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Instructions

Online Ramsey theory (ORT) is a modern branch of the well-developed classical Ramsey theory, which studies the existence of homogeneous substructures in various large combinatorial structures. We focus on ORT problems regarding graphs: Builder and Painter play in turns, where Builder draws an edge and Painter colors it either red or blue. Builder wins if there is a monochromatic copy of a graph H, otherwise Painter wins; Builder must win as fast as possible. Recently, ORT gained a lot of attention in the international community. Moreover, there exists a lot of interesting open questions to be solved, which we want to address by this thesis.

1) Survey previous work in the field of graph ORT.
2) Investigate the differences of online Ramsey number and size Ramsey number for certain graph classes, like paths, cycles, and various subclasses of trees.
3) Try to attack the problem of investigating the online Ramsey number of trees.

References

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Master's thesis

Online Ramsey Theory

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Declaration

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Abstrakt

Tato práce se zabývá online Ramseyovou teorií. Problém je definován jako kombinatorická hra Buidera a Paintera. Je dán libovolný graf H a hrací plocha nekonečně mnoha nezávislých vrcholů. Každé kolo Builder postaví novou hranu do grafu hrací plochy a Painter ji obarví červeně nebo modře. Online Ramseyovo číslo grafu H je minimální počet kol, které Builder potřebuje, aby vynutil vznik jednobarevného podgrafu isomorfního s H uvnitř hrací plochy.

Online Ramseyovo číslo se často srovnává se size-Ramsey číslem, což je nejmenší počet hran grafu takového, že libovolné jeho obravení dvěma barvama obsahuje jednobarevnou kopii H. Size-Ramsey číslo shora omezuje online Ramseyovo číslo, nicméně zdá se obtížné dokázat, že je mezi nimi asymptoticky významný rozdíl.

Existuje pouze jeden výsledek takového typu, od Conlona [On-line Ramsey Numbers, SIAM J. Discrete Math. 2009], který dokázal, že pro nekonečně mnoho úplných grafů je online Ramseyovo číslo asymptoticky menší než size-Ramsey číslo.

V této diplomové práci je popsána nekonečná rodina stromů, pro které je online Ramseyovo číslo asymptoticky menší než size-Ramsey číslo. Také jsou v ní dokázány horní meze pro online Ramseyovo číslo cyklů a k-podrozdělených hvězd. A nakonec je přesně určena hodnota omezeného online Ramseyova čísla pro trojúhelníky versus hvězdy na třídě souvislých grafů.

Klíčová slova online Ramseyova čísla, size-Ramsey čísla, Ramseyova grafová teorie

Abstract

In this thesis we study the online Ramsey theory. The problem is defined as a game between Builder and Painter. They are given an arbitrary graph H and a playground of infinite number of independent vertices. On each round, Builder builds a new edge to the playground and Painter colors it either red or blue. The online Ramsey number of a graph H is the minimum number of rounds Builder needs to force a monochromatic H to appear as a subgraph of the playground.

We compare the online Ramsey number to the size-Ramsey number, which is the minimum number of edges in a graph, that for arbitrary 2-edge-coloring contains a monochromatic copy of H. The size-Ramsey number upper bounds the online Ramsey number, however it seems to be difficult to show that there is an asymptotic gap between them.

There is only one known result of this type, by Conlon [On-line Ramsey Numbers, SIAM J. Discrete Math. 2009], who showed that for an infinite number of complete graphs, the online Ramsey number is asymptotically smaller than the size-Ramsey number.

In this thesis we describe an infinite family of trees for which the online Ramsey number is asymptotically smaller than size-Ramsey number. We also prove upper bounds for online Ramsey numbers of cycles and k-subdivided stars. And finally we provide an exact value of online Ramsey numbers of triangles versus stars restricted to connected graphs.

Keywords online Ramsey number, size Ramsey number, Ramsey graph theory

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Chapter 1

Introduction and Notation

Let us suppose we are given a graph and its edges are colored arbitrarily red or blue. Results in the field of Ramsey graph theory suggest that if the initial graph satisfies some conditions, then a certain monochromatic subgraph will always be present. As an example, given people are always either mutually friends or strangers, it is certain that in a group of six people there are three who know each other or three who are mutually strangers (also known as Theorem on friends and strangers). The same notion can be expressed in graph theory by a complete graph with six vertices and each edge colored either red or blue. Vertices correspond to people and edge color to their relationship, red for friends and blue for strangers. The statement that three of these people are in the same relationship with each other is analogous to saying that a 2-edge-colored complete graph on six vertices contains a monochromatic triangle.

The theorem on friends and strangers can be generalized such that we ask a question: How many people do we need to guarantee that n of those people are in a same mutual relationship? This generalization allows more types of relationships than two, however it is not clear whether there is always a solution. This problem was studied by F. P. Ramsey and in 1930 [1] he proved that the solution exists. From a graph theoretic view, he showed that when given k complete graphs with respective orders and colors, the order of an edge-colored complete graph, which contains at least one monochromatic clique of respective order and color, is finite. The minimal order of such a complete graph is called the (classical) Ramsey number. This result was followed by an explicit upper bound on Ramsey number by Erdős and Szekeres [2] and lower bound by Erdős [3]. It should be noted that there are only 9 known precise Ramsey numbers for non-trivial cases, which demonstrates the difficulty of evaluating Ramsey numbers [4].

Given k arbitrary graphs in their unique colors, the generalized Ramsey number is the minimal order of a complete graph such that it contains one of those arbitrary graphs as a monochromatic subgraph in its color. Since any graph is contained as a subgraph in a complete graph on the same number of vertices, it follows that the generalized Ramsey number for arbitrary graphs is finite and bounded by the Ramsey number of its respective complete graph. There are many results for generalized Ramsey numbers, some of which we will recall in this thesis.

Later, a version called size-Ramsey number was introduced by Erdős et al. [5]. Size-Ramsey number minimizes the number of created edges instead of the order of the complete graph. Clearly, the size-Ramsey number is bounded by the number of edges in the complete graph of respective Ramsey number. This version was studied for example in [6, 7, 8, 9, 10].

The online Ramsey number, which combines the notion of Ramsey numbers and combinatorial games, was introduced independently by Beck [11] and Friedgut et al. [12]. It is best defined as a game of two players called Builder and Painter, playing over an infinite set of vertices. They are given k arbitrary graphs and one color for each of these arbitrary graphs. Each round, Builder creates an edge and Painter colors it one of k colors immediately. Builder's goal is to force one of the arbitrary graphs to appear as a monochromatic subgraph in its color, regardless of Painter's decisions. Painter's goal is to deny Builder achieving his goal for as many rounds as possible. The online Ramsey number is the minimum number of rounds such that Builder has a winning strategy, assuming both players play optimally.

The online Ramsey number is guaranteed to exist, because Builder can simply create a big complete graph, which by the Ramsey theorem contains a smaller monochromatic clique. Therefore, similarly to previous versions of Ramsey numbers, our only goal is to determine its value. This led to the creation of a new version of online Ramsey number we call restricted online Ramsey number. It was introduced in 2004 by Grytczuk et al. [13]. Unlike versions we presented so far, this version does not allow Builder to create arbitrary graphs. Instead, Builder is only allowed moves which keep the graph in the given class of graphs. This might restrict Builder in a way such that he needs significantly more moves to win. If the class of graphs is too strict, it is impossible for Builder to win, meaning such online Ramsey number does not exist. The focus of study in the restricted online Ramsey number variant is to define a class of graphs which allow Builder to win. Some results for this variant can be found in [14, 13, 15, 16].

Although we know, that the online Ramsey number is not bigger than the size-Ramsey number, it is not clear whether there is an asymptotically significant difference. As far as we know, there is only one known result for nontrivial graphs which proves that there is an asymptotic gap between the size-Ramsey number and the online Ramsey number. The proof was made in 2009 by Conlon [17] showing that there is an infinite number of complete graphs for which the online Ramsey number is smaller than their size-Ramsey number. Therefore, we define the main goal of this thesis is to study differences between size-Ramsey numbers and online Ramsey numbers and to characterize cases for which the online Ramsey number is asymptotically smaller than the size-Ramsey number.

In this thesis we present the following results. First, we describe the second nontrivial infinite family of graphs, for which the online Ramsey number $\widetilde{r}$ is asymptotically smaller than the size-Ramsey number $\overline{r}$.

Theorem 1. *There is an infinite sequence of trees $T_1, T_2, \ldots$ such that $|T_i| < |T_{i+1}|$ for each $i \geq 1$ and*

$$\lim_{i \to \infty} \frac{\widetilde{r}(T_i)}{\overline{r}(T_i)} = 0.$$

Second, we show a result for the online Ramsey number of stars with subdivided edges which matches the lower bound for size-Ramsey number, however in a constructive way. Similarly, we show a constructive proof that the online Ramsey number of even cycles is no more than $23n/2 - 20$, and for odd cycles $24n - 20$, lowering the known upper bounds.

Theorem 2. *Let C_n be a cycle on n vertices. Then $\widetilde{r}(C_n) \leq 23n/2 - 20$ if n is even, and $\widetilde{r}(C_n) \leq 24n - 20$ if n is odd.*

And last, we provide an exact value of a restricted online Ramsey number of triangles versus stars, when Builder is allowed to make only connected graphs.

Theorem 3. *The online Ramsey number for a red C_3 versus a blue S_n, given Builder is allowed to create only connected graphs, is $\widetilde{r}_{\mathcal{C}}(C_3, S_n) = 3n - 1$.*

The rest of the thesis is arranged as follows. After the introduction we continue with a brief notation overview. In Section 2, we survey work done so far in Ramsey graph theory, generalized Ramsey numbers, size-Ramsey number, and online Ramsey number and we reiterate on results which are relevant to our work. Then in Section 3 we elaborate on our own results. In Section 3.1, we start by proving Theorem 1, presenting a family of trees for which the online Ramsey number is asymptotically smaller than their size-Ramsey number. Then we show our results for subdivided stars and cycles in Sections 3.2 and 3.3 respectively. And last, we solve restricted online Ramsey number of triangles versus stars by proving Theorem 2 in Section 3.4.

1.1 Notation

We define a graph G as an ordered pair (V, E) of its vertices $V(G)$ and edges $E(G)$. The order of G is the number of its vertices $|V(G)|$. A tree T is a graph which is connected and $|E(T)| = |V(T)| - 1$. A forest is a union of disjoint trees. We use $\overline{G}$ to denote the complement of graph G,

$$\overline{G} = (V(G), \{\{u, v\} \mid u, v \in V(G)\} \setminus E(G)).$$

Let K_t be a complete graph of order t, $K_{n,m}$ be a complete bipartite graph, P_n a path of length n, and $S_k = K_{1,k}$ a star with k vertices connected to a common center vertex, i.e.,

$$\begin{aligned} K_t &= (\{v_1, v_2, \dots, v_k\}, \{\{u, w\} \mid \text{for all } u, w \in v_1, v_2, \dots, v_k; u \neq w\}), \\ K_{n,m} &= (V(K_n) \cup V(K_m), E(\overline{K_n \cup K_m})) \text{ assuming } V(K_n) \cap V(K_m) = \emptyset, \\ P_n &= (\{v_1, v_2, \dots, v_{n+1}\}, \{\{v_i, v_{i+1}\} \mid i \in \{1, 2, \dots, n\}\}), \\ S_k &= (\{u, v_1, v_2, \dots, v_k\}, \{\{u, v\} \mid v \in v_1, v_2, \dots, v_k\}). \end{aligned}$$

We will use $\Delta(G) = \max_{u \in V(G)} |\{u, v\} \in E(G)|$ as the maximum degree of graph G. Let the k-edge-coloring, or k-coloring, of a graph G be a function $f : E(G) \to \{1, 2, \dots, k\}$, and note that it does not have to be a proper edge-coloring.

In Section 3.1 we show results for the following graphs. Let $S_{k,l}$ be a tree consisting of a vertex v which has k neighbors $x_1, \dots, x_k$ and each x_i is connected to l leaves. Let *spider* $\sigma_{k,l}$ for $k \geq 3$ and $l \geq 1$ be a union of k paths of length l that share exactly one common endpoint.

We denote Ramsey number by $r(n_1, n_2, \dots, n_k)$, similarly generalized Ramsey number as $r(G_1, G_2, \dots, G_k)$, size-Ramsey number as $\overline{r}(G_1, G_2, \dots, G_k)$ and online Ramsey number as $\widetilde{r}(G_1, G_2, \dots, G_k)$ (for proper definitions see Section 2). We also use $\overline{r}_{\mathcal{G}}(G_1, G_2, \dots, G_k)$ and $\widetilde{r}_{\mathcal{G}}(G_1, G_2, \dots, G_k)$ for the restricted size-Ramsey number and restricted online Ramsey number respectively. For the 2-color version with two identical target graphs, we use $r(n) = r(n, n)$ and similarly $r(G), \overline{r}(G)$ and $\widetilde{r}(G)$ respectively for each Ramsey number version.

Chapter **2**

Survey of related results

Survey of the field of Ramsey theory would take several hundred pages, so in this section we cover only results essential to Ramsey theory and results related to this thesis. Broader survey of Ramsey theory can be found for example in Handbook of Graph Theory [4] and Ramsey Theory [18].

The majority of results in Ramsey graph theory use only 2 colors. For the sake of consistency, we chose the first color to be red and the second to be blue throughout this thesis.

We divided this section based on variants of the Ramsey number. We first define the classical Ramsey number and show few essential results. We follow with the definition of generalized Ramsey number and present few results as well. After that we define size-Ramsey number and establish known results which are related to our goal of finding an asymptotic gap between the size-Ramsey number and the online Ramsey number. In the last part, we formally introduce an online Ramsey number. We will reiterate on few known results which bound the online Ramsey number and we will remind results made in variant where Builder is restricted. We also include two known results in more detail. The first is the only result in showing that online Ramsey number is asymptotically smaller than size-Ramsey number by Conlon [17]. The second is proof of upper bound for paths by Grytczuk et al. [19], which we use to establish our upper bounds on cycles and subdivided stars.

2.1 Results for classical Ramsey numbers

Given a number t, what is the minimum number $r(t)$ such that a $K_{r(t)}$ with edges colored arbitrarily red or blue, such that there is either a red K_t or a blue K_t as a subgraph? The $r(t)$ is called Ramsey number and even though it answers such a simple question, determining its value proved to be quite difficult.

Definition 1. *Let the* Ramsey number $r(t_1, t_2, \ldots, t_k)$ *be the minimum number, such that k-edge-colored complete graph of such order is guaranteed to contain at least one K_i as a monochromatic subgraph in i-th color.*

By definition, order of parameters does not have any influence on the Ramsey number. Also trivially for $n \geq 2, r(2, n) = r(n, 2) = n$.

The first nontrivial result which is associated with Ramsey numbers was made in 1930 by F. P. Ramsey [1] (and re-released in 1987 [20]).

Theorem 4 (Ramsey's Theorem [1], simplified)**.** *Given positive integers $k, n_1, n_2, \ldots, n_k$, there is a least positive integer $r(n_1, n_2, \ldots, n_k)$ such that, for any partition $C_1, C_2, \ldots, C_k$ of the edges of a complete graph K_p with $p \geq r(n_1, n_2, \ldots, n_k)$, there is for some i a complete subgraph K_{n_i} all of whose edges are in C_i.*

The partitions of the edges are commonly described as colors. Partitioning edges of G into k sets is analogous to coloring those edges with k colors, which we refer to as k-edge-coloring of G. Compared to the introductory example, here we have k colors instead of 2 and each complete graph can have different order.

The Ramsey numbers in the form $r(t) = r(t, t)$ are called *diagonal* Ramsey numbers and those with $r(n, m)$ where $n \neq m$ are called *off-diagonal* Ramsey numbers.

The Ramsey numbers were upper bounded by Erdős and Szekeres [2] proving, that $r(m, n) \leq r(m-1, n) + r(m, n-1)$, for all $m, n \geq 3$, which can be used inductively to prove

$$r(m+1, n+1) \leq \binom{m+n}{m}, \tag{2.1}$$

which was used to get an upper bound

$$r(t) \leq [1 + o(1)] \frac{4^{t-1}}{\sqrt{\pi t}}.$$

Several years later, Erdős [3] came up with the first lower bound on Ramsey number, showing that

$$r(t) \geq 2^{t/2}, \tag{2.2}$$

using probabilistic method. Since then, there have been many results refining these bounds. The best bounds as of today are as follows. The bounds on diagonal Ramsey numbers $r(t)$ are

$$r(t) \geq t2^{t/2}[(\sqrt{2}/e) + o(1)],$$
$$r(t+1) \leq t^{-c\frac{\log t}{\log \log t}} \binom{2t}{t},$$

for some constant c. The lower bound is due to Spencer [21], and the upper bound is due to Conlon [22]. For off-diagonal Ramsey numbers we have

$$c'_s \frac{t^{\frac{s+1}{2}}}{(\log t)^{\frac{s+1}{2} - \frac{1}{s-2}}} \leq r(s, t) \leq c_s \frac{t^{s-1}}{(\log t)^{s-2}}.$$

The lower bound is due to Bohman and Keevash [23] and the upper bound due to Ajtai et al. [24]

2.2 Results for generalized Ramsey numbers

Ramsey's theorem implies that for any given set of *target graphs* $G_1, G_2, \ldots, G_k$ the minimal order of k-edge-colored complete graph which contains G_i in i-th color is finite.

Definition 2. *Let the* generalized Ramsey number $r(G_1, G_2, \ldots, G_k)$ *be the minimum number, such that arbitrarily k-edge-colored complete graph of such order contains at least one G_i as a monochromatic subgraph in i-th color.*

Clearly for any graphs G_n and G_m of order n and m respectively, $r(G_n, G_m) \leq r(n, m)$. However generalized Ramsey number can be significantly smaller than Ramsey number, as demonstrated by Gerencsér and Gyárfás [25]

$$r(P_n, P_m) = n + \left\lfloor \frac{m}{2} \right\rfloor - 1,$$

which has only linear order compared to the exponential lower bound for classical Ramsey number.

In 1973 Bondy and Erdős [26] investigated Ramsey number for red cycles and either blue cycles or blue cliques, showing the exact values for many cases. This was followed by Rosta [27, 28] and by Faudree and Schelp [29] leading to a complete solution for cycles. For $3 \leq m \leq n$ and $(m, n) \neq (3, 3), (4, 4)$,

$$r(C_m, C_n) = \begin{cases} 2n - 1 & \text{when } m \text{ is odd,} \\ n + \frac{m}{2} - 1 & \text{when } m \text{ and } n \text{ are even, and} \\ \max\{n + \frac{m}{2} - 1, 2m - 1\} & \text{when } m \text{ is even and } n \text{ is odd.} \end{cases}$$

This was followed by an increased interest in the 3 color version of the Ramsey number for cycles.

In 1977 Chvátal [30] proved that the Ramsey number of cliques versus trees of orders m and n is

$$r(K_m, T_n) = (m - 1)(n - 1) + 1.$$

Since the generalized Ramsey number is the most studied of the variants we describe in this thesis, we had to skip many inspiring results in the sake of brevity.

2.3 Results for size-Ramsey numbers

The graphs created for generalized Ramsey numbers are always complete graphs. However the number of required edges to reach the same goal might be asymptotically smaller than the number of edges in a complete graph. This notion is captured in the definition of the size-Ramsey number.

Definition 3. *Let the* size-Ramsey number $\overline{r}(G_1, G_2, \ldots, G_k)$ *be the minimum number of edges of a graph, for which arbitrary k-edge-coloring contains at least one G_i as a monochromatic subgraph in i-th color.*

Its value has trivial lower bound and is upper bounded by the number of edges of the complete graph for respective generalized Ramsey number.

$$1 + \sum_{i=1}^{k} (E(G_i) - 1) \leq \overline{r}(G_1, G_2, \ldots, G_k) \leq \binom{r(G_1, G_2, \ldots, G_k)}{2}$$

Both of these cases occur, so we are interested in characterization of specific bounds for various classes of graphs.

A different view on the size-Ramsey number is through a combinatorial game of two players. The players, Builder and Painter, are given k target graphs $G_1, G_2, \ldots, G_k$. Builder first creates the whole graph and then Painter colors all the edges with k colors, attempting to avoid G_i in i-th color. The size-Ramsey number $\overline{r}(G)$ describes the minimum number of edges the Builder must construct in order to ensure that regardless of Painter's decisions a monochromatic copy of G_i in i-th color will be created somewhere in the graph. We will draw comparisons to this interpretation in Section 2.4.

There are several known results for size-Ramsey numbers, however they frequently rely on probabilistic method which in turn makes those results non-constructive.

In 1978 by Erdős et al. [5] showed that the size-Ramsey number for cliques is the same as number of edges of the clique for the respective classical Ramsey number.

$$\overline{r}(K_m, K_n) = \binom{r(m,n)}{2} \tag{2.3}$$

Interestingly, results concerning size-Ramsey number of trees are well developed. Let T_0 and T_1 be partitions of the unique bipartitioning of the given tree T. Let $\beta(T) = |T_0|\Delta(T_0) + |T_1|\Delta(T_1)$. In 1990 Beck [6, 7] showed bounds on the size-Ramsey number of trees

$$\beta(T)/4 \leq \overline{r}(T) \leq O(\beta(T)\log(|T|)^{12}).$$

In 1995 Haxell and Kohayakawa [9] refined the upper bound of Beck and proved that

$$\overline{r}(T) = O(\beta(T)\log\Delta(T)).$$

Recently, in 2012 Dellamonica [8] showed that Beck's lower bound is asymptotically tight, proving Beck's conjecture

$$\overline{r}(T) = \Theta(\beta(T)).$$

He proved existence of a graph G with special properties using the probabilistic method and provided an algorithm to embed any T into such G.

In 1983 Beck [6] showed that the size-Ramsey number of paths is linear in their length by upper bounding them with $\overline{r}(P_n) < 900n$. This bound was improved in 2015 by Dudek and Prałat [31] to $\overline{r}(P_n) < 137n$.

There are also some results in variation which requires the target graph to appear as an induced monochromatic subgraph.

Definition 4. *Let the* induced size-Ramsey number *be the minimum number of edges of a graph, for which arbitrary k-edge-coloring contains at least one G_i in i-th color as an induced monochromatic subgraph.*

In 1995 Haxell, Kohayakawa and Łuczak [10] showed that induced size-Ramsey number for cycles $\overline{r}(C_n)$ is linear in n. This result naturally applies to size-Ramsey numbers, since induced variant implies non-induced one.

2.4 Results for online Ramsey numbers

The notion of an *online Ramsey number* was introduced independently by Beck [11] and by Friedgut et al. [12]. The online Ramsey number is often introduced in terms of the following combinatorial game, called an *online Ramsey game*.

The game is played in rounds between Builder and Painter. Each round Builder creates an edge and Painter colors it immediately with one of k colors. These edges induce gradually growing *background graph*. The game also contains a set of *target graphs* $G_1, G_2, \ldots, G_k$. Builder wins the game whenever the background graph contains a monochromatic subgraph in i-th color isomorphic to the target graph G_i. Painters goal is to play the game for as many rounds as possible.

Definition 5. *The* online Ramsey number *is the minimum number of rounds such that Builder has a winning strategy in the online Ramsey game if both players play optimally.*

The connection to classical Ramsey number and Ramsey theorem implies that Builder will always win, when given enough rounds. However, the online Ramsey number is also bounded by the size-Ramsey number.

In Section 2.3 we mentioned that the size-Ramsey number can be defined in terms of a combinatorial game of two players. First one creates m edges in order to ensure that any k-coloring of those edges of the graph G yields G_i in i-th color as a subgraph. However, in that variant Builder first created the whole graph and then Painter picked one of k colors for each edge. The online Ramsey game gives Builder an advantage of knowing the edge-coloring of the graph built so far. This means Builder has more information than in the game for size-Ramsey number and can alter his strategy accordingly. But he can still use the same moves he would use in the game for size-Ramsey number, therefore

$$\widetilde{r}(G_1, G_2, \ldots, G_k) \leq \overline{r}(G_1, G_2, \ldots, G_k).$$

This means that many results from size-Ramsey numbers translate as upper bounds to online Ramsey numbers.

The best known bounds for online Ramsey number of complete graphs $\widetilde{r}(K_t)$ are

$$\frac{r(t)-1}{2} \leq \widetilde{r}(K_t) \leq t^{-c\frac{\log t}{\log\log t}} 4^t$$

where c is a positive constant. The lower bound is due to Alon (and was first published by Beck [11]), and the upper bound is due to Conlon [17].

In 2008 Grytczuk, Kierstead and Prałat [19] came up with an upper bound for paths

$$\widetilde{r}(P_n) \leq 4n - 3$$

which we present in detail later in this section. The cases of the online Ramsey number of paths versus various graphs have been recently investigated by Cyman et al. [32]. They showed that $\widetilde{r}(P_3, P_{l+1}) = \widetilde{r}(P_3, C_l) = \lceil 5l/4 \rceil$ for all $l \geq 5$. They also determined $\widetilde{r}(P_4, P_{l+1})$ up to an additive constant for all $l \geq 3$, and proved some general lower bounds for online Ramsey numbers of the form $\widetilde{r}(P_k, H)$.

Besides the value of the online Ramsey number, we are also interested in structure of the created background graph. Particularly we are interested in a class of graphs which allow Builder to win. This led to the definition of a variant by Grytczuk, Hałuszczak and Kierstead [13].

Definition 6. *Given a class of graphs $\mathcal{G}$ and a set of graphs $G_1, G_2, \ldots, G_k$, let the* restricted online Ramsey number $\widetilde{r}_{\mathcal{G}}(G_1, G_2, \ldots, G_k)$ *be the minimum number of rounds such that the background graph is always in $\mathcal{G}$ and Builder has a winning strategy in the online Ramsey game if both players play optimally.*

It was proved by Grytczuk et al. [13] that for a class of forests $\mathcal{F}$ and an arbitrary forest F, the online Ramsey number $\widetilde{r}_{\mathcal{F}}(F)$ exists. Their proof is constructive and gives an exponential upper bound on $\widetilde{r}_{\mathcal{F}}(F)$. Note that in the size-Ramsey number variant the forest F cannot be forced on the class of forests, because Painter can always color the edges in a way that the biggest monochromatic subgraph is a star.

We say the graph G is *avoidable* on class $\mathcal{G}$ if $\widetilde{r}_{\mathcal{G}}(G)$ does not exist. If $\widetilde{r}_{\mathcal{G}}(G)$ does exist, then G is *unavoidable* on class $\mathcal{G}$. We also use these terms for whole class of graphs when it is true for all graphs from that class. If every graph from class $\mathcal{G}$ is avoidable on $\mathcal{G}$, then $\mathcal{G}$ is *self-avoidable*, and if every graph from class $\mathcal{G}$ is unavoidable on $\mathcal{G}$, then $\mathcal{G}$ is *self-unavoidable*.

Besides self-unavoidability of forests, Grytczuk et al. [13] proved that the class of k-colorable graphs is self-unavoidable. They showed that outerplanar graphs are self-avoidable, and that cycles are unavoidable on planar graphs, and few other results for small graphs. They also generalized self-unavoidability of 3-colorable graphs to the online Ramsey game with 3 colors. They conjectured that the class of graphs unavoidable on planar graphs is exactly the class of outerplanar graphs.

Recently, in 2014 Petříčková [15] showed that outerplanar graphs are unavoidable on planar graphs. She also showed an infinite subclass of non-outerplanar graphs which are unavoidable on planar graphs, disproving the conjecture of Grytczuk et al. [13].

In 2011 Butterfield et al. [14] investigated $\widetilde{r}_{\mathcal{B}}(G)$ of graphs $\mathcal{B}$ with bounded degree. They got several results, one of which is that G is unavoidable on $\mathcal{B}$ which has maximum degree at most 3 if and only if G is a linear forest or each component lies inside $K_{1,3}$. Shortly after, Rolnick [16] provided a complete classification for trees T on the class of graphs with maximum degree 4.

In 2009 Conlon [17] studied the online Ramsey number of cliques. He proved that there is a constant c such that for infinitely many values of t, Builder needs to draw no more than $c^{-t}\binom{r(t)}{2}$ edges to obtain a monochromatic K_t. Therefore, for infinitely many complete graphs the online Ramsey number is asymptotically smaller than their size-Ramsey number. As far as we know, this is the only result of its kind, showing an asymptotic gap between the size-Ramsey number and the online Ramsey number.

We will now remind the proof of Conlon [17], and right after that we show an upper bound on paths by Grytczuk, Hałuszczak and Kierstead [13].

2.4.1 Online Ramsey number of Complete graphs

Our main goal is to compare online and size-Ramsey numbers. In some cases, it is clear that online Ramsey number will not be asymptotically smaller than its size-Ramsey number counterpart (e.g., paths and cycles). However, there is a well known conjecture by Kurek and Ruciński [33] regarding complete graphs, saying that

$$\lim_{t\to\infty} \frac{\widetilde{r}(K_t)}{\overline{r}(K_t)} = 0.$$

In 2009 Conlon [17] made a step in proving this conjecture by showing that there are infinitely many values of t for which the online Ramsey number of K_t is asymptotically smaller than its size-Ramsey number.

Theorem 5 (Conlon 2009)**.** *There exists a constant $c > 1$ such that, for infinitely many t,*

$$\widetilde{r}(t) \le c^{-t}\binom{r(t)}{2}.$$

Informally said, the proof is based on creating one vertex of a monochromatic clique at a time. Each such vertex either contributes to the red clique or the blue clique. When we meet some threshold we finish all edges of the complete subgraph which is big enough to contain the rest of the monochromatic clique. Then we bound the total number of created edges and show it is asymptotically smaller than respective size-Ramsey number.

Proof. Let $\alpha = 0.01$, $\mu = 0.99$, $\nu = 0.01$, and let R and B be counters for red and blue colors which are initially set to 0, and let V_0 be n isolated vertices. Let $N_R(v, V)$ and $N_B(v, V)$ denote the red and blue neighborhood of the vertex v in the set of vertices V. We repeat the following for i from 0 increasing by 1, until either $R \geq \mu t$ or $B \geq \mu t$ or $R \geq \nu t$ and $B \geq \nu t$ at the same time. Select an arbitrary vertex v_{i+1} of V_i, and create $|V_i| - 1$ edges from vertex v_{i+1} to all other vertices of $|V_i|$. If $R = B$ we set V_{i+1} to $N_C(v_i, V_i)$ where C is the majority color. If $R > B$ we set V_{i+1} to be $N_R(v_i, V_i)$ only $|N_R(v_i, V_i)| \geq (1 - \alpha)(|V_i| - 1)$ and $N_B(v_i, V_i)$ otherwise. Similarly, if $R < B$ we set V_{i+1} to be $N_B(v_i, V_i)$ only $|N_B(v_i, V_i)| \geq (1 - \alpha)(|V_i| - 1)$ and $N_R(v_i, V_i)$ otherwise. Now we increase counter R by one if we chose V_{i+1} to be $N_R(v_i, V_i)$ or we increase counter B by one if $N_R(v_i, V_i)$ was chosen.

Suppose that either $R \geq \mu t$ or $B \geq \mu t$ or $R \geq \nu t$ and $B \geq \nu t$ is true. We finish by creating K_p over vertices of V_m in $\binom{p}{2}$ rounds.

The number of iterations where we select the majority color is at most νt. Such iterations can be paired up with iterations where we balance values of counters B and R, in which we choose at least α vertices. When we get close to bound νt we can continue increasing either R or B. This time we make at most μt iterations, and each one we choose at least $(1 - \alpha)$ of vertices. It follows that the final number of vertices in the last iteration is $p = n(1/2)^{\nu t} \alpha^{\nu t} (1 - \alpha)^{\mu t}$. To ensure that p ends up big enough we set

$$n = \left(\frac{\alpha}{2}\right)^{-\nu t} (1 - \alpha)^{-\mu t} p. \tag{2.4}$$

Since we use $n - 1$ rounds during the first iteration and each successive iteration will use less than $n - 1$ rounds, we can bound the number of rounds during all iterations by mn. Therefore the final number of rounds is no more than $mn + \binom{p}{2}$.

We set $p = \max(r((1 - \mu)t, t), r((1 - \nu)t))$, which ensures that a monochromatic K_t will appear. By setting $p \geq r((1 - \mu)t, t)$ we either win by a blue K_t or we have a red clique of order $(1 - \mu)t$. If we finished by condition $R \geq \mu t$, the $(1 - \mu)t$ red vertices which are connected to each other and to all the vertices of V_m will form a red clique of order t. The same argument works if we finished by condition $B \geq \mu t$. On the other hand, by setting $p \geq r((1 - \nu)t)$ we ensure at least one monochromatic clique of order $(1 - \nu)t$. And since we ended up with $R \geq \nu t$ and $B \geq \nu t$, vertices which are connected to each other and to the $(1 - \nu)t$ clique in one color, forming a monochromatic clique of order t.

We know that this process will create a monochromatic K_t. To prove that the necessary number of rounds to complete a K_t is small, we first show bound on $p \leq 1.001^{-t} r(t)$.

If we reached either $R \geq \mu t$ or $B \geq \mu t$, then

$$r((1 - \mu)t, t) = r(0.01t, t) \leq \binom{1.01t}{t} = \binom{1.01t}{0.01t} \leq \left(\frac{1.01et}{0.01t}\right)^{0.01t} \leq 1.06^t \leq 1.25^{-t} r(t).$$

We obtained these expressions in the following way. The first comes from substitution, then we used Inequation 2.1, next is true by definition, next follows from the fact that $\binom{n}{k} \leq (en/k)^k$, then we evaluated all the constants, and the last comes from $r(t) \geq 2^{t/2}$ (Inequation 2.2).

We also want to show similar results for the case where we reached $R \geq \nu t$ and $B \geq \nu t$ instead. Our goal is to show $r((1 - \nu)t) = r(0.99t) \leq 1.001^{-t} r(t)$. Suppose for the contradiction that there is some t_0, such that for all $t \geq t_0$,

$$\frac{r(t)}{r(0.99t)} \leq 1.001^t.$$

We use telescoping to get

$$r(0.99^{-A}t_0) \leq (1.001)^{(0.99^{-1}+\cdots+0.99^{-A})t_0} r(t_0) \leq (1.001)^{100(0.99)^{-A}t_0} r(t_0).$$

Now, by setting $t = 0.99^{-A}t_0$ and $C = r(t_0)$, we get

$$r(t) \leq C(1.001)^{100t} \leq C(1.106)^t.$$

Which is false, because $r(t) \geq 2^{t/2}$ (Inequation 2.2) and $C(1.106)^t = o(\sqrt{2}^t)$.

Proofs of both cases led to the conclusion, that there are infinite different values of t such that $p \leq 1.001^{-t}r(t)$. The last step we need to do is to show, that all of this can be done in reasonable number of rounds. By the previous discussion, the number of rounds is no more than

$$mn + \binom{p}{2} \leq t\left(\frac{\alpha}{2}\right)^{-\nu t}(1-\alpha)^{-\mu t}p + \binom{p}{2},$$

which comes from $m \leq \mu t + \nu t = t$ and n from Equation 2.4. We continue by substitution, and get the following inequality.

$$mn + \binom{p}{2} \leq t\left(\frac{\alpha}{2}\right)^{-\nu t}(1-\alpha)^{-\mu t}p + \binom{p}{2} \leq t(200)^{0.01t}(0.99)^{-0.99t}p + \binom{p}{2} \leq$$

$$\leq t(1.06497)^t p + \binom{p}{2} \leq \frac{r(t)-1}{4}p + \binom{p}{2} \leq 1.001^{-t}\binom{r(t)}{2} = 1.001^{-t}\overline{r}(K_t)$$

Proving that there is an infinite family of cliques for which online Ramsey number is asymptotically smaller than their size-Ramsey number. □

2.4.2 Online Ramsey number of Paths

In 2008 Grytczuk, Kierstead and Prałat [19] showed an upper bound on $\widetilde{r}(P_n)$. In particular, they showed that for a path P_n of length n, $\widetilde{r}(P_n) \leq 4n-3$. Note that in the original paper P_n denotes path of length $n-1$, which leads to $\widetilde{r}(P_n) \leq 4n-7$, however to make our notation consistent throughout this thesis we use P_n to denote path of length n. Now we will present their result.

Theorem 6. *The online Ramsey number for paths of length n has an upper bound $\widetilde{r}(P_n) \leq 4n-3$.*

Proof. Assume we have two disjoint monochromatic red and blue paths. A red path of length m ending in vertex u and a blue path of length k ending in vertex v. We create an edge $f = \{u, v\}$ which is without loss of generality blue. Now we create an edge $e = \{u, w\}$ where w is previously unused vertex. If the edge e is red, we make the red path longer by adding e to it. If that is not the case and e is blue, we make the blue path longer by adding f and e, and shorten the red path by one, so that the paths end up disjoint again (see Fig. 2.1).

If the paths have zero lengths, we create an edge and get one path of length 1 in only one move. If one path has length 0 and its only vertex would have been removed from it, we pick a different isolated vertex as its new endpoint.

The sum of lengths $m+k$ is increased by 1 every two rounds. To get a monochromatic P_n it suffices to make $m+k = 2n-1$. Thus we need at most $2(m+k)-1 = 2(2n-1)-1 = 4n-3$ rounds to create a monochromatic path P_n. (Note that if we had denoted paths by number of vertices as in the original paper, we would have had at most $2(2(n-1)-1)-1 = 4n-7$ rounds.) □

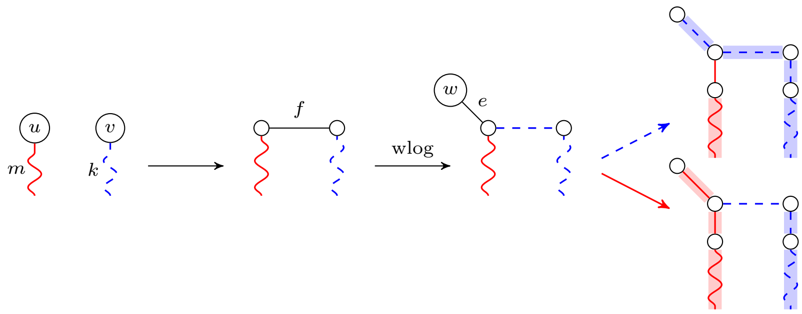

Figure 2.1: One step in creating a monochromatic P_n

Chapter **3**

Our results

Our main goal is to characterize for which classes of graphs there is an asymptotic gap between the size-Ramsey number and the online Ramsey number. In this part of the thesis we present our results.

First and foremost, in Section 3.1 we present an infinite family of trees for which the online Ramsey number is asymptotically smaller than their size-Ramsey number.

Then, in Section 3.2 we explore online Ramsey number of stars with subdivided edges (spiders) and show an upper bound, which asymptotically matches the lower bound for their size-Ramsey number.

In Section 3.3, we show that online Ramsey number of cycles is no more than $23n/2-20$ for even cycles, and $24n-20$ for odd cycles, providing new upper bounds for the $\widetilde{r}(C_n)$.

And last, in Section 3.4, we show an exact value of $\widetilde{r}_{\mathcal{C}}(C_3, S_n)$ where $\mathcal{C}$ is class of connected graphs.

3.1 Online Ramsey Number of Trees

Let us denote $S_{k,\ell}$ a tree consisting of a root vertex c which has k neighbors $x_1, x_2, \ldots, x_k$ and each x_i is connected to ℓ leaves. Note that $S_{k,\ell}$ has $1+k+k\ell$ vertices and $k\ell$ leaves.

In the following theorem we show that any tree $S_{k,\ell}$ exhibits small online Ramsey number.

Theorem 7. $\widetilde{r}(S_{k,\ell}) \leq k^2 + 2k\ell^2 + 8k\ell + 2k - 4\ell - 3 = O(k^2 + k\ell^2)$.

Proof. To force a monochromatic $S_{k,\ell}$, we start by creating a star $S_{2(p+k-1)-1}$, where $p = k(\ell+1)+k-1$, with center in c. without loss of generality, let the majority color of created edges be blue. It follows that we have a blue S_{p+k-1}. We continue by creating $p+k-1$ stars with centers in leaves of S_{p+k-1}, each with ℓ leaves of its own. If k of these stars are blue, we win immediately. If that is not the case, we know that at least ℓ of those stars are red. We will denote these red stars and their centers by $\mathcal{T}^0 = \{T_1^0, T_2^0, \ldots, T_p^0\}$ and $C^0 = \{c_1^0, c_2^0, \ldots, c_p^0\}$ respectively. Now for $i = 1, \ldots, k$ we do as follows. First we create $k+\ell-1$ new edges E^i between an arbitrary center $u \in C^{i-1}$ and $k+\ell-1$ other centers in C^{i-1}. If there are k red edges among E^i we win. If this is not the case there must be at least ℓ blue edges in E^i from u to vertices of $B^i \subseteq C^{i-1}$. Let $X \subseteq \mathcal{T}^{i-1}$ be stars such that their centers are in $B^i \cup \{u\}$. At the end of each step let $\mathcal{T}^i = \mathcal{T}^{i-1} \setminus X$ and $C^i = C^{i-1} \setminus (B^i \cup \{u\})$.

Note that in each step we either win or create a blue S_ℓ which is connected with a blue edge to vertex c. If we did not win by building a red $S_{k,\ell}$ in k steps we know that there are k blue stars S_ℓ each connected with a blue edge to c forming a blue $S_{k,\ell}$.

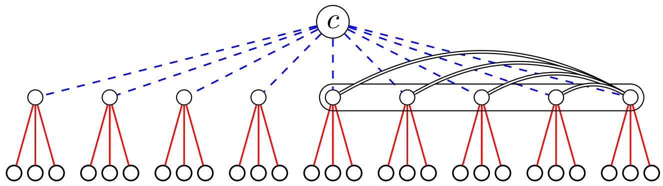

Figure 3.1: One step in building $S_{2,3}$

We started by creating a monochromatic S_{p+k-1} in $2(p+k-1)-1$ rounds. Then we made $p+k-1$ disjoint monochromatic stars S_ℓ for which we needed $(p+k-1)(2l-1)$ rounds. After that, we connected centers of stars in $k+\ell-1$ rounds k times which required $k(k+\ell-1)$ rounds in total. This gives us that the final number of necessary rounds is no more than $2(p+k-1)-1+(p+k-1)(2l-1)+k(k+\ell-1) = k^2+2k\ell^2+8k\ell+2k-4\ell-3$. □

Recall that due to Beck [7] we have a lower bound for trees T which is $\overline{r}(T) \geq \beta(T)/4$ where $\beta(T)$ is defined as

$$\beta(T) = |T_0|\Delta(T_0) + |T_1|\Delta(T_1),$$

where T_0 and T_1 are partitions of the unique bipartitioning of the tree T. The β for our family of trees is $\beta(S_{k,\ell}) = (1+k\ell)k + k(\ell+1) = \Theta(k^2\ell)$, whis gives us the lower bound in size-Ramsey number $\overline{r}(S_{k,\ell}) = \Omega(k^2\ell)$.

Since by Theorem 7 we have $\widetilde{r}(S_{k,\ell}) = O(k^2+k\ell^2)$, the online Ramsey number for $S_{k,\ell}$ is asymptotically smaller than its size-Ramsey number if $k = \omega(\ell)$.

Corollary 1. *There is an infinite sequence of trees $T_1, T_2, \ldots$ such that $|T_i| < |T_{i+1}|$ for each $i \geq 1$ and*

$$\lim_{i\to\infty} \frac{\widetilde{r}(T_i)}{\overline{r}(T_i)} = 0.$$

3.2 Family of trees of arbitrary depth

Let us define a *spider* $\sigma_{k,\ell}$ for $k \geq 3$ and $\ell \geq 1$ as a union of k paths of length ℓ that share exactly one common endpoint. Let a *center* of $\sigma_{k,\ell}$ denote the only vertex with degree equal to k.

In the following theorem we obtain an upper bound on $\widetilde{r}(\sigma_{k,\ell})$ that asymptotically matches the lower bound on $\overline{r}(\sigma_{k,\ell})$.

Theorem 8. $\widetilde{r}(\sigma_{k,\ell}) \leq k^2\ell + 32k\ell - 10k - 24\ell + 8 = O(k^2\ell)$.

Proof. We describe the Builder's strategy for obtaining a monochromatic $\sigma_{k,\ell}$. We first create monochromatic paths $P' = \{P'_1, P'_2, \ldots, P'_{4k-3}\}$ of length 2ℓ using the strategy by Grytczuk et al. [19]. Let the majority color among these path without loss of generality be blue, and denote these blue paths $P_1, P_2, \ldots, P_{2k-1}$. Let $P_{i,j}$ denote a j-th vertex on path P_i. We create a new vertex u and edges $E' = \{\{u,v\} \mid v \in \{P_{1,1}, P_{2,1}, \ldots, P_{2k-1,1}\}\}$. If there are k blue edges among E' there is $\sigma_{k,\ell}$ with center in u. If this is not the case there are at least k red edges $E^1 \subseteq E'$. Let vertices $V^1 = \{v \mid \{u,v\} \in E^1, \{u,v\} \text{ is red}\}$. Note that we defined E^1 and V^1 such that $|E^1| = |V^1| = k$ and they form a red S_k.

For i from 1 to $\ell - 1$ we do as follows. Let $U_1 = \{P_{q,i+1} \mid q \in \{1, \dots, 2k-1\}\}$. For each $v_j \in V^i$ for j from 1 to k create k edges $D_j = \{\{v_j, w\} \mid w \in U_j\}$. If D_j has k blue edges, they connect k blue paths of length $2l$ to a center v_j forming a blue $\sigma_{k,\ell}$. If that is not the case there is one red edge $\{v_j, q\} \in D_j$. Let $U_{j+1} = U_j \setminus \{q\}$ and continue with next iteration of j.

If there is no blue $\sigma_{k,\ell}$ after the last iteration for j, we prepare for the next iteration of i. Let edges $E^{i+1} = \{\{v,q\} \mid v \in V^i, q \in U_1, \{v,q\} \text{ is red}\}$ and vertices $V^{i+1} = \{q \mid \{v,q\} \in E^{i+1}\}$. Note that $|E^{i+1}| = |V^{i+1}| = k$ and that u is the center vertex of a red $\sigma_{k,i+1}$.

When we finish the last iteration of i and we did not win by obtaining a blue $\sigma_{k,\ell}$ in the process, there is a red $\sigma_{k,\ell}$ with vertices $V(\sigma_{k,\ell}) = \{u\} \cup \{v \mid v \in V^i, 1 \le i \le \ell\}$ and edges $E(\sigma_{k,\ell}) = \{e \mid e \in E^i, 1 \le i \le \ell\}$.

We built $4k-3$ paths of length 2ℓ using strategy by Grytczuk et al. [19] in $(4k-3)(4(2\ell)-3)$ rounds. We created $2k-1$ edges from u and then we used $k^2\ell$ rounds to ℓ times repeat process which created k^2 edges. We either got a blue $\sigma_{k,\ell}$ in the process or a red $\sigma_{k,\ell}$ after using not more than $k^2\ell + 32k\ell - 10k - 24\ell + 8$ rounds, which concludes the proof. □

Note that $|V(\sigma_{k,\ell})| = 1 + k\ell$, $|E(\sigma_{k,\ell})| = k\ell$ and $\Delta(\sigma_{k,\ell}) = k$. The Beck's [7] lower bound for spiders is $\beta(\sigma_{k,\ell}) = k(1 + k\lfloor \ell/2 \rfloor) + 2(k\lceil \ell/2 \rceil) = \Theta(k^2\ell)$, so our result for online Ramsey number of spiders asymptotically matches their lower bound for size-Ramsey number.

3.3 Online Ramsey Number of Cycles

The Ramsey number for cycles was investigated by Bondy and Erdős [26]. Also generally more complicated version of size-Ramsey numbers, where the target monochromatic graph has to be induced on subset of vertices of the background graph, was investigated by Haxell et al. [10], proving that the induced size-Ramsey number of C_n is linear in n. This naturally bounds the online variant with as well.

Since $n \le \widetilde{r}(C_n) \le \overline{r}(C_n)$ we know that the online Ramsey number is linear, however the exact value of $\widetilde{r}(C_n)$ is still unknown. In this section we will present an upper bounds for $\widetilde{r}(C_n)$.

3.3.1 Upper bound for even cycles

Theorem 9. *Let C_n be a cycle on n vertices, where n is even. Then $\widetilde{r}(C_n) \le 23n/2 - 20$.*

Proof. To create an even cycle C_n we first obtain 13 monochromatic paths of length $2n$ (using the strategy by Grytczuk et al. [19]) and choose 7 paths $\rho_1, \rho_2, \dots, \rho_7$ of the same color which are without loss of generality red. Let $\rho_{i,j}$ denote j-th vertex of path ρ_i. Note that connecting vertex $u \notin \rho_i$ to vertices $w_1 = \rho_{i,j}$ and $w_2 = \rho_{i,j+n-2}$ either creates a red C_n or at least one edge $\{u, w_1\}$ or $\{u, w_2\}$ is blue. We will now enforce a C_n on these paths in the following way.

Let u be an unused vertex and let $Q = \rho_1$ and $W = \rho_2$. To create one blue path of length $n/2$ we start by creating edges from u to Q_1 and Q_{n-1}. If both of these edges are red, we win. If that is not the case then at least one edge $\{u, v_1\}$ where $v_1 \in \{Q_1, Q_{n-1}\}$ is blue. Now for i from 1 to $n/2 - 1$ we do this. Let $X = W$ if i is odd and $X = Q$ of i is even. Create edges $\{v_i, X_{i+1}\}$ and $\{v_i, X_{i+n-1}\}$ and again by the same argument we either win, or one of these edges $\{v_i, v_{i+1}\}$ where $v_{i+1} \in \{X_{i+1}, X_{i+n-1}\}$ is blue. After all iterations finish, we have a blue path of length $n/2-1$ starting in u and ending in $v_{n/2}$. Finish the path by creating edges $\{v_{n/2}, \rho_{7,1}\}$ and $\{v_{n/2}, \rho_{7,n-1}\}$.

Using this procedure we built a blue path of length $n/2$ starting in u and ending in either $\rho_{7,1}$ or $\rho_{7,n-1}$. We repeat this procedure two more times using $Q=\rho_3, W=\rho_4$ and $Q=\rho_5, W=\rho_6$ as initial settings. All of these paths start in the same vertex and two of these paths will necessarily have a common endpoint, creating a blue C_n as a result.

We will now further improve the upper bound on even cycles by optimizing length of initial paths. Path ρ_7 needs to be $n-2$ edges long. For paths $\rho_1, \rho_2, \ldots, \rho_6$ we use $\rho_{1,1}$ and $\rho_{1,n-1}$ in first iteration, $\rho_{2,2}$ and $\rho_{2,n}$ in second iteration, $\rho_{1,3}$ and $\rho_{1,n+1}$ in third iteration, $\ldots$, and similarly for paths ρ_3, ρ_4 and ρ_5, ρ_6. Since there are $n/2-1$ iterations in building a blue path, we require these paths to be at least $(n-2)+(n/2-1)=3n/2-3$ long. (We chose $2n$ to make the basic idea clearer.) This can be reduced even more with changing value of i only every even iteration, which we call *foding* (Fig. 3.2).

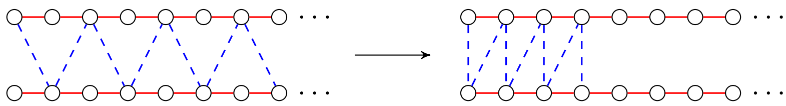

Figure 3.2: Improving upper bound for even cycles with folding

Using this method, we require paths $\rho_1, \rho_2, \ldots, \rho_6$ to have the length at least

$$(n-2)+\left(\left\lfloor \frac{\frac{n}{2}-1}{2} \right\rfloor - 1\right) \leq \frac{5n}{4}-\frac{7}{2}.$$

It would be easy to create seven paths of the same length, however ρ_7 is shorter. Let $\mathcal{Q}=\frac{5n}{4}-\frac{7}{2}$. We now have the choice to either build seven paths of length $\mathcal{Q}$ (which is longer than $n-2$ for all $n \geq 2$) or we will make one path of length $6(\mathcal{Q}+1)+(n-2)$. If we choose the former method, we get

$$13\widetilde{r}\left(P_{\mathcal{Q}}\right)=13\left(4\left(\mathcal{Q}\right)-3\right)=65n-221.$$

If we choose the latter method, we will be finished in at most

$$4\left(6\left(\mathcal{Q}+1\right)+\left(n-2\right)\right)-3=\frac{17n}{2}-20$$

rounds. This means the latter method for creating all necessary initial paths is superior for $n>\frac{402}{113}$, which holds true for $n \geq 4$, and hence for any reasonable even cycle.

After folding blue paths we decreased number of rounds to create initial paths to $\widetilde{r}(P_q)=17n/2-20$, for $q=6(5n/4-7/2+1)+(n-2)$. Then we used $3(2(n/2))=3n$ rounds to make three blue paths. This gives us the final upper bound for online Ramsey number of even cycles $\widetilde{r}(C_n) \leq 3n+17n/2-20=23n/2-20$. □

On Figure 3.3 we see 7 red paths and 3 blue paths. Notice that each blue edge is created with its counterpart (pattern demonstrated on ρ_5, ρ_6), both starting in same vertex and ending $n-2$ edges apart (see ρ_1), forcing one of them to be blue or a red C_n would appear immediately. Note that counterpart edges would be red, because if they were blue, we would not need to create another edge to force a blue edge.

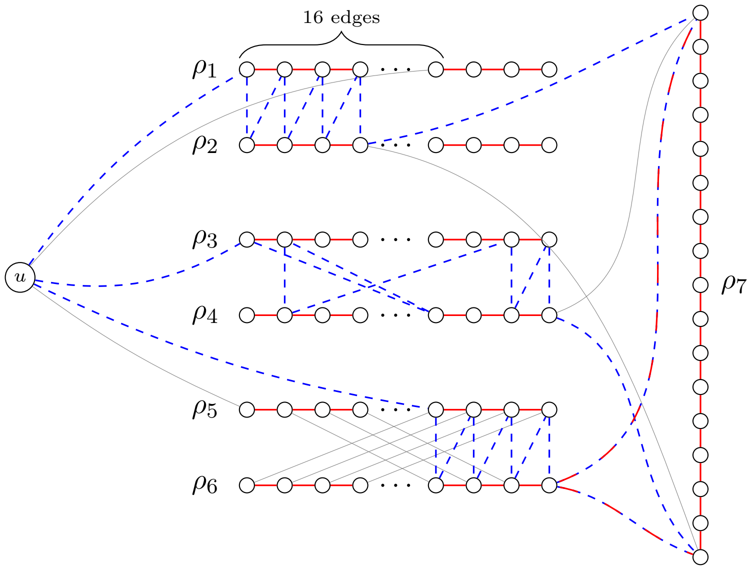

Figure 3.3: Whole process of making C_{18}

3.3.2 Upper bound for odd cycles

Theorem 10. *Let C_n be a cycle on n vertices, where n is odd. Then $\widetilde{r}(C_n) \leq 24n - 20$.*

Proof. An upper bound for odd cycles C_n is based on upper bound for even cycles. Let $c_0, c_1, \ldots, c_{2n-1}$ denote vertices on the cycle C_{2n} modulo $2n$, therefore c_i for $i \geq 2n$ denotes vertex c_j, $j = i \bmod 2n$. We start with building without loss of generality a blue C_{2n} using strategy described in Section 3.3.1. Then we add edges $E = \{\{c_i, c_{i+n-1}\} \mid i \in \{1 + j(n-1)\}, j \in \{0, 1, \ldots, (n-1)\}\}$. Each edge in E connects two vertices which are $n-1$ apart on C_{2n}, and since $\gcd(n-1, 2n) = 2$ it will take exactly n edges to complete a cycle $C_n^{'} = (\{c_0, c_2, \ldots, c_{2n-2}\}, E)$.

Note that if any edge of $C_n^{'}$ is blue, it forms a blue cycle using that edge and part of the C_{2n}. However if every edge of E is red, then $C_n^{'}$ forms a red cycle (see Fig. 3.4).

We first used the strategy for even cycle C_{2n} which took $23(2n)/2 - 20 = 23n - 20$ rounds. Then we simply added n edges to form the $C^{'}$. This gives us the upper bound for odd cycles $\widetilde{r}(C_n) \leq \widetilde{r}(C_{2n}) + n \leq 24n - 20$. □

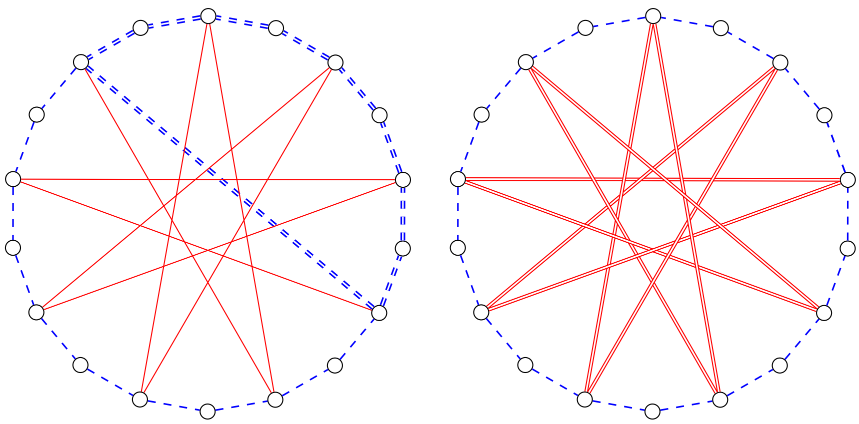

Figure 3.4: Final step of building C_9

3.4 Online Ramsey Number of Stars vs Triangles

Upper bound of $\widetilde{r}_{\mathcal{C}}(C_3, S_n)$

Theorem 11. $\widetilde{r}_{\mathcal{C}}(C_3, S_n) \leq 3n - 1$, *where $\mathcal{C}$ is class of connected graphs.*

Proof. We start with creating $2n - 1$ edges from a vertex u forming a star S_{2n-1}. We know that one of the following three cases will occur:

1. A blue S_n occurred, which wins the game immediately.

2. A red S_{n+1} occurred, which can be exploited to force a blue S_n on its leaves in n rounds.

3. A blue S_{n-1} and a red S_n occurred.

In any case we used at most $2n-1$ rounds so far. The first case is winning position already. In the second case we create a S_n on leaves of the red star. If every edge is blue, we win by a blue S_n, otherwise there is a red C_3 formed by one red edge and edges of the original red S_{n+1}. The second case therefore takes at most $3n - 1$ rounds. If the third case occurs we do as follows. Let v denote a leaf of the red star, which is guaranteed to exist in this case. We create an edge $e = \{v, w\}$, where w is previously unused vertex. If Painter decides to color e red, we create an edge $\{w, u\}$ which either completes a red C_3 on vertices u, v, w, or it completes a blue star with center in u. In either case we completed one of the desired graphs in at most $2n + 1$ rounds. If Painter decides to color e blue, we build $n - 1$ edges between v and other leaves of the original red star S_n. This either completes a red C_3 in similar manner as in the second case, or we get a blue S_{n-1} on leaves of the original red star with the center in v. This blue S_{n-1} together with the blue edge e forms a S_n with the center in v. This case can be performed in no more than $3n - 1$ rounds. See Fig. 3.5 for an overview of the whole strategy. □

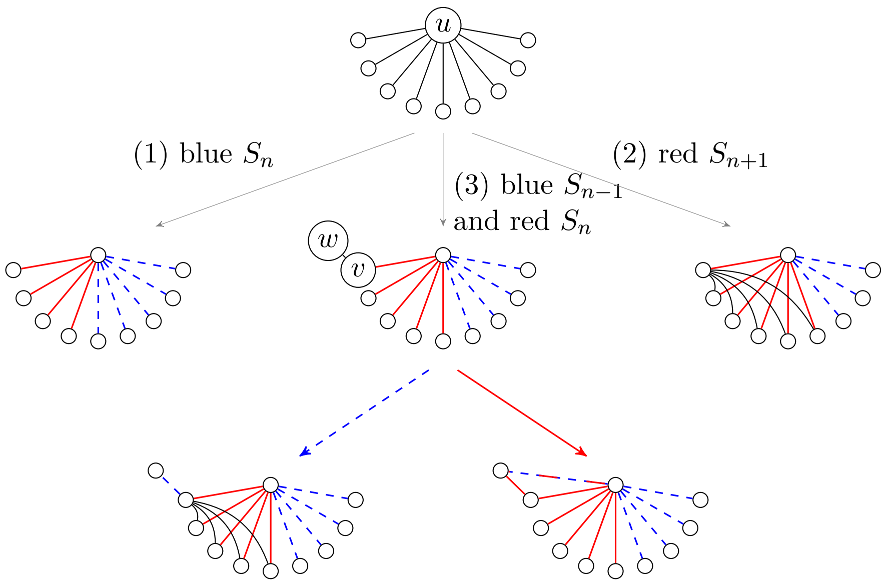

Figure 3.5: Builders strategy for $\widetilde{r}(C_3, S_5)$ on connected graphs

Lower bound of $\widetilde{r}_{\mathcal{C}}(C_3, S_n)$

We proved that the Builder is able to force either a red C_3 or a blue S_n in at most $3n-1$ rounds. However, to prove the tight bound, we also have to show that the Painter is able to defend for at least $3n-2$ rounds.

Theorem 12. *$\widetilde{r}_{\mathcal{C}}(C_3, S_n) > 3n-2$, where $\mathcal{C}$ is class of connected graphs.*

Proof. To prevent construction of C_3 we will divide vertices of the background graph G into partitions $\mathcal{P}_1$ and $\mathcal{P}_2$, which are initially empty. Let partition of vertex v be $\mathcal{P}(v) = \{\mathcal{P}_i \mid v \in \mathcal{P}_i\}$ and partition without vertex v be $\mathcal{P}'(v) = \{\mathcal{P}_i \mid v \notin \mathcal{P}_i\}$. Whenever Builder adds a new vertex to the background graph G, we will decide its partition. We will decide color of an edge only from partitions of incident vertices in the following way. The edge $\{u, v\}$ will be painted blue if $u \in \mathcal{P}(v)$ and red if $u \in \mathcal{P}'(v)$. Let number of blue edges of partition $\mathcal{P}$ be $B(\mathcal{P}) = |\{\{u, v\} \mid u, v \in \mathcal{P}, \{u, v\} \in E(G)\}|$, meaning all blue edges which are between vertices of P. This coloring scheme ensures, that Builder cannot enforce a red C_3, because the subgraph of the background graph formed by the red edges is bipartite. Note that if Builder creates a new edge between two vertices which are already used in the background graph, the color of that edge is known to him in advance.

Let a round be *active* if Builder either introduced a new vertex to the background graph, or he connected two components of the background graph, and let all other rounds be *passive*. We will denote edges created in active and passive rounds as *active* and *passive* edges respectively. Let a Builders strategy be called *delayed* strategy, if all feasible games played by that strategy all active rounds were played before all passive rounds. Note that any Builders strategy which plays against our Painters strategy can be transformed to an equivalent delayed strategy, because the color of passive edges is already decided, so Builder can postpone all passive rounds until the very end of his strategy.

At this point both players know that the only way to win is by forming/denying a blue S_n. Our goal as Painter will be to restrict the number of vertices and number of blue edges in each partition in a way that the blue S_n cannot be formed for as long as possible.

We assume Builders strategy is delayed and restricted to connected graphs. Note that there will be two new vertices in the first move, one in each other active move, and none in all passive moves. We assign the first edge $\{u, v\}$ vertices to partitions P_1 and P_2 respectively and color the first edge red. Now we have to decide the partition of each introduced vertex of respective active edge. Let the new edge be $e = \{u, v\}$, where u is already in the background graph and v is the newly introduced vertex. We will choose the partition of v in the following way:

1. if $|P_1| \geq n$ and $|P_2| \geq n$ put v into partition with less blue edges,
2. if $|P_i| = n$ and $|P_b| < n$ then put v to P_{2-i},
3. otherwise put v to $P'(u)$ (edge will be colored red).

These rules clearly ensure that no blue edge can be created until one partition has n vertices in it, which in turn means that first n created edges will be red (Fig. 3.6 (1)). After one partition P_i has n vertices then all new vertices will be put in the partition with less vertices (Fig. 3.6 (2)). Edges created in this process might be colored blue, however note that there is no chance for Builder to make an S_n in any partition until there are at least $n+1$ vertices in it. He adds new vertices in active rounds and both partitions will have exactly n vertices after $2n-1$ rounds. Now any new vertex will be put into partition with less blue edges, which is P_i with zero blue edges (Fig. 3.6 (3)). This is last round which needs to be active for the strategy to work. Now builder will have to use at least n rounds to finish S_n on P_i (Fig. 3.6 (4)). If he wanted to switch, and build S_n on P_{2-i} he would have to ensure that $B(P_i) \leq B(P_{2-i})$ which would take $n-1$ rounds, rendering such strategy not better than building S_n on P_i. □

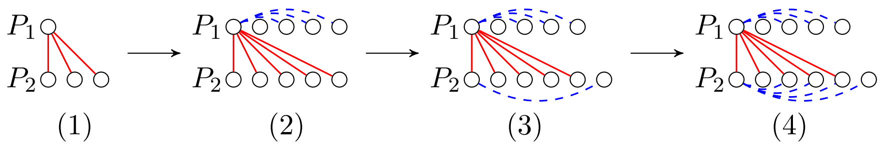

Figure 3.6: Painters strategy for $\widetilde{r}(C_3, S_5)$ on connected graphs

Corollary 2. *$\widetilde{r}_{\mathcal{C}}(C_3, S_n) = 3n - 1$, where $\mathcal{C}$ is class of connected graphs.*

Note that the condition of connectivity for the background graph is important for analysis of Painter's defending strategy. The described Painter's strategy could be exploited by the Builder in the following way. First we create star S_n, which will be by the Painter's strategy red. Then create an isolated red edge, which would be also red by the Painter's strategy. We draw an edge between any vertex of the isolated edge and any vertex of the star. The resulting graph will have $n+1$ vertices in one component, now we only need to build the blue S_n star on vertices of that component and we win. This means that for disconnected graphs the lower bound of presented Painter's strategy is $2n$.

We presented Builder's way to force either a red C_3 or a blue S_5 in no more than $3n-1$ rounds, and also we showed Painter's defending strategy to avoid $\widetilde{r}(C_3, S_5)$ in $3n-2$ rounds. This implies that the bound is tight, proving Theorem 11.

Future work

There are many open problems in the online Ramsey graph theory. The main questions that directly follow from this thesis are: First, to determine the online Ramsey number of triangles versus stars $\overline{r}(C_3, S_n)$ without restricting class of graphs. And second, to expand the class of trees for which the online Ramsey number is asymptotically smaller than the size-Ramsey number, possibly characterizing the class of such trees completely.

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Appendix **A**

Contents of enclosed CD

```
readme.txt ................................ the file with CD contents description
src-thesis ...................... the directory of LaTeX source codes of the thesis
text ................................................. the thesis text directory
  thesis.pdf ................................... the thesis text in PDF format
  thesis.ps ...................................... the thesis text in PS format
```