Bachelor's Thesis

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F3

Faculty of Electrical Engineering
Department of Electromagnetic Field

Local Perturbation in Method of Moments

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Field of study: Electronics and Communications

ZADÁNÍ BAKALÁŘSKÉ PRÁCE

I. OSOBNÍ A STUDIJNÍ ÚDAJE

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Studijní program: **Elektronika a komunikace**

II. ÚDAJE K BAKALÁŘSKÉ PRÁCI

Název bakalářské práce:

Lokální perturbace v metodě momentů

Název bakalářské práce anglicky:

Local perturbation in method of moments

Pokyny pro vypracování:

Seznamte se s možnostmi blokové inverze matice a využitelnosti této techniky v rámci metody momentů. V prostředí MATLAB implementujte Sherman-Morrison-Woodbury identitu a porovnejte rychlost blokové a konvenční inverze. Následně implementujte inverzi matice v případě drobných lokálních úprav elektrického obvodu, popsané pomocí impedanční či admitanční matice. Uvažte vliv velikosti matice a různý počet úprav a jejich vliv na efektivitu inverze. Zaveďte topologickou senzitivitu pro vybraný elektrický obvod a odvoďte vztah pro její výpočet při elementární změně topologie obvodu. Implementujte gradientní algoritmus, který umožní topologickou optimalizaci obvodů s ohledem na vybrané kritérium. Pro vybranou počáteční topologii proveďte optimalizaci a zhodnoťte její výsledky.
Specialista: doc. Ing. Miloslav Čapek, Ph.D.

Seznam doporučené literatury:

[1] Hager, W. W.: Updating the Inverse of a Matrix, SIAM, vol. 31, no. 2, pp. 221-239, June 1989.
[2] Capek, M., Jelinek, L., Gustafsson, M.: Shape Synthesis Based on Topology Sensitivity, IEEE Trans. AP, submitted, available online: https://arxiv.org/abs/1808.02479
[3] Wing, O.: Classical Circuit Theory, Springer, 2008.
[4] Chen, X., Gu, Ch., Zhang, Y., Mittra, R.: Analysis of Partial Geometry Modification Problems Using the Partitioned-Inverse Formula and Sherman–Morrison–Woodbury Formula-Based Method, IEEE Trans. AP, Vol. 66, No. 10, pp. 5425-5431, October 2018.
[5] MATLAB, Mathworks. (2019)

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Datum zadání bakalářské práce: **08.02.2019** Termín odevzdání bakalářské práce: **24.05.2019**

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III. PŘEVZETÍ ZADÁNÍ

Student bere na vědomí, že je povinen vypracovat bakalářskou práci samostatně, bez cizí pomoci, s výjimkou poskytnutých konzultací.
Seznam použité literatury, jiných pramenů a jmen konzultantů je třeba uvést v bakalářské práci.

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Datum převzetí zadání

Podpis studenta

Acknowledgements

I would like to express my deep gratitude to Lukas Jelinek and Miloslav Capek, my thesis supervisors, for their patient guidance, enthusiastic encouragement and useful criticism.

I also thank my parents for the unceasing encouragement, support and attention. I am also grateful to my partner who supported me through this venture.

Declaration

I hereby declare that I have completed this thesis individually and that I have listed all used information sources in accordance with "Metodický pokyn o dodržování etických principů při přípravě vysokoškolských závěrečných prací".

In Prague, 22. May 2019

Abstract

An efficient method for evaluating the sensitivity of an electric circuit to topological changes is proposed. Nodal voltages are used as degrees of freedom. Impedance matrix of a circuit after an elementary perturbation is derived through the inversion-free formulas. This leads to a computationally efficient gradient optimization procedure.

The goal of this bachelor's thesis is to demonstrate the feasibility of the proposed optimization procedure. The validity is verified using two examples. It is shown that implemented optimization algorithm can be used as a local step in global optimizers. Developed procedures serve as a proof of concept and form the basis for future work in circuit optimization.

Keywords: Circuit optimization, Sherman-Morrison-Woodbury formulas, topology sensitivity.

Supervisor: doc. Ing. Lukáš Jelínek, Ph.D.

Abstrakt

Práce představuje efektivní metodu pro vyhodnocení citlivosti elektrického obvodu vůči zvolenému parametru při změně jeho topologie. Napětí v uzlu reprezentuje jeden stupeň volnosti obvodu. Impedanční matice modifikovaného obvodu je odvozena pomocí blokové inverze matice. To umožňuje implementaci výpočetně efektivního optimalizačního algoritmu.

Cílem bakalářské práce je ukázat proveditelnost navrhovaného optimalizačního postupu. Správnost implementace je ověřena na dvou příkladech. Vyvinutý optimalizační algoritmus lze použít jako lokální krok v globálních optimalizátorech. Vyvinuté metody slouží jako ověření koncepce a jsou podkladem pro budoucí práci v optimalizaci obvodů.

Klíčová slova: Optimalizace obvodů, Sherman-Morrison-Woodbury identita, topologická citlivost.

Překlad názvu: Lokální perturbace v metodě momentů

Contents

Figures

Tables

Chapter 1

Introduction and motivation

Throughout the history of electrical engineering, essential laws of circuit analysis have been developed. Circuit analysis is the process of finding electrical quantities for a given topology. The problem of the circuit analysis has already been mastered, and many advanced automated circuit simulators do exist. On the other hand, circuit synthesis is the process of finding a circuit shape with the predetermined electrical quantities. Although, there is a noticeable development of topology optimization, *e.g.*, of optical circuits [1] or conductors in circuit [2], the process of shape synthesis is far from being mastered, since we encounter serious obstacles. An objective function defined by the user cannot be set without restrictions. Furthermore, a solution is non-unique. Generally, shape synthesis algorithms have high computational cost and suffer from the curse of dimensionality [3].

In the past few years, a procedure is sought for making shape synthesis more efficient. This thesis is highly motivated by a few papers [4], [5], [6], in which efficient way of antenna synthesis is described. The goal of this thesis is to develop a topology optimization scheme for lumped element circuits with the same algorithmic properties. The thesis proposes efficient incorporation of the topology sensitivity evaluation into an optimization scheme. The key step is the utilization of inversion-free formulas, which have shown their strengths in the other applications [7], [8] as well.

The gradient-based algorithm is implemented and used to optimize a topology of an arbitrarily sized grid containing lumped elements with regards to predetermined behaviour.

Chapter 2

Method of moments

Method of moments (MoM) is a technique frequently used in engineering to solve electromagnetic problems, *e.g.*, radiation and scattering problems. The scheme of computation within the MoM paradigm can be found in [9] or with examples in [10]. The MoM generally recasts an integro-differential equation into a system of linear equations

$$\mathbf{Z}\mathbf{I} = \mathbf{V}, \tag{2.1}$$

where $\mathbf{Z}$ is an impedance matrix [11], $\mathbf{V}$ is a known voltage feeding vector, and $\mathbf{I}$ is an unknown current vector (expressed in terms of expansion coefficients [9]). In order to evaluate current $\mathbf{I}$, an inversion of impedance matrix

$$\mathbf{I} = \mathbf{Z}^{-1}\mathbf{V} = \mathbf{Y}\mathbf{V}, \tag{2.2}$$

is needed, where $\mathbf{Y}$ is called admittance matrix. Hence, the impedance matrix $\mathbf{Z}$ presents the electromagnetic behaviour of an actual shape.

A system of linear equations (2.1) can also describe an electrical circuit. The behaviour of an actual shape/topology of a circuit is reflected in the impedance matrix.

2.1 Shape synthesis within MoM paradigm

Shape synthesis attempts to extract a particular shape with respect to the desired behaviour. An important part of the synthesis problem is the understand-

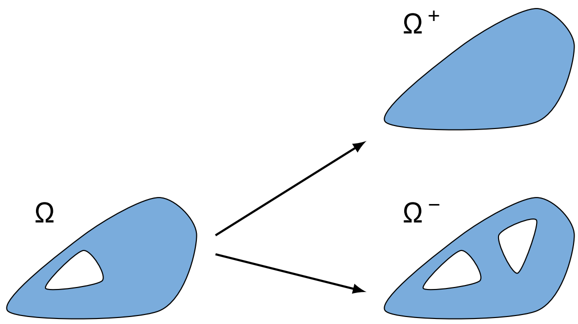


Figure 2.1: Comparison of possible modification of the initial structure Ω. Either subtracting or adding small part is considered.

ing of how the observed metric changes after small structural modifications are performed. Figure 2.1 presents possible topology modification of the original shape Ω. Employing the conventional MoM for this task is time-consuming since the set of complex computations, *e.g.*, construction of the impedance matrix $\mathbf{Z}$, must be repeated each time the structure is modified. Furthermore, equation (2.2) is usually solved by LU-decomposition algorithm [12], which has $O(N^3)$ computational complexity, where N is the number of basis function of the initial structure. Computational complexity is further increased to $O(N^4)$ if each basis function is assumed to be removed or added. This evaluation has to be repeated iteratively while removing or adding basis functions, *i.e.*, degrees of freedom (DOFs). Consequently, the optimization procedure is proportional to $O(N^5)$ which is a considerable computational burden.

The elementary topology modification can be also defined within circuit synthesis with primary quantity being the voltage at each node. This application is described in chapter 3.

Since the inversion of a matrix is a time-consuming operation, an inversion-free formula is sought for. The key step is the employment of the block matrix inversion and Sherman-Morisson-Woodbury formula [13]. According to the block matrix inversion formula, the solutions for all possible structures can be derived by reusing the inverse of the impedance matrix of the initial shape Ω. These inversion-free formulas are presented in the next section.

2.2 Block matrix inversion

Prior to the introduction of block matrix inversion (also called partitioned-inverse formula), the submatrix and block matrix are defined as follows:

Definition 2.1. A **submatrix** $\mathbf{A}[\alpha, \beta]$ of matrix $\mathbf{A}$ is a matrix with indices α and β being mappings on rows and columns of matrix $\mathbf{A}$. The set of row indices $\{1, \dots, m\}$ of matrix $\mathbf{A}$ is partitioned into the subsets $\alpha_1, \dots, \alpha_r$, so that $\alpha_i \cap \alpha_j = \emptyset$, for all $i \neq j, 1 \leq i, j \leq r$, and $\alpha_1 \cup \cdots \cup \alpha_r = \{1, \dots, m\}$. Similarly, column indices $\{1, \dots, n\}$ are partitioned into the subsets $\beta_1, \dots, \beta_s$.

Definition 2.2. A **block matrix** is a matrix that is partitioned into the submatrices $\mathbf{A}[\alpha_i, \beta_j]$ with the row and column indices partitioned into subsets, *i.e.*, $\alpha_1 = \{1, \dots, i_1\}$, $\alpha_2 = \{i_1 + 1, \dots, i_2\}$, etc. For simplicity, we will write $\mathbf{A}_{ij} = \mathbf{A}[\alpha_i, \beta_j]$.

Matrix operations can be performed block-wise. Some basic properties of block matrices and operations among them can be found in [14] and [15]. Matrix inversion is an essential operation for this thesis, hence it can be useful to know the inversion of a partitioned matrix. For simplicity, let matrix $\mathbf{A} \in \mathsf{F}^{n \times n}$ be partitioned as

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}, \tag{2.3}$$

where $\mathbf{A}_{ii} \in \mathsf{F}^{n_i \times n_i}$, $i = 1, 2$ and $n_1 + n_2 = n$, are submatrices. A useful formula for the corresponding blocks of the partitioned matrix representation of $\mathbf{A}^{-1}$ [16] is

$$\begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}_{11}^{-1} + \mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{S}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{S}^{-1} \\ -\mathbf{S}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \mathbf{S}^{-1} \end{pmatrix}, \tag{2.4}$$

where $\mathbf{S} = \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}$ is so-called Schur complement [16]. If and only if both the matrix $\mathbf{A}_{11}$ and the Schur complement $\mathbf{S}$ are nonsingular, then $\mathbf{A}$ is nonsingular [17].

Equation (2.4) provides a recursive algorithm involving two inverses of $n_1 \times n_1$ and $n_2 \times n_2$ matrices ($\mathbf{A}_{11}$ and S) and four multiplications. It is proven, that matrix inversion is equivalent to matrix multiplication [18], *i.e.*, if $t(n)$ denotes the time of multiplication of two $n \times n$ matrices, then the time to invert an $n \times n$ matrix is $O(t(n))$. The computational complexity of mentioned algorithm is $O(n^{2.807})$, [19]. Table 2.1 summarizes matrix multiplication algorithms and their computational complexities.

If matrix $\mathbf{A}_{11}^{-1}$ is known and the three other blocks are small in size, (2.4) constitutes efficient formula for matrix inversion as compared to full inversion of matrix $\mathbf{A}$.

Matrix multiplication algorithm	Year	Computational complexity
Naive algorithm	1950	$O(n^3)$
Strassen's algorithm	1969	$O(n^{2.807})$
Coppersmith-Winograd algorithm	1990	$O(n^{2.376})$
Gall's algorithm	2014	$O(n^{2.373})$

Table 2.1: Computational complexity of matrix multiplication algorithms throughout the past decades.

2.2.1 Sherman-Morrison-Woodbury formula

The Sherman-Morrison-Woodbury formula (SWM) [13], also known as Woodbury matrix identity, expresses the inversion of a matrix after a small rank modification in terms of the inversion of the original matrix. This modification formula comes from studies of block matrices and can be derived from the block matrix inversion [16], which is presented in the previous section 2.2.

Theorem 2.3. *Assume that a nonsingular matrix* $\mathbf{A}\quad \mathsf{F}^{n\times n}$ *has a known inverse* $\mathbf{A}^{-1}$ *and consider matrix* $\mathbf{C} = \mathbf{A} + \mathbf{EBF}$*, in which* $\mathbf{E}$ *is* $n \times r$*,* $\mathbf{F}$ *is* $r \times n$*, and* $\mathbf{B}$ *is* $r \times r$ *and nonsingular. If* $\mathbf{C}$ *and* $\mathbf{B}^{-1} + \mathbf{FA}^{-1}\mathbf{E}$ *are nonsingular, then*

$$\mathbf{C}^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{E}(\mathbf{B}^{-1} + \mathbf{FA}^{-1}\mathbf{E})^{-1}\mathbf{FA}^{-1}. \tag{2.5}$$

If $r \quad n$, then $\mathbf{B}$ and $\mathbf{B}^{-1} + \mathbf{FA}^{-1}\mathbf{E}$ are much faster to invert than $\mathbf{C}$. As an example, let us, consider a modification of a $n \times n$ matrix, in which a $r \times r$ block is modified. Final matrix is then multiplied by a correctly sized column vector. Figure 2.2 shows performance of the formula (2.5) compared to the built-in MATLAB functions $inv()$ and $mldivide$ [20]. It is apparent, that SMW formula is e□cient for low rank correction of the original matrix.

As a special case of major importance for this thesis, if $\mathbf{u}, \mathbf{v} \quad \mathsf{F}^n$ are column vectors, $\mathbf{E} = \mathbf{u}, \mathbf{F} = \mathbf{v}^\mathsf{T}$ and $\mathbf{B} = b$ is a scalar, then (2.5) becomes a formula for the inversion of a matrix after it is modified by a rank 1 correction:

$$(\mathbf{A} + \mathbf{u}b\mathbf{v}^\mathsf{T})^{-1} = \mathbf{A}^{-1} - \frac{b\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^\mathsf{T}\mathbf{A}^{-1}}{1 + b\mathbf{v}^\mathsf{T}\mathbf{A}^{-1}\mathbf{u}}. \tag{2.6}$$

Equation (2.6) is called Sherman-Morrison (SM) formula [13]. Various applications of the SMW formulas to statistics, networks, asymptotic analysis, optimization and partial di□erential equations are summarized in [13].

Let us consider a specific rank 1 modification of a matrix $\mathbf{A}$, where $b \to \infty$. In this case the inversion formula reads

$$\lim_{b\to\infty} (\mathbf{A} + \mathbf{u}b\mathbf{v}^\mathsf{T})^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^\mathsf{T}\mathbf{A}^{-1}}{\mathbf{v}^\mathsf{T}\mathbf{A}^{-1}\mathbf{u}}. \tag{2.7}$$

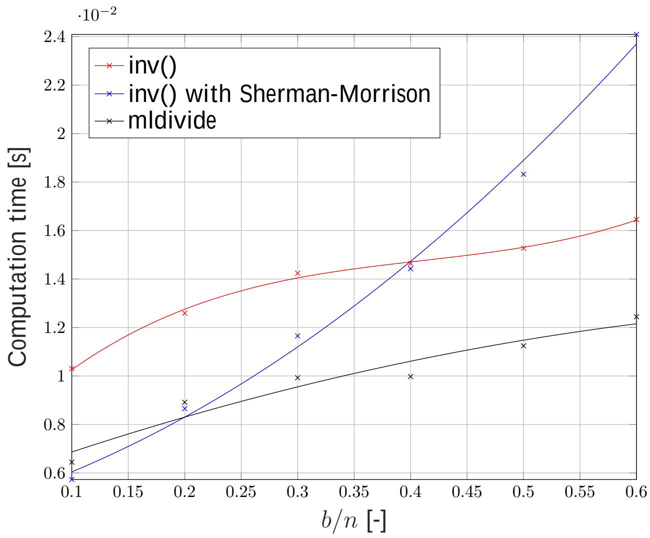


Figure 2.2: Performance comparison of the SMW formula and the MATLAB built-in functions in dependence on the size of a modification. The size of a perturbation is denoted as b, while the size of a matrix is n.

If vectors $\mathbf{u} = \mathbf{v}$ are simple indexing vectors, the matrix multiplications in (2.7) are reduced to computational inexpensive indexing. The corresponding column and row given by the indexing vector is zeroed in a final matrix, which allows to dynamically change the size of a matrix (zeroed columns and rows are discarded).

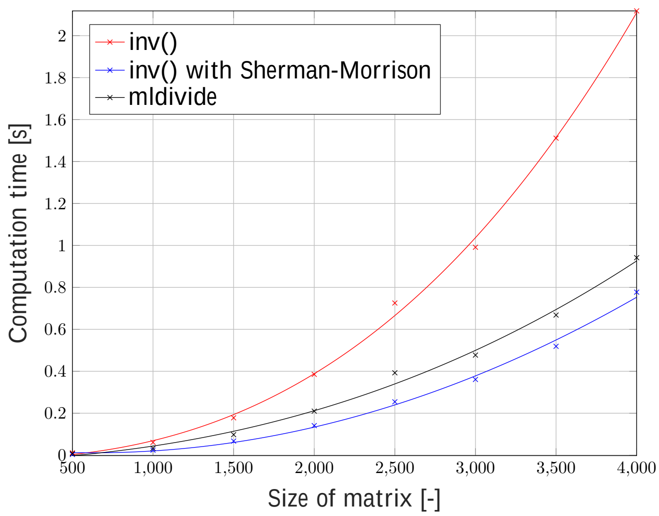


Figure 2.3: Performance comparison of SM formula and MATLAB built-in functions. Modified matrix is further multiplied by a dense vector.

Figure 2.3 shows the performance of the SM formula and the MATLAB built-in functions $inv()$ and $mldivide$, where the resulting matrix is further

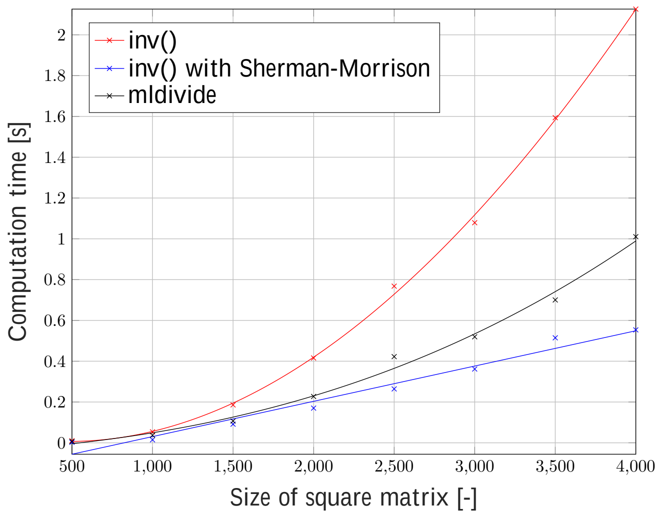


Figure 2.4: Performance comparison of SM formula and MATLAB built-in functions. Modified matrix is further multiplied by a vector with only one non-zero entry.

multiplied by a dense vector. The employment of the SM formula reduces computational complexity from $O(N^3)$ ($inv()$) to $O(N^2)$. Since $mldivide$ is able to take advantage of symmetries in the problem by dispatching to an appropriate solver [20], the computation time is reduced.

Furthermore, A resulting matrix is multiplied by a vector with only one non-zero entry. Employing the SM formula further reduces the computational complexity to $O(N)$. The performance is shown in figure 2.4.

As will be shown in the next chapter, an elementary modification in a circuit topology with the same algorithmic properties can be found.

Chapter 3

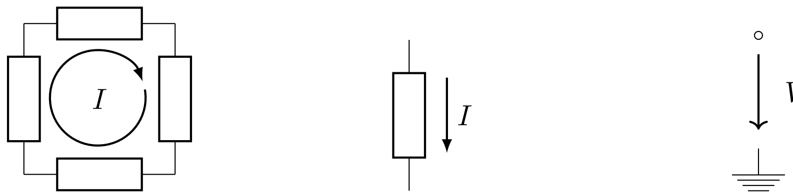
Circuit analysis

Let us consider an electromagnetic problem, *e.g.*, antenna design. The initial shape Ω is discretized into triangles $\Omega \to \Omega_T$, see figure 3.1. Hence, the system is reduced to a system having a finite number of degrees-of-freedom (DOFs) and all calculations are performed over the discretized domain, *i.e.*, we deal solely with the impedance matrix and corresponding current and voltage column vectors. A removal or an addition of a set of DOFs naturally modifies the impedance and admittance matrix, *i.e.*, the behaviour of the antenna changes. Using inversion-free formulas, see section 2.2, the impedance matrix of perturbed system $\overline{\mathbf{Z}}$ is acquired [4], [5].

(a) : Original structure **(b) :** Discretized structure

Figure 3.1: Illustration of discretization of a continuous shape (3.1a). Electromagnetic behaviour of a structure is approximated using its discretized model (3.1b).

The scheme mentioned above can readily be applied on a trivial structure, *e.g.*, an electrical circuit. Consequently, the shape synthesis of circuits becomes analogue problem to the antenna design [4], [5], [6]. The question is: What shall be used to represent one DOF in a circuit?



(a) : Rectangle cell. **(b) :** Single impedance. **(c) :** Voltage node.

Figure 3.2: Possible representations of a degree of freedom for a lumped element circuit.

When studying shape optimization it is often desirable to know how the objective function p alters, when the structure is modified, *e.g.*, removal or addition of DOF is performed. In electrical engineering, this function can be a properly defined metric such as the input impedance of a circuit. The objective function p will be a function of the prime quantity, *e.g.*, in the node-oriented system it is explicitly dependent on the voltage in the each node.

In this chapter, a descriptive matrix for a circuit is chosen together with a suitable modification among the chosen description of a circuit. It is shown that the selected modification allows the employment of the inversion-free formulas presented in section 2.2 if just one DOF is to be removed or added. The following section presents topology sensitivity $\tau(p, \Omega_T)$ of the objective function p for the circuit topology Ω and define its evaluation as a simple matrix product.

3.1 Impedance or admittance matrix?

For purposes of circuit analysis, a descriptive matrix for a circuit needs to be defined together with an elementary DOF. Also, we have to keep in mind that suitable elementary modification needs to be defined afterwards. We consider three possible choices:

1. rectangle cell in figure 3.2a,
2. single impedance in figure 3.2b,
3. voltage node in figure 3.2c.

First, let us consider a rectangular cell consisting of four lumped elements with a mesh current I. Applying mesh current law and Kirchho�'s voltage law (KVL) [21] on a circuit composed of multiple rectangle cells, leads to the impedance matrix $\mathbf{Z}$. Mesh current method leads to a small amount of equation, *i.e.*, the small size of the impedance matrix $\mathbf{Z}$. But the method cannot evaluate circuits with current sources.

Second, consider a single lumped element with a current I. Application of the Kirchhoff's laws clearly leads to the impedance matrix $\mathbf{Z}$. With this approach the analysis of any circuit is possible. This approach is similar to the conventional MoM applied in the antenna design. But the method leads to a big amount of equations.

The last consideration belongs to a voltage node with a voltage V. Assume that nodes are connected with an arbitrary admittance Y. Applying branch current method with Kirchhoff's current law (KCL) [21] on a circuit with N nodes leads to the $N \times N$ admittance matrix $\mathbf{Y}$. The assembly algorithm is relatively straightforward to implement, but the method cannot analyze circuits with a voltage source. A modified version of this method is used in many circuits simulators [21]. For purposes of this thesis, the node oriented system has been chosen since the construction of the admittance matrix is straightforward.

3.2 Construction of the admittance matrix

Consider a circuit with N nodes. In general, node-voltage equations can be written in matrix form. For any node m, KCL states

$$\sum_{n=m} Y_{nm}(V_m - V_n) = 0, \tag{3.1}$$

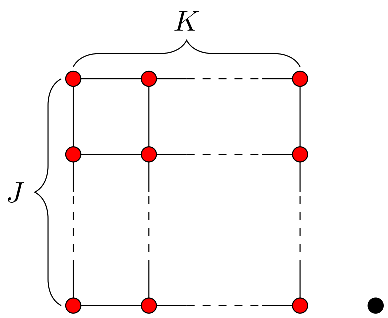
where Y_{nm} is the sum of the admittances between nodes m and n, and V_m is the voltage at m-th node. The equation (3.1) further implies

$$0 = \sum_{n=m} Y_{nm}V_m - \sum_{n=m} Y_{nm}V_n = Y_{mm}V_m - \sum_{n=m} Y_{nm}V_n, \tag{3.2}$$

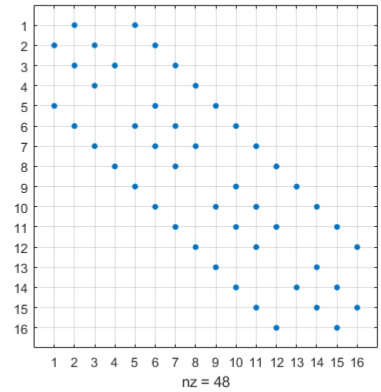
where Y_{mm} is the sum of all admittances connected to node m. The first term contributes linearly to the node voltage via Y_{mm}, while the second term contributes to each node n connected to the m node. Assuming a current source I_m is connected to the m-th node, (3.2) is generalized to

$$I_m = Y_{mm}V_m - \sum_{n=m} Y_{nm}V_n, \tag{3.3}$$

which is a matrix equation in the form $\mathbf{YV} = \mathbf{I}$. The matrix $\mathbf{Y}$ is singular. The reference (ground) node is considered to be the last node, *i.e.*, $V_N = 0$. The admittance matrix $\mathbf{Y}$ becomes non-singular after discarding the corresponding column and the row. The rest of the admittance matrix $\mathbf{Y}$ remains the same.



(a) : A graph composed of $N = KJ$ voltage nodes (red) and one ground node (black).



(b) : Sparsity pattern of the adjacency matrix $\mathbf{A}$ of a grid graph.

Figure 3.3: A grid circuit represented as a graph (left) and sparsity pattern of the adjacency matrix $\mathbf{A}$ of a 4×4 grid graph (right).

Furthermore, let us assume a grid circuit of an arbitrary size. Any circuit can be considered as an graph $G = (V, E)$ [22]. Figure 3.3a presents a grid graph G with $N = JK + 1$ vertices and $J(K-1) + (J-1)K$ edges. The admittance between two nodes is considered to be a weight of the corresponding edge, *i.e.*, $w_{ij} = Y_{ij}$. The adjacency matrix A of the undirected weighted graph G is defined as

$$A_{ij} = \begin{cases} w_{ij} & \text{if there is some edge } \{v_i, v_j\} \quad E, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

For every node $v_i \quad V$, the degree $d(v_i)$ of the node v_i is the sum of weights of the edges connected to node v_i

$$d(v_i) = \sum_{j=1}^{N} w_{ij}. \tag{3.5}$$

The degree matrix $\mathbf{D}$ is defined as

$$\mathbf{D} = \mathrm{diag}(d(v_1), \cdots, d(v_N)). \tag{3.6}$$

Since the uniform grid is considered as an initial topology, the maximum number of connections to one node is limited to four. Then, the adjacency matrix $\mathbf{A}$ of the graph is sparse and has a specific form, see figure 3.3b. Notice that the matrix entries above diagonal and further from diagonal present the horizontal and vertical connections in the graph, respectively.

Consequently, the admittance matrix of an arbitrarily sized grid circuit can be constructed as follows

$$\mathbf{Y} = \mathbf{D} - \mathbf{A}. \tag{3.7}$$

Diagonal entries of the admittance matrix $\mathbf{Y}$ are equal to the total sum of all admittances connected to the corresponding node. Non-diagonal entries show an interaction between two nodes, *i.e.*, the value of admittance between two nodes. Furthermore, the connection of an impedance between a node and ground node modifies only corresponding diagonal entry of the admittance matrix $\mathbf{Y}$.

This reduction to the graph representation allows the effective construction of an admittance matrix for an arbitrarily sized grid network. Furthermore, it is possible to distinguish horizontally and vertically placed lumped elements.

Impedance matrix $\mathbf{Z}$ fully describes a circuit and with appropriately defined feeding current vector $\mathbf{I}$, voltage vector $\mathbf{V}$ is computed. As compared to the antenna design [4], [5], [6], where admittance matrix $\mathbf{Y}$ and appropriately defined feeding voltage vector $\mathbf{V}$ are used to evaluate current vector $\mathbf{I}$. Consequently, algorithmic properties are the same.

3.3 The elementary perturbation in a circuit topology

In this section, the possible elementary modifications in a node-oriented grid network are considered.

3.3.1 Change of the admittance between two nodes

The first relevant elementary modification is a replacement of the admittance Y_{mn} by the admittance $\overline{Y}_{mn}$ between the m-th and the n-th node [13], see figure 3.4. Let us suppose that the admittance matrix $\mathbf{Y}$ of the system

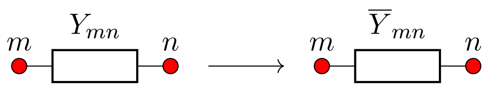


Figure 3.4: Modification is performed by changing the admittance between nodes.

was constructed and node-voltages vector $\mathbf{V}$ computed, *i.e.*, $\mathbf{V} = \mathbf{Y}^{-1}\mathbf{I} = \mathbf{ZI}$. Afterwards, the network is perturbed appropriately. The admittance matrix $\mathbf{Y}^{\text{new}}$ of the altered network is expressed as

$$\mathbf{Y}^{\text{new}} = \mathbf{Y} - \mathbf{UDU}^{\mathsf{T}}, \tag{3.8}$$

where $\mathbf{U}$ is a projection matrix, which consists of the columns of the identity matrix comparable to the modified nodes, and $\mathbf{D}$ is

$$\mathbf{D} = d\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad d = \overline{Y}_{mn} - Y_{mn}. \tag{3.9}$$

Consider the modification between the first and second node. The admittance Y_{12} is to be removed. This removal corresponds to the definition of a lumped element

$$Y_0 = \frac{1}{Z_\infty}, \tag{3.10}$$

where Z_∞ is the impedance with infinite value, and Y_0 is the admittance with zero value. Then, the form $\mathbf{UDU}^{\mathsf{T}}$ is given by

$$\mathbf{UDU}^{\mathsf{T}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \end{pmatrix} \begin{pmatrix} -Y_{12} & Y_{12} \\ Y_{12} & -Y_{12} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \end{pmatrix}. \tag{3.11}$$

The example above is called node-oriented modification [23]. Note that the original admittance matrix of the network is modified at four positions and the corresponding impedance matrix $\mathbf{Z}^{\text{new}}$ of the modified structure is a $N \times N$ matrix (N is the number of nodes). Consequently, this type of modification will not dynamically change the size of the impedance matrix $\mathbf{Z}$.

3.3.2 Connection of a node to the ground node

The second possible elementary modification is the connection of m-th node to the reference (ground) node. We claim that the connection to the ground node corresponds to the introduction of a lumped element

$$Y_\infty = \frac{1}{Z_0}, \tag{3.12}$$

where Y_∞ is the admittance with infinity value and Z_0 is the impedance approaching zero value. Assume that the set Σ of the nodes is connected to the ground node. Then, the correction term $\mathbf{Y}_{\text{C}}$ is given by

$$\mathbf{Y}_{\text{C}} = \mathbf{C}_\Sigma Y_\infty \mathbf{C}_\Sigma^{\mathsf{T}}, \tag{3.13}$$

where the matrix $\mathbf{C}_\Sigma$ is defined as

$$C_{nn,\Sigma} = \begin{cases} 1 & n \quad \Sigma, \\ 0 & \text{otherwise}, \end{cases} \tag{3.14}$$

and where the columns containing only zeros are discarded. Consequently, the admittance matrix $\overline{\mathbf{Y}}$ becomes

$$\overline{\mathbf{Y}} = \mathbf{Y} + \mathbf{C}_\Sigma Y_\infty \mathbf{C}_\Sigma^\mathsf{T}. \tag{3.15}$$

This "removal" procedure for $n \quad \Sigma = \{m\}$ is illustrated in figure 3.5. The number of DOF of the system is reduced by one, since the connection to the ground node zeros the voltage V_m, which leads to the impedance matrix $\mathbf{Z} = \mathbf{Y}^{-1}$ in which the m-th row and column have been zeroed, see figure 3.5. All other entries are modified according to the formula further described. This is a very favourable feature to accelerate all the underlying matrix algebra since it is basically dependent on N. The inversion of $\overline{\mathbf{Y}}$ can

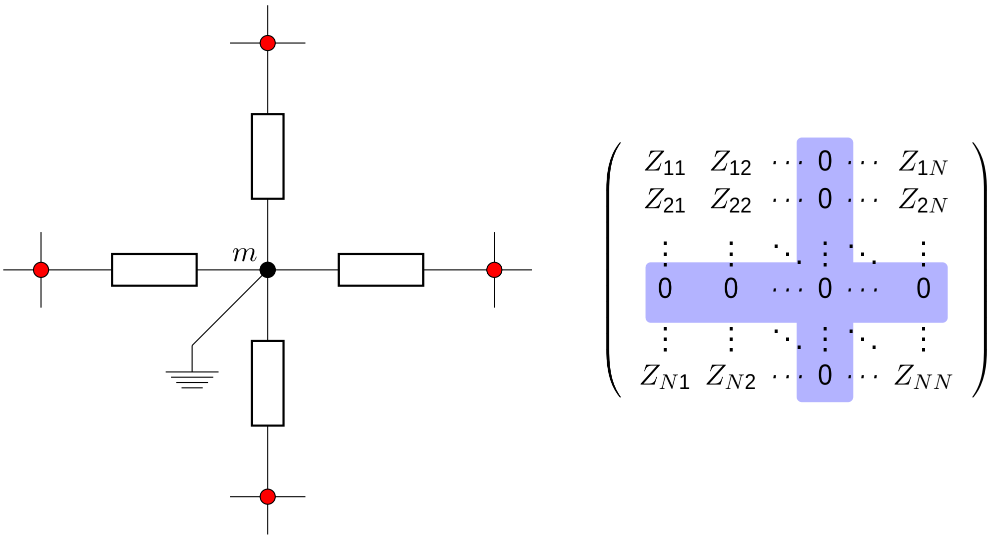


Figure 3.5: Illustration of connection the m-th node to the ground with corresponding modification of impedance matrix $\mathbf{Z}$.

be evaluated by the SMW formula, described in section 2.2.1. Its application to (3.15) gives

$$\overline{\mathbf{Z}} = \overline{\mathbf{Y}}^{-1} = \mathbf{Y}^{-1} - \mathbf{Y}^{-1}\mathbf{C}\left(\frac{1}{Y_\infty}\mathbf{1}_D + \mathbf{C}^\mathsf{T}\mathbf{Y}^{-1}\mathbf{C}\right)^{-1}\mathbf{C}^\mathsf{T}\mathbf{Y}^{-1}, \tag{3.16}$$

where $\mathbf{1}_D$ is an $D \times D$ identity matrix. Using the limit $Y_\infty \rightarrow \infty$ and $\mathbf{Z} = \mathbf{Y}^{-1}$, (3.16) is simplified to

$$\overline{\mathbf{Z}} = \mathbf{Z} - \mathbf{Z}\mathbf{C}\left(\mathbf{C}^\mathsf{T}\mathbf{Z}\mathbf{C}\right)^{-1}\mathbf{C}^\mathsf{T}\mathbf{Z}. \tag{3.17}$$

Although a matrix inversion is still required, the impedance matrix $\mathbf{Z}$ is calculated only once at the beginning and an inversion of a $D \times D$ matrix is performed individually. The outer matrix multiplication might be implemented as computationally cheap indexing, *e.g.*, in MATLAB [20], (matrix $\mathbf{C}$ contains only a single non-zero entry). Let us consider a single node ($D = 1$) to be removed. The formula (3.17) can be simplified to

$$\overline{\mathbf{Z}} = \mathbf{Z} - \frac{\mathbf{z}_n \mathbf{z}_n^{\mathsf{T}}}{Z_{nn}}, \tag{3.18}$$

where Z_{nn} is the n-th diagonal element of the impedance matrix $\mathbf{Z}$, $\mathbf{z}_n$ is the n-th column of impedance matrix $\mathbf{Z}$. The formula (3.18) modifies elements in the impedance matrix $\overline{\mathbf{Z}}$ so that the voltage in the n-th node is zeroed and all interaction with this node is eliminated, *i.e.*, the n-th row and the n-th column of impedance matrix $\overline{\mathbf{Z}}$ is zeroed.

Finally, let us consider an initial grid circuit (with N nodes) fed by a single current source at f-th node

$$\mathbf{I}_f = [0 \quad \cdots \quad I_0 \quad \cdots \quad 0]^{\mathsf{T}}, \tag{3.19}$$

where I_0 is the feeding current. This current source generates a corresponding voltage vector

$$\mathbf{V}_f = \mathbf{Z}\mathbf{I}_f. \tag{3.20}$$

Assume that n-th node is connected to ground. The perturbed voltage vector $\overline{\boldsymbol{V}}_{fn}$ is computed using equation (3.18) and (3.20)

$$\overline{\mathbf{V}}_{fn} = \overline{\mathbf{Z}}\mathbf{I}_f = \left(\mathbf{Z} - \frac{\mathbf{z}_n \mathbf{z}^{\mathsf{T}}{}_n}{Z_{nn}}\right)\mathbf{I}_f = \mathbf{z}_f I_0 - \frac{Z_{fn}}{Z_{nn}}\mathbf{z}_n I_0 = \mathbf{V}_f + \xi_{fn}\mathbf{V}_n, \tag{3.21}$$

where

$$\xi_{ij} = -\frac{Z_{ij}}{Z_{jj}}, \tag{3.22}$$

and where subscript denotes the position in the impedance matrix. Equation (3.21) shows that node removal, *i.e.*, connecting to the ground node, is equivalent to a two-node feeding via

$$\mathbf{I}_f = [0 \quad \cdots \quad I_0 \quad \cdots \quad \xi_{fn} \quad \cdots \quad 0]^{\mathsf{T}}, \tag{3.23}$$

which forces zero voltage on the n-th node.

Equation (3.21) allows the alignment of removals of all nodes into a matrix

$$\overline{\mathbf{V}}_{f\Gamma^-} = [\mathbf{V}_f + \xi_{f1}\mathbf{V}_1 \quad \cdots \quad \mathbf{V}_f + \xi_{fN}\mathbf{V}_N], \tag{3.24}$$

where $\Gamma^- = \{1, \cdots, f-1, f+1, \cdots, N\}$ denotes a set of nodes to be removed one by one. Since $\overline{\mathbf{V}}_{ff}$ is identically zero, it is not a part of the set. For compact notation, we will only use $\mathbf{V}_{f\Gamma^-} \equiv \overline{\mathbf{V}}_{f\Gamma^-}$.

3.3.3 Disconnection of a node from the ground node

The disconnection of a node from the ground node, *i.e.*, node addition, is introduced by further applying the block-wise matrix inversion. Consider a set of already removed nodes denoted as Σ^-. Then, the information about these nodes is included in the admittance matrix $\mathbf{Y}_{\text{init}}$ for the initial circuit topology, which has been preserved. Assume that $\mathbf{y}_a$ is a column vector, which corresponds to the a-th node, which is to be added. Hence, the new admittance matrix is

$$\mathbf{Y}^{\text{NEW}} = \begin{pmatrix} \mathbf{Y}^{\text{OLD}} & \mathbf{y}_a \\ \mathbf{y}_a^{\text{T}} & Y_{aa} \end{pmatrix}, \tag{3.25}$$

where $\mathbf{Y}^{\text{OLD}}$ is admittance matrix for actual topology and Y_{aa} is a-th diagonal term in initial admittance matrix. The goal is to deal only with the impedance matrix $\mathbf{Z}$. Hence, the inversion-free formula is employed. According to the block-wise matrix inversion, see section 2.2, the updated impedance matrix of the circuit is defined as

$$\mathbf{Z}^{\text{NEW}} = \left(\mathbf{Y}^{\text{NEW}}\right)^{-1} = \frac{1}{y_a}\begin{pmatrix} y_a\mathbf{Z}^{\text{OLD}} + \mathbf{Z}^{\text{OLD}}\mathbf{y}_a\mathbf{x}_a & -\mathbf{Z}^{\text{OLD}}\mathbf{y}_a \\ -\mathbf{x}_a & 1 \end{pmatrix}, \tag{3.26}$$

where

$$\begin{aligned} \mathbf{x}_a &= \mathbf{y}_a^{\text{T}}\mathbf{Z}^{\text{OLD}}, \\ y_a &= Y_{aa} - \mathbf{x}_a\mathbf{y}_a. \end{aligned} \tag{3.27}$$

The block-wise matrix formula demands that the node added must be the last one. Consequently, sorted impedance matrix is

$$\mathbf{Z}^{\text{NEW}}_{\text{sort}} = \mathbf{C}_a^{\text{T}}\mathbf{Z}^{\text{NEW}}\mathbf{C}_a, \tag{3.28}$$

where $\mathbf{C}_a$ is a permutation matrix defined as

$$C_{a,mn} = \begin{cases} 1 & n = \Sigma^-(m), \\ 0 & \text{otherwise}, \end{cases} \tag{3.29}$$

and where Σ^- is a set of target indices. Permutation matrix $\mathbf{C}_a$ provides a correct ordering and is easy to implement. Furthermore, entries in voltage vector $\mathbf{V}$ are also modified as

$$\mathbf{V}^{\text{NEW}} = \mathbf{Z}^{\text{NEW}}\begin{pmatrix} \mathbf{I}^{\text{OLD}} \\ \mathbf{I}^{\text{NEW}} \end{pmatrix}. \tag{3.30}$$

Since the number of current sources is fixed and the node, which is fed by one source, cannot be removed, the vector $\mathbf{I}^{\text{NEW}}$ is identically zero and formula (3.30) is further reduced to

$$\mathbf{V}^{\text{NEW}} = \frac{1}{y_a}\mathbf{C}_a^{\text{T}}\begin{pmatrix} y_a\mathbf{V}^{\text{OLD}} + \mathbf{Z}^{\text{OLD}}\mathbf{y}_a\mathbf{x}_a\mathbf{I}^{\text{OLD}} \\ -\mathbf{x}_a\mathbf{I}^{\text{OLD}} \end{pmatrix}, \tag{3.31}$$

with the auxiliary variables defined in (3.27).

This formulation also allows for the accumulation of all possible nodes addition into a matrix $\mathbf{V}_{\Gamma^+}$, where Γ^+ denotes a set of all possible node addition.

3.4 Topology optimization

Topology optimization is a method which addresses a fundamental engineering problem stated as follows: How to place material within a design domain in order to obtain the best performance? Originally, this concept was invented for mechanical problems but has spread to other physical disciplines, *e.g.*, acoustics or electromagnetics.

For purposes of this thesis, the node-oriented circuit system was chosen. Hence, an objective function p [24] will be explicitly dependent on the voltage vector $\mathbf{V}$, *i.e.*, $p = p(\mathbf{V})$. In general, a topology optimization problem can be written as: For a given admittance matrix $\mathbf{Y} \in \mathbb{C}^{N\times N}$, objective function p, constrains G_i, G_j and a given current vector $\mathbf{I}$, find a vector $\mathbf{g}$ such that

$$\begin{aligned} \text{minimize} \quad & p(\mathbf{g}) \\ \text{subject to} \quad & G_i(\mathbf{g}) = p_i \\ & G_j(\mathbf{g}) \leq p_j \\ & \mathbf{Y}\mathbf{V} = \mathbf{I} \\ & \mathbf{g} \in \{0,1\}^N, \end{aligned} \tag{3.32}$$

where the vector $\mathbf{g}$ denotes a set of enabled ($g_n = 1$) and disabled ($g_n = 0$) nodes, respectively. Since the observables are power-base quantities, some metrics, say p, are quadratic-dependent on voltage and can be expressed via quotients of the quadratic form [9] as

$$p(\mathbf{V}) = \frac{\mathbf{V}^{\mathsf{H}}\mathbf{A}\mathbf{V}}{\mathbf{V}^{\mathsf{H}}\mathbf{B}\mathbf{V}}, \tag{3.33}$$

where H denotes Hermitian transpose and matrices $\mathbf{A}$, $\mathbf{B}$ are general matrix operators. A question of how much a metric p changes after performing a small perturbation is investigated. The formula (3.24) readily provides an answer. The effect of all individual removals or additions is computed as

$$p(\mathbf{V}_{f\Gamma^\pm}) = \text{diag}\left(\mathbf{V}^{\mathsf{H}}_{f\Gamma^\pm}\mathbf{A}\mathbf{V}_{f\Gamma^\pm}\right) \oslash \text{diag}\left(\mathbf{V}^{\mathsf{H}}_{f\Gamma^\pm}\mathbf{B}\mathbf{V}_{f\Gamma^\pm}\right), \tag{3.34}$$

where symbol $\oslash$ denotes the Hadamard division [25]. The topology sensitivity $\tau(p, \Omega)$ then measures the sensitivity of studied metric with respect to small topological changes. The topology sensitivity [4] is defined as

$$\tau(p, \Omega) = p(\mathbf{V}_{f\Gamma^{\pm}}) - p(\mathbf{V}_f) \approx \quad p(\mathbf{V}_f). \tag{3.35}$$

Note that, the topology sensitivity is defined generally, hence, it works in the same way both for the removal and the addition of a node. The removals or additions are performed as long as a node with negative topology sensitivity $\tau(p, \Omega)$ occurs, *i.e.*, as long as an improvement of the structure exists.

The computation scheme for adjusting a shape in order to get a minimum of the objective function p can be understood as a discrete version of a gradient algorithm. Although this definition does not ensure convergence to the global minimum, its computational cost is reasonably low [26]. The gradient-based algorithm described above is further implemented in the next chapter. Since many problems have multiple optima, it would be more convincing to implement global optimizer and let it cooperate with the gradient one, which is considered as a local step, but this approach is out of the scope of this thesis.

Chapter 4

Implementation

In this chapter, the implemented algorithm based on methods developed earlier is presented. A subsequent section employs the implemented algorithm in two examples.

The optimization algorithm is implemented in MATLAB [20]. MATLAB provides an efficient implementation of matrix manipulations, thus, developed procedures can be fully vectorized, *i.e.*, scripted loops can be eliminated. This provides a considerable speed-up as compared to the loop-based code.

The flowchart of the implemented algorithm is depicted in figure 4.1. Firstly, the properties of a circuit, *e.g.*, size, load, feeding current vector $\mathbf{I}$, are specified. System matrix of a circuit, *i.e.*, admittance matrix $\mathbf{Y}$, is constructed. Subsequently, the computation of impedance matrix $\mathbf{Z}$ and node voltage vector $\mathbf{V}$ is performed. Furthermore, the effect of all node removals and addition (one by one) on the node voltage vector is accumulated to matrices $\mathbf{V}_{f\Gamma^-}$ and $\mathbf{V}_{f\Gamma^+}$. Topology sensitivity is evaluated afterwards with respect to desired objective function p, which is to be minimized. Each removal or addition is then represented by one entry in vectors τ_{Γ^-} and τ_{Γ^+}. If both vectors contain only non-negative entries, a local minimum of an objective function was found and optimization is terminated. If the termination criterion is not fulfilled, the vectors are compared and it is decided what action is to be performed *i.e.*, the node removal or addition. Consequently, impedance matrix $\mathbf{Z}$ is updated by inversion-free formulas. Corresponding node voltage vector $\mathbf{V}$ is included in matrices $\mathbf{V}_{f\Gamma^-}$ and $\mathbf{V}_{f\Gamma^+}$, thus, re-computation is not necessary. Afterwards, the algorithm is looped and all preceding steps are evaluated until the minimum is reached.

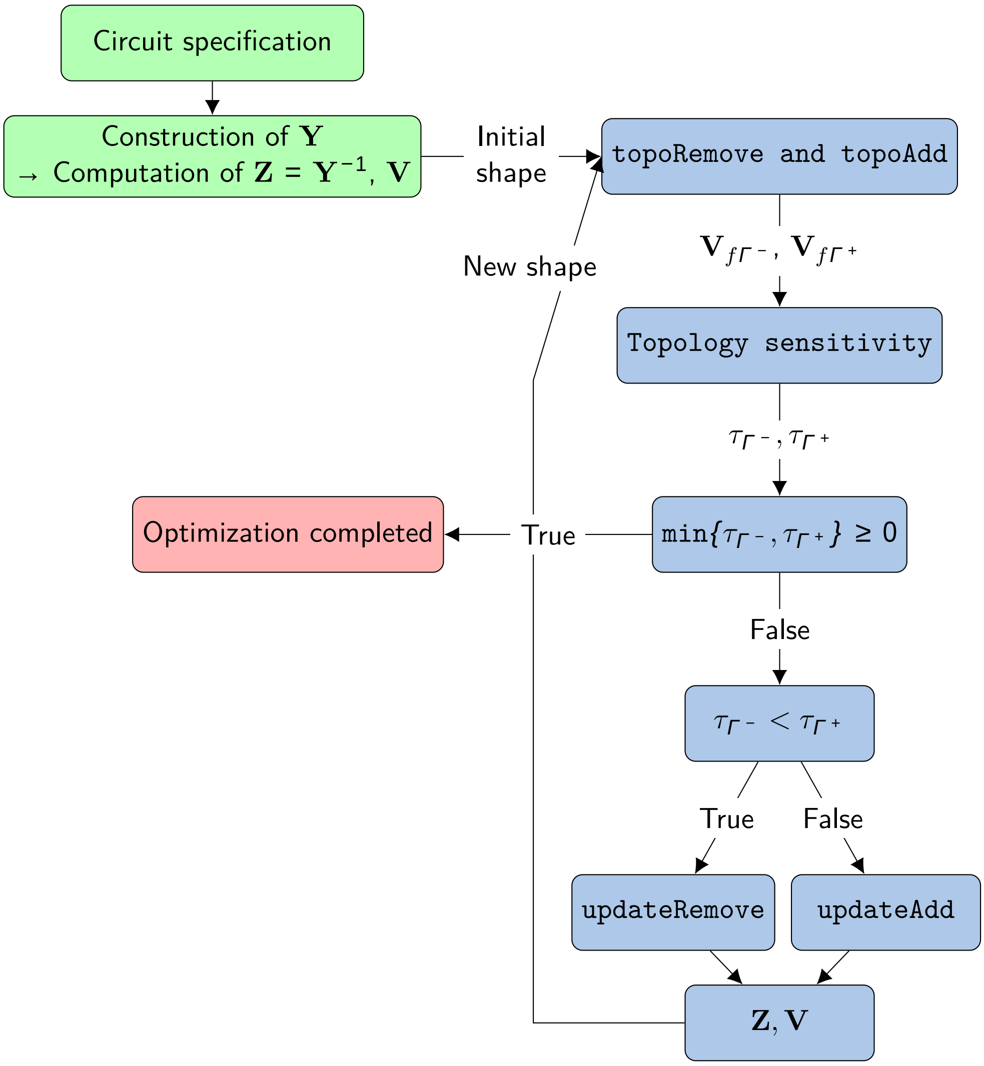


Figure 4.1: Flowchart of implemented algorithm.

4.1 Current divider

Consider an initial resistive circuit in a form of a mesh of $R = 1\ \Omega$ resistors and let us construct a current divider from it with current ratio

$$k = I_{\text{out}}/I_0, \tag{4.1}$$

where I_{out} and I_0 is current flowing through the load and source, respectively. The objective function p is defined as

$$p = \left| \frac{I_{\text{out}}}{I_0} - k \right|, \tag{4.2}$$

which is to be minimized. The initial setup of the grid circuit with load $R_{\text{L}} = 1\ \Omega$ and optimized circuit by the algorithm described in the previous

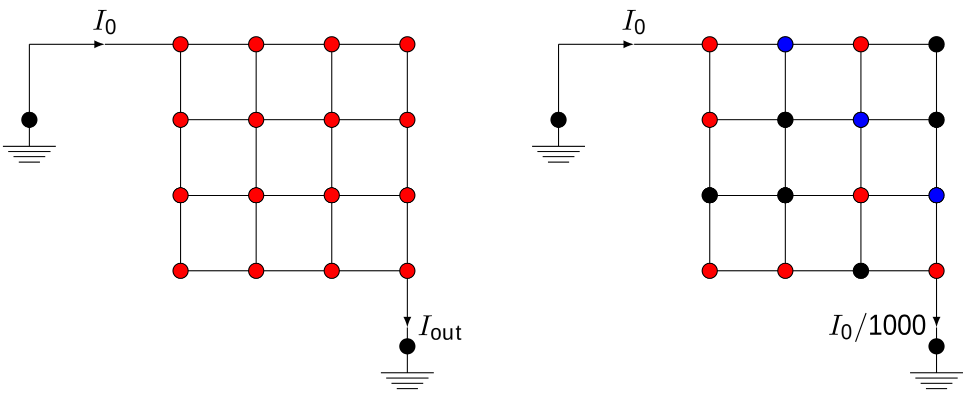


Figure 4.2: Initial topology of a 4 × 4 circuit (left) with edges representing conductance $G = 1$ S. Circuit is optimized (right) to obtain current divider with current ratio $k = 1/1000$. Black dots depict grounded nodes, red dots depict active nodes and blue dots depict once removed nodes, which were added back to the mesh.

section are depicted in figure 4.2 reduced to theirs graph representation. If the denominator of the current ratio (4.1) is rising, the number of iteration, *i.e.*, modification, is most likely to be higher, since the unnecessary current has to be drained.

The optimized circuit was simulated in a commercial circuit simulator Micro-Cap [27] to test the validity of the proposed implementation. Figure 4.3 shows that the initial topology was optimized so as to realized current ratio $k = 1/1000$.

4.2 Impedance matching

The topic of impedance matching is often part of the larger design process for a microwave system. Figure 4.4 illustrates the basic idea of impedance matching, which is a placing matching circuit between a transmission line (or another network) and a load impedance. The matching circuit is to be lossless in order to avoid loss of power and is designed so that input impedance is Z_0. Consequently, reflection coe□cient [28] is zero. If a load impedance is matched to the transmission line, maximum power is delivered to the load.

The matching circuit can in many cases be realized as a mesh of reactances, which is well adapted to the optimization procedure proposed in this thesis.

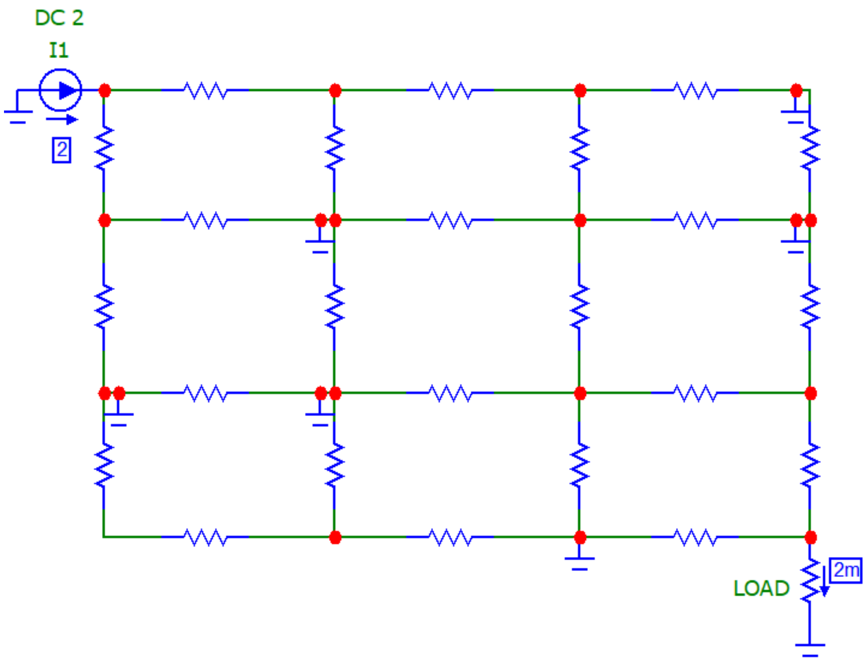


Figure 4.3: Simulation of the optimized current divider in Micro-Cap.

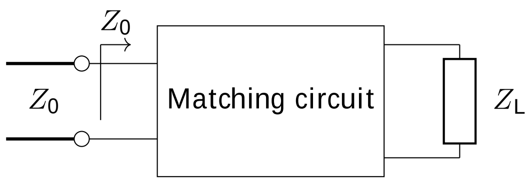


Figure 4.4: A lossless circuit matching a load impedance to the transmission line.

The objective function p is defined as

$$p = \left| 1 - \frac{|Z_{\text{in}}|}{|Z_0|} \right|, \tag{4.3}$$

where input impedance Z_{in} seen by a current source connected to f-th node is expressed as

$$Z_{\text{in}} = \frac{\mathbf{V}^{\text{H}}\mathbf{Y}\mathbf{V}}{|I_f|^2}, \tag{4.4}$$

where I_f is a feeding current at the f-th node.

Consider a reactive network that matches load impedance $Z_L = 100\ \Omega$ to the transmission line with impedance $Z_0 = 50\ \Omega$. Figure 4.5 presents the grid circuit, which consists of ideal lumped elements, *i.e.*, lossless inductors and conductors, and which has been optimized to match load impedance. Note that the second node was removed and further added back to the set. Addition of the second node move the input impedance of the circuit closer to the matched one, see red line in figure 4.6. The algorithm ends with input

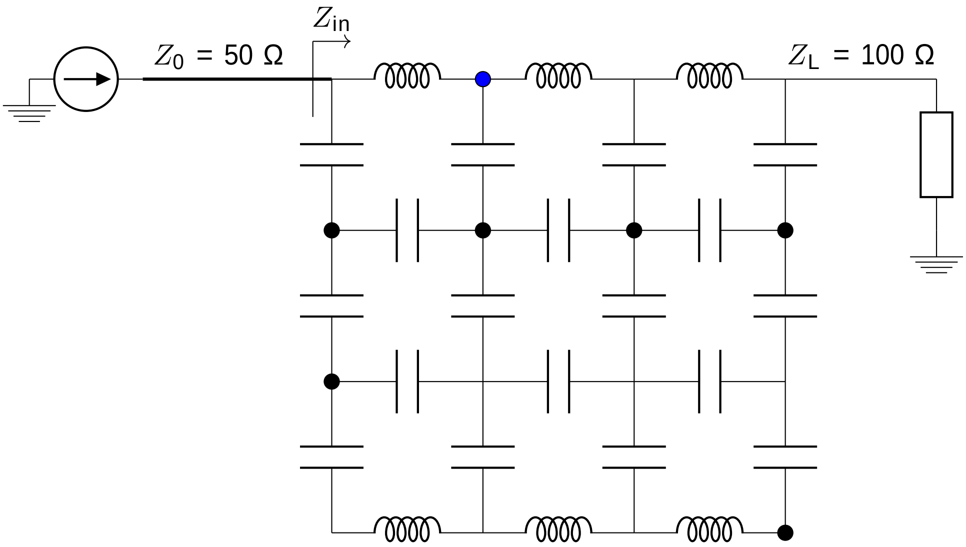


Figure 4.5: The optimized matching circuit connecting the Z_{L} = 100 Ω load to the Z_0 = 50 Ω. The reactances of underlying inductors and capacitors are X_L = 31.4j Ω and X_C = −159j Ω, respectively. Black dots depict grounded nodes and blue dots depict removed noded, which were added back to the network.

impedance Z_{in} in the middle of normalized smith-chart, see figure 4.6, which corresponds to Z_{in} = 50 Ω. Concretely, The input impedance was optimized to the value Z_{in} = 50 − 0.10j Ω.

Assume now, that capacitive reactances X_C = −159j Ω are not available and instead, capacitive reactances X_C = −79.5j Ω are at hand. Is it possible to provide matching with these new reactances? Figure 4.7 shows the answer. If the value of capacitive reactance varies the algorithm traverse a different path in the solution space which typically contains many local minima of the objective function. Since the implemented algorithm is a local optimization algorithm, once it finds a local minimum, it terminates, ignoring that better minima might be available. Employment of a global optimization algorithm and alteration of admittances in the network would be more convenient, but this approach is out of the scope of this thesis.

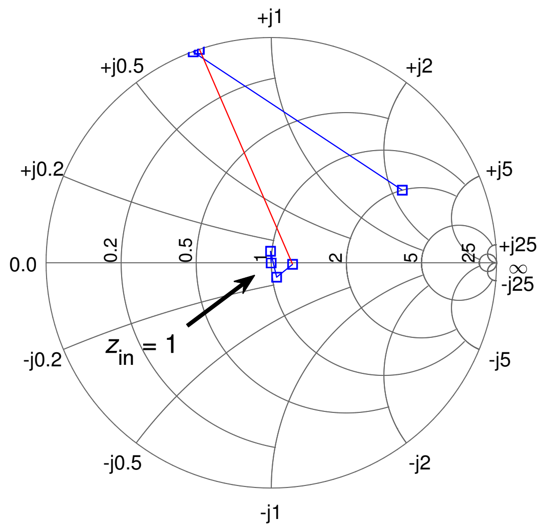


Figure 4.6: Input impedance Z_{in} of the circuit in each iteration of the algorithm plotted in normalized smith-chart. Removal and addition of a node is depicted by the blue and red line, respectively.

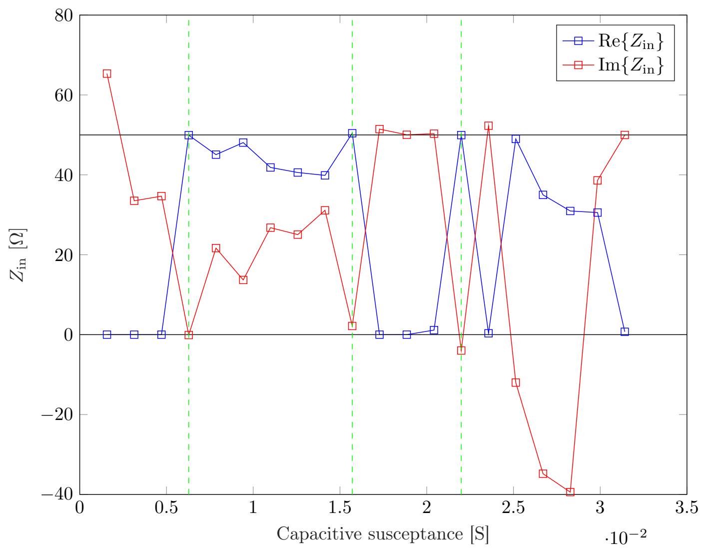


Figure 4.7: Resulting input impedance Z_{in} after optimization procedure found a local minimum. The horizontal axis shows a capacitive susceptance used. Green dashed lines depicts an acceptable solution.

Chapter 5

Conclusion

An optimization scheme was proposed to optimize an arbitrarily sized grid circuit with regards to determined criteria and constraints. The node removal and the node addition in the node-oriented circuit were derived and used for the fast computation of the topology sensitivity. Due to the employment of the inversion-free formulas and the favourable properties of the smallest modification of circuit topology, the computational complexity of the optimization routine was tremendously reduced. The implementation also heavily employs vectorization for which the presented formulations are well-suited. A gradient-based algorithm was employed to synthesize locally optimal circuit topology. This approach was demonstrated in two examples, which can be found in the attachment of this thesis.

In circuit synthesis, the defined elementary modification, *i.e.*, connection/disconnection of a node from the reference node, is a strange operation with questionable practical impact. The proposed procedure is more likely to be called proof of concept since a rough prototype of a new idea was constructed in order to demonstrate its feasibility.

Future work will aim at the complete redefinition of the system and the elementary modification. A single impedance will be probably used as a basis function and a current flowing through the impedance as a primary quantity. This approach will hopefully have a bigger practical impact since its concept is brought nearer to the classical 2D method of moment paradigm. Incorporation of a gradient-based algorithm into global optimization routines, *e.g.*, genetic algorithms, shall be done. Since the global methods overcome local extremes to find the global optimum.

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