\infty} (a_n^0 h_n^{(1)}(k_+r) + b_n h_n^{(2)}(k_+r))\varphi_n(\theta) \ +$$

$$+ \sum_{n=|m|}^{+\infty} (n\cos\theta)\left(c_n^0 \frac{1}{r}\frac{d}{dr}(rh_n^{(1)}(k_+r)) + d_n \frac{1}{r}\frac{d}{dr}(rh_n^{(2)}(k_+r)) \right)\varphi_n(\theta) -$$

$$- \sum_{n=|m|}^{+\infty} (n+m+1)\left(d_{n+1}\frac{1}{r}\frac{d}{dr}(rh_{n+1}^{(2)}(k_+r)) + c_{n+1}^0 \frac{1}{r}\frac{d}{dr}(rh_{n+1}^{(1)}(k_+r)) \right)\varphi_n(\theta) \ =$$

$$= (|m|+m)\left(d_{|m|}\frac{1}{r}\frac{d}{dr}(rh_{|m|}^{(2)}(k_+r)) + c_{|m|}^0 \frac{1}{r}\frac{d}{dr}(rh_{|m|}^{(1)}(k_+r)) \right)\varphi_{|m|-1}(\theta).$$

Regarding the fact that $\varphi_n(\theta) = P_n^{(m)}(\cos\theta)$, $n = |m|, |m|+1,...$ and that in the considered system one of the summand has the multiplicand $\varphi_{|m|-1}(\theta)$, the corresponding summand do not exists in the context of the given problem. Since that the system could be modified as following:

$$-\omega\mu_0\mu m \sum_{n=|m|}^{+\infty} (a_n^0 h_n^{(1)}(k_+r) + b_n h_n^{(2)}(k_+r))\varphi_n(\theta) \ +$$

$$+ \sum_{n=|m|}^{+\infty} (n\cos\theta)\left(c_n^0 \frac{1}{r}\frac{d}{dr}(rh_n^{(1)}(k_+r)) + d_n \frac{1}{r}\frac{d}{dr}(rh_n^{(2)}(k_+r)) \right)\varphi_n(\theta) -$$

$$- \sum_{n=|m|}^{+\infty} (n+m+1)\left(d_{n+1}\frac{1}{r}\frac{d}{dr}(rh_{n+1}^{(2)}(k_+r)) + c_{n+1}^0 \frac{1}{r}\frac{d}{dr}(rh_{n+1}^{(1)}(k_+r)) \right)\varphi_n(\theta) \ = 0.$$

Thus the functions $\varphi_n(\theta)$ form a complete system, the coefficients of the corresponding functions must be equal:

$$-\omega\mu_0\mu m(a_n^0 h_n^{(1)}(k_+r) + b_n h_n^{(2)}(k_+r)) \ +$$
$$+ (n\cos\theta)\left(c_n^0 \frac{1}{r}\frac{d}{dr}(rh_n^{(1)}(k_+r)) + d_n \frac{1}{r}\frac{d}{dr}(rh_n^{(2)}(k_+r)) \right) -$$
$$-(n+m+1)\left(d_{n+1}\frac{1}{r}\frac{d}{dr}(rh_{n+1}^{(2)}(k_+r)) + c_{n+1}^0 \frac{1}{r}\frac{d}{dr}(rh_{n+1}^{(1)}(k_+r)) \right) \ = 0$$

Thus, a recurrent formula to find the coefficient d_n is obtained.

Similarly, consideration of the second equation from equation set (31) will obtain a recurrent formula for the coefficient b_n. Which is enough to find the expansion of the reflected wave on the metal sphere.

Chapter 4: Implementation

This part of the work considers implementation of the electromagnetic wave diffraction by spherical surface. The applied development process, as well as all optimizations and parallelization algorithms are described in this part of the work.

4.1. Development process

The development and testing process is one of the most important aspects of a software development. A process is a sequence of actions that leads to a predictable result. Development without a process will be chaotic and will not always lead to the desired result.

This work is not a large software product, but even here it is necessary to adhere to some stages of software development. Since this work is research project, its first (and significant in volume) stage is the theoretical study of the problem. The first stage of development process, in turn, is developing a prototype of a future system.

The prototype must be tested for reliability, according to the physical phenomena and laws. After that, the prototype is improving in terms of performance and memory consumption. And various optimizations are applied, such as more advanced algorithms and other more complex data structures that allow the use of other algorithms.

Finally the resulting solution can be parallelized using various algorithms and technologies in order to improve computing performance.

4.2. Implementation of calculations of spherical Bessel functions

The first and simplest implementation of calculating the functions described in the paragraph 2.5. and 2.5.2. in particular is the following functions containing recursion (hereinafter, the code provided in the C ++ programming language):

```
double j_n(int n, double z)
{
    if (n == 0)
        return sin(z) / z;
    if (n == 1)
        return sin(z) / (z*z) - cos(z) / z;

    return (2*n - 1)/z * j_n(n-1, z) - j_n(n-2, z);
}

double y_n(int n, double z)
{
    if (n == 0)
        return - cos(z) / z;
    if (n == 1)
        return - cos(z) / (z*z) - sin(z) / z;

    return (2*n - 1)/z * y_n(n-1, z) - y_n(n-2, z);
}


std::complex< double > h_n(int n, double z, int i)
{
    double j = j_n(n, z);
    double y = y_n(n, z);
```

```
    std::complex< double > res( j, y );

    if (i == 1)
        return res;
    else if (i == 2)
        return std::conj( res );
    else return 0;
}
```

This code snippet contains three functions that calculate the spherical Bessel functions of the first, second and third kind, respectively.

The arguments of these functions are: n – function order, z – зthe value for which the function is calculated, i – kind of the Hankel function (could only take 2 values: 0 or 1).

Nota that this implementation does not work for complex values as arguments, which can be easily fixed by changing the data type of the local variable z.

The main disadvantage of this approach to the implementation of calculations is the fact that in order to solve the initially posed problem (system of equations), it is necessary to calculate not one function, but all functions with an order in the range from 0 to some.

This implementation, with multiple sequential calls of these functions, will recursively recalculate all values from the current to zero each time. These calculations take a lot of time and memory.

The solution to this problem is a creation of data structure that stores the previous, already calculated, values. Thus, to calculate each subsequent value, it will be necessary to perform significantly fewer operations.

Here is the data structure described above. For example, it can be a small structure containing two arrays:

```
struct Data
{
    double *j;
    double *y;
};
```

Only 2 arrays are needed, not 3 or 4, since the values of the Hankel functions of the first and second kind could be obtained from the already available data with the usage a small number of computational operations, using the following expression:

$$h_n^{(1)}(z) = j_n(z) + i\,y_n(z),$$
$$h_n^{(2)}(z) = j_n(z) - i\,y_n(z),$$

It is also worth noting that, according to (20) and (25), functions are calculated only for two arguments; therefore, two similar structures with data must be stored in memory. In addition, for one of these arguments, only the Bessel function of the first kind is calculated, respectively, the second array with data can also not be kept in memory.

As a result, when calculating the Bessel functions of the first and second kind, the call to the function `j_n(n-1, z)` can be replaced by a memory access

Here are the modified functions for calculating Hankel functions taking into account the use of this data structure:

```
std::complex<double> h_1(int n, Data &data)
{
    double j = data.j[n];
    double y = data.y[n];

    std::complex<double> res(j,y);

    return res;
}

std::complex<double> h_2(int n, Data &data)
{
    double j = data.j[n];
    double y = data.y[n];

    std::complex<double> res(j,-y);

    return res;
}
```

The arguments of these functions are: n – function order, $data$ – link to the structure which contain values of the Bessel function of the first and the second kind for the corresponding argument.

4.3. Implementation of the case of the dielectric surface of the sphere

In order to use formula (26), which is an explicit solution to the system of 2n equations, it is necessary to calculate all 12 coefficients described in (20) and (25).

Let's consider these coefficients. The expressions describing the coefficients (20) and (25) are simple linear functions containing, in addition to the coefficients, spherical Bessel functions of the first and third kind. Besides that, the spherical Bessel functions of the third kind (Hankel functions) are defined by the following expressions:

$$h_n^{(1)}(z) = j_n(z) + i\,y_n(z),$$
$$h_n^{(2)}(z) = j_n(z) - i\,y_n(z),$$

where $j_n(z)$ and $y_n(z)$ – spherical Bessel functions of the first and the second kind respectively, according to 10.1.1. from [13].

Thus, the problem of solving a system of $2n$ linear equations with $2n$ unknown variables is reduced to the problem of calculating the first n Bessel functions of the first and second kind.

4.3.1. First implementation

In order to increase the efficiency of memory access, the calculations can be divided into 2 stages: filling the array data with the values of the Bessel functions and the direct solution of the system of equations using (26).

After dividing the calculations into two stages, each of them can be considered and optimized independently.

Let’s first consider the second stage, more specifically the direct solution of the system of equations. It is an iterative process where 4 values of the parameters a_n, b_n, c_n and d_n. are calculated at each iteration. Taking into account the already calculated values of the Bessel functions of the first and second kind, each of these iterations is independent from the previous ones and allows the usage of parallel algorithms and technologies.

Let's start with the code for the sequential implementation of the calculation of parameters a_n, b_n, c_n and d_n. It is just simple arithmetic calculations:

```
for (int i = 1; i < n; ++i)
{
    double param = i+1;
    X = data_minus.j[i];
    Y = -h_2(i, data_plus);
    V = pr.a_0[i]*h_1(i,data_plus);

    Z = pr.k_minus*pr.R*data_minus.j[i-1] -
        param*data_minus.j[i];
    T = param*h_2(i,data_plus)-
        pr.k_plus*pr.R*h_2(i-1,data_plus);
    W = pr.a_0[i] * ( pr.k_plus*pr.R*h_1(i-1,data_plus) -
        param*h_1(i,data_plus) );

    _X = pr.eps1/pr.eps2 * X;
    _Y = Y;
    _V = pr.c_0[i]*h_1(i,data_plus);

    _Z = (1 - pr.eps1/pr.eps2) * data_minus.j[i] + Z;
    _T = T;
    _W = pr.c_0[i]*( pr.k_plus*pr.R*h_1(i-1,data_plus) -
         param*h_1(i,data_plus));

    result.a_n[i] =  V/X - Y/X * ((W*X-V*Z)/(T*X-Y*Z));
    result.b_n[i] = (W*X-V*Z)/(T*X-Y*Z);

    result.c_n[i] =  _V/_X - _Y/_X *
                    ((_W*_X-_V*_Z)/(_T*_X-_Y*_Z));
    result.d_n[i] = (_W*_X-_V*_Z)/(_T*_X-_Y*_Z);
}
```

The data type for the calculated parameters described in (20), (25) is the complex number: `std::complex<double>`.

Now let’s go back to the first stage of the solution required to solve the problem posed, the computation of the Bessel functions. The program code for the implementation of these calculations in provided below:

```
{
    double z = pr.k_minus*pr.R;
    data_k_minus.j[0] = sin(z) / z;
    data_k_minus.j[1] = sin(z)/ (z*z) - cos(z) / z;

    for (int i = 1; i < n; ++i)
    {
        data_k_minus.j[i+1] = (2*i-1)/z *  data_k_minus.j[i] -
                             data_k_minus.j[i-1];
    }
}
```

```
{
    double z = pr.k_plus*pr.R;
    data_k_plus.j[0] = sin(z) / z;
    data_k_plus.j[1] = sin(z)/ (z*z) - cos(z) / z;

    data_k_plus.y[0] = - cos(z) / z;
    data_k_plus.y[1] = - cos(z)/ (z*z) - sin(z) / z;

    for (int i = 1; i < n; ++i)
    {
        data_k_plus.j[i+1] = (2*i-1)/z *  data_k_plus.j[i] -
                             data_k_plus.j[i-1];
        data_k_plus.y[i+1] = (2*i-1)/z *  data_k_plus.y[i] -
                             data_k_plus.y[i-1];
    }
}
```

4.3.2. Simple testing

The result of the implemented simulation should be tested for reliability, according to the physical phenomena and laws, such as the law of conservation of incident and reflected energy (which is address the Poynting vector) and the fact the tangential component of **E** must be continuous on the sphere (the sum of the total field **E** must be zero, or the same that the sum of the tangent component of the field on the sphere must be zero).

To check these physical laws described above, it is enough to summarize the vector components obtained as a result of the modeling.

4.3.3. Optimization

The first implementation has two parts, since the task assumes the existence of two areas of space: internal and external relative to the sphere. First, let's combine the calculations into one block in order to combine two loops into one:

```
double z1 = pr.k_minus*pr.R;
double z2 = pr.k_plus*pr.R;

data_k_minus.j[0] = sin(z1) / z1;
data_k_minus.j[1] = sin(z1)/ (z1*z1) - cos(z1) / z1;

data_k_plus.j[0] = sin(z2) / z2;
data_k_plus.j[1] = sin(z2)/ (z2*z2) - cos(z2) / z2;

for (int i = 1; i < n; ++i)
{
    data_k_minus.j[i+1] = (2*i-1)/z1 *  data_k_minus.j[i] -
                          data_k_minus.j[i-1];
    data_k_plus.j[i+1] = (2*i-1)/z2 *  data_k_plus.j[i] -
                         data_k_plus.j[i-1];
    data_k_plus.y[i+1] = (2*i-1)/z2 *  data_k_plus.y[i] -
                         data_k_plus.y[i-1];
}
```

Calculations in the above code fragment are implemented as a loop, but these calculations are recurrent, which does not allow direct application of various parallel algorithms to calculate values.

However, calculations within this cycle perform absolutely identical operations on different data, which means that these calculations can be easily vectorized.

With an increase in the amount of computations, there is also a variant of dividing these computations into three independent processes using MPI, however, further optimizations, more than 3 times, are impossible when using this algorithm for calculating the values of the Bessel functions.

4.3.4. Other options for calculating Bessel functions

For further optimization of calculations within the framework of the stated problem, it is necessary to use other methods for calculating the values of the Bessel functions.

There are several quick methods for calculating Bessel functions based on their various expansions. One of them is presented in [16] and can be one of the options for further optimizations within the framework of this problem.

4.3.5. OpenMP for the solution

For a sequential implementation, the variables storing the values for intermediate calculations, specifically the parameters described in (20), (25) were reused for each subsequent implementation. To use the parallel algorithm, these variables must be local to each thread.

To calculate parameters a_n, b_n, c_n and d_n using OpenMP, the only which is need is to make small changes to the program code, specifically add the following line of code before the loop that performs calculations:

```
#pragma omp parallel for private(X,Y,V,Z,T,W,_X,_Y,_V,_Z,_T,_W):
```

4.3.6. MPI for the solution

Since each of the loop iterations can be performed independently of the previous and subsequent ones, it is possible to divide the data into several processes. And since the amount of calculations at each of the iterations is identical, it is advisable to divide it into equal parts.

To perform the calculation of the parameters on a separate process, the following data are required: the values of the parameters of the problem, such as the values of the dielectric constant and the wavenumber, the radius of the sphere and the initial values of the incident wave.

The task parameter data is stored in one data structure and can be easily sent to processes. Here is the described data structure:

```
struct Parameters
{
    double k_minus; // k_
    double k_plus; // k+
    double R;
    double eps1; // eps_
    double eps2; // eps+

    double* a_0;
    double* c_0;
};
```

It is also worth noting that to calculate a specific iteration, not all initial values of the incident wave are needed, but only those whose index coincides with the iteration number. Accordingly, it is not necessary to store all the data of the initial values in each of the processes, they can be shared between the processes.

In addition to the parameters described above, calculations also require Bessel values with the current and previous index, which means that it is also necessary to send the corresponding parts of the data stored in the structure `Data` to the processes.

Here are code snippets for sending the values of Bessel functions across processes:

```
MPI_Scatter(data_minus.j, n/ProcNum, MPI_DOUBLE, data_m_local.j, n/ProcNum, MPI_DOUBLE, 0, MPI_COMM_WORLD);

MPI_Scatter(data_plus.j, n/ProcNum, MPI_DOUBLE, data_p_local.j, n/ProcNum, MPI_DOUBLE, 0, MPI_COMM_WORLD);

MPI_Scatter(data_plus.y, n/ProcNum, MPI_DOUBLE, data_p_local.y, in/ProcNum, MPI_DOUBLE, 0, MPI_COMM_WORLD);
```

Similarly, the rest of the necessary data is sent to the processes. As a result of calculations, each of the processes locally has its own part of the parametersa_n, b_n, c_n and d_n, which must be collected together on one of the processes for subsequent output. In addition, since the data type of the received parameters is complex numbers, to simplify the data transfer, it is necessary to replace the previously used data type `std::complex<double>` to `double _Complex` from the library `<complex.h>`.

Here is a data structure for storing the results of calculations, taking into account the changed data type:

```
struct Result
{
    double _Complex *a_n;
    double _Complex *b_n;
    double _Complex *c_n;
    double _Complex *d_n;
};
```

Here is a piece of code that performs a data collection operation on one process:

```
MPI_Gather(result_local.a_n, n/ProcNum, MPI_C_DOUBLE_COMPLEX, result.a_n, n/ProcNum, MPI_C_DOUBLE_COMPLEX, 0, MPI_COMM_WORLD);

MPI_Gather(result_local.b_n, n/ProcNum, MPI_C_DOUBLE_COMPLEX, result.b_n, n/ProcNum, MPI_C_DOUBLE_COMPLEX, 0, MPI_COMM_WORLD);

MPI_Gather(result_local.c_n, n/ProcNum, MPI_C_DOUBLE_COMPLEX, result.c_n, n/ProcNum, MPI_C_DOUBLE_COMPLEX, 0, MPI_COMM_WORLD);

MPI_Gather(result_local.d_n, n/ProcNum, MPI_C_DOUBLE_COMPLEX, result.d_n, n/ProcNum, MPI_C_DOUBLE_COMPLEX, 0, MPI_COMM_WORLD);
```

As a result of executing this piece of code, the result of solving the original system of equations will be saved in the `Result` data structure described above.

4.3.7. Combining MPI and OpenMP to solve the system

As a result of using MPI, the code that calculates the parameters a_n, b_n, c_n and d_n, will look like this:

```
int N = n/ProcNum
```

```
for (int i = 1; i < N; ++i)
{
    double param = i+1;
    X = data_m_local.j[i];
    Y = -h_2(i, data_p_local);
    V = pr.a_0[i]*h_1(i,data_p_local);

    Z = pr.k_minus*pr.R*data_m_local.j[i-1] -
        param*data_m_local.j[i];
    T = param*h_2(i,data_p_local)-
        pr.k_plus*pr.R*h_2(i-1,data_p_local);
    W = pr.a_0[i]*( pr.k_plus*pr.R*h_1(i-1,data_p_local) -
        param*h_1(i,data_p_local) );

    _X = pr.eps1/pr.eps2 * X;
    _Y = Y;
    _V = pr.c_0[i]*h_1(i,data_p_local);

    _Z = (1 - pr.eps1/pr.eps2) * data_m_local.j[i] + Z;
    _T = T;
    _W =  pr.c_0[i]*( pr.k_plus*pr.R*h_1(i-1,data_p_local) -
          param*h_1(i,data_p_local));

    result_local.a_n[i] =  V/X - Y/X * ((W*X-V*Z)/(T*X-Y*Z));
    result_local.b_n[i] = (W*X-V*Z)/(T*X-Y*Z);

    result_local.c_n[i] = _V/_X - _Y/_X *
                          ((_W*_X-_V*_Z)/(_T*_X-_Y*_Z));
    result_local.d_n[i] = (_W*_X-_V*_Z)/(_T*_X-_Y*_Z);
}
```

In this case, the calculations performed at each separate iteration are also independent, which allows usage of the OpenMP technology inside the MPI process. To do this, it is necessary to add the following code fragment before the loop:

```
#pragma omp parallel for private(X,Y,V,Z,T,W,_X,_Y,_V,_Z,_T,_W)
```

Thus, the joint usage of MPI and OpenMP technologies implemented in order to solve the system, described in this work.

Chapter 5: Results

This part of the work contains the results of computational experiments for various versions of the algorithm described above.

Computational experiments were carried out for various values of the parameter n (the number of spherical harmonics in the expansion) on the following hardware and software:

Operating system: MacOS Catalina 10.15.5 (19F101);

CPU: 2,3 GHz Quad-Core Intel Core i5;

RAM: 8 GB 2133 MHz LPDDR3;

Threads: 8;

Processes: 8.

Experiments were also carried out on other different configurations of cluster-type hardware, but since the meaningful picture of the experimental results does not differ, this work presents the results obtained on personal hardware, since they can be easily repeated by anyone.

The measurements of the running time of the algorithm in milliseconds, obtained as a result of the experiments, are presented in Table 1.

N	Sequential	OpenMP	MPI	OpenMP + MPI
1000000	1271	387	798	587
2000000	2550	756	1381	1070
3000000	3871	1143	1922	1491
4000000	5126	1519	2424	1889
5000000	6317	1878	2883	2259
6000000	7629	2256	3304	2603
7000000	8936	2610	3693	2915

Table 1 – Algorithms runtime

The runtimes of the program in milliseconds for sequential and various parallel implementations of the algorithm for different values of the parameter n, obtained as a result of the experiments, are also presented in the form of a graph in Figure 5.

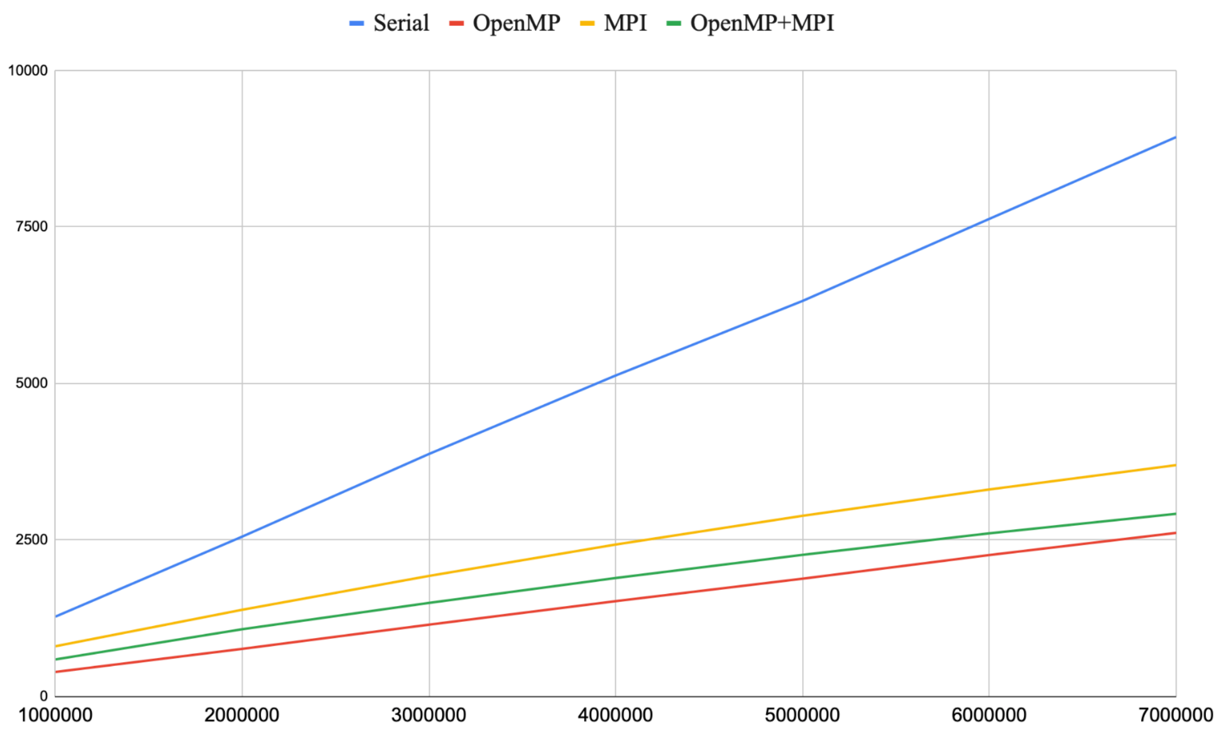

Figure 5 – . Algorithms runtime

Table 2 presents data with the acceleration values obtained as a result of the experiments performed. The same acceleration data is plotted in Figure 6.

N	OpenMP	MPI	OpenMP + MPI
1000000	3.284237726	1.592731830	2.165247019
2000000	3.373015873	1.846488052	2.383177570
3000000	3.386701662	2.014047867	2.596244131
4000000	3.374588545	2.114686469	2.713605082
5000000	3.363684771	2.191120361	2.796370075
6000000	3.381648936	2.309019370	2.930849020
7000000	3.423754789	2.419712970	3.065523156

Table 2 – Achieved acceleration values

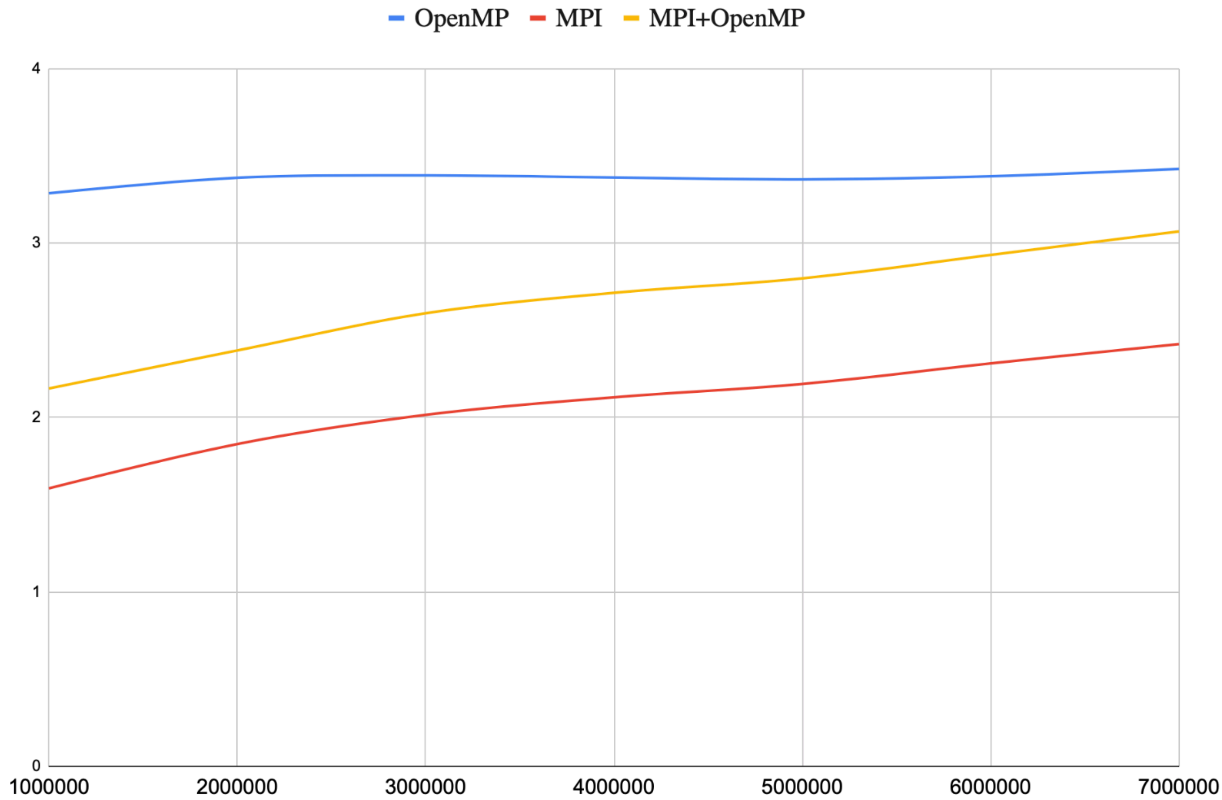

Figure 6 – Achieved acceleration values

Thus, when this algorithm is parallelized by 8 threads, the maximum acceleration obtained is 3.42 times relative to the sequential implementation.

Such results are explained by the fact that the operations performed in the parallel section of the algorithm are the simplest arithmetic operations, in addition, the complexity of the algorithm itself is linear.

It should also be noted that the greatest acceleration of the algorithm was obtained when using only OpenMP technology. Implementations using MPI and sharing MPI with OpenMP are slower than the OpenMP implementation. This feature is also explained by the complexity of the algorithm itself and the nature of the operations performed in the parallel domain. And since the use of MPI technology carries additional overheads for transferring data between processes, therefore, these implementations work slower than without using MPI. With an increase in the number of harmonics (parameter), the acceleration of implementations with MPI approaches OpenMP, since at the same time the calculations begin to take up an increasing proportion of the program runtime.

Chapter 6: Summary and conclusions

6.1. Conclusions

Within the framework of this work, the existing approaches to solving the problem of diffraction of electromagnetic waves were studied. Various technologies of parallel computing were studied and considered, as well as the possibilities of their application to solving diffraction problems.

A system of equations that describes the diffraction of an incident plane electromagnetic wave on a spherical surface of a given radius was obtained. The resulting system was transformed and expressions were obtained to calculate the expansion coefficients of the components of the electromagnetic field. A numerical solution was implemented that calculates these components.

Three different versions of the numerical solution were implemented using parallel computing technologies OpenMP, MPI and their joint use.

Experiments were carried out as a result of which an acceleration of 3.4 times was obtained using 8 computational threads.

Analysis of the experimental results shows that with this approach to solving the problem, parallel technologies can be successfully applied, however, due to the specifics of computations and the linear complexity of the implemented algorithm, there are restrictions on the maximum achievable performance by methods of parallel computations.

In this work a new method is applied to calculate the coefficients of the expansion in terms of the Bessel functions of the components of the vectors of the electromagnetic field. Also in this work a parallel solution of this system is implemented, in addition, three different versions of a parallel solution of this problem are implemented using parallel computations.

6.2. Future work

The results enable to draw the following directions for further work:

– Improvement of the existing implementation of the considered method in the way of implementation alternative data structured.

– Implementation of other parallel versions of the provided solution, for example with the usage of TBB library or similar to it.

– Consideration of other methods for the solution of the described problem.

– Consideration of a more complex problems, for example more complex surfaces.

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