Determination of Uncertainties for Correlated Input Quantities by the Monte Carlo Method

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Abstract

This paper presents the calculation of the uncertainty for distribution propagation by the Monte Carlo method for a measurement model with one output quantity. The procedure is shown on the basis of an example of the calculation of a rectangle by direct measurement of length by the same caliper. The measurements are correlated, and the uncertainties are calculated for three values of the correlation coefficients. Another part of the paper presents a validation of the law of propagation of uncertainties for distribution propagation by the Monte Carlo method.

Keywords: uncertainty of measurement, Monte Carlo method, correlation.

1 Introduction

The numerical method that implements the law of propagation of distribution, specifically Monte Carlo simulation, is used especially in cases when linearization by Taylor series cannot be implemented, the probability distribution of the input quantities is asymmetric, and it is difficult to determine the partial derivatives. An instruction given in Annex 1 to GUM (Guide to the expression of uncertainty in measurement) [2] includes a general alternative procedure in accordance with GUM [1] for the numerical evaluation of uncertainty in measurement that is suitable for computer processing. The procedure applies for model having a single output quantity, where the values of the input quantities are associated with any probability density function, including asymmetric probability density functions. When calculating the uncertainty of the correlated input quantities, it is necessary to determine the covariance matrix, the correlation coefficients and the associated probability density of the input quantities. The result of the Monte Carlo simulation is 95 % reference interval, estimate and standard uncertainty for the output quantity [2, 3].

2 Propagation of distribution by the Monte Carlo

The Monte Carlo method provides a general approach for numerical approximation to the distribution function $g_Y(\eta)$ for the output quantity $Y = f(\mathbf{X})$. The input quantities of the model are $\mathbf{X} = (X_1, X_2, \ldots, X_N)^{\mathrm{T}}$. The core approach is repeated selection of the values of probability density functions for input quantities X_i [4, 5]. The distribution function for output quantity Y obtained from Monte Carlo simulation is defined as

$$G_Y(\eta) = \int_{-\infty}^{\eta} g_Y(z)\,\mathrm{d}z \tag{1}$$

The probability density function is defined [3] as

$$g_Y(\eta) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} g_X(\xi)\delta(\eta - f(\xi))\,\mathrm{d}\xi_N \ldots \mathrm{d}\xi_1 \tag{2}$$

where δ is the Dirac function, $g_{Xi}(\xi_i)$, and $i = 1, \ldots, N$, are the probability density function of the input quantities X_i, $i = 1, \ldots, N$.

The dependence of the input quantities can be expressed by the covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{11} & \ldots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \ldots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \ldots & \sigma_{nn} \end{bmatrix} \tag{3}$$

The Monte Carlo simulation as an implementation of the propagation of distribution can conveniently be stated as a step-by-step procedure

- Select the number M of Monte Carlo trials to be made,
- Generate M samples of input quantities,
- Evaluate the model to give the output quantity X_i,
- Sort these M values of the output quantity into non-decreasing order,

- Form an estimate of the output quantity and the associated standard uncertainty,
- Form the shortest 95 % coverage interval for the output quantity [2, 3].

3 Determination of uncertainties by Monte Carlo

To validate the law of propagation of uncertainties using the Monte Carlo method, the same caliper is used to calculate the rectangle by direct measurement of its sides, so the measurements are correlated. The lengths of the sides of the rectangle have nominal values of 100 mm and 50 mm. According to the manufacturer's certificate, error of measurement 0.01 mm is admissible. This error applies at 20 °C [5], and we neglect other effects.

Measurement model

$$P = a \cdot b = (a_m + \Delta a_M) \cdot (b_m + \Delta a_M) \qquad (4)$$

where a_m is the measurement error in measuring the length of side a (mm), Δa_M is measurement error (mm), b_m is measurement error in measuring the length of side b (mm).

The model for calculating the sides of a rectangle by measuring the input quantities for the Gaussian probability density function and one an output quantity is shown in Figure 2.

Calculation of uncertainties according to the law of propagation of uncertainties, we can view in table of balance uncertainty, which is used to compare two methods for evaluation of uncertainties.

The input quantities used in the Monte Carlo simulation are shown in Table 2. $M = 10^7$ Monte Carlo trials are used in calculating the Monte Carlo simulation. The input quantities are generated using a pseudo-random number generator. The 32-bit Mersenne Twister generator with a period of $2^{1\,019\,937} - 1$ is used as a pseudo-random number generator.

Figure 2 shows the probability density functions and the frequency distributions (histograms) of the rectangle obtained by the distribution propagation by Monte Carlo simulation for the three correlation coefficients. The vertical lines define the 95 % confidence intervals for the three correlation coefficients, with estimates and expanded uncertainties. The width of the coverage intervals increases with increasing value of the correlation coefficient.

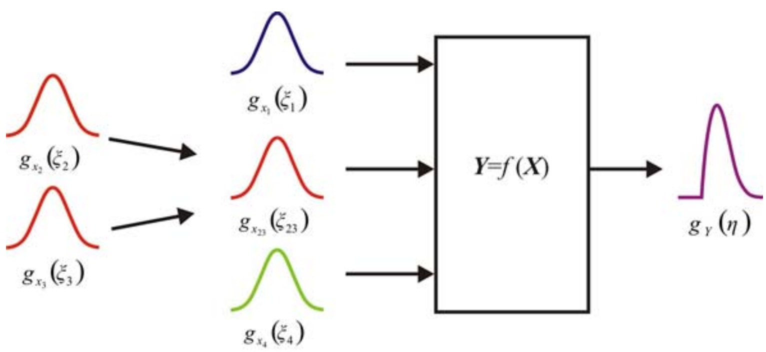

Figure 1: Propagation of the distribution of input quantities for the measurement model

Table 1: Balance of uncertainties

Input quantity X_i	Estimate x_i/mm	Standard deviation $u(x_i)$/mm	Distribution	Sensitivity coefficient /mm	Contribution to the standard uncertainty $u_i(y)/\mathrm{mm}^2$
The arithmetic mean of the measured values a_m	100.097	0.016 3	normal	50.096	0.82
Measurement error Δa_M	0.000	0.010	normal	50.096	0.50
The arithmetic mean of the measured values b_m	50.096	0.0164	normal	100.097	1.65
Measurement error Δb_M	0.000	0.010	normall	100.097	1.00
P	5 014.46	Correlation coefficient $r = 0$			2.15
P	5 014.46	Correlation coefficient $r = 0.5$			2.27
P	5 014.46	Correlation coefficient $r = 0.9$			2.35

Table 2: Inputs to the Monte Carlo simulation

Quantity	Estimate	Standard deviation	Distribution
The arithmetic mean of the measured values a_m	100.097 mm	0.0163 mm	normal
Measurement error Δa_M	0.000 mm	0.010 mm	normal
The arithmetic mean of the measured valuesb_m	50.096 mm	0.0164 mm	normal

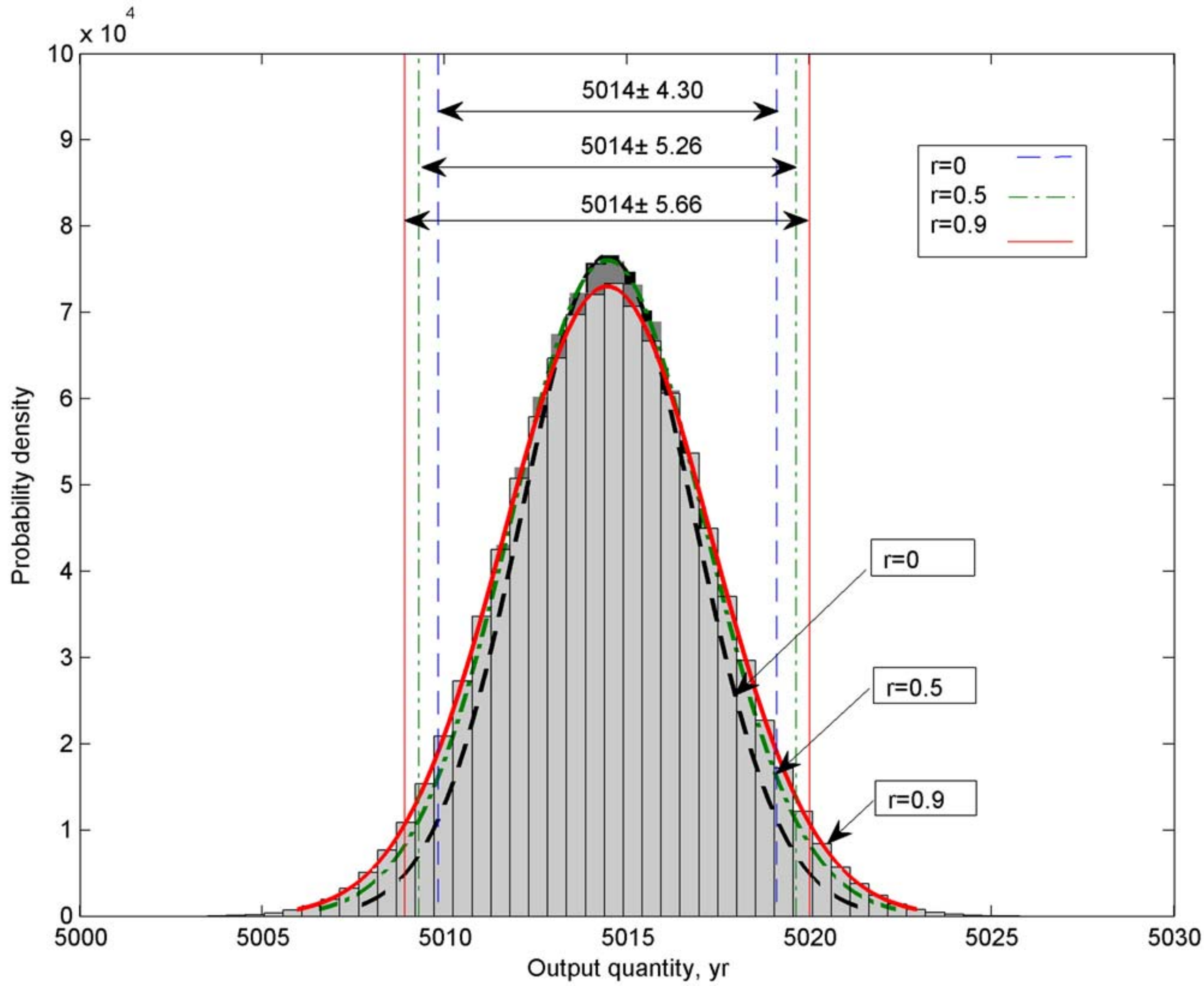

Figure 2: Probability density functions of the rectangle for the output quantity

Figure 3 shows the distribution function which is made of values of measurement model which are sorting into non-decresing order. Vertical lines mark the endpoints of the probabilistically symmetric 95 % coverage interval for the three correlation coefficients. The standard uncertainty increases with the increase of the correlation coefficient.

Estimates and the combined uncertainties of the two methods for evaluating the uncertainties of output quantity Y are given in Table 3 [2]. In order to validate the law of propagation of uncertainty using Monte Carlo simulation it is necessary to determine δ [3].

$$u(y) = 2.15\ \text{mm}^2 = 21 \cdot 10^{-1}\ \text{mm}^2, \quad a = 21,$$
$$r = -1, \quad \delta = \frac{1}{2} \cdot 10^{-1}\ \text{mm}^2 = 0.05\ \text{mm}^2 \tag{5}$$

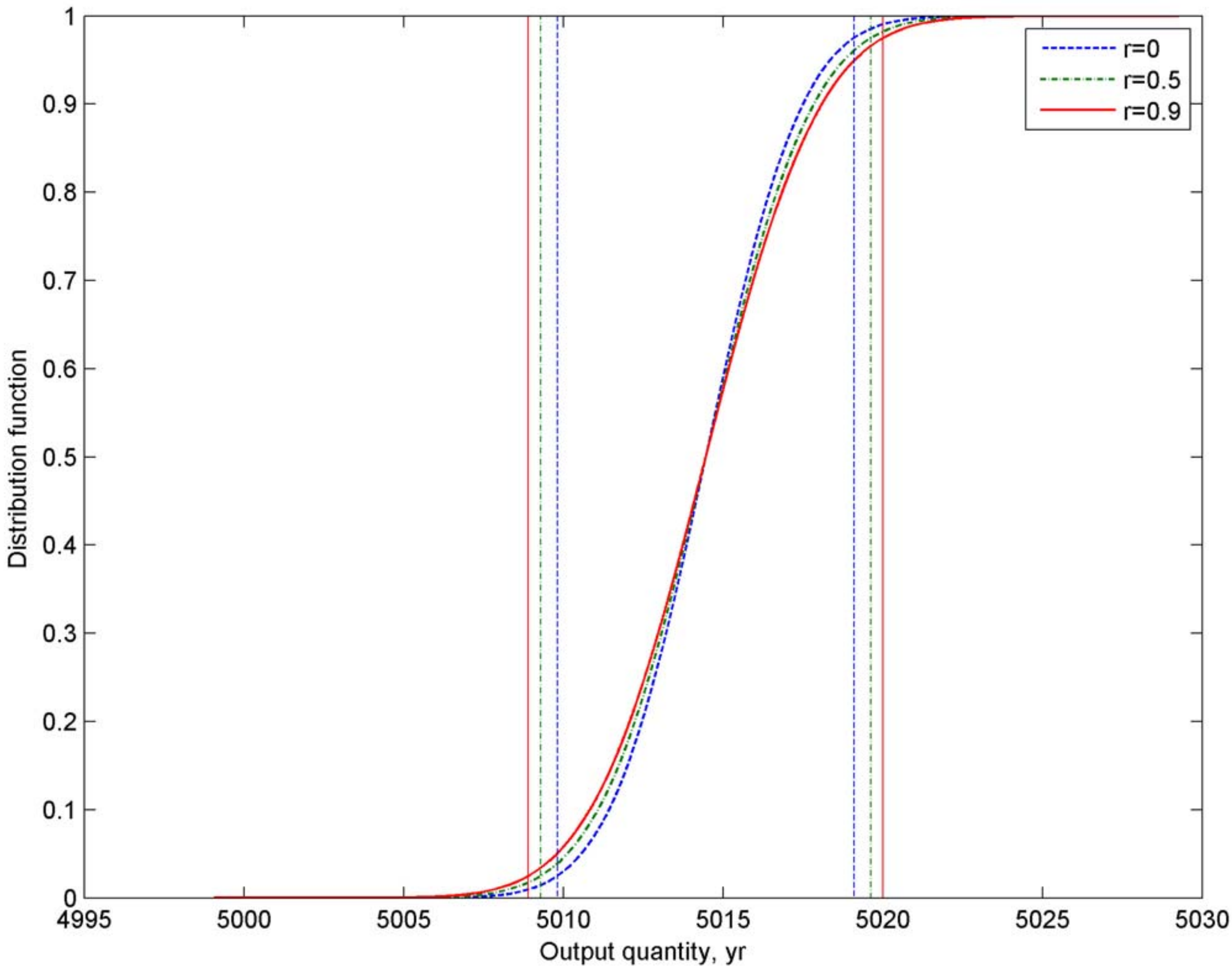

Figure 3: The distribution functions of the rectangle for the output quantity

Table 3: Validation of the law of propagation of uncertainty by Monte Carlo simulation

Method	r	M	y/mm	$u(y)$/mm	95 % coverage interval/mm	d_{low}, d_{high}/mm	Validation $\delta = 0.05$
Propagation of uncertainty	0		5 014.46	2.15	[5 010.15; 5 018.77]		
	0.5			2.27	[5 009.92; 5 019.00]		
	0.9			2.35	[5 009.76; 5 019.16]		
Propagation of distribution (MCM)	0	10^7	5 014.46	2.15	[5 010.25; 5 018.67]	0.1; 0.1	Nie
	0.5			2.64	[5 009.29; 5 019.63]	0.6; 0.6	Nie
	0.9			2.83	[5 008.91; 5 020.02]	0.8; 0.8	Nie

4 Conclusion

In this paper, the Monte Carlo method has been used for estimating the uncertainty of correlated input quantities by direct measurement of the sides of a rectangle using the same caliper for three different correlation coefficients [2]. The measurement result was also determined by the propagation of uncertainty in accordance with [5]. When calculating the uncertainties, it is necessary to consider the correlation between the input quantities of the measurement model, because the correlation affects the final element of the uncertainty. The calculations have shown that the use of the law of propagation of uncertainties for the model considered here is not acceptable, and that the higher members of the Taylor series should be taken into consideration.

Acknowledgement

The authors wish to thank the Slovak University of Technology in Bratislava and the VEGA grant agency, grant No. 1/0120/12, and APPV-grant No. 0096-10, for their support.

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