Czech Technical University in Prague
Faculty of Electrical Engineering
Department of Control Engineering

bachelor's thesis

Modelling of acoustic plane waves in gases with variable temperature

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BACHELOR PROJECT ASSIGNMENT

Student: **Hanna Chaika**

Study programme: Cybernetics and Robotics
Specialisation: Systems and Control

Title of Bachelor Project: **Modelling of acoustic plane waves in gases with variable temperature**

Guidelines:

1. Study problems concerning description of acoustic waves in fluids with spatially variable temperature, including derivation of model equations. Take into account one-dimensional temperature distribution.
2. Find analytical solutions of linear model equations for given temperature distributions and realize their analysis.
3. Derive model equation describing nonlinear acoustic plane waves for small temperature gradients. Solve this equation numerically by means of a suitable numerical code in the programming language C. Analyze the numerical results.

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Declaration

I declare that I completed the presented thesis independently and that all used sources are quoted in accordance with the Methodological Instructions that cover the ethical principles for writing an academic thesis.

In Prague on ....

Signature

Abstrakt

Bakalářská práce se zabývá modelováním rovinných akustických vln v plynech s proměnnou teplotou. Pro šíření rovinných vln v teplotně nehomogenním prostředí byla odvozena vlnová rovnice s proměnnými koeficienty. Pro vybrané teplotní distribuce jsou v práci prezentována přesná analytická řešení této rovnice. Pomocí nalezených obecných řešení byly vypočteny koeficienty transmise a reflexe. Pro případ vln konečných amplitud (nelineární vlny) šířících se tekutinou s malým teplotním gradientem byla odvozena modifikovaná Burgersova rovnice. Tato rovnice byla řešena numericky v kmitočtové oblasti pomocí Runge-Kuttovy metody 4. řádu v programovacím jazyce C.

Klíčová slova

Vlnová rovnice s proměnnými koeficienty; koeficienty transmise a reflexe; Burgersova rovnice.

Abstract

The thesis presents the modelling of acoustic plane waves in gases with variable temperature. The wave equation with variable coefficients for the propagation of plane waves in a thermally inhomogeneous medium was derived. For chosen temperature distributions the exact analytical solutions of this equation are presented in this thesis. The coefficients of transmission and reflection were calculated using found general solutions. For the case of finite amplitude waves (nonlinear waves), propagating in a fluid with a low temperature gradient, the modified Burgers equation was derived. This equation was solved numerically in the frequency domain using the fourth-order Runge-Kutta method written in the C programming language.

Keywords

Wave equation with variable coefficients; transmission and reflection coefficients; the Burgers equation.

Contents

Abbreviations

Latin capital letters

A	constant in Eqs. (3.19), (3.24) and (3.71)
C_1, C_2	integration constants in Eqs. (3.34), (3.53), (3.55), (3.88) and (3.116)
L	characteristic length
P	acoustic pressure amplitude
R	specific gas constant
T_A, T_B	characteristic temperatures of temperature-homogeneous regions A, B resp.
T_0	absolute temperature in temperature-inhomogenious region
V	acoustic velocity amplitude
W	dimensionless acoustic velocity

Latin lowercase letters

b	constant in Eq. (3.88)
c_0	linear sound speed in temperature-inhomogenious region
c_A	$= \sqrt{\varkappa R T_A}$, sound speed in temperature-homogeneous region A with absolute temperature T_A
c_B	$= \sqrt{\varkappa R T_B}$, sound speed in temperature-homogeneous region B with absolute temperature T_B
k_A	$= \omega / c_A$, wave number at $x = 0$
k_B	$= \omega / c_B$, wave number at $x = L$
p	pressure
p_0	atmospheric pressure
p'	acoustic pressure
q	constant in Eq. (3.44)
r	constant in Eq. (3.17)
t	time
v	acoustic velocity (the particle velocity of the medium)
x	distance

Greek capital letters

Δ	$= c_A^2 q^2 - 4\omega^2$, discriminant of Eq. (3.49)
Ξ	$= T_0 / T_A$, dimensionless absolute temperature
Π	$= p' / \left(\rho_A c_A^2\right)$, dimensionless acoustic pressure
Υ	dimensionless acoustic velocity amplitude
Φ	dimensionless acoustic pressure amplitude

Greek lowercase letters

α	sound diffusivity
β	coefficient in Eq. (5.2a)
γ	constant in Eq. (3.90)
ϵ	a shear viscosity

ζ	a bulk viscosity
η	variable, given by Eq. (3.3)
θ	dimensionless time
κ	thermal conduction coefficient
$\varkappa$	ratio of specific heats
μ	small dimensionless parameter
ν	variable, given by Eq. (3.32)
ξ	$=\sqrt{\omega^2 - r^2}$, coefficient in Eq. (3.27)
ρ	density of the medium
ρ_0	density of the medium before sound propagation
ρ_A, ρ_B	densities of the medium in regions A, B resp.
σ	$= x/L$, dimensionless length
τ	$= t - x/c_A$, retarded time
ϕ	$= 2/\left(\varkappa R \gamma^2\right)$, coefficient in Eq. (3.112)
ω	angular frequency

1 Introduction

The description and analysis of acoustic waves in ducts with a region containing temperature-inhomogeneous fluid represents a significant problem of scientific and practical interest. This interest is induced by the need to understand how temperature fields affect acoustic processes. This would lead to the possibility of more effective design and control systems in which interactions between acoustic and temperature fields occur. This includes, for instance, thermo-acoustic devices and engines, combustors, automotive mufflers, measuring methods of impedance of high-temperature systems, the investigation of thermo-acoustics and combustion instabilities among other possible applications.

This thesis presents modelling of plane acoustic waves in gases with variable temperature. The whole work can be divided into two parts, the first part is devoted to linear model equations, and the second describes nonlinear acoustic plane waves for small temperature gradients.

The analysis consists of seven chapters. Within Chapter 2 the basic linear one-dimensional model equations for fluids in temperature-inhomogeneous regions are derived. Chapter 3 is devoted to the exact analytical solutions of linear model equations for varoius temperature distributions. Chapter 4 deals with an application of the found solutions for calculation of transmission and reflection coefficients. In Chapter 5 is presented a derivation of the Burgers equation for temperature-inhomogeneous fluids. The numerical method for the Burgers equation for small temperature gradients is shown in Chapter 6. Chapter 7 states the conclusion.

2 Derivation of model equations

2.1 Fundamental equations of fluid mechanics

To describe acoustic waves in fluids the following system of equations is considered (see e.g. [1], [2]) :

1. The Navier – Stokes equation (Momentum equation)
   This equation describes the motion of fluid substances and the general form is

$$\rho\left[\frac{\partial \boldsymbol{v}}{\partial t}+\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}\right]=-\boldsymbol{\nabla}p+\left(\zeta+\frac{\epsilon}{3}\right)\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\boldsymbol{v}\right)+\epsilon\boldsymbol{\nabla}^2\mathbf{v}\,, \tag{2.1}$$

   where ρ is the density of the medium, $\boldsymbol{v}$ is the particle velocity of the medium, t is time, p is the pressure, ζ is a bulk viscosity, ϵ is a shear viscosity.

2. Equation of continuity
   This equation describes the transport of a conserved quantity, i.e. fluid.

$$\frac{\partial \rho}{\partial t}+\boldsymbol{\nabla}\cdot(\rho\mathbf{v})=0\,. \tag{2.2}$$

3. Energy equation

$$\rho T\left(\frac{\partial \varepsilon}{\partial t}+\boldsymbol{v}\cdot\boldsymbol{\nabla}\varepsilon\right)=\frac{\epsilon}{2}\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}-\frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right)^2+\zeta\left(\boldsymbol{\nabla}\cdot\boldsymbol{v}\right)^2+\boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}T)\ , \tag{2.3}$$

   where ε is entropy per unit mass, T is absolute temperature, κ is the thermal conduction coefficient, δ_{ik} is the Kronecker delta.

4. Equation of state

$$p=p\left(\rho,\varepsilon\right)\ . \tag{2.4}$$

- If a perfect gas is considered then the Navier – Stokes equation (2.1) is reduced to the form

$$\frac{\partial \mathbf{v}}{\partial t}+\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v}=-\frac{1}{\rho}\boldsymbol{\nabla}p\,, \tag{2.5}$$

  which is called the Euler equation.
- Equation of state (2.4) for a perfect gas is (see e.g. [2])

$$p=R\rho T\,, \tag{2.6}$$

  where R is the specific gas constant, i.e. the (molar) gas constant divided by the molar mass of the gas.

2.2 Derivation of basic model equations for temperature-inhomogeneous region

In order to derive model equations it is necessary to take into consideration four equations from the previous section and make some assumptions, which enable us to neglect nonlinear terms. Assuming a perfect, inviscid and non-heat-conducting gas with one-dimensional temperature distribution, Eqs. (2.2), (2.3), (2.5) and (2.6) can be transformed in the following ways (see e.g. [2]) :

- Energy equation
  In a perfect gas no energy is dissipated, so $\mathrm{d}\varepsilon/\mathrm{d}t = 0$, and hence a specific entropy ε is a constant. From this *an isentropic process* and *an adiabatic gas law* take place.
- Equation of state
  From Eq. (2.4) it follows

$$\mathrm{d}p = \left(\frac{\partial p}{\partial \rho}\right)_{\varepsilon} \mathrm{d}\rho + \left(\frac{\partial p}{\partial \varepsilon}\right)_{\rho} \mathrm{d}\varepsilon \; . \tag{2.7}$$

As the process is *isentropic* then state equation can be written as $p = p(\rho)$ and

$$\mathrm{d}p = \left(\frac{\partial p}{\partial \rho}\right)_{\varepsilon} \mathrm{d}\rho \; . \tag{2.8}$$

As *an adiabatic gas* law takes place then

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\varkappa} , \tag{2.9}$$

where p and p_0 are different specific pressures, ρ and ρ_0 are different specific densities, $\varkappa$ is the ratio of specific heats.

By assuming the gas to be inviscid, it is permissible to express the sound speed c

$$c^2 = \frac{\mathrm{d}p}{\mathrm{d}\rho} \; . \tag{2.10}$$

Taking into account Eqs. (2.9) and (2.6), the equation above can be written as

$$c^2 = \frac{\varkappa p}{\rho} = \varkappa R T \; . \tag{2.11}$$

Equation (2.10) lets us take into account the following equality

$$\frac{\mathrm{d}p}{\mathrm{d}t} = c^2 \frac{\mathrm{d}\rho}{\mathrm{d}t} \; . \tag{2.12}$$

The total derivative is

$$\frac{\mathrm{d}(\cdot)}{\mathrm{d}t} = \frac{\partial(\cdot)}{\partial t} + \upsilon \frac{\partial(\cdot)}{\partial x} \; . \tag{2.13}$$

Then Eq. (2.12) can be written as

$$\frac{\partial p}{\partial t} + \upsilon \frac{\partial p}{\partial x} = c^2 \left(\frac{\partial \rho}{\partial t} + \upsilon \frac{\partial \rho}{\partial x}\right) \; . \tag{2.14}$$

- Equation of continuity
  One-dimensional equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \; . \tag{2.15}$$

  From this equation the equality $v\partial\rho/\partial x = -\partial\rho/\partial t - \rho\partial v/\partial x$ can be obtained. After substitution into Eq. (2.14) the resulting equation is

$$\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x} = -c^2\rho\frac{\partial v}{\partial x} \; . \tag{2.16}$$

  From Eq. (2.11) the above equation can be written as

$$\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x} + \varkappa p\frac{\partial v}{\partial x} = 0 \; . \tag{2.17}$$

- Euler equation
  By assuming one-dimensional temperature distribution, Eq. (2.5) takes the form of the one-dimensional Euler equation

$$\rho\frac{\partial v}{\partial t} + \rho v\frac{\partial v}{\partial x} + \frac{\partial p}{\partial x} = 0 \; . \tag{2.18}$$

Each of the dependent variables can be expressed in the following way (see e.g. [3])

1. the density $\rho(x,t) = \rho_0(x) + \rho'(x,t)$ is the sum of the ambient density ρ_0 and an acoustic density ρ',
2. the velocity $v = v(x,t)$ is the particle velocity of the medium, resp. acoustic velocity,
3. the pressure $p(x,t) = p_0 + p'(x,t)$ is the sum of the atmospheric pressure p_0, which is supposed to be constant (see e.g. [3]) and the acoustic pressure p'.

The assumptions can be taken into account $|p'|/p_0 \sim |v|/c_0 \sim \rho'/\rho \sim \mu \ll 1$, where c_0 is the linear sound speed, $\mu < 0$ is a small dimensionless parameter. It is considered that $c_0 = c_0(x)$ and temperature $T_0 = T_0(x)$ are dependent on the distance x, where T_0 is the ambient temperature of the fluid.

The substitution of the above dependent variables into the equation of continuity (2.17)

$$\frac{\partial(p_0 + p')}{\partial t} + v\frac{\partial(p_0 + p')}{\partial x} + \varkappa\,(p_0 + p')\,\frac{\partial v}{\partial x} = 0 \tag{2.19}$$

and linearization of Eq. (2.19) leads us to the following linear form of the equation of continuity

$$\frac{\partial p'}{\partial t} + \varkappa p_0\frac{\partial v}{\partial x} = 0 \; . \tag{2.20}$$

The same substitution into the Euler equation (2.18)

$$(\rho_0 + \rho')\,\frac{\partial v}{\partial t} + (\rho_0 + \rho')\,v\frac{\partial v}{\partial x} + \frac{\partial(p_0 + p')}{\partial x} = 0 \tag{2.21}$$

and linearization of Eq. (2.21) leads us to the linear form of the Euler equation

$$\rho_0\frac{\partial v}{\partial t} + \frac{\partial p'}{\partial x} = 0 \; . \tag{2.22}$$

Differentiating of Eq. (2.20) with respect to time t and Eq. (2.22) with respect to coordinate x and eliminating the cross-derivative term by their combination yields

$$\frac{\partial^2 p'}{\partial x^2} + \frac{\mathrm{d}\rho_0}{\mathrm{d}x}\frac{\partial v}{\partial t} - \frac{\rho_0}{\varkappa p_0}\frac{\partial^2 p'}{\partial t^2} = 0 \,. \tag{2.23}$$

Expressing the derivative $\partial v/\partial t$ from Eq. (2.22) and substituting it into Eq. (2.23) obtains the equation

$$\frac{\partial^2 p'}{\partial x^2} - \frac{\mathrm{d}\rho_0}{\mathrm{d}x}\frac{1}{\rho_0}\frac{\partial p'}{\partial x} - \frac{\rho_0}{\varkappa p_0}\frac{\partial^2 p'}{\partial t^2} = 0 \,. \tag{2.24}$$

From the perfect gas law $p_0 = R\rho_0 T_0$ the total derivative of p_0 with respect to x is

$$0 = \frac{\mathrm{d}p_0}{\mathrm{d}x} = R\frac{\mathrm{d}\rho_0}{\mathrm{d}x}T_0 + R\rho_0\frac{\mathrm{d}T_0}{\mathrm{d}x} \,. \tag{2.25}$$

After multiplying the above equation by $1/(R\rho_0 T_0)$ the resulting expression is

$$\frac{1}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}x} + \frac{1}{T_0}\frac{\mathrm{d}T_0}{\mathrm{d}x} = 0 \,. \tag{2.26}$$

From Eq. (2.11) follows

$$c_0^2 = \frac{\varkappa p_0}{\rho_0} \,, \tag{2.27}$$

and using Eq. (2.26) hence is obtained the wave equation with variable coefficients (see e.g. [3])

$$\frac{\partial^2 p'}{\partial x^2} + \frac{1}{T_0}\frac{\mathrm{d}T_0}{\mathrm{d}x}\frac{\partial p'}{\partial x} - \frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} = 0 \,. \tag{2.28}$$

Assuming a time periodic source of sound, where ω is an angular frequency, it is possible to consider

$$p'(x,t) = P(x)e^{-\mathrm{j}\omega t} \,. \tag{2.29}$$

Equation (2.28) can be written now as

$$\frac{\mathrm{d}^2 P}{\mathrm{d}x^2} + \frac{1}{T_0}\frac{\mathrm{d}T_0}{\mathrm{d}x}\frac{\mathrm{d}P}{\mathrm{d}x} + \frac{\omega^2}{c_0^2}P = 0 \,. \tag{2.30}$$

Now, let us differentiate vice versa - Eq. (2.20) with respect to x and Eq. (2.22) with respect to t, then the cross-derivative term can be eliminated. This manipulation leads to the wave equation

$$\frac{\partial^2 v}{\partial t^2} - c_0^2\frac{\partial^2 v}{\partial x^2} = 0 \,. \tag{2.31}$$

Substituting

$$v(x,t) = V(x)e^{-\mathrm{j}\omega t} \,, \tag{2.32}$$

into Eq. (2.31) obtains the equation

$$\frac{\mathrm{d}^2 V}{\mathrm{d}x^2} + \frac{\omega^2}{c_0^2(x)}V(x) = 0 \,. \tag{2.33}$$

This is the Helmholtz equation.

Let us rewrite the model equations (2.28) and (2.31) in dimensionless form (see e.g. [4])

$$\frac{\partial^2 \Pi}{\partial \sigma^2} + \frac{1}{\Xi(\sigma)} \frac{\mathrm{d}\Xi(\sigma)}{\mathrm{d}\sigma} \frac{\partial \Pi}{\partial \sigma} - \frac{1}{C_0^2(\sigma)} \frac{\partial^2 \Pi}{\partial \theta^2} = 0 \,, \tag{2.34}$$

$$\frac{\partial^2 W}{\partial \theta^2} - C_0^2(\sigma) \frac{\partial^2 W}{\partial \sigma^2} = 0 \,. \tag{2.35}$$

Here

$$\begin{aligned} \Pi &= \frac{p'}{\rho_A c_A^2} = \frac{p'}{\varkappa p_0} \,, & \sigma &= \frac{x}{L} \,, \quad \theta = \omega t \,, \\ C_0^2(\sigma) &= \frac{c_0^2(\sigma)}{\omega^2 L^2} = \frac{\varkappa R T_0(\sigma)}{\omega^2 L^2} = \frac{\Xi(\sigma)}{h^2} \,, & \Xi &= \frac{T_0}{T_A} \,, \quad W = \frac{v}{c_A} \,, \end{aligned} \tag{2.36}$$

where T_A is a characteristic temperature, $c_A = c_0(T_A) = \sqrt{\varkappa R T_A}$, $\rho_A = \rho_0(T_A) = p_0/(RT_A)$, L is a characteristic length and $h = \omega L / c_A$.

Assuming that the solutions of Eqs. (2.34) and (2.35) have a periodic time dependence, i. e.

$$\Pi(\sigma, \theta) = \Phi(\sigma) e^{-\mathrm{j}\theta} \tag{2.37}$$

and

$$W(\sigma, \theta) = \Upsilon(\sigma) e^{-\mathrm{j}\theta} \,, \tag{2.38}$$

then equations can be written as

$$\frac{\mathrm{d}^2 \Phi}{\mathrm{d}\sigma^2} + \frac{1}{\Xi(\sigma)} \frac{\mathrm{d}\Xi(\sigma)}{\mathrm{d}\sigma} \frac{\mathrm{d}\Phi}{\mathrm{d}\sigma} + \frac{1}{C_0^2(\sigma)} \Phi(\sigma) = 0 \,, \tag{2.39}$$

$$\frac{\mathrm{d}^2 \Upsilon}{\mathrm{d}\sigma^2} + \frac{\Upsilon(\sigma)}{C_0^2(\sigma)} = 0 \,. \tag{2.40}$$

3 Exact analytical solutions for various temperature functions

This chapter is devoted to exact analytical solutions. These solutions can be divided into two cases. The first case represents solving equations where the temperature distribution $T_0(x)$ is unknown, so it is necessary to find both the temperature distribution $T_0(x)$ and the solution of the equation. The second case deals with equations where it is supposed the function $T_0(x)$ must be known. For this reason only the equation solution is to be found in this case. Linear and exponential temperature distributions are assumed.

3.1 Transformation of derivatives

Suppose that a function $f(x)$ is known then can be derived the following spatial derivatives

$$\frac{\mathrm{d}(\cdot)}{\mathrm{d}x} = \frac{\mathrm{d}(\cdot)}{\mathrm{d}f}\frac{\mathrm{d}f}{\mathrm{d}x} \,, \tag{3.1}$$

and

$$\begin{aligned}\frac{\mathrm{d}^2(\cdot)}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}(\cdot)}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}(\cdot)}{\mathrm{d}f}\frac{\mathrm{d}f}{\mathrm{d}x}\right) = \left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}(\cdot)}{\mathrm{d}f}\right)\right]\frac{\mathrm{d}f}{\mathrm{d}x} + \left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)\right]\frac{\mathrm{d}(\cdot)}{\mathrm{d}f} \\ &= \left[\frac{\mathrm{d}}{\mathrm{d}f}\left(\frac{\mathrm{d}(\cdot)}{\mathrm{d}x}\right)\right]\frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\frac{\mathrm{d}(\cdot)}{\mathrm{d}f} = \left[\frac{\mathrm{d}}{\mathrm{d}f}\left(\frac{\mathrm{d}(\cdot)}{\mathrm{d}f}\frac{\mathrm{d}f}{\mathrm{d}x}\right)\right]\frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\frac{\mathrm{d}(\cdot)}{\mathrm{d}f} \\ &= \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2\frac{\mathrm{d}^2(\cdot)}{\mathrm{d}f^2} + \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\frac{\mathrm{d}(\cdot)}{\mathrm{d}f} \,.\end{aligned} \tag{3.2}$$

Taking into account a new variable (see e.g. [5])

$$\eta = \int_0^x \frac{1}{c_0(x_1)}\,\mathrm{d}x_1 \,, \tag{3.3}$$

then

$$\frac{\mathrm{d}\eta}{\mathrm{d}x} = \frac{1}{c_0(x)} \,, \tag{3.4}$$

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{c_0(x)}\right) \,. \tag{3.5}$$

Putting $\eta(x) \equiv f(x)$ can be used the transformation relations (3.1) and (3.2) as follows

$$\frac{\mathrm{d}(\cdot)}{\mathrm{d}x} = \frac{\mathrm{d}(\cdot)}{\mathrm{d}\eta}\frac{\mathrm{d}\eta}{\mathrm{d}x} \,, \tag{3.6}$$

$$\frac{\mathrm{d}^2(\cdot)}{\mathrm{d}x^2} = \left(\frac{\mathrm{d}\eta}{\mathrm{d}x}\right)^2\frac{\mathrm{d}^2(\cdot)}{\mathrm{d}\eta^2} + \frac{\mathrm{d}^2\eta}{\mathrm{d}x^2}\frac{\mathrm{d}(\cdot)}{\mathrm{d}\eta} = \frac{1}{c_0^2(x)}\frac{\mathrm{d}^2(\cdot)}{\mathrm{d}\eta^2} + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{c_0(x)}\right)\frac{\mathrm{d}(\cdot)}{\mathrm{d}\eta} \,. \tag{3.7}$$

3.2 Analytical solutions for unknown temperature functions

It is assumed that a temperature distribution $T_0(x)$ in a temperature-inhomogeneous region of the length L in a waveguide separates two regions of different constant temperatures T_A and T_B, see Fig. 1 (see e.g. [4]) .

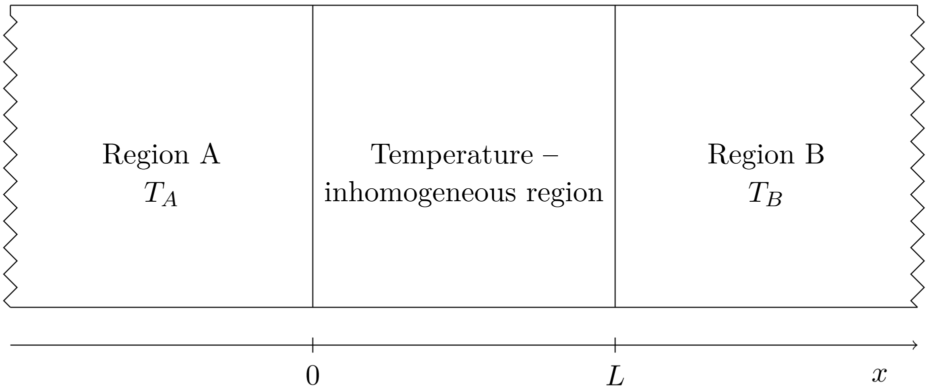

Fig. 1 Temperature regions in a waveguide.

In dimensionless equations the temperature-inhomogeneous region has the length equal to 1 and separates two temperature-homogeneous regions with constant dimensionless temperatures $\Xi_A = 1$ and $\Xi_B = T_B/T_A$.

3.2.1 First method of finding analytical solution

To solve the Helmholtz equation (2.33) the first step is to introduce a new function $F(x)$ (see e.g. [6])

$$V(x) = \sqrt{c_0(x)}F(x) \ . \tag{3.8}$$

Its derivatives are

$$\frac{\mathrm{d}(\sqrt{c_0}F)}{\mathrm{d}x} = F\frac{\mathrm{d}\sqrt{c_0}}{\mathrm{d}x} + \sqrt{c_0}\frac{\mathrm{d}F}{\mathrm{d}x} \ , \tag{3.9}$$

$$\frac{\mathrm{d}^2(\sqrt{c_0}F)}{\mathrm{d}x^2} = \frac{\mathrm{d}F}{\mathrm{d}x}\frac{\mathrm{d}\sqrt{c_0}}{\mathrm{d}x} + F\frac{\mathrm{d}^2\sqrt{c_0}}{\mathrm{d}x^2} + \frac{\mathrm{d}\sqrt{c_0}}{\mathrm{d}x}\frac{\mathrm{d}F}{\mathrm{d}x} + \sqrt{c_0}\frac{\mathrm{d}^2F}{\mathrm{d}x^2} \ , \tag{3.10}$$

where

$$\frac{\mathrm{d}\sqrt{c_0}}{\mathrm{d}x} = \frac{1}{2\sqrt{c_0}}\frac{\mathrm{d}c_0}{\mathrm{d}x} \ , \tag{3.11}$$

$$\frac{\mathrm{d}^2\sqrt{c_0}}{\mathrm{d}x^2} = -\frac{1}{4c_0\sqrt{c_0}}\left(\frac{\mathrm{d}c_0}{\mathrm{d}x}\right)^2 + \frac{1}{2\sqrt{c_0}}\frac{\mathrm{d}^2c_0}{\mathrm{d}x^2} \ . \tag{3.12}$$

Substituting into Eq. (2.33)

$$-\frac{F}{4c_0^2}\left(\frac{\mathrm{d}c_0}{\mathrm{d}x}\right)^2 + \frac{F}{2c_0}\frac{\mathrm{d}^2c_0}{\mathrm{d}x^2} + \frac{1}{2c_0}\frac{\mathrm{d}c_0}{\mathrm{d}x}\frac{\mathrm{d}F}{\mathrm{d}x} + \frac{1}{2c_0}\frac{\mathrm{d}c_0}{\mathrm{d}x}\frac{\mathrm{d}F}{\mathrm{d}x} + \frac{\mathrm{d}^2F}{\mathrm{d}x^2} + \frac{\omega^2}{c_0^2}F = 0 \ . \tag{3.13}$$

And rearranging leads to the equation

$$\frac{\mathrm{d}^2F}{\mathrm{d}x^2} + \frac{1}{c_0}\frac{\mathrm{d}c_0}{\mathrm{d}x}\frac{\mathrm{d}F}{\mathrm{d}x} - \frac{1}{4c_0^2}\left(\frac{\mathrm{d}c_0}{\mathrm{d}x}\right)^2 F + \frac{1}{2c_0}\frac{\mathrm{d}^2c_0}{\mathrm{d}x^2}F + \frac{\omega^2}{c_0^2}F = 0 \ . \tag{3.14}$$

Imposing the variable η from Eq. (3.3) then it is possible to write Eq. (3.14) in the following form

$$\frac{1}{c_0^2}\frac{\mathrm{d}^2F}{\mathrm{d}\eta^2}+\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{c_0}\right)\frac{\mathrm{d}F}{\mathrm{d}\eta}+\frac{1}{c_0^2}\frac{\mathrm{d}c_0}{\mathrm{d}x}\frac{\mathrm{d}F}{\mathrm{d}\eta}-\frac{1}{4c_0^2}\left(\frac{\mathrm{d}c_0}{\mathrm{d}x}\right)^2F+\frac{1}{2c_0}\frac{\mathrm{d}^2c_0}{\mathrm{d}x^2}F+\frac{\omega^2}{c_0^2}F=0\ . \tag{3.15}$$

After modification Eq. (3.15) takes the form

$$\frac{\mathrm{d}^2F}{\mathrm{d}\eta^2}+\omega^2F=F\left[\frac{1}{4}\left(\frac{\mathrm{d}c_0}{\mathrm{d}x}\right)^2-\frac{c_0}{2}\frac{\mathrm{d}^2c_0}{\mathrm{d}x^2}\right]\ . \tag{3.16}$$

Equation (3.16) is an ordinary differential equation with varible coefficients. In order to get an ordinary differential equation with constant coefficients, it is necessary to equalise the expression in the square brackets using a constant r^2. After this step is obtained

$$\frac{\mathrm{d}^2F}{\mathrm{d}\eta^2}+\omega^2F=r^2F \tag{3.17}$$

and

$$r^2=\frac{1}{4}\left(\frac{\mathrm{d}c_0}{\mathrm{d}x}\right)^2-\frac{c_0}{2}\frac{\mathrm{d}^2c_0}{\mathrm{d}x^2}\ . \tag{3.18}$$

In solving the differential equation (3.18) its solution can be written as

$$c_0(x)=\frac{1}{4}\frac{\left(A^2-4r^2\right)x^2}{B}+Ax+B\ , \tag{3.19}$$

where A and B are integration constants.

After some small algebraic manipulation the relation (3.19) can be expressed in the convenient form

$$c_0(x)=c_A\left[\left(A^2-\frac{r^2}{c_A^2}\right)x^2+2Ax+1\right]\ . \tag{3.20}$$

Now, let us remember some relations with characteristic temperature T_A. From previous chapter

$$c_0(T_A)=c_A\ ,\quad \rho_0(T_A)=\rho_A\ . \tag{3.21}$$

From Eqs. (2.11) and (2.27) it follows that

$$c_0(x)=\sqrt{\varkappa RT_0(x)}\ . \tag{3.22}$$

As Eq. (3.22) takes place then with the help of Eq. (3.20) it is possible to write a temperature relation

$$T_0(x)=\frac{c_A^2}{\varkappa R}\left[\left(A^2-\frac{r^2}{c_A^2}\right)x^2+2Ax+1\right]^2\ . \tag{3.23}$$

According to Eqs. (2.36) and (3.23)

$$T_0(x)=T_A\left[\left(A^2-\frac{r^2}{c_A^2}\right)x^2+2Ax+1\right]^2\ , \tag{3.24}$$

where $T_A=c_A^2/(\varkappa R)$.

Using expressions (2.36) temperature-inhomogeneous region T_0 can be expressed in dimensionless form (see e.g. [4])

$$\Xi(\sigma) = \frac{T_0(\sigma)}{T_A} = \left[\left(A^2 - \frac{r^2}{c_A^2} \right) L^2\sigma^2 + 2AL\sigma + 1 \right]^2 . \tag{3.25}$$

Let characteristic length L be equal to 1 m. By choosing different values of constants A, r and velocity c_A it is possible to see exemplary solutions for temperature Ξ among many instances.

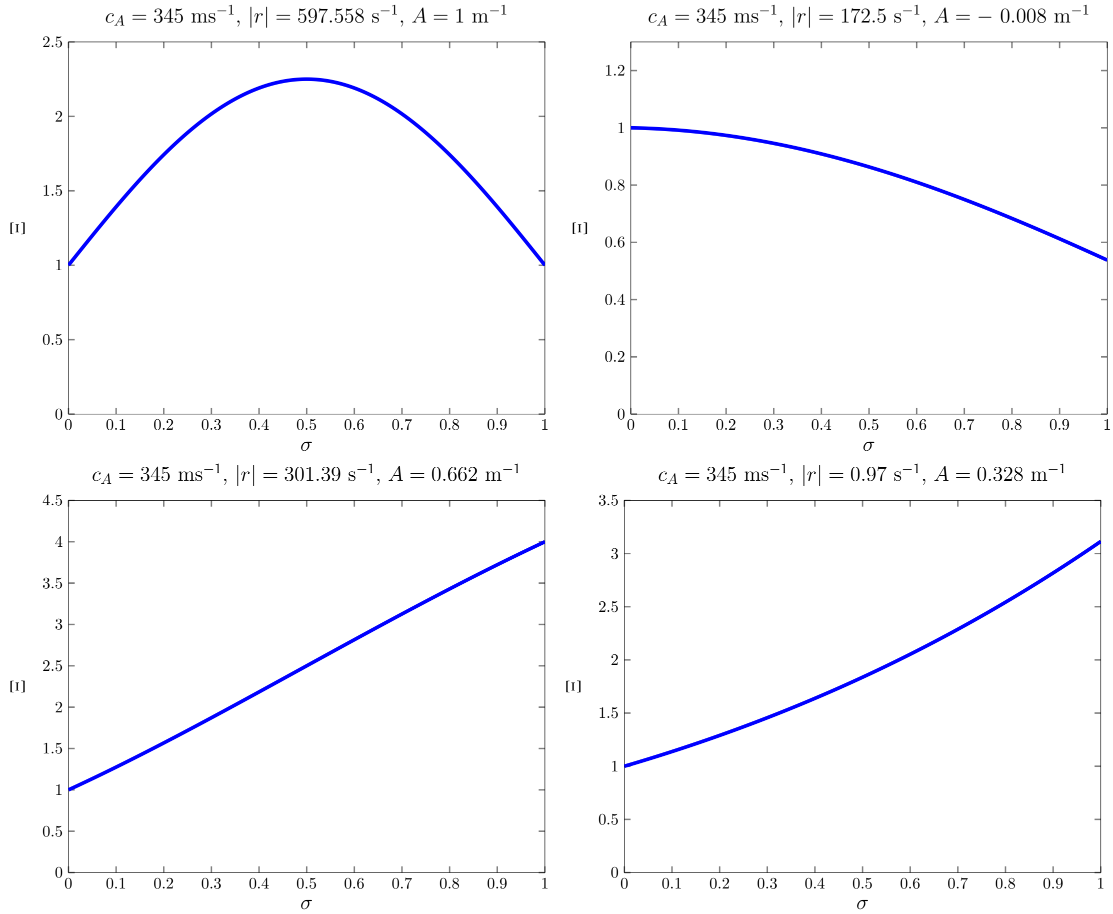

Fig. 2 Dimensionless temperature function $\Xi(\sigma)$ for different coefficients of Eq. (3.25).

To solve Eq. (3.17) let us use the following substitution

$$\xi^2 = \omega^2 - r^2 , \tag{3.26}$$

where $\omega^2 - r^2 \geqslant 0$, because an evanescent wave is not considered. After this substitution Eq. (3.17) can be written as

$$\frac{\mathrm{d}^2 F}{\mathrm{d}\eta^2} + \xi^2 F = 0 . \tag{3.27}$$

The above equation is a second-order linear ordinary differential equation, where the discriminant is less than zero. Then the roots of the characteristic equation of Eq. (3.27) are complex

$$\lambda_{1,2} = \pm \mathrm{j}\xi . \tag{3.28}$$

The solution of Eq. (3.27) is then

$$F(\eta) = C_1 \cos(\xi\eta) + C_2 \sin(\xi\eta) \; , \tag{3.29}$$

where C_1 and C_2 are constants.

According to Eqs. (3.3) and (3.20) the variable η can be written as

$$\eta = \int_0^x \frac{1}{c_0(x_1)} \mathrm{d}x_1 = \int_0^x \frac{1}{c_A\left[\left(A^2 - \frac{r^2}{c_A^2}\right) x_1^2 + 2Ax_1 + 1\right]} \mathrm{d}x_1 \; . \tag{3.30}$$

After integration the variable η can be written as

$$\eta = -\frac{1}{|r|} \tanh^{-1} \left[\frac{c_A}{|r|} \left(A + \left(A^2 - \frac{r^2}{c_A^2} \right) x_1 \right) \right] \Bigg|_0^x = -\frac{1}{|r|} \nu(x) \; , \tag{3.31}$$

where

$$\nu(x) = \tanh^{-1} \left[\frac{c_A}{|r|} \left(A + \left(A^2 - \frac{r^2}{c_A^2} \right) x \right) \right] - \tanh^{-1} \left(\frac{c_A}{|r|} A \right) \; . \tag{3.32}$$

The absolute value of c_A was omitted, because T_A is an indoor temperature when $c_A > 0$.

Now the function $F(x)$ can be written as

$$F(x) = C_1 \cos\left(\frac{\xi}{|r|} \nu(x) \right) - C_2 \sin\left(\frac{\xi}{|r|} \nu(x) \right) \; . \tag{3.33}$$

Taking into account Eq. (3.8), the solution of the Helmholtz equation (2.33) is

$$\begin{aligned} V(x) = \sqrt{c_0(x)} F(x) = &\sqrt{c_A \left[\left(A^2 - \frac{r^2}{c_A^2} \right) x^2 + 2Ax + 1 \right]} \\ &\times \left[C_1 \cos\left(\frac{\xi}{|r|} \nu(x) \right) - C_2 \sin\left(\frac{\xi}{|r|} \nu(x) \right) \right] \; . \end{aligned} \tag{3.34}$$

The dimensionless form of the velocity is (see e.g. [4])

$$\begin{aligned} \Upsilon(\sigma) = \frac{V(\sigma)}{c_A} = &\sqrt{\frac{1}{c_A} \left[\left(A^2 - \frac{r^2}{c_A^2} \right) L^2\sigma^2 + 2AL\sigma + 1 \right]} \\ &\times \left[C_1 \cos\left(\frac{\xi}{|r|} \nu(\sigma) \right) - C_2 \sin\left(\frac{\xi}{|r|} \nu(\sigma) \right) \right] \; , \end{aligned} \tag{3.35}$$

where

$$\nu(\sigma) = \tanh^{-1} \left[\frac{c_A}{|r|} \left(A + \left(A^2 - \frac{r^2}{c_A^2} \right) L\sigma \right) \right] - \tanh^{-1} \left(\frac{c_A}{|r|} A \right) \; . \tag{3.36}$$

3.2.2 Second method of finding an analytical solution

Now let us solve Eq. (2.30). It can be rewritten once more

$$\frac{\mathrm{d}^2 P}{\mathrm{d}x^2} + \frac{1}{T_0} \frac{\mathrm{d}T_0}{\mathrm{d}x} \frac{\mathrm{d}P}{\mathrm{d}x} + \frac{\omega^2}{c_0^2} P = 0 \; . \tag{3.37}$$

Using Eqs. (3.1)–(3.7) the above equation can be written as

$$\frac{1}{c_0^2}\frac{\mathrm{d}^2P}{\mathrm{d}\eta^2} + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{c_0}\right)\frac{\mathrm{d}P}{\mathrm{d}\eta} + \frac{1}{T_0}\frac{\mathrm{d}T_0}{\mathrm{d}x}\frac{1}{c_0}\frac{\mathrm{d}P}{\mathrm{d}\eta} + \frac{\omega^2}{c_0^2}P = 0\ . \tag{3.38}$$

From Eq. (3.22) can be derived

$$\frac{\mathrm{d}T_0}{\mathrm{d}x} = \frac{1}{\varkappa R}\frac{\mathrm{d}c_0^2}{\mathrm{d}x}\ . \tag{3.39}$$

Multiplying Eq. (3.39) by $1/T_0$ leads to

$$\frac{1}{T_0}\frac{\mathrm{d}T_0}{\mathrm{d}x} = \frac{2c_0}{c_0^2}\frac{\mathrm{d}c_0}{\mathrm{d}x}\ . \tag{3.40}$$

So, Eq. (3.37) can be modified as follows

$$\frac{1}{c_0^2}\frac{\mathrm{d}^2P}{\mathrm{d}\eta^2} - \frac{1}{c_0^2}\frac{\mathrm{d}c_0}{\mathrm{d}x}\frac{\mathrm{d}P}{\mathrm{d}\eta} + \frac{2c_0}{c_0^3}\frac{\mathrm{d}c_0}{\mathrm{d}x}\frac{\mathrm{d}P}{\mathrm{d}\eta} + \frac{\omega^2}{c_0^2}P = 0\ , \tag{3.41}$$

and after rearranging and reducing it becomes

$$\frac{\mathrm{d}^2P}{\mathrm{d}\eta^2} + \frac{\mathrm{d}c_0}{\mathrm{d}x}\frac{\mathrm{d}P}{\mathrm{d}\eta} + \omega^2P = 0\ . \tag{3.42}$$

In order to solve Eq. (3.42) as an ordinary differential equation with constant coefficients, it is necessary take into account that $\mathrm{d}c_0/\mathrm{d}x = const.$ Let this constant be q_1. Thus, c_0 has the form

$$c_0(x) = q_1x + q_2\ , \tag{3.43}$$

where q_1 and q_2 are integration constants and $q_2 \geq 0$.

By choosing $q_2 = c_A$ and denoting $q1/q2 = q$ the form is

$$c_0(x) = c_A(qx + 1)\ . \tag{3.44}$$

According to Eq. (3.22), function $T_0(x)$ is equal to

$$T_0(x) = T_A\left(qx + 1\right)^2\ , \tag{3.45}$$

where

$$T_A = \frac{c_A^2}{\varkappa R}\ . \tag{3.46}$$

Using expressions (2.36) the dimensionless form of temperature-inhomogeneous region T_0 (see e.g. [4])

$$\Xi(\sigma) = \frac{T_0(\sigma)}{T_A} = \left(qL\sigma + 1\right)^2\ . \tag{3.47}$$

By choosing different values of constant q and letting $L = 1$ m, it is possible to see exemplary solutions for temperature Ξ among many instances.

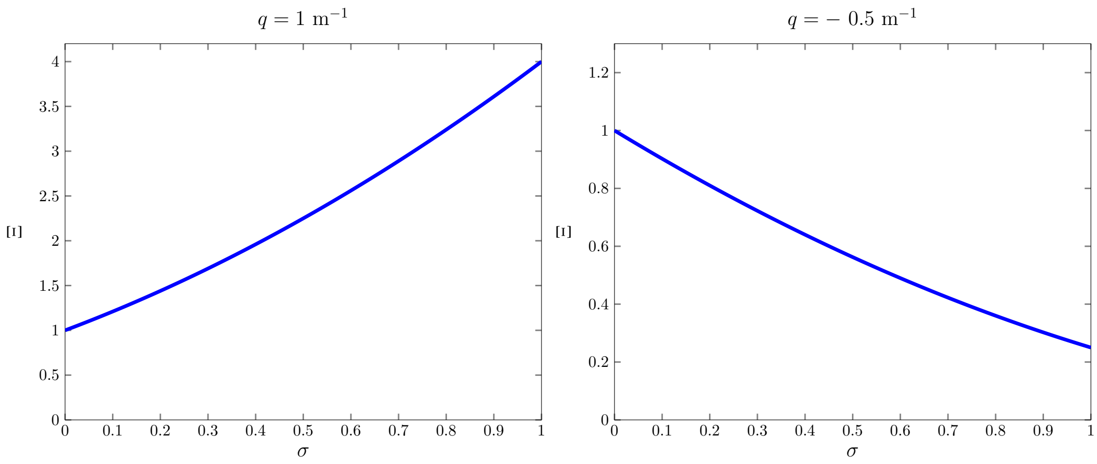

Fig. 3 Dimensionless temperature function $\Xi(\sigma)$ for different coefficients of Eq. (3.47).

To solve Eq. (3.42) let us substitute the expression of sound velocity (3.44). Then Eq. (3.42) takes the form

$$\frac{\mathrm{d}^2 P}{\mathrm{d}\eta^2} + c_A q \frac{\mathrm{d}P}{\mathrm{d}\eta} + \omega^2 P = 0\ . \tag{3.48}$$

The solution of the above equation can be found through the characteristic equation of an appropriate equation.

$$\lambda^2 + c_A q \lambda + \omega^2 = 0\ . \tag{3.49}$$

The discriminant of Eq. (3.49) is

$$\Delta = c_A^2 q^2 - 4\omega^2\ . \tag{3.50}$$

There are different possible solutions according to the sign of the discriminant. The solutions are represented below in the table, where C_1 and C_2 are constants.

$\Delta > 0$	$\Delta = 0$	$\Delta < 0$
$\lambda_{1,2} = \frac{-c_A q \pm \sqrt{\Delta}}{2}$	$\lambda_1 = \lambda_2 = -\frac{1}{2} c_A q$	$\lambda_{1,2} = \frac{-c_A q \pm \mathrm{i}\sqrt{\Delta}}{2}$
$P(\eta) = C_1 e^{\left(-\frac{1}{2}c_A q + \frac{1}{2}\sqrt{\Delta}\right)\eta} + C_2 e^{\left(-\frac{1}{2}c_A q - \frac{1}{2}\sqrt{\Delta}\right)\eta}$	$P(\eta) = e^{-\frac{1}{2}c_A q \eta} \times [C_1 + C_2\eta]$	$P(\eta) = e^{-\frac{1}{2}c_A q \eta}\left[C_1 \cos\left(\frac{1}{2}\sqrt{\Delta}\eta\right) + C_2 \sin\left(\frac{1}{2}\sqrt{\Delta}\eta\right)\right]$

Tab. 1 The solution of Eq. (3.48).

According to Eqs. (3.3) and (3.44) the variable η can be written as

$$\eta = \int_0^x \frac{1}{c_0(x_1)}\, \mathrm{d}x_1 = \int_0^x \frac{1}{c_A(q x_1 + 1)}\, \mathrm{d}x_1\ . \tag{3.51}$$

Notice that x_1 cannot be equal to $-1/q$.
After integration the variable η can be written as

$$\eta = \left.\frac{\ln(q x_1 + 1)}{c_A q}\right|_0^x = \frac{1}{c_A q} \ln(q x + 1)\ . \tag{3.52}$$

Now it is possible to write the solution of Eq. (2.30) according to the discriminant of Eq. (3.49)

- If $\Delta > 0$, an evanescent wave is not considered.
- If $\Delta = 0$,

$$P(x) = \frac{1}{\sqrt{qx+1}} \left[C_1 + C_2 \ln(qx+1)\right] \ , \tag{3.53}$$

where C_1 and C_2 are integration constants. According to Eqs. (2.36) and (3.21) the dimensionless form of the pressure is (see e.g. [4])

$$\Phi(\sigma) = \frac{1}{\sqrt{qL\sigma+1}} \left[C_1 + C_2 \ln(qL\sigma+1)\right] \ , \tag{3.54}$$

where C_1 and C_2 are new constants.
- If $\Delta < 0$,

$$P(x) = \frac{1}{\sqrt{qx+1}} \left[C_1 \cos\left(\sqrt{\frac{1}{4} - \frac{\omega^2}{c_A^2 q^2}} \ln(qx+1)\right) \right. \\ \left. + C_2 \sin\left(\sqrt{\frac{1}{4} - \frac{\omega^2}{c_A^2 q^2}} \ln(qx+1)\right)\right] \ , \tag{3.55}$$

where C_1 and C_2 are integration constants. The dimensionless form of this solution is (see e.g. [4])

$$\Phi(\sigma) = \frac{1}{\sqrt{qL\sigma+1}} \left[C_1 \cos\left(\sqrt{\frac{1}{4} - \frac{\omega^2}{c_A^2 q^2}} \ln\left(qL\sigma+1\right)\right) \right. \\ \left. + C_2 \sin\left(\sqrt{\frac{1}{4} - \frac{\omega^2}{c_A^2 q^2}} \ln\left(qL\sigma+1\right)\right)\right] \ , \tag{3.56}$$

where C_1 and C_2 are new constants.

3.2.3 Third method of finding an analytical solution

There is one more possible way to solve Eq. (2.28). Let us start with multiplying this equation by c_0^2

$$\frac{\partial^2 p'}{\partial t^2} - \frac{c_0^2}{T_0} \frac{\mathrm{d}T_0}{\mathrm{d}x} \frac{\partial p'}{\partial x} - c_0^2 \frac{\partial^2 p'}{\partial x^2} = 0 \ . \tag{3.57}$$

With reference to the following expression

$$\frac{c_0^2}{T_0} \frac{\partial}{\partial x} \left(T_0 \frac{\partial p'}{\partial x}\right) = \frac{c_0^2}{T_0} \frac{\mathrm{d}T_0}{\mathrm{d}x} \frac{\partial p'}{\partial x} + c_0^2 \frac{\partial^2 p'}{\partial x^2} \ , \tag{3.58}$$

Eq. (3.57) can be written as (see e.g. [6])

$$\frac{\partial^2 p'}{\partial t^2} - \frac{c_0^2}{T_0} \frac{\partial}{\partial x} \left(T_0 \frac{\partial p'}{\partial x}\right) = 0 \ . \tag{3.59}$$

Now let us substitute a new function $F(x,t)$ instead of the pressure

$$p'(x,t) = F(x,t)/\sqrt{T_0(x)} \ . \tag{3.60}$$

Equation (3.59) takes the form

$$\frac{\partial^2 (F/\sqrt{T_0})}{\partial t^2} - \frac{c_0^2}{T_0} \frac{\partial}{\partial x} \left(T_0 \frac{\partial (F/\sqrt{T_0})}{\partial x}\right) = 0 \ . \tag{3.61}$$

Applying all simplifications, the above equation takes the form

$$\frac{\partial^2 F}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 F}{\partial t^2} = \frac{1}{\sqrt{T_0}}\frac{\mathrm{d}^2\sqrt{T_0}}{\mathrm{d}x^2}F \, . \tag{3.62}$$

Let us remove from Eq. (3.62) derivatives dependent on t. The function F can be represented as $F = F(x,t) = \hat{F}(x)e^{-\mathrm{j}\omega t}$ for this purpose. After substitution and rearranging the resulting equation is

$$\frac{\mathrm{d}^2\hat{F}}{\mathrm{d}x^2} = \left(\frac{1}{\sqrt{T_0}}\frac{\mathrm{d}^2\sqrt{T_0}}{\mathrm{d}x^2} - \frac{\omega^2}{c_0^2}\right)\hat{F} \, . \tag{3.63}$$

Concerning Eq. (3.22) and the right part in brackets of the equation above, which is a constant d, it is possible to derive the following equation

$$\frac{1}{c_0}\frac{\mathrm{d}^2 c_0}{\mathrm{d}x^2} - \frac{\omega^2}{c_0^2} = d \, . \tag{3.64}$$

This is a nonlinear second-order ordinary differential equation, that can be solved by an appropriate numerical method, but unfortunately its analytical solution is not known and for this reason Eq. (2.28) cannot be solved analytically.

3.3 Finding analytical solutions for known temperature distributions

Let us write Eq. (3.37) once more

$$\frac{\mathrm{d}^2 P}{\mathrm{d}x^2} + \frac{1}{T_0}\frac{\mathrm{d}T_0}{\mathrm{d}x}\frac{\mathrm{d}P}{\mathrm{d}x} + \frac{\omega^2}{c_0^2}P = 0 \, . \tag{3.65}$$

According to the transformations (3.1) and (3.7) it is possible to express the following derivatives

$$\frac{\mathrm{d}P}{\mathrm{d}x} = \frac{\mathrm{d}P}{\mathrm{d}T_0}\frac{\mathrm{d}T_0}{\mathrm{d}x} \, , \tag{3.66}$$

$$\frac{\mathrm{d}^2 P}{\mathrm{d}x^2} = \left(\frac{\mathrm{d}T_0}{\mathrm{d}x}\right)^2\frac{\mathrm{d}^2 P}{\mathrm{d}T_0^2} + \frac{\mathrm{d}^2 T_0}{\mathrm{d}x^2}\frac{\mathrm{d}P}{\mathrm{d}T_0} \, . \tag{3.67}$$

Substituting expressions (3.66) and (3.67) into Eq. (3.65), using the equality $c_0^2 = \varkappa R T_0$ and rearranging the terms leads to the equation

$$\left(\frac{\mathrm{d}T_0}{\mathrm{d}x}\right)^2\frac{\mathrm{d}^2 P}{\mathrm{d}T_0^2} + \left[\frac{1}{T_0}\left(\frac{\mathrm{d}T_0}{\mathrm{d}x}\right)^2 + \frac{\mathrm{d}^2 T_0}{\mathrm{d}x^2}\right]\frac{\mathrm{d}P}{\mathrm{d}T_0} + \frac{\omega^2}{\varkappa R}\frac{P}{T_0} = 0 \, . \tag{3.68}$$

Keeping in mind the equality

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(T_0\frac{\mathrm{d}T_0}{\mathrm{d}x}\right) = \frac{\mathrm{d}T_0}{\mathrm{d}x}\frac{\mathrm{d}T_0}{\mathrm{d}x} + T_0\frac{\mathrm{d}^2 T_0}{\mathrm{d}x^2} = \left(\frac{\mathrm{d}T_0}{\mathrm{d}x}\right)^2 + T_0\frac{\mathrm{d}^2 T_0}{\mathrm{d}x^2} \, , \tag{3.69}$$

Eq. (3.68) can be written as (see e.g. [3])

$$\left(\frac{\mathrm{d}T_0}{\mathrm{d}x}\right)^2\frac{\mathrm{d}^2 P}{\mathrm{d}T_0^2} + \frac{1}{T_0}\frac{\mathrm{d}}{\mathrm{d}x}\left(T_0\frac{\mathrm{d}T_0}{\mathrm{d}x}\right)\frac{\mathrm{d}P}{\mathrm{d}T_0} + \frac{\omega^2}{\varkappa R}\frac{P}{T_0} = 0 \, . \tag{3.70}$$

3.3.1 Linear temperature distribution

In this section an acoustic pressure of a duct with a linear temperature distribution is studied. The linear temperature distribution can be given by the expression (see e.g. [3])

$$T_0(x) = T_A(Ax+1)\ , \tag{3.71}$$

where A is constant.

The dimensionless form of a temperature function Ξ is according to Eq. (2.36) (see e.g. [4])

$$\Xi(\sigma) = \frac{T_0(\sigma)}{T_A} = AL\sigma + 1\ . \tag{3.72}$$

Some dimensionless linear temperature distributions are made possible by choosing different values of constant A and letting $L = 1$ m are shown in Fig. 4.

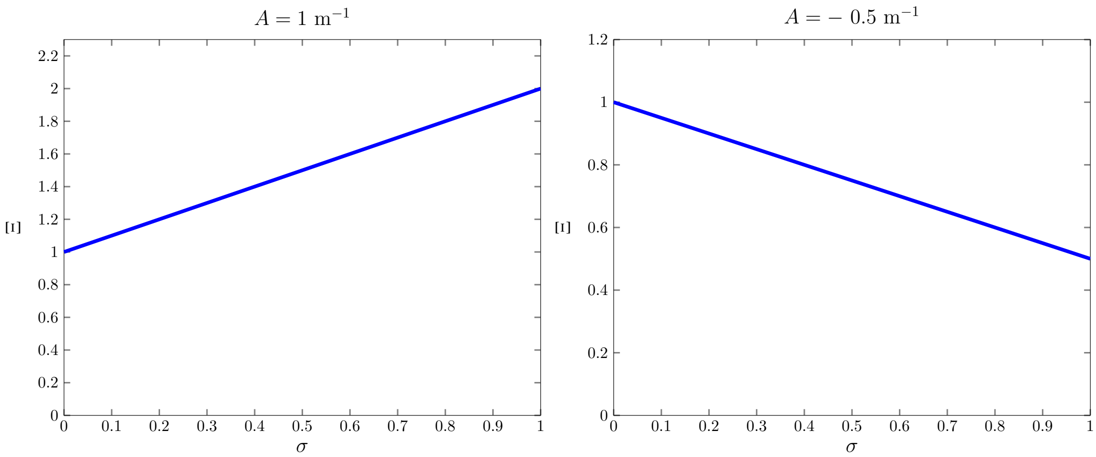

Fig. 4 Some possible linear dimensionless temperature distributions Ξ.

As derivatives of the temperature function (3.71) are

$$\frac{\mathrm{d}T_0}{\mathrm{d}x} = T_A A \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(T_0\frac{\mathrm{d}T_0}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(T_A^2A^2x + T_A^2A\right) = T_A^2A^2\ , \tag{3.73}$$

then multiplying Eq. (3.70) by $1/T_A^2A^2$ and substituting the above derivatives into this equation obtains the following equation (see e.g. [3])

$$\frac{\mathrm{d}^2P}{\mathrm{d}T_0^2} + \frac{1}{T_0}\frac{\mathrm{d}P}{\mathrm{d}T_0} + \frac{\omega^2/\left(T_A^2A^2\right)}{\varkappa R}\frac{P}{T_0} = 0\ . \tag{3.74}$$

To simplify Eq. (3.74) a new independent variable s is introduced

$$s^2 = aT_0\ , \tag{3.75}$$

where the constant a is given by

$$a = \frac{4\omega^2}{T_A^2A^2\varkappa R}\ . \tag{3.76}$$

It is necessary to know the first and the second derivatives of a new variable s with respect to T_0

$$2s\mathrm{d}s = a\mathrm{d}T_0 \qquad \Rightarrow \qquad \frac{\mathrm{d}s}{\mathrm{d}T_0} = \frac{a}{2s}\ , \tag{3.77}$$

$$2\left(\frac{\mathrm{d}s}{\mathrm{d}T_0}\right)^2 + 2s\frac{\mathrm{d}^2s}{\mathrm{d}T_0^2} = 0 \qquad \Rightarrow \qquad \frac{\mathrm{d}^2s}{\mathrm{d}T_0^2} = -\frac{a^2}{4s^3}\ . \tag{3.78}$$

According to transformation of derivatives the derivatives of Eq. (3.74) then

$$\frac{\mathrm{d}P}{\mathrm{d}T_0} = \frac{\mathrm{d}P}{\mathrm{d}T_0}\frac{\mathrm{d}s}{\mathrm{d}T_0} = \frac{a}{2s}\frac{\mathrm{d}P}{\mathrm{d}T_0}\ , \tag{3.79}$$

$$\frac{\mathrm{d}^2P}{\mathrm{d}T_0^2} = \left(\frac{\mathrm{d}s}{\mathrm{d}T_0}\right)^2\frac{\mathrm{d}^2P}{\mathrm{d}s^2} + \frac{\mathrm{d}^2s}{\mathrm{d}T_0^2}\frac{\mathrm{d}P}{\mathrm{d}s} = \frac{a^2}{4s^2}\frac{\mathrm{d}^2P}{\mathrm{d}s^2} - \frac{a^2}{4s^3}\frac{\mathrm{d}P}{\mathrm{d}s}\ , \tag{3.80}$$

and Eq. (3.74) transforms to Eq. (3.83). Some steps of transforming are included

$$\frac{a^2}{4s^2}\frac{\mathrm{d}^2P}{\mathrm{d}s^2} - \left(\frac{a^2}{4s^3} - \frac{a^2}{2s^3}\right)\frac{\mathrm{d}P}{\mathrm{d}s} + \frac{a^2}{4s^2}P = 0\ , \tag{3.81}$$

$$\frac{a^2}{4s^2}\frac{\mathrm{d}^2P}{\mathrm{d}s^2} + \frac{a^2}{4s^3}\frac{\mathrm{d}P}{\mathrm{d}s} + \frac{a^2}{4s^2}P = 0\ . \tag{3.82}$$

By multiplying Eq. (3.82) by $4s^2/a^2$ the final equation after the transformation is

$$\frac{\mathrm{d}^2P}{\mathrm{d}s^2} + \frac{1}{s}\frac{\mathrm{d}P}{\mathrm{d}s} + P = 0\ . \tag{3.83}$$

Equation (3.83) is the zeroth order Bessel equation. The solution to the equation is well known and given by

$$P(s) = C_1\mathrm{J}_0(s) + C_2\mathrm{Y}_0(s)\ , \tag{3.84}$$

where C_1 and C_2 are complex integration constants, J_0 and Y_0 are the Bessel and Neumann functions of the order zero.

Let us express variable s

$$s = \sqrt{aT_0} = \sqrt{\frac{4\omega^2}{T_A^2A^2\varkappa R}}\sqrt{T_0} = \frac{\omega}{b}\sqrt{T_0}\ , \tag{3.85}$$

where

$$b = \frac{T_A|A|}{2}\sqrt{\varkappa R}\ . \tag{3.86}$$

Then acoustic pressure can be rewritten as

$$P(T_0) = C_1\mathrm{J}_0\left(\frac{\omega}{b}\sqrt{T_0}\right) + C_2\mathrm{Y}_0\left(\frac{\omega}{b}\sqrt{T_0}\right)\ . \tag{3.87}$$

Let us substitute temperature distribution (3.71)

$$P(x) = C_1\mathrm{J}_0\left(\frac{\omega}{b}\sqrt{T_A(Ax+1)}\right) + C_2\mathrm{Y}_0\left(\frac{\omega}{b}\sqrt{T_A(Ax+1)}\right)\ . \tag{3.88}$$

The dimensionless form of the pressure is (see e.g. [4])

$$\Phi(\sigma) = C_1\mathrm{J}_0\left(B\sqrt{AL\sigma+1}\right) + C_2\mathrm{Y}_0\left(B\sqrt{AL\sigma+1}\right)\ , \tag{3.89}$$

where C_1, C_2 are new constants, $B = \left(\omega\sqrt{T_A}\right)/b$.

3.3.2 Exponential temperature distribution

This section investigates the acoustic pressure of a duct with an exponential temperature distribution that is given by the expression (see e.g. [3])

$$T_0(x) = T_A e^{-\gamma x} \,, \tag{3.90}$$

where T_A and γ are constants.

The dimensionless form Ξ of the temperature-inhomogeneous region T_0 is (see e.g. [4])

$$\Xi(\sigma) = \frac{T_0(\sigma)}{T_A} = e^{-\gamma L \sigma} \,. \tag{3.91}$$

Some dimensionless exponential temperature distributions made possible by choosing different values of constant γ and letting $L = 1$ m are shown in Fig. 5.

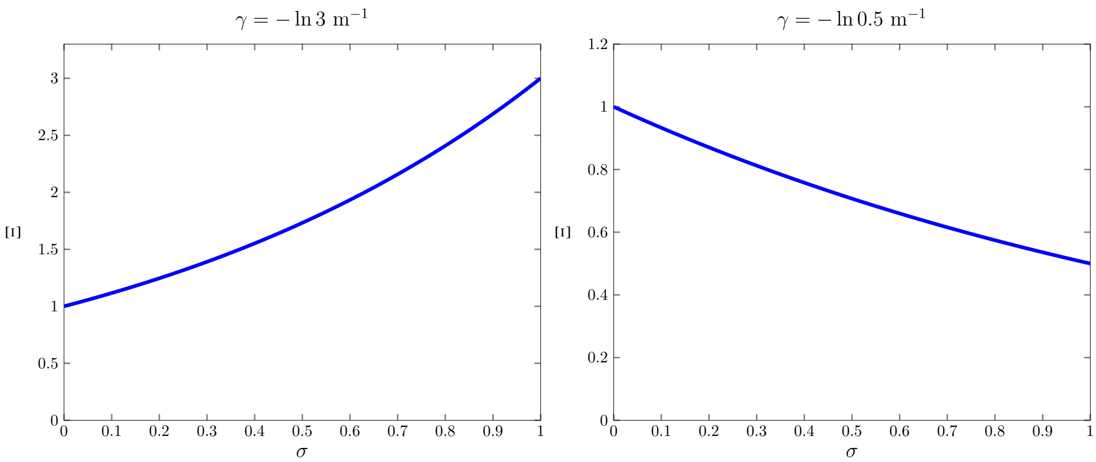

Fig. 5 Some possible exponential dimensionless temperature distributions Ξ.

Let us find derivatives to substitute into Eq. (3.70)

$$\frac{\mathrm{d}T_0}{\mathrm{d}x} = -T_A\gamma e^{-\gamma x} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(T_0\frac{\mathrm{d}T_0}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(-T_A^2\gamma e^{-2\gamma x}\right) = 2T_A^2\gamma^2 e^{-2\gamma x} \,. \tag{3.92}$$

Substituting into Eq. (3.70) and reducing gives the equation

$$(T_A\gamma e^{-\gamma x})^2\frac{\mathrm{d}^2P}{\mathrm{d}T_0^2} + 2T_A\gamma^2 e^{-\gamma x}\frac{\mathrm{d}P}{\mathrm{d}T_0} + \frac{\omega^2}{\varkappa R}\frac{P}{T_A e^{-\gamma x}} = 0 \,. \tag{3.93}$$

Multiplying Eq. (3.93) by $1/\gamma^2$

$$(T_A e^{-\gamma x})^2\frac{\mathrm{d}^2P}{\mathrm{d}T_0^2} + 2T_A e^{-\gamma x}\frac{\mathrm{d}P}{\mathrm{d}T_0} + \frac{\omega^2}{\varkappa R\gamma^2}\frac{P}{T_A e^{-\gamma x}} = 0 \tag{3.94}$$

and using backward substitition $T_A e^{-\gamma x} = T_0$ leads to the equation (see e.g. [3])

$$T_0^2\frac{\mathrm{d}^2P}{\mathrm{d}T_0^2} + 2T_0\frac{\mathrm{d}P}{\mathrm{d}T_0} + \frac{\omega^2}{\varkappa R\gamma^2}\frac{P}{T_0} = 0 \,. \tag{3.95}$$

To simplify the previous equation two new variables w and z are introduced

$$w = P\sqrt{T_0} \qquad \text{and} \qquad z^2 = \varpi\frac{1}{T_0} \,, \tag{3.96}$$

where constant ϖ is

$$\varpi = \frac{4\omega^2}{\varkappa R\gamma^2} \, . \tag{3.97}$$

1. Let us start to work with variable w in the following three steps:
   - expressing the derivatives to substitute into Eq. (3.95)

$$\frac{\mathrm{d}P}{\mathrm{d}T_0} = \frac{\mathrm{d}}{\mathrm{d}T_0}\left(\frac{w}{T_0^{1/2}}\right) = \frac{1}{T_0^{1/2}}\frac{\mathrm{d}w}{\mathrm{d}T_0} - \frac{1}{2}\frac{w}{T_0^{3/2}} \, . \tag{3.98}$$

$$\begin{aligned}\frac{\mathrm{d}^2P}{\mathrm{d}T_0^2} &= -\frac{1}{2}\frac{1}{T_0^{3/2}}\frac{\mathrm{d}w}{\mathrm{d}T_0} + \frac{1}{T_0^{1/2}}\frac{\mathrm{d}^2w}{\mathrm{d}T_0^2} - \frac{1}{2}\left[\frac{T_0^{3/2}\mathrm{d}w/\mathrm{d}T_0 - 3/2wT_0^{1/2}}{T_0^3}\right] \\ &= -\frac{1}{T_0^{3/2}}\frac{\mathrm{d}w}{\mathrm{d}T_0} + \frac{1}{T_0^{1/2}}\frac{\mathrm{d}^2w}{\mathrm{d}T_0^2} + \frac{3}{4}\frac{w}{T_0^{5/2}} \, .\end{aligned} \tag{3.99}$$

   - substitution into Eq. (3.95)

$$\begin{aligned}T_0^2\left[-\frac{1}{T_0^{3/2}}\frac{\mathrm{d}w}{\mathrm{d}T_0} + \frac{1}{T_0^{1/2}}\frac{\mathrm{d}^2w}{\mathrm{d}T_0^2} + \frac{3}{4}\frac{w}{T_0^{5/2}}\right] + 2T_0\left[\frac{1}{T_0^{1/2}}\frac{\mathrm{d}w}{\mathrm{d}T_0} - \frac{1}{2}\frac{w}{T_0^{3/2}}\right] \\ + \frac{\omega^2}{\varkappa R\gamma^2}\frac{w}{T_0^{3/2}} = 0 \, .\end{aligned} \tag{3.100}$$

   - multiplying and rearranging terms in the previous equation gives the equation below

$$T_0^{3/2}\frac{\mathrm{d}^2w}{\mathrm{d}T_0^2} + T_0^{1/2}\frac{\mathrm{d}w}{\mathrm{d}T_0} - \frac{1}{4}\frac{w}{T_0^{1/2}} + \frac{\omega^2}{\varkappa R\gamma^2}\frac{w}{T_0^{3/2}} = 0 \, . \tag{3.101}$$

2. Now let us work with the second variable z:
   - finding the first and the second derivatives of a new variable z with respect to the temperature T_0

$$2z\mathrm{d}z = -\varpi\frac{1}{T_0^2}\mathrm{d}T_0 \qquad \Rightarrow \qquad \frac{\mathrm{d}z}{\mathrm{d}T_0} = -\frac{\varpi}{2z}\frac{1}{T_0^2} \, , \tag{3.102}$$

$$\begin{aligned}\frac{\mathrm{d}^2z}{\mathrm{d}T_0^2} &= -\frac{\varpi}{2}\left(-\frac{1}{z^2}\frac{1}{T_0^2}\frac{\mathrm{d}z}{\mathrm{d}T_0} - 2\frac{1}{T_0^3}\frac{1}{z}\right) = \frac{\varpi}{2}\left(2\frac{1}{z}\frac{1}{T_0^3} - \frac{\varpi}{2z^3}\frac{1}{T_0^4}\right) \\ &= \frac{\varpi}{z}\frac{1}{T_0^3} - \frac{\varpi^2}{4z^3}\frac{1}{T_0^4}\frac{\mathrm{d}z}{\mathrm{d}T_0} \, .\end{aligned} \tag{3.103}$$

   - according to the transformation of derivatives the derivatives of Eq. (3.101) then

$$\frac{\mathrm{d}w}{\mathrm{d}T_0} = \frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}T_0} = -\frac{\varpi}{2z}\frac{1}{T_0^2}\frac{\mathrm{d}w}{\mathrm{d}z} \, , \tag{3.104}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}T_0^2} = \left(\frac{\mathrm{d}z}{\mathrm{d}T_0}\right)^2\frac{\mathrm{d}^2w}{\mathrm{d}z^2} + \frac{\mathrm{d}^2z}{\mathrm{d}T_0^2}\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\varpi^2}{4z^2}\frac{1}{T_0^4}\frac{\mathrm{d}^2w}{\mathrm{d}z^2} + \left(\frac{\varpi}{z}\frac{1}{T_0^3} - \frac{\varpi^2}{4z^3}\frac{1}{T_0^4}\right)\frac{\mathrm{d}w}{\mathrm{d}z} \, . \tag{3.105}$$

- substituting the above derivatives into Eq. (3.101)

$$T_0^{3/2}\left[\frac{\varpi^2}{4z^2}\frac{1}{T_0^4}\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+\left(\frac{\varpi}{z}\frac{1}{T_0^3}-\frac{\varpi^2}{4z^3}\frac{1}{T_0^4}\right)\frac{\mathrm{d}w}{\mathrm{d}z}\right]-T_0^{1/2}\frac{\varpi}{2z}\frac{1}{T_0^2}\frac{\mathrm{d}w}{\mathrm{d}z}-\frac{1}{4}\frac{w}{T_0^{1/2}}$$
$$+\frac{\varpi}{4}\frac{w}{T_0^{3/2}}=0\,. \tag{3.106}$$

- reducing and rearranging of terms

$$\frac{\varpi^2}{4z^2}\frac{1}{T_0^{5/2}}\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+\left(\frac{\varpi}{2z}\frac{1}{T_0^{3/2}}-\frac{\varpi^2}{4z^3}\frac{1}{T_0^{5/2}}\right)\frac{\mathrm{d}w}{\mathrm{d}z}-\frac{1}{4}\frac{w}{T_0^{1/2}}+\frac{\varpi}{4}\frac{w}{T_0^{3/2}}=0\,. \tag{3.107}$$

- from Eqs. (3.96) it follows that $T_0=\varpi/z^2$, substitution of this equality into Eq. (3.107)

$$\frac{\varpi^2}{4z^2}\frac{z^5}{\varpi^{5/2}}\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+\left(\frac{\varpi}{2z}\frac{z^3}{\varpi^{3/2}}-\frac{\varpi^2}{4z^3}\frac{z^5}{\varpi^{5/2}}\right)\frac{\mathrm{d}w}{\mathrm{d}z}-\frac{1}{4}\frac{wz}{\varpi^{1/2}}+\frac{\varpi}{4}\frac{wz^3}{\varpi^{3/2}}=0\,. \tag{3.108}$$

- after some reductions and rearranging some terms

$$\frac{z^3}{4\varpi^{1/2}}\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+\frac{1}{4}\frac{z^2}{\varpi^{1/2}}\frac{\mathrm{d}w}{\mathrm{d}z}-\frac{1}{4}\frac{wz}{\varpi^{1/2}}+\frac{1}{4}\frac{wz^3}{\varpi^{1/2}}=0\,. \tag{3.109}$$

- multiplying the above equation by $4\varpi^{1/2}/z^3$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+\frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z}+\left(1-\frac{1}{z^2}\right)w=0\,. \tag{3.110}$$

Equation (3.110) is the first order Bessel differential equation. The solution is given as

$$w(z)=C_1\mathrm{J}_1(z)+C_2\mathrm{Y}_1(z)\,, \tag{3.111}$$

where C_1 and C_2 are integration complex constants and J_1 and Y_1 are the Bessel and Neumann functions of the first order. Let us express variable z

$$z=\sqrt{\frac{\varpi}{T_0}}=\sqrt{\frac{4\omega^2}{\varkappa R\gamma^2}}\sqrt{\frac{1}{T_0}}=\omega\phi\frac{1}{\sqrt{T_0}}\,, \tag{3.112}$$

where

$$\phi=\frac{2}{\sqrt{\varkappa R\gamma^2}}\,. \tag{3.113}$$

So, the solution of Eq. (3.110) can be written in the form

$$w(T_0)=C_1\mathrm{J}_1\left(\omega\phi\frac{1}{\sqrt{T_0}}\right)+C_2\mathrm{Y}_1\left(\omega\phi\frac{1}{\sqrt{T_0}}\right)\,. \tag{3.114}$$

According to Eqs. (3.96) the acoustic pressure then is

$$P(T_0)=\frac{w(T_0)}{\sqrt{T_0}}=\frac{1}{\sqrt{T_0}}\left[C_1\mathrm{J}_1\left(\omega\phi\frac{1}{\sqrt{T_0}}\right)+C_2\mathrm{Y}_1\left(\omega\phi\frac{1}{\sqrt{T_0}}\right)\right]\,. \tag{3.115}$$

Let us substitute temperature distribution (3.90)

$$P(x)=\frac{1}{\sqrt{T_Ae^{-\gamma x}}}\left[C_1\mathrm{J}_1\left(\omega\phi\frac{1}{\sqrt{T_Ae^{-\gamma x}}}\right)+C_2\mathrm{Y}_1\left(\omega\phi\frac{1}{\sqrt{T_Ae^{-\gamma x}}}\right)\right]\,. \tag{3.116}$$

The dimensionless form of the pressure is (see e.g. [4])

$$\Phi(\sigma)=\sqrt{e^{\gamma L\sigma}}\left[C_1\mathrm{J}_1\left(B\sqrt{e^{\gamma L\sigma}}\right)+C_2\mathrm{Y}_1\left(B\sqrt{e^{\gamma L\sigma}}\right)\right]\,, \tag{3.117}$$

where C_1, C_2 are new constants, $B=(\omega\phi)/\sqrt{T_A}$.

4 Transmission and reflection coefficients

This chapter deals with the calculation of transmission and reflection coefficients for exact analytical solutions derived in the previous chapter.

4.1 Sound velocity

The first exact analytical solution was derived for an acoustic velocity. Consider the reflection and transmission problem through the temperature-inhomogeneous region for an incident plane wave sketched in Fig. 6. The wave is partly reflected and partly transmitted (see e.g. [2]).

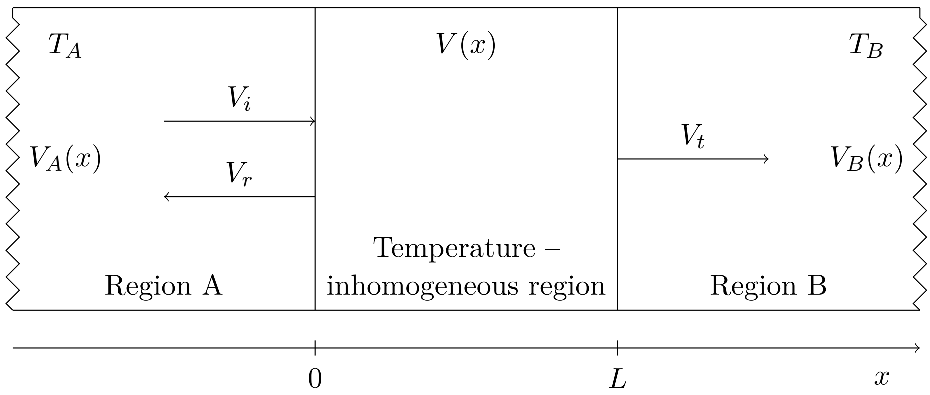

Fig. 6 Reflection and transmission of a sound velocity in a waveguide (see e.g. [2]).

It is essential to know the amplitudes V_r and V_t of the reflected and transmitted waves respectively and their integral constants.

In the region A can be written

$$V_A = V_i e^{\mathrm{j}k_A x} + V_r e^{-\mathrm{j}k_A x} \,, \tag{4.1}$$

where the quantities V_i and V_r are the complex velocity amplitudes of the incident and reflected waves, and

$$k_A = \frac{\omega}{c_A} \,. \tag{4.2}$$

In the temperature-inhomogeneous region the velocity amplitude is given by Eq. (3.34). Let us write it once more

$$V(x) = \sqrt{c_0(x)}F(x) = \sqrt{c_A\left[\left(A^2 - \frac{r^2}{c_A^2}\right)x^2 + 2Ax + 1\right]} \times \left[C_1 \cos\left(\frac{\xi}{|r|}\nu(x)\right) - C_2 \sin\left(\frac{\xi}{|r|}\nu(x)\right)\right] \,, \tag{4.3}$$

where

$$\nu(x) = \tanh^{-1}\left[\frac{c_A}{|r|}\left(A + \left(A^2 - \frac{r^2}{c_A^2}\right)x\right)\right] - \tanh^{-1}\left(\frac{c_A}{|r|}A\right) \;, \quad \xi^2 = \omega^2 - r^2 \;. \tag{4.4}$$

In the region B can be written

$$V_B = V_t e^{\mathrm{j}k_B(x-L)} \;, \tag{4.5}$$

where V_t is the complex acoustic velocity amplitude of the transmitted wave and

$$c_B = \sqrt{\varkappa R T_B} \;, \tag{4.6a}$$

$$k_B = \frac{\omega}{c_B} \;. \tag{4.6b}$$

The acoustic velocities must be the same on the interface of temperature-homogeneous and temperature-inhomogeneous regions. This is true for the both interfaces that have coordinates $x = 0$ and $x = L$. Consequently, there are two conditions in the waveguide

1. condition

$$V_A(0) = V(0) \;, \tag{4.7}$$

2. condition

$$V_B(L) = V(L) \;. \tag{4.8}$$

But these conditions contain four unknown variables V_r, V_t, C_1 and C_2. So, it is necessary to impose two more conditions.

From the equation of continuity (2.20) and Eq. (2.27) the following equation can be written

$$\frac{\partial p'}{\partial t} + \rho_0 c_0^2 \frac{\partial v}{\partial x} = 0 \;. \tag{4.9}$$

Substitution of the equalities (2.29) and (2.32) into Eq. (4.9)

$$P(x) = \frac{\rho_0 c_0^2}{\mathrm{j}\omega}\frac{\mathrm{d}V}{\mathrm{d}x}. \tag{4.10}$$

Let us distinguish densities and velocities in temperature-homogeneous regions A and B

$$\rho_0(x=0) = \rho_A \;, \qquad c_0(x=0) = \sqrt{\varkappa R T_A} = c_A \;, \tag{4.11}$$

$$\rho_0(x=L) = \rho_B \;, \qquad c_0(x=L) = \sqrt{\varkappa R T_B} = c_B \;. \tag{4.12}$$

From Eqs. (4.1), (4.5), (4.10), (4.11), (4.12) the acoustic pressures in regions A and B are

$$P_A = \frac{\rho_A c_A^2}{\mathrm{j}\omega}\mathrm{j}k_A\left(V_i e^{\mathrm{j}k_A x} - V_r e^{-\mathrm{j}k_A x}\right) \;, \tag{4.13}$$

$$P_B = \frac{\rho_B c_B^2}{\mathrm{j}\omega}\mathrm{j}k_B V_t e^{\mathrm{j}k_B(x-L)} \;. \tag{4.14}$$

The pressures also must be the same on the interface of temperature-homogeneous and temperature-inhomogeneous regions. For the interface at coordinate $x = 0$ the following steps are introduced

$$P_A|_{x=0} = P(x)|_{x=0} \;, \tag{4.15}$$

$$\left.\frac{\rho_A c_A^2}{\mathrm{j}\omega}\mathrm{j}k_A\left(V_i e^{\mathrm{j}k_A x} - V_r e^{-\mathrm{j}k_A x}\right)\right|_{x=0} = \left.\frac{\rho_A c_A^2}{\mathrm{j}\omega}\frac{\mathrm{d}V}{\mathrm{d}x}\right|_{x=0} . \tag{4.16}$$

After reducing the term $\rho_A c_A^2/\mathrm{j}\omega$ in Eq. (4.16) the left part $\mathrm{j}k_A\left(V_i e^{\mathrm{j}k_A x} - V_r e^{-\mathrm{j}k_A x}\right)$ then is the acoustic velocity derivative $\mathrm{d}V_A/\mathrm{d}x$. So,

$$\left.\frac{\mathrm{d}V_A}{\mathrm{d}x}\right|_{x=0} = \mathrm{j}k_A\left(V_i - V_r\right) = \left.\frac{\mathrm{d}V}{\mathrm{d}x}\right|_{x=0} . \tag{4.17}$$

Similarly, the acoustic velocity derivative at coordinate $x = L$ is

$$\left.\frac{\mathrm{d}V_B}{\mathrm{d}x}\right|_{x=L} = \mathrm{j}k_B V_t = \left.\frac{\mathrm{d}V}{\mathrm{d}x}\right|_{x=L} . \tag{4.18}$$

As a consequence, the acoustic velocity derivatives must be the same on the interface of temperature-homogeneous and temperature-inhomogeneous regions. For both interfaces at coordinates $x = 0$ and $x = L$ the two following boundary conditions are defined

3. condition

$$\left.\frac{\mathrm{d}V_A}{\mathrm{d}x}\right|_{x=0} = \left.\frac{\mathrm{d}V}{\mathrm{d}x}\right|_{x=0} , \tag{4.19}$$

4. condition

$$\left.\frac{\mathrm{d}V_B}{\mathrm{d}x}\right|_{x=L} = \left.\frac{\mathrm{d}V}{\mathrm{d}x}\right|_{x=L} . \tag{4.20}$$

Substituting Eqs. (4.1), (4.3), (4.5) into the four derived boundary conditions (4.7), (4.8), (4.19), (4.20) produces the following system of equations

$$V_i + V_r = C_1 G_1 , \tag{4.21}$$

$$V_t = C_1 H_1 + C_2 H_2 , \tag{4.22}$$

$$\mathrm{j}k_A\left(V_i - V_r\right) = C_1 M_1 + C_2 M_2 , \tag{4.23}$$

$$\mathrm{j}k_B V_t = C_1 N_1 + C_2 N_2 , \tag{4.24}$$

where

$$G_1 = \sqrt{c_A} , \quad \psi(L) = A + \left(A^2 - \frac{r^2}{c_A^2}\right) L , \quad \xi = \sqrt{\omega^2 - r^2} , \tag{4.25}$$

$$\chi(L) = \sqrt{c_A\left[\left(A^2 - \frac{r^2}{c_A^2}\right) L^2 + 2AL + 1\right]} , \tag{4.26}$$

$$\nu(L) = \tanh^{-1}\left[\frac{c_A}{|r|}\psi(L)\right] - \tanh^{-1}\left(\frac{c_A}{|r|}A\right) , \tag{4.27}$$

$$H_1 = \chi(L)\cos\left(\frac{\xi}{|r|}\nu(L)\right) , \quad H_2 = -\chi(L)\sin\left(\frac{\xi}{|r|}\nu(L)\right) , \tag{4.28}$$

$$M_1 = \sqrt{c_A}A , \quad M_2 = \frac{\xi}{\sqrt{c_A}} , \tag{4.29}$$

$$\varrho(L) = \frac{\xi c_A\left(A^2 - \frac{r^2}{c_A^2}\right)\chi(L)}{c_A^2\psi^2(L) - r^2} , \quad k(L) = \frac{c_A\psi(L)}{\chi(L)} , \tag{4.30}$$

$$N_1 = \varrho(L)\sin\left(\frac{\xi}{|r|}\nu(L)\right) + k(L)\cos\left(\frac{\xi}{|r|}\nu(L)\right) , \tag{4.31}$$

$$N_2 = \varrho(L)\cos\left(\frac{\xi}{|r|}\nu(L)\right) - k(L)\sin\left(\frac{\xi}{|r|}\nu(L)\right) . \tag{4.32}$$

Solving the system of Eqs. (4.21)–(4.24) obtains

$$C_1 = \frac{2k_A\left(k_B H_2 + \mathrm{j}N_2\right)V_i}{k_A k_B G_1 H_2 + M_1 N_2 - M_2 N_1 + \mathrm{j}\left[k_A G_1 N_2 + k_B\left(H_1 M_2 - H_2 M_1\right)\right]}\,, \tag{4.33}$$

$$C_2 = -\frac{2k_A\left(k_B H_1 + \mathrm{j}N_1\right)V_i}{k_A k_B G_1 H_2 + M_1 N_2 - M_2 N_1 + \mathrm{j}\left[k_A G_1 N_2 + k_B\left(H_1 M_2 - H_2 M_1\right)\right]}\,, \tag{4.34}$$

$$V_r = \frac{\left(k_A k_B G_1 H_2 - M_1 N_2 + M_2 N_1 + \mathrm{j}\left[k_A G_1 N_2 - k_B\left(H_1 M_2 - H_2 M_1\right)\right]\right)V_i}{k_A k_B G_1 H_2 + M_1 N_2 - M_2 N_1 + \mathrm{j}\left[k_A G_1 N_2 + k_B\left(H_1 M_2 - H_2 M_1\right)\right]}\,, \tag{4.35}$$

$$V_t = \frac{\mathrm{j}2k_A\left(H_1 N_2 - H_2 N_1\right)V_i}{k_A k_B G_1 H_2 + M_1 N_2 - M_2 N_1 + \mathrm{j}\left[k_A G_1 N_2 + k_B\left(H_1 M_2 - H_2 M_1\right)\right]}\,. \tag{4.36}$$

As V_i is optional, the reflection coefficient $R^{(v)}$ and transmission coefficient $Tr^{(v)}$ can be calculated on the basis of solutions of the system equations. The coefficients (see e.g. [2], [4]) are defined by

$$R^{(v)} = \frac{V_r}{V_i} \tag{4.37}$$

and

$$Tr^{(v)} = \frac{V_t}{V_i}\,. \tag{4.38}$$

By supposing values of air constants as $\varkappa = 7/5$ and $R = 287.058$ Jkg^{-1}K^{-1}, setting the characteristic length L to 1 m and choosing different values of constants A, r and velocity c_A it is possible to see a frequency dependence of the wave reflection and transmission coefficients of the exact analytical solutions given by Eq. (4.3). Here attention must be paid to one serious condition

$$\xi^2 = \omega^2 - r^2 \geq 0\,. \tag{4.39}$$

That is why the graphs start at point $\omega = |r|$.

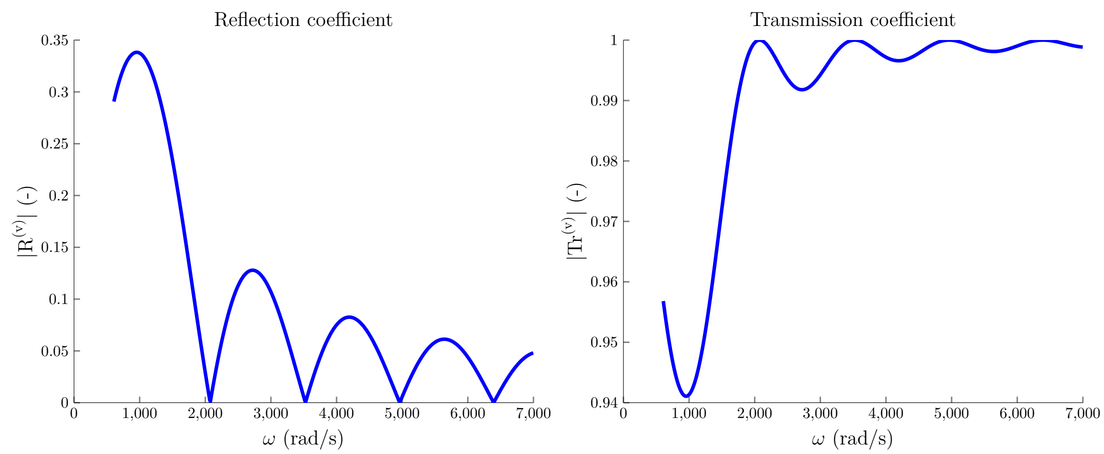

Fig. 7 Dependence of modulii of reflection and transmission coefficients for $c_A = 345$ ms^{-1}, $|r| = 597.558$ s^{-1}, $A = 1$ m^{-1} in Eq. (4.3) on angular frequency.

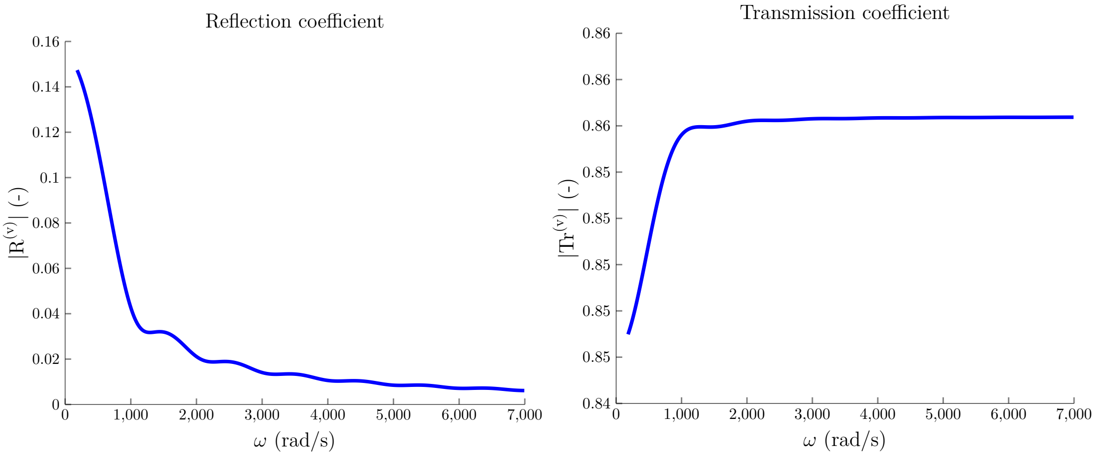

Fig. 8 Dependence of modulii of reflection and transmission coefficients for $c_A = 345\ \mathrm{ms}^{-1}$, $|r| = 172.5\ \mathrm{s}^{-1}$, $A = -0.008\ \mathrm{m}^{-1}$ in Eq. (4.3) on angular frequency.

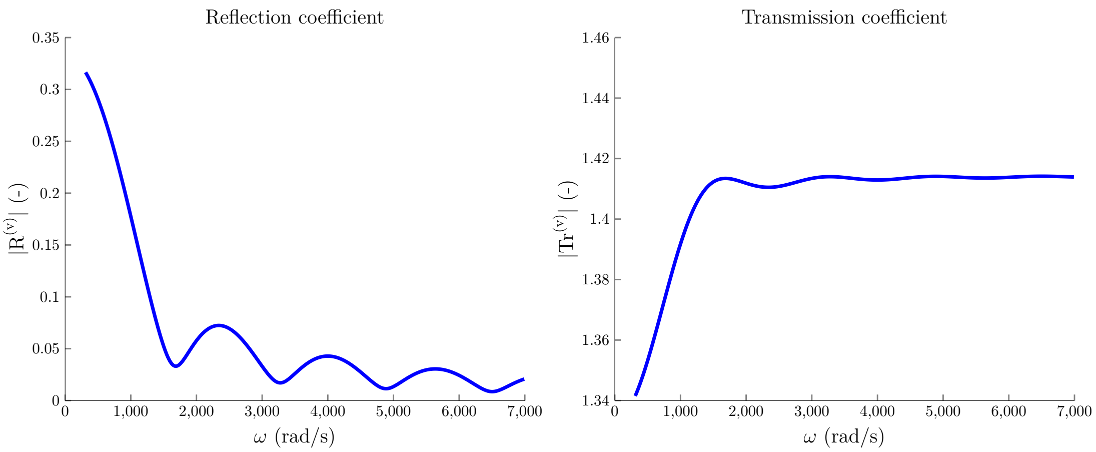

Fig. 9 Dependence of modulii of reflection and transmission coefficients for $c_A = 345\ \mathrm{ms}^{-1}$, $|r| = 301.39\ \mathrm{s}^{-1}$, $A = -0.662\ \mathrm{m}^{-1}$ in Eq. (4.3) on angular frequency.

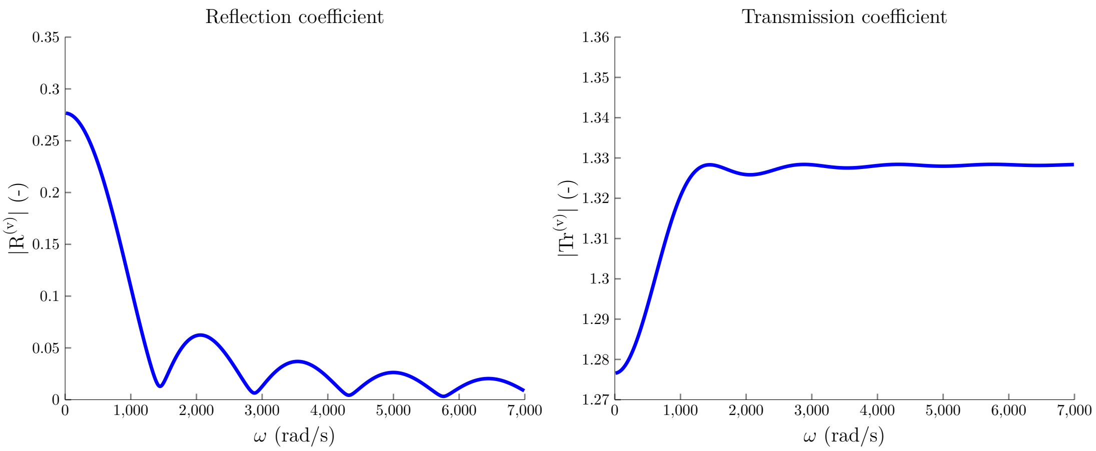

Fig. 10 Dependence of modulii of reflection and transmission coefficients for $c_A = 345\ \mathrm{ms}^{-1}$, $|r| = 0.97\ \mathrm{s}^{-1}$, $A = 0.328\ \mathrm{m}^{-1}$ in Eq. (4.3) on angular frequency.

4.2 Sound pressure

The following exact analytical solutions are derived for sound pressure.

Consider the reflection and transmission problem through the temperature-inhomogeneous region for an incident plane wave sketched in Fig. 11. The wave is partly reflected and partly transmitted as it was for sound velocity, but now the object of interest is sound pressure (see e.g. [2], [4]).

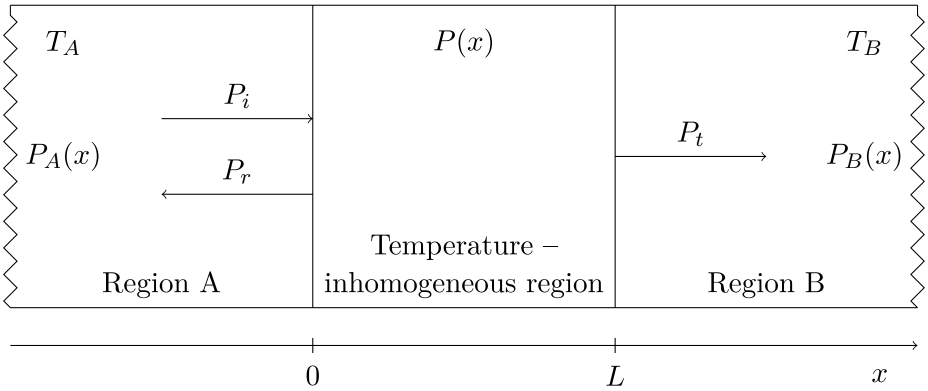

Fig. 11 Reflection and transmission of a sound pressure in a waveguide (see e.g. [2], [4]).

Now, it is necessary to know the amplitudes P_r and P_t of reflected and transmitted waves respectively and also their integral constants.

In the region A can be written

$$P_A = P_i e^{\mathrm{j}k_A x} + P_r e^{-\mathrm{j}k_A x} \ , \tag{4.40}$$

where the quantities P_i and P_r are the complex velocity amplitudes of the incident and reflected waves, and the same Eq. (4.2) is applied.

In the region B can be written

$$P_B = P_t e^{\mathrm{j}k_B (x-L)} \ , \tag{4.41}$$

where V_t is the complex velocity amplitude of the transmitted wave, and Eqs. (4.6) are true.

In the temperature-inhomogeneous region pressure amplitude is given by appropriate equalities, derived in the previous chapter, and all of these equalities are discussed below.

To calculate P_r, P_t, C_1 and C_2 it is necessary to deduce boundary conditions. These conditions can be imposed in a similar manner to boundary conditions of sound velocity.

From the linear form of the Euler equation (2.22) it follows that

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \ . \tag{4.42}$$

After substitution of equalities (2.29) and (2.32) into Eq. (4.42)

$$\frac{\partial}{\partial t}\left(V(x)e^{-\mathrm{j}\omega t}\right) = -\frac{1}{\rho_0}\frac{\partial}{\partial x}\left(P(x)e^{-\mathrm{j}\omega t}\right) \tag{4.43}$$

the amplitude of sound velocity then is

$$V(x) = \frac{1}{\mathrm{j}\rho_0\omega}\frac{\mathrm{d}P}{\mathrm{d}x} \ . \tag{4.44}$$

From Eqs. (4.11), (4.12), (4.40), (4.41), (4.44) the acoustic velocities at regions A and B are

$$V_A = \frac{\mathrm{j}k_A}{\mathrm{j}\rho_A\omega}\left(P_i e^{\mathrm{j}k_A x} - P_r e^{-\mathrm{j}k_A x}\right) \ , \tag{4.45}$$

$$V_B = \frac{\mathrm{j}k_B}{\mathrm{j}\rho_B\omega} P_t e^{\mathrm{j}k_B(x-L)} \ . \tag{4.46}$$

The velocities also must be the same on the interface of temperature-homogeneous and temperature-inhomogeneous regions. For an interface at coordinate $x = 0$ the following steps are introduced

$$V_A|_{x=0} = V(x)|_{x=0} \ , \tag{4.47}$$

$$\left.\frac{\mathrm{j}k_A}{\mathrm{j}\rho_A\omega}\left(P_i e^{\mathrm{j}k_A x} - P_r e^{-\mathrm{j}k_A x}\right)\right|_{x=0} = \left.\frac{1}{\mathrm{j}\rho_A\omega}\frac{\mathrm{d}P}{\mathrm{d}x}\right|_{x=0} \ . \tag{4.48}$$

After reducing the term $1/\mathrm{j}\rho_A\omega$ in Eq. (4.48) the left part $\mathrm{j}k_A\left(P_i e^{\mathrm{j}k_A x} - P_r e^{-\mathrm{j}k_A x}\right)$ is then an acoustic pressure derivative $\mathrm{d}P_A/\mathrm{d}x$. So

$$\left.\frac{\mathrm{d}P_A}{\mathrm{d}x}\right|_{x=0} = \mathrm{j}k_A\left(P_i - P_r\right) = \left.\frac{\mathrm{d}P}{\mathrm{d}x}\right|_{x=0} \ . \tag{4.49}$$

Similarly, the acoustic pressure derivative at coordinate $x = L$ is

$$\left.\frac{\mathrm{d}P_B}{\mathrm{d}x}\right|_{x=L} = \mathrm{j}k_B P_t = \left.\frac{\mathrm{d}P}{\mathrm{d}x}\right|_{x=L} \ . \tag{4.50}$$

Finally, the four boundary conditions for the rest analytical solutions can be introduced

1. condition

$$P_A(0) = P(0) \ , \tag{4.51}$$

2. condition

$$P_B(L) = P(L) \ , \tag{4.52}$$

3. condition

$$\left.\frac{\mathrm{d}P_A}{\mathrm{d}x}\right|_{x=0} = \left.\frac{\mathrm{d}P}{\mathrm{d}x}\right|_{x=0} \ , \tag{4.53}$$

4. condition

$$\left.\frac{\mathrm{d}P_B}{\mathrm{d}x}\right|_{x=L} = \left.\frac{\mathrm{d}P}{\mathrm{d}x}\right|_{x=L} \ . \tag{4.54}$$

4.2.1 Second exact analytical solution

In the temperature-inhomogeneous region the pressure amplitude is given by Eqs. (3.53) and (3.55).

- $\Delta = c_A^2 q^2 - 4\omega^2 = 0$
  Let us write Eq. (3.53) once more

$$P(x) = \frac{1}{\sqrt{qx+1}}\left[C_1 + C_2 \ln(qx+1)\right] \ . \tag{4.55}$$

  For this solution only one frequency can be found, that is why we are not interested in this solution.

- $\Delta = c_A^2 q^2 - 4\omega^2 < 0$
According to Eq. (4.2) the exact analytical solution (3.55) can be rewritten as

$$P(x) = \frac{1}{\sqrt{qx+1}} \left[C_1 \cos\left(\sqrt{\frac{1}{4} - \frac{k_A^2}{q^2}} \ln(qx+1) \right) \right. \\ \left. + C_2 \sin\left(\sqrt{\frac{1}{4} - \frac{k_A^2}{q^2}} \ln(qx+1) \right) \right] \ . \quad (4.56)$$

According to the system of Eqs. (4.51)–(4.54), the system of equations for the exact analytical solution (4.56) is then

$$P_i + P_r = C_1 \ , \quad (4.57)$$

$$P_t = C_1 H_1 + C_2 H_2 \ , \quad (4.58)$$

$$\mathrm{j} k_A (P_i - P_r) = C_1 M_1 + C_2 M_2 \ , \quad (4.59)$$

$$\mathrm{j} k_B P_t = C_1 N_1 + C_2 N_2 \ , \quad (4.60)$$

where

$$\chi(L) = \sqrt{\frac{1}{4} - \frac{k_A^2}{q^2}} \ln(qL+1) \ , \quad (4.61)$$

$$H_1 = \frac{1}{\sqrt{qL+1}} \cos(\chi(L)) \ , \quad H_2 = \frac{1}{\sqrt{qL+1}} \sin(\chi(L)) \ , \quad (4.62)$$

$$M_1 = -\frac{1}{2} q \ , \quad M_2 = q \sqrt{\frac{1}{4} - \frac{k_A^2}{q^2}} \ , \quad (4.63)$$

$$N_1 = -q(qL+1)^{-\frac{3}{2}} \left[\frac{1}{2} \cos(\chi(L)) + \sqrt{\frac{1}{4} - \frac{k_A^2}{q^2}} \sin(\chi(L)) \right] \ , \quad (4.64)$$

$$N_2 = -q(qL+1)^{-\frac{3}{2}} \left[\frac{1}{2} \sin(\chi(L)) - \sqrt{\frac{1}{4} - \frac{k_A^2}{q^2}} \cos(\chi(L)) \right] \ . \quad (4.65)$$

The solution of the system of Eqs. (4.57)–(4.60) is

$$C_1 = \frac{2k_A (k_B H_2 + \mathrm{j} N_2) P_i}{k_A k_B H_2 + M_1 N_2 - M_2 N_1 + \mathrm{j}\left[k_A N_2 - k_B (H_2 M_1 - H_1 M_2)\right]} \ , \quad (4.66)$$

$$C_2 = -\frac{2k_A (k_B H_1 + \mathrm{j} N_1) P_i}{k_A k_B H_2 + M_1 N_2 - M_2 N_1 + \mathrm{j}\left[k_A N_2 - k_B (H_2 M_1 - H_1 M_2)\right]} \ , \quad (4.67)$$

$$P_r = \frac{\left(k_A k_B H_2 - M_1 N_2 + M_2 N_1 + \mathrm{j}\left[k_A N_2 + k_B (H_2 M_1 - H_1 M_2)\right]\right) P_i}{k_A k_B H_2 + M_1 N_2 - M_2 N_1 + \mathrm{j}\left[k_A N_2 - k_B (H_2 M_1 - H_1 M_2)\right]} \ , \quad (4.68)$$

$$P_t = \frac{\mathrm{j} 2k_A (H_1 N_2 - H_2 N_1) P_i}{k_A k_B H_2 + M_1 N_2 - M_2 N_1 + \mathrm{j}\left[k_A N_2 - k_B (H_2 M_1 - H_1 M_2)\right]} \ . \quad (4.69)$$

The calculation of reflection and transmission coefficients (P_i is optional) is the same as for sound velocity (see e.g. [2], [4])

$$R^{(P)} = \frac{P_r}{P_i} \quad (4.70)$$

and

$$Tr^{(P)} = \frac{P_t}{P_i} \ . \quad (4.71)$$

By supposing values of air constants as $\varkappa = 7/5$ and $R = 287.058$ Jkg^{-1}K^{-1}, setting the characteristic length L to 1 m and choosing different values of constants q, T_A it is possible to see a frequency dependence of the wave reflection and transmission coefficients of the exact analytical solutions given by Eq. (4.56). $\Delta = c_A^2 q^2 - 4\omega^2 < 0$, $c_A > 0$ and from Eq. (4.56) it can be concluded that $\omega > -1/q$, then only $q > 0$ can be examined. Let us choose $T_A = 296$ K and some different values of q to see these frequency dependencies.

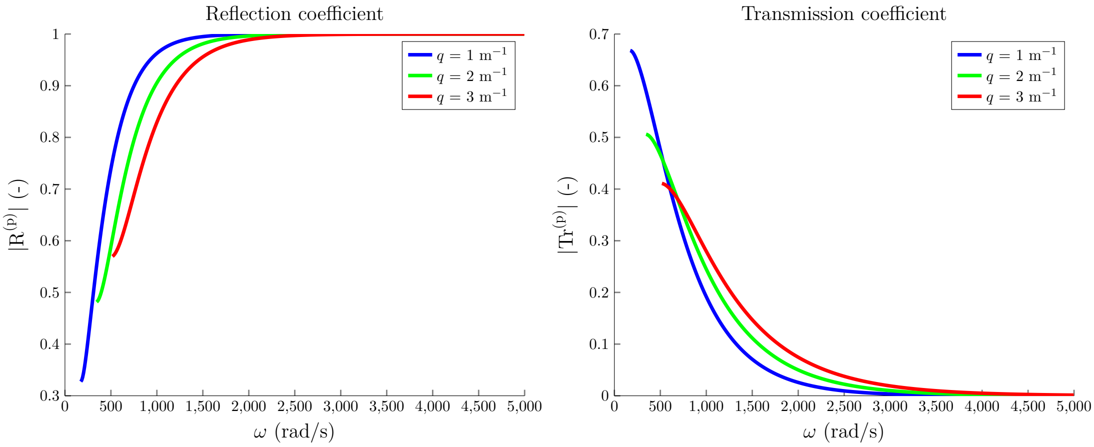

Fig. 12 Dependence of modulii of reflection and transmission coefficients for $T_A = 296$ K and different values of q in Eq. (4.56) on angular frequency.

4.2.2 Exact analytical solution for a linear temperature distribution

The exact analytical solution for a linear temperature distribution from Chapter 3 is

$$P(x) = C_1 \mathrm{J}_0\left(\frac{\omega}{b}\sqrt{T_A(Ax+1)}\right) + C_2 \mathrm{Y}_0\left(\frac{\omega}{b}\sqrt{T_A(Ax+1)}\right) , \tag{4.72}$$

where $b = T_A|A|\sqrt{\varkappa R}/2$.

After all calculations according to the system of Eqs. (4.51)–(4.54) the following system of equations is produced

$$P_i + P_r = C_1 G_1 + C_2 G_2 , \tag{4.73}$$
$$P_t = C_1 H_1 + C_2 H_2 , \tag{4.74}$$
$$\mathrm{j}k_A (P_i - P_r) = C_1 M_1 + C_2 M_2 , \tag{4.75}$$
$$\mathrm{j}k_B P_t = C_1 N_1 + C_2 N_2 , \tag{4.76}$$

where

$$G_1 = \mathrm{J}_0\left(\frac{\omega}{b}\sqrt{T_A}\right) , \quad G_2 = \mathrm{Y}_0\left(\frac{\omega}{b}\sqrt{T_A}\right) , \tag{4.77}$$
$$H_1 = \mathrm{J}_0\left(\frac{\omega}{b}\sqrt{T_A(AL+1)}\right) , \quad H_2 = \mathrm{Y}_0\left(\frac{\omega}{b}\sqrt{T_A(AL+1)}\right) , \tag{4.78}$$
$$M_1 = -\frac{\omega A}{2b}\sqrt{T_A}\mathrm{J}_1\left(\frac{\omega}{b}\sqrt{T_A}\right) , \quad M_2 = -\frac{\omega A}{2b}\sqrt{T_A}\mathrm{Y}_1\left(\frac{\omega}{b}\sqrt{T_A}\right) , \tag{4.79}$$
$$N_1 = -\frac{\omega A}{2b}\sqrt{\frac{T_A}{AL+1}}\mathrm{J}_1\left(\frac{\omega}{b}\sqrt{T_A(AL+1)}\right) , \tag{4.80}$$
$$N_2 = -\frac{\omega A}{2b}\sqrt{\frac{T_A}{AL+1}}\mathrm{Y}_1\left(\frac{\omega}{b}\sqrt{T_A(AL+1)}\right) . \tag{4.81}$$

Solution of the system of Eqs. (4.73)–(4.76) is

$$C_1 = \frac{2k_A\left(k_B H_2 + \mathrm{j}N_2\right)P_i}{k_A k_B\left(G_1 H_2 - G_2 H_1\right) + M_1 N_2 - M_2 N_1 + \mathrm{j}\left[k_A\left(G_1 N_2 - G_2 N_1\right) + k_B\left(H_1 M_2 - H_2 M_1\right)\right]}, \tag{4.82}$$

$$C_2 = -1\times \frac{2k_A\left(k_B H_1 + \mathrm{j}N_1\right)P_i}{k_A k_B\left(G_1 H_2 - G_2 H_1\right) + M_1 N_2 - M_2 N_1 + \mathrm{j}\left[k_A\left(G_1 N_2 - G_2 N_1\right) + k_B\left(H_1 M_2 - H_2 M_1\right)\right]}, \tag{4.83}$$

$$P_r = P_i\times \frac{k_A k_B\left(G_1 H_2 - G_2 H_1\right) - M_1 N_2 + M_2 N_1 + \mathrm{j}\left[k_A\left(G_1 N_2 - G_2 N_1\right) - k_B\left(H_1 M_2 - H_2 M_1\right)\right]}{k_A k_B\left(G_1 H_2 - G_2 H_1\right) + M_1 N_2 - M_2 N_1 + \mathrm{j}\left[k_A\left(G_1 N_2 - G_2 N_1\right) + k_B\left(H_1 M_2 - H_2 M_1\right)\right]}, \tag{4.84}$$

$$P_t = P_i\times \frac{\mathrm{j}2k_A\left(H_1 N_2 - H_2 N_1\right)}{k_A k_B\left(G_1 H_2 - G_2 H_1\right) + M_1 N_2 - M_2 N_1 + \mathrm{j}\left[k_A\left(G_1 N_2 - G_2 N_1\right) + k_B\left(H_1 M_2 - H_2 M_1\right)\right]}. \tag{4.85}$$

The reflection and transmission coefficients can be calculated according to Eqs. (4.70), (4.71) (see e.g. [2], [4]).

By supposing values of constants for air as $\varkappa = 7/5$ and $R = 287.058$ Jkg^{-1}K^{-1}, setting the characteristic length L to 1 m and choosing different values of constants A, T_A it is possible to see a frequency dependence of the wave reflection and transmission coefficients of the exact analytical solution given by Eq. (4.72). Let us choose $T_A = 296$ K for a positive temperature gradient, $T_A = 752$ K for a negative temperature gradient and some different values of A to see these frequency dependencies.

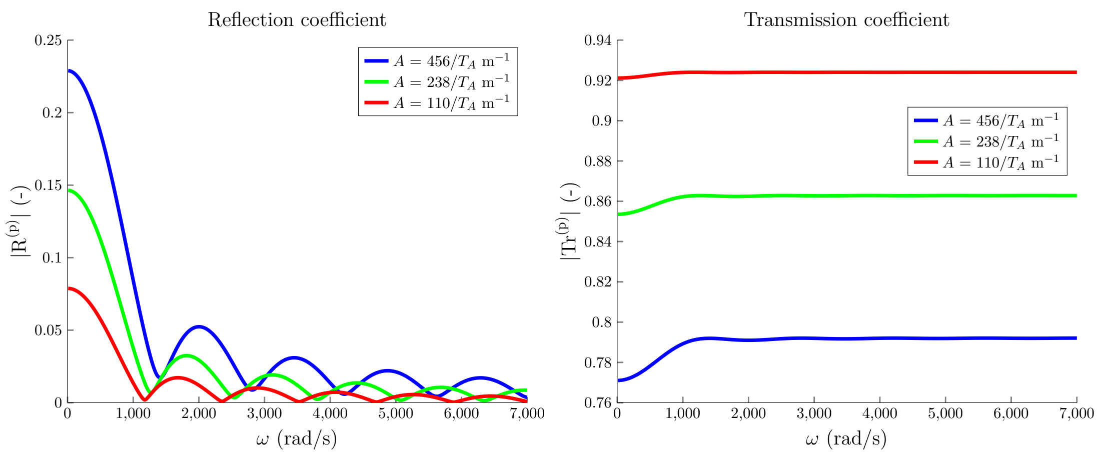

Fig. 13 Dependence of modulii of reflection and transmission coefficients for $T_A = 296$ K and different values of A (positive gradient) in Eq. (4.72) on angular frequency.

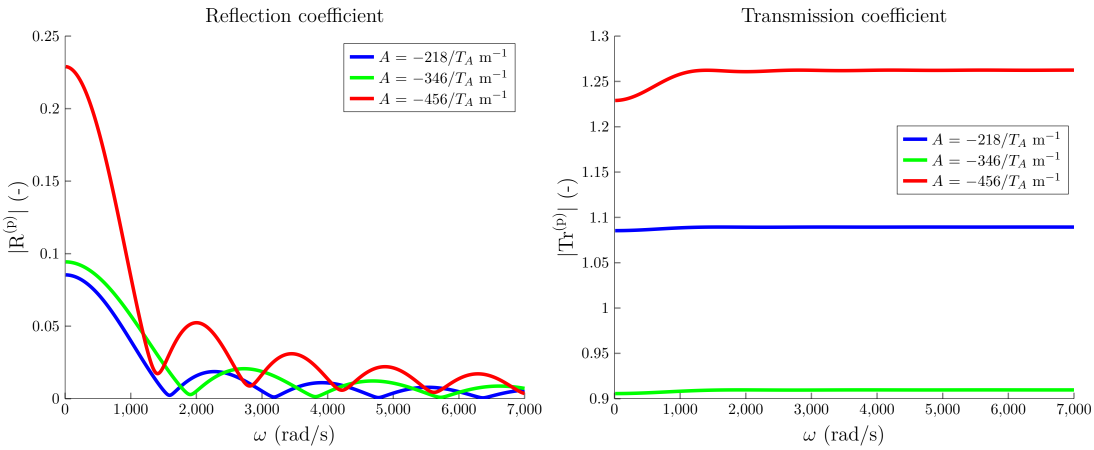

Fig. 14 Dependence of modulii of reflection and transmission coefficients for $T_A = 752$ K and different values of A (negative gradient) in Eq. (4.72) on angular frequency.

4.2.3 Exact analytical solution for an exponential temperature distribution

The exact analytical solution for an exponential temperature distribution was derived in the previous chapter

$$P(x) = \frac{1}{\sqrt{T_A e^{-\gamma x}}} \left[C_1 \mathrm{J}_1 \left(\omega\phi \frac{1}{\sqrt{T_A e^{-\gamma x}}} \right) + C_2 \mathrm{Y}_1 \left(\omega\phi \frac{1}{\sqrt{T_A e^{-\gamma x}}} \right) \right] , \tag{4.86}$$

where $\phi = 2/\sqrt{\varkappa R \gamma^2}$.

According to the system of Eqs. (4.51)–(4.54), the system of equations for an exact analytical solution (4.86) is the same as the system of Eqs. (4.73)–(4.76), therefore the solution has the form of Eqs. (4.82)–(4.85), but notations G_1, G_2, H_1, H_2, M_1, M_2, N_1 and N_2 have different meanings

$$G_1 = \frac{1}{\sqrt{T_A}} \mathrm{J}_1 \left(\omega\phi \frac{1}{\sqrt{T_A}} \right) , \qquad G_2 = \frac{1}{\sqrt{T_A}} \mathrm{Y}_1 \left(\omega\phi \frac{1}{\sqrt{T_A}} \right) , \tag{4.87}$$

$$H_1 = \frac{1}{\sqrt{T_A e^{-\gamma L}}} \mathrm{J}_1 \left(\omega\phi \frac{1}{\sqrt{T_A e^{-\gamma L}}} \right) , \qquad H_2 = \frac{1}{\sqrt{T_A e^{-\gamma L}}} \mathrm{Y}_1 \left(\omega\phi \frac{1}{\sqrt{T_A e^{-\gamma L}}} \right) , \tag{4.88}$$

$$M_1 = \frac{\gamma\omega\phi}{2T_A} \mathrm{J}_0 \left(\omega\phi \frac{1}{\sqrt{T_A}} \right) , \qquad M_2 = \frac{\gamma\omega\phi}{2T_A} \mathrm{Y}_0 \left(\omega\phi \frac{1}{\sqrt{T_A}} \right) , \tag{4.89}$$

$$N_1 = \frac{\gamma\omega\phi}{2T_A e^{-\gamma L}} \mathrm{J}_0 \left(\omega\phi \frac{1}{\sqrt{T_A e^{-\gamma L}}} \right) , \qquad N_2 = \frac{\gamma\omega\phi}{2T_A e^{-\gamma L}} \mathrm{Y}_0 \left(\omega\phi \frac{1}{\sqrt{T_A e^{-\gamma L}}} \right) . \tag{4.90}$$

Reflection and transmission coefficients can be calculated according to Eqs. (4.70), (4.71) (see e.g. [2], [4]).

By supposing values of air constants as $\varkappa = 7/5$ and $R = 287.058$ Jkg^{-1}K^{-1}, setting the characteristic length L to 1 m and choosing different values of constants γ, T_A it is possible to see a frequency dependence of the wave reflection and transmission coefficients of the exact analytical solution given by Eq. (4.86). Let us choose $T_A = 296$ K for both positive and negative temperature gradients and some different values of γ to see frequency dependencies.

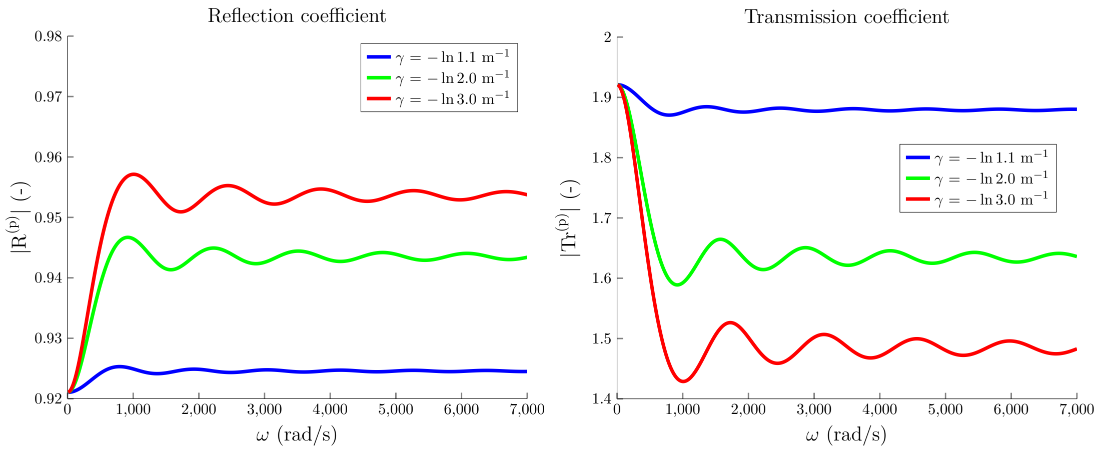

Fig. 15 Dependence of modulii of reflection and transmission coefficients for $T_A = 296$ K and different values of γ (positive gradient) in Eq. (4.86) on angular frequency.

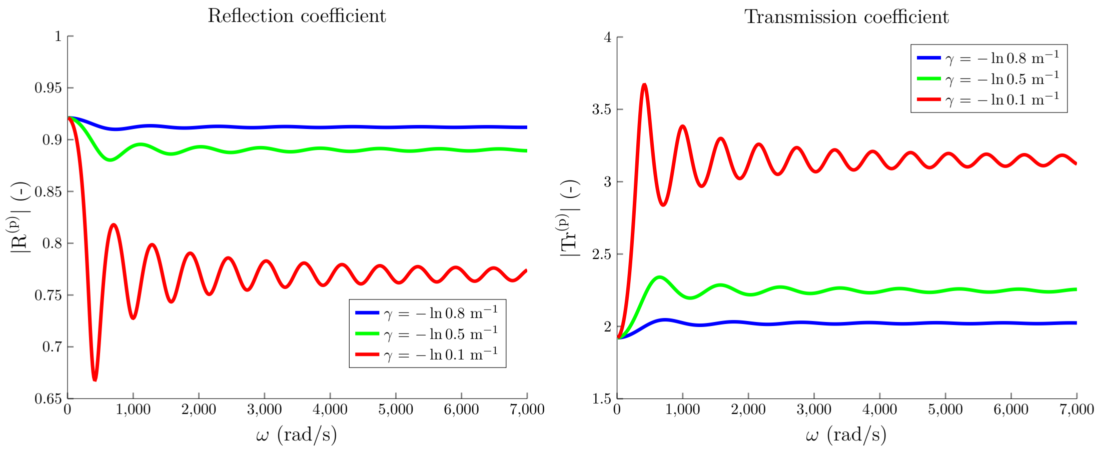

Fig. 16 Dependence of modulii of reflection and transmission coefficients for $T_A = 296$ K and different values of γ (negative gradient) in Eq. (4.86) on angular frequency.

5 Derivation of the Burgers-type equation for temperature-inhomogeneous fluids

In this section the Burgers-type equation for nonlinear acoustic waves in the temperature-inhomogeneous fluids and its dimensionless form is derivated. The classical Burgers equation is the most widely used model equation for studying the combined effects of dissipation and nonlinearity on progressive plane waves (see e.g. [7]).

5.1 Westervelt equation

Let us begin with the nonlinear acoustic wave equation for progressive waves, which is called the Westervelt equation (see e.g. [1], [8], [9])

$$\Box^2 p' = -\frac{\alpha}{\rho_0 c_0^4}\frac{\partial^3 p'}{\partial t^3} - \frac{\beta}{\rho_0 c_0^4}\frac{\partial^2 p'^2}{\partial t^2}\ , \tag{5.1}$$

where β is the nonlinearity coefficient, α is the sound diffusivity and $\Box^2$ is the d'Alembertion operator for plane waves. These notations mean

$$\beta = \frac{\varkappa + 1}{2}\ , \tag{5.2a}$$

$$\Box^2(\cdot) = \frac{\partial^2(\cdot)}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial(\cdot)}{\partial t}\ , \tag{5.2b}$$

$$\alpha = \frac{4}{3}\epsilon + \zeta + \kappa\left(\frac{1}{c_V} - \frac{1}{c_p}\right) \sim \mu\ , \tag{5.2c}$$

where c_V and c_p are specific heats at constant volume and pressure respectively and $\mu < 1$ a small dimensionless parameter.

The left hand side of Eq. (5.1) represents the canonical wave equation for acoustic waves in homogeneous fluids. For temperature-inhomogeneous media it was derived the wave equation (2.24) which can be expressed as

$$\Box_i^2 p' = 0\ , \tag{5.3}$$

where $\Box_i^2(\cdot) = \frac{\partial^2(\cdot)}{\partial x^2} - \frac{1}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}x}\frac{\partial(\cdot)}{\partial x} - \frac{1}{c_0^2}\frac{\partial^2(\cdot)}{\partial t^2}$ is the d'Alembertion operator for plane waves in temperature-inhomogeneous media. On the basis of this operator the Westervelt equation (5.1) can be modified for a temperature inhomogeneous media

$$\Box_i^2 p' = -\frac{\alpha}{\rho_0 c_0^4}\frac{\partial^3 p'}{\partial t^3} - \frac{\beta}{\rho_0 c_0^4}\frac{\partial^2 p'^2}{\partial t^2}\ . \tag{5.4}$$

The validity of this equation is restricted to regions with low temperature gradients.

5.2 Burgers-type equation

Given that the spatial variation of a plane progressive acoustic wave is small enough in proportion to one wavelength it is possible to apply the *multiple-scale* method to simplify Eq. (5.4). The retarded time $\tau = t - x/c_A$ is introduced.

Let us find the solution of Eq. (5.4) in the form (see e.g. [7])

$$p = p(x_1, \tau) \; , \tag{5.5}$$

where

$$x_1 = \mu x \; , \quad (5.6a) \qquad\qquad \tau = t - \frac{x}{c_A} \; . \quad (5.6b)$$

In the retarded time frame (i.e., for an observer in a reference frame that moves at speed c_A), nonlinearity and absorption separately produce only slow variations as functions of distance. Moreover, the relative order of the variations due to each effect is the same, i.e., it is $O(\mu)$. Thus, it can be anticipated that the combined effects of nonlinearity and absorbtion will introduce variations of the same order. The coordinate x_1 is referred to as the *slow scale* corresponding to the retarded time frame τ.

To derive a simplified progressive-wave equation that accounts for both absorption and nonlinearity, let us first rewrite Eq. (5.4) in the new coordinate system (x_1, τ). Transformations of the partial derivatives are

$$\frac{\partial(\cdot)}{\partial x} = \mu \frac{\partial(\cdot)}{\partial x_1} - \frac{1}{c_A}\frac{\partial(\cdot)}{\partial \tau} \; , \tag{5.7}$$

$$\frac{\partial^2(\cdot)}{\partial x^2} = -\mu \frac{2}{c_A}\frac{\partial^2(\cdot)}{\partial \tau \partial x_1} + \frac{1}{c_A^2}\frac{\partial^2(\cdot)}{\partial \tau^2} \; , \tag{5.8}$$

$$\frac{\partial(\cdot)}{\partial t} = \frac{\partial(\cdot)}{\partial \tau} \; , \qquad \frac{\partial^2(\cdot)}{\partial t^2} = \frac{\partial^2(\cdot)}{\partial \tau^2} \; , \qquad \frac{\partial^3(\cdot)}{\partial t^3} = \frac{\partial^3(\cdot)}{\partial \tau^3} \; , \tag{5.9}$$

applying the introduced notations

$$\begin{gathered}
\frac{\partial^2 p'}{\partial x^2} = -\mu \frac{2}{c_A}\frac{\partial^2 p'}{\partial \tau \partial x_1} + \frac{1}{c_A^2}\frac{\partial^2 p'}{\partial \tau^2} \; , \\
\frac{\mathrm{d}\rho_0}{\mathrm{d}x} = \mu \frac{\mathrm{d}\rho_0}{\mathrm{d}x_1} \; , \qquad \frac{\partial p'}{\partial x} = \mu \frac{\partial p'}{\partial x_1} - \frac{1}{c_A}\frac{\partial p'}{\partial \tau} \; , \\
\frac{\partial^2 p'}{\partial t^2} = \frac{\partial^2 p'}{\partial \tau^2} \; , \qquad \frac{\partial^3 p'}{\partial t^3} = \frac{\partial^3 p'}{\partial \tau^3} \; .
\end{gathered} \tag{5.10}$$

Substitution of derivatives (5.10) into Eq. (5.4) gives

$$-\mu \frac{2}{c_A}\frac{\partial^2 p'}{\partial \tau \partial x_1} + \frac{1}{c_A^2}\frac{\partial^2 p'}{\partial \tau^2} - \frac{\mu^2}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}x_1}\frac{\partial p'}{\partial x_1} + \frac{\mu}{\rho_0 c_A}\frac{\mathrm{d}\rho_0}{\mathrm{d}x_1}\frac{\partial p'}{\partial \tau} - \frac{1}{c_0^2}\frac{\partial^2 p'}{\partial \tau^2} = -\frac{\alpha}{\rho_0 c_0^4}\frac{\partial^3 p'}{\partial \tau^3} - \frac{2\beta}{\rho_0 c_0^4} p' \frac{\partial^2 p'}{\partial \tau^2} \; . \tag{5.11}$$

We are interested in the second approximation of the above equation. The third term in Eq. (5.11) is $O(\mu^3)$ and is therefore discarded. After that and small rearranging Eq. (5.11) can be written as

$$-\mu \frac{2}{c_A}\frac{\partial^2 p'}{\partial \tau \partial x_1} + \left(\frac{1}{c_A^2} - \frac{1}{c_0^2}\right)\frac{\partial^2 p'}{\partial \tau^2} + \frac{\mu}{\rho_0 c_A}\frac{\mathrm{d}\rho_0}{\mathrm{d}x_1}\frac{\partial p'}{\partial \tau} = -\frac{\alpha}{\rho_0 c_0^4}\frac{\partial^3 p'}{\partial \tau^3} - \frac{2\beta}{\rho_0 c_0^4} p' \frac{\partial^2 p'}{\partial \tau^2} \; . \tag{5.12}$$

Integration of Eq. (5.12) with respect to τ, multiplication of the resulting equation by $-c_A/2$ and having noted

$$-\frac{c_A}{2}\left(\frac{1}{c_A^2} - \frac{1}{c_0^2}\right) = \frac{1}{2c_A}\frac{c_A^2 - c_0^2}{c_0^2} = \frac{1}{2c_A}\frac{T_A - T_0}{T_0} = \frac{1 - \Xi(\sigma)}{2c_A \Xi(\sigma)} \tag{5.13}$$

leads to equation

$$\mu\frac{\partial p'}{\partial x_1} + \frac{1-\Xi(\sigma)}{2c_A\Xi(\sigma)}\frac{\partial p'}{\partial\tau} - \frac{\mu}{2\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}x_1}p' = \frac{\alpha c_A}{2\rho_0 c_0^4}\frac{\partial^2 p'}{\partial\tau^2} + \frac{\beta c_A}{\rho_0 c_0^4}p'\frac{\partial p'}{\partial\tau}\ . \tag{5.14}$$

Applying Eq. (2.26) and returning to the physical coordinate x in place of x_1 the above equation can be written as

$$\frac{\partial p'}{\partial x} + \frac{1-\Xi(\sigma)}{2c_A\Xi(\sigma)}\frac{\partial p'}{\partial\tau} + \frac{1}{2T_0}\frac{\mathrm{d}T_0}{\mathrm{d}x}p' - \frac{\alpha c_A}{2\rho_0 c_0^4}\frac{\partial^2 p'}{\partial\tau^2} - \frac{\beta c_A}{\rho_0 c_0^4}p'\frac{\partial p'}{\partial\tau} = 0\ , \tag{5.15}$$

and this is the Burgers-type equation for a temperature-inhomogeneous fluid.

5.3 Dimensionless Burgers-type equation

Using expressions (2.36) the Burgers-type equation (5.15) can be written as

$$\frac{\partial\Pi}{\partial\sigma} + \frac{[1-\Xi(\sigma)]\,\omega L}{2c_A\Xi(\sigma)}\frac{\partial\Pi}{\partial\theta} + \frac{1}{2\Xi(\sigma)}\frac{\mathrm{d}\Xi(\sigma)}{\mathrm{d}\sigma}\Pi - \frac{\alpha\omega^2 L}{2\Xi(\sigma)\,\rho_A c_A^3}\frac{\partial^2\Pi}{\partial\theta^2} - \frac{\beta\omega L}{\Xi(\sigma)\,c_A}\Pi\frac{\partial\Pi}{\partial\theta} = 0\ . \tag{5.16}$$

Introducing new constants

$$Q = \frac{\omega L}{2c_A}\ , \qquad G = \frac{\alpha\omega^2 L}{2\rho_A c_A^3}\ , \qquad N = \frac{\beta\omega L}{c_A} \tag{5.17}$$

and remembering

$$\sigma = \frac{x}{L}\ , \qquad \theta = \omega\tau\ , \tag{5.18}$$

the Burgers-type equation (5.16) can be rewritten into the form

$$\frac{\partial\Pi}{\partial\sigma} + \frac{1-\Xi(\sigma)}{\Xi(\sigma)}Q\frac{\partial\Pi}{\partial\theta} + \frac{1}{2\Xi(\sigma)}\Pi\frac{\mathrm{d}\Xi(\sigma)}{\mathrm{d}\sigma} - \frac{G}{\Xi(\sigma)}\frac{\partial^2\Pi}{\partial\theta^2} - \frac{N}{\Xi(\sigma)}\Pi\frac{\partial\Pi}{\partial\theta} = 0\ . \tag{5.19}$$

Equation (5.19) is the dimensionless Burgers-type equation for plane progressive nonlinear waves in temperature-inhomogeneous fluids.

6 Numerical solution of the Burgers-type equation for temperature-inhomogeneous fluids

In this chapter, a numerical method for solving the Burgers-type equation was implemented in the C programming language and the numerical solutions obtained from this method are presented.

6.1 Description of the numerical method

Let us write the dimensionless Burgers-type equation (5.19) for plane progressive nonlinear waves in temperature-inhomogeneous fluids once more

$$\frac{\partial \Pi}{\partial \sigma} + \frac{1-\Xi(\sigma)}{\Xi(\sigma)} Q \frac{\partial \Pi}{\partial \theta} + \frac{1}{2\Xi(\sigma)} \Pi \frac{\mathrm{d}\Xi(\sigma)}{\mathrm{d}\sigma} - \frac{G}{\Xi(\sigma)} \frac{\partial^2 \Pi}{\partial \theta^2} - \frac{N}{\Xi(\sigma)} \Pi \frac{\partial \Pi}{\partial \theta} = 0 \,, \tag{6.1}$$

where $Q = \omega L / \left(2c_A\right)$, $G = \alpha\omega^2 L / \left(2\rho_A c_A^3\right)$, $N = \beta\omega L / c_A$, $\sigma = x/L$, $\theta = \omega\tau$.

Let us suppose acoustic pressure in the form (see e.g. [1], [10])

$$\Pi\left(\sigma, \theta\right) = \sum_{n=-\infty}^{\infty} \Phi_n\left(\sigma\right) e^{\mathrm{j}n\theta} \,, \tag{6.2}$$

where

$$\Phi_{-n}\left(\sigma\right) = \Phi_n^*\left(\sigma\right) \,. \tag{6.3}$$

Here, the asterisk sign * means complex conjugate.

Remembering dependence of functions $\Pi\left(\sigma, \theta\right)$, $\Xi\left(\sigma\right)$, $\Phi\left(\sigma\right)$ let us omit writing these dependencies in the following equations to simplify expressions.

Substituting Eq. (6.2) into Eq. (6.1)

$$\sum_{n=-\infty}^{\infty} \left[\frac{\mathrm{d}\Phi_n}{\mathrm{d}\sigma} + \mathrm{j}n \frac{1-\Xi}{\Xi} Q \Phi_n + \frac{1}{2\Xi} \Phi_n \frac{\mathrm{d}\Xi}{\mathrm{d}\sigma} + \frac{n^2 G}{\Xi} \Phi_n \right] e^{\mathrm{j}n\theta} = \frac{1}{2} \frac{N}{\Xi} \frac{\partial}{\partial \theta} \sum_{i=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \Phi_i \Phi_m e^{\mathrm{j}(i+m)\theta} \,. \tag{6.4}$$

The right-hand side of Eq. (6.4) can be derivated, thus Eq. (6.4) can be rewritten

$$\sum_{n=-\infty}^{\infty} \left[\frac{\mathrm{d}\Phi_n}{\mathrm{d}\sigma} + \mathrm{j}n \frac{1-\Xi}{\Xi} Q \Phi_n + \frac{1}{2\Xi} \Phi_n \frac{\mathrm{d}\Xi}{\mathrm{d}\sigma} + \frac{n^2 G}{\Xi} \Phi_n \right] e^{\mathrm{j}n\theta} = \sum_{n'=-\infty}^{\infty} \left(\mathrm{j} \frac{n'}{2} \frac{N}{\Xi} \sum_{m=-\infty}^{\infty} \Phi_m \Phi_{n'-m} \right) e^{\mathrm{j}n'\theta} \,, \tag{6.5}$$

where $n' = i + m$.

The left-hand and right-hand sides of Eq. (6.5) are equal for arbitrary dimensionless time θ only for $n' = n$ and when terms in brackets are equal, so

$$\frac{\mathrm{d}\Phi_n}{\mathrm{d}\sigma} + \mathrm{j}n\frac{1-\Xi}{\Xi}Q\Phi_n + \frac{1}{2\Xi}\Phi_n\frac{\mathrm{d}\Xi}{\mathrm{d}\sigma} + \frac{n^2G}{\Xi}\Phi_n = \mathrm{j}\frac{n}{2}\frac{N}{\Xi}\sum_{m=-\infty}^{\infty}\Phi_m\Phi_{n-m} \ . \tag{6.6}$$

To apply convolution to the right-hand side of Eq. (6.6) it is convenient to express the summation in the following step using the expression (6.3)

$$\sum_{m=-\infty}^{\infty}\Phi_m\Phi_{n-m} = \sum_{m=1}^{n-1}\Phi_m\Phi_{n-m} + 2\sum_{m=n+1}^{\infty}\Phi_m\Phi^*_{m-n} \ . \tag{6.7}$$

Thus, there are only terms Φ_n with positive n in summation (6.7). Thanks to this and expression (6.7) it is possible to rewrite Eq. (6.6)

$$\begin{aligned}\frac{\mathrm{d}\Phi_n}{\mathrm{d}\sigma} = &-\mathrm{j}n\frac{1-\Xi}{\Xi}Q\Phi_n - \frac{1}{2\Xi}\Phi_n\frac{\mathrm{d}\Xi}{\mathrm{d}\sigma} - \frac{n^2G}{\Xi}\Phi_n \\ &+\mathrm{j}\frac{n}{2}\frac{N}{\Xi}\left(\sum_{m=1}^{n-1}\Phi_m\Phi_{n-m} + 2\sum_{m=n+1}^{\infty}\Phi_m\Phi^*_{m-n}\right) \ .\end{aligned} \tag{6.8}$$

By placing a limitation on the M terms of the Fourier series, Eq. (6.8) can be rewritten as

$$\begin{aligned}\frac{\mathrm{d}\Phi_n}{\mathrm{d}\sigma} = &-\mathrm{j}n\frac{1-\Xi}{\Xi}Q\Phi_n - \frac{1}{2\Xi}\Phi_n\frac{\mathrm{d}\Xi}{\mathrm{d}\sigma} - \frac{n^2G}{\Xi}\Phi_n \\ &+\mathrm{j}\frac{n}{2}\frac{N}{\Xi}\left(\sum_{m=1}^{n-1}\Phi_m\Phi_{n-m} + 2\sum_{m=n+1}^{M}\Phi_m\Phi^*_{m-n}\right) \ .\end{aligned} \tag{6.9}$$

Equation (6.9) is a simultaneous system of ordinary nonlinear differential equations with M complex independent variables Φ_1, Φ_2, ..., Φ_M, which can be written as

$$\Phi_n = \frac{1}{2}\left(\Phi R_n - \mathrm{j}\Phi I_n\right) \ . \tag{6.10}$$

The system of equations, which is represented by Eq. (6.9), can be solved numerically by the standard fourth-order Runge-Kutta method (see e.g. [11]). Let us denote Eq. (6.9) by function $f(\sigma_n, \Phi_n)$. To have an initial value problem just a boundary initial condition should be added. The organization of the numerical method then can be written as

$$\begin{aligned} k_1 &= hf(\sigma_n, \Phi_n) \ , \\ k_2 &= hf(\sigma_n + \frac{h}{2}, \Phi_n + \frac{k_1}{2}) \ , \\ k_3 &= hf(\sigma_n + \frac{h}{2}, \Phi_n + \frac{k_2}{2}) \ , \\ k_4 &= hf(\sigma_n + h, \Phi_n + k_3 \ , \\ \Phi_{n+1} &= \Phi_n + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) + O\left(h^5\right) \ , \end{aligned} \tag{6.11}$$

where h is a step size and $O\left(h^5\right)$ is an error term of the 4th order.

This Runge-Kutta method was written in C language, as a result a boundary value problem is resolved.

To find acoustic pressure, according to Eqs. (6.2), (6.10) and the result from C code, the following equation should be evaluated

$$\Pi(\sigma, \theta) = \sum_{m=1}^{M}\left[\Phi R_m \cos\left(m\theta\right) + \Phi I_m \sin\left(m\theta\right)\right] \ . \tag{6.12}$$

6.2 Representation of an artificial attenuation filter

The sequential generation of higher harmonics leads to the transfer of acoustic energy from the first harmonic components to the components higher, where the acoustic energy is damped more effectively, because attenuation increases with frequency. This leads to a phenomenon which is called the nonlinear attenuation. Considering the finite number of Fourier series terms leads to the interruption of acoustic energy flow from the lower harmonics to higher, i.e. the disruption of nonlinear attenuation, as a result acoustic energy starts accumulating at the highest harmonics and therefore higher harmonics grow abnormally. Due to this fact the higher harmonics have more effect than they really have during nonlinear interaction. For this reason even lower harmonics start rising, that is why numerical instability occurs in the solution. In a way this instability influence affects the solution by causing unwanted and gradually rising oscillations. In order to avoid this phenomenon, it is necessary, especially while considering a lower number of harmonics, to introduce artificial attenuation which replaces dissipation of acoustic energy that occurs at ignored higher harmonic components. Therefore, artificial attenuation should influence higher harmonics to a greater extent than lower ones. As an example, attenuation can be implemented so that with each integration step calculated harmonic components can be multiplied by the following function

$$\frac{\sin\left(\frac{n}{D}\right)}{\frac{n}{D}}\,, \tag{6.13}$$

where n is a number of particular harmonic component, D is a selected constant.

This filter is implemented in C code.

6.3 Graphical representation of the Burgers-type equation solution for a linear temperature distribution

Let us see possible solutions of the Burgers-type equation. A linear temperature distribution is given by Eq. (3.71).

6.3.1 Positive temperature gradient

Let us set the angular frequency ω to 10000 s^{-1}, the characteristic temperature T_A to 298.15 K, the complex pressure amplitude of incident wave is -2000j Pa. The first point of interest is a linear temperature distribution with a positive gradient. The gradient is set by constant A. The solution is represented in Fig. 17.

The following statement can be made: as the temperature gradient is higher, pressure amplitude is lower. The same result could be seen from the transmission coefficient graph in Fig. 13. In Fig. 17 distortion of a wave profile can be also observed. There is no a big difference between constant temperature T_A and nonzero positive gradients since these gradients are small.

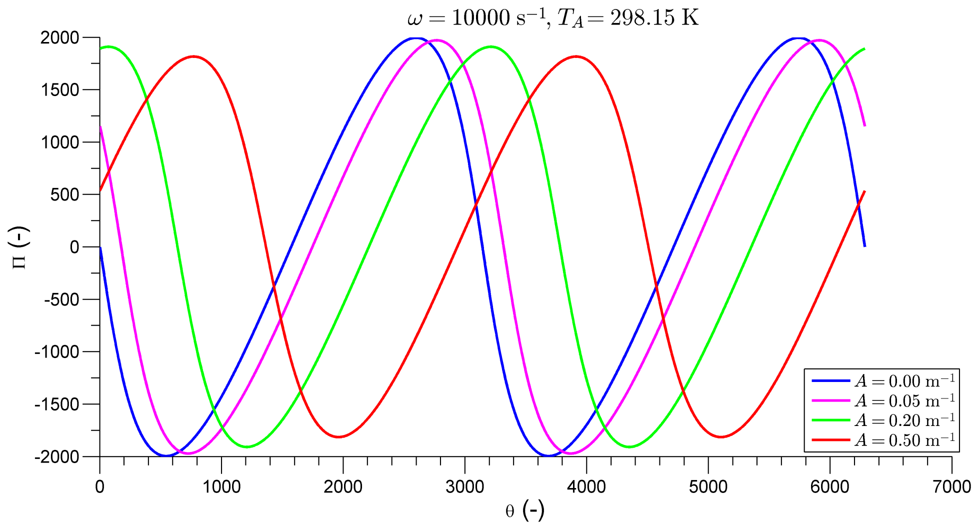

Fig. 17 The solution of the Burgers-type equation for temperature inhomogeneous fluids with positive temperature gradient A.

6.3.2 Negative temperature gradient

The second point of interest is a linear temperature distribution with a negative gradient A. Let us leave the angular frequency ω with the same value of 10000 s^{-1} and the complex pressure amplitude is -2000j Pa, the characteristic temperature T_A set to 357.78 K.

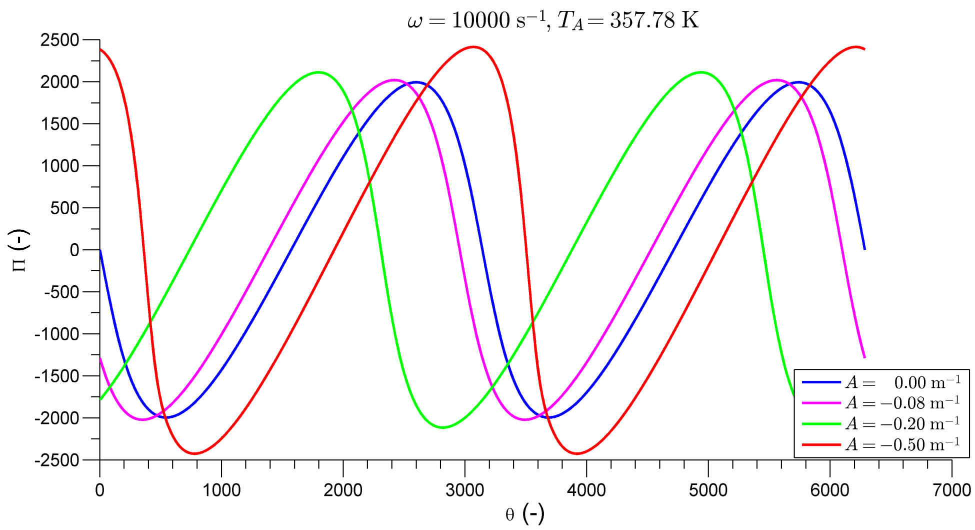

Fig. 18 The solution of the Burgers-type equation for temperature inhomogeneous fluids with negative temperature gradient A.

As the temperature gradient is lower, pressure amplitude is higher. The same result could be seen from the transmission coefficient graph in Fig. 14. As for a positive temperature gradient the difference of phases can be seen, the nonlinearity of a wave profile can be also observed, the difference between constant temperature T_A and nonzero negative gradients are also small enough due to small gradients.

6.3.3 Different lengths of a duct with a positive temperature gradient

The third point of interest is a linear temperature distribution with a positive gradient A for different duct lengths. Let us leave the angular frequency ω the same value of 10000 s^{-1} and the complex pressure amplitude is -2000j Pa, the characteristic temperature T_A is set to 298.15 K, a temperature gradient A has a positive value of 0.5 m^{-1}. The size of a step length is 0.25 m.

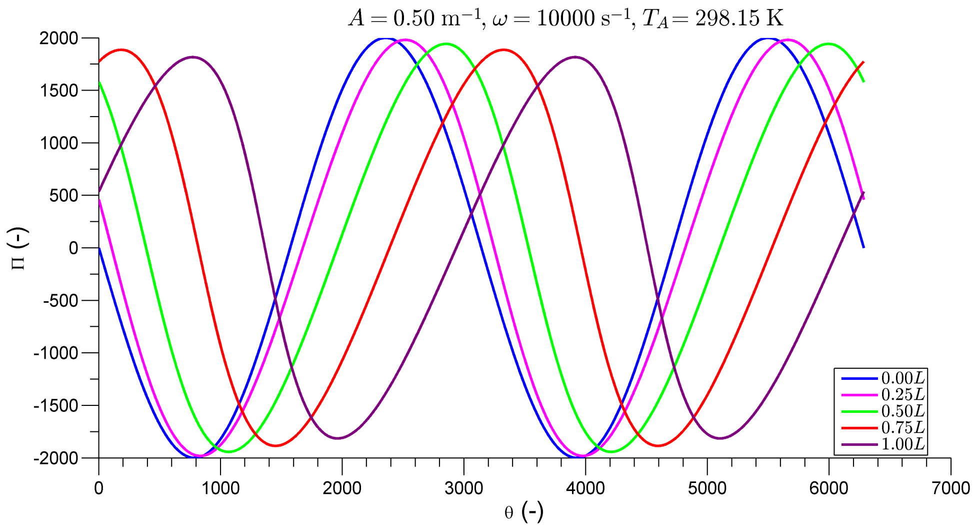

Fig. 19 The solution of the Burgers-type equation for temperature inhomogeneous fluids with a positive temperature gradient $A = 0.5$ m^{-1} but different lengths of a duct.

In Fig. 19 can be observed how the wave-profile is gradually distorted.

7 Conclusion

In this bachelor thesis I have studied descriptions of acoustic waves in fluids with spatially variable temperatures, including derivation of one-dimensional model equations.

Analytical solutions of linear model equations for chosen temperature distributions were found. For these solutions transmission and reflection coefficients were calculated. The dependencies of these coefficients on angular frequency were shown in graphs.

In order to describe nonlinear acoustic plane waves for small temperature gradients the Burgers-type equation was derived. The obtained Burgers-type equation was solved numerically in the frequency domain by the fourth-order Runge-Kutta method since the solution is unknown. A numerical code was written in the programming language C. The numerical solutions were used for plotting nonlinear wave-profiles and discussed.

All assignment points were accomplished.

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Appendix A

Attached CD contents

Matlab calculation scripts. These scripts contain calculations of some complex expressions and generations of graphs represented in the thesis.
Maple calculation script, which holds some calculations from the thesis.
Fourth-order Runge-Kutta numerical method written in the C programming language.
Electronic version of the thesis in PDF.
All Figures from the thesis.