ORIGINAL PAPER

ZAMM

Damage model for plastic materials at finite strains

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We introduce a model for elastoplasticity at finite strains coupled with damage. The internal energy of the deformed elastoplastic body depends on the deformation, the plastic strain, and the unidirectional isotropic damage. The main novelty is a dissipation distance allowing the description of coupled dissipative behavior of damage and plastic strain. Moving from time-discretization, we prove the existence of energetic solutions to the quasistatic evolution problem.

KEYWORDS
damage, energetic solution, finite plasticity, nonlinear, quasistatic evolution

MSC (2010)
74A45, 74C15, 49J40, 49J45

1 | INTRODUCTION

Failure in ductile materials, such as metals or polymers, proceeds from the initiation of micro-defects, followed by their diffuse growth accompanied by large irreversible deformations, up to the formation of localized macroscopic cracks. Altogether, these phenomena constitute *ductile fracture*, e.g., [23] or [32, Section 1.1.3], and are of primary concern in predictive modeling of forming processes in industrial practice.

Continuum-based models for ductile fracture must involve two dissipative mechanisms: *damage* and *plasticity*; see, e.g., [7] for an overview. Damage accounts for the stiffness reduction due to initiation, growth, and coalescence of defects, e.g., [32, Chapter 7], whereas plasticity quantifies the development of permanent strains within the material, e.g., [32, Chapter 7]. Moreover, the two mechanisms interact, resulting in the need for *coupled* damage-plasticity models, e.g., [32, Section 7.4.1].

In what follows, we adopt the format of *generalized standard materials*[28] and assume that the material behavior is governed by a stored energy density and a dissipation potential. Under small strains, the first local energy-based model for coupled damage-plasticity was introduced by JU.[29] Later on, ALESSI et al.[2–4] developed its non-local extension by including gradients of a damage variable into the stored energy, in the spirit of variational models for regularized brittle fracture and gradient damage developed in the mathematical, e.g., [9,10,24,40] and engineering, e.g., [22,33,44,45] literature. Such enrichment introduces an additional length scale into the energy functional to characterize the regions to which damage localizes; see also [1] for an overview and comparison of available formulations. Very recently, this class of models has been extended to a finite-strain regime independently by AMBATI et al.,[5] BORDEN et al.,[8] and MIEHE et al.[35] The last formulation involves additional regularization with gradients of plastic strains to control the localization of permanent strains, too. We invite an interested Reader to [5,8,35] for illustration of predictive power of these models, including their experimental validation.

Apart from providing a convenient approach to constitutive modeling, the framework of generalized standard materials naturally leads to the notion of *energetic solutions* – a solution concept for rate-independent problems developed by MIELKE and co-workers[38,41] that characterizes the evolution of state variables by conditions of global stability and energy conservation. The existence of an energetic solution for small-strain damage-plastic models has recently been shown by CRISMALE,[13] who further extended his result to gradient plasticity coupled with damage[14] (see also [15,16]). However, existence results for finite-strain models are currently lacking, although finite-strain damage[40] and gradient plasticity[34,36,37,39,42] were successfully addressed within the energetic solution concept.

In the current work, we prove the existence of an energetic solution to models of incomplete damage coupled with gradient plasticity at finite strains, under structural assumptions that comply with contemporary engineering models.[1] More specifically, we consider an elastoplastic body $\Omega \subset \mathbb{R}^d$ subjected to a deformation $\varphi : \Omega \to \mathbb{R}^d$. In nonlinear plasticity it is commonly assumed that the deformation gradient $\nabla\varphi$ complies for the multiplicative decomposition $\nabla\varphi = FP$ where $F : \Omega \to \mathbb{R}^{d \times d}$ and $P : \Omega \to SL(d)$ stand for the elastic and plastic strains. Moreover, we introduce an internal scalar variable $z : \Omega \to [0, 1]$ describing the damage of the medium, where the value $z(x) = 1$ corresponds to an undamaged status of Ω at x, while values close to zero mean that the body is highly damaged. The *internal stored energy* of a material state (φ, P, z) is given by

$$\mathcal{W}(\varphi, P, z) = \int_\Omega W_{\mathrm{el}}(\nabla\varphi P^{-1}, z) + W_{\mathrm{h}}(P, z) + \frac{\nu}{r_{\mathrm{p}}}|\nabla P|^{r_{\mathrm{p}}} + \frac{\mu}{r_{\mathrm{z}}}|\nabla z|^{r_{\mathrm{z}}} \, \mathrm{d}x,$$

see Section 2.5. In this expression, the first term is the elastic energy, the second represents the energy related to hardening effects[1], the third and fourth are regularization terms which from a physical point of view can be viewed as surface energies penalizing spatial variations of the internal variables P and z. More precisely, we can expect P and z, to change values on length scales of order $\mu^{1/(r_{\mathrm{p}}-1)}$ and $\nu^{1/(r_{\mathrm{z}}-1)}$, respectively. This would schematically correspond to the observation of the emergence of lower dimensional substructures in plasticity and damage, namely plastic shear bands and cracks. The correlation between the variables P and z partly relies on the behavior of the internal energy, which is monotone increasing in z, see Section 2.5. The evolution is driven by a time-dependent external loading ℓ which completes the *total energy* of the system given by

$$\mathcal{E}(t, \varphi, P, z) = \mathcal{W}(\varphi, P, z) - \langle \ell(t), \varphi \rangle,$$

where the dual product $\langle \ell(t), \varphi \rangle$ is defined in (26). We consider rate-independent evolution of the energy $\mathcal{E}$ coupled with a dissipation distance between internal states given by

$$\mathcal{D}(P, z, \widehat{P}, \widehat{z}) = \int_\Omega D(P(x), z(x), \widehat{P}(x), \widehat{z}(x)) \, \mathrm{d}x.$$

The latter depends on the joint behavior of plastic strain and damage. This coupling is implemented in the non-symmetric distance

$$D(P, z, \widehat{P}, \widehat{z}) = \begin{cases} \kappa|z - \widehat{z}| + \rho(\widehat{z})D_{\mathrm{p}}(P, \widehat{P}), & \text{if } z \geq \widehat{z}, \\ \infty, & \text{else}, \end{cases}$$

see Section 2.3. Here, ρ is a positive, monotone increasing function and D_{p} is the classical plastic dissipation distance introduced by MIELKE.[36,37] The function ρ models the fact that the material plasticizes more easily once it is damaged.

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[1] From the physical viewpoint, the term W_{h} represents the energy stored in interlocked and blocked dislocations during the plastic flow and corresponds to *kinematic* hardening; see, e.g., [32, Section 5.4] and [31, Section 7.4] for further details. As such, it should be distinguished from *strain hardening* in the sense of [3, Equation (19)], which arises from the combined effect of stored energy and dissipation of the damage part of the model.

We rely on the concept of *energetic solutions*, and consider a quasistatic evolution, namely a trajectory $[0, T] \ni t \mapsto (\varphi(t), P(t), z(t))$ on a time interval $[0, T]$ satisfying at every time t a stability condition and an energy balance, see Definition 3.1. Our main result, Theorem 3.2 in Section 3, asserts the existence of an energetic solution for any compatible (stable) initial datum. To prove this result we apply a standard time-discretization scheme introduced by MIELKE and co-workers.[38,41] This scheme has shown to be very successful in order to achieve existence of energetic solution to rate-independent systems. It is versatile, as it has been employed in many different settings, like in problems of nonlinear plasticity (e.g., [34,39]), damage,[40,50,51] cracks growth (e.g. [18,30]), delamination,[46] dislocations evolution,[49] and many others (see [41] and references therein for a more detailed discussion and a more exhaustive bibliography).

In many applications, thanks to the solid theory of MIELKE, the existence of energetic solutions is easily obtained by checking a series of standard hypotheses. In the present paper, due to our particular model which couples plasticity with damage, we borrow ingredients coming from both these fields. The proof of our main result relies on checking that the coupled dissipation defined in Section 2.3 is a lower semicontinuous quasidistance (C1-C2), the loading power is energetically controlled (C3), sublevels of the energy are compact (C4), and the set of stable states is closed (C5). In order to verify these conditions, we need to proof the linear growth of the plastic dissipation (Proposition 2.2) and a formula showing that the (abstract) coupled dissipation distance (21) splits additively into a damage-dissipation and a damage-weighted plastic dissipation (Proposition 2.4). We consider this part the main novelty of the present work.

The paper is organized as follows. Section 2 introduces our model, emphasizing the treatment of the dissipation potential that accounts for damage and plastic processes. The existence proof, based on incremental energy minimization, is presented in Section 3. We note in passing that our analysis rests on the conditions of global stability; alternative solution concepts like viscous approximation, employed in a similar context by CRISMALE and LAZZARONI,[15] or semistability, used by ROUBÍČEK and VALDMAN,[47,48] are excluded from consideration. Finally, in Section 4, we discuss possible extensions and generalizations.

2 | THE MODEL

2.1 | Preliminaries

We first describe the setting of our model and then introduce some basic concepts of linear algebra and geodesic calculus which help to understand the model.

Reference configuration. In the sequel we work on a bounded connected open set $\Omega \subset \mathbb{R}^d$, $d \geq 2$, with Lipschitz boundary representing the reference configuration of an elastoplastic body. We assume that the boundary of Ω is the union of a Dirichlet and Neumann part, namely $\partial \Omega := \Gamma_D \cup \Gamma_N$, and suppose Γ_D has strictly positive $(d-1)$-Hausdorff measure. Once we have fixed a Dirichlet boundary condition for the deformation $\varphi : \Omega \to \mathbb{R}^d$, we can make use of the Poincaré inequality

$$\|\varphi\|_{W^{1,p}} \leq C \|\nabla \varphi\|_{L^p},$$

which holds true for this domain since $\mathcal{H}^{d-1}(\Gamma_D) > 0$. Throughout the paper we use the letter C to denote a generic positive constant that may change from line to line.

Matrices and groups. We denote by $\mathbb{R}^{d \times d}$ the vector space of $d \times d$ matrices with real entries. The standard Euclidean inner product is denoted by double dots, namely $A : B = A_{ij} B_{ij}$ (summation convention). The symbols $\mathbb{R}^{d \times d}_{\text{sym}}$ and $\mathbb{R}^{d \times d}_{\text{anti}}$ denote the subspaces of $\mathbb{R}^{d \times d}$ consisting of symmetric and anti-symmetric matrices, respectively. The symbol $\mathbb{R}^{d \times d}_{\text{dev}}$ stands for deviatoric matrices, where deviatoric means tracefree. We employ the following notation for common matrix groups

$$GL(d) := \{A \in \mathbb{R}^{d \times d} : \det A \neq 0\},$$

$$GL^+(d) := \{A \in \mathbb{R}^{d \times d} : \det A > 0\},$$

$$O(d) := \{A \in \mathbb{R}^{d \times d} : A^\top A = A A^\top = \mathbb{I}\},$$

$$SL(d) := \{A \in \mathbb{R}^{d \times d} : \det A = 1\},$$

$$SO(d) := \{A \in SL(d) : A^\top A = A A^\top = \mathbb{I}\}$$

where $\mathbb{I} \in \mathbb{R}^{d \times d}$ denotes the identity matrix.

Norms. We consistently use the notation $|\cdot|$ for norms of tensors and scalars, e.g. $|A| = (A : A)^{1/2}$. This notation is employed in general for k-tensors of every order. On the other hand, we make use of the double-bar notation $\|\cdot\|$ for norms on function spaces, e.g. $\|f\|_{L^1} = \int_\Omega |f(x)|dx$.

Polar decomposition. For all $A \in GL(d)$ there exists a unique decomposition

$$A = RT, \tag{1}$$

with $R \in O(d)$ and $T \in \mathbb{R}^{d\times d}_{\text{sym}}$ positive definite. If, moreover, $A \in SL(d)$ then it is easy to see that both T and R must have determinant equal to 1. Furthermore, as T is symmetric, there exists an orthogonal matrix Q and a diagonal matrix Λ such that

$$T = Q\Lambda Q^\top. \tag{2}$$

The diagonal matrix Λ has the positive eigenvalues λ_i of T on the diagonal. The matrix $\xi = \text{diag}(\log \lambda_1, \dots, \log \lambda_d)$ then satisfies

$$\Lambda = e^\xi \quad \text{and} \quad T = Qe^\xi Q^\top = e^{Q\xi Q^\top}, \tag{3}$$

where the last equality follows from the fact that Q is invertible and $Q^{-1}e^A Q = e^{Q^{-1}AQ}$ for all $A \in \mathbb{R}^{d\times d}$.

Geodesic exponential map vs. matrix exponential. Let G be a (matrix) Lie group, e.g. $SO(d)$ or $SL(d)$. The (geodesic) exponential map is defined by

$$\text{Exp} : T_e G \to G$$

$$v \mapsto \gamma_v(1)$$

where γ_v is the unique geodesic starting from the identity $e \in G$ with initial velocity v lying in the tangent space to G at the identity. It is easy to show that the tangent space of $SL(d)$ (resp. $SO(d)$) at the identity is $\mathbb{R}^{d\times d}_{\text{dev}}$ (resp. $\mathbb{R}^{d\times d}_{\text{anti}}$), see [11, Example I.9.4., Exercise I.17(b)]. It is important to remark that in general the geodesic exponential map defined above differs from the algebraic exponential of a matrix ξ used above and denoted by e^ξ. In fact it was shown in [36, Theorem 6.1] that for the left-invariant metric induced by the standard Euclidean scalar product $A : B$ the geodesics on $SL(d)$ starting from $P(0)$ in direction of $\xi \in \mathbb{R}^{d\times d}_{\text{dev}}$ are given by

$$P(t) = P(0)e^{t\xi^\top}e^{t(\xi-\xi^\top)}.$$

Notice that for tracefree matrices in general $\xi^\top\xi \neq \xi\xi^\top$, that is why $\text{Exp}(\xi) \neq e^\xi$. For antisymmetric matrices however, the product commutes. This implies that on $SO(d)$ the geodesics are exactly given by $P(t) = P(0)e^{t\xi}$ for $\xi \in \mathbb{R}^{d\times d}_{\text{anti}}$.

Rotations. Since the set of rotations $SO(d)$ is a compact connected Lie group, the exponential map

$$\text{Exp} : \mathbb{R}^{d\times d}_{\text{anti}} \to SO(d)$$

$$\xi \mapsto e^\xi$$

is surjective [27, Corollary 11.10.]. Therefore, for every $R \in SO(d)$ there exists $\xi \in \mathbb{R}^{d\times d}_{\text{anti}}$ such that $R = e^\xi$. We can use the spectral theory for real skew-symmetric matrices to bring ξ to a block diagonal form. Namely, there exists an orthogonal matrix Q such that $\xi = Q\Sigma Q^\top$ with

$$\Sigma = \begin{pmatrix} \alpha_1 L_1 & & & \\ & \alpha_2 L_2 & & \\ & & \ddots & \\ & & & \alpha_p L_p \end{pmatrix}, \tag{4}$$

where either $L_j = 0 \in \mathbb{R}$ or $L_j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbb{R}^{2\times2}_{\text{anti}}$ and $\alpha_j \in \mathbb{R}$. Since Σ is block diagonal its exponential is easily computed as

$$e^\Sigma = \begin{pmatrix} e^{\alpha_1 L_1} & & & \\ & e^{\alpha_2 L_2} & & \\ & & \ddots & \\ & & & e^{\alpha_p L_p} \end{pmatrix}, \quad e^{\alpha_j L_j} = \begin{pmatrix} \cos\alpha_j & \sin\alpha_j \\ -\sin\alpha_j & \cos\alpha_j \end{pmatrix} \text{ or } e^0 = 1.$$

Using the periodicity of Sinus and Cosinus, the rotation R can be written as

$$R = e^{Q\Sigma Q^\top}, \tag{5}$$

where Σ is defined as in (4), but with $\alpha_j \in [0, 2\pi)$.

2.2 | Plastic dissipation

The *(plastic) dissipation potential* is a mapping

$$\Delta : \Omega \times SL(d) \times \mathbb{R}^{d\times d} \to [0, +\infty],$$

which is measurable in $x \in \Omega$ and convex and positively 1-homogeneous in the rate, i.e.,

$$\Delta(x, P, \lambda\dot{P}) = \lambda\Delta(x, P, \dot{P}) \quad \text{for all } \lambda \geq 0. \tag{6}$$

We further assume plastic indifference which corresponds to requiring that

$$\Delta(x, PQ, \dot{P}Q) = \Delta(x, P, \dot{P}) \quad \text{for all } Q \in SL(d). \tag{7}$$

This property implies that there exists a measurable, 1-homogeneous function $\widehat{\Delta} : \Omega \times \mathbb{R}^{d\times d} \to [0, +\infty]$ such that

$$\Delta(x, P, \dot{P}) = \widehat{\Delta}(x, \dot{P}P^{-1}),$$

see [37] or [41, Section 4.2.1.1]. We assume there exist constants $c_0, c_1 > 0$, independent of $x \in \Omega$, such that

$$c_0|Q| \leq \widehat{\Delta}(x, Q) \leq c_1|Q| \quad \text{for every } Q \in SL(d). \tag{8}$$

With the potential at disposal, we define the induced *plastic dissipation distance* on $SL(d)$ for any pair $P_1, P_2 \in SL(d)$ by

$$D_{\text{p}}(x, P_1, P_2) = \inf\left\{ \int_0^1 \Delta(x, P(s), \dot{P}(s))\,\mathrm{d}s \ : \ P \in W^{1,\infty}([0,1]; SL(d)), \ P(0) = P_1, \ P(1) = P_2 \right\}. \tag{9}$$

Notice that due to plastic indifference we have that $D_{\text{p}}(x, P_1, P_2) = \widehat{D}_{\text{p}}(x, P_2 P_1^{-1})$ with

$$\widehat{D}_{\text{p}}(x, P) = \inf\left\{ \int_0^1 \widehat{\Delta}(x, \dot{P}(s)P(s)^{-1})\,\mathrm{d}s \ : \ P \in W^{1,\infty}([0,1]; SL(d)), \ P(0) = \mathbb{I}, \ P(1) = P \right\}.$$

Due to (8), the dissipation distance $\widehat{D}_{\text{p}}$ is equivalent to the standard Riemannian distance induced by the Euclidean scalar product on the Lie algebra $\mathbb{R}^{d\times d}_{\text{dev}}$. In particular, we have that for every $x \in \Omega$

$$c_0\widehat{d}_{\text{SL}}(P) \leq \widehat{D}_{\text{p}}(x, P) \leq c_1\widehat{d}_{\text{SL}}(P), \tag{10}$$

where

$$\widehat{d}_{\text{SL}}(P) = \inf\left\{ \int_0^1 |\dot{P}(s)P^{-1}(s)|\,\mathrm{d}s \ : \ P \in W^{1,\infty}([0,1]; SL(d)), \ P(0) = \mathbb{I}, \ P(1) = P \right\}.$$

As it was pointed out in [36], the geodesics with respect to $\widehat{\Delta}$ in direction ξ are in general not known and even in the specific Riemannian case geodesics of $\widehat{d}_{SL}$ connecting the identity to ξ are not given by $t \mapsto e^{t\xi}$. In particular, it might happen that $\widehat{d}_{SL}(e^\xi) < |\xi|$. However, from standard theory of Riemannian manifolds it is known that

$$d_{SL}(P_0, P_1) := \widehat{d}_{SL}(P_1 P_0^{-1})$$

is a metric on $SL(d)$, see pp. 19-20 of [11]. We conclude this introduction with the following results which are employed in Section 3:

Lemma 2.1 (D_p is a quasi-distance). *For every $P_1, P_2, P_3, Q \in SL(d)$ and all $x \in \Omega$ the following properties hold:*

(i) $D_p(x, P_1, P_2) = 0$ *if and only if* $P_1 = P_2$,

(ii) $D_p(x, P_1, P_3) \le D_p(x, P_1, P_2) + D_p(x, P_2, P_3)$,

(iii) $D_p(x, P_1 Q, P_2 Q) = D_p(x, P_1, P_2)$.

Proof. The implication (i) follows from the previous remark that d_{SL} is a metric on $SL(d)$ which by (10) is equivalent to D_p. Condition (ii) is easily checked, while (iii) follows from (7). $\qquad\square$

Notice that D_p might *not* be symmetric. We now show that the quasi-distance $\widehat{D}_p$ has sublinear growth. To prove this upper bound the most important observation is that, if $P \in SL(d)$ is such that $P = e^\xi$ for some ξ, then we may test the definition of $\widehat{D}_p(P)$ with the path $s \mapsto e^{s\xi}, s \in [0, 1]$ and get

$$\widehat{D}_p(x, P) \le \int_0^1 \widehat{\Delta}(e^{s\xi} \xi e^{-s\xi}) \, ds = \int_0^1 \widehat{\Delta}(\xi) \, ds = \widehat{\Delta}(\xi). \tag{11}$$

Proposition 2.2. *There exists a positive constant $C = C(d) > 0$ such that for every $P_1, P_2 \in SL(d)$ and all $x \in \Omega$*

$$D_p(x, P_1, P_2) \le C(1 + |P_1| + |P_2|). \tag{12}$$

Proof. In the following, not to overburden notation, we drop the explicit dependence on $x \in \Omega$. For the Reader's convenience the proof is split into several steps. In Steps 1-3 we show that for every $P \in SL(d)$

$$\widehat{D}_p(P) \le C(1 + |P|) \tag{13}$$

for some constant $C = C(c_1, d) > 0$ (c_1 being the constant in (8)). In Step 4 we deduce the general statement of the proposition.

Step 1. Let $P \in SL(d)$ be arbitrary. The decompositions (1) and (2) entail

$$P = RT = RQ\Lambda Q^\top, \tag{14}$$

where $R \in SO(d)$, $Q \in O(d)$ and Λ is diagonal and can be written as the exponential of $\xi = \log \Lambda$ as in (3). We estimate the dissipation relative to the positive definite symmetric matrix $T = Q\Lambda Q^\top$ using (11), and writing $T = Qe^\xi Q^\top = e^{Q\xi Q^\top}$,

$$\widehat{D}_p(T) \le \widehat{\Delta}(Q\xi Q^\top) \le c_1 |Q\xi Q^\top| = c_1 |\xi|, \tag{15}$$

where the second inequality follows from assumption (8).

Step 2. We now show that $|\xi|$ can be estimated in terms of $|T|$. The eigenvalues of Λ satisfy $\prod_{i=1}^d \lambda_i = 1$, and thus $\sum_{i=1}^d \log(\lambda_i) = 0$. Assume $\lambda_1, \dots, \lambda_m > 1$ for some m and $\lambda_{m+1}, \dots, \lambda_d \le 1$. Let $\ell = \sum_{i=1}^m \log \lambda_i$, so that $\sum_{i=m+1}^d \log \lambda_i = -\ell$. Since $\log \lambda_i > 0$ for all $i = 1, \dots, m$ and $\log \lambda_i \le 0$ for all $i = m + 1, \dots, d$ we can write

$$\sum_{i=1}^m (\log \lambda_i)^2 \le \ell^2, \qquad \sum_{i=m+1}^d (\log \lambda_i)^2 \le \ell^2.$$

Then, Jensen's inequality implies that

$$|\xi|^2 = \sum_{i=1}^{d} (\log \lambda_i)^2 \le 2\ell^2 = 2 \left(\sum_{i=1}^{m} \log \lambda_i \right)^2 \le 2m \sum_{i=1}^{m} (\log \lambda_i)^2$$

$$\le 2m \sum_{i=1}^{m} (\lambda_i)^2 \le 2(d-1) \sum_{i=1}^{d} \lambda_i^2 = 2(d-1)|T|^2.$$

In particular, estimate (15) leads to

$$\widehat{D}_p(T) \le C_1 |T|, \tag{16}$$

where $C_1 := c_1 \sqrt{2(d-1)}$.

Step 3. Let us now give an estimate for the rotation R in the decomposition (14).

We use decomposition (5) to estimate

$$\widehat{D}_p(R) \le \widehat{\Delta}(Q \Sigma Q^\top) \le c_1 |Q \Sigma Q^\top| = c_1 |\Sigma| \le C_2, \tag{17}$$

where $C_2 := c_1 \sqrt{d}\pi$.

Step 4. By Lemma 2.1(ii),(iii) we have

$$\widehat{D}_p(PQ) \le \widehat{D}_p(Q) + \widehat{D}_p(P), \quad \text{for all } P, Q \in SL(d). \tag{18}$$

Now let $P_1, P_2 \in SL(d)$ and use the polar decomposition (1)-(3) to write

$$P_i = R_i T_i = R_i Q_i \Lambda_i Q_i^\top, \quad i = 1, 2.$$

Using (16), (17) and (18) we obtain

$$D_p(P_1, P_2) = \widehat{D}_p(P_2 P_1^{-1}) = \widehat{D}_p(R_2 T_2 T_1^{-1} R_1^{-1})$$

$$\le \widehat{D}_p(R_1^{-1}) + \widehat{D}_p(T_1^{-1}) + \widehat{D}_p(T_2) + \widehat{D}_p(R_2)$$

$$\le 2C_2 + C_1 |T_2| + \widehat{D}_p(T_1^{-1}).$$

Now $T_1^{-1} = Q_1 \Lambda_1^{-1} Q_1^\top$ and

$$\widehat{D}_p(T_1^{-1}) = \widehat{\Delta}(Q_1 \log \Lambda_1^{-1} Q_1^\top) \le c_1 |\log \Lambda_1^{-1}| = c_1 |\log \Lambda_1|.$$

As in Step 2 we deduce that $c_1 |\log \Lambda_1| \le C_1 |T_1|$. Altogether we have shown that

$$D_p(P_1, P_2) \le 2C_2 + C_1(|T_1| + |T_2|) \le 2C_2 + C_1 C_3(|P_1| + |P_2|), \tag{19}$$

where $C_1 = c_1 \sqrt{2(d-1)}$, $C_2 = c_1 \sqrt{d}\pi$ and $C_3 = \sup\{|R| : R \in SO(d)\}$. □

Lemma 2.3. *The dissipation distance* $D_p : \Omega \times SL(d) \times SL(d) \to [0, \infty)$ *is a Carathéodory function (i.e. measurable in the first variable and continuous in the other variables for a.e. fixed $x \in \Omega$).*

Proof. The measurability of $D_p(\cdot, P_0, P_1)$ follows from measurability of $\widehat{\Delta}$. To show continuity let $P^*, P \in SL(d)$ be fixed and let $P_k \to P$ in $SL(d)$. Then (dropping the x-variable dependence) we use the triangle inequality to estimate

$$|D_p(P^*, P_k) - D_p(P^*, P)| \le D_p(P, P_k) = \widehat{D}_p(P_k P^{-1}).$$

Therefore it suffices to show that $\widehat{D}_p(\widehat{P}_k) \to 0$ for any sequence $\widehat{P}_k \to \mathbb{I}$. Since $\widehat{P}_k \in SL(d)$ we can use the decompositions (1) and (3) as well as the spectral theory (5) to write

$$\widehat{P}_k = e^{Q_k \Sigma_k Q_k^\top} e^{S_k \xi_k S_k^\top},$$

where Q_k, S_k are orthogonal and $\Sigma_k, \xi_k \to 0$ as $\widehat{P}_k \to \mathbb{I}$. We again use the triangle inequality and (11) to estimate

$$\widehat{D}_{\mathrm{p}}(\widehat{P}_k) \leq \widehat{\Delta}(Q_k \Sigma_k Q_k^\top) + \widehat{\Delta}(S_k \xi_k S_k^\top) \leq c_1 \left(|\Sigma_k| + |\xi_k| \right) \to 0 \quad \text{as } k \to \infty.$$

This proves the claimed continuity. $\qquad\square$

2.3 | Coupled damage-plastic dissipation

Let $\rho : \Omega \times \mathbb{R} \to \mathbb{R}^+$ be a Carathéodory function (measurable in x for every $t \in \mathbb{R}$, continuous in t for a.e. $x \in \Omega$). We make the following assumption:

$$\rho(x, \cdot) \text{ is non-decreasing and constant on the intervals } (-\infty, 0] \text{ and } [1, +\infty). \tag{20}$$

Let $\kappa \in L^\infty(\Omega; \mathbb{R}^+)$ be such that $\kappa(x) \geq \kappa_0 > 0$ for a.e. $x \in \Omega$. Given $x \in \Omega$, $z_1, z_2 \in [0, 1]$ and $P_1, P_2 \in SL(d)$ we define the (coupled damage-plastic) dissipation distance between (P_1, z_1) and (P_2, z_2) at x as

$$D(x, P_1, z_1, P_2, z_2) = \inf \left\{ \int_0^1 \Psi(x, \dot{z}(s)) + \rho(x, z(s))\Delta(x, P(s), \dot{P}(s)) \, ds : \right.$$

$$(P, z) \in W^{1,\infty}([0, 1]; SL(d) \times [0, 1]),$$

$$\left. P(0) = P_1, \ P(1) = P_2, \ z(0) = z_1, \ z(1) = z_2 \right\}, \tag{21}$$

where

$$\Psi(x, \dot{z}) := \begin{cases} \kappa(x)|\dot{z}| & \text{if } \dot{z} \leq 0, \\ \infty & \text{else.} \end{cases}$$

Thanks to the monotonicity assumption (20) we can prove the following.

Proposition 2.4. *Let $x \in \Omega$, $z_1, z_2 \in [0, 1]$ and $P_1, P_2 \in SL(d)$. Then*

$$D(x, P_1, z_1, P_2, z_2) = \Psi(x, z_2 - z_1) + \rho(x, z_2)D_{\mathrm{p}}(x, P_1, P_2). \tag{22}$$

Proof. We consider the following two cases.

Case $z_1 < z_2$: In this case the right-hand side of (22) is infinite. So we need to show that D is infinite too. This follows as, for every path $z \in W^{1,\infty}([0, 1]; [0, 1])$ connecting z_1 to z_2, the measure $\mathcal{L}^1(\{\dot{z} > 0\})$ is strictly positive. By definition of Ψ the path has infinite dissipation length.

Case $z_1 \geq z_2$: Since ρ is non-decreasing, by definition of Ψ every path of finite dissipation satisfies $\dot{z} \leq 0$ a.e. on [0,1]. By monotonicity (20),

$$D(x, P_1, z_1, P_2, z_2) = \inf \left\{ \int_0^1 \Psi(x, \dot{z}(s))ds + \int_0^1 \rho(x, z(s))\Delta(x, P(s), \dot{P}(s)) \, ds : \dot{z} \leq 0 \right\}$$

$$\geq \Psi(x, z_2 - z_1) + \inf \left\{ \int_0^1 \rho(x, z(s))\Delta(x, P(s), \dot{P}(s)) \, ds : z_2 \leq z \leq z_1 \right\}$$

$$\geq \Psi(x, z_2 - z_1) + \rho(x, z_2)D_{\mathrm{p}}(x, P_1, P_2).$$

To show the opposite inequality let $P_k \in W^{1,\infty}([0, 1]; SL(d))$ be a sequence with $P_k(0) = P_1$, $P_k(1) = P_2$ such that, for any k,

$$\int_0^1 \Delta(x, P_k(s), \dot{P}_k(s)) \, ds \leq D_{\mathrm{p}}(x, P_1, P_2) + \frac{1}{k}. \tag{23}$$

Let $z_k \in W^{1,\infty}([0,1];[z_2,z_1])$ be the function

$$
z_k(s) = \begin{cases} k(z_2 - z_1)\left(s - \dfrac{1}{k}\right) + z_2, & \text{if } 0 \le s \le \dfrac{1}{k}, \\[2mm] z_2, & \text{else.} \end{cases}
$$

Moreover let $\zeta : [\frac{1}{k}, 1] \to [0,1]$ be the unique affine function such that $\zeta(\frac{1}{k}) = 0$, $\zeta(1) = 1$, and let

$$
\widetilde{P}_k(t) = \begin{cases} P_1 & \text{for } t \in \left[0, \dfrac{1}{k}\right], \\[2mm] P_k(\zeta(t)) & \text{for } t \in \left(\dfrac{1}{k}, 1\right]. \end{cases}
$$

Notice that $\widetilde{P}_k$ is Lipschitz continuous as well. Since $\widetilde{P}_k$ is constant on $[0, 1/k]$ it follows that $R(x, \widetilde{P}_k, \dot{\widetilde{P}}_k) = 0$ on $[0, 1/k]$, and by 1-homogeneity, we have

$$
D(x, P_1, z_1, P_2, z_2) \le \Psi(x, z_2 - z_1) + \int_{1/k}^{1} \rho(x, z_k(t)) \Delta(x, \widetilde{P}_k(t), \dot{\widetilde{P}}_k(t))\, dt
$$

$$
= \Psi(x, z_2 - z_1) + \rho(x, z_2) \int_{1/k}^{1} \Delta(x, P_k(\zeta(t)), \dot{P}_k(\zeta(t))) \dot{\zeta}(t)\, dt
$$

$$
= \Psi(x, z_2 - z_1) + \rho(x, z_2) \int_{0}^{1} \Delta(x, P_k(s), \dot{P}_k(s))\, ds
$$

$$
\overset{(23)}{\le} \Psi(x, z_2 - z_1) + \rho(x, z_2)\left(D_{\mathrm{p}}(x, P_1, P_2) + \dfrac{1}{k}\right)
$$

where we used the change of variables $s = \zeta(t)$. We conclude by taking the limit $k \to \infty$ on the right-hand side. $\qquad\square$

We now define the *dissipation* between two internal states $(P_1, z_1), (P_2, z_2) : \Omega \to SL(d) \times [0,1]$ as

$$
\mathcal{D}(P_1, z_1, P_2, z_2) = \int_{\Omega} D(x, P_1(x), z_1(x), P_2(x), z_2(x))\, dx,
$$

and the *total dissipation* of a damage-plastic process, given $(P, z) : [s, t] \to L^1(\Omega; SL(d)) \times L^1(\Omega; [0,1])$, as

$$
\mathrm{Diss}(P, z; s, t) := \sup \sum_{i=1}^{N} \mathcal{D}(P(r_{i-1}), z(r_{i-1}), P(r_i), z(r_i)), \tag{24}
$$

where the supremum is computed over all partitions $s = r_0 < r_1 < \cdots < r_{N-1} < r_N = t$, and all $N \in \mathbb{N}$.

2.4 | State space

In order to deal with time-dependent boundary conditions of the form

$$
\varphi = g_{\mathrm{Dir}}(t) \text{ on } \Gamma_D,
$$

where $g_{\mathrm{Dir}} : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ represents a Dirichlet datum, we use the so-called multiplicative splitting technique[19,21,30,34] replacing the variable φ by $g_{\mathrm{Dir}}(t) \circ y$, see e.g. [21, Section 5]. More precisely, we set

$$
\varphi(x) = g_{\mathrm{Dir}}(t, y(x)), \quad \text{where } y(x) = x \text{ on } \Gamma_D.
$$

This results in a multiplicative split of the deformation gradient

$$
\nabla \varphi(x) = \nabla g_{\mathrm{Dir}}(t, y(x)) \nabla y(x).
$$

We refer to the next section for the hypotheses on g_{Dir}. The space of admissible states, denoted by $\mathcal{Q}$, is the triple $\mathcal{Y} \times \mathcal{P} \times \mathcal{Z}$, where

$$\mathcal{Y} := W_D^{1,q}(\Omega; \mathbb{R}^d) := \{y \in W^{1,q}(\Omega; \mathbb{R}^d) : y = \mathrm{id} \text{ on } \Gamma_D\},$$

$$\mathcal{P} := W^{1,r_{\mathrm{p}}}(\Omega; SL(d)),$$

$$\mathcal{Z} := W^{1,r_z}(\Omega; [0,1]),$$

for some coefficients $q > d$ and $r_{\mathrm{p}}, r_z > 1$. The space $\mathcal{Q}$ is endowed with the weak topologies of the Sobolev spaces, namely,

$$P_k \rightharpoonup P \text{ in } \mathcal{P} \quad \text{if and only if} \quad P_k \rightharpoonup P \text{ weakly in } W^{1,r_{\mathrm{p}}}(\Omega; \mathbb{R}^{d \times d}),$$

$$z_k \rightharpoonup z \text{ in } \mathcal{Z} \quad \text{if and only if} \quad z_k \rightharpoonup z \text{ weakly in } W^{1,r_z}(\Omega; \mathbb{R}).$$

By Poincaré's inequality, weak convergence in $\mathcal{Y}$ is equivalent to weak convergence of gradients, i.e.,

$$y_k \rightharpoonup y \text{ in } \mathcal{Y} \text{ if and only if } \nabla y_k \rightharpoonup \nabla y \text{ weakly in } L^q(\Omega; \mathbb{R}^{d \times d}).$$

Notice that the space $\mathcal{P}$ is not a linear subspace of $W^{1,r_{\mathrm{p}}}(\Omega; \mathbb{R}^{d \times d})$ because the target space is the manifold $SL(d)$. Nevertheless weak limits of sequences $(P_k)_{k \in \mathbb{N}} \subset \mathcal{P}$ are again in $\mathcal{P}$. This follows since weak convergence in $\mathcal{P}$ implies strong convergence in $L^{r_{\mathrm{p}}}(\Omega; \mathbb{R}^{d \times d})$.

We introduce the short notation $q = (y, P, z)$ for elements in $\mathcal{Q}$ and occasionally use the variable q in the dissipation distance $\mathcal{D}$ although it depends only on the internal variables and is independent of y.

2.5 | Energy

We consider the following total energy for the system:

$$\mathcal{E}(t, y, P, z) = \int_{\Omega} W_{\mathrm{el}}(x, \nabla g_{\mathrm{Dir}}(t, y(x))\nabla y(x)(P(x))^{-1}, z(x)) + W_{\mathrm{h}}(x, P(x), z(x))\, \mathrm{d}x$$

$$+ \frac{\nu}{r_{\mathrm{p}}} \int_{\Omega} |\nabla P(x)|^{r_{\mathrm{p}}}\, \mathrm{d}x + \frac{\mu}{r_z} \int_{\Omega} |\nabla z(x)|^{r_z}\, \mathrm{d}x - \langle \ell(t), g_{\mathrm{Dir}}(t, y) \rangle, \tag{25}$$

for some material parameters $\nu, \mu > 0$, where the mapping $t \mapsto \ell(t)$ represents external loading of the mechanical system and is defined as

$$\langle \ell(t), \varphi \rangle = \int_{\Omega} f(x, t) \cdot \varphi(x)\, \mathrm{d}x + \int_{\Gamma_N} \tau(x, t) \cdot \varphi(x)\, \mathrm{d}\mathcal{H}^{d-1}(x), \tag{26}$$

where f is a prescribed bulk force and τ is a prescribed traction on the Neumann boundary Γ_N. The quantity

$$\mathcal{W}_{\mathrm{el}}(t, y, P, z) = \int_{\Omega} W_{\mathrm{el}}(x, \nabla g_{\mathrm{Dir}}(t, y(x))\nabla y(x)(P(x))^{-1}, z(x))\, \mathrm{d}x,$$

is the elastic energy of the system and the term

$$\mathcal{W}_{\mathrm{h}}(P, z) = \int_{\Omega} W_{\mathrm{h}}(x, P(x), z(x))\, \mathrm{d}x,$$

represents the energy related to kinematic hardening instead. The terms in (25) involving ∇P and ∇z are higher order energetic terms which have the role of regularizations introducing internal length scales. Notice that the elastic energy density depends on the elastic strain $\nabla g_{\mathrm{Dir}}(t, y)\nabla y P^{-1}$ whereas the hardening energy depends on the plastic strain P. It is convenient to denote the total bulk energy (density) without regularization by

$$W(x, F, P, z) = W_{\mathrm{el}}(x, F, z) + W_{\mathrm{h}}(x, P, z),$$

and

$$\mathcal{W}(t, y, P, z) = \mathcal{W}_{\mathrm{el}}(t, y, P, z) + \mathcal{W}_{\mathrm{h}}(P, z).$$

The presence of the time-dependent Dirichlet datum is reflected in the power of external forces given by

$$\partial_t \mathcal{E}(t, y, P, z) = \partial_t \mathcal{W}_{\mathrm{el}}(t, y, P, z) - \langle \dot{\ell}(t), g_{\mathrm{Dir}}(t, y) \rangle - \langle \ell(t), \dot{g}_{\mathrm{Dir}}(t, y) \rangle, \tag{27}$$

where

$$\partial_t \mathcal{W}_{\mathrm{el}}(t, y, P, z) = \int_\Omega \partial_F W_{\mathrm{el}}(x, F(t, x), z(x))(F(t, x))^\top : \nabla \dot{g}_{\mathrm{Dir}}(t, y(x))(\nabla y(x))^{-1} \, \mathrm{d}x$$

with $F(t, x) = \nabla g_{\mathrm{Dir}}(t, y(x)) \nabla y(x)(P(x))^{-1}$. This motivates the assumptions on the *Kirchhoff stress* $\partial_F W_{\mathrm{el}}(x, F, z)F^\top$ explained below and used in [34].

For our analysis, we ask the following conditions to hold:

- *Control on the Kirchhoff stress:*

  $$\exists \, c_0^W \in \mathbb{R}, c_1^W > 0, \delta > 0, \text{ a modulus of continuity } \omega : (0, \delta) \to (0, \infty) \text{ such that}$$

  $$\forall (x, F, z) \text{ s.t. } W_{\mathrm{el}}(x, F, z) < \infty, \, N \in \mathcal{N}_\delta := \{N \in \mathbb{R}^{d \times d} : |N - \mathbb{I}| < \delta\} :$$

  $$W_{\mathrm{el}}(x, \cdot, z) \text{ is differentiable on } \mathcal{N}_\delta F,$$

  $$|\partial_F W_{\mathrm{el}}(x, F, z)F^\top| \le c_1^W (W_{\mathrm{el}}(x, F, z) + c_0^W), \tag{28}$$

  $$|\partial_F W_{\mathrm{el}}(x, F, z)F^\top - \partial_F W_{\mathrm{el}}(x, NF, z)(NF)^\top| \le \omega(|N - \mathbb{I}|)(W_{\mathrm{el}}(x, F, z) + c_0^W). \tag{29}$$

- *Polyconvexity:* We assume that W_{h} is a normal integrand, meaning $W_{\mathrm{h}}(\cdot, P, z)$ is measurable for every $P \in SL(d), z \in [0, 1]$ and $W_{\mathrm{h}}(x, \cdot, \cdot)$ is lower semicontinuous for a.e. $x \in \Omega$. Moreover, we assume that the elastic energy density W_{el} is finite just on $GL_+(d)$ and *polyconvex*,[6] namely

  $$W_{\mathrm{el}}(x, F, z) = \mathbb{W}_{\mathrm{conv}}(x, \mathbb{M}(F), z), \tag{30}$$

  where $\mathbb{W}_{\mathrm{conv}}$ is a normal integrand, $\mathbb{W}_{\mathrm{conv}}(x, \cdot, z)$ is convex for a.e. $x \in \Omega$ and every $z \in [0, 1]$, and $\mathbb{M}(F)$ denotes the vector of all minors of the elastic strain F. In dimension $d = 3$, for instance,

  $$\mathbb{M}(F) = (F, \mathrm{cof}\, F, \det F).$$

- *Coercivity:* Furthermore, we assume the *coercivity* bounds

  $$W_{\mathrm{el}}(x, F, z) \ge C_1 |F|^{q_{\mathrm{e}}} - C_2, \tag{31a}$$

  $$W_{\mathrm{h}}(x, P, z) \ge C_1 |P|^{q_{\mathrm{p}}} - C_2, \tag{31b}$$

  for some constants $C_1, C_2 > 0$ and exponents satisfying

  $$\frac{1}{q_{\mathrm{e}}} + \frac{1}{q_{\mathrm{p}}} \le \frac{1}{q} < \frac{1}{d}, \tag{32}$$

  see [41, Section 4.1.3].

- *Monotonicity and continuity:* We further assume continuity and monotonicity in z. More precisely, we ask

  $$W(x, F, P, \cdot) \in C^0([0, 1]; \mathbb{R}) \tag{33}$$

and

$$z \leq \hat{z} \quad \Rightarrow \quad W(x, F, P, z) \leq W(x, F, P, \hat{z}) \tag{34}$$

for every $F \in GL^+(d)$, $P \in SL(d)$ and a.e. $x \in \Omega$.

- *Regularity of Dirichlet data and loading:* Moreover, one needs to assume that g_{Dir} and ℓ are sufficiently regular. Precisely, one requires

$$g_{\text{Dir}} \in C^1([0, T] \times \mathbb{R}^d; \mathbb{R}^d) \text{ with}$$

$$\nabla g_{\text{Dir}}, \nabla \dot{g}_{\text{Dir}}, (\nabla g_{\text{Dir}})^{-1} \in C^0([0, T] \times \mathbb{R}^d; \mathbb{R}^{d \times d}) \cap L^\infty([0, T] \times \mathbb{R}^d; \mathbb{R}^{d \times d}), \tag{35}$$

$$\ell \in W^{1,1}(0, T; (W^{1,q}(\Omega; \mathbb{R}^d))^*), \tag{36}$$

see [41, Condition (4.1.32), Remark 4.2.5].

Here, the *multiplicative stress control* (28) is needed to bound the power of external forces (27) by the energy, whereas the *uniform continuity condition* (29) together with (35) and (36) ultimately guarantees the convergence $\partial_t \mathcal{E}(t, q_k) \to \partial_t \mathcal{E}(t, q)$ as stable sequences (t_k, q_k) tend to (t, q) [41, Prop. 2.1.17]. In particular, it was shown in [34, Theorem 5.3] that there exist constants $c_0^E \in \mathbb{R}, c_1^E > 0$ and a modulus of continuity ω such that

$$|\partial_t \mathcal{W}_{\text{el}}(t, q)| \leq c_1^E(\mathcal{W}_{\text{el}}(t, q) + c_0^E), \tag{37a}$$

$$|\partial_t \mathcal{W}_{\text{el}}(s, q) - \partial_t \mathcal{W}_{\text{el}}(t, q)| \leq \omega(|t - s|)(\mathcal{W}_{\text{el}}(t, q) + c_0^E). \tag{37b}$$

Moreover, polyconvexity (30), coercivity (31), and continuity (33) are used to show lower semicontinuity and compactness of the energy, whereas monotonicity (34) is needed for the construction of recovery sequences in Section 3.1. We would like to emphasize that these conditions are compatible with frame-indifference (objectivity) and non-interpenetrability of matter, namely

- *Objectivity:*

$$\mathcal{W}_{\text{el}}(x, QF, z) = \mathcal{W}_{\text{el}}(x, F, z) \quad \forall Q \in SO(d), F \in GL^+(d). \tag{38}$$

- *Non-interpenetrability:*

$$\mathcal{W}_{\text{el}}(x, F, z) \to \infty \text{ as } \det F \to 0^+, \tag{39}$$

see Example 2.6. As these conditions are essential in modeling continuous media, it is certainly desirable to include them into the model. However, they are not needed for the analysis.

Remark 2.5 (Ciarlet-Nečas condition). With assumption (39) it is clear that a finite energy solution satisfies the *local* non-interpenetration $\det \nabla \varphi > 0$ a.e. in Ω. It is possible to guarantee *global* non-self-interpenetration involving the so-called Ciarlet-Nečas condition,[12] which reads

$$\int_\Omega \det \nabla \varphi(x) \, dx \leq \mathcal{L}^d(\varphi(\Omega)),$$

where $\mathcal{L}^d$ denotes the Lebesgue measure on $\mathbb{R}^d$. In order to achieve this we would change the state space $\mathcal{Y}$ to

$$\mathcal{Y}_{CN} := \left\{ y \in \mathcal{Y} : \int_\Omega \det \nabla y(x) \, dx \leq \mathcal{L}^d(y(\Omega)) \right\}.$$

It can be shown that supposing Ciarlet-Nečas condition for y instead of φ is equivalent under the assumption that $g_{\text{Dir}}(t)$ is an orientation-preserving diffeomorphism [41, Lemma 4.1.1]. Moreover, due to the condition $q > d$, convergence of $\nabla y_k \rightharpoonup \nabla y$ in $L^q(\Omega)$ implies convergence of $\det(\nabla y_k) \rightharpoonup \det(\nabla y)$ in $L^1(\Omega)$. This shows that $\mathcal{Y}_{CN}$ is weakly closed in $\mathcal{Y}$.

Example 2.6 (Ogden-type materials with kinematic hardening). Let us give an example for an energy satisfying the assumptions above. We choose

$$W(F, P, z) = g(z)\widetilde{W}_{\mathrm{el}}(F) + \widetilde{W}_{\mathrm{h}}(P),$$

where $g(z) = (1 + z)/4$. Then clearly assumptions (33) and (34) are satisfied. For the hardening energy density $\widetilde{W}_{\mathrm{h}}$, we consider a kinematic hardening law of the form

$$\widetilde{W}_{\mathrm{h}}(P) = |P - \mathbb{1}|^{q_{\mathrm{p}}}.$$

Possible examples for polyconvex (30), frame-indifferent (38) elastic energy densities $\widetilde{W}_{\mathrm{el}}$ satisfying non-interpenetrability (39) can be found in the class of *Ogden-type materials*

$$\widetilde{W}_{\mathrm{el}}(F) = \sum_{i=1}^{n} a_i \operatorname{tr}(F^\top F - \mathbb{1})^{\alpha_i/2} + \sum_{j=1}^{m} b_j \operatorname{tr}(\operatorname{cof} F^\top F - \mathbb{1})^{\beta_j/2} + \Gamma(\det F),$$

for $F \in GL_+(d)$ and $\widetilde{W}_{\mathrm{el}} = +\infty$ otherwise, where $n, m \geq 1, \alpha_i, \beta_i \geq 1, a_i, b_i > 0$ and $\Gamma : (0, \infty) \to (0, \infty)$ is convex satisfying $\Gamma(\varepsilon) \to +\infty$ as $\varepsilon \to 0^+$. The coercivity condition (31) is met, for instance, if $\alpha_i \geq q_{\mathrm{e}}$ for some i and q_{p} is large enough such that $1/q_{\mathrm{e}} + 1/q_{\mathrm{p}} < 1/d$.

3 | QUASISTATIC EVOLUTION

We follow the concept of *energetic solutions*, which is solely based on the energy functional $\mathcal{E}$, the dissipation distance $\mathcal{D}$ and the state space $\mathcal{Q}$ introduced above. Given initial conditions $(y_0, P_0, z_0) \in \mathcal{Q}$ we look for an energetic solution $(y, P, z) : [0, T] \to \mathcal{Q}$. We first introduce the concept of stable states at a given time $t \in [0, T]$: this is defined via the subset $\mathcal{S}(t)$ of $\mathcal{Q}$ defined as

$$\mathcal{S}(t) = \left\{ (y, P, z) \in \mathcal{Q} : \mathcal{E}(t, y, P, z) \leq \mathcal{E}(t, \hat{y}, \hat{P}, \hat{z}) + \mathcal{D}(P, z, \hat{P}, \hat{z}) \ \forall (\hat{y}, \hat{P}, \hat{z}) \in \mathcal{Q} \right\}.$$

An energetic solution is asked to satisfy the following energy balance (E) and global stability condition (S).

Definition 3.1. We say that $(y, P, z) : [0, T] \to \mathcal{Q}$ is an energetic solution with initial conditions $(y_0, P_0, z_0) \in \mathcal{Q}$ if $(y(0), P(0), z(0)) = (y_0, P_0, z_0)$, the map $s \mapsto \partial_t \mathcal{E}(s, y(s), P(s), z(s))$ belongs to $L^1(0, T)$, $\mathcal{E}(t, y(t), P(t), z(t)) < \infty$ for all $t \in [0, T]$, and the two following conditions are satisfied for all $t \in [0, T]$:

$$(y(t), P(t), z(t)) \in \mathcal{S}(t), \tag{S}$$

$$\mathcal{E}(t, y(t), P(t), z(t)) + \operatorname{Diss}(P, z; 0, t) = \mathcal{E}(0, y_0, P_0, z_0) + \int_0^t \partial_s \mathcal{E}(s, y(s), P(s), z(s)) \, ds, \tag{E}$$

where $\operatorname{Diss}(P, z; 0, t)$ is defined in (24).

We now formulate the main results of the paper.

Theorem 3.2 (Existence of energetic solutions). *Let* $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ *be the triple introduced in Section 2. Let* $q_0 = (y_0, P_0, z_0) \in \mathcal{S}(0)$ *be a stable initial state. Suppose the hypotheses given in Section 2.5 are satisfied, i.e. the bulk energy satisfies* (28), (29), (30), (31), (33), *and* (34), *whereas the data of the problem satisfy* (35) *and* (36). *Then there exists an energetic solution* $q = (y, P, z) :$ $[0, T] \to \mathcal{Q}$ *with initial condition* q_0.

Given an interval $[0, T]$ and a positive natural number n, we denote by Π_n the family of partitions of $[0, T]$ into n intervals, namely the family of n-tuples of real numbers satisfying $0 = t_0 < t_1 < \cdots < t_n = T$. We define the family of partitions of arbitrary length as

$$\Pi = \bigcup_{n=1}^{\infty} \Pi_n.$$

The fineness of a partition is defined as $\max_k |t_k - t_{k-1}|$.

Theorem 3.3 (Existence via incremental minimization). *For every stable initial data $q_0 \in S(0)$ and every sequence of partitions $\sigma_n \in \Pi$ of $[0, T]$ with fineness tending to zero as $n \to \infty$, we can find a trajectory $q_n : [0, T] \to Q$ with $q(0) = q_0$ which is piecewise constant on the partition, right-continuous, and satisfies*

$$q_n(t) \in S(t), \tag{40}$$

$$\mathcal{E}(t, q_n(t)) + \mathrm{Diss}(P_n, z_n; s, t) - \mathcal{E}(s, q_n(s)) \leq \int_s^t \partial_r \mathcal{E}(r, q_n(r)) \, \mathrm{d}r \tag{41}$$

for every $s, t \in \sigma_n$. Moreover, there exists a subsequence and an energetic solution $q = (y, P, z) : [0, T] \to Q$ for the initial conditions q_0 with the following properties:

$$\forall t \in [0, T]: \qquad P_{n_k}(t) \rightharpoonup P(t) \text{ in } \mathcal{P},$$

$$\forall t \in [0, T]: \qquad z_{n_k}(t) \rightharpoonup z(t) \text{ in } \mathcal{Z},$$

$$\forall s, t \in [0, T]: \qquad \mathrm{Diss}(P_{n_k}, z_{n_k}; s, t) \to \mathrm{Diss}(P, z; s, t),$$

$$\forall t \in [0, T]: \qquad \mathcal{E}(t, q_{n_k}(t)) \to \mathcal{E}(t, q(t)),$$

and

$$\partial_r \mathcal{E}(\cdot, q_{n_k}(\cdot)) \to \partial_r \mathcal{E}(\cdot, q(\cdot)) \quad \text{in } L^1(0, T). \tag{42}$$

Notice that the statement of Theorem 3.3 is actually stronger than that of Theorem 3.2 because it additionally provides a way to construct energetic solutions using incremental minimization and convergence results.

In order to show existence of energetic solutions, we resort in applying the theory introduced and developed by MIELKE and co-authors in a series of papers and books (see [38] or more recently[41] and references therein). Along the existence proof in Section 3.2 below we use that, under the assumptions stated in Section 2, the following conditions are satisfied:

(C1) The dissipation $\mathcal{D}$ satisfies the following two properties:

  (i) $\forall (P_1, z_1), (P_2, z_2) \in \mathcal{P} \times \mathcal{Z}$:

$$\mathcal{D}(P_1, z_1, P_2, z_2) = 0 \iff P_1 = P_2, \; z_1 = z_2.$$

  (ii) $\forall (P_i, z_i) \in \mathcal{P} \times \mathcal{Z}, i = 0, 1, 2$:

$$\mathcal{D}(P_0, z_0, P_2, z_2) \leq \mathcal{D}(P_0, z_0, P_1, z_1) + \mathcal{D}(P_1, z_1, P_2, z_2).$$

(C2) $\mathcal{D} : (\mathcal{P} \times \mathcal{Z})^2 \to [0, +\infty]$ is lower semicontinuous.

(C3) There exists a function $\lambda \in L^1(0, T)$ such that for all $q \in Q$ the following implication holds true:

$$\mathcal{E}(0, q) < \infty \Rightarrow \partial_t \mathcal{E}(\cdot, q) : [0, T] \to \mathbb{R} \text{ is integrable and}$$

$$|\partial_t \mathcal{E}(t, q)| \leq \lambda(t)(1 + \mathcal{E}(t, q)).$$

(C4) For all $t \in [0, T]$, the map $q \mapsto \mathcal{E}(t, q)$ has compact sublevels.

(C5) The set of stable states is closed on $[0, T] \times Q$: namely, for every sequence (t_k, q_k) such that $q_k \in S(t_k)$ for every k, $t_k \to t$, and $q_k \rightharpoonup q$ in Q, we have $q \in S(t)$.

Notice that (C1) and (C2) imply that for any bounded sequence $(P_k, z_k)_{k \in \mathbb{N}} \in \mathcal{P} \times \mathcal{Z}$ we have

$$\min\{\mathcal{D}(P, z, P_k, z_k), \mathcal{D}(P_k, z_k, P, z)\} \to 0 \quad \Rightarrow \quad (P_k, z_k) \rightharpoonup (P, z) \text{ in } \mathcal{P} \times \mathcal{Z}$$

as was observed in [34, Lemma 4.1], because bounded sets in $\mathcal{P} \times \mathcal{Z}$ are precompact (with respect to the weak topologies). As a consequence, once (C1) and (C2) are established, we may use the generalized version of *Helly's selection principle* stated in [41, Theorem 2.1.24].

Aiming to prove Theorem 3.3, we start by checking that conditions (C1)-(C4) are indeed satisfied by our model introduced in Section 2. The proof of (C5) is typically the hardest part and we establish it separately in Section 3.1 by arguing as in THOMAS.[50,51] The process in finding mutual recovery sequences used therein is directly applicable to our setting.

Proof of (C1): The dissipation $\mathcal{D}$ is defined as an integral over Ω of the non-negative function D. By Proposition 2.4 for almost every $x \in \Omega$ we have

$$D(x, P_0(x), z_0(x), P_1(x), z_1(x)) = \kappa(x)(z_0(x) - z_1(x)) + \rho(z_1(x))D_p(x, P_0(x), P_1(x)),$$

with $\rho(z_1(x))D(x, P_0(x), P_1(x)) \geq 0$, and $\kappa(x) \geq \kappa_0 > 0$. It is thus easily seen that if $D(\cdot, P_0, z_0, P_1, z_1) = 0$ a.e. in Ω it must be $z_0 = z_1$ a.e. on Ω. Now, since ρ is strictly positive, $D_p(x, P_0(x), P_1(x)) = 0$ for a.e. $x \in \Omega$, which in turn implies $P_0 = P_1$ a.e. by Lemma 2.1(i). This proves point (i) of (C1). We now prove the triangle inequality (ii). Let $(P_i, z_i) \in \mathcal{P} \times \mathcal{Z}$ for $i = 1, 2, 3$. We can assume without loss of generality that $z_1 \geq z_2 \geq z_3$ a.e. on Ω. Otherwise the right-hand side of the triangle inequality is $+\infty$. Fix $x \in \Omega$ and for simplicity drop the x-dependence in the next formulas. We use Lemma 2.1(ii), Proposition 2.4, and the monotonicity of ρ to estimate

$$
\begin{aligned}
D(P_1, z_1, P_3, z_3) &= \kappa(z_1 - z_3) + \rho(z_3)D_p(P_1, P_3) \\
&= \kappa(z_1 - z_2) + \kappa(z_2 - z_3) + \rho(z_3)D_p(P_1, P_3) \\
&\leq \kappa(z_1 - z_2) + \kappa(z_2 - z_3) + \rho(z_3)(D_p(P_1, P_2) + D_p(P_2, P_3)) \\
&\leq \kappa(z_1 - z_2) + \rho(z_2)D_p(P_1, P_2) + \kappa(z_2 - z_3) + \rho(z_3)D_p(P_2, P_3) \\
&= D(P_1, z_1, P_2, z_2) + D(P_2, z_2, P_3, z_3).
\end{aligned}
$$

We conclude by integrating over Ω.

Proof of (C2): We have to show that whenever $(P_k, z_k, \widehat{P}_k, \widehat{z}_k) \rightharpoonup (P, z, \widehat{P}, \widehat{z})$ in $(\mathcal{P} \times \mathcal{Z})^2$ then

$$\mathcal{D}(P, z, \widehat{P}, \widehat{z}) \leq \liminf_{k \to \infty} \mathcal{D}(P_k, z_k, \widehat{P}_k, \widehat{z}_k).$$

By compactness the convergence of $(P_k, z_k, \widehat{P}_k, \widehat{z}_k)$ to $(P, z, \widehat{P}, \widehat{z})$ above is strong in $L^1(\Omega)$. By Proposition 2.4 it suffices to show that

$$\int_\Omega \Psi(x, z(x) - \widehat{z}(x))\, dx \leq \liminf_{k \to \infty} \int_\Omega \Psi(x, z_k(x) - \widehat{z}_k(x))\, dx \tag{43}$$

and

$$\int_\Omega \rho(x, \widehat{z})D_p(x, P, \widehat{P})\, dx = \lim_{k \to \infty} \int_\Omega \rho(x, \widehat{z}_k)D_p(x, P_k, \widehat{P}_k)\, dx. \tag{44}$$

The implication (43) simply follows from Fatou's Lemma since Ψ is non-negative and lower semicontinuous in the second component. In order to prove (44) we use that $\rho(x, \cdot)$ is continuous. As shown in Lemma 2.3, $D_p(x, \cdot, \cdot)$ is continuous as well and using the sublinear growth proved in Proposition 2.2 we conclude by Dominated Convergence Theorem.

Proof of (C3): From the very definition of the energy we recall

$$\partial_t \mathcal{E}(t, y, P, z) = \partial_t \mathcal{W}_{el}(t, y, P, z) - \langle \dot{\ell}(t), g_{Dir}(t, y) \rangle - \langle \ell(t), \dot{g}_{Dir}(t, y) \rangle$$

and use estimate (37a) to control

$$|\partial_t \mathcal{W}_{el}(t, y, P, z)| \leq c_1^E(\mathcal{W}_{el}(t, y, P, z) + c_0^E) \leq c_1^E(\mathcal{E}(t, y, P, z) + \tilde{c}_0^E)$$

by assumptions (35) and (36). Moreover,

$$|\langle \dot{\ell}(t), g_{Dir}(t, y) \rangle + \langle \ell(t), \dot{g}_{Dir}(t, y) \rangle| \overset{(35)}{\leq} C(\|\dot{\ell}(t)\|_{(W^{1,q})^*} + \|\ell(t)\|_{(W^{1,q})^*})\|\nabla y\|_{L^q}$$

$$\overset{(*)}{\leq} C\lambda(t)(1 + \mathcal{E}(t, y, P, z)),$$

where $\lambda(t) := \|\dot{\ell}(t)\|_{(W^{1,q})^*} + \|\ell(t)\|_{(W^{1,q})^*} \in L^1(0, T)$ by assumption (36) and $(*)$ follows from (48) shown below.

Proof of (C4): To assert that all sublevels of the energy are compact is equivalent to saying that sublevels are precompact and closed. We start by showing (sequential) precompactness. Let $t \in [0, T]$ and assume that we have a sequence $q_k = (y_k, P_k, z_k) \in \mathcal{Q}$ which satisfies $\mathcal{E}(t, q_k) \leq C$. Using coercivity (31) we see that

$$
\begin{aligned}
\mathcal{E}(t, q_k) \geq c \Big(\|\nabla y_k P_k^{-1}\|_{L^{q_e}}^{q_e} + \|P_k\|_{L^{q_p}}^{q_p} + \|\nabla P_k\|_{L^{r_p}}^{r_p} + \|\nabla z_k\|_{L^{r_z}}^{r_z} \Big) \\
- C_1 \|\nabla y_k\|_{L^q} - C_2,
\end{aligned}
\tag{45}
$$

by assumptions (35) and (36). By Young's inequality we deduce that, for any $\eta > 0$,

$$
C_1 \|\nabla y_k\|_{L^q} \leq \eta^{-1} C + \frac{\eta \|\nabla y_k\|_{L^q}^q}{q}.
\tag{46}
$$

Additionally, using Hölder's and Young's inequality in view of (32), we have

$$
\|\nabla y_k\|_{L^q}^q \leq \|\nabla y_k P_k^{-1}\|_{L^{q_e}}^q \|P_k\|_{L^{q_p}}^q \leq C \Big(\|\nabla y_k P_k^{-1}\|_{L^{q_e}}^{q_e} + \|P_k\|_{L^{q_p}}^{q_p} \Big).
\tag{47}
$$

Combining (45), (46), and (47) and choosing $\eta > 0$ suitably small, we readily see that

$$
\|\nabla y_k\|_{L^q}^q + \|P_k\|_{L^{q_p}}^{q_p} + \|\nabla P_k\|_{L^{r_p}}^{r_p} + \|\nabla z_k\|_{L^{r_z}}^{r_z} \leq C(1 + \mathcal{E}(t, q_k)) \leq C.
\tag{48}
$$

Then, there exists a (not relabeled) subsequence such that

$$
z_k \to z^* \text{ in } L^1(\Omega) \text{ and pointwise a.e.},
$$

$$
z_k \rightharpoonup z^* \text{ weakly in } W^{1, r_z}(\Omega).
$$

Notice that $z_k \in [0, 1]$ a.e. on Ω, thus z_k stays uniformly bounded in $L^\infty(\Omega)$, so that by Vitali's Convergence Theorem we infer

$$
z_k \to z^* \text{ in } L^\sigma(\Omega)
$$

for all $\sigma \geq 1$. Similarly, we argue for P_k which is uniformly bounded in $L^{\bar{q}_p}(\Omega)$ with $\bar{q}_p := \max\{q_p, r_p^*\}$, r_p^* being the Sobolev exponent to r_p, and extract another (not relabeled) subsequence such that

$$
P_k \rightharpoonup P^* \text{ weakly in } W^{1, r_p}(\Omega),
$$

$$
P_k \to P^* \text{ in } L^s(\Omega),
$$

for every $s \in [1, \bar{q}_p)$. Furthermore, we infer that

$$
\nabla y_k \rightharpoonup \nabla y^* \text{ weakly in } L^q(\Omega).
$$

In particular, we have checked that

$$
q_k \rightharpoonup q^* \text{ in } \mathcal{Q},
$$

which is nothing but sequential *pre*compactness.

It remains to show the lower semicontinuity of $\mathcal{E}$, which is equivalent to closedness of sublevels. Take a sequence $q_k \rightharpoonup q$ in $\mathcal{Q}$ where $q_k = (y_k, P_k, z_k)$ and assume without loss of generality that $\sup_k \mathcal{E}(t, q_k) \leq C$. We can use estimate (48) and choose a (not relabeled) subsequence such that $\mathcal{E}(t, q_k)$ converges to $\liminf_{k \to \infty} \mathcal{E}(t, q_k)$ and

$$
\nabla y_k \rightharpoonup \nabla y \text{ weakly in } L^q(\Omega),
$$

$$
\nabla P_k \rightharpoonup \nabla P \text{ weakly in } L^{r_p}(\Omega),
$$

$$
\nabla z_k \rightharpoonup \nabla z \text{ weakly in } L^{r_z}(\Omega),
$$

$$
P_k \to P \text{ in } L^s(\Omega),
$$

$$
z_k \to z \text{ in } L^\sigma(\Omega)
$$

for every $s \in [1, \bar{q}_{\mathrm{p}}), \sigma \in [1, \infty)$. Now in order to use polyconvexity (30) we need to show that

$$\mathbb{M}(\nabla g_{\mathrm{Dir}}(t, y_k) \nabla y_k P_k^{-1}) \rightharpoonup \mathbb{M}(\nabla g_{\mathrm{Dir}}(t, y) \nabla y P^{-1}) \quad \text{weakly in } L^1(\Omega).$$

This result was established in [39] and applied in [34, Proposition 5.1] (see also [41, Lemma 4.1.3]). The convergence is proven under the assumption that $q > d$ and

$$\frac{1}{q} + \frac{d-1}{s} \leq 1$$

which is indeed satisfied here since $\bar{q}_{\mathrm{p}} > q > d$ and therefore s can be chosen larger than d. The lower semicontinuity of $(y, P, z) \mapsto \mathcal{W}(t, y, P, z)$ now follows from classical theory due to the polyconvexity assumptions in Section 2.5. It was pointed out in [34] that the classical assumption of W being a Carathéodory function can be relaxed to the one of a normal integrand using a Yosida-Moreau regularization.

3.1 | Closedness of stable states (C5)

This closedness relies on finding suitable recovery sequences for the damage variable z. In [40] this was achieved in the framework of damage in nonlinear elasticity for $r_z > d$, in which case damage is continuous in space. In the papers,[50,51] it was generalized to $1 < r_z < d$. We apply the strategy of [50,51] to our model. In particular, the choice of recovery sequences is the same as in the mentioned works.

We want to prove that, if (t_k, q_k) is a sequence such that $q_k \in \mathcal{S}(t_k)$, $t_k \to t$, and $q_k \rightharpoonup q$ in $\mathcal{Q}$, then $q \in \mathcal{S}(t)$. Thus we need to ensure that for every $\hat{q} \in \mathcal{Q}$

$$0 \leq \mathcal{E}(t, \hat{q}) + \mathcal{D}(q, \hat{q}) - \mathcal{E}(t, q).$$

In order to show this we provide a so-called *mutual recovery sequence* $\hat{q}_k$ (see [40,43]) satisfying

$$\limsup_{k \to \infty} \left(\mathcal{E}(t_k, \hat{q}_k) + \mathcal{D}(q_k, \hat{q}_k) - \mathcal{E}(t_k, q_k) \right) \leq \mathcal{E}(t, \hat{q}) + \mathcal{D}(q, \hat{q}) - \mathcal{E}(t, q). \tag{49}$$

Indeed, by stability of q_k, we have for every $\hat{q}_k \in \mathcal{Q}$

$$0 \leq \mathcal{E}(t_k, \hat{q}_k) + \mathcal{D}(q_k, \hat{q}_k) - \mathcal{E}(t_k, q_k). \tag{50}$$

Then the lim sup bound (49) together with (50) implies $q \in \mathcal{S}(t)$.

Notice that if the dissipation $\mathcal{D}$ was continuous (not only lower semicontinuous) then (49) would hold true even for the constant recovery sequence $\hat{q}_k = \hat{q}$ because $\mathcal{E}$ is lower semicontinuous and $\mathcal{E}(\cdot, \hat{q})$ is continuous. In the present case however, the dissipation $\mathcal{D}$ is not (globally) continuous but only on its domain. The same is true for dissipation distances in finite plasticity with isotropic hardening, see [34, Conditions 3.5].

Lemma 3.4. *Let*

$$\mathbb{D} := \left\{ (P, z, \hat{P}, \hat{z}) \in (\mathcal{P} \times \mathcal{Z})^2 : \mathcal{D}(P, z, \hat{P}, \hat{z}) < \infty \right\}.$$

Then $\mathcal{D} : \mathbb{D} \to [0, \infty)$ is continuous.

Proof. By Proposition 2.4,

$$\mathcal{D}(P, z, \hat{P}, \hat{z}) = \int_\Omega \Psi(x, \hat{z} - z) + \rho(x, \hat{z}) D_{\mathrm{p}}(x, P, \hat{P}) \, dx.$$

Now take a sequence $(P_k, z_k, \hat{P}_k, \hat{z}_k) \in \mathbb{D}$ such that $(P_k, z_k, \hat{P}_k, \hat{z}_k) \rightharpoonup (P, z, \hat{P}, \hat{z})$ in $(\mathcal{P} \times \mathcal{Z})^2$. Then the convergence is strong in $L^1(\Omega)$, so that up to not relabeled subsequence we might assume $(P_k, z_k, \hat{P}_k, \hat{z}_k) \to (P, z, \hat{P}, \hat{z})$ a.e. in Ω. Observe that

$$\mathbb{D} = \left\{ (P, z, \hat{P}, \hat{z}) \in (\mathcal{P} \times \mathcal{Z})^2 : z(x) \geq \hat{z}(x) \text{ for a.e. } x \in \Omega \right\}.$$

Thus,

$$\mathcal{D}(P_k, z_k, \widehat{P}_k, \widehat{z}_k) = \int_\Omega \kappa(x)(z_k(x) - \widehat{z}_k(x)) + \rho(x, \widehat{z}_k) D_{\mathrm{p}}(x, P_k, \widehat{P}_k) \, \mathrm{d}x$$

for every $k \in \mathbb{N}$. By Lemma 2.3 the integrand converges pointwise a.e. in Ω to the corresponding limit. We can further estimate the integrand, using Proposition 2.2, by

$$2\|\kappa\|_{L^\infty} + C\|\rho\|_{L^\infty}(1 + |P_k(x)| + |\widehat{P}_k(x)|).$$

This bound allows us to use the Dominated Convergence Theorem. Hence,

$$\lim_{k\to\infty} \int_\Omega \kappa(z_k - \widehat{z}_k) + \rho(\widehat{z}_k) D_{\mathrm{p}}(P_k, \widehat{P}_k) \, \mathrm{d}x = \int_\Omega \kappa(z - \widehat{z}) + \rho(\widehat{z}) D_{\mathrm{p}}(P, \widehat{P}) \, \mathrm{d}x, \tag{51}$$

where, for simplicity, we have again omitted the x-dependence. Noticing that $\kappa(x)(z(x) - \widehat{z}(x)) = \Psi(x, z(x) - \widehat{z}(x))$ for a.e. $x \in \Omega$, the right-hand side of (51) is nothing but $\mathcal{D}(P, z, \widehat{P}, \widehat{z})$ and the statement follows. □

Lemma 3.5. *Let $q_k \in S(t_k)$ such that $t_k \to t$ and $q_k \rightharpoonup q$ in Q. Then for every $\widehat{q} \in Q$ there exists a mutual recovery sequence $\widehat{q}_k$ in the sense of* (49).

Proof. For sake of completeness, we give the full proof following the steps in [51, Theorem 3.14].

Step 1. Let (t_k, q_k) be as in the statement and $\widehat{q} = (\widehat{y}, \widehat{P}, \widehat{z}) \in Q$ be arbitrary. We first set $\widehat{y}_k := \widehat{y}$ and $\widehat{P}_k = \widehat{P}$ for all k. From this choice it is possible to reduce to the case $t_k = t$ for every k. Indeed, we claim that the lim sup in (49) coincides with

$$\limsup_{k\to\infty} \left(\mathcal{E}(t, \widehat{q}_k) + \mathcal{D}(q_k, \widehat{q}_k) - \mathcal{E}(t, q_k) \right).$$

Consider the difference

$$\left| \mathcal{E}(t_k, \widehat{q}_k) - \mathcal{E}(t, \widehat{q}_k) + \mathcal{E}(t, q_k) - \mathcal{E}(t_k, q_k) \right| \leq \left| \mathcal{W}_{\mathrm{el}}(t_k, \widehat{q}_k) - \mathcal{W}_{\mathrm{el}}(t, \widehat{q}_k) + \mathcal{W}_{\mathrm{el}}(t, q_k) - \mathcal{W}_{\mathrm{el}}(t_k, q_k) \right|$$

$$+ \left| \langle \ell(t_k), g_{\mathrm{Dir}}(t_k, \widehat{y}) \rangle - \langle \ell(t), g_{\mathrm{Dir}}(t, \widehat{y}) \rangle \right|$$

$$+ \left| \langle \ell(t_k), g_{\mathrm{Dir}}(t_k, y_k) \rangle - \langle \ell(t), g_{\mathrm{Dir}}(t, y_k) \rangle \right|. \tag{52}$$

We want to show that this is infinitesimal as $t_k \to t$. The first line in the right-hand side is

$$\left| \mathcal{W}_{\mathrm{el}}(t_k, \widehat{q}_k) - \mathcal{W}_{\mathrm{el}}(t, \widehat{q}_k) + \mathcal{W}_{\mathrm{el}}(t, q_k) - \mathcal{W}_{\mathrm{el}}(t_k, q_k) \right|$$

$$= \left| \int_t^{t_k} \partial_s \mathcal{W}_{\mathrm{el}}(s, \widehat{q}_k) - \partial_s \mathcal{W}_{\mathrm{el}}(s, q_k) \, \mathrm{d}s \right|$$

$$\leq \int_t^{t_k} |\partial_s \mathcal{W}_{\mathrm{el}}(t, \widehat{q}_k)| + |\partial_s \mathcal{W}_{\mathrm{el}}(t, q_k)| + |\partial_s \mathcal{W}_{\mathrm{el}}(s, \widehat{q}_k) - \partial_s \mathcal{W}_{\mathrm{el}}(t, \widehat{q}_k)| + |\partial_s \mathcal{W}_{\mathrm{el}}(t, q_k) - \partial_s \mathcal{W}_{\mathrm{el}}(s, q_k)| \, \mathrm{d}s.$$

Now, by (37a) and (37b) we can bound this by $C|t - t_k|(1 + \mathcal{E}(t, \widehat{q}_k))$, see also (62) below. Let us continue with the estimate of the second and third lines in (52), that are less or equal to

$$\left| \langle \ell(t_k) - \ell(t), g_{\mathrm{Dir}}(t_k, \widehat{y}) \rangle \right| + \left| \langle \ell(t), g_{\mathrm{Dir}}(t_k, \widehat{y}) - g_{\mathrm{Dir}}(t, \widehat{y}) \rangle \right| + \left| \langle \ell(t_k) - \ell(t), g_{\mathrm{Dir}}(t_k, y_k) \rangle \right| + \left| \langle \ell(t), g_{\mathrm{Dir}}(t_k, y_k) - g_{\mathrm{Dir}}(t, y_k) \rangle \right|$$

$$\leq C\|\ell(t_k) - \ell(t)\|_{(W^{1,q})^*} + \|\ell(t)\|_{(W^{1,q})^*} \|g_{\mathrm{Dir}}(t_k, y_k) - g_{\mathrm{Dir}}(t, y_k)\|_{W^{1,q}} \to 0$$

as $k \to \infty$ by (35) and (36). Altogether,

$$\left| \mathcal{E}(t_k, \widehat{q}_k) - \mathcal{E}(t, \widehat{q}_k) + \mathcal{E}(t, q_k) - \mathcal{E}(t_k, q_k) \right| \leq o(1) + o(1)\mathcal{E}(t, \widehat{q}_k).$$

This shows the claim provided $\mathcal{E}(t, \widehat{q}_k) \leq C < \infty$, which is satisfied by the recovery sequence chosen below.

Step 2. If $\mathcal{E}(t, \widehat{q}) + \mathcal{D}(q, \widehat{q}) = +\infty$ then (49) holds trivially. Let us therefore assume $\mathcal{E}(t, \widehat{q}) < +\infty$ and $\mathcal{D}(q, \widehat{q}) < +\infty$. This in particular implies

$$\widehat{z} \le z, \tag{53}$$

a.e. on Ω. We define the recovery sequence as $\widehat{q}_k := (\widehat{y}, \widehat{P}, \widehat{z}_k)$ where

$$\widehat{z}_k := \min\{(\widehat{z} - \delta_k)^+, z_k\} \tag{54}$$

and $\delta_k > 0$ is a sequence that will be chosen later (tending to zero as $k \to \infty$).

We now claim that $\widehat{z}_k \rightharpoonup \widehat{z}$ weakly in $W^{1,r_z}(\Omega)$. Indeed, by construction, $\widehat{z}_k$ is bounded in $W^{1,r_z}(\Omega)$. So for every subsequence $\widehat{z}_{k_l}$ there exists a further subsequence $\widehat{z}_{k_{l_j}}$ and a limit z^* (a priori depending on the subsequence we choose) such that

$$\widehat{z}_{k_{l_j}} \rightharpoonup z^* \text{ weakly in } W^{1,r_z}(\Omega),$$

$$\widehat{z}_{k_{l_j}} \to z^* \text{ in } L^{r_z}(\Omega),$$

$$\widehat{z}_{k_{l_j}} \to z^* \text{ a.e. on } \Omega.$$

But by definition of $\widehat{z}_k$, it follows that it converges to $\widehat{z}$ a.e. on Ω. Thus, $z^* = \widehat{z}$ independently of the subsequence and we have shown

$$\widehat{z}_k \rightharpoonup \widehat{z} \text{ weakly in } W^{1,r_z}(\Omega).$$

Notice that $(q_k, \widehat{q}_k) \in \mathbb{D}$ because $\widehat{z}_k \le z_k$. Therefore, by Lemma 3.4

$$\limsup_{k \to \infty} \mathcal{D}(q_k, \widehat{q}_k) = \lim_{k \to \infty} \mathcal{D}(q_k, \widehat{q}_k) = \mathcal{D}(q, \widehat{q}).$$

Step 3. It remains to show that

$$\limsup_{k \to \infty} \left(\mathcal{E}(t, \widehat{q}_k) - \mathcal{E}(t, q_k) \right) \le \mathcal{E}(t, \widehat{q}) - \mathcal{E}(t, q). \tag{55}$$

To achieve this we need to choose the sequence δ_k in such a way that $\mathcal{L}^d(\{z_k < (\widehat{z} - \delta_k)^+\})$ goes to zero as $k \to \infty$. This particularly implies $\widehat{z}_k \to \widehat{z}$ in L^σ for all $\sigma \ge 1$. Recall that

$$\mathcal{E}(t, y, P, z) = \mathcal{W}(t, y, P, z) + \frac{\nu}{r_p} \int_\Omega |\nabla P|^{r_p} \, dx + \frac{\mu}{r_z} \int_\Omega |\nabla z|^{r_z} \, dx - \langle \ell(t), g_{\mathrm{Dir}}(t, y) \rangle.$$

By lower semicontinuity of $\mathcal{E}$, to establish (55), it suffices to show

$$\lim_{k \to \infty} \mathcal{W}(t, \widehat{y}, \widehat{P}, \widehat{z}_k) = \mathcal{W}(t, \widehat{y}, \widehat{P}, \widehat{z}) \tag{56}$$

and

$$\limsup_{k \to \infty} \int_\Omega \left(|\nabla \widehat{z}_k|^{r_z} - |\nabla z_k|^{r_z} \right) dx \le \int_\Omega \left(|\nabla \widehat{z}|^{r_z} - |\nabla z|^{r_z} \right) dx. \tag{57}$$

Up to subsequences we can assume that $\widehat{z}_k \to \widehat{z}$ a.e. on Ω. Since W is continuous in z, this implies

$$W(\cdot, \widehat{y}(\cdot), \widehat{P}(\cdot), \widehat{z}_k(\cdot)) \to W(\cdot, \widehat{y}(\cdot), \widehat{P}(\cdot), \widehat{z}(\cdot)) \text{ a.e. on } \Omega.$$

By using $\widehat{z}_k \le \widehat{z}$ and monotonicity (34) we get the uniform bound

$$W(\widehat{y}, \widehat{P}, \widehat{z}_k) \le W(\widehat{y}, \widehat{P}, \widehat{z}) \in L^1(\Omega).$$

Therefore, (56) follows from the Dominated Convergence Theorem. We are left with showing (57). We define

$$B_k = \{z_k < (\hat{z} - \delta_k)^+\},$$

$$A_k = \Omega \setminus B_k.$$

Since $B_k \subset \{|z - z_k| \geq \delta_k\}$ thanks to (53), we can use Markov's inequality to show that

$$\mathcal{L}^d(B_k) \leq \frac{1}{\delta_k^{r_z}} \int_\Omega |z - z_k|^{r_z} dx.$$

As we want this to go to 0 we impose that

$$\delta_k = \|z - z_k\|_{L^{r_z}}^{1/r_z}.$$

Now we can write, by the definition of $\hat{z}_k$,

$$\frac{\mu}{r_z} \int_\Omega \left(|\nabla \hat{z}_k|^{r_z} - |\nabla z_k|^{r_z}\right) dx = \frac{\mu}{r_z} \int_{A_k} \left(|\nabla \hat{z}|^{r_z} - |\nabla z_k|^{r_z}\right) dx.$$

We take the lim sup as $k \to \infty$ and use that $I_{A_k} \nabla z_k \rightharpoonup \nabla z$ weakly in $L^{r_z}(\Omega)$ (here I_{A_k}, the characteristic function of A_k, converges to 1 strongly in $L^q(\Omega)$ for any $q \in [1, +\infty)$, while ∇z_k tends to ∇z weakly in $L^p(\Omega)$ for all $p < r_z$; the equiboundedness of $I_{A_k} \nabla z_k$ on L^{r_z} implies the claim) to get

$$\limsup_{k \to \infty} \frac{\mu}{r_z} \int_{A_k} \left(|\nabla \hat{z}|^{r_z} - |\nabla z_k|^{r_z}\right) dx = \frac{\mu}{r_z} \left(\int_\Omega |\nabla \hat{z}|^{r_z} dx - \liminf_k \int_{A_k} |\nabla z_k|^{r_z} dx\right)$$

$$\leq \frac{\mu}{r_z} \left(\int_\Omega |\nabla \hat{z}|^{r_z} dx - \int_\Omega |\nabla z|^{r_z} dx\right)$$

by weak lower semicontinuity of the norm. □

3.2 | Proof of Theorem 3.3

We are now in position to prove the main result. We proceed in several steps following the general scheme showed in [38] (see also, e.g. [34] for the treatment of the boundary datum). Since many steps are standard we do not enter into much detail and refer to [38,41]. Nevertheless, for completeness all crucial steps of the proof are mentioned.

Step 1: Approximation via incremental minimization. Let $\sigma_n = \{0 = t_0^n < t_1^n < \cdots < t_{N(n)}^n = T\} \in \Pi, n \in \mathbb{N}$, be a sequence of partitions such that the fineness tends to zero as n tends to ∞. For fixed n we iteratively solve for

$$(y_j, P_j, z_j) \in \underset{(\hat{y}, \hat{P}, \hat{z}) \in Q}{\operatorname{argmin}} \left\{\mathcal{E}(t_j, \hat{y}, \hat{P}, \hat{z}) + \mathcal{D}(P_{j-1}, z_{j-1}, \hat{P}, \hat{z})\right\}, \quad j \in \{1, \ldots, N(n)\}. \tag{58}$$

Note that (C2) and (C4) guarantee the existence of minimizers. This selection satisfies $q_j = (y_j, P_j, z_j) \in \mathcal{S}(t_j)$. This can be seen by using the minimum property in (58) and the triangle inequality (C1 (ii)). Arguing in a standard way (testing the minimum in (58) by q_{j-1}) we arrive at the inequality

$$\mathcal{E}(t, q_n(t)) + \operatorname{Diss}(P_n, z_n; s, t) \leq \mathcal{E}(s, q(s)) + \int_s^t \partial_r \mathcal{E}(r, q_n(r)) \, dr, \tag{59}$$

for every $s, t \in \sigma_n$, where we have defined the right-continuous piecewise constant approximation

$$q_n(t) := q_{j-1}, \quad \text{for } t \in [t_{j-1}, t_j).$$

We have just established (40) and (41). The next goal is to pass this inequality to the limit.

Step 2: A priori estimates. Using (59) in combination with (C3), and a standard application of Gronwall's inequality, entails

$$\mathcal{E}(t, q_n(t)) \leq (1 + \mathcal{E}(0, q_0)) \exp\left(\int_0^t \lambda(s) \, ds\right) \leq C$$

for every $t \in [0, T]$. This leads to

$$\sup_{t \in [0,T]} \mathcal{E}(t, q_n(t)) + \text{Diss}(P_n, z_n; 0, T) \leq C. \tag{60}$$

In this step we have exploited the uniform continuity of $\mathcal{E}(\cdot, q)$ guaranteed by conditions (35) and (36).

Step 3: Selection of subsequences. The dissipation distance satisfies (C1) and (C2) and due to (C4) the sequence (P_n, z_n) takes values in a compact subset of $\mathcal{P} \times \mathcal{Z}$. Moreover, its dissipation is bounded uniformly in n. Therefore we can use Helly's selection principle [41, Theorem 2.1.24] and find a subsequence (not relabeled) and functions $P, z : [0, T] \to \mathcal{P} \times \mathcal{Z}, \delta : [0, T] \to [0, C]$ such that the following hold:

$$\forall t \in [0, T] : \qquad P_n(t) \rightharpoonup P(t) \text{ in } \mathcal{P}, \tag{61a}$$

$$\forall t \in [0, T] : \qquad z_n(t) \rightharpoonup z(t) \text{ in } \mathcal{Z}, \tag{61b}$$

$$\forall t \in [0, T] : \qquad \delta_n(t) := \text{Diss}(P_n, z_n; 0, t) \to \delta(t), \tag{61c}$$

$$\forall s, t \in [0, T] : \qquad \text{Diss}(P, z; s, t) \leq \delta(t) - \delta(s). \tag{61d}$$

Let us define the sequence

$$\theta_n(t) := \partial_t \mathcal{E}(t, q_n(t)) = \partial_t \mathcal{W}_{\text{el}}(t, q_n(t)) - \langle \dot{\ell}(t), g_{\text{Dir}}(t, y_n(t)) \rangle - \langle \ell(t), \dot{g}_{\text{Dir}}(t, y_n(t)) \rangle,$$

Notice that, thanks to (37b) and the fact that

$$\mathcal{W}_{\text{el}}(t, q) \leq C(1 + \mathcal{E}(t, q)), \tag{62}$$

the power $\partial_t \mathcal{W}_{\text{el}}(\cdot, q)$ satisfies the following uniform-continuity property

$$\forall E > 0, \varepsilon > 0, \quad \exists \delta > 0 \text{ such that} \tag{63}$$

$$\mathcal{E}(0, q) \leq E, \ |t - s| < \delta \Rightarrow |\partial_t \mathcal{W}_{\text{el}}(t, q) - \partial_t \mathcal{W}_{\text{el}}(s, q)| < \varepsilon.$$

It is easy to check that θ_n is bounded in $L^1(0, T)$ and equi integrable. Actually, the term $\partial_t \mathcal{W}_{\text{el}}(t, q_n(t))$ belongs to $L^\infty(0, T)$ thanks to (37a) and the a-priori estimates. Furthermore, by (35), for every interval $I \subset [0, T]$, we have

$$\int_I |\langle \dot{\ell}(s), g_{\text{Dir}}(s) \circ y_n(s) \rangle| \, ds \leq C \int_I \|\dot{\ell}(s)\|_{(W^{1,q})^*} \|\nabla y_n(s)\|_{L^q} \, ds \leq C \int_I \|\dot{\ell}(s)\|_{(W^{1,q})^*} \, ds,$$

and, since $\dot{\ell} \in L^1(0, T; (W^{1,q}(\Omega))^*)$, for every $\varepsilon > 0$ there exists a $\eta > 0$ such that, if $\mathcal{L}^d(I) < \eta$, we have

$$\int_I \|\dot{\ell}(s)\|_{\mathcal{Y}^*} \, ds < \varepsilon/C.$$

The estimate of the term $\langle \ell(t), \dot{g}_{\text{Dir}}(t, y_n(t)) \rangle$ is similar. Thus, we can use Dunford-Pettis Theorem[20] or [41, Theorem B.3.8] and find a further (not relabeled) subsequence such that

$$\theta_n \rightharpoonup \theta \quad \text{weakly in } L^1(0, T). \tag{64}$$

Notice that we did not construct a limit for the deformation yet because we are only able to use Helly's selection principle on the dissipative variables. We can still use the fact that y_n is controlled by the energy for every fixed time t. We define the limit deformation $y : [0, T] \to \mathcal{Y}$ as follows. Fix $t \in [0, T]$ and use (C4) and (60) to select a *t-dependent* subsequence $(n_k^t)_{k \in \mathbb{N}}$ such that

$$\theta_{n_k^t}(t) \to \limsup_{n \to \infty} \theta_n(t) =: \theta_{\sup}(t)$$

and $y_{n_k^t}(t)$ converges weakly to some limit $\tilde{y}$ in $\mathcal{Y}$. We now define

$$y(t) := \tilde{y}.$$

Notice that such $\tilde{y}$ may not be unique and may depend on the chosen subsequence. From the definition of $\theta_{\sup}$ together with regularity (35) and continuity (63), we can deduce

$$\theta_{\sup}(t) = \lim_{k \to \infty} \left(\partial_t \mathcal{W}_{\text{el}}(t, q_{n_k^t}(t)) - \langle \dot{\ell}(t), g_{\text{Dir}}(t, y_{n_k^t}(t)) \rangle - \langle \ell(t), \dot{g}_{\text{Dir}}(t, y_{n_k^t}(t)) \rangle \right)$$

$$= \partial_t \mathcal{W}_{\text{el}}(t, q(t)) - \langle \dot{\ell}(t), g_{\text{Dir}}(t, y(t)) \rangle - \langle \ell(t), \dot{g}_{\text{Dir}}(t, y(t)) \rangle \tag{65}$$

for every $t \in [0, T]$. We refer to [21, Proposition 3.3] for further details on the convergence of the first term.

Step 4: Stability. We define $\tau_n^t := \max\{\tau \in \sigma_n : \tau \le t\}$. Then, by definition, $\tau_n^t \to t$, $q_n(t) \rightharpoonup q(t)$ as $n = n_k^t \to \infty$ and $q_n(t) = q_n(\tau_n^t) \in \mathcal{S}(\tau_n^t)$ for every $n \in \mathbb{N}$. Therefore $q(t) \in \mathcal{S}(t)$ by (C5).

Step 5: Upper energy estimate. Let $t \in [0, T]$ be fixed and τ_n^t be as above. Our goal is to pass to the limit in (41) for $s = 0$ which reads

$$\mathcal{E}(\tau_n^t, q_n(\tau_n^t)) + \text{Diss}(P_n, z_n; 0, \tau_n^t) \le \mathcal{E}(0, q_0) + \int_0^{\tau_n^t} \partial_r \mathcal{E}(r, y_n(r)) \, dr. \tag{66}$$

This is a standard procedure. Indeed, first one proves

$$\mathcal{E}(t, q(t)) \le \liminf_{n \to \infty} \mathcal{E}(\tau_n^t, q_n(\tau_n^t)), \tag{67}$$

then uses (61c), (61d), (66) and (67) to get

$$\mathcal{E}(t, q(t)) + \text{Diss}(P, z; 0, t) \le \liminf_{n \to \infty} \mathcal{E}(\tau_n^t, q_n(\tau_n^t)) + \lim_{n \to \infty} \text{Diss}(P_n, z_n; 0, t)$$

$$\le \limsup_{n \to \infty} \mathcal{E}(\tau_n^t, q_n(\tau_n^t)) + \lim_{n \to \infty} \text{Diss}(P_n, z_n; 0, t)$$

$$\le \mathcal{E}(0, q_0) + \int_0^t \theta(r) \, dr \tag{68}$$

$$\le \mathcal{E}(0, q_0) + \int_0^t \theta_{\sup}(r) \, dr$$

$$= \mathcal{E}(0, q_0) + \int_0^t \partial_r \mathcal{E}(r, y(r)) \, dr.$$

Step 6: Lower energy estimate. Since this part is established differently compared to the standard scheme,[38] we discuss it more deeply. Take any partition $\sigma = \{0 = r_0 < r_1 < \cdots < r_N = t\} \in \Pi$ of $[0, t]$. By the stability of the limit (Step 4) one has that

$$\mathcal{E}(r_{i-1}, q(r_{i-1})) \le \mathcal{E}(r_i, q(r_i)) - \mathcal{E}(r_i, q(r_i)) + \mathcal{E}(r_{i-1}, q(r_i)) + \mathcal{D}(q(r_{i-1}), q(r_i))$$

Summing this over $i = 1, \ldots, N$ we get

$$\mathcal{E}(t, q(t)) - \mathcal{E}(0, q(0)) + \sum_{i=1}^N \mathcal{D}(q(r_{i-1}), q(r_i)) \ge \sum_{i=1}^N \int_{r_{i-1}}^{r_i} \partial_r \mathcal{E}(r, y(r_i)) \, dr. \tag{69}$$

We use

$$\sum_{i=1}^N \mathcal{D}(q(r_{i-1}), q(r_i)) \le \text{Diss}(P, z; 0, t)$$

to estimate the left hand side of (69) as desired. It remains to show that there exists a sequence of partitions

$$\sigma_n = \{0 = r_0^n < r_1^n < \cdots < r_{N(n)}^n = t\} \in \Pi, \quad n \in \mathbb{N} \tag{70}$$

such that

$$\int_0^t \partial_r \mathcal{E}(r, y(r)) \, dr = \lim_{n \to \infty} \sum_{i=1}^{N(n)} \int_{r_{i-1}^n}^{r_i^n} \partial_r \mathcal{E}(r, y(r_i)) \, dr. \tag{71}$$

The difficulty here is that we cannot assume measurability of y in time. However, $t \mapsto \partial_t \mathcal{E}(t, y(t))$ is integrable. Thus, we can find a sequence of partitions such that the integral is approximated by its Riemann sums,[26] i.e.

$$\int_0^t \partial_r \mathcal{E}(r, y(r)) \, dr = \lim_{n \to \infty} \sum_{i=1}^{N(n)} \partial_r \mathcal{E}(r_i^n, y(r_i^n))(r_i^n - r_{i-1}^n). \tag{72}$$

Now, in order to get (71), we prove that there exists a sequence of partitions simultaneously satisfying (72) and

$$\lim_{n \to \infty} \left| \sum_{i=1}^{N(n)} \int_{r_{i-1}^n}^{r_i^n} \partial_r \mathcal{W}_{\mathrm{el}}(r, q(r_i^n)) - \partial_r \mathcal{W}_{\mathrm{el}}(r_i^n, q(r_i^n)) \, dr \right| = 0,$$

$$\lim_{n \to \infty} \left| \sum_{i=1}^{N(n)} \int_{r_{i-1}^n}^{r_i^n} \langle \dot{\ell}(r), g_{\mathrm{Dir}}(r, y(r_i^n)) \rangle - \langle \dot{\ell}(r_i^n), g_{\mathrm{Dir}}(r_i^n, y(r_i^n)) \rangle \, dr \right| = 0,$$

$$\lim_{n \to \infty} \left| \sum_{i=1}^{N(n)} \int_{r_{i-1}^n}^{r_i^n} \langle \ell(r), \dot{g}_{\mathrm{Dir}}(r, y(r_i^n)) \rangle - \langle \ell(r_i^n), \dot{g}_{\mathrm{Dir}}(r_i^n, y(r_i^n)) \rangle \, dr \right| = 0. \tag{73}$$

The condition in the first line is straightforwardly achieved for every sequence with fineness tending to zero using equiboundeness of the energy and uniform continuity (63). To treat the second condition we estimate

$$\left| \sum_{i=1}^{N(n)} \int_{r_{i-1}^n}^{r_i^n} \langle \dot{\ell}(r), g_{\mathrm{Dir}}(r, y(r_i^n)) \rangle - \langle \dot{\ell}(r_i^n), g_{\mathrm{Dir}}(r_i^n, y(r_i^n)) \rangle \, dr \right|$$

$$\leq \left| \sum_{i=1}^{N(n)} \int_{r_{i-1}^n}^{r_i^n} \langle \dot{\ell}(r), g_{\mathrm{Dir}}(r, y(r_i^n)) - g_{\mathrm{Dir}}(r_i^n, y(r_i^n)) \rangle \, dr \right| + \left| \sum_{i=1}^{N(n)} \int_{r_{i-1}^n}^{r_i^n} \langle \dot{\ell}(r) - \dot{\ell}(r_i^n), g_{\mathrm{Dir}}(r_i^n, y(r_i^n)) \rangle \, dr \right|$$

$$\leq C \sum_{i=1}^{N(n)} \int_{r_{i-1}^n}^{r_i^n} \| g_{\mathrm{Dir}}(r) - g_{\mathrm{Dir}}(r_i^n) \|_{L^\infty} + \| \dot{\ell}(r) - \dot{\ell}(r_i^n) \|_{(W^{1,q})^*} \, dr, \tag{74}$$

where we used conditions (35) and (36) together with the fact that $\sup_{r \in [0,T]} \| \nabla y(r) \|_{L^q} \leq C$. The third line can be bounded analogously by

$$C \sum_{i=1}^{N(n)} \int_{r_{i-1}^n}^{r_i^n} \| \dot{g}_{\mathrm{Dir}}(r) - \dot{g}_{\mathrm{Dir}}(r_i^n) \|_{L^\infty} + \| \ell(r) - \ell(r_i^n) \|_{(W^{1,q})^*} \, dr.$$

Since $g_{\mathrm{Dir}}, \dot{g}_{\mathrm{Dir}} : [0, T] \to L^\infty(\Omega)$, $\ell, \dot{\ell} : [0, T] \to (W^{1,q}(\Omega))^*$ and $\partial_t \mathcal{E}(\cdot, y(\cdot)) : [0, T] \to \mathbb{R}$ are integrable, we can apply Lemma 3.6 below to the mentioned functions to find a sequence of partitions satisfying (71).

Lemma 3.6. *Let $s < t$. Assume we have a countable family of Bochner integrable functions*

$$f_k : [s, t] \to X_k, \quad k \in \mathbb{N},$$

*where X_k are Banach spaces. Then there exists a **k-independent** sequence of partitions*

$$\sigma_n = \{ s = r_0^n < r_1^n < \cdots < r_{N(n)}^n = t \}, \quad n \in \mathbb{N},$$

with fineness $\Delta(\sigma_n) \to 0$ such that

$$\lim_{n \to \infty} \sum_{i=1}^{N(n)} \int_{r_{i-1}^n}^{r_i^n} \| f_k(r) - f_k(r_i^n) \| \, dr = 0$$

for every $k \in \mathbb{N}$.

The proof can be found in [18, Lemma 4.12 & Remark 4.13].

Step 7: Conclusion. A combination of the upper estimate (68) with the lower energy estimate in Step 6 gives the following chain of inequalities

$$\mathcal{E}(t, q(t)) + \mathrm{Diss}(P, z; 0, t) \leq \liminf_{k \to \infty} \mathcal{E}(t, q_{n_k}(t)) + \lim_{k \to \infty} \mathrm{Diss}(P_{n_k}, z_{n_k}; 0, t)$$

$$\leq \limsup_{k \to \infty} \mathcal{E}(t, q_{n_k}(t)) + \delta(t)$$

$$\leq \mathcal{E}(0, q_0) + \int_0^t \theta(r) \, dr$$

$$\leq \mathcal{E}(0, q_0) + \int_0^t \partial_r \mathcal{E}(r, q(r)) \, dr$$

$$\leq \mathcal{E}(t, q(t)) + \mathrm{Diss}(P, z; 0, t).$$

Hence, equality holds everywhere, implying

$$\theta(r) = \theta_{\sup}(r) = \partial_r \mathcal{E}(r, q(r)) \text{ for a.e. } r \in [0, T],$$

$$\mathrm{Diss}(P_{n_k}, z_{n_k}; 0, t) \to \mathrm{Diss}(P, z; 0, t), \text{ and}$$

$$\mathcal{E}(t, q_{n_k}(t)) \to \mathcal{E}(t, q(t)).$$

To show convergence (42) we argue as in [21, Lemma 3.5]. We know that $\theta_{n_k} \rightharpoonup \theta$ in $L^1(0, T)$ and $\theta(t) = \limsup_{k \to \infty} \theta_{n_k}(t)$ for a.e. $t \in [0, T]$. Now,

$$\| \theta_{n_k} - \theta \|_{L^1} = \int_0^T (\theta - \theta_{n_k}) \, dt + 2 \int_0^T (\theta_{n_k} - \theta)^+ \, dt \tag{75}$$

where $f^+ := \max\{0, f\}$. The first integral converges to zero by weak convergence and the second integrand satisfies $0 \leq (\theta_{n_k} - \theta)^+ \leq \Theta_k := \sup_{l \geq k} \theta_{n_l} - \theta$. Due to equiboundedness of θ_{n_k} in L^1 we know that $\Theta_1 \in L^1(0, T)$. Therefore we can use Levi's Monotone Convergence Theorem for the monotone decreasing sequence Θ_k and conclude that also the second term in (75) converges to zero entailing (42).

4 | EXTENSIONS AND GENERALIZATIONS

4.1 | Nonlinear loading

In our discussion, for simplicity and not to overburden notation, we assume that the external loading acts linearly on the system, see (26). In this case, the force densities per unit volume (or area) in the reference configuration are independent of the deformation. Such loads are also called *dead loads* and are quite standard and commonly used also in nonlinear settings, see e.g. [21,34]. They describe external loads that are for instance determined by experimental devices. Nevertheless, it is possible to extend Theorem 3.2 to the case of nonlinear loading functionals, in the spirit of [18].

To allow for more general loads one replaces $\ell(t)$ by a nonlinear, bounded operator $\mathcal{L}(t) : \mathcal{Y} \to \mathbb{R}$ defined as

$$\mathcal{L}(t)[y] = \int_\Omega F(x, t, y(x)) \, dx + \int_{\Gamma_N} T(x, t, y(x)) \, d\mathcal{H}^{d-1}(x),$$

where $F : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}$ and $T : \Gamma_N \times [0,T] \times \mathbb{R}^d \to \mathbb{R}$ are bulk and traction force densities of suitable growth and regularity such that $\mathcal{L} \in W^{1,1}(0,T)$ satisfies:

$$\exists c_0^L \in \mathbb{R}, \lambda \in L^1(0,T), \delta > 0, \text{ modulus of continuity } \omega : (0,\delta) \to (0,\infty)$$

$$\forall y \in \mathcal{Y}, t, t_1, t_2 \in [0,T] \text{ s.t. } |t_1 - t_2| < \delta :$$

$$|\dot{\mathcal{L}}(t)[y]| \leq \lambda(t)(c_0^L + \|\nabla y\|_{L^q}), \tag{76}$$

$$|\dot{\mathcal{L}}(t_1)[y] - \dot{\mathcal{L}}(t_2)[y]| \leq \omega(|t_1 - t_2|)(c_0^L + \|\nabla y\|_{L^q}). \tag{77}$$

We do not enter into a detailed proof of the existence result for energetic solution with these new hypotheses, but restrict ourselves to observing that assumption (76) is needed for (C3) and condition (77) guarantees uniform continuity in the sense of (63), namely

$$\forall E > 0, \varepsilon > 0, \quad \exists \delta > 0 \text{ such that}$$

$$\mathcal{E}(0, y, P, z) \leq E, \ |t - s| < \delta \Rightarrow |\dot{\mathcal{L}}(t)[y] - \dot{\mathcal{L}}(s)[y]| < \varepsilon. \tag{78}$$

4.2 | BV-regularizations

For consistency, we fixed the condition $r_p, r_z > 1$ throughout this paper. However, without any further assumptions, one can allow $r_p = 1$ or $r_z = 1$ or both. This extension for the setting of damage in elastic materials has been introduced and dealt with by THOMAS.[52] In plasticity theory, this has already been studied in the linear and nonlinear settings; see e.g. [25] for a quasistatic evolution to the Gurtin-Anand model, at small deformations, where the BV-control of the plastic strain is compensated by an L^2-control of its curl (the macroscopic Burgers tensor). In this contribution the authors show that, as the regularization term coefficients vanish, the solutions approach a quasistatic evolution to perfect plasticity.[17] The Gurtin-Anand model has been coupled with damage in [13,14], where the regularization term for the z variable controls its H^1-norm; this is necessary to ensure lower semicontinuity of the plastic potential. In spirit of [25], the quasistatic perfectly plastic limit is achieved with the aid of a higher order regularization for the damage, which was later improved in [16].

In the nonlinear setting, it appears to be much harder to proof existence of energetic solutions without regularization terms, even for solely elastoplastic models without damage. We refer to [34] for the use of regularization terms controlling the $W^{1,r}$-norm, if $r > 1$, or the BV-norm, if $r = 1$. The main idea to deal with the case $r = 1$ is based on the contribution,[52] which can indeed also be used to cover the case of BV-regularization in our model, see discussion below. It remains an open problem to show existence of quasistatic evolutions to the model introduced in [39], which uses the term $(\operatorname{curl} P)P^\top$ as a regularization.

We can consider the general cases $r_p \geq 1$, $r_z \geq 1$; however, for simplicity of discussion we restrict to detail a bit the special case $r_p = r_z = 1$ (the cases when only one exponent is 1 is treated similarly). We define

$$Q = \mathcal{Y} \times BV(\Omega; SL(d)) \times BV(\Omega; [0,1]), \tag{79}$$

and

$$\mathcal{E}(t, y, P, z) = \int_\Omega W_{\mathrm{el}}(x, \nabla g_{\mathrm{Dir}}(t, y(x))\nabla y(x)(P(x))^{-1}, z(x)) + W_{\mathrm{h}}(x, P(x), z(x)) \, \mathrm{d}x$$

$$+ \nu |DP(\Omega)| + \mu |Dz(\Omega)| - \langle \ell(t), y \rangle, \tag{80}$$

where $|DP(\Omega)|$ and $|Dz(\Omega)|$ denote the total variation of P and z, respectively.

Theorem 4.1 (Existence of energetic solutions with BV-regularizations). *Let Q be the triple (79), $\mathcal{E}$ as in (80), and $\mathcal{D}$ as in Section 2.3. Let $q_0 = (y_0, P_0, z_0) \in S(0)$ be a stable initial state. Under the same hypotheses as in Theorem 3.2, there exists an energetic solution $q = (y, P, z) : [0,T] \to Q$ with initial condition q_0.*

The proof only changes slightly compared to the one we presented in Section 3; instead of Sobolev embedding theorems, we use that $BV(\Omega)$ compactly embeds into $L^1(\Omega)$. To proof the existence of mutual recovery sequences (C5) as in Lemma 3.5, we notice that *Step 1* of the proof can exactly be copied, whereas for *Steps 2-3* we argue as in [52].

Let $q_k \rightharpoonup q$ in Q and $\hat{q} \in Q$ be arbitrary. Set $\hat{q}_k = (\hat{y}, \hat{P}, \hat{z}_k)$, with $\hat{z}_k$ defined, like in (54), as

$$\hat{z}_k = \begin{cases} \hat{z} - \delta_k & \text{on } A_k := \{0 \le \hat{z} - \delta_k \le z_k\} \\ z_k & \text{on } B_k := \{0 \le z_k < \hat{z} - \delta_k\} \\ 0 & \text{on } C_k := \Omega \setminus (A_k \cup B_k), \end{cases}$$

where $\delta_k > 0$ is chosen such that $\delta_k \in [m_k^{1/2}, m_k^{1/4}]$ with $m_k := \max\{k^{-1}, \|z - z_k\|_{L^1}\}$. With this choice, it can be shown that $0 \le \hat{z}_k \le z_k$ a.e., $\mathcal{L}^d(B_k) + \mathcal{L}^d(C_k) \to 0$, and $\hat{z}_k \to \hat{z}$ strongly in $L^1(\Omega)$. Together with Lemma 3.4, the latter implies

$$\lim_{k \to \infty} \mathcal{D}(q_k, \hat{q}_k) = \mathcal{D}(q, \hat{q}).$$

By the same arguments as in *Step 3* of Lemma 3.5, it can be shown that

$$\lim_{k \to \infty} \mathcal{W}(t, \hat{y}, \hat{P}, \hat{z}_k) = \mathcal{W}(t, \hat{y}, \hat{P}, \hat{z}).$$

Notice that we do not need to show weak convergence of $\hat{z}_k$ to $\hat{z}$ in $BV(\Omega)$, but only strong convergence in $L^1(\Omega)$. It remains to proof the analogue of inequality (57) in the BV-setting, namely

$$\limsup_{k \to \infty} |D\hat{z}_k|(\Omega) - |Dz_k|(\Omega) \le |D\hat{z}|(\Omega) - |Dz|(\Omega).$$

This result is the main novelty of [52] compared to the Sobolev-setting.[50,51] Its proof uses BV-decomposition techniques and a careful study of the different BV-traces resulting from the case distinction in the definition of $\hat{z}_k$. We refer to [52, Lemma 2.13] for the details.

ACKNOWLEDGEMENTS

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