

# LAPLACE EQUATIONS, CONFORMAL SUPERINTEGRABILITY AND BÔCHER CONTRACTIONS

ERNEST KALNINS<sup>a</sup>, WILLARD MILLER, JR.<sup>b,\*</sup>, EYAL SUBAG<sup>c</sup>

<sup>a</sup> Department of Mathematics, University of Waikato, Hamilton, New Zealand

<sup>b</sup> School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455, USA

<sup>c</sup> Department of Mathematics, Pennsylvania State University, State College 16802, Pennsylvania USA

\* corresponding author: [miller@ima.umn.edu](mailto:miller@ima.umn.edu)

ABSTRACT. Quantum superintegrable systems are solvable eigenvalue problems. Their solvability is due to symmetry, but the symmetry is often “hidden”. The symmetry generators of 2nd order superintegrable systems in 2 dimensions close under commutation to define quadratic algebras, a generalization of Lie algebras. Distinct systems and their algebras are related by geometric limits, induced by generalized Inönü-Wigner Lie algebra contractions of the symmetry algebras of the underlying spaces. These have physical/geometric implications, such as the Askey scheme for hypergeometric orthogonal polynomials. The systems can be best understood by transforming them to Laplace conformally superintegrable systems and using ideas introduced in the 1894 thesis of Bôcher to study separable solutions of the wave equation. The contractions can be subsumed into contractions of the conformal algebra  $so(4, \mathbb{C})$  to itself. Here we announce main findings, with detailed classifications in papers submitted and under preparation.

KEYWORDS: conformal superintegrability; contractions; Laplace equations.

## 1. INTRODUCTION

A quantum superintegrable system is an integrable Hamiltonian system on an  $n$ -dimensional Riemannian/pseudo-Riemannian manifold with potential:  $H = \Delta_n + V$  that admits  $2n - 1$  algebraically independent partial differential operators  $L_j$  commuting with  $H$ , the maximum possible.  $[H, L_j] = 0$ ,  $j = 1, 2, \dots, 2n - 1$ . Superintegrability captures the properties of quantum Hamiltonian systems that allow the Schrödinger eigenvalue problem (or Helmholtz equation)  $H\Psi = E\Psi$  to be solved exactly, analytically and algebraically [1–5]. A system is of order  $K$  if the maximum order of the symmetry operators, other than  $H$ , is  $K$ . For  $n = 2$ ,  $K = 1, 2$  all systems are known, see, e.g., [6, 7]

We review quickly the facts for *free* 2nd order superintegrable systems, (i.e., no potential,  $K = 2$ ) in the case  $n = 2, 2n - 1 = 3$ . The complex spaces with Laplace-Beltrami operators admitting at least three 2nd order symmetries were classified by Koenigs in 1896 [8]. They are:

- The two constant curvature spaces (flat space and the complex sphere), six linearly independent 2nd order symmetries and three 1st order symmetries,
- The four Darboux spaces (one with a parameter), four 2nd order symmetries and one 1st order symmetry [9],

$$ds^2 = 4x(dx^2 + dy^2),$$

$$ds^2 = \frac{x^2 + 1}{x^2}(dx^2 + dy^2),$$

$$ds^2 = \frac{e^x + 1}{e^{2x}}(dx^2 + dy^2),$$

$$ds^2 = \frac{2 \cos 2x + b}{\sin^2 2x}(dx^2 + dy^2).$$

- Eleven 4-parameter Koenigs spaces. No 1st order symmetries. An example is

$$ds^2 = \left( \frac{c_1}{x^2 + y^2} + \frac{c_2}{x^2} + \frac{c_3}{y^2} + c_4 \right) (dx^2 + dy^2).$$

For 2nd order systems with non-constant potential,  $K = 2$ , the following is true [6, 7, 10–12].

- The symmetry operators of each system close under commutation to generate a quadratic algebra, and the irreducible representations of this algebra determine the eigenvalues of  $H$  and their multiplicity.
- All the 2nd order superintegrable systems are limiting cases of a single system: the generic 3-parameter potential on the 2-sphere,  $S_9$  in our listing [13], or are obtained from these limits by a Stäckel transform (an invertible structure preserving mapping of superintegrable systems [6]). Analogously all quadratic symmetry algebras of these systems are limits of that of  $S_9$ .

$$S_9: \quad H = \Delta_2 + \frac{a_1}{s_1^2} + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2}, \quad s_1^2 + s_2^2 + s_3^2 = 1,$$

$$L_1 = (s_2 \partial_{s_3} - s_3 \partial_{s_2})^2 + \frac{a_3 s_2^2}{s_3^2} + \frac{a_2 s_3^2}{s_2^2}, \quad L_2, L_3,$$

- 2nd order superintegrable systems are multiseparable.

Here we consider only the nondegenerate superintegrable systems: Those with 4-parameter potentials (the maximum possible):

$$V(\mathbf{x}) = a_1V_{(1)}(\mathbf{x}) + a_2V_{(2)}(\mathbf{x}) + a_3V_{(3)}(\mathbf{x}) + a_4,$$

where  $\{V_{(1)}(\mathbf{x}), V_{(2)}(\mathbf{x}), V_{(3)}(\mathbf{x}), 1\}$  is a linearly independent set. For these the symmetry algebra generated by  $H, L_1, L_2$  always closes under commutation and gives the following quadratic algebra structure: Define 3rd order commutator  $R$  by  $R = [L_1, L_2]$ . Then

$$\begin{aligned} [L_j, R] &= A_1^{(j)}L_1^2 + A_2^{(j)}L_2^2 + A_3^{(j)}H^2 \\ &+ A_4^{(j)}\{L_1, L_2\} + A_5^{(j)}HL_1 + A_6^{(j)}HL_2 + A_7^{(j)}L_1 \\ &\quad + A_8^{(j)}L_2 + A_9^{(j)}H + A_{10}^{(j)}, \\ R^2 &= b_1L_1^3 + b_2L_2^3 + b_3H^3 + b_4\{L_1^2, L_2\} + b_5\{L_1, L_2^2\} \\ &+ b_6L_1L_2L_1 + b_7L_2L_1L_2 + b_8H\{L_1, L_2\} + b_9HL_1^2 \\ &\quad + b_{10}HL_2^2 + b_{11}H^2L_1 + b_{12}H^2L_2 + b_{13}L_1^2 \\ &\quad + b_{14}L_2^2 + b_{15}\{L_1, L_2\} + b_{16}HL_1 + b_{17}HL_2 \\ &\quad + b_{18}H^2 + b_{19}L_1 + b_{20}L_2 + b_{21}H + b_{22}, \end{aligned}$$

where  $\{L_1, L_2\} = L_1L_2 + L_2L_1$  and  $A_i^{(j)}, b_k$  are constants.

All 2nd order 2D superintegrable systems with potential and their quadratic algebras are known. There are 44 nondegenerate systems, on a variety of manifolds (just the manifolds classified by Koenigs), but under the Stäckel transform they divide into 6 equivalence classes with representatives on flat space and the 2-sphere [14]. Every 2nd order symmetry operator on a constant curvature space takes the form

$$L = K + W(\mathbf{x}),$$

where  $K$  is a 2nd order element in the enveloping algebra of  $o(3, \mathbb{C})$  or  $e(2, \mathbb{C})$ . An example is  $S_9$  where

$$H = J_1^2 + J_2^2 + J_3^2 + \frac{a_1}{s_1^2} + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2},$$

where  $J_3 = s_1\partial_{s_2} - s_2\partial_{s_1}$  and  $J_2, J_3$  are obtained by cyclic permutations of indices. Basis symmetries are ( $J_3 = s_2\partial_{s_1} - s_1\partial_{s_2}, \dots$ )

$$\begin{aligned} L_1 &= J_1^2 + \frac{a_3s_2^2}{s_3^2} + \frac{a_2s_3^2}{s_2^2}, \\ L_2 &= J_2^2 + \frac{a_1s_3^2}{s_1^2} + \frac{a_3s_1^2}{s_3^2}, \\ L_3 &= J_3^2 + \frac{a_2s_1^2}{s_2^2} + \frac{a_1s_2^2}{s_1^2}. \end{aligned}$$

**Theorem 1.** *There is a bijection between quadratic algebras generated by 2nd order elements in the enveloping algebra of  $o(3, \mathbb{C})$ , called free, and 2nd order nondegenerate superintegrable systems on the complex 2-sphere. Similarly, there is a bijection between quadratic algebras generated by 2nd order elements in the enveloping algebra of  $e(2, \mathbb{C})$  and 2nd order nondegenerate superintegrable systems on the 2D complex flat space.*

*Remark.* This theorem is constructive [15]. Given a free quadratic algebra  $\tilde{Q}$  one can compute the potential  $V$  and the symmetries of the quadratic algebra  $Q$  of the nondegenerate superintegrable system.

Special functions arise from these systems in two distinct ways: 1) As separable eigenfunctions of the quantum Hamiltonian. Second order superintegrable systems are multiseparable [6]. 2) As interbasis expansion coefficients relating distinct separable coordinate eigenbases [16, 17, 19, 20]. Most of the classical special functions in the Digital Library of Mathematical Functions, as well as Wilson polynomials, appear in these ways [21].

### 1.1. THE BIG PICTURE: CONTRACTIONS AND SPECIAL FUNCTIONS

- Taking coordinate limits starting from quantum system  $S_9$  we can obtain other superintegrable systems.
- These coordinate limits induce limit relations between the special functions associated as eigenfunctions of the superintegrable systems.
- The limits induce contractions of the associated quadratic algebras, and via the models, limit relations between the associated special functions.
- For constant curvature systems the required limits are all induced by Inönü-Wigner-type Lie algebra contractions of  $o(3, \mathbb{C})$  and  $e(2, \mathbb{C})$  [22–24]
- The Askey scheme for orthogonal functions of hypergeometric type fits nicely into this picture [25].

*Lie algebra contractions.* Let  $(A; [; ]_A), (B; [; ]_B)$  be two complex Lie algebras. We say that  $B$  is a contraction of  $A$  if for every  $\epsilon \in (0, 1]$  there exists a linear invertible map  $t_\epsilon : B \rightarrow A$  such that for every  $X, Y \in B, \lim_{\epsilon \rightarrow 0} t_\epsilon^{-1}[t_\epsilon X, t_\epsilon Y]_A = [X, Y]_B$ . Thus, as  $\epsilon \rightarrow 0$  the 1-parameter family of basis transformations can become nonsingular but the structure constants go to a finite limit.

*Contractions of  $e(2, \mathbb{C})$  and  $o(3, \mathbb{C})$ .* These are the symmetry Lie algebras of free (zero potential) systems on constant curvature spaces. Their contractions have long since been classified [15]. There are 6 nontrivial contractions of  $e(2, \mathbb{C})$  and 4 of  $o(3, \mathbb{C})$ . They are each induced by coordinate limits.

*Example — an Inönü-Wigner-contraction of  $o(3, \mathbb{C})$ .* We use the classical realization for  $o(3, \mathbb{C})$  acting on the 2-sphere, with basis  $J_1 = s_2\partial_{s_3} - s_3\partial_{s_2}, J_2 = s_3\partial_{s_1} - s_1\partial_{s_3}, J_3 = s_1\partial_{s_2} - s_2\partial_{s_1}$ , commutation relations  $[J_2, J_1] = J_3, [J_3, J_2] = J_1, [J_1, J_3] = J_2$ , and Hamiltonian  $H = J_1^2 + J_2^2 + J_3^2$ . Here  $s_1^2 + s_2^2 + s_3^2 = 1$ . We introduce the basis change:

$$\{J'_1, J'_2, J'_3\} = \{\epsilon J_1, \epsilon J_2, J_3\}, \quad 0 < \epsilon \leq 1,$$

with coordinate implementation  $x = \frac{s_1}{\epsilon}, y = \frac{s_2}{\epsilon}, s_3 \approx 1$ . The structure relations become

$$[J'_2, J'_1] = \epsilon^2 J'_3, \quad [J'_3, J'_2] = J'_1, \quad [J'_1, J'_3] = J'_2,$$

As  $\epsilon \rightarrow 0$  these converge to

$$[J'_2, J'_1] = 0, \quad [J'_3, J'_2] = J'_1, \quad [J'_1, J'_3] = J'_2,$$

the Lie algebra  $e(2, \mathbb{C})$ .

*Contractions of quadratic algebras.* Just as for Lie algebras we can define a contraction of a quadratic algebra in terms of 1-parameter families of basis changes in the algebra: As  $\epsilon \rightarrow 0$  the 1-parameter family of basis transformations becomes singular but the structure constants go to a finite limit [15].

*Motivating idea — Lie algebra contractions induce quadratic algebra contractions.* For constant curvature spaces we have the following theorem.

**Theorem 2** [15]. *Every Lie algebra contraction of  $A = e(2, \mathbb{C})$  or  $A = o(3, \mathbb{C})$  induces a contraction of a free (zero potential) quadratic algebra  $\tilde{Q}$  based on  $A$ , which in turn induces a contraction of the quadratic algebra  $Q$  with potential. This is true for both classical and quantum algebras.*

### 1.2. THE PROBLEMS AND THE PROPOSED SOLUTIONS

The various limits of 2nd order superintegrable systems on constant curvature spaces and their applications, such as to the Askey-Wilson scheme, can be classified and understood in terms of generalized Inönü-Wigner contractions [15]. However, there are complications for spaces not of constant curvature. For Darboux spaces the Lie symmetry algebra is only 1-dimensional so limits must be determined on a case-by-case basis. There is no Lie symmetry algebra at all for Koenigs spaces. Furthermore, there is the issue of finding a more systematic way of classifying the 44 distinct Helmholtz superintegrable eigenvalue systems on different manifolds, and their relations. These issues can be clarified by considering the Helmholtz systems as Laplace equations (with potential) on flat space. This point of view was introduced in the paper [26] and applied in [27] to solve important classification problems in the case  $n = 3$ . It is the aim of this paper to describe the Laplace equation mechanism and how it can be applied to systematize the classification of Helmholtz superintegrable systems and their relations via limits. The basic idea is that families of (Stäckel-equivalent) Helmholtz superintegrable systems on a variety of manifolds correspond to a single conformally superintegrable Laplace equation on flat space. We exploit this relation in the case  $n = 2$ , but it generalizes easily to all dimensions  $n \geq 2$ . The conformal symmetry algebra for Laplace equations with constant potential on flat space is the conformal algebra  $so(n + 2, \mathbb{C})$ . In his 1894 thesis [28] Bôcher introduced a limit procedure based on the roots of quadratic forms to find families of R-separable solutions of the ordinary (zero potential) flat space Laplace equation  $\Delta_n \Psi = 0$  in  $n$  dimensions. (That is, he constructed separable solutions of the

form  $\Psi = \mathcal{R}(\mathbf{u}) \prod_{j=1}^n \psi_j(u_j)$  where  $\mathcal{R}$  is a fixed gauge function and  $\psi_j$  depends only on the variable  $u_j$  and the separation constants.) We show that his limit procedure can be interpreted as constructing generalized Inönü-Wigner Lie algebra contractions of  $so(4, \mathbb{C})$  to itself. We call these Bôcher contractions and show that all of the limits of the Helmholtz systems classified before for  $n = 2$  [15] are induced by the larger class of Bôcher contractions. Here we present the main constructions and findings. Detailed proofs and the lengthy classifications are in [32].

## 2. THE LAPLACE EQUATION

Systems of Laplace type are of the form  $H\Psi \equiv \Delta_n \Psi + V\Psi = 0$ . Here  $\Delta_n$  is the Laplace-Beltrami operator on a conformally flat  $nD$  Riemannian or pseudo-Riemannian manifold. A conformal symmetry of this equation is a partial differential operator  $S$  in the variables  $\mathbf{x} = (x_1, \dots, x_n)$  such that  $[S, H] \equiv SH - HS = R_S H$  for some differential operator  $R_S$ . The system is maximally conformally superintegrable (or Laplace superintegrable) for  $n \geq 2$  if there are  $2n - 1$  functionally independent conformal symmetries,  $S_1, \dots, S_{2n-1}$  with  $S_1 = H$  [26]. It is second order conformally superintegrable if each symmetry  $S_i$  can be chosen to be a differential operator of at most second order. Every  $2D$  Riemannian manifold is conformally flat, so we can always find a Cartesian-like coordinate system with coordinates  $(x, y) \equiv (x_1, x_2)$  such that the Helmholtz eigenvalue equation takes the form

$$\tilde{H}\Psi = \left( \frac{1}{\lambda(x, y)} (\partial_x^2 + \partial_y^2) + \tilde{V}(\mathbf{x}) \right) \Psi = E\Psi. \quad (1)$$

However, this equation is equivalent to the flat space Laplace equation

$$H\Psi \equiv (\partial_x^2 + \partial_y^2 + V(\mathbf{x}))\Psi = 0, \quad V(\mathbf{x}) = \lambda(\mathbf{x})(\tilde{V}(\mathbf{x}) - E). \quad (2)$$

In particular, the symmetries of (1) correspond to the conformal symmetries of (2). Indeed, if  $[S, \tilde{H}] = 0$  then

$$[S, H] = [S, \lambda(\tilde{H} - E)] = [S, \lambda](\tilde{H} - E) = [S, \lambda]\lambda^{-1}H.$$

Conversely, if  $S$  is an  $E$ -independent conformal symmetry of  $H$  we find that  $[S, \tilde{H}] = 0$ . Further, the conformal symmetries of the system  $(\tilde{H} - E)\Psi = 0$  are identical with the conformal symmetries of (2). Thus without loss of generality we can assume the manifold is flat space with  $\lambda \equiv 1$ .

*The conformal Stäckel transform.* Suppose we have a second order conformal superintegrable system

$$H = \partial_{xx} + \partial_{yy} + V(x, y) = 0, \quad H = H_0 + V,$$

where  $V(x, y) = W(x, y) - E$   $U(x, y)$  for arbitrary parameter  $E$ .

**Theorem 3** [26]. *The transformed (Helmholtz) system  $\tilde{H}\Psi = E\Psi$ ,  $\tilde{H} = \frac{1}{U}(\partial_{xx} + \partial_{yy}) + \tilde{V}$  is superintegrable (not just conformally superintegrable), where  $\tilde{V} = \frac{W}{U}$ .*

There is a similar definition of ordinary Stäckel transforms of Helmholtz superintegrable systems  $H\Psi = E\Psi$  which takes superintegrable systems to superintegrable systems, essentially preserving the quadratic algebra structure [29].

Thus any second order conformal Laplace superintegrable system admitting a nonconstant potential  $U$  can be Stäckel transformed to a Helmholtz superintegrable system. By choosing all possible special potentials  $U$  associated with the fixed Laplace system we generate the equivalence class of all Helmholtz superintegrable systems obtainable through this process.

**Theorem 4.** *There is a one-to-one relationship between flat space conformally superintegrable Laplace systems with nondegenerate potential and Stäckel equivalence classes of superintegrable Helmholtz systems with nondegenerate potential.*

Indeed, for a Stäckel transform induced by the function  $U^{(1)}$ , we can take the original Helmholtz system to have Hamiltonian

$$H = H_0 + V = H_0 + U^{(1)}\alpha_1 + U^{(2)}\alpha_2 + U^{(3)}\alpha_3 + \alpha_4, \quad (3)$$

where  $\{U^{(1)}, U^{(2)}, U^{(3)}, 1\}$  is a basis for the 4-dimensional potential space. A 2nd order symmetry  $S$  would have the form

$$S = S_0 + W^{(1)}\alpha_1 + W^{(2)}\alpha_2 + W^{(3)}\alpha_3,$$

where  $S_0$  is a symmetry of the potential free Hamiltonian,  $H_0$ . The Stäckel transformed symmetry and Hamiltonian take the form  $\tilde{S} = S - W^{(1)}\tilde{H}$  and

$$\tilde{H} = \frac{1}{U^{(1)}}H_0 + \frac{U^{(1)}\alpha_1 + U^{(2)}\alpha_2 + U^{(3)}\alpha_3 + \alpha_4}{U^{(1)}}.$$

Note that the parameter  $\alpha_1$  cancels out of the expression for  $\tilde{S}$ ; it is replaced by a term  $-\alpha_4 W^{(1)}/U^{(1)}$ . Now suppose that  $\Psi$  is a formal eigenfunction of  $H$  (not required to be normalizable):  $H\Psi = E\Psi$ . Without loss of generality we can absorb the energy eigenvalue into  $\alpha_4$  so that  $\alpha_4 = -E$  in (3) and, in terms of this redefined  $H$ , we have  $H\Psi = 0$ . It follows immediately that  $\tilde{S}\Psi = S\Psi$ . Thus, for the 3-parameter system  $H'$  and the Stäckel transform  $\tilde{H}'$ ,

$$H' = H_0 + V' = H_0 + U^{(1)}\alpha_1 + U^{(2)}\alpha_2 + U^{(3)}\alpha_3, \\ \tilde{H}' = \frac{1}{U^{(1)}}H_0 + \frac{-U^{(1)}E + U^{(2)}\alpha_2 + U^{(3)}\alpha_3}{U^{(1)}},$$

we have  $H'\Psi = E\Psi$  and  $\tilde{H}'\Psi = -\alpha_1\Psi$ . The effect of the Stäckel transform is to replace  $\alpha_1$  by  $-E$  and  $E$  by  $-\alpha_1$ . Further,  $S$  and  $\tilde{S}$  must agree on eigenspaces of  $H'$ .

We know that the symmetry operators of all 2nd order nondegenerate superintegrable systems in 2D generate a quadratic algebra of the form

$$[R, S_j] = f^{(j)}(S_1, S_2, \alpha_1, \alpha_2, \alpha_3, H'), \quad j = 1, 2, \\ R^2 = f^{(3)}(S_1, S_2, \alpha_1, \alpha_2, \alpha_3, H'), \quad (4)$$

where  $\{S_1, S_2, H\}$  is a basis for the 2nd order symmetries and  $\alpha_1, \alpha_2, \alpha_3$  are the parameters for the potential [6]. It follows from the above considerations that the effect of a Stäckel transform generated by the potential function  $U^{(1)}$  is to determine a new superintegrable system with structure

$$[\tilde{R}, \tilde{S}_j] = f^{(j)}(\tilde{S}_1, \tilde{S}_2, -\tilde{H}', \alpha_2, \alpha_3, -\alpha_1), \quad j = 1, 2, \\ R^2 = f^{(3)}(\tilde{S}_1, \tilde{S}_2, -\tilde{H}', \alpha_2, \alpha_3, -\alpha_1), \quad (5)$$

Of course, the switch of  $\alpha_1$  and  $H'$  is only for illustration; there is a Stäckel transform that replaces any  $\alpha_j$  by  $-H'$  and  $H'$  by  $-\alpha_j$  and similar transforms that apply to any basis that we choose for the potential space.

Formulas (4) and (5) are just instances of the quadratic algebras of the superintegrable systems belonging to the equivalence class of a single nondegenerate conformally superintegrable Hamiltonian

$$\hat{H} = \partial_{xx} + \partial_{yy} + \sum_{j=1}^4 \alpha_j V^{(j)}(x, y). \quad (6)$$

Let  $\hat{S}_1, \hat{S}_2, \hat{H}$  be a basis of 2nd order conformal symmetries of  $\hat{H}$ . From the above discussion we can conclude the following.

**Theorem 5.** *The symmetries of the 2D nondegenerate conformal superintegrable Hamiltonian  $\hat{H}$  generate a quadratic algebra*

$$[\hat{R}, \hat{S}_1] = f^{(1)}(\hat{S}_1, \hat{S}_2, \alpha_1, \alpha_2, \alpha_3, \alpha_4), \\ [\hat{R}, \hat{S}_2] = f^{(2)}(\hat{S}_1, \hat{S}_2, \alpha_1, \alpha_2, \alpha_3, \alpha_4), \\ \hat{R}^2 = f^{(3)}(\hat{S}_1, \hat{S}_2, \alpha_1, \alpha_2, \alpha_3, \alpha_4), \quad (7)$$

where  $\hat{R} = [\hat{S}_1, \hat{S}_2]$  and all identities hold mod( $\hat{H}$ ). A conformal Stäckel transform generated by the potential  $V^{(j)}(x, y)$  yields a nondegenerate Helmholtz superintegrable Hamiltonian  $\tilde{H}$  with quadratic algebra relations identical to (7), except that we make the replacements  $\hat{S}_\ell \rightarrow \tilde{S}_\ell$  for  $\ell = 1, 2$  and  $\alpha_j \rightarrow -\tilde{H}$ . These modified relations (6) are now true identities, not mod( $\hat{H}$ ).

Every 2nd order conformal symmetry is of the form  $S = S_0 + W$  where  $S_0$  is a 2nd order element of the enveloping algebra of  $so(4, \mathbb{C})$ . The dimension of this space of 2nd order elements is 21 but there is an 11-dimensional subspace of symmetries congruent to 0 mod( $H_0$ ) where  $H_0 = P_1^2 + P_2^2$ . Thus mod( $H_0$ ) the space of 2nd order symmetries is 10-dimensional.

### 3. THE BÔCHER METHOD

In his 1894 thesis Bôcher [28], developed a geometrical method for finding and classifying the R-separable orthogonal coordinate systems for the flat space Laplace equation  $\Delta_n \Psi = 0$  in  $n$  dimensions. It was based on the conformal symmetry of these equations. The conformal Lie symmetry algebra of the flat space complex Laplacian is  $so(n + 2, \mathbb{C})$ . We will use his ideas for  $n = 2$ , but applied to the Laplace equation with potential  $H\Psi \equiv (\partial_x^2 + \partial_y^2 + V)\Psi = 0$ . The  $so(4, \mathbb{C})$  conformal symmetry algebra in the case  $n = 2$  has the basis  $P_1 = \partial_x$ ,  $P_2 = \partial_y$ ,  $J = x\partial_y - y\partial_x$ ,  $D = x\partial_x + y\partial_y$ ,  $K_1 = (x^2 - y^2)\partial_x + 2xy\partial_y$ ,  $K_2 = (y^2 - x^2)\partial_y + 2xy\partial_x$ . Bôcher linearizes this action by introducing tetraspherical coordinates. These are 4 projective complex coordinates  $(x_1, x_2, x_3, x_4)$  confined to the null cone  $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$ . They are related to complex Cartesian coordinates  $(x, y)$  via

$$x = -\frac{x_1}{x_3 + ix_4}, \quad y = -\frac{x_2}{x_3 + ix_4},$$

$$H = \partial_{xx} + \partial_{yy} + \tilde{V} = (x_3 + ix_4)^2 \left( \sum_{k=1}^4 \partial_{x_k}^2 + V \right),$$

where  $\tilde{V} = (x_3 + ix_4)^2 V$ . We define  $L_{jk} = x_j \partial_{x_k} - x_k \partial_{x_j}$ ,  $1 \leq j, k \leq 4$ ,  $j \neq k$ , where  $L_{jk} = -L_{kj}$ . These operators are clearly a basis for  $so(4, \mathbb{C})$ . The generators for flat space conformal symmetries are related to these via

$$\begin{aligned} P_1 = \partial_x = L_{13} + iL_{14}, & \quad P_2 = \partial_y = L_{23} + iL_{24}, \\ D = iL_{34}, & \quad J = L_{12}, \\ K_1 = L_{13} - iL_{14}, & \quad K_2 = L_{23} - iL_{24}. \end{aligned}$$

#### 3.1. RELATION TO SEPARATION OF VARIABLES

Bôcher uses symbols of the form  $[n_1, n_2, \dots, n_p]$  where  $n_1 + \dots + n_p = 4$ , to define coordinate surfaces as follows. Consider the quadratic forms

$$\begin{aligned} \Omega &= x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0, \\ \Phi &= \frac{x_1^2}{\lambda - e_1} + \frac{x_2^2}{\lambda - e_2} + \frac{x_3^2}{\lambda - e_3} + \frac{x_4^2}{\lambda - e_4}. \end{aligned}$$

If the parameters  $e_j$  are pairwise distinct, the elementary divisors of these two forms are denoted by  $[1, 1, 1, 1]$ . Given a point in 2D flat space with Cartesian coordinates  $(x^0, y^0)$ , there corresponds a set of tetraspherical coordinates  $(x_1^0, x_2^0, x_3^0, x_4^0)$ , unique up to multiplication by a nonzero constant. If we substitute into  $\Phi$  we see that there are exactly 2 roots  $\lambda = \rho, \mu$  such that  $\Phi = 0$ . (If  $e_4 \rightarrow \infty$  these correspond to elliptic coordinates on the 2-sphere.) They are orthogonal with respect to the metric  $ds^2 = dx^2 + dy^2$  and are R-separable for the Laplace equations  $(\partial_x^2 + \partial_y^2)\Theta = 0$  or  $(\sum_{j=1}^4 \partial_{x_j}^2)\Theta = 0$ .

*Example.* Consider the potential  $V_{[1,1,1,1]} = \frac{a_1}{x_1^2} + \frac{a_2}{x_2^2} + \frac{a_3}{x_3^2} + \frac{a_4}{x_4^2}$ . It is the only potential  $V$  such that equation  $(\sum_{j=1}^4 \partial_{x_j}^2 + V)\Theta = 0$  is R-separable in elliptic coordinates for all choices of the parameters  $e_j$ .

The separation is characterized by 2nd order conformal symmetry operators that are linear in the parameters  $e_j$ . In particular the symmetries span a 3-dimensional subspace of symmetries, so the system  $H\Theta = (\sum_{j=1}^4 \partial_{x_j}^2 + V_{[1,1,1,1]})\Theta = 0$  must be conformally superintegrable.

#### 3.2. BÔCHER LIMITS

Suppose some of the  $e_i$  become equal. To obtain separable coordinates we cannot just set them equal in  $\Omega, \Phi$  but must take limits, Bôcher develops a calculus to describe this. Thus the process of making  $e_1 \rightarrow e_2$  is described by the mapping, which in the limit as  $\epsilon \rightarrow 0$  takes the null cone to the null cone:

$$\begin{aligned} e_1 = e_2 + \epsilon^2, \quad x_1 &\rightarrow \frac{i(x'_1 + ix'_2)}{\sqrt{2}\epsilon}, \\ x_2 &\rightarrow \frac{(x'_1 + ix'_2)}{\sqrt{2}\epsilon} + \epsilon \frac{(x'_1 - ix'_2)}{\sqrt{2}}, \\ x_3 &\rightarrow x'_3, \quad x_4 \rightarrow x'_4, \end{aligned}$$

In the limit we have

$$\begin{aligned} \Omega &= x_1'^2 + x_2'^2 + x_3'^2 + x_4'^2 = 0, \\ \Phi &= \frac{(x'_1 + ix'_2)^2}{2(\lambda - e_2)^2} + \frac{x_1'^2 + x_2'^2}{\lambda - e_2} + \frac{x_3'^2}{\lambda - e_3} + \frac{x_4'^2}{\lambda - e_4}, \end{aligned}$$

which has elementary divisors  $[2, 1, 1]$ , see [30, 31]. In the same way as for  $[1, 1, 1, 1]$ , these forms define a new set of orthogonal coordinates R-separable for the Laplace equations. We can show that the coordinate limit induces a contraction of  $so(4, \mathbb{C})$  to itself:

$$\begin{aligned} L'_{12} &= L_{12}, \quad L'_{13} = -\frac{i}{\sqrt{2}\epsilon}(L_{13} - iL_{23}) - \frac{i\epsilon}{\sqrt{2}}L_{13}, \\ L'_{23} &= -\frac{i}{\sqrt{2}\epsilon}(L_{13} - iL_{23}) - \frac{\epsilon}{\sqrt{2}}L_{13}, \quad L'_{34} = L_{34}, \\ L'_{14} &= -\frac{i}{\sqrt{2}\epsilon}(L_{14} - iL_{24}) - \frac{i\epsilon}{\sqrt{2}}L_{14}, \\ L'_{24} &= -\frac{i}{\sqrt{2}\epsilon}(L_{14} - iL_{24}) - \frac{\epsilon}{\sqrt{2}}L_{14}. \end{aligned}$$

We call this the Bôcher contraction  $[1, 1, 1, 1] \rightarrow [2, 1, 1]$ . There are analogous Bôcher contractions of  $so(4, \mathbb{C})$  to itself corresponding to limits from  $[1, 1, 1, 1]$  to  $[2, 2]$ ,  $[3, 1]$ ,  $[4]$ . Similarly, there are Bôcher contractions  $[2, 1, 1] \rightarrow [2, 2]$ , etc.

If we apply the contraction  $[1, 1, 1, 1] \rightarrow [2, 1, 1]$  to the potential  $V_{[1,1,1,1]}$  we get a finite limit

$$V_{[2,1,1]} = \frac{b_1}{(x'_1 + ix'_2)^2} + \frac{b_2(x'_1 - ix'_2)}{(x'_1 + ix'_2)^3} + \frac{b_3}{x_3'^2} + \frac{b_4}{x_4'^2}, \quad (8)$$

provided the parameters transform as

$$\begin{aligned} a_1 &= -\frac{1}{2} \left( \frac{b_1}{\epsilon^2} + \frac{b_2}{2\epsilon^4} \right), & a_2 &= -\frac{b_2}{4\epsilon^4}, \\ a_3 &= b_3, & a_4 &= b_4. \end{aligned}$$

Note: We know from theory that the 4-dimensional vector space of potentials  $V_{[1,1,1,1]}$  maps to the 4-dimensional vector space of potentials  $V_{[2,1,1]}$  1-1 under the contraction [15]. The reason for the  $\epsilon$ -dependence of the parameters is the arbitrariness of choosing a basis. If we had chosen a basis for  $V_{[1,1,1,1]}$  specially adapted to this contraction, we could have achieved  $a_j = b_j$ ,  $1 \leq j \leq 4$ .

Bôcher contractions obey a composition law:

**Theorem 6.** *Let*

$$\begin{aligned} A: (\Delta_{\mathbf{x}} + V_A(\mathbf{x}))\Psi = 0, \quad B: (\Delta_{\mathbf{y}} + V_B(\mathbf{y}))\Psi = 0 \\ C: (\Delta_{\mathbf{z}} + V_C(\mathbf{z}))\Psi = 0, \end{aligned}$$

be Bôcher superintegrable systems such that A Bôcher-contracts to B and B Bôcher-contracts to C. Then there is a one-parameter contraction of A to C.

A fundamental advantage in recognizing Bôcher's limit procedure as contractions is that whereas the Bôcher limits had a fixed starting and ending point, say  $[1, 1, 1, 1] \rightarrow [2, 1, 1]$ , contractions can be applied to any nondegenerate conformally superintegrable system and are guaranteed to result in another nondegenerate conformally superintegrable system. This greatly increases the range of applicability of the limits.

**4. THE 8 CLASSES OF NONDEGENERATE CONFORMALLY SUPERINTEGRABLE SYSTEMS**

The possible Laplace equations (in tetraspherical coordinates) are  $(\sum_{j=1}^4 \partial_{x_j}^2 + V)\Psi = 0$  with potentials:

$$\begin{aligned} V_{[1,1,1,1]} &= \sum_{j=1}^4 \frac{a_j}{x_j^2}, \tag{9} \\ V_{[2,1,1]} &= \frac{a_1}{x_1^2} + \frac{a_2}{x_2^2} + \frac{a_3(x_3 - ix_4)}{(x_3 + ix_4)^3} + \frac{a_4}{(x_3 + ix_4)^2}, \\ V_{[2,2]} &= \frac{a_1}{(x_1 + ix_2)^2} + \frac{a_2(x_1 - ix_2)}{(x_1 + ix_2)^3} \\ &\quad + \frac{a_3}{(x_3 + ix_4)^2} + \frac{a_4(x_3 - ix_4)}{(x_3 + ix_4)^3}, \\ V_{[3,1]} &= \frac{a_1}{(x_3 + ix_4)^2} + \frac{a_2x_1}{(x_3 + ix_4)^3} \\ &\quad + \frac{a_3(4x_1^2 + x_2^2)}{(x_3 + ix_4)^4} + \frac{a_4}{x_2^2}, \\ V_{[4]} &= \frac{a_1}{(x_3 + ix_4)^2} + a_2 \frac{x_1 + ix_2}{(x_3 + ix_4)^3} \\ &\quad + a_3 \frac{3(x_1 + ix_2)^2 - 2(x_3 + ix_4)(x_1 - ix_2)}{(x_3 + ix_4)^4}, \\ V_{[0]} &= \frac{a_1}{(x_3 + ix_4)^2} + \frac{a_2x_1 + a_3x_2}{(x_3 + ix_4)^3} \\ &\quad + a_4 \frac{x_1^2 + x_2^2}{(x_3 + ix_4)^4}, \\ V(1) &= a_1 \frac{1}{(x_1 + ix_2)^2} + a_2 \frac{1}{(x_3 + ix_4)^2} \end{aligned}$$

$$\begin{aligned} &+ a_3 \frac{(x_3 + ix_4)}{(x_1 + ix_2)^3} + a_4 \frac{(x_3 + ix_4)^2}{(x_1 + ix_2)^4}, \\ V(2) &= a_1 \frac{1}{(x_3 + ix_4)^2} + a_2 \frac{(x_1 + ix_2)}{(x_3 + ix_4)^3} \\ &+ a_3 \frac{(x_1 + ix_2)^2}{(x_3 + ix_4)^4} + a_4 \frac{(x_1 + ix_2)^3}{(x_3 + ix_4)^5}. \end{aligned}$$

(The last 3 systems do not correspond to elementary divisors; they appear as Bôcher contractions of systems that do correspond to elementary divisors.) Each of the 44 Helmholtz nondegenerate superintegrable (i.e., 3-parameter) eigenvalue systems is Stäckel equivalent to exactly one of these systems. Thus, with one caveat, there are exactly 8 equivalence classes of Helmholtz systems. The caveat is the singular family of systems with potentials  $V_S = (x_3 + ix_4)^{-2} h(\frac{x_1 + ix_2}{x_3 + ix_4})$  where  $h$  is an arbitrary analytic function except that  $V_S \neq V(1), V(2)$ . This family is unrelated to the other systems.

Expressed as flat space Laplace equations  $(\partial_x^2 + \partial_y^2 + \tilde{V})\Psi = 0$  in Cartesian coordinates, the potentials are

$$\begin{aligned} \tilde{V}_{[1,1,1,1]} &= \frac{a_1}{x^2} + \frac{a_2}{y^2} + \frac{4a_3}{(x^2 + y^2 - 1)^2} \\ &\quad - \frac{4a_4}{(x^2 + y^2 + 1)^2}, \\ \tilde{V}_{[2,1,1]} &= \frac{a_1}{x^2} + \frac{a_2}{y^2} - a_3(x^2 + y^2) + a_4, \\ \tilde{V}_{[2,2]} &= \frac{a_1}{(x + iy)^2} + \frac{a_2(x - iy)}{(x + iy)^3} \\ &\quad + a_3 - a_4(x^2 + y^2), \\ \tilde{V}_{[3,1]} &= a_1 - a_2x + a_3(4x^2 + y^2) + \frac{a_4}{y^2}, \\ \tilde{V}_{[4]} &= a_1 - a_2(x + iy) \\ &\quad + a_3(3(x + iy)^2 + 2(x - iy)) \\ &\quad - a_4(4(x^2 + y^2) + 2(x + iy)^3), \\ \tilde{V}_{[0]} &= a_1 - (a_2x + a_3y) + a_4(x^2 + y^2), \\ \tilde{V}(1) &= \frac{a_1}{(x + iy)^2} + a_2 - \frac{a_3}{(x + iy)^3} + \frac{a_4}{(x + iy)^4}, \\ \tilde{V}(2) &= a_1 + a_2(x + iy) + a_3(x + iy)^2 \\ &\quad + a_4(x + iy)^3. \tag{10} \end{aligned}$$

**4.1. SUMMARY OF BÔCHER CONTRACTIONS OF LAPLACE SUPERINTEGRABLE SYSTEMS**

Table 1 contains a partial list of contractions. The full list is presented in [32]. We have omitted some contractions, such as  $[3, 1] \rightarrow [4]$ , because they are consequences of other contractions in the table.

**5. HELMHOLTZ CONTRACTIONS FROM BÔCHER CONTRACTIONS**

We describe how Bôcher contractions of conformal superintegrable systems induce contractions of Helmholtz superintegrable systems.

We consider the conformal Stäckel transforms of the conformal system  $[1, 1, 1, 1]$  with potential  $V_{[1,1,1,1]}$ .

$[1, 1, 1, 1] \rightarrow [2, 1, 1]$	$V_{[1,1,1,1]} \downarrow V_{[2,1,1]}$ $V_{[4]} \downarrow V_{[0]}$	$V_{[2,1,1]} \downarrow V_{[2,1,1]}$ $V_{[0]} \downarrow V_{[0]}$	$V_{[2,2]} \downarrow V_{[2,2]}$ $V(1) \downarrow V(1)$	$V_{[3,1]} \downarrow V_{(1)}$ $V(2) \downarrow V(2)$
$[1, 1, 1, 1] \rightarrow [2, 2]$	$V_{[1,1,1,1]} \downarrow V_{[2,2]}$ $V_{[4]} \downarrow V(2)$	$V_{[2,1,1]} \downarrow V_{[2,2]}$ $V_{[0]} \downarrow V_{[0]}$	$V_{[2,2]} \downarrow V_{[2,2]}$ $V(1) \downarrow V(1)$	$V_{[3,1]} \downarrow V(1)$ $V(2) \downarrow V(2)$
$[2, 1, 1] \rightarrow [3, 1]$	$V_{[1,1,1,1]} \downarrow V_{[3,1]}$ $V_{[4]} \downarrow V_{[0]}$	$V_{[2,1,1]} \downarrow V_{[3,1]}$ $V_{[0]} \downarrow V_{[0]}$	$V_{[2,2]} \downarrow V_{[0]}$ $V(1) \downarrow V(2)$	$V_{[3,1]} \downarrow V_{[3,1]}$ $V(2) \downarrow V(2)$
$[1, 1, 1, 1] \rightarrow [4]$	$V_{[1,1,1,1]} \downarrow V_{[4]}$ $V_{[4]} \downarrow V_{[0]}$	$V_{[2,1,1]} \downarrow V_{[4]}$ $V_{[0]} \downarrow V_{[0]}$	$V_{[2,2]} \downarrow V_{[0]}$ $V(1) \downarrow V(2)$	$V_{[3,1]} \downarrow V_{[4]}$ $V(2) \downarrow V(2)$
$[2, 2] \rightarrow [4]$	$V_{[1,1,1,1]} \downarrow V_{[4]}$ $V_{[4]} \downarrow V(2)$	$V_{[2,1,1]} \downarrow V_{[4]}$ $V_{[0]} \downarrow V_{[0]}$	$V_{[2,2]} \downarrow V_{[4]}$ $V(1) \downarrow V(2)$	$V_{[3,1]} \downarrow V(2)$ $V(2) \downarrow V(2)$
$[1, 1, 1, 1] \rightarrow [3, 1]$	$V_{[1,1,1,1]} \downarrow V_{[3,1]}$ $V_{[4]} \downarrow V_{[0]}$	$V_{[2,1,1]} \downarrow V_{[3,1]}$ $V_{[0]} \downarrow V_{[0]}$	$V_{[2,2]} \downarrow V_{[0]}$ $V(1) \downarrow V(2)$	$V_{[3,1]} \downarrow V_{[3,1]}$ $V(2) \downarrow V(2)$

TABLE 1. Bôcher contractions of Laplace superintegrable systems.

As we show explicitly in [32], the various possibilities are  $S_9$  above and 2 more Helmholtz systems on the sphere,  $S_7$  and  $S_8$ , 2 Darboux systems D4B and D4C, and a family of Koenigs systems.

**Example 1.** Using Cartesian coordinates  $x, y$ , we consider the  $[1, 1, 1, 1]$  Hamiltonian

$$H = \partial_x^2 + \partial_y^2 + \frac{a_1}{x^2} + \frac{a_2}{y^2} + \frac{4a_3}{(x^2 + y^2 - 1)^2} + \frac{4a_4}{(x^2 + y^2 + 1)^2}.$$

Dividing on the left by  $1/x^2$  we obtain

$$\hat{H} = x^2(\partial_x^2 + \partial_y^2) + a_1 + a_2 \frac{x^2}{y^2} + 4a_3 \frac{x^2}{(x^2 + y^2 - 1)^2} - 4a_4 \frac{x^2}{(x^2 + y^2 + 1)^2},$$

the Stäckel transform corresponding to the case  $(a_1, a_2, a_3, a_4) = (1, 0, 0, 0)$ . This becomes more transparent if we introduce variables  $x = e^{-a}, y = r$ . The Hamiltonian  $\hat{H}$  can be written

$$\hat{H} = \partial_a^2 + e^{-2a} \partial_r^2 + a_1 + a_2 \frac{e^{-2a}}{r^2} + a_3 \frac{4}{(e^{-a} + e^a(r^2 - 1))^2} - a_4 \frac{4}{(e^{-a} + e^a(r^2 + 1))^2}.$$

Recalling horospherical coordinates on the complex two sphere, viz.

$$s_1 = \frac{i}{2}(e^{-a} + (r^2 + 1)e^a), \quad s_2 = re^a, \\ s_3 = \frac{1}{2}(e^{-a} + (r^2 - 1)e^a)$$

we see that the Hamiltonian  $\hat{H}$  can be written as

$$\hat{H} = \partial_{s_1}^2 + \partial_{s_2}^2 + \partial_{s_3}^2 + a_1 + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2} + \frac{a_4}{s_1^2},$$

and this is explicitly the superintegrable system  $S_9$ .

More generally, let  $H$  be the initial Hamiltonian. In terms of tetraspherical coordinates a general conformal Stäckel transformed potential will take the form

$$V = \frac{\frac{a_1}{x_1^2} + \frac{a_2}{x_2^2} + \frac{a_3}{x_3^2} + \frac{a_4}{x_4^2}}{\frac{A_1}{x_1^2} + \frac{A_2}{x_2^2} + \frac{A_3}{x_3^2} + \frac{A_4}{x_4^2}} = \frac{V_{[1,1,1,1]}}{F(\mathbf{x}, \mathbf{A})},$$

where

$$F(\mathbf{x}, \mathbf{A}) = \frac{A_1}{x_1^2} + \frac{A_2}{x_2^2} + \frac{A_3}{x_3^2} + \frac{A_4}{x_4^2},$$

and the transformed Hamiltonian will be

$$\hat{H} = \frac{1}{F(\mathbf{x}, \mathbf{A})} H,$$

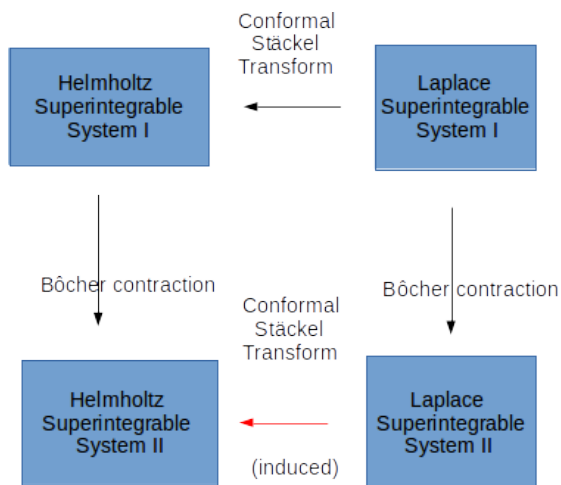
where the transform is determined by the fixed vector  $(A_1, A_2, A_3, A_4)$ . Now we apply the Bôcher contraction  $[1, 1, 1, 1] \rightarrow [2, 1, 1]$  to this system. In the limit as  $\epsilon \rightarrow 0$  the potential  $V_{[1,1,1,1]} \rightarrow V_{[2,1,1]}$ , (8), and  $H \rightarrow H'$  the  $[2, 1, 1]$  system. Now consider

$$F(\mathbf{x}(\epsilon), \mathbf{A}) = V'(\mathbf{x}', A)\epsilon^\alpha + O(\epsilon^{\alpha+1}),$$

where the the integer exponent  $\alpha$  depends upon our choice of  $\mathbf{A}$ . We will provide the theory to show that the system defined by Hamiltonian

$$\hat{H}' = \lim_{\epsilon \rightarrow 0} \epsilon^\alpha \hat{H}(\epsilon) = \frac{1}{V'(\mathbf{x}', A)} H'$$

is a superintegrable system that arises from the system  $[2, 1, 1]$  by a conformal Stäckel transform induced by the potential  $V'(\mathbf{x}', A)$ . Thus the Helmholtz superintegrable system with potential  $V = V_{[1,1,1,1]}/F$  contracts to the Helmholtz superintegrable system with potential  $V_{[2,1,1]}/V'$ . The contraction is induced by a generalized Inönü-Wigner Lie algebra contraction of the conformal algebra  $so(4, \mathbb{C})$ . Always the  $V'$  can be identified with a specialization of the  $[2, 1, 1]$  potential. Thus a conformal Stäckel transform of  $[1, 1, 1, 1]$  has been contracted to a conformal Stäckel



The diagram commutes

FIGURE 1. Relationship between conformal Stäckel transforms and Bôcher contractions.

transform of  $[2, 1, 1]$ . The results follow and generalize to all Laplace systems.

The basic idea is that the procedure of taking a conformal Stäckel transform of a conformal superintegrable system, followed by a Helmholtz contraction yields the same result as taking a Bôcher contraction followed by an ordinary Stäckel transform: The diagrams commute. The possible Helmholtz contractions obtainable from these Bôcher contractions number well over 100; they will be classified in another paper. All quadratic algebra contractions are induced by Lie algebra contractions of  $so(4, \mathbb{C})$ , even those for Darboux and Koenigs spaces.

### 6. CONCLUSIONS AND DISCUSSION

We have pointed out that the use of Lie algebra contractions based on the symmetry groups of constant curvature spaces to construct quadratic algebra contractions of 2nd order 2D Helmholtz superintegrable systems is incomplete, because it doesn't satisfactorily account for Darboux and Koenigs spaces, and because even for constant curvature spaces there are abstract quadratic algebra contractions that cannot be obtained from the Lie symmetry algebras. However, this gap is filled in when one extends these systems to 2nd order Laplace conformally superintegrable systems with conformal symmetry algebra. Classes of Stäckel equivalent Helmholtz superintegrable systems are now recognized as corresponding to a single Laplace superintegrable system on flat space with underlying conformal symmetry algebra  $so(4, \mathbb{C})$ . The conformal Lie algebra contractions are induced by Bôcher limits associated with invariants of quadratic forms. They generalize all of the Helmholtz contractions derived earlier. In particular, contractions of Darboux

### Schematic of Laplace & Helmholtz superintegrable systems

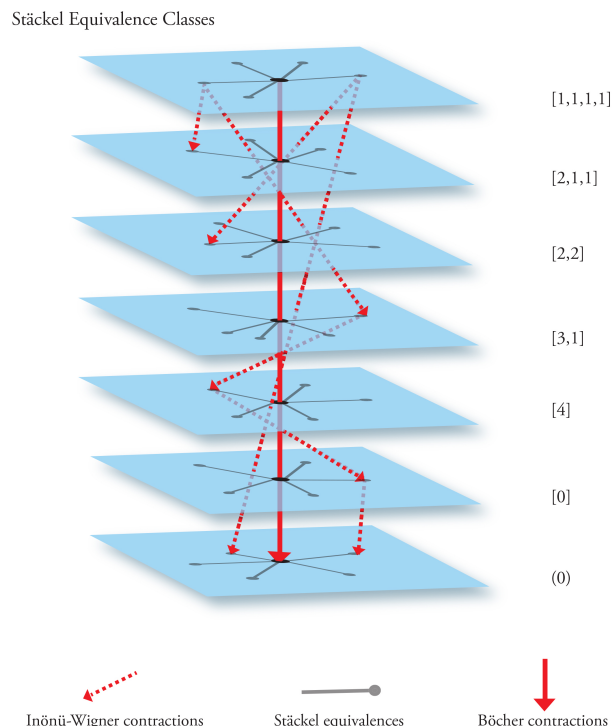


FIGURE 2. The bigger picture.

and Koenigs systems can be described easily. All of the concepts introduced in this paper are clearly also applicable for dimensions  $n \geq 3$  [33].

In papers submitted [32], and under preparation we will:

- (1.) give a complete detailed classification of 2D non-degenerate 2nd order conformally superintegrable systems and their relation to Bôcher contractions;
- (2.) present a detailed classification of all Bôcher contractions of 2D nondegenerate 2nd order conformally superintegrable systems;
- (3.) present tables describing the contractions of non-degenerate 2nd order Helmholtz superintegrable systems and how they are induced by Bôcher contractions;
- (4.) introduce  $so(4, \mathbb{C}) \rightarrow e(3, \mathbb{C})$  contractions of Laplace systems and show how they produce conformally 2nd order superintegrable 2D time-dependent Schrödinger equations.

From Theorem 1 we know that the potentials of all Helmholtz superintegrable systems are completely determined by their free quadratic algebras, i.e., the symmetry algebra that remains when the parameters in the potential are set equal to 0. Thus for classification purposes it is enough to classify free abstract quadratic algebras. In a second paper under preparation we will:

- (1.) apply the Bôcher construction to degenerate



- (1-parameter) Helmholtz superintegrable systems (which admit a 1st order symmetry);
- (2.) give a classification of free abstract degenerate quadratic algebras and identify which of those correspond free 2nd order superintegrable systems;
- (3.) classify abstract contractions of degenerate quadratic algebras and identify which of those correspond to geometric contractions of Helmholtz superintegrable systems;
- (4.) classify free abstract nondegenerate quadratic algebras and identify those corresponding to free nondegenerate Helmholtz 2nd order superintegrable systems;
- (5.) classify the abstract contractions of nondegenerate quadratic algebras.

We note that by taking contractions step-by-step from a model of the  $S_9$  quadratic algebra we can recover the Askey Scheme [25]. However, the contraction method is more general. It applies to all special functions that arise from the quantum systems via separation of variables, not just polynomials of hypergeometric type, and it extends to higher dimensions. The functions in the Askey Scheme are just those hypergeometric polynomials that arise as the expansion coefficients relating two separable eigenbases that are *both* of hypergeometric type. Thus, there are some contractions which do not fit in the Askey scheme since the physical system fails to have such a pair of separable eigenbases. In a third paper under preparation we will analyze the Laplace 2nd order conformally superintegrable systems, determine which of them is exactly solvable or quasi-exactly solvable and identify the spaces of polynomials that arise. Again, multiple Helmholtz superintegrable systems will correspond to a single Laplace system. This will enable us to apply our results to characterize polynomial eigenfunctions not of Askey type and their limits.

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#### REFERENCES

- [1] Evans N.W., Super-Integrability of the Winternitz System; *Phys. Lett. V.A* 147, 483–486, (1990), doi:10.1016/0375-9601(90)90611-Q
- [2] Tempesta P., Turbiner A. and Winternitz P., Exact solvability of superintegrable systems, *J. Math. Phys.*, **42**, 4248–4257 (2001), doi:10.1063/1.1386927
- [3] Superintegrability in Classical and Quantum Systems, Tempesta P., Winternitz P., Miller W., Pogosyan G., editors, AMS, vol. 37, 2005, ISBN-10: 0-8218-3329-4, ISBN-13: 978-0-8218-3329-2
- [4] Fordy A. P., Quantum Super-Integrable Systems as Exactly Solvable Models, *SIGMA* **3** 025, (2007), doi:10.3842/SIGMA.2007.025
- [5] Miller W. Jr., Post S. and Winternitz P., Classical and Quantum Superintegrability with Applications, *J. Phys. A: Math. Theor.*, **46**, 423001, (2013)
- [6] Kalnins E. G., Kress J. M., and Miller W. Jr., Second order superintegrable systems in conformally flat spaces. I: 2D classical structure theory. *J. Math. Phys.*, **46**, 053509, (2005); II: The classical 2D Stäckel transform. *J. Math. Phys.*, **46**, 053510, (2005); III. 3D classical structure theory, *J. Math. Phys.*, **46**, 103507, (2005), IV. The classical 3D Stäckel transform and 3D classification theory,, *J. Math. Phys.*, **47**, 043514, (2006) ; V: 2D and 3D quantum systems. *J. Math. Phys.*, **47**, 09350, (2006); Nondegenerate 2D complex Euclidean superintegrable systems and algebraic varieties, *J. Phys. A: Math. Theor.*, **40**, 3399-3411, (2007), doi:10.1088/1751-8113/40/13/008
- [7] Daskaloyannis C. and Tanoudis Y., Quantum superintegrable systems with quadratic integrals on a two dimensional manifold. *J. Math Phys.*, **48**, 072108 (2007).
- [8] Koenigs, G., Sur les géodésiques a intégrales quadratiques. A note appearing in “Lecons sur la théorie générale des surfaces”. G. Darboux. Vol 4, 368-404, 1896, *Chelsea Publishing* 1972.
- [9] Kalnins E. G., Kress J. M., Miller W. Jr. and Winternitz P., *Superintegrable systems in Darboux spaces. J. Math. Phys.*, V.44, 5811–5848, (2003), doi:10.1063/1.1619580
- [10] Granovskii Ya. I., Zhedanov A. S., and Lutsenko I. M., Quadratic algebras and dynamics in curved spaces. I. Oscillator, *Theoret. and Math. Phys.*, 1992, **91**, 474-480, doi:10.1007/BF01018846; Quadratic algebras and dynamics in curved spaces. II. The Kepler problem, *Theoret. and Math. Phys.*, **91**, 604-612, (1992), doi:10.1007/BF01017335
- [11] Bonatos D., Daskaloyannis C. and Kokkotas K., Deformed Oscillator Algebras for Two-Dimensional Quantum Superintegrable Systems; *Phys. Rev.*, V.A 50, 3700–3709, (1994), doi:10.1103/PhysRevA.50.3700
- [12] Letourneau P. and Vinet L., Superintegrable systems: Polynomial Algebras and Quasi-Exactly Solvable Hamiltonians. *Ann. Phys.*, V.243, 144–168, (1995), doi:10.1006/aphy.1995.1094
- [13] Kalnins E. G., Kress J. M., Miller W. Jr. and Pogosyan G. S., Completeness of superintegrability in two-dimensional constant curvature spaces. *J. Phys. A: Math Gen.* **34**, 4705–4720 (2001), doi:10.1088/0305-4470/34/22/311
- [14] Kress J. M., Equivalence of superintegrable systems in two dimensions. *Phys. Atomic Nuclei*, **70**, 560-566, (2007), doi:10.1088/0305-4470/34/22/311
- [15] Kalnins E. G. and Miller W. Jr., Quadratic algebra contractions and 2nd order superintegrable systems, *Anal. Appl.* **12**, 583-612, (2014), doi:10.1142/S0219530514500377
- [16] Kalnins E. G., Miller W. Jr. and Post S., Wilson polynomials and the generic superintegrable system on the 2-sphere, *J. Phys. A: Math. Theor.* **40**, 11525-11538 (2007), doi:10.1088/1751-8113/40/38/005

- [17] Kalnins E. G., Miller W. Jr. and Post S., Models for quadratic algebras associated with second order superintegrable systems, *SIGMA* 4, 008, 21 pages; [arXiv:0801.2848](https://arxiv.org/abs/0801.2848), (2008), DOI:10.3842/SIGMA.2008.008
- [18] Kalnins E. G., Miller W. Jr. and Post S., Two-variable Wilson polynomials and the generic superintegrable system on the 3-sphere, <http://www.emis.de/journals/SIGMA/2011/051/math-ph>], *SIGMA*, 7, 051 (2011) 26 pages, DOI:10.3842/SIGMA.2011.051
- [19] Post, S., Models of quadratic algebras generated by superintegrable systems in 2D. *SIGMA* 7 (2011), 036, 20 pages [arXiv:1104.0734](https://arxiv.org/abs/1104.0734), DOI:10.3842/SIGMA.2011.036
- [20] Li Q, and Miller W. Jr, Wilson polynomials/functions and intertwining operators for the generic quantum superintegrable system on the 2-sphere, 2015 *J. Phys.: Conf. Ser.* 597 012059 (<http://iopscience.iop.org/1742-6596/597/1/012059>)
- [21] Digital Library of Mathematical Functions (<http://dlmf.nist.gov>).
- [22] İnönü E. and Wigner E. P., On the contraction of groups and their representations. *Proc. Nat. Acad. Sci. (US)*, **39**, 510-524, (1953), DOI:10.1073/pnas.39.6.510
- [23] Weimar-Woods E., The three-dimensional real Lie algebras and their contractions, *J. Math. Phys.*, **32**, 2028-2033 (1991), doi:10.1063/1.529222
- [24] Nesterenko M. and Popovych R., Contractions of low-dimensional Lie algebras, *J. Math. Phys.*, **47** 123515, (2006).
- [25] Kalnins E. G., Miller W. Jr and Post S., Contractions of 2D 2nd order quantum superintegrable systems and the Askey scheme for hypergeometric orthogonal polynomials *SIGMA*, **9** 057, 28 pages, (2013), DOI:10.3842/SIGMA.2013.057
- [26] Kalnins E. G., Kress J.M, Miller W. Jr and Post, S., Laplace-type equations as conformal superintegrable systems, *Advances in Applied Mathematics* 46 (2011) 396416.
- [27] Capel J.J. and Kress J.M., Invariant Classification of Second-order Conformally Flat Superintegrable Systems, *J. Phys. A: Math. Theor.* 47 (2014), 495202.
- [28] Bôcher M., Ueber die Reihenentwicklungen der Potentialtheorie, B. G. Teubner, Leipzig 1894.
- [29] Kalnins E.G., Miller W. Jr. and Post S., Coupling constant metamorphosis and Nth order symmetries in classical and quantum mechanics, *J. Phys. A: Math. Theor.* 43 (2010) 035202. (20 pages) , DOI:10.1088/1751-8113/43/3/035202
- [30] Kalnins E.G., Miller W. Jr., and Reid G.J., Separation of variables for complex Riemannian spaces of constant curvature. I. Orthogonal separable coordinates for Snc and Enc, *Proc. R. Soc. Lond. A* 394 (1984), pp. 183-206, DOI:10.1098/rspa.1984.0075
- [31] Bromwich T. J. A., Quadratic forms and their classification by means of invariant factors. Cambridge tract no. 3. Cambridge University Press, 1906, reprint Hafner, New York, 1971.
- [32] E.G. Kalnins, W. Miller, Jr., and E. Subag, Bôcher contractions of conformally superintegrable Laplace equations, (submitted), [arXiv:1512.09315](https://arxiv.org/abs/1512.09315), 2016; E.G. Kalnins, W. Miller, Jr., and E. Subag, Bôcher contractions of conformally superintegrable Laplace equations: Detailed computations, [arXiv:1601.02876](https://arxiv.org/abs/1601.02876), 2016
- [33] Capel J.J., Kress J.M. and Post S., Invariant Classification and Limits of Maximally Superintegrable Systems in 3D, *SIGMA*, **11** (2015), 038, 17 pages [arXiv:1501.06601](https://arxiv.org/abs/1501.06601), DOI:10.3842/SIGMA.2015.038