# ZADÁNÍ DIPLOMOVÉ PRÁCE

## I. OSOBNÍ A STUDIJNÍ ÚDAJE

Příjmení: **Uhrín** Jméno: **Matej** Osobní číslo: **393042**

Fakulta/ústav: **Fakulta elektrotechnická**

Zadávající katedra/ústav: **Katedra počítačů**

Studijní program: **Otevřená informatika**

Studijní obor: **Softwarové inženýrství**

## II. ÚDAJE K DIPLOMOVÉ PRÁCI

Název diplomové práce:

**Systém pro autonomní sázení s optimální alokací sázek**

Název diplomové práce anglicky:

**System for autonomous betting with optimal wealth allocation**

Pokyny pro vypracování:

Advances in predictive modeling in the domain of sports analytics enable development of systems for autonomous bet staking. While most of the systems focus on the predictive part of the problem, the goal of this thesis is to investigate the problem in broader context with focus on wealth allocation.
1) Research state of the art in statistical approaches to sports betting.
2) Focus on the subproblem of allocating bettors wealth across individual opportunities.
3) Review mathematical aspects of portfolio optimization from the field of econometrics, and identify key concepts for the use in the context of bet placing.
4) Propose an architecture combining statistical modeling with the bet placing strategy.
5) Select a suitable sport domain and collect relevant historical data.
6) Develop a system simulating autonomous betting in the selected domain.
7) Evaluate overall performance of your system from different perspectives.

Seznam doporučené literatury:


Williams, Leighton Vaughan, ed. Information efficiency in financial and betting markets. Cambridge University Press, 2005.
Hausch, Donald B., Victor SY Lo, and William T. Ziemba. Efficiency of racetrack betting markets. Vol. 2. World Scientific, 2008.
Thorp, Edward O. "The Kelly criterion in blackjack, sports betting and the stock market." Handbook of asset and liability management 1 (2006).


Jméno a pracoviště vedoucí(ho) diplomové práce:

**Ing. Ondřej Hubáček, Intelligent Data Analysis FEL**

Jméno a pracoviště druhé(ho) vedoucí(ho) nebo konzultanta(ky) diplomové práce:

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Ing. Ondřej Hubáček
podpis vedoucí(ho) práce

podpis vedoucí(ho) ústavu/katedry

prof. Ing. Pavel Ripka, CSc.
podpis děkana(ky)

## III. PŘEVZETÍ ZADÁNÍ

Diplomant bere na vědomí, že je povinen vypracovat diplomovou práci samostatně, bez cizí pomoci, s výjimkou poskytnutých konzultací.
Seznam použité literatury, jiných pramenů a jmen konzultantů je třeba uvést v diplomové práci.

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Datum převzetí zadání

Podpis studenta

Insert here your thesis' task.

Master's thesis

# System for autonomous betting with optimal wealth allocation

***Bc. Matej Uhrín***

Department of Computer Science
Supervisor: Ing. Ondřej Hubáček

May 25, 2018

# Acknowledgements

First and foremost, my thank you goes to the following three people. My supervisors Ondřej Hubáček, Gustav Šourek and my good friend and colleague Juraj Pohanka. Thank you, I have been very fortunate to receive a significant amount of your help, your time and your advice.

Second, I would like to thank my school and faculty for the very opportunity to be writing this work.

Last but not least, I would like to thank my family for their continued support.

# Declaration

I hereby declare that the presented thesis is my own work and that I have cited all sources of information in accordance with the Guideline for adhering to ethical principles when elaborating an academic final thesis.

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In Prague on May 25, 2018 . . . . . . . . . . . . . . . . . . . . .

## Citation of this thesis

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# Abstrakt

Cílem naší práce bylo najít optimální investiční strategii a ověřit její funkčnost na vhodné doméně sportovního sázení. Zjistili jsme, že pravidlo maximálního geometrického průměru (tzv. “Kellyho kritérium”) je takovou obecně optimální strategií, a v práci ji analyzujeme jak v matematické tak experimentální rovině. Byť matematicky optimální, její předpoklady jsou jen zřídka splněny, což v praxi vytváří četné problémy plynoucí z nejistoty v odhadech pravděpodobností a výpočetní složitosti při řešení pro množinu souběžných her. V práci představujeme nejrůznější návrhy řešení všech takových omezení, se kterými jsme se setkali při reálných aplikacích této strategie ve sportovním sázení. Navíc představujeme framework pro zátěžové testování sázecích srategií, který umožňuje experimentální analýzu různých sázecích scénářů. Nakonec naše zjištění ověřujeme na reálných datech ze třech různých domén sportovního sázení: dostihy, basketbal a fotbal.

**Klíčová slova** sázkařské trhy, sportovní analýza, dostihy, basketbal, fotbal, Kellyho kritérium

# Abstract

The goal of our work was to find an optimal wealth allocation policy and to verify its functionality on a suitable sports betting domain. We found that the geometric mean policy ("Kelly Criterion") is a generally optimal strategy and we discuss its optimality both mathematically and experimentally. While mathematically optimal, its assumptions are rarely fully met which presents a set of challenges such as how to deal with errors stemming from innacurate probability estimates and computational difficulty of solving many simultaneous games. We present solution proposals to all the limitations we encountered in applications of the geometric mean policy to sports betting. Moreover, we introduce a stress testing framework for betting strategies which allows for testing of various sports betting scenarios. Finally, we verify our findings on real data from three different domains of sports betting: horse racing, basketball and football.

**Keywords** betting markets, sports analytics, horse racing, basketball, futbal, Kelly criterion

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# Introduction

From the ancient times, gambling has always been part of society. While word gambling is often said with a negative connotation, it can hardly be argued against that games of chance sparked some of greatest discoveries ever made.

> The old ones say we Spartans are descended from Hercules himself...

Similarly, the old ones say that probability theory descended from gambling itself. History of gambling tells a fascinating story of games motivating the greatest mathematicians, engineers, that put their minds together to “beat” those games. Using the first US telephone line not to call their mothers, but to hear results of horse races before the bookies. To bring first pocket computer to Las Vegas casino and calculate how “tilted” the roulette is. To withhold the broadcast of a football match by a few seconds and use the information of a possible scored goal to bet before the bookies know it.

What these stories have in common is that great thinkers put their minds together to gain some kind of advantage over the casino, over the bookie. In this text we will investigate what it means to have an advantage and what is the best course of action to take when we have that advantage.

We will start from the most simple and progress to the more advanced definitions. The text is divided into four chapters. In chapter 1 we will assume player knows the true probability of the game, (blackjack, coin toss etc.). In this chapter we will introduce the optimal strategy and prove it’s optimality both mathematically and experimentally. In chapter 2 we will discuss the case when player has an estimation of the true game probability (football, horse racing), whether the accuracy of estimation is important and if so how important. How much the randomness of the real world affects the optimality of our strategy and whether such randomness can be at least partially tamed. In chapter 3 we will look into a case of multiple simultaneous games and the challenges it presents. Finally, in chapter 4 we present experiments on multiple datasets of multiple essentially different games where randomness of the real world may have the last word. Buckle up and enjoy the ride.

Chapter 1

# Fortune's Formula

In this chapter we will introduce the world's most famous betting paradigm, Kelly criterion, often dubbed as the fortune's formula. Initially used to exploit roulette game, later blackjack, then horse racing and finally stock market.

## 1.1 Coin toss

Assume a simple coin toss game. We win if the coin comes up heads, if it is tails we lose. If we win, we receive $o_g$ times the amount that we bet in addition to our original bet money. The definitions are:

- $p$ winning probability.
- $q$ loosing probability.
- $w$ number of wins.
- $l$ number of loses.
- $t = w + l$ number of all trials.
- $o$ odds, sometimes also called dividends.
- $o_g$ odds gain. Payoff from the winning bet in addition to the original amount.
- $b$ is the fraction of our wealth that we decide to bet.
- $W_0$ is our starting wealth
- $W_t$ is our final wealth after $t$ trials

Next please assume we play a slightly profitable game of coin toss. Probability of winning the game is $p = 0.4$. Odds are $o = 3.0$, hence our odds gain is $o_g = o - 1 = 2.0$. We start with initial bank $W_0$ of 100,- CZK and we bet $b = 0.05$ , 5 percent fraction of our bank every time. We play this game $t = 3$ times, we win some $w = 2$ and we lose some $l = 1$. Our final wealth $W_3$ can hence be calculated as follows:

$$W_3 = (1 + 2 \cdot 0.05)^2 \cdot (1 - 0.05)^1 \cdot 100 \tag{1.1}$$

We make a gain of 2.0 times our decided "bank" fraction 0.05 twice and we loose 0.05 of our bank once.

$$W_3 = 114.95 \tag{1.2}$$

What should be noted is that the order of our wins and losses does not matter as long as the respective counts of winning $w$ and loosing games $l$ follows the problem definition.

### 1.1.1 Growth rate

The example from 1.1 can be generalized in the following definition of wealth after $t$ trials.

$$W_t = (1 + o_g \cdot b)^w \cdot (1 - b)^l \cdot W_0 \tag{1.3}$$

Next we will define the average growth rate with given fraction $b$ over $t$ trials to be $g(b)$:

$$g(b) = \frac{1}{t} \cdot \frac{W_t}{W_0} = \frac{1}{t} \cdot (1 + o_g \cdot b)^w \cdot (1 - b)^l \tag{1.4}$$

Clearly any racional player wishes for his final bank to be as large as possible. He should therefore wish for his wealth to grow as quickly as possible. What we will next refer to as the optimal strategy is a strategy that maximizes the growth rate defined in 1.4.

## 1.2 Search For Optimality

Maximization of the average growth rate can be expressed as optimization problem.

$$\underset{\boldsymbol{b}}{\text{maximize}} \quad g(b) = \frac{1}{t} \cdot (1 + o_g \cdot b)^w \cdot (1 - b)^l$$

We differentiate $g(b)$ with respect to $b$

$$\frac{\partial g}{\partial b} = \frac{o_g w (1-b)^l \cdot (b o_g + 1)^{w-1}}{t} - \frac{l (1-b)^{l-1} \cdot (b o_g + 1)^w}{t} \tag{1.5}$$

We set the derivative equal to zero and solve for $b$.

$$b^\star \ such \ that \quad \frac{o_g w(1-b)^l \cdot (bo_g+1)^{w-1}}{t} - \frac{l(1-b)^{l-1} \cdot (bo_g+1)^w}{t} = 0 \quad (1.6)$$

The only root that makes sense in our context is

$$b^\star = 1 \quad (1.7)$$

And it is indeed a maximum as proved by Edward Thorp in Thorp, 2008

To maximize the growth rate, player should be betting the whole bank in every single trial. Would that be rational way of bank management? We can easily see where the problem lies.

$$P(ruin) = 1 - p^t \quad (1.8)$$

$$\lim_{t \to \infty} (1 - p^t) = 1 \quad (1.9)$$

where *ruin* is a state where player's wealth reaches zero. $W_i = 0$ for some $i$.

Clearly, player would surely come to ruin with such strategy. It would only take a single lost game for it to happen.

A different approach would be to define the above problem as a minimization of risk. In that case however the optimal strategy would be to withhold all the money and never bet or to make minimum allowed bets as shown in Feller, 1968, unless the winning probability is $p = 1.0$, which would mean that player is not playing a game anymore, but receiving free money. Therefore even though such strategy minimizes risk, it unfortunately minimizes growth as well.

We conclude this section with a statement that both strategies are infeasible for us. There is however a perfect way to balance both growth rate and risk in a single strategy.

## 1.3 Utility

A completely different idea is to value money using utility functions. In the problem from the previous section. 1.2 we used linear utility function $U$. Simply:

$$U(W) = W \quad (1.10)$$

We have shown that it is insufficient for our purpose of wealth allocation. The idea of many great thinkers such as *Danielle Bernoulli* in *Exposition of a new* *theory on the measurement of risk* Bernoulli, 2011 is to use logarithmic utility function to value our money.

$$U(W) = \log(W) \tag{1.11}$$

Intuitively, it makes perfect sense. For a player who re-invests his money in every single game, who's wealth is compounding, reaching a ruin situation means complete stop to his operation. Hence it should be penalized accordingly.

$$\lim_{W \to 0^+} \log(W) = -\infty \tag{1.12}$$

Is there a situation where using linear utility function would be sensible? (Kelly Jr, 2011) Provides a great example.

> Suppose the situation were different; for example, suppose the gambler's wife allowed him to bet one dollar each week but not to reinvest his winnings. He should then maximize his expectation (expected value of capital) on each bet. He would bet all his available capital (one dollar) on the event yielding the highest expectation. With probability one he would get ahead of anyone dividing his money differently.

Such player should therefore forget about the Kelly criterion. The reason is that his winnings do not compound, they simply accumulate (Poundstone, 2010).

## 1.4 Kelly Criterion

The fortune's formula is based on the of idea evaluating the growth rate using logarithmic utility function.

$$G(b) = \frac{1}{t} \log(\frac{W_t}{W_0})) \tag{1.13}$$

$$G(b) = \log[(\frac{W_t}{W_0})^{\frac{1}{t}}] \tag{1.14}$$

$$G(b) = \log[(\frac{(1 + o_g \cdot b)^w \cdot (1 - b)^l \cdot W_0}{W_0})^{\frac{1}{t}}] \tag{1.15}$$

$$= \log[(1 + o_g \cdot b)^{\frac{w}{t}} \cdot (1 - b)^{\frac{l}{t}}] \tag{1.16}$$

$$G(b) = \frac{w}{t} \log(1 + o_g \cdot b) + \frac{l}{t} \log(1 - b) \tag{1.17}$$

$\frac{w}{t}$ and $\frac{l}{t}$ stand for our probabilities of winning $p$ and loosing $q$. Therefore our final formula for average logarithmic growth rate looks as follows:

$$\mathbb{E}[G(b)] = p\log(1 + o_g \cdot b) + q\log(1 - b)) \tag{1.18}$$

We now repeat the same process as in 1.2

$$\underset{\boldsymbol{b}}{\text{maximize}} \quad \mathbb{E}[G(b)] = p\log(1 + o_g \cdot b) + q\log(1 - b)$$

This time differentiation should yield a very different result.

$$\frac{\partial\,\mathbb{E}[G(b)]}{\partial b} = \frac{p \cdot o_g}{1 + o_g \cdot b} - \frac{q}{1 - b} \tag{1.19}$$

$$b^\star \;\; such\;\; that \quad \frac{p \cdot o_g}{1 + o_g \cdot b} - \frac{q}{1 - b} = 0 \tag{1.20}$$

We follow through with the calculation.

$$\frac{p \cdot o_g}{1 + o_g \cdot b} = \frac{q}{1 - b} \tag{1.21}$$

$$p \cdot o_g(1 - b) = q(1 + o_g \cdot b) \tag{1.22}$$

$$po_g - q = o_g b(p + q) \tag{1.23}$$

Where $p + q = 1$ and our optimal strategy is hence defined as:

$$b^\star = \frac{po_g - q}{o_g} \tag{1.24}$$

It is indeed a maximum as shown in Latane, 2011. It is also a well known formula sometimes written as

$$\frac{edge}{odds} \tag{1.25}$$

or using different notation where $b$ stands for $o_g = odds - 1$

$$\frac{pb - q}{b} \tag{1.26}$$

What happens if we choose optimal fraction $b^\star$ according to 1.24 in our original problem 1.1

$$b^\star = \frac{po_g - q}{o_g} = \frac{0.4 \cdot 2 - 0.6}{2} = 0.1 \tag{1.27}$$

The following experiment provides an illustration why it is not reasonable to bet higher than the Kelly optimal fraction $b^\star$.

Assume we play the game 1.1 $t = 30000$ times, $b_1 = 0.05$, our initial guess is the green player, $b^\star = 0.1$, the optimal Kelly fraction is the blue player. Red player is betting $b_h = 0.25$.

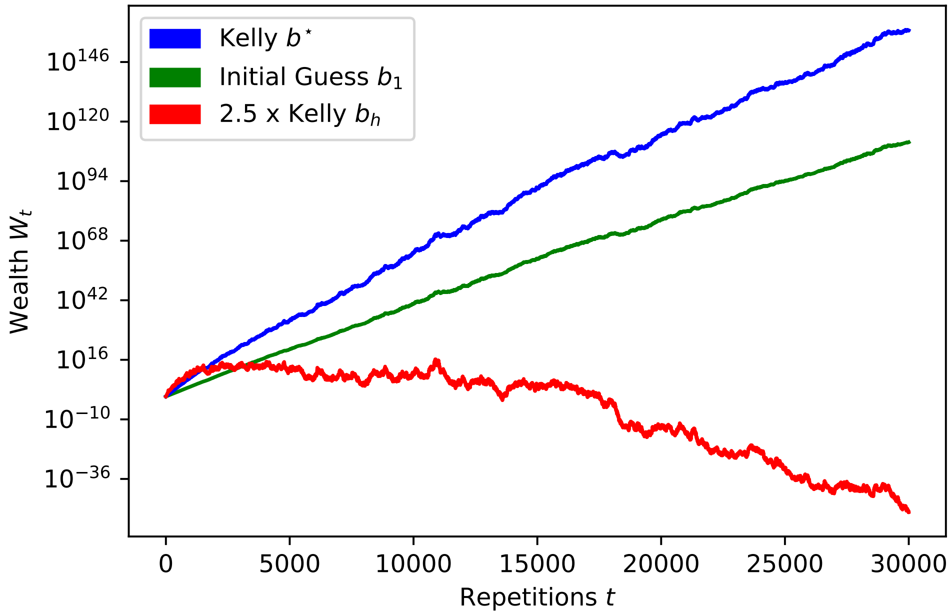


Figure 1.1: Mean trajectories of 1000 parallel histories for $b^\star$, $b_1$, $b_h$

Note that the red trajectory $b_h$ does indeed "beat" Kelly on the first 500 or so trials, but eventually it leads to ruin and the red player leaving the casino in tears.

Both blue and green player would go home smiling with blue player being the richer of the two.

The important conclusion is that Kelly fraction $b^\star$ acts like the upper bound of the achievable wealth growth. Betting anything less than $b^\star$ leads to sub-optimal final wealth. Betting anything higher lowers returns and increases risk.

> For true long-term investors, the Kelly criterion is the boundary between aggressive and insane risk-taking. Like most boundaries, it is an invisible line. You can be standing right on it, and you

> won't see a neat dotted line painted on the ground. Nothing dramatic happens when you cross the line. Yet the situation on the ground is treacherous because the risk-taker, though heading for doom, is liable to find things getting better before they get worse. (Poundstone, 2010)

We close this section with a statement that $b^\star = \frac{p \cdot o_g - q}{o_g}$ is the optimal strategy for binary game such as coin toss. What can we do when the game is more complex remains to be answered.

## 1.5 Exclusive Games

In our first game 1.1, we discussed a situation where player has a binary choice of betting fraction of his wealth on a single outcome and the other option of leaving some of the money aside.

Next we assume a game with $K$ outcomes that are exclusive, e.g. horse race where only a single horse can win.

### 1.5.1 Growth Rate

We have already discussed how Kelly criterion uses the idea of evaluating our money with logarithmic utility function. Given fractional strategy $b$ and probability $p$ the Kelly criterion is usually expressed as follows from (Cover et al., 2012).

$$W(\boldsymbol{b}, \boldsymbol{p}) = E[\log(\boldsymbol{b}^T \boldsymbol{o})] \tag{1.28}$$

It again restates the expected logarithm of growth rate. This time however it is for general K exclusive outcome game.

$$W(b, p) = \sum_{i=1}^{K} p_i \log(b_i \cdot o_i) \tag{1.29}$$

Please note that $1 + o_g \cdot b$ transformed into $b_i \cdot o_i$, because $o$ stands for odds $o = o_g + 1$ as defined, hence it already includes the original bet amount.

### 1.5.2 Kelly Proof

In the previous section we defined Kelly criterion for exclusive games with $K$ outcomes.

The maximization of the above defined growth rate 1.29 looks as follows from

(Cover et al., 2012).

$$\begin{aligned} \underset{\boldsymbol{b}}{\text{maximize}} \quad & \sum_{i=1}^{K} p_i \log(b_i \cdot o_i) \\ \text{subject to} \quad & \sum_{i=1}^{K} b_i = 1.0 \\ & b_i \geq 0 \end{aligned}$$

Using the method of lagrange multipliers we expand the above problem into

$$\ell(b) = \sum_{i=1}^{K} p_i \log(b_i \cdot o_i) + \lambda \cdot \sum_{i=1}^{K} b_i \tag{1.30}$$

We differentiate with respect to $b$

$$\frac{\partial \ell}{\partial b} = \sum_{i=1}^{K} \frac{p_i}{b_i \cdot o_i} + \lambda \tag{1.31}$$

Bookie's estimated probability distribution sums up to 1.0.

$$\sum_{i=1}^{K} \frac{1}{o_i} = 1.0 \tag{1.32}$$

We simplify our derivation into

$$\frac{\partial \ell}{\partial b} = \frac{p_i}{b_i} + \lambda \qquad i = 1, 2, ..., K \tag{1.33}$$

To find the optimum of $b_i$ we set the above equation equal to 0

$$b_i = -\frac{p_i}{\lambda} \tag{1.34}$$

Now we substitue back into the constraint $\sum_{i=1}^{K} b_i = 1.0$ we find that

$$\lambda = -1 \qquad b_i^\star = p_i \tag{1.35}$$

This only tells us that the $b_i = p_i$ is only a stationary point. In the next section we will prove that it is indeed a maximum.

### 1.5.3 Maximum

We will prove that probability proportional strategy $b_i = p_i$ is the maximum of the above defined problem. Before we proceed, a few additional definitions are necessary.

First we recall the formula for entropy:

$$H(P) = -\sum_{i=1}^{n} p_i \log(p_i) \tag{1.36}$$

Next we define Kullback–Leibler divergence from definition:

$$D(P||Q) = -\sum_{i=0}^{n} p_i \log \frac{q_i}{p_i} \tag{1.37}$$

We start with a standard formula for growth rate.

$$W(b,p) = \sum_{i=1}^{K} p_i \log(b_i \cdot o_i) \tag{1.38}$$

In this step we use a convenient trick from (Cover et al., 2012).

$$W(b,p) = \sum_{i=1}^{K} p_i \log(\frac{b_i}{p_i} p_i \cdot o_i) \tag{1.39}$$

$$W(b,p) = \sum_{i=1}^{K} p_i \log(\frac{b_i}{p_i}) + \sum_{i=1}^{K} p_i \log(p_i) + \sum_{i=1}^{K} p_i \log(o_i) \tag{1.40}$$

We can now transform our formula into:

$$W(b,p) = -D(p||b) - H(p) + \sum_{i=1}^{K} p_i \log(o_i) \tag{1.41}$$

KL-divergence being non negative we can safely conclude:

$$W(b,p) \leq -H(p) + \sum_{i=1}^{K} p_i \log(o_i) \tag{1.42}$$

And we can only achieve equality,(maximum growth rate) if distance,("KL-divergence") is 0.

$$D(p||b) = 0 \tag{1.43}$$

that holds for:

$$b^\star = p \tag{1.44}$$

Probability proportional gambling achieves maximum growth rate when bookie's probabilities $\frac{1}{o_i}$ sum up to 1.0.

Interesting finding is that in such a case, odds are completely ignored by the growth optimal strategy. All that matters is the probability. In other texts this strategy is often referred to as "betting your beliefs".

What should be noted from this section is that Kelly growth optimal betting is closely linked to Kullback–Leibler divergence, which is a fact we will later investigate in chapter 2.

### 1.5.4 Dividends

In the previous section we proved that probability proportional strategy is optimal when bookie's probabilities sum up to 1. This happens only if the dividends are fair. From the perspective of dividends,("odds") we can distinguish three cases. Fair odds, super-fair and sub-fair odds, (Cover et al., 2012).

#### 1.5.4.1 Fair

The dividend implied probabilities $\frac{1}{o_i}$ sum up to 1.0

$$\sum_{i=1}^{K} \frac{1}{o_i} = 1.0 \tag{1.45}$$

Optimal strategy is probability proportional. Intuitively, if odds are fair, they do not provide any more information.

#### 1.5.4.2 Super-Fair

In this case odds are even better than fair. It is an arbitrage situation, that in real life happens very rarely, if so, it is only by a mistake of bookie.

$$\sum_{i=1}^{K} \frac{1}{o_i} \leq 1.0 \tag{1.46}$$

#### 1.5.4.3 Sub-Fair

This is the case we will be investigating in this text. It represents most of the betting situations in real life. Subfair odds usually lowered by some margin or "track-take".

$$(1 - tt) \cdot o_i \qquad tt \in (0, 1) \tag{1.47}$$

where $tt$ stands for track-take. Hence

$$\sum_{i=1}^{K} \frac{1}{o_i} \geq 1.0 \tag{1.48}$$

probability proportional gambling is no longer growth optimal. For answer we will have to look into Kelly Jr, 2011 and the legendary problem of a *Gambler with a private wire.*

In (Kelly Jr, 2011) story begins with a gambler who owns a "private-wire", through which he receives insider tips on which horse will win the race. Received tips are not 100% reliable, though gambler always knows how "unreliable" the tips are. He knows the true probability distribution.

### 1.5.5 Life Is Not Fair

What can we do if odds are not fair? It is the most common real life situation, when the odds implied probabilities sum up to over 1.0.

$$\sum_{i=1}^{K} \frac{1}{o_i} \geq 1.0 \tag{1.49}$$

Clearly, probability proportional strategy is no longer optimal. Because of the existing "track-take", betting on all the horses (outcomes) is no longer sensible.

Thankfully, in Kelly Jr, 2011 a waterfall algorithm is presented for such a case. It is later rediscovered in Smoczynski et al., 2010.

**Algorithm 1** Kelly exclusive algorithm

```
procedure KELLY-EXCLUSIVE(i, p, o)
    chosen = [ ]
    fractions = [0, 0, ..., 0]
    R = 1.0
    ev = p · o
    (ĩ, p̃, õ) = order_descending((i, p, o), by=ev)
    for (i, p_i, o_i) in (ĩ, p̃, õ) do
        if p_i · o_i > R then
            chosen.add((i, p_i, o_i))
            R = (1 − sum(chosen.p)) / (1 − sum(1 / chosen.o))
        end if
    end for

    for (i, p_i, o_i) in chosen do
        fractions[i] = p_i − R / o_i
    end for

    return fractions
end procedure
```

Where $R$ stands for reserved rate. $\boldsymbol{i}$ is the vector of event identifiers $\boldsymbol{p}$ is the vector of probabilities, $\boldsymbol{o}$ is the vector of odds. In simple terms, the algorithm can be explained as:

1. Order all of the possible bets from most to least profitable (highest to lowest $ev$).
2. For each event, see if the $ev$ for that event exceeds the "reserve rate" for your existing set of bets, (The reserve rate is initially 1.0 when your set

of planned bets is empty). If the *ev* is higher, then add that bet to your chosen set of bets.

3. After each addition of a bet, update the reserved rate according to. R = (1 – (sum of each probability bet on)) / (1 – (sum of each odds implied probabilities))

4. Once the optimal set of outcomes is discovered. The fractions are calculated as

$$b_i^\star = p_i - \frac{R}{o_i} \tag{1.50}$$

Next we define the expected gain to be.

$$\mu = p \cdot o - 1 = ev - 1 \tag{1.51}$$

The important finding, that can be looked at as counter intuitive is that Kelly may decide to bet on an outcome with negative expected gain $\mu$ if certain conditions are met. The reason behind this is that such diversified betting portfolio has higher geometric mean return than non diversified.

Assume a horse race of 3 horse with the following definitions.

$$\boldsymbol{p} = [0.08, 0.5, 0.42] \tag{1.52}$$

$$\boldsymbol{o} = [19, 1.99, 1.3] \tag{1.53}$$

Hence, $\mu$ from definition:

$$\boldsymbol{\mu} = [0.52, -0.005, -0.454] \tag{1.54}$$

Using Kelly exclusive algorithm on this problem yields the following optimal fractions $\boldsymbol{b}^\star$

$$\boldsymbol{b}^\star = [0.03030916, 0.0255648, 0.0] \tag{1.55}$$

The second horse,(betting opportunity) is of interest to us. Clearly it has negative expected gain $\mu$, why exactly did Kelly exclusive algorithm decided to bet on this outcome?

The following experiment gives a clear answer. Blue player plays according to Kelly suggested strategy, red player decided to bet according to Kelly but only where he can expect positive gain. Very sound decision, one could say. We repeat the game 10000 times for 10000 parallel histories and we take the mean history for each one.

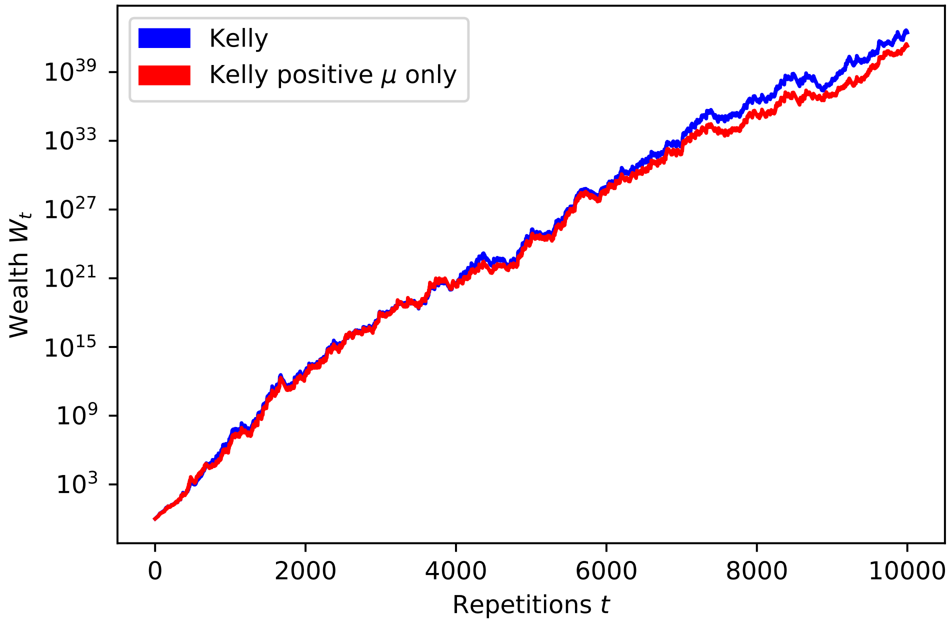


Figure 1.2: Blue-Kelly player, Red-Positive $\mu$ Kelly player, mean trajectory of 10000 parallel histories over 10000 games

The important idea to remember from this section is that it sometimes pays to bet on negative expectation bets in combination with other positive expectation ones. The second important thing is that modifying Kelly fractions in any way results in somewhat sub-optimal strategy.

## 1.6 K-outcome Games

Question that we also need to answer is: What can we do if the game is non-exclusive? A game where multiple bets pay off after a single outcome. Assume football outcome 5:1 for team A, then bets defined as: team A will score over 2 goals, team A will score over 4 goals, both pay off.

To be able to proceed with a solution, first we need be able to formulate such games. Taking inspiration from (Busseti et al., 2016) we define a return matrix $\boldsymbol{R}$ such that columns represent different assets available to us and rows represent different probabilities of our world. Each "box" hence represents a single payoff from a single asset.

We include additional asset to our representation. The risk-free cash asset which allows our strategy to put money aside. In addition it also allows us to model that leaving a large money aside can cost us small amounts of money in every betting turn ("inflation") or possibility to keep our "cash" in a bank with some interest rate.

All in all, our model allocates wealth among $n+1$ assets, $n$ risky assets and 1 risk-free cash asset. Our "world" has $K$ possible probabilistic outcomes.

$$\boldsymbol{R} = \begin{bmatrix}\boldsymbol{a_1} & \boldsymbol{a_2} & \dots & \boldsymbol{a_{n-1}} & \boldsymbol{c}\end{bmatrix} \tag{1.56}$$

$r_{i,j}$ stands for a single return in our matrix $\boldsymbol{R}$. Each asset column vector $\boldsymbol{a_i}$ is defined as follows.

$$\boldsymbol{a_i} = \begin{bmatrix} r_{i,1} \\ r_{i,2} \\ \dots \\ r_{i,K} \end{bmatrix} \tag{1.57}$$

$\boldsymbol{b}$ stands for chosen bet fractions.

$$\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_{n-1} \\ b_c \end{bmatrix} \tag{1.58}$$

and of course probability vector $\boldsymbol{p}$

$$\boldsymbol{p} = \begin{bmatrix} p_1 & p_2 & \dots & p_K \end{bmatrix} \tag{1.59}$$

### 1.6.1 2-asset game

The most basic case of a game we divide our bank between a single asset and cash. We redefine our previously used example of a fair coin toss that either pays off $o = 3.0$ or nothing.

$$\boldsymbol{R} = \begin{bmatrix} 3.0 & 1.0 \\ 0.0 & 1.0 \end{bmatrix} \tag{1.60}$$

$$\boldsymbol{p} = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix} \tag{1.61}$$

### 1.6.2 3-asset game

A great example is basketball, because interstingly basketball has no draw state. For two teams A,B our 3 assets are $WINA$, $WINB$, $CASH$. Assume a slightly less profitable game where probability of winning for team A is

$p_A = 0.6$ and payoff is $o_a = 1.8$, probability of winning for team B is $p_b = 0.4$ and payoff is $o_b = 2.01$.

$$\boldsymbol{R} = \begin{bmatrix} 1.8 & 0.0 & 1.0 \\ 0.0 & 2.1 & 1.0 \end{bmatrix} \tag{1.62}$$

$$\boldsymbol{p} = \begin{bmatrix} 0.6 & 0.4 \end{bmatrix} \tag{1.63}$$

This format allows us to formulate a much more complex game. Imagine you are faced with a choice of allocating your money between 3 wheel's of fortune. Taken from (Poundstone, 2010).

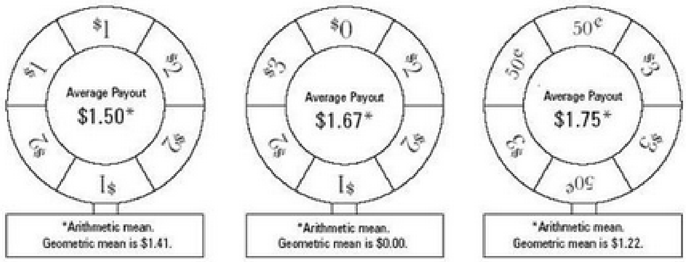


Figure 1.3: 3 wheels of fortune

We can easily represent such problem using our matrix $\boldsymbol{R}$.

$$\boldsymbol{R} = \begin{bmatrix} 1 & 3 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ 2 & 2 & 3 \\ 2 & 2 & 3 \\ 1 & 1 & \frac{1}{2} \\ 2 & 2 & 3 \end{bmatrix} \tag{1.64}$$

Note that in this particular problem we are not using cash asset.

$$\boldsymbol{p} = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \dots & \frac{1}{6} \end{bmatrix} \tag{1.65}$$

### 1.6.3 N-asset game

Assume horse race with 16 running horses. Bet type quinella denoted $QNL(i, j)$ pays off if pair of horses $(i, j)$ win the race. Order does not matter. There are hence 120 different pairs, 121 different assets including cash asset and 120 probabilities in the vector $\boldsymbol{p}$. $o_{i,j}$ denotes posted odds for given $QNL(i, j)$.

$$\boldsymbol{R} = \begin{bmatrix} o_{1,1} & 0 & 0 & ... & 1 \\ 0 & o_{1,2} & 0 & ... & 1 \\ 0 & 0 & o_{1,3} & ... & 1 \\ ... & ... & ... & ... & 1 \end{bmatrix} \tag{1.66}$$

This is a bet on an exclusive outcome, hence $R$ matrix is almost completely made up of zeros and odds diagonally.

$$\boldsymbol{p} = \left[p_{1,1}, p_{1,2}, ..., p_{15,16}\right] \tag{1.67}$$

One may argue that it would be wiser to solve such problem using Kelly exclusive algorithm and he would be correct. This example is here to display that using our $\boldsymbol{R}$ matrix in combination with probability vector $\boldsymbol{p}$, we are able to express any real world complex game.

### 1.6.4 General definition

Taking the matrix $\boldsymbol{R}$ and probability distribution $\boldsymbol{p}$. We proceed with general definition of the Kelly strategy.

$$\begin{aligned} \underset{\boldsymbol{b}}{\text{maximize}} \quad & \mathbb{E}[U(\mathbf{R} \cdot \mathbf{b})] \\ \text{subject to} \quad & \sum_{i=1}^{K} b_i = 1.0, \ b_i \geq 0 \end{aligned}$$

where $U$ is general non-decreasing utility function(more money is always at least as good as less money). In this text we will focus on the logarithmic utility function. Mainly for it's mathematical properties which we discussed above.

$$U(W) = \log(W) \tag{1.68}$$

## 1.7 Modern Portfolio Theory

The idea behind Modern Portfolio Theory is that portfolio $\boldsymbol{b_1}$ is superior to $\boldsymbol{b_2}$ if the expected gain $\mathbb{E}[\boldsymbol{b}]$ is at least as great.

$$\mathbb{E}[\boldsymbol{b_1}] \geq \mathbb{E}[\boldsymbol{b_2}] \tag{1.69}$$

and the risk, here general risk measure denoted $r$ is no greater, (Markowitz, 1952).

$$r(\boldsymbol{b_1}) \leq r(\boldsymbol{b_2}) \tag{1.70}$$

This creates a partial ordering on the set of all available portfolios. Taking the portfolios that no portfolio is superior gives us the set of efficient portfolios $\Theta$.

Markowitz, 1952 proposes measures of dispersion,(risk measures) that can possibly be used such as variance $Var$, standard deviation $\sigma$ and "coefficient of variation" $CV$.

$$Var[\boldsymbol{b}] \tag{1.71}$$

$$\sigma(\boldsymbol{b}) = \sqrt{Var[\boldsymbol{b}]} \tag{1.72}$$

$$CV(\boldsymbol{b}) = \frac{\sigma(\boldsymbol{b})}{\mathbb{E}[\boldsymbol{b}]} \tag{1.73}$$

In our case, portfolio $\boldsymbol{b}$ is actually a wealth allocation across different betting opportunities.

### 1.7.1 Definition

MPT can be expressed as a maximization problem:

$$\begin{aligned} \underset{\boldsymbol{b}}{\text{maximize}} \quad & \boldsymbol{\mu}^T\boldsymbol{b} - \gamma \boldsymbol{b}^T \Sigma \boldsymbol{b} \\ \text{subject to} \quad & \sum_{i=1}^{K} b_i = 1.0,\ b_i \geq 0 \end{aligned}$$

where $\boldsymbol{b}$ is fraction vector, $\gamma$ is risk aversion parameter and $\boldsymbol{\mu}$ is the expected values vector of offered opportunities. In layman terms we maximize the following:

$$return - \gamma \cdot risk \tag{1.74}$$

In the most general set up risk is defined as variance $\Sigma$.

### 1.7.2 MPT and Kelly

To understand the difference between MPT and Kelly, please recall our example of the three fortune wheels in 1.3. We had three wheels $\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}$ with uniform probability distribution and the following returns.

$$\boldsymbol{a_1}^T = \begin{bmatrix} 1.0 & 1.0 & 2.0 & 2.0 & 1.0 & 2.0 \end{bmatrix} \tag{1.75}$$

$$\boldsymbol{a_2}^T = \begin{bmatrix} 3.0 & 0.0 & 2.0 & 2.0 & 1.0 & 2.0 \end{bmatrix} \tag{1.76}$$

$$\boldsymbol{a_3}^T = \begin{bmatrix} 0.5 & 0.5 & 3.0 & 3.0 & 0.5 & 3.0 \end{bmatrix} \tag{1.77}$$

$$\boldsymbol{p} = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix} \tag{1.78}$$

Assume that we are faced with a decision of choosing a single wheel and letting all of our wealth run on a chosen wheel, instead of allocating across all three as we did in 1.3. Which wheel should we choose, which one should we choose according to the Kelly criterion and which one according to the MPT?

Additional information includes arithmetic mean for each:

$$A(\boldsymbol{a_1}) = 1.50 \quad A(\boldsymbol{a_2}) = 1.67 \quad A(\boldsymbol{a_3}) = 1.75 \tag{1.79}$$

and geometric mean for each:

$$GM(\boldsymbol{a_1}) = 1.41 \quad GM(\boldsymbol{a_2}) = 0.00 \quad GM(\boldsymbol{a_3}) = 1.22 \tag{1.80}$$

Kelly criterion says we should never choose wheel $\boldsymbol{a_2}$ for such an investment. Because running all of our money through such a wheel, we would definitely be loosing everything in the long run, Poundstone, 2010. According to the Kelly strategy we should choose $\boldsymbol{a_1}$, the wheel with the highest geometric mean. This wheel would yield us the maximum compound return.

MPT on the other hand would resort from choosing a specific wheel. All three wheels are valid choices for different risk parameter $\gamma$. Using variance as a risk measure, asset $\boldsymbol{a_1}$ is perfect for people with low risk preference, $\boldsymbol{a_3}$ for people with desire for high returns and $\boldsymbol{a_2}$ for people somewhere in the middle.

Interestingly wheel $\boldsymbol{a_2}$ has lower risk than $\boldsymbol{a_3}$ even though there is a chance of loosing everything. Which hints at the imperfection of variance as a risk measure Poundstone, 2010. On the other hand, goal of the Kelly strategy is to avoid the chance of ruin, however small it is.

Obviously, this is quite a specific example. However, more complex definitions and such wheels can easily represents assets on the stock market.

MPT can be understood as a framework. Note that the following idea is very modular:

$$Maximize \quad return - risk\_parameter * risk \tag{1.81}$$

It is exactly this modularity that we will find useful in the next chapters.

We conclude this section with a statement that MPT is one of the frameworks we will be using in the next chapters. We will show a direct connection between MPT and the Kelly strategy in 3.1.

## 1.8 The Flat Stake

In this section we will define the flat strategy and compare it to the reinvestment startegy using the wheels example, (Poundstone, 2010).

Assume a player who bets using flat strategy meaning that for a year he gets \$1 every week from his wife. He would do best by choosing the wheel with the highest arithmetic mean $\boldsymbol{a_3}$. After 52 weeks his expected earnings look as follows:

$$52 \cdot 1.75 = 87 \tag{1.82}$$

Next assume a player who starts with \$1 and reinvests his winnings every week. Here's a comparison of how he would fare choosing each wheel.

52 weeks using Kelly advised wheel $\boldsymbol{a_1}$.

$$1.41^{52} = 67,108,864 \tag{1.83}$$

Now for the second wheel $\boldsymbol{a_2}$

$$0^{52} = 0 \tag{1.84}$$

and finally for the third wheel $\boldsymbol{a_3}$

$$1.22^{52} = 37,877 \tag{1.85}$$

In case of reinvestment we see a big difference between Kelly suggested wheel and any other wheel when player reinvests his winnings.

We conclude this chapter with a few key notes. Kelly criterion is nothing more than an upper bound of how much of gambler's wealth are the presented betting opportunities worth.

Latane, 2011 used the name "geometric mean policy" instead. In short, geometric mean policy assumes gambler can not predict what the future will bring and the best thing for him to do "right-now" is to choose a portfolio with the highest geometric mean.

Chapter 2

# Uncertainty

The compounding nature of the reinvestment strategies can often be a double edge sword. Kelly is the most aggresive form of reinvestment strategy that is still sane. Hence clearly, when Kelly wins, it wins big, when it looses, it is also in a big way.

In this chapter, things get a lot more serious as our estimate of probability becomes inaccurate. Not only our estimate, but the estimate of bookie, our adversary, as well. How inaccurate and most importantly what do we mean by inaccurate?

First we present an intuitive way to grasp the uncertainty and explain how it connects to the world of betting. Second we define statistical measures used in the context of uncertainty. Third we present ways of what we previously referred to as "taming" the uncertainty, multiple modifications that make our decision more robust to error of overbetting.

Systematic overbetting hurts any betting strategy and is fatal in case of Kelly. Our ultimate goal is therefore to avoid over-valuing the betting opportunities presented to us, while still maintaining as much growth as possible. We shall see whether our proposed strategies will reign supreme or the uncertain game will void all of our assumptions.

## 2.1 Kullback-Leibler Divergence

We recall the proof from section 1.5.3 where we show that maximum growth rate in a fair odds game is achieved if we "bet our beliefs", we bet according to the probability paradoxically ignoring published odds. Expressed in the language of Kullback-Leibler divergence, (KL-divergence):

$$D(p||b) = 0 \tag{2.1}$$

Holds for.

$$b^\star = p \tag{2.2}$$

We redefine KL-divergence without the minus sign as in 1.5.3:

$$D(P||Q) = \sum_{i=0}^{n} p_i \log \frac{p_i}{q_i} \tag{2.3}$$

This is the first connection of KL-divergence to fractional betting.

### 2.1.1 KL-Advantage

In this section we show the second, most important connection of KL-divergence to fractional betting. Citing the great (Cover et al., 2012) we once again write down the formula for growth rate as follows.

$$W(b,p) = \sum_{i=1}^{K} p_i \log(b_i \cdot o_i) \tag{2.4}$$

This time we employ a slighlty different trick than in the "Maximum" proof. We define bookie's dividend implied probabilities. $d_i = \frac{1}{o_i}$

$$W(b,p) = \sum_{i=1}^{K} p_i \log(\frac{b_i}{p_i} \cdot \frac{p_i}{d_i}) \tag{2.5}$$

$$W(b,p) = \sum_{i=1}^{K} p_i \log(\frac{b_i}{p_i}) + \sum_{i=1}^{K} p_i \log(\frac{p_i}{d_i}) \tag{2.6}$$

From definition of KL-divergence we simplify as follows.

$$W(b,p) = -D(p||b) + D(p||d) \tag{2.7}$$

Or better.

$$W(b,p) = D(p||d) - D(p||b) \tag{2.8}$$

Here $p$ denotes real probability distribution $b$ denotes bettor's estimate, $d$ denotes dividend implied probabilities, hence bookie's estimate. What follows is that Kelly fractional bettor has positive growth rate if and only if his estimate is better than the dividend implied one.

Note of importance is that here we are speaking about a single game. If such a game were repeated indefinitely, fractional player has to have such an avantage or "edge" if he is to have a positive growth rate. As we will experimentally show in 4.1.

In real life it becomes almost impossible to have a consistent edge on every single game. Even in his black-jack system, Thorp, 1966 had to place "waiting" bets in disadvantageous plays. Hence over multiple games it is not required to have the edge every single time or on every single bet. The options are numerous: To have a big advantage sometimes, (often enough) and small disadvantage otherwise. To know where our edge is on the subset of bets and do a pre-selection of such subset and many other methods. One thing is for certain however, if there is no edge anywhere, Kelly strategy will be very difficult to implement successfully.

After all, the Kelly strategy as known by practicioners has always been the following.

$$\frac{edge}{odds} \tag{2.9}$$

The only way to beat the market (of stocks or horse wagers) is by knowing something of significance that other people do not. A gambler who wants to beat the market must have an edge, a more accurate view of what bets are really worth, (Poundstone, 2010).

This is the important connection between fractional betting and KL-divergence that we spoke about. Kelly betting bounds the highest achievable growth rate using fractional strategy, KL-advantage bounds Kelly betting. Such duality of problems between maximization of logarithmic utility and minimization of KL-divergence has been investigated in depth by Nau et al., 2009. Logically we decided to use these ideas in our analysis.

The previous notation might be slightly confusing. We will clear up any possible confusion and set the following notation for the rest of the thesis.

- $P_R$ real probability
- $P_B$ bookie's estimated probability, (dividend implied)
- $P_M$ model estimated probability, (gambler's estimate)

We proceed with the definition of KL-advantage.

$$A_{KL} = D(P_R||P_B) - D(P_R||P_M) \tag{2.10}$$

### 2.1.2 KL-Divergence Upper Bound

In gambling context, it is clear that the KL-divergence strongly depends on the number of states our game has. In the case $K = 3$, game has 3 possible outcomes a $D(P_R||P_M) = 0.9$ is quite “far” from the true probability distribution. In the case of $K = 220$, e.g. triplets in a horse race $D(P_R||P_M) = 0.9$ is a very good estimate. Thankfully, Cover et al., 2012 also provides us with upper bound of this distance measure.

$$D(P_R||\frac{1}{K}) = \log(K) - H(P) \tag{2.11}$$

Where did it come from? First lets consider a formula for entropy

$$H(P) = -\sum_{i=1}^{K} p_i \log(p_i) \tag{2.12}$$

Next we consider KL-divergence between $P_R$ and uniform distribution, (random guessing estimate).

$$D(P_R||\frac{1}{K}) = \sum_{i}^{K} p_i \log(\frac{p_i}{\frac{1}{K}}) = \sum_{i}^{K} p_i \log(p_i) - \sum_{i}^{K} p_i \log(\frac{1}{K}) \tag{2.13}$$

$$D(P_R||\frac{1}{K}) = -H(P_R) + \log(K) \tag{2.14}$$

At worst, our probability estimation should be a random guess. Hence $\log(K) - H(P)$, where $K$ is the number of outcomes is our worst case distance. Considering any distance that is greater than that makes no sense in the gambling context.

### 2.1.3 KL Random Spill Algorithm

In this section we present an algorithm to distance player’s estimate from the real probability, or in general to generate probability distribution $Q$ from given distribution $P$ such that $D(P||Q) = d$.

To the best of author’s knowledge, there is no such algorithm published in the existing literature. We hence first present intuitive idea behind the algorithm, second we present mathematical background, third we present pseudo-code. Finally in chapter A we present a real implementation.

#### 2.1.3.1 Intuition

The main idea behind the algorithm comes as the name suggests from “randomly spilling” the distance $d$ across the probability distribution $P$.

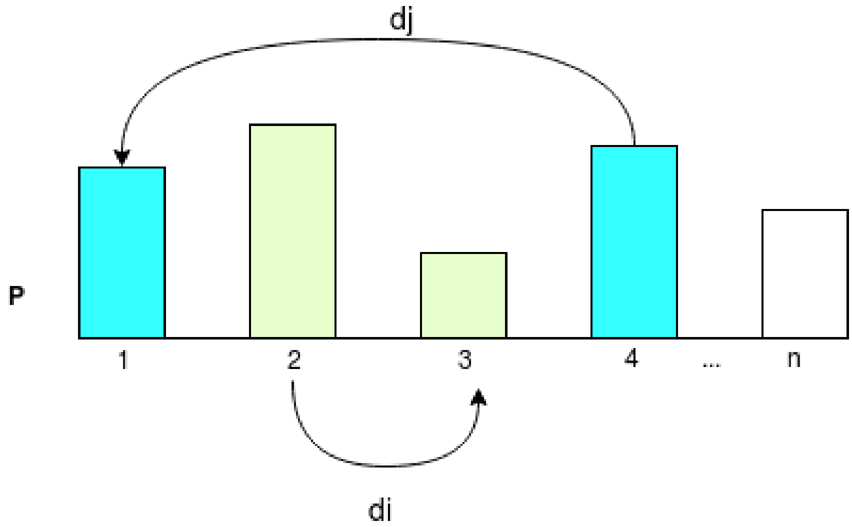


Figure 2.1: Randomly "spilling" the distance across the distribution P

Intuitively we randomly divide distance $d$ into pieces $d_1, d_2 \ldots$. We non-repetively generate random pairs $(p_i, p_j)$ from probability distribution $P$. If the number of states $n$ is an odd number, a single state, single probability is left out and has no pair. Finally we randomly "spill", add these pieces of KL-divergence across generated pairs to generate $Q$ such that $D(P||Q) = d$ and also that $Q$ stays a probability distribution $\sum_{i=1}^{K} q_i = 1$.

#### 2.1.3.2 Principle

Assume we have chosen one of the random pairs $(p_i, p_j)$, to which we have attributed one of the chunks of the desired KL-divergence $d_i$. Our goal is to generate modified pair of probabilities $(\tilde{p_i}, \tilde{p_j})$ such that the formula for KL divergence between the pairs $(p_i, p_j)$ and $(\tilde{p_i}, \tilde{p_j})$ equals the attributed distance $d_k$.

$$p_i \log(\frac{p_i}{\tilde{p_i}}) + p_j \log(\frac{p_j}{\tilde{p_j}}) = d_k \tag{2.15}$$

We express modified probabilities in the following way.

$$\tilde{p_i} = p_i \cdot \exp(\delta_i) \qquad \delta_i = \log(\frac{\tilde{p_i}}{p_i}) \tag{2.16}$$

$$\tilde{p_j} = p_j \cdot \exp(\delta_j) \qquad \delta_j = \log(\frac{\tilde{p_j}}{p_j}) \tag{2.17}$$

No loss of probability happens during our transformation.

$$p_i + p_j = \tilde{p_i} + \tilde{p_j} \tag{2.18}$$

We can hence modify the initial equation 2.15 as follows.

$$p_i(-\delta_i) + p_j(-\delta_j) = d_k \tag{2.19}$$

Our goal now is to express $\delta_j$.

$$p_j(-\delta_j) = d_k - p_i(-\delta_i) \tag{2.20}$$

$$(-\delta_j) = \frac{d_k - p_i(-\delta_i)}{p_j} \tag{2.21}$$

$$\delta_j = \frac{-d_k - p_i\delta_i}{p_j} \tag{2.22}$$

Please recall that.

$$p_i + p_j = \tilde{p_i} + \tilde{p_j} \tag{2.23}$$

Therefore $\tilde{p_j}$ can be expressed as.

$$\tilde{p_j} = p_i + p_j - \tilde{p_i} \tag{2.24}$$

$$\tilde{p_j} = p_j + p_i - (p_i \cdot \exp(\delta_i)) \tag{2.25}$$

$$\tilde{p_j} = p_j + p_i(1 - \exp(\delta_i)) \tag{2.26}$$

$$\tilde{p_j} = p_j(1 + \frac{p_i}{p_j}(1 - \exp(\delta_i))) \tag{2.27}$$

$$\tilde{p_j} = p_j(1 - \frac{p_i}{p_j}(\exp(\delta_i) - 1)) \tag{2.28}$$

$$\tilde{p_j} = p_j \cdot \exp(\log(1 - \frac{p_i}{p_j}(\exp(\delta_i) - 1))) \tag{2.29}$$

That of course looks a lot like one of our initial equations

$$\tilde{p_j} = p_j \cdot \exp(\delta_j) \tag{2.30}$$

It follows.

$$\delta_j = \log(1 - \frac{p_i}{p_j}(\exp(\delta_i) - 1)) \tag{2.31}$$

We can now simplify our original equation.

$$p_i(-\delta_i) + p_j(-\delta_j) = d_k \tag{2.32}$$

Into:

$$p_i(-\delta_i) + p_j(-\log(1 - \frac{p_i}{p_j}(\exp(\delta_i) - 1))) = d_k \tag{2.33}$$

$$-p_i\delta_i - p_j \log(1 - \frac{p_i}{p_j}(\exp(\delta_i) - 1)) = d_k \tag{2.34}$$

Finally we can drop the index from $\delta_i$ as there is only a single $\delta$ now. We define function $f$ as follows.

$$f(p_i, p_j, \delta, d_k) = -p_i\delta - p_j \log(1 - \frac{p_i}{p_j}(\exp(\delta) - 1)) - d_k \tag{2.35}$$

For a specific pair of probabilities $(p_i, p_j)$, distance $d_k$, we are looking for a $\delta$ such that

$$f(p_i, p_j, d_k, \delta) = 0 \tag{2.36}$$

$$\delta^\star \ \ such\ \ that \quad -p_i\delta - p_j \log(1 - \frac{p_i}{p_j}(\exp(\delta) - 1)) - d_k = 0 \tag{2.37}$$

We then generate modified pair of probabilities $(\tilde{p_i}, \tilde{p_j})$ as follows

$$\tilde{p_i} = p_i \cdot \exp(\delta^\star) \tag{2.38}$$

From $p_i + p_j = \tilde{p_i} + \tilde{p_j}$ we express $\tilde{p_j}$.

$$\tilde{p_j} = p_i + p_j - \tilde{p_i} \tag{2.39}$$

We repeat this procedure for every pair of probabilities $(p_i, p_j)$ and every assigned distance $d_k$ to generate new pair $(\tilde{p_i}, \tilde{p_j})$. Please note that every probability $p_i$ exists in only a single pair, (they are paired with non-repetition). If there is a single probability $p_k$ such that it has no pair, it is copied with no modification.

This way we replace every single probability in the original probability distribution $P$ to generate a new distribution $Q$ such that

$$D(P||Q) = \sum_{k=1}^{c} d_k = d \tag{2.40}$$

Where $c$ stands for number of pairs we can split probability distribution $P$ into.

#### 2.1.3.3 Pseudocode

**Algorithm 2** KL random spill algorithm

```
procedure RandomSpill(P = [p_1, p_2, ..., p_n], d)
    pairs, q_i = random_pairs([p_1, p_2, ..., p_n])    ▷ randomly pair states of P

    distances = random_split(d, count = length(pairs))
    Q = [0, ..., 0]                                   ▷ initialize

    if q_i then                                       ▷ If there is odd number of states
        Q[i] = q_i                                    ▷ The state with no pair is copied unmodified
    end if

    assigned = random_assign(pairs, distances)
    for [(p_i, p_j), d_k] in assigned do
        Find δ* such that  − p_i δ − p_j log(1 − (p_i/p_j)(exp(δ) − 1)) − d_k = 0

        p̃_i = p_i · exp(δ*)
        p̃_j = p_i + p_j − p̃_i
        Q[i] = p̃_i
        Q[j] = p̃_j
    end for
    return Q
end procedure
```

### 2.1.4 Divergence Spread

KL divergence is in it's essence a very difficult measure to grasp. We hence define the following for the two distances.

$$KL_s = \frac{D(P_R||P_M) - D(P_R||P_B)}{\frac{1}{2}(D(P_R||P_M) + D(P_R||P_B))} \tag{2.41}$$

intuitively it is

$$KL_s = \frac{advantage}{average} \tag{2.42}$$

### 2.1.5 Market Efficiency

There are many definitions for market efficiency. We will use the following inspired by (Cover et al., 2012).

$$EFF = 1 - \frac{D(P_R||P_B)}{\log(K) - H(P_R)} \tag{2.43}$$

intuitively it is how much potential there is for growth in the given market.

$$EFF = 1 - \frac{bookie's\ \ KL\ \ distance}{random\ \ guess} \tag{2.44}$$

If bookie is random guessing, efficiency is zero. Market is very inefficient. There is a lot of "inefficiency" to make money from. If bookie has his estimate so accurate as it is equal to the real probability, $EFF = 1$, there is almost no opportunity to make money systematically.

## 2.2 Statistical KL Measures

We have seen how we can calculate KL-advantage $A_{KL}$ given real probability distribution $P_R$, model probability $P_M$ and bookie's odds implied probability $P_B$. Question that remains is how do we calculate our KL measures on real data where $P_R$ is unknown? We do so statistically.

In real data our $P_R$ for given game comes in the form of vector consisting of zeros and ones(a single one if the game is exclusive). Simple example of $P_R$ for arbitrary exclusive game looks as follows.

$$P_{R_i} = \begin{bmatrix} 0 & 1 & 0 & 0 & ... & 0 \end{bmatrix} \tag{2.45}$$

Where 1 marks the outcomes that realized and 0 marks the outcomes that did not realize. The zeros would lead to problems in our calculation.

$$D(P_R||P_M) = 0 \cdot \log(\frac{0}{p_{M_1}}) + 1 \cdot \log(\frac{1}{p_{M_2}}) + ... \tag{2.46}$$

We conveniently leave out the zeroed terms from our calculation. Then take calculate the average distance from the dataset.

$$D(P_R||P_M)S = \frac{\sum_{i=1}^{game\_count} D(P_{R_i}||P_{M_i})}{game\_count} \tag{2.47}$$

### 2.2.1 Statistical KL-advantage

It follows that calculating the KL-advantage from the data, we will use the previously defined statistical distance measures.

$$A_{KL_S} = D_S(P_R||P_B) - D_S(P_R||P_M) \tag{2.48}$$

### 2.2.2 Statistical Efficiency

We calculate the previously defined market efficiency $EFF$ from the data in a similar manner. We only add minor fool proof modifications for when we calculate from the data.

Our $P_R$ is now an identity vector, hence it's entropy can be removed from our efficiency formula.

$$H(P_R) = 0 \tag{2.49}$$

We also put a bound on the distance using minimum function. We do so as there may be samples where the distance goes over our upper bound $\log(K)$. We have mentioned, anything worse than the random guess distance $D(P_R||\frac{1}{K}) = \log(K)$ makes no sense in the gambling context. Our average statistical efficiency $EFF_S$ is hence a slightly transformed definition of the standard efficiency $EFF$.

$$EFF_S = 1 - \frac{min(D_S(P_R||P_B), \log(K))}{\log(K)} \tag{2.50}$$

We can also transform into.

$$EFF_S = 1 - min(1, \frac{D_S(P_R||P_B)}{\log(K)}) \tag{2.51}$$

Note that it is required to take group games according to the number of outcomes, calculate average and then divide by the upper bound. Bookie can possibly be systematically more accurate in the race of 10 horses than in the race of 16 horses. Hence resulting efficiency in each subset of races will differ.

Intuitively, bookie is at best, (from player's perspective) making his odds up randomly, his KL-distance from reality is then $\log(K)$, where $K$ is the number of outcomes the game has. Efficiency of the market is then 0. We can easily make money.

At worst he is making his odds up according to the true probability in which case $D_S(P_R||P_B) = 0$ and hence $EFF_S = 1$. Market is then fully efficient.

## 2.3 Fractional Kelly

The basic idea behind the fractional Kelly is that we bet only a fraction of suggested Kelly optimal fractions. The most famous being "Half-Kelly", where as the name suggest we multiply optimal fractions $\boldsymbol{b}^\star$ by $\frac{1}{2}$.

We have seen that betting less than Kelly results in sub-optimal strategy 1.4 . Why exactly is it good idea to bet "Half-Kelly" then?

Fractional Kelly allows us to trade the optimality of our strategy for more security. When our model estimation $P_M$ is inaccurate, we must at all cost avoid over betting the presented betting opportunities. We rather trade the optimality of growth for security regarding our bank.

We recall that betting more than Kelly, as we have seen in 1.4 results in ruin Thorp, 1975.

The next picture taken from (Poundstone, 2010) provides a good overview of what a different fraction means for Kelly.

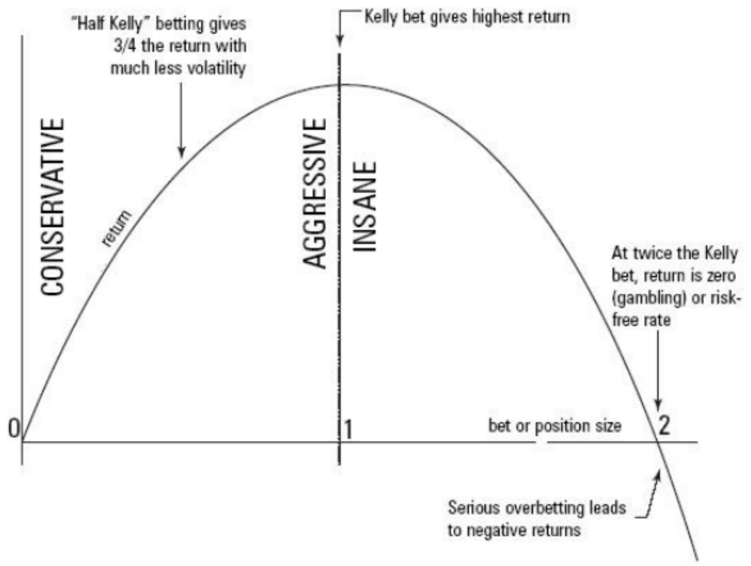


Figure 2.2: Fractional Kelly

We define trade-off index $\lambda$ and investment model $M$. $M_k$ stands for Kelly model and $M_0$ is a model where the only investment is a risk-free, “cash“ investment. We will investigate all models generated by convex combinations.

$$M(\lambda) = \lambda M_k + (1 - \lambda)M_0 \tag{2.52}$$

$\lambda$ or trade-off index is now a lever between “growth“ and “security“ (MacLean et al., 2011).

In the next section we will present a method of managing the bank with a more elegant solution than the ad-hoc of Half-Kelly.

## 2.4 Drawdown

Assume that $W^{MIN}$ is the lowest our wealth goes. The drawdown is then defined as follows from (Busseti et al., 2016).

$$1 - W^{MIN} = 0.3 \tag{2.53}$$

Meaning our wealth dropped down by 30%. We can then define the risk of drawdown as follows

$$P(W^{MIN} < \alpha) = \beta \tag{2.54}$$

Probability of experiencing $1 - \alpha$ drawdown equals *beta*. For example:

$$P(W^{MIN} < 0.7) = 0.1 \tag{2.55}$$

is a probability of experiencing more than 30% drawdown is 0.1.

We then include this additional constraint in our model definition Busseti et al., 2016.

$$\begin{aligned} \underset{\boldsymbol{b}}{\text{maximize}} \quad & \mathbb{E}[U(\boldsymbol{R} \cdot \boldsymbol{b})] \\ \text{subject to} \quad & \sum_{i=1}^{K} b_i = 1.0 \\ & b_i \geq 0 \\ & P(W^{MIN} < \alpha) < \beta,\ \alpha, \beta \in (0,1) \end{aligned}$$

This constraint is very difficult to implement. It is far wiser to consider an approximation.

$$P(W^{MIN} < \alpha) = \beta \tag{2.56}$$

Can be approximated as Busseti et al., 2016

$$\mathbb{E}[\boldsymbol{R} \cdot \boldsymbol{b}]^{-\lambda} \leq 1 \tag{2.57}$$

Where

$$\lambda = \frac{\log(\beta)}{\log(\alpha)} \tag{2.58}$$

We can say that if bet satisfies $\mathbb{E}[\boldsymbol{R} \cdot \boldsymbol{b}]^{-\lambda} \leq 1$ it's drawdown risk satisfies $P(W^{MIN} < \alpha) = \beta$.

Our drawdown constraint then becomes

$$\mathbb{E}(\boldsymbol{R} \cdot \boldsymbol{b})^{-\lambda} \leq 1 \tag{2.59}$$

Which we can reformat the following way.

$$\log(\sum_{i=1}^{K} p_i \cdot (\boldsymbol{r_i} \cdot b_i)^{-\lambda}) \leq \log(1) \tag{2.60}$$

we know that

$$x = \exp(\log(x)) \tag{2.61}$$

therefore we can further simplify into the following.

$$\log(\sum_{i=1}^{K} \exp(\log(p_i \cdot (\boldsymbol{r_i} \cdot b_i)^{-\lambda}))) \leq 0 \tag{2.62}$$

$$\log(\sum_{i=1}^{K} \exp(\log(p_i) + \log(\boldsymbol{r_i} \cdot b_i)^{-\lambda})) \leq 0 \tag{2.63}$$

$$\log(\sum_{i=1}^{K} \exp(\log(p_i) - \lambda \log(\boldsymbol{r_i} \cdot b_i))) \leq 0 \tag{2.64}$$

This log-sum-exp constraint is convex Busseti et al., 2016 and we will discuss it's implementation in A. Our Kelly model now includes the drawdown constraint.

$$\begin{aligned} \underset{\boldsymbol{b}}{\text{maximize}} \quad & \sum_{i=1}^{K} p_i log(\boldsymbol{r_i} \cdot b_i) \\ \text{subject to} \quad & \sum_{i=1}^{K} b_i = 1.0 \\ & b_i \geq 0 \\ & \log(\sum_{i=1}^{K} \exp(\log(p_i) - \lambda \log(\boldsymbol{r_i} \cdot b_i))) \leq 0 \end{aligned}$$

where $\lambda = \frac{\log(\beta)}{\log(\alpha)}$ for some $\alpha, \beta \in (0, 1)$

Both the fractional and the drawdown constraint can be used to gain more of overall security in our selected portfolios.

In this chapter we successfully defined uncertainty in the context of gambling, it's upper bound and measures. The key notes from this chapter are that the KL-advantage directly bounds the Kelly growth optimal strategy. We will propose a testing framework based on this idea in the final chapter.

Chapter 3

# Simultaneous Games

We would like to begin the chapter about simultaneous games with the following example. Please assume a game of football between Hrozenkov and Drnovice with three available betting opportunities.

1. WIN Hrozenkov
2. WIN Drnovice
3. DRAW

Next please assume a quite realistic scenario that today there are 30 of such football games. Hence there are 90 betting opportunities, an acceptable number. To express this problem in the language of Kelly gambling, all we need now is to generate matrix $\boldsymbol{R}$, where each row stands for each possible outcome of our world. The number of possible outcomes is $3^{30}$ and this is exactly where our problem begins. If we have $3^{30}$ possible outcomes and 90 betting opportunities + 1 cash asset. No more needs to be said about the size of matrix $\boldsymbol{R}$.

This chapter is all about the realization that such large scale bet aggregations are impossible to solve using the Kelly strategy. To be able to proceed, we will once again be forced to trade optimality for reality. First the logarithm in our model will be dealt with. In the second part, we will be forced to cut the state space and finally in the third section we will discuss easier solutions that are available under a very specific set of circumstances.

## 3.1 Quadratic Kelly

There are not too many solvers which can deal with logarithms both in problem definitions and also in constraints, (drawdown constraint). If they do, they usually use quite a bit of computational power. Therefore if we want to solve larger bet aggregation problems we have to deal with the logarithm first.

We do so by using the Taylor expansion. Let us first recall that.

$$\log(1+x) = x - \frac{x^2}{2} + O(x^3) \tag{3.1}$$

We define a modification of our matrix $\boldsymbol{R}$ to be $\boldsymbol{\rho}$ as follows.

$$\boldsymbol{R} - 1 = \boldsymbol{\rho} \tag{3.2}$$

Next according to Busseti et al., 2016, we make assumption that $\boldsymbol{R} \cdot \boldsymbol{b} \approx \mathbf{1}$ and we express the logarithmic part of our Kelly problem as follows.

$$\log(\boldsymbol{R} \cdot \boldsymbol{b}) = \log(1 + \boldsymbol{\rho} \cdot \boldsymbol{b}) \tag{3.3}$$

Our transformation can then proceed.

$$\log(1 + \boldsymbol{\rho} \cdot \boldsymbol{b}) = \boldsymbol{\rho} \cdot \boldsymbol{b} - \frac{(\boldsymbol{\rho} \cdot \boldsymbol{b})^2}{2} + ... \tag{3.4}$$

Taking only the first two terms, we arrive at our new problem definition. We transform from the now well understood Kelly formula.

$$\begin{aligned} \underset{\boldsymbol{b}}{\text{maximize}} \quad & \mathbb{E}[\log(\boldsymbol{R} \cdot \boldsymbol{b})] \\ \text{subject to} \quad & \sum_{i=1}^{K} b_i = 1.0 \\ & b_i \geq 0 \end{aligned}$$

Into a new formula.

$$\begin{aligned} \underset{\boldsymbol{b}}{\text{maximize}} \quad & \mathbb{E}[\boldsymbol{\rho} \cdot \boldsymbol{b} - \frac{(\boldsymbol{\rho} \cdot \boldsymbol{b})^2}{2}] \\ \text{subject to} \quad & \sum_{i=1}^{K} b_i = 1.0 \\ & b_i \geq 0 \end{aligned}$$

### 3.1.1 Quadratic Drawdown

Similarly we modify the drawdown constraint according to (Busseti et al., 2016).

$$\mathbb{E}[\boldsymbol{R} \cdot \boldsymbol{b}] \leq 1 \tag{3.5}$$

$$(\boldsymbol{R} \cdot \boldsymbol{b})^{-\lambda} = (1 + \boldsymbol{\rho} \cdot \boldsymbol{b})^{-\lambda} \tag{3.6}$$

First we recall the following.

$$(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n \tag{3.7}$$

Next we follow through with the expansion according to the formula.

$$(1 + \boldsymbol{\rho} \cdot \boldsymbol{b})^{-\lambda} = 1(\boldsymbol{\rho} \cdot \boldsymbol{b})^0 - \lambda(\boldsymbol{\rho} \cdot \boldsymbol{b})^1 - \lambda(-\lambda - 1)\frac{(\boldsymbol{\rho} \cdot \boldsymbol{b})^2}{2} \tag{3.8}$$

If we take only first two terms, our quadratic approximation for drawdown constraint is as follows.

$$1(\boldsymbol{\rho} \cdot \boldsymbol{b})^0 - \lambda(\boldsymbol{\rho} \cdot \boldsymbol{b})^1 - \lambda(-\lambda - 1)\frac{(\boldsymbol{\rho} \cdot \boldsymbol{b})^2}{2} \leq 1 \tag{3.9}$$

$$-\lambda(\boldsymbol{\rho} \cdot \boldsymbol{b})^1 + \lambda(\lambda + 1)\frac{(\boldsymbol{\rho} \cdot \boldsymbol{b})^2}{2} \leq 0 \tag{3.10}$$

$$\lambda(\lambda + 1)\frac{(\boldsymbol{\rho} \cdot \boldsymbol{b})^2}{2} \leq \lambda(\boldsymbol{\rho} \cdot \boldsymbol{b}) \tag{3.11}$$

Quadratic approximation of our Kelly model is as follows.

$$\begin{aligned} \underset{\boldsymbol{b}}{\text{maximize}} \quad & \mathbb{E}[\boldsymbol{\rho} \cdot \boldsymbol{b}] - \frac{1}{2}\,\mathbb{E}[(\boldsymbol{\rho} \cdot \boldsymbol{b})^2] \\ \text{subject to} \quad & \sum_{i=1}^{K} b_i = 1.0 \\ & b_i \geq 0 \\ & \lambda(\lambda + 1)\frac{(\boldsymbol{\rho} \cdot \boldsymbol{b})^2}{2} \leq \lambda(\boldsymbol{\rho} \cdot \boldsymbol{b}) \end{aligned}$$

We note that our model now follows the MPT framework.

$$Maximize \quad return - \frac{1}{2} * risk \tag{3.12}$$

Or as previously mentioned.

$$\begin{aligned} \underset{\boldsymbol{b}}{\text{maximize}} \quad & \boldsymbol{\mu}^T\boldsymbol{b} - \gamma\boldsymbol{b}^T\Sigma\boldsymbol{b} \\ \text{subject to} \quad & \sum_{i=1}^{K} b_i = 1.0,\ b_i \geq 0 \end{aligned}$$

Where our $\mathbb{E}[\boldsymbol{\rho} \cdot \boldsymbol{b}]$ is in the basic MPT definition defined as $\boldsymbol{\mu}^T\boldsymbol{b}$ or excess return. $\mathbb{E}[(\boldsymbol{\rho} \cdot \boldsymbol{b})^2]$ is the risk measure $\boldsymbol{b}^T\Sigma\boldsymbol{b}$ (Variance). $\frac{1}{2} = \gamma$ is the risk aversion parameter.

Similarly to Markowitz, 1952 we arrive at the conclusion that geometric mean is approximately the artihmetic mean minus $\frac{1}{2}$ of variance.

What will be important in the following sections is that our covariance matrix $\boldsymbol{\Sigma}$ is calculated as follows.

$$\boldsymbol{\Sigma} = \mathbb{E}[\boldsymbol{\rho} \cdot \boldsymbol{\rho}^T] \tag{3.13}$$

$$\mathbf{\Sigma} = \sum_{i=1}^{K} p_i(\rho_i \cdot \rho_i^T) \tag{3.14}$$

Where $K$ is the number of outcomes our world has and $p_i$ is from joint probability distribution.

It is hence still required to generate the joint probability distribution and a modified matrix $\boldsymbol{\rho}$ of all the possible outcomes. Number of outcomes from our initial example is $K = 3^{30}$. Clearly this is numerically quite difficult task especially for subsequent simulation of parallel histories.

We conclude this section with a statement that quadratically approximating the Kelly strategy is an absolute necessity when dealing with many simultaneous games as we will see in experiment 4.3.2.

## 3.2 State Space

Following our findings from the previous section. If we want to solve large bet aggregations we will often be forced to somehow select the betting opportunities we are interested in from the ones that we can possibly ignore.

The first distinction that often comes to mind is the positive expected return. As we have already seen in one of our examples in chapter 1. Removing the negative expectation bets has an effect of a very small decrease in optimality.

In the next few subsections we will discuss bet selection criteria we experimented with.

### 3.2.1 Expected Return

We select only the betting opportunities, where our expected return exceeds some specified value. Often the most natural filter is to only include only the positive expected value bets.

$$p \cdot o \geq 1.0 \tag{3.15}$$

Or in alternative notation of excess return $\mu$

$$\mu \geq 0 \tag{3.16}$$

### 3.2.2 Sharpe Ratio

Sharpe ratio is also known as a “reward-to-variability“ ratio and the definition is as follows.

$$\frac{\mu - rf}{\sigma} \tag{3.17}$$

Where $\mu$ stands for excess return of the bet, $\sigma$ for it's standard deviation and $rf$ is a risk free rate. In the case of betting we will always assume $rf = 0$.

In the next section we will look into a separate strategy based on maximizing this value across the whole portfolio.

## 3.3 Maximum Sharpe Ratio

Our previous modifications grealty helped us. We no longer have to solve the logarithm in our problem definition and constraints, we also reduced our state space significantly, reducing every typical three state game(WIN, LOSS, DRAW) into often two betting opportunities, sometimes into a single bet. We can assume that the typical three state game will be reduced into two bets, sometimes a single bet, hence $2^{30}$. That is still a very respectable number of elements in our joint probability distribution.

The idea is to select only a single bet in the entire game (or none). Obviously this leads to a massive reduction of state space, however, there is one more thing that can be safely assumed and that is independence of such selected outcomes. With such assumption, we no longer need to generate the joint probability distribution. Our covariance matrix now looks as follows. The idea of opportunity preselection is taken from Hubáček, 2017.

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & 0 & 0 & ... & 0 \\ 0 & \sigma_2^2 & 0 & ... & 0 \\ 0 & 0 & \sigma_3^2 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & \sigma_n^2 \end{bmatrix} \tag{3.18}$$

where $\sigma_i^2$ stands for variance of a single betting opportunity $i$. Covariance matrix $\boldsymbol{\Sigma}$ is hence a diagonal matrix that is easily generated.

Variance of a single betting opportunity is calculated as follows.

$$\sigma^2 = p(1-p) \cdot o^2 \tag{3.19}$$

From the MPT definition we recall that accross the porfolio, $\boldsymbol{\sigma}$ is defined as follows.

$$\boldsymbol{\sigma} = \sqrt{\boldsymbol{b}\boldsymbol{\Sigma}\boldsymbol{b}} \tag{3.20}$$

We can now define a separate strategy based on the maximum Sharpe selection criteria.

$$\begin{aligned} \underset{\boldsymbol{b}}{\text{maximize}} \quad & \frac{\boldsymbol{\mu b}}{\sqrt{\boldsymbol{b \Sigma b}}} \\ \text{subject to} \quad & \sum_{i=1}^{K} b_i = 1.0 \\ & b_i \geq 0 \end{aligned}$$

One might object that not all games can be simplified into a single outcome. For instance combination bets in a horse race can go up to 220 different betting opportunities per single race. If we were to always pick a single one, we might not win once in a year.

We hence believe that there are games where aggregation is a great idea such as basketball or football. Where we do not over simplify the games and generally, the more we aggregate, the better. There are also games such as horse racing, athletics, swimming where we should not over-simplify.

## 3.4 Parlays

We include this section only informatively because in a real world, bookie often offers accumulator bets also referred to as "parlays" on some sports, between some games.

Assume a perfect world that he indeed offers such bets and he does so on every single sport and between all possible games. Additionaly assume that all payoffs of such accumulator bets are no higher and especially no lower than the product of single bet payoffs included in the accumulator bet.

$$o_{1,2,3} = o_1 \cdot o_2 \cdot o_3 \tag{3.21}$$

If all such requirements are met. According to Grant et al., 2008 only individual games have to be considered. Overall optimal fractions come from the products of the individual game fractions and their complements.

As an example we consider 2 game, 2 outcome scenario. $b^*_{(1,1)}$ optimal fractions for individual game 1 outcome 1. $b^*_{(2,1)}$ respectively.

$$\begin{aligned} b_{(1,1)(2,1)} &= b^*_{(1,1)} \cdot c^*_{(2,1)} \\ b_{(1,1)} &= b^*_{(1,1)} \cdot c^*_2 \\ b_{(2,1)} &= b^*_{(2,1)} \cdot c^*_1 \end{aligned}$$

Where $c^*_1$,$c^*_2$ are resources left in the cash asset.

Chapter **4**

# Experiments

In this chapter, we will finally put our findings to a test. First we will conduct a verification experiment using generated data. Then we will test defined strategies on three different games intentionally selected to be from three different domains where each game presents a different set of challenges.

In the first section we will look at a hypothetical game and experiment with three different scenarios. We will show the connection between KL-advantage, Kelly strategy and compound growth rate.

The sections onward present real world experiments. We will first briefly introduce the domain or the dataset available to us. In the second step we will "map the territory" by measuring the KL measures across the dataset. This will give us a first guess of what we can expect from the problem.

We will then proceed with simulation of appropriate strategies. The discussed strategies will be Kelly, Quadratic Kelly (QKelly) and the Max-Sharpe strategy (MSharpe). Finally in the third part, we will propose what we believe to be the best solution for given problem. We will compare the markets according to the following measures.

- *m-acc* model count accuracy.
- *b-acc* bookie count accuracy.
- $n$ number of risky assets in a single game.
- *odds* odds range
- *tt* track take, (margin) taken by bookie.
- $A_{KL}/log(K)$ KL-advantage of our model over the upper bound.
- $E$ KL-efficiency of the market.

## 4.1 Generated Data

In this section we present a convenient testing framework which allows for simulations of arbitrary betting scenarios. Additionally, we will experimentally verify the correctness of our previous definitions from chapter 2.

### 4.1.1 Betting Scenario

In a game with $K$ outcomes we assume two entities. Bookie($B$) and a gambler($M$) for "model". Both bookie and gambler have their belief of probability distribution. Our ultimate goal is to model the test as close to a real world as possible. We hence further assume that both $B$ and $M$ are incorrect in their respective assumptions at least to some degree. What it means is that $B$ and $M$ are somewhat "KL-distanced" from the real probability distribution($R$) of the event.

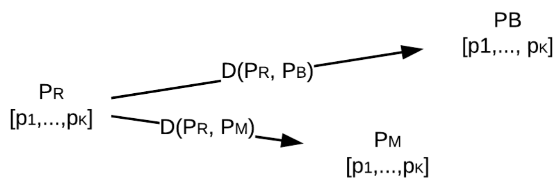


Figure 4.1: Intuitive understanding of the betting scenario

### 4.1.2 Framework Definition

What follows is that we are going to be working with triplets.

$$(P_R, P_B, P_M) \tag{4.1}$$

Where $P_R$ will be used to execute a game. $P_B$ will be transformed into odds and together with $P_M$ it will then be used to make betting decisions. The single game triplet generation process is as follows:

1. Generate $P_R$ for $n$ outcome game from Dirichlet distribution
2. Generate $P_B$ such that $D(P_R||P_B) = d_B$
3. Generate $P_M$ such that $D(P_R||P_M) = d_M$

4. Generate fair odds $O$ from bookie's estimate $P_B$ such that $o_i = \frac{1}{p_{b,i}}$
5. Introduce possible track-take, (margin) into the odds $O$ and generate $\tilde{O}$
6. Combine the previous steps into triplet $(P_R, \tilde{O}, P_M)$

Choosing $d_B, d_M$ will decide what kind of advantage or disadvantage gambler is playing with. It can be a constant or it can come from arbitrary distribution, allowing us to model situations of having e.g. big edge sometimes and a small disadvantage otherwise.

#### 4.1.2.1 Track Take

We introduce track take $tt$ into the framework using simple formula.

$$\tilde{o_i} = (1 - tt) \cdot o_i \tag{4.2}$$

Usually this makes the situation even harder for the player (M).

### 4.1.3 Experiment

Our experiment will feature a 4 outcome game with no track-take, hence $K = 4$ and $tt = 0$.

We will generate and compare three different scenarios from the viewpoint of KL-advantage. Gambler in advantage, no advantage, gambler in disadvantage.

The three scenarios are as follows.

1. Advantage: Bookie is on average less precise

$$d_M \sim \mathcal{N}^+(0.020,\ 0.0015)$$

$$d_B \sim \mathcal{N}^+(0.021,\ 0.0015)$$

2. Both are equally precise

$$d_M \sim \mathcal{N}^+(0.020,\ 0.0015)$$

$$d_B \sim \mathcal{N}^+(0.020,\ 0.0015)$$

3. Disadvantage: Model is less precise

$$d_M \sim \mathcal{N}^+(0.021,\ 0.0015)$$

$$d_B \sim \mathcal{N}^+(0.020,\ 0.0015)$$

One might object that there is now a non-zero probability of having a negative distance $d_M, d_B$, which would surely cause trouble in our subsequent calculations. For simplicity sake we hence assure positivity of $d_M, d_B$ by using log-normal distribution and the following transformation

$$\mu = \log(\frac{E[X]^2}{\sqrt{Var[X] + E[X]^2}}) \tag{4.3}$$

$$\sigma^2 = \log(1 + \frac{Var[X]}{E[X]^2}) \tag{4.4}$$

$$\mathcal{N}^+ \sim Lognormal(\mu, \sigma^2) \tag{4.5}$$

Where as $E[X]$ we input the required mean for our non-negative "normal" distribution and $Var[X]$ the required variance. For simplicity sake please assume all the "normal" distributed distances were generated in such a way to be non-negative. We use the notation $\mathcal{N}^+$.

We generate 1000 parallel Kelly betting histories of 1000 games, take the mean history for each scenario and the results are as shown in the following figure.

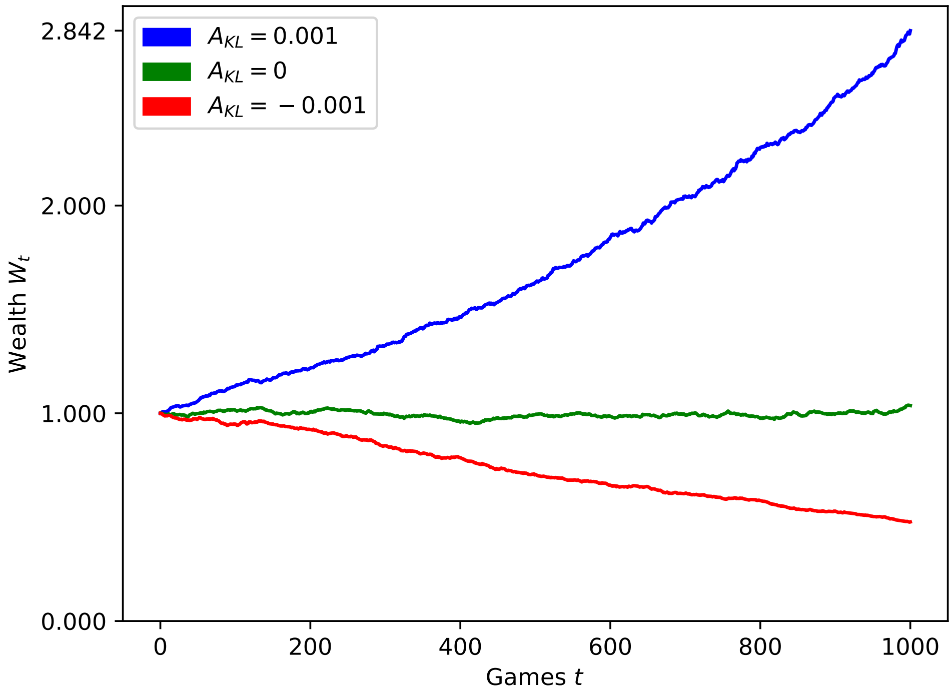


Figure 4.2: Testing framework experiment. Blue - advantageous mean history, Green - equal, Red - disadvantageous mean history.

It is clear that the equal scenario(green) fluctuates around the intitial wealth. The relatively small disadvantage produces clear loss and advantage clear profit. The longer our generated histories, the smoother results we get. In the picture 4.2 advantageous scenario clearly shows almost exponential growth, equal scenario is close to being linear at the initial wealth.

### 4.1.4 KL-advantage

Next we will show a direct connection between KL-advantage and the compound growth rate.

$$A_{KL} = d_B - d_M \tag{4.6}$$

$$A_{KL} \approx 0.001 \tag{4.7}$$

$$CGR = (\frac{end \;\; value}{start \;\; value})^{\frac{1}{\#reinvestments}} - 1 \tag{4.8}$$

We started on 1.0 and ended on approximately 3.0 after 1000 games where we reinvested.

$$CGR = (\frac{2.842}{1.0})^{\frac{1}{1000}} - 1 \tag{4.9}$$

$$CGR \approx 0.001 \tag{4.10}$$

$$CGR \approx A_{KL} \tag{4.11}$$

Compound growth rate is truly equal to the KL-advantage.

We conclude this experiment with a statement that the KL random spill algorithm works correctly. Our testing framework is sufficient for our purpose as it is now possible to model uncertainty in probability predictions and it's variance for both advantageous and disadvantageous scenarios in general K-outcome games.

## 4.2 Horse Race

In South Korea, geographically there are three main cities. Seoul the capital, the port city Busan and the capital of Jeju island, Jeju City. Every single one of those cities organizes horse races. Interestingly, every one of those races uses different kinds of horses. Especially in the case of Jeju, where they use the traditional Jeju pony.

We collected relevant data from the Korean Racing Agency,(KRA) to build a model using conditional logistic regression and provide approximately 2500 horse races of our model probability estimation $P_M$, odds $O$ transformed into bookie's estimate $P_B$ and $P_R$ in the form of realized games, hence vectors of zeros and ones. We will investigate Seoul, WIN pool. Meaning bets represent horse winning the race.

### 4.2.1 Market

The Korean horse racing market has the following properties.

| $m\text{-}acc$ | $b\text{-}acc$ | $n$ | $odds$ | $tt$ | $A_{KL}/log(n)$ | $E$ |
|---|---|---|---|---|---|---|
| $\approx 0.537$ | $\approx 0.523$ | $\in [6, 16]$ | $\in [1.0, 931.3]$ | $\approx 0.2$ | 0.0057 | 0.317 |

In horse racing and many other sports the market is pari-mutuel, meaning that money is put into a shared pool from which a heavy track-take is usually taken. "Bookie's" estimate is hence made up entirely according to the public opinion.

Public is well informed by the KRA website which provides predictions based on a similar system like ours. However we believe that people's thinking is always somewhat biased.

The problem is therefore specific in a way that our model needs to beat a much higher track take than in the case of bookie's odds. On the other hand our empirical evidence suggests that getting systematic KL-advantage against public opinion will be easier.

What is important is that in the overall evaluation of the betting opportunities our model is closer to estimating their true value as shown by the $A_{KL}/log(n)$ value.

Another factor of specificity comes in the form of number of outcomes. Whereas football game has 3, the WIN bet we will be looking at often up to 16 outcomes. Higher number of outcomes leads to infrequent, often much higher payoffs than a game with less outcomes.

### 4.2.2 Strategy

In a single exclusive game such as horse race, the optimal strategy for flat staking as has been discussed by Kelly Jr, 2011, is simply choosing the highest expected value bet in each race. We hence directly proceed with discussion about the reinvestment strategies.

So far we always presented a mean trajectories of the Kelly strategy. Which may give the impression that Kelly strategy is in general quite stable. In reality however, unless risk constrained, it often produces trajectory that swings from disaster into lottery winning. We present a single run of Kelly and QKelly to show how it usually behaves on a real data.

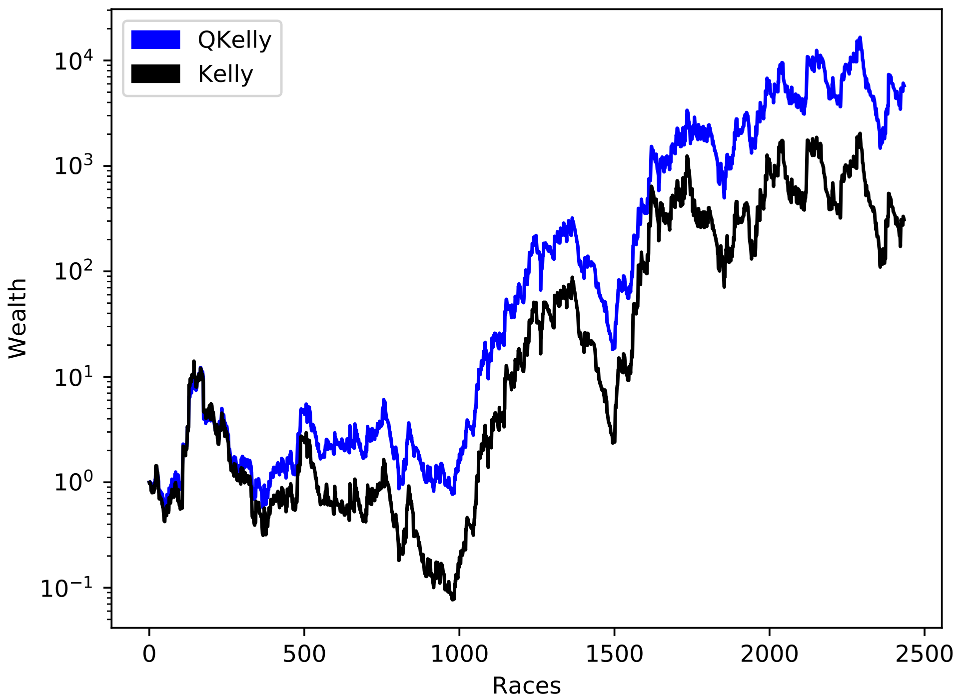


Figure 4.3: A single history for Kelly and QKelly on horse racing data

Kelly strategy is well known for it's large deviations. Interestingly in this particular dataset QKelly does better than Kelly. Additionally we observe, that even though we posses a systematic KL advantage, a true "privilege" in the world of betting, Kelly, being the growth optimal strategy is able to yield very high returns on one side and dangerously low $\frac{1}{10}$ of our bank on the other. This brings us to a discussion of what makes a truly reasonable strategy.

Reasonable strategy can be loosely defined as a middle ground between one risky strategy that is aiming to win a lottery and a second highly conservative strategy that makes any returns impossible.

To propose a reasonable strategy we will always present a boxplot comparison of strategy performance on the training dataset and testing dataset in addition to a selection criterion.

#### 4.2.2.1 Proposal

When a systematic KL-advantage is achieved, we do not need to overly constraint the Kelly strategy. Any additional constraints will make the strategy more reasonable, but less optimal. Our risk constraint for a reasonable strategy is hence as follows.

$$P(W^{MIN} < 0.4) = 0.1 \tag{4.12}$$

Meaning that our wealth will go below 40% with only 10% probability.

The following boxplot is generated from 500 random iterations through both training and testing dataset. The reinvestment trajectory reaches the same wealth no matter the order of the games. To display the high variance of returns in horse racing we shuffled the order of the races in both the training and testing datasets. The wealth trajectory starts and ends at the same point no matter the order of the races. The high variance may be surprising, but such is the nature of horse racing where both model and bookie are quite "far" from reality in their estimates and infrequent wins pay very high dividends. Please recall that the odds range from 1.0 to 931.9 in our dataset.

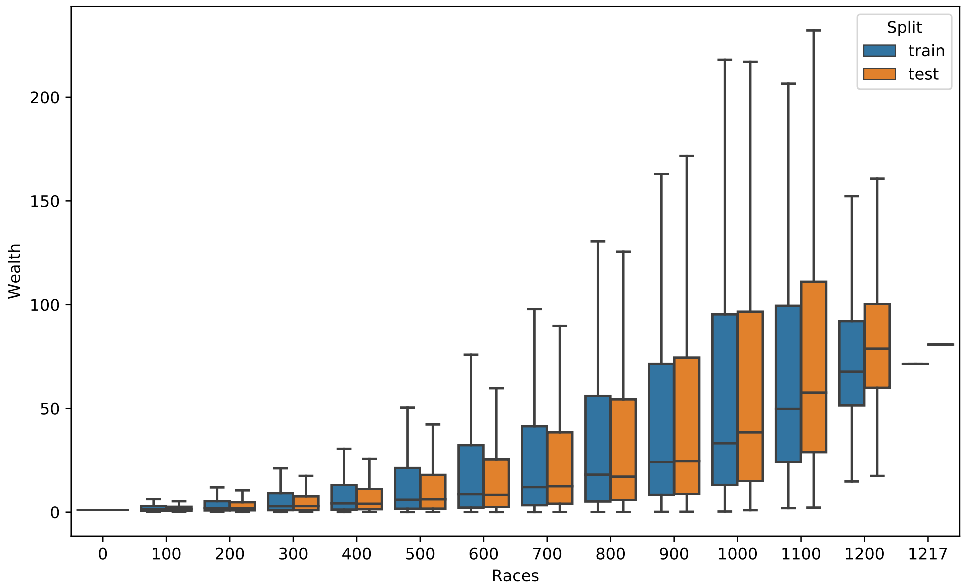


Figure 4.4: Proposed strategy performance on the training and testing dataset

We can now safely conclude this section. Kelly strategy is the optimal strategy and for a more reasonable one we can add the risk constraint 4.12.

## 4.3 Basketball

The second domain we will discuss is the NBA. Dataset consists of close to 15000 games ranging from year 2000 all the way to 2014.

### 4.3.1 Market

The basketball market has the following properties.

| $m\text{-}acc$ | $b\text{-}acc$ | $n$ | $tt$ | $odds$ | $A_{KL}/log(n)$ | $E$ |
|---|---|---|---|---|---|---|
| $\approx 0.68$ | $\approx 0.7$ | 2 | $\approx 0.038$ | $\in [1.01, 41.]$ | $-0.0146$ | 0.58 |

We observe a significant KL disadvantage. It is hence guaranteed that we would be overbetting the opportunities, shall we decide to use the Kelly strategy as is. Which as we have seen guarantees ruin. It will be required to choose our strategy far more wisely than in the previous section.

The specificity of this problem is exactly the measured disadvantage. It is much harder to beat the bookie than it was to beat the public in our previous problem. Another factor of specificity comes in the form of simultaneous games. There are often multiple basketball games played at once, which is rarely the case in Korean horse racing.

In the reinvestment experiments, we assume a round of 10 games hapenning in "parallel". Mainly for the feasibility of calculating the many trajectories required which we discuss in the next section.

### 4.3.2 Simultaneous Games

The necessity of quadratic approximation becomes apparent the moment we try aggregating the basketball games. Calculating single sample round of 10 simultaneous basketball games takes 0.00028 seconds using QKelly, whereas Kelly takes 223 seconds.

We split the dataset into testing and training datasets. To display the variance of our dataset we simulate through randomly shuffled data in each set. The result is different trajectories of many different 10 game rounds. There are always 8 random games that do not fit into any 10-round. The 8 games are always randomly selected and excluded from the respective dataset. To be able to draw some conclusion from such a simulations we needed at least 100 of such independent trajectories through the datasets. Using QKelly instead of Kelly is hence an absolute must.

### 4.3.3 Flat Stake

Our first experiment conducted is a flat staking simulation. Whereas in horse racing, choosing highest expectation bet is the optimal strategy. In simultaneous games, things are not completely clear. We definitely should choose the highest expectation bets. In this case however, the games are exclusive on their own, grouped together they are not. We hence also need to compare and evaluate them in relation to each other. In our experiments, the MSharpe strategy proved to be the best solution.

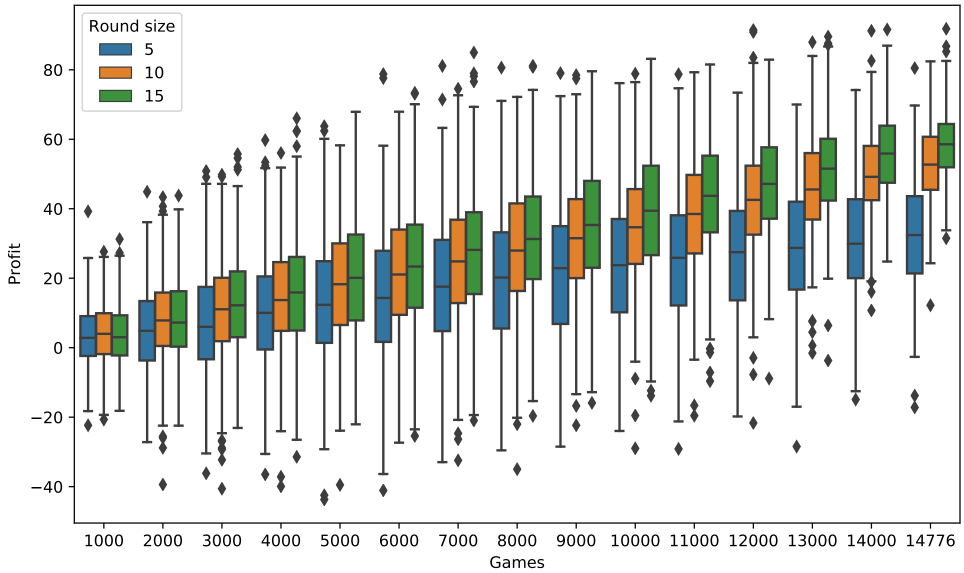


Figure 4.5: MSharpe flat trajectories, comparison of different round sizes.

### 4.3.4 Reinvestment

In this scenario, any reinvestment betting strategy will require a heavy risk constraint. In case of fractional MSharpe we are looking for a fraction $f^\star$ and in case of QKelly, we are looking for values $\alpha$ and $\beta$ in the dradown constraint.

$$P(W^{MIN} < \alpha) = \beta \tag{4.13}$$

$$\lambda = \frac{\log(\alpha)}{\log(\beta)} \tag{4.14}$$

We are looking for such $f^\star$ and $\lambda$, (or combinations of $\alpha$, $\beta$) that satisfy the following across all of our simulated trajectories.

$$\begin{aligned} \text{maximize} \quad & median(\boldsymbol{W_F}) \\ \text{subject to} \quad & Q_5 > 0.95 \end{aligned}$$

We are hence looking for strategy that reached the maximum median final wealth across all trajectories. The 5th percentile being the value below which 5% of all the wealth positions may be found. We select from all the strategies that we simulated. For fractional MSharpe strategy it yields fraction $f = 0.11$. For QKelly risk constraint it is $\lambda = 21.3$. Which represents approximately $\alpha = 0.87$ and $\beta = 0.05$.

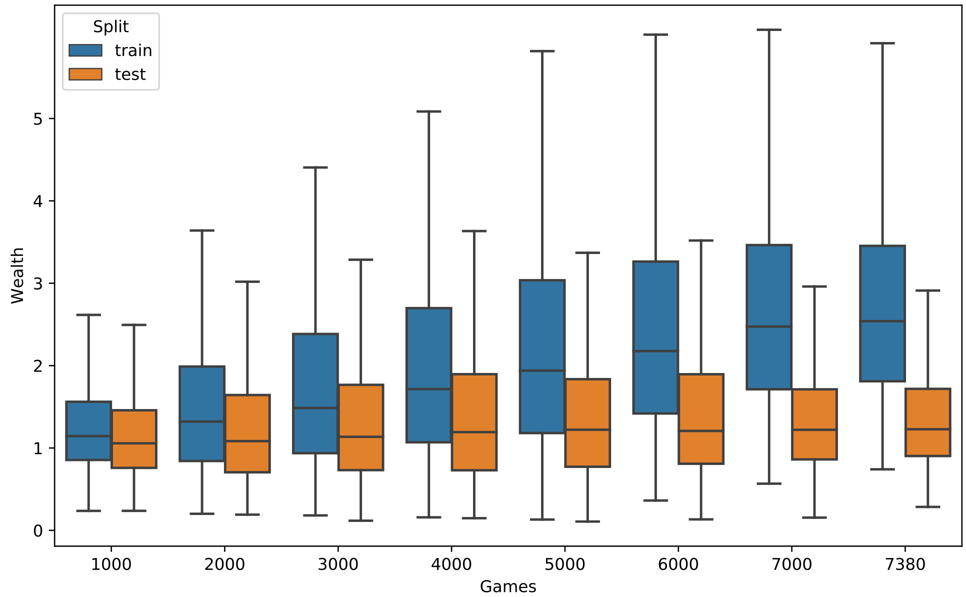


Figure 4.6: 1000 MSharpe trajectories across the testing and training datasets.

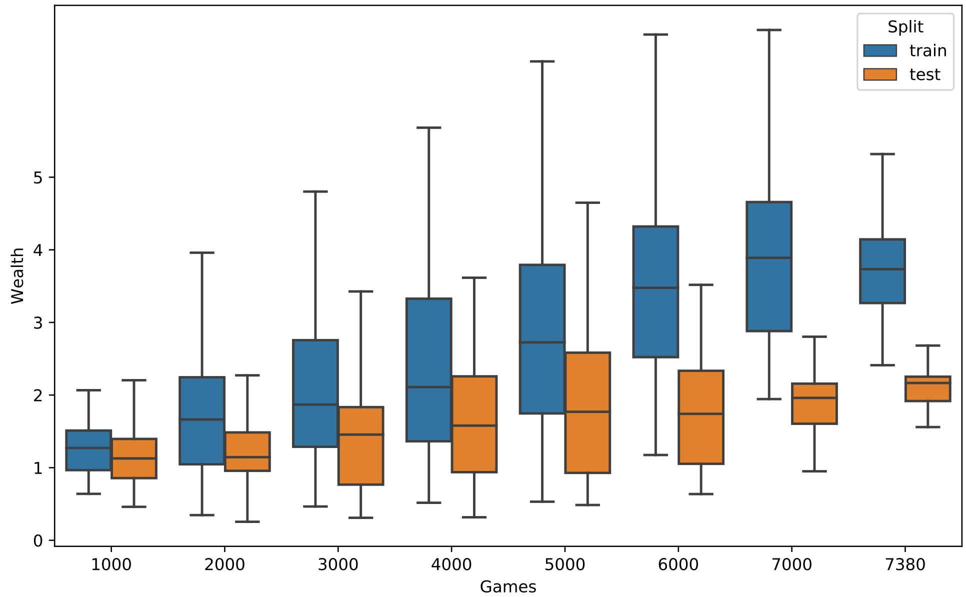


Figure 4.7: 200 QKelly trajectories across the testing and training datasets.

The conclusions of the basketball experiment are as follows.

The round size indeed does matter. Suppose that there are too few games taking place in a given day, (e.g. 5). It is so that the strategy has too few opportunities to compare against each other which leads to overbetting the ones presented. However, the more games we are able to aggregate, the smaller the difference, which as a result leads to a higher stability of the betting strategy.

Devising a reinvestment strategy in a disadvantageous scenario is a challenging task. Both fractional MSharpe and QKelly reach very similar results over the basketball dataset. Whereas MSharpe is much easier to calculate. Finally as we discussed in our generated data experiment 4.1, any Kelly based strategy is very sensitive to the KL advantage. In some cases it may be more appropriate to use a flat strategy, which is unarguably more robust to error.

## 4.4 Football

The Football dataset consists of close to 30000 football games from all over the world. It provides both the opening and closing odds offered by the bookmaker.

### 4.4.1 Market

The football betting market has the following properties in relation to the model prediction.

| $m\text{-}acc$ | $b\text{-}acc$ | $n$ | $tt$ | $odds$ | $A_{KL_O}/log(n)$ | $A_{KL_C}/log(n)$ | $E_O$ |
|---|---|---|---|---|---|---|---|
| 0.523 | 0.537 | 3 | 0.03 | $[1.03, 66]$ | $-0.012$ | $-0.016$ | 0.37 |

Where $A_{KL_O}$ stands for opening odds advantage and $A_{KL_C}$ for closing odds advantage.

The important this is that the advantage is higher on the opening odds. We should hence time our betting accordingly.

Out of all the three of our experiments on real data, this is somewhere in the middle. There are three outcomes instead of two as was the case for basketball which is still significantly less than 16 as in the horse race. Efficiency of the market is higher than in horse racing experiment, but lower than the basketball experiment. Which tells us that there is more opportunity for us to devise a profitable reinvestment strategy.

### 4.4.2 Flat Stake

Just like in the previous case, we first conduct a flat staking experiment across the data. The best solution again proved to be the MSharpe strategy both from the perspective of profitability and the speed of calculation.

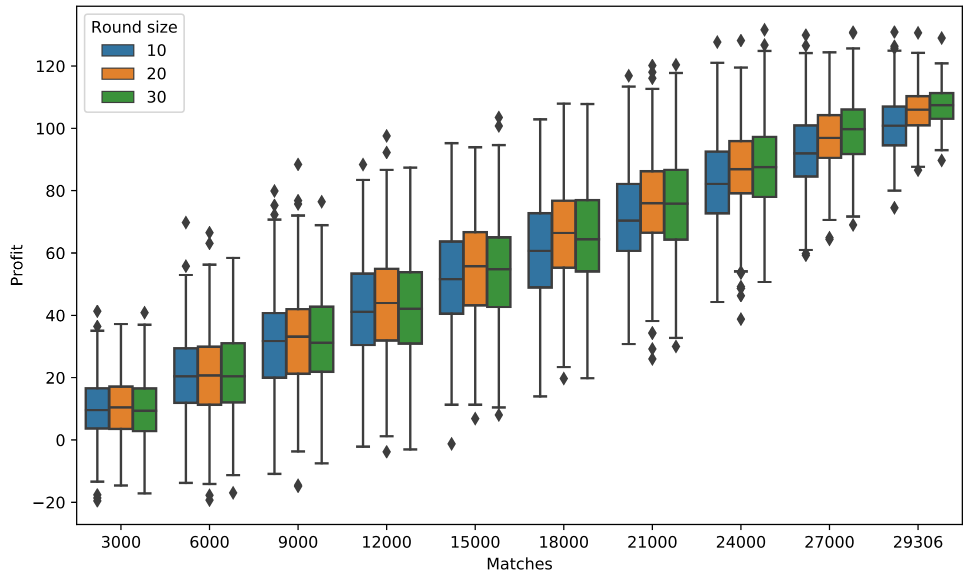


Figure 4.8: MSharpe flat trajectories. Effect of the round size dissipates when rounds consist of more games.

Round size 5 is not included in this case. We observe a much smaller differences between the round sizes of $10, 20, 30$ and onward. This hints that if there are at least 10 games in a given day, our strategy has enough presented opportunities and will not significantly overbet.

We hence decided to use the 10 games as a round size in our reinvestment experiment.

### 4.4.3 Reinvestment

There are 3 outcomes to each game and 10 games in parallel. In a Kelly based strategy, the solver will have to maximize the objective function of $3^{10}$ terms over $30 + 1$ betting opportunities. This has to be done for close to 30 thousand games, hence 3000 such 10-game rounds. To be able to conclude any properties of our strategy we need at least 100, (possibly more) of shuffled trajectories similar to the basketball experiment.

For the above mentioned reasons we decided to simplify every game with

Sharpe ratio pre-selection. For a Kelly based strategies, we hence always choose the maximum Sharpe ratio bet in a given game and then also include an "inverse" bet of the selected bet not happening, which we evaluate with a zero payoff $o = 0$. This transforms the 3-outcome game into a 2-outcome game very similar to the Basketball experiment.

In case of MSharpe strategy, the calculation is much faster and we can simulate close to 1000 trajectories.

We are once again looking for a reinvestment strategy that satisfies the following maximization problem across all the testing dataset trajectories.

$$\begin{aligned} &\text{maximize} && median(\boldsymbol{W_F}) \\ &\text{subject to} && Q_5 > 0.95 \end{aligned}$$

For fractional MSharpe strategy it yields fraction $f = 0.10$. For risk constrained QKelly the parameter is $\lambda = 9.4$.

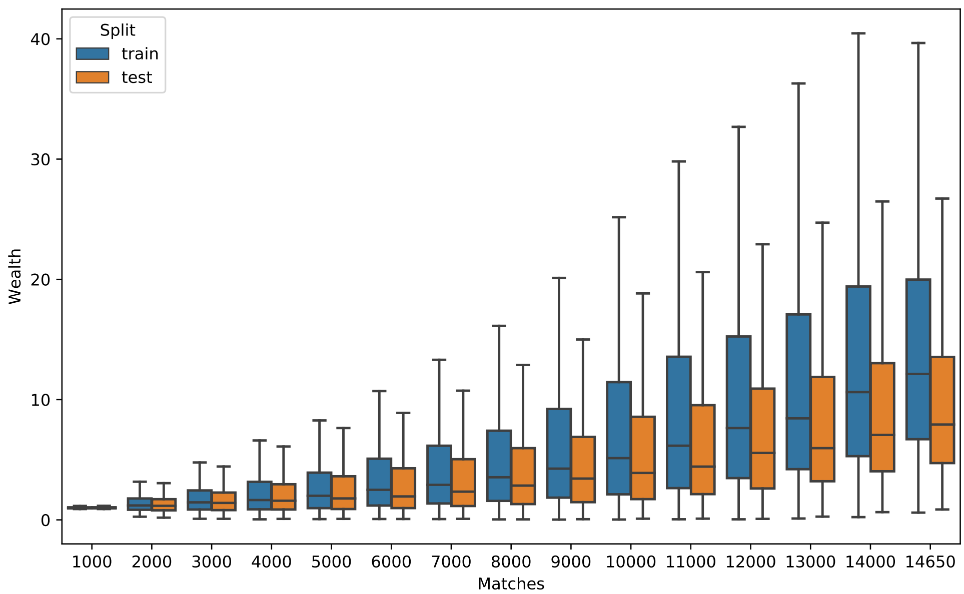


Figure 4.9: 1000 MSharpe fractional reinvestment trajectories comparison on training and testing datasets.

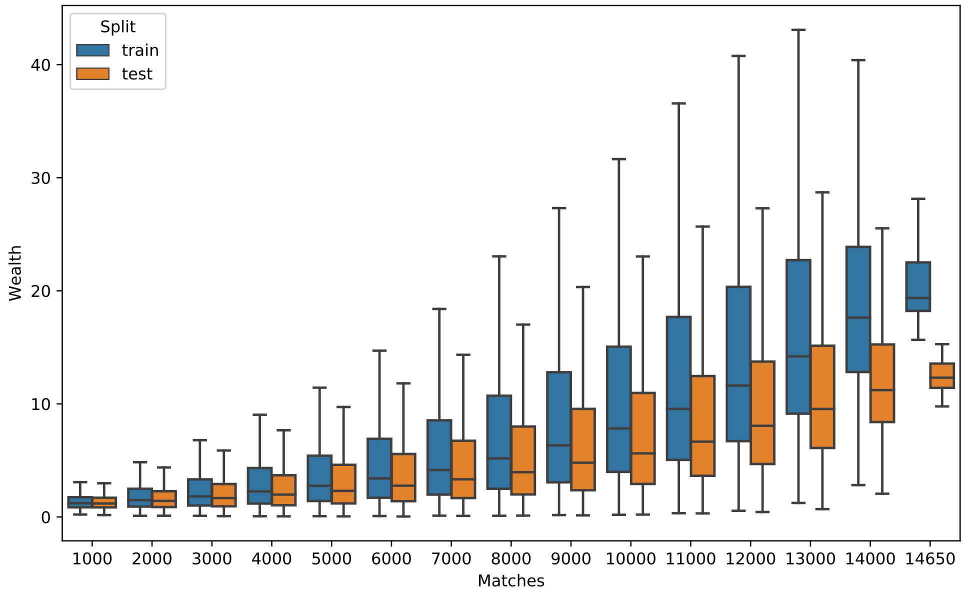


Figure 4.10: 200 QKelly fractional reinvestment trajectories comparison on the training and testing datasets

Results for both reinvestment strategies are again very similar, where QKelly reaches slightly higher final wealth on average than MSharpe.

The important finding is that if expressed as a correct optimization problem, it is possible to find profitable reinvestment strategy even in generally disadvantageous situation.

Chapter 5

# Conclusion

The real goal of our work was to answer the age old question of what is the truly optimal betting strategy? The question is very simple yet the answer is quite complex. In this thesis we aimed to answer this question both theoretically and practically.

Theoretically the question of the optimal strategy has already been answered by many great thinkers. It is the Kelly criterion, also known as the geometric mean policy. In practice however, the assumptions of Kelly criterion are rarely met.

First practical challenge is that most existing literature focuses on simple games that are two outcome, are exclusive or have other additional pre-assumptions. We found an explanation and provide a simple way to express any K-outcome game in the language of Kelly criterion.

Second and possibly the biggest problem of Kelly criterion is the assumption of having a correct probability estimate. That is rarely the case. We hence provide an intuitive framework to grasp the uncertainty in the context of sports betting. The framework is based on our mathematical findings and naturally connects to measures of econometrics. We have shown in one of our experiments that together with the defined measures, it can be used to stress test any arbitrary strategy under various conditions and under various levels of uncertainty.

We provide a cython based version of the KL random spill algorithm. Algorithm we devised to generate uncertain estimates for general K-outcome games in our testing framework.

From the perspective of practical experiments we collected relevant data for Korean horse racing and were provided data for football and basketball. For all three different domains we conducted a separate experiment. All three domains showcased different techniques that were discussed and in every case we were able to propose a profitable strategy.

In our experiments we show that it is indeed possible to find a profitable reinvestment strategy even when player has significant disadvantage. Which experimentally extends the findings of Hubáček, 2017 where in such a case, they were able to devise a profitable flat based strategy.

Another interesting finding is that in simultaneous games it is generally good idea to aggregate as many games as numerically possible. Too few games in combination with uncertain estimates lead to significant overbetting in our experiments.

## 5.1 Future Work

The topic provides numerous ideas for future work. We plan to extend our testing framework with the ability to set the third distance between Bookie and Model as well. This will allow us to investigate what kind of influence does this third distance have on the overall profitability in a general case. For instance what effect does the model decorrelation from bookies dividends have on the overall KL measures of the scenario. Maybe even more appropriate measures can be found with a more direct connection to profitability. We hence plan to extend the system with additional distance measures such as Jensen–Shannon divergence and others.

The main goal however is to devise what we call a “robust” Kelly strategy. A betting strategy somewhere in between the flat strategies that are very robust to error and the growth optimal but very error sensitive Kelly criterion.

# Bibliography

BERNOULLI, Daniel, 2011. Exposition of a new theory on the measurement of risk. In: *Exposition of a new theory on the measurement of risk. The Kelly Capital Growth Investment Criterion: Theory and Practice*. World Scientific, pp. 11–24.

BUSSETI, Enzo; RYU, Ernest K; BOYD, Stephen, 2016. Risk-constrained Kelly gambling. *arXiv preprint arXiv:1603.06183*.

COVER, Thomas M; THOMAS, Joy A, 2012. *Elements of Information Theory*. John Wiley & Sons.

DIAMOND, Steven; BOYD, Stephen, 2016. CVXPY: A Python-Embedded Modeling Language for Convex Optimization. *Journal of Machine Learning Research*. To appear.

DOMAHIDI, Alexander; CHU, Eric; BOYD, Stephen, 2013. ECOS: An SOCP solver for embedded systems. In: *ECOS: An SOCP solver for embedded systems. Control Conference (ECC), 2013 European*, pp. 3071–3076.

FELLER, William, 1968. *An Introduction to Probability Theory and Its Applications* [Hardcover]. Wiley. ISBN 0471257087.

GRANT, Andrew; JOHNSTONE, David; KWON, Oh Kang, 2008. Optimal betting strategies for simultaneous games. *Decision Analysis*. Vol. 5, no. 1, pp. 10–18.

HUBÁČEK, Ondřej, 2017. *Exploiting betting market inefficiencies with machine learning*. Master's thesis. Czech Technical University, Faculty of Electrical Engineering.

KELLY JR, John L, 2011. A new interpretation of information rate. In: *A new interpretation of information rate. The Kelly Capital Growth Investment Criterion: Theory and Practice*. World Scientific, pp. 25–34.

LATANE, Henry Allen, 2011. Criteria for choice among risky ventures. In: *Criteria for choice among risky ventures. The Kelly Capital Growth Investment Criterion: Theory and Practice*. World Scientific, pp. 35–46.

MACLEAN, LC; ZIEMBA, William T; BLAZENKO, George, 2011. Growth versus security in dynamic investment analysis. In: *Growth versus security in dynamic investment analysis. The Kelly Capital Growth Investment Criterion: Theory and Practice*. World Scientific, pp. 331–354.

MARKOWITZ, Harry, 1952. Portfolio Selection. *The journal of finance*. Vol. 7, no. 1, pp. 77–91.

NAU, Robert F; JOSE, Victor Richmond R; WINKLER, Robert L, 2009. Duality between maximization of expected utility and minimization of relative entropy when probabilities are imprecise. In: *Duality between maximization of expected utility and minimization of relative entropy when probabilities are imprecise. ISIPTA*. Vol. 9, pp. 337–346.

O'DONOGHUE, B; CHU, E; PARIKH, N; BOYD, S, 2016. *SCS: Splitting conic solver, version 1.2. 6*.

POUNDSTONE, William, 2010. *Fortune's Formula: The untold story of the scientific betting system that beat the casinos and wall street*. Hill and Wang.

SMOCZYNSKI, Peter; TOMKINS, Dave, 2010. An explicit solution to the problem of optimizing the allocations of a bettor's wealth when wagering on horse races. *Mathematical Scientist*. Vol. 35, no. 1.

THORP, Edward O, 1966. Beat the dealer. In: *Beat the dealer*.

THORP, Edward O, 1975. Portfolio Choice and the Kelly Criterion. In: *Portfolio Choice and the Kelly Criterion. Stochastic Optimization Models in Finance*. Elsevier, pp. 599–619.

THORP, Edward O, 2008. The Kelly criterion in blackjack sports betting, and the stock market. In: *The Kelly criterion in blackjack sports betting, and the stock market. Handbook of asset and liability management*. Elsevier, pp. 385–428.

Appendix A

# Implementation

In this section we present the implementation details and notes of concepts that were discussed in the previous chapters.

## A.1 Solvers

We have chosen the cvxpy by Diamond et al., 2016 as our main optimization framework. There are numerous reasons, one of them being that it is a very clear framework that results in good code readability.

### A.1.1 Kelly

There really is no need to format any specific matrices, for instance the following:

```
goal = cvx.Maximize(p * cvx.log(R * b))
constraints =    [cvx.sum_entries(b) == 1,
                  b >= 0]

problem = cvx.Problem(goal, constraints)
problem.solve()
```

is a representation of our Kelly problem on six lines of code.

The most important reason however is that it provides solvers that can solve logarithms in both the problem definition and constraints. We first solve given problem using Splitting Conic Solver(SCS) by O'Donoghue et al., 2016, should a numerical problem arise we proceed with solving the problem using Embedded Conic Solver (ECOS) by Domahidi et al., 2013.

The following is our Kelly drawdown constraint.

```
lambda_risk = cvx.Parameter(sign='positive',
```

```
                                        value=l_val)
cvx.log_sum_exp(
cvx.log(p) - lambda_risk * cvx.log(R * b)
) <= 0
```

### A.1.2 Quadratic Kelly

Respectively our quadratic Kelly problem is expressed as follows.

```
growth = b.T * mu
risk = cvx.quad_form(b, Sigma)
goal = cvx.Maximize(growth - (1/2) * risk)

constraints = [cvx.sum_entries(b) == 1,
               b >= 0]

problem = cvx.Problem(goal, constraints)
problem.solve(solver = 'SCS')
```

## A.2 KL Random Spill

When implementing the KL random spill algorithm, one needs to put a bound on generated probabilities. Specifically for Python, where a very small probability will manifest itself to all numerical calculations. We solve it simply by rejecting any sample(triplet) where any probability from Dirichlet generated $P_R$ is lower than this specified bound.

Chapter 6

# Acronyms

**MPT** Modern Portfolio Theory

**NBA** National Basketball Association

**KRA** Korean Racing Agency

**QKelly** Quadratic Kelly

**MSharpe** Maximum Sharpe

Chapter 7

# Contents of enclosed CD

```
├── src ........................................... implementation sources
└── text ......................................... the thesis text directory
    ├── thesis.pdf ......................... the thesis text in PDF format
    └── thesis.tex ....................... the thesis text in LaTeX format
```