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THE SHMUSHKEVICH METHOD FOR HIGHER SYMMETRY GROUPS OF INTERACTING PARTICLES

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Abstract. About 60 years ago, I. Shmushkevich presented a simple ingenious method for computing the relative probabilities of channels involving the same interacting multiplets of particles, without the need to compute the Clebsch-Gordan coefficients. The basic idea of Shmushkevich is "isotopic non-polarization" of the states before the interaction and after it. Hence his underlying Lie group was $SU(2)$. We extend this idea to any simple Lie group. This paper determines the relative probabilities of various channels of scattering and decay processes following from the invariance of the interactions with respect to a compact simple a Lie group. Aiming at the probabilities rather than at the Clebsch-Gordan coefficients makes the task easier, and simultaneous consideration of all possible channels for given multiplets involved in the process, makes the task possible. The probability of states with multiplicities greater than 1 is averaged over. Examples with symmetry groups $O(5)$, $F(4)$, and $E(8)$ are shown.

Keywords: isospin, particle collisions, Lie group representation.

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1. Introduction

The method of Shmushkevich [1, 2] was conceived as a simpler alternative to computing the relative probabilities of various channels of scattering and decay processes under strict isospin invariance ($SU(2)$ invariance). The traditional alternative method for calculating the probabilities of the same channels is to calculate first all pertinent Clebsch-Gordan coefficients (CGC) for the channel.

The most remarkable feature of the Shmushkevich method is the complete avoidance of the need to calculate the Clebsch-Gordan coefficients. The underlying idea is to consider isotopically unpolarized states before and after the interaction, assuming that each possible state came with equal probability.

The simplicity of the idea has attracted the attention of many physicists [3–6].

Technically, the two methods differ in their objective: Shmushkevich's method calculates just the probabilities. The conventional alternative method calculates the CGC, their squares, and then provides the probabilities. Neither of the tasks is easy for higher ranks of the representations.

From the point of view of symmetries the two methods differ by the symmetry group that they exploit: Shmushkevich uses just the Weyl group of the Lie group, while CGCs are built using the symmetry group of Demazure-Tits, see [7, 8].

The difficulty of generalizing of Shmushkevich's method to higher rank groups lies in the frequent occurrence of multiple states with the same quantum numbers, equivalently labeled by the same weights of irreducible representations [9], as well as the sheer number of channels that need to be written down.

It is likely that practical exploitation of Shmushkevich's idea for higher groups and possibly representations of much higher dimensions, will not proceed by spelling out the large number of channels for each case and counting the number of occurrences of each state in all the channels. Instead, one would start from one known channel and use the symmetry group, the Weyl group in this case, to produce other channels with the same probability. This is a routine operation which, however does not produce all possible channels. This will relate only the states which are situated on the same Weyl group orbit. This may be all that one needs as long as only the channels defined by individual orbits are studied. However, if the equal probability of all the states of the Lie group is to be involved, the link between different orbits present in the same representation has to be imposed independently. For the probabilities, a natural link is provided by the requirement that the probabilities add up to one. If an orbit is present in irreducible representation more than once, say m times, we count them as equally probable m channels. In this paper

C_2	Decomposition into irreducible representations with multiplicities			Multiplicity	Orbit size
$(20) \times (10)$	(30)	(11)	(10)		
10×4	[30]			1	4
	[11]	[11]		2	8
	2[10]	2[10]	[10]	5	4
	Decomposition into orbits with multiplicities				
	[30]	2[11]	5[10]	40	

TABLE 1. Decomposition of the product in the C_2 example. Decomposition is given in the weight system of irreducible representations (the first line) and in terms of orbits (bottom line). The dimensions of the representations and the sizes of the orbits are shown together with the multiplicities.

we provide an illustration of this approach, for the symmetry groups $O(5)$, $F(4)$, and $E(8)$. The process that consider is the simplest, where three multiplets are interacting, more precisely the interaction of two particles yields a third one. Our aim is to show how to average over different particles/states which carry the same Lie group representation labels.

We write the highest weights of the representations in round brackets, and the dominant weights of the Weyl group orbits in square brackets.

In the examples, we show that there are many states which have the group labels (weights) identical although they label different particle states. In order to avoid the almost impossible task of distinguishing these states, we add them up and count their total probability.

2. Symmetry group $O(5)$

Consider the example where the underlying symmetry group is the Weyl reflection group of the Lie group $O(5)$, or equivalently of the Lie algebra C_2. We label the representations by their unique highest weight (relative to the basis of the fundamental weights). The product of representations of dimensions 10 and 4 decomposes as follows,

$$\begin{aligned}(20) \times (10) &= (30) + (11) + (10),\\ 10 \times 4 &= 20 + 16 + 4 = 40,\end{aligned}$$

where the second line shows the dimensions of representations, see [10]. Labeling the Weyl group orbits by their unique dominant weights, the product of the weight systems decomposes into the Weyl group orbits as follows,

$$\begin{aligned}(20) \times (10) &= [30] + 2[11] + 5[10],\\ 10 \times 4 &= 4 + 2 \cdot 8 + 5 \cdot 4 = 40,\end{aligned}$$

where the integers in front of the square brackets are the multiplicities of the occurrence of the respective orbits in the decomposition.

If only the product of the Weyl group orbits were to be considered, the decomposition would be simpler:

$$\begin{aligned}[20] \times [10] &= [30] + [11] + [10],\\ 4 \times 4 &= 4 + 8 + 4 = 16.\end{aligned}$$

There are 40 states in the product. If equal probability is assumed, each of the channels comes with the probability $\frac{1}{40}$. Consequently, we have the probabilities:

- 4 states from [30], each present once: 1/40;
- 8 states from [11], each present twice: 1/20;
- 4 states from [10], each present $5\times$: 1/8.

The results of the example are summarized in Table 1.

3. Symmetry group $F(4)$

Consider decomposition of the product of representations in terms of their weight systems,

$$\begin{aligned}(0001) \times (0001) &= (0002) + (0010) + (1000)\\ &\quad + (0001) + (0000),\\ 26 \times 26 &= 324 + 273 + 52 + 26 + 1 = 676.\end{aligned}$$

Decomposition of the same product in terms of Weyl group orbits,

$$\begin{aligned}(0001) \times (0001) &= [0002] + 2[0010] + 6[1000]\\ &\quad + 12[0001] + 28[0000]\end{aligned}$$

and the corresponding equality of the dimensions

$$26 \times 26 = 24 + 2 \cdot 96 + 6 \cdot 24 + 12 \cdot 24 + 28 \cdot 1.$$

Suppose that we want to decompose only the product of the orbits of the highest weights

$$\begin{aligned}[0001] \times [0001] &= [0002] + 2[0010] + 6[1000]\\ &\quad + 8[0001] + 24[0000],\\ 24 \times 24 &= 24 + 2 \cdot 96 + 6 \cdot 24 + 8 \cdot 24 + 24 \cdot 1\end{aligned}$$

If equal probability of the 676 states is assumed we have the following probabilities of the channels:

$F(4)$	**Decomposition into irreducible representations with multiplicities**					**Multiplicity**	**Orbit size**
$(0001)\times(0001)=$	(0002)	(0010)	(1000)	(0001)	(0000)		
26×26	[0002]					1	24
	[0010]	[0010]				2	96
	3[1000]	2[1000]	[1000]			6	24
	5[0001]	5[0001]	[0001]	[0001]		12	24
	12[0000]	9[0000]	4[0000]	2[0000]	[0000]	28	1
	Decomposition into orbits with multiplicities						
	[0002]	2[0010]	6[1000]	12[0001]	28[0000]	$676=26^2$	

Table 2. Decomposition of the product in the $F(4)$ example. The decomposition is given in the weight system of irreducible representations (the first line) and in terms of orbits (bottom line). The dimensions of the representations and the sizes of the orbits are shown together with the multiplicities.

- 24 states from [0002], each present once: 1/676;
- 96 states from [0010], each present twice: 2/676;
- 24 states from [1000], each present 6 ×: 6/676;
- 24 states from [0001], each present 12 ×: 12/676;
- 1 state from [0000], each present 28 ×: 28/676.

The results of the example are summarized in Table 2.

4. Symmetry group $E(8)$

Consider decomposition of the product of the representations in terms of their weight systems

$$\begin{aligned}(1000\overset{0}{0}00)\times(1000\overset{0}{0}00) &= (2000\overset{0}{0}00)+(0100\overset{0}{0}00)\\ &\quad+(0000\overset{0}{0}01)+(1000\overset{0}{0}00)+(0000\overset{0}{0}00)\end{aligned}$$

with the respective dimensions

$$\begin{aligned}248\times 248 &= 27000+30380+3875+248+1\\ &= 61504.\end{aligned}$$

We write the components of $E(8)$ weights as they would be attached to the corresponding Dynkin diagram.

The same product decomposed into the sum of the Weyl group orbits has very different multiplicities,

$$\begin{aligned}(1000\overset{0}{0}00)\times(1000\overset{0}{0}00) &= [2000\overset{0}{0}00]+2[0100\overset{0}{0}00]\\ &\quad+14[0000\overset{0}{0}01]+72[1000\overset{0}{0}00]+304[0000\overset{0}{0}00],\end{aligned}$$

and the equality of the dimensions in the decomposed product,

$$\begin{aligned}248\times 248 &= 240+2\cdot 6720+14\cdot 2160\\ &\quad+72\cdot 240+304\cdot 1=61504.\end{aligned}$$

If only the product of the orbits is to be calculated, the result is much simpler,

$$\begin{aligned}[1000\overset{0}{0}00]\times[1000\overset{0}{0}00] &= [2000\overset{0}{0}00]+2[0100\overset{0}{0}00]\\ &\quad+14[0000\overset{0}{0}01]+56[1000\overset{0}{0}00]+240[0000\overset{0}{0}00],\end{aligned}$$

and the corresponding orbit sizes with the appropriate multiplicities are

$$\begin{aligned}240\times 240 &= 240+2\cdot 6720\\ &\quad+14\cdot 2160+56\cdot 240+240\cdot 1=57600.\end{aligned}$$

If equal probability of the 61504 states is assumed, we have the following probabilities of the channels:

- 240 states from $[2000\overset{0}{0}00]$, each present once: 1/61504;
- 6720 states from $[0100\overset{0}{0}00]$, each present twice: 2/61504;
- 2160 states from $[0000\overset{0}{0}01]$, each present 14 times: 14/61504;
- 240 states from $[1000\overset{0}{0}00]$, each present 72 times: 72/61504;
- 1 state from $[0000\overset{0}{0}00]$, each present 304 times: 304/61504.

The results of the example are summarized in Table 3.

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$E(8)$	**Decomposition into irreducible representations with multiplicities**					**Multi-plicity**	**Orbit size**
$\left(\begin{smallmatrix}0\\1000000\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0\\1000000\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\2000000\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0100000\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0000001\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\1000000\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0000000\end{smallmatrix}\right)$		
248×248	$\left[\begin{smallmatrix}0\\2000000\end{smallmatrix}\right]$					1	240
	$\left[\begin{smallmatrix}0\\0100000\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0\\0100000\end{smallmatrix}\right]$				2	6720
	$6\left[\begin{smallmatrix}0\\0000001\end{smallmatrix}\right]$	$7\left[\begin{smallmatrix}0\\0000001\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0\\0000001\end{smallmatrix}\right]$			14	2160
	$29\left[\begin{smallmatrix}0\\1000000\end{smallmatrix}\right]$	$35\left[\begin{smallmatrix}0\\1000000\end{smallmatrix}\right]$	$7\left[\begin{smallmatrix}0\\1000000\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0\\1000000\end{smallmatrix}\right]$		72	240
	$120\left[\begin{smallmatrix}0\\0000000\end{smallmatrix}\right]$	$140\left[\begin{smallmatrix}0\\0000000\end{smallmatrix}\right]$	$35\left[\begin{smallmatrix}0\\0000000\end{smallmatrix}\right]$	$8\left[\begin{smallmatrix}0\\1000000\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0\\0000000\end{smallmatrix}\right]$	304	1
	Decomposition into orbits with multiplicities						
	$\left[\begin{smallmatrix}0\\2000000\end{smallmatrix}\right]$	$2\left[\begin{smallmatrix}0\\0100000\end{smallmatrix}\right]$	$14\left[\begin{smallmatrix}0\\0000001\end{smallmatrix}\right]$	$72\left[\begin{smallmatrix}0\\1000000\end{smallmatrix}\right]$	$304\left[\begin{smallmatrix}0\\0000000\end{smallmatrix}\right]$	$61504 = 248^2$	

Table 3. Decomposition of the product in the $E(8)$ example. The decomposition is given in the weight system of irreducible representations (the first line) and in terms of orbits (bottom line). The dimensions of the representations and the sizes of the orbits are shown together with the multiplicities.

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