

CONCRETE QUANTUM LOGICS WITH GENERALISED
COMPATIBILITY

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(Received April 16, 1997)

Abstract. We present three results stating when a concrete (= set-representable) quantum logic with covering properties (generalization of compatibility) has to be a Boolean algebra. These results complete and generalize some previous results [3, 5] and answer partially a question posed in [2].

Keywords: Boolean algebra, concrete quantum logic, covering, Jauch–Piron state, ortho-completeness

MSC 1991: 03G12, 81P10

1. BASIC NOTIONS

Let us recall the main notion we shall deal with in this paper.

Definition 1.1. A *concrete logic* is a pair (X, L) , where $X \neq \emptyset$ and $L \subset \exp X$ such that

- (1) $\emptyset \in L$;
- (2) $A^c = X \setminus A \in L$ whenever $A \in L$;
- (3) $\bigcup M \in L$ whenever $M \subset L$ is a finite set of mutually disjoint elements.

A *concrete σ -logic* is a concrete logic (X, L) such that

- (3 σ) $\bigcup M \in L$ whenever $M \subset L$ is a countable set of mutually disjoint elements.

Let us note that the above definition is not given in the most efficient way. Indeed, since \emptyset is a finite set of mutually disjoint elements and $\bigcup \emptyset = \emptyset$, condition (1) follows from condition (3). Moreover, it is obvious that condition (3) follows from condition (3 σ).

The author acknowledges the support by the grant 201/96/0117 of the Czech Grant Agency.

The following lemma will be useful in the sequel. First, let us observe that if $A, B \in L$ and $A \subset B$, then $B \setminus A = (A \cup B^c)^c \in L$ for every concrete logic (X, L) .

Lemma 1.2. *Let (X, L) be a concrete σ -logic and let $A_i \in L$ ($i = 1, 2, \dots$) be such that $A_1 \supset A_2 \supset \dots$. Then $\bigcap_{i=1}^{\infty} A_i \in L$.*

Proof. The elements $A_i \setminus A_{i+1} \in L$ ($i = 1, 2, \dots$) are mutually orthogonal, hence $\bigcup_{i=1}^{\infty} (A_i \setminus A_{i+1}) \in L$ and $\bigcap_{i=1}^{\infty} A_i = A_1 \setminus \bigcup_{i=1}^{\infty} (A_i \setminus A_{i+1}) \in L$. \square

2. COVERING PROPERTIES

Definition 2.1. Let (X, L) be a concrete logic, $Y \subset X$ and let n be a natural number. A *covering* of Y is a set $M \subset L$ such that $Y = \bigcup M$. A covering M is an *n -covering* if $\text{card } M \leq n$.

We say that (X, L) has the *n -covering property* (*finite covering property*, resp.) if for every $A, B \in L$ there is an n -covering (finite covering, resp.) of $A \cap B$.

It is well-known that a concrete logic (X, L) is a Boolean algebra if and only if $A \cap B \in L$ for every $A, B \in L$, i.e., if and only if (X, L) has the 1-covering property. Thus, the notions of n -covering property (finite covering property), introduced in [3], are generalizations of compatibility in Boolean algebras.

The next lemma will be used in the sequel.

Lemma 2.2. *Let (X, L) be a concrete logic with the finite covering property. Then for every finite set $F \subset L$ there is a finite covering $G \subset L$ of $\bigcap F$.*

Proof. Let us proceed by induction. First, if F is a one-element subset of L (empty set, resp.), then we can put $G = F$ ($G = \{X\}$, resp.).

Now, let us suppose that there is a natural number $n \geq 1$ such that the lemma holds for every $F \subset L$ with $\text{card } F = n$. Let $F \subset L$ with $\text{card } F = n + 1$ and let $A \in F$. According to the previous assumption, there is a finite covering $G \subset L$ of $\bigcap (F \setminus \{A\})$. According to the finite covering property, for every $B \in G$ there is a finite covering $G_B \subset L$ of $A \cap B$. Thus, $\bigcup_{B \in G} G_B \subset L$ is a finite covering of $\bigcap F$. \square

Before we present the main result of this section, let us prove the following technical lemma.

Lemma 2.3. *Let (X, L) be a concrete σ -logic and let $m, n \geq 2$ be natural numbers such that $m \leq n + 1$. Let us suppose that for every set $F \subset L$ with $\text{card } F \leq n$ there is an m -covering $G \subset L$ of $\bigcap F$. Then for every set $F \subset L$ with $\text{card } F \leq n$ there is an $(m - 1)$ -covering $G \subset L$ of $\bigcap F$.*

Proof. Let $F \subset L$ with $\text{card } F \leq n$. Let us define by induction sequences $(A_{i1}, \dots, A_{in}) \in L^n$, $(B_{i0}, \dots, B_{in}) \in L^{n+1}$ ($i = 1, 2, \dots$) as follows: Let (A_{11}, \dots, A_{1n}) be such that $F = \{A_{11}, \dots, A_{1n}\}$. If $(A_{i1}, \dots, A_{in}) \in L^n$ is defined for a natural number $i \geq 1$ then let us take $(B_{i0}, \dots, B_{in}) \in L^{n+1}$ such that $B_{ij} = \emptyset$ for $j \geq m$ and $\bigcap_{j=1}^n A_{ij} = \bigcup_{j=0}^n B_{ij}$ and let us put $A_{i+1,j} = A_{ij} \setminus B_{ij}$ ($j \in \{1, \dots, n\}$).

Let us denote

$$B_0 = \bigcap_{i=1}^{\infty} B_{i0}, \quad B_j = \bigcup_{i=1}^{\infty} B_{ij}, \quad j \in \{1, \dots, n\}.$$

It is easy to see that the elements B_{1j}, B_{2j}, \dots ($j \in \{1, \dots, n\}$) are mutually disjoint, hence $B_j \in L$ for every $j \in \{1, \dots, n\}$. Moreover, $B_m = \dots = B_n = \emptyset$. Further,

$$B_{i0} \supset \bigcap_{j=1}^n A_{i+1,j} \supset B_{i+1,0} \quad (i = 1, 2, \dots).$$

Hence, according to Lemma 1.2, $B_0 \in L$, too. Since

$$\bigcap F = B_0 \cup B_1 \cup \dots \cup B_{m-1}$$

and since $B_0 \cup B_1 \in L$ ($B_0 \cap B_1 = \emptyset$), the proof is complete. \square

Theorem 2.4. *Let (X, L) be a concrete σ -logic. Let us suppose that there is a natural number $n \geq 2$ such that for any set $F \subset L$ with $\text{card } F \leq n$ there is an $(n + 1)$ -covering of $\bigcap F$. Then (X, L) is a Boolean algebra.*

Proof. Using Lemma 2.3 n -times, we obtain that (X, L) has the 1-covering property, i.e., (X, L) is a Boolean algebra. \square

Corollary 2.5. *Every concrete σ -logic with the 3-covering property is a Boolean algebra.*

This corollary generalizes [3, Proposition 4.6], where an analogous result is stated for concrete σ -logics with the 2-covering property.

3. COVERING PROPERTIES AND JAUCH–PIRON STATES

Definition 3.1. Let (X, L) be a concrete logic. A *state* on (X, L) is a mapping $s: L \rightarrow [0, 1]$ such that

- (1) $s(X) = 1$;
- (2) $s(\bigcup M) = \sum_{A \in M} s(A)$ whenever $M \subset L$ is a finite set of mutually disjoint elements.

A state s on (X, L) is called *Jauch–Piron* if for every $A, B \in L$ with $s(A) = s(B) = 1$ there is a $C \in L$ such that $C \subset A \cap B$ and $s(C) = 1$.

It is easy to see that $s(\emptyset) = 0$ and $s(A^c) = 1 - s(A)$ for every state s on a concrete logic (X, L) and for every $A \in L \setminus \{\emptyset\}$. Further, for every concrete logic (X, L) , every point $x \in X$ carries a two-valued state s_x on (X, L) defined by

$$s_x(A) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Before we present the main result of this section, we need the following definition.

Definition 3.2. Let (X, L) be a concrete logic and let $M, N \subset L$ be two coverings of $Y \subset X$. We say that N is a *coarsing* of M if for every $A \in M$ there is a $B \in N$ such that $A \subset B$.

Theorem 3.3. *Let (X, L) be a concrete logic such that every state on it is Jauch–Piron. Let us suppose that for every $A, B \in L$ every covering of $A \cap B$ admits a countable coarsing. Then L is a Boolean algebra.*

Proof. It suffices to prove that $A \cap B \in L$ for every $A, B \in L$. Let $A, B \in L$. If $A \cap B = \emptyset$, the proof is complete. Let us suppose that $A \cap B \neq \emptyset$. Then $S_{A,B} = \{s; s \text{ is a state on } (X, L) \text{ with } s(A) = s(B) = 1\}$ is nonempty (every point $x \in A \cap B$ carries a two-valued state $s_x \in S_{A,B}$). Since every state on (X, L) is Jauch–Piron, for every $s \in S_{A,B}$ there is a $C_s \in L$ such that $s(C_s) = 1$. Let us take a countable coarsing M of the covering $\{C_s; s \in S_{A,B}\}$ of $A \cap B$, a countable set $Y \subset A \cap B$ such that $Y \cap (C \setminus D) \neq \emptyset$ for every $C, D \in M$ with $C \setminus D \neq \emptyset$ and, finally, a state s that is a σ -convex combination (with non-zero coefficients) of all s_y ($y \in Y$). Since $s \in S_{A,B}$, there is a $D_s \in M$ such that $s(D_s) = 1$. Thus, $D_s \supset Y$ and therefore $A \cap B = \bigcup M = D_s \in L$. \square

Theorem 3.3 seems to be independent of the previous results in [3, 4, 7], nevertheless it has corollaries that were obtained using quite a different techniques. (Let us note that a unifying look at these attempts is presented in [8].) The following corollary was obtained (in a more general form) in [4].

Corollary 3.4. *Every countable concrete logic such that every state on it is Jauch–Piron is a Boolean algebra.*

The next corollary of Theorem 3.3 was obtained (in a more general form) in [7].

Corollary 3.5. *Let (X, L) be a concrete logic such that every state on it is Jauch–Piron. Let us suppose that (X, L) contains only countably many maximal Boolean subalgebras and these are complete. Then (X, L) is a Boolean algebra.*

Proof. It is easy to see that for every $A, B \in L$ every covering of $A \cap B$ admits a countable coarsening. \square

4. COVERING PROPERTIES AND ORTHOCOMPLETENESS

Definition 4.1. Let α be a cardinal number. A concrete logic (X, L) is called α -*orthocomplete* if $\bigvee M \in L$ (supremum with respect to inclusion) whenever $M \subset L$ is a set of mutually disjoint elements with $\text{card } M \leq \alpha$.

It is obvious that condition (3σ) of Definition 1.1 implies that a concrete σ -logic is ω_0 -orthocomplete (ω_0 denotes the countable cardinal) — this is usually denoted as σ -*orthocompleteness*.

The following theorem generalizes a result from [5] and answers partially a question posed in [2].

Theorem 4.2. *Every c -orthocomplete (c denotes the cardinality of real numbers) concrete σ -logic with the finite covering property is a Boolean algebra.*

Proof. Let (X, L) be a concrete σ -logic with the finite covering property and let $A, B \in L$. It suffices to prove that $A \cap B \in L$. Let us define by induction finite subsets F_i ($i = 1, 2, \dots$) of L as follows: First, $F_1 \subset L$ is a finite covering of $A \cap B$. Now, let a finite set $F_i = \{A_1, \dots, A_n\} \subset L$ be defined for a natural number $i \geq 1$. Let us denote by G_i the set of all intersections of the form $A_1^{e_1} \cap \dots \cap A_n^{e_n}$, where $(e_1, \dots, e_n) \in \{-1, 1\}^n \setminus \{-1\}^n$ and $A_j^1 = A_j$, $A_j^{-1} = X \setminus A_j$ ($j = 1, \dots, n$). G_i is a finite set of mutually disjoint subsets of X such that $\bigcap F_i = \bigcup G_i$. According to Lemma 2.2, for every $Y \in G_i$ there is a finite covering $G_Y \subset L$ of Y . Let us put $F_{i+1} = \bigcup_{Y \in G_i} G_Y$.

Let us consider all sequences C_1, C_2, \dots such that $C_i \in F_i$ ($i = 1, 2, \dots$) and $C_1 \supset C_2 \supset \dots$. According to Lemma 1.2, $\bigcap_{i=1}^{\infty} C_i \in L$ for each such sequence. Hence, we have obtained at most the continuum of mutually disjoint elements of L such that their union is $A \cap B$. Since their supremum exists, it is equal to $A \cap B$. Thus, $A \cap B \in L$. \square

Before we present a corollary of Theorem 4.2, let us recall a result connecting the covering properties with Jauch–Piron states [3, Theorem 3.5].

Theorem 4.3. *Let (X, L) be a concrete logic such that every two-valued state on it is Jauch–Piron. Then (X, L) has the finite covering property.*

Corollary 4.4. *Every c -orthocomplete concrete σ -logic such that every two-valued state on it is Jauch–Piron is a Boolean algebra.*

Proof. It follows from Theorem 4.3 and Theorem 4.2. □

Remark 4.5. The above corollary can be stated in the following (more general) way: Every c -orthocomplete quantum σ -logic with a closed full set of two-valued Jauch–Piron σ -states is a Boolean algebra. Indeed, concrete σ -logics are exactly representations of quantum σ -logics with a full set of two-valued σ -states (see e.g. [1, 6]) and Theorem 4.3 can be stated for quantum logics with a closed full set of two-valued Jauch–Piron states (the set of two-valued states is closed in the product topology in $[0, 1]^L$).

The following question (posed in [2]) remains open. Here we have given the negative answer in the case that the concrete logic in question is also c -orthocomplete.

Question 4.6. *Is there a concrete σ -logic that is not a Boolean algebra such that every state on it is Jauch–Piron?*

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