Unified homogenization theory for magnetoinductive and electromagnetic waves in split-ring metamaterials

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A unified homogenization procedure for split-ring metamaterials taking into account time and spatial dispersion is introduced. It is shown that electromagnetic and magnetoinductive waves propagating in the metamaterial are obtained from this analysis. Therefore, the proposed time and spatially dispersive permeability accounts for the characterization of the complete spectrum of waves of the metamaterial. Finally, it is shown that the proposed theory is in good quantitative and qualitative agreement with full wave simulations.

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I. INTRODUCTION

Diamagnetic properties of systems of conducting rings have long been known by physicists. In 1852 Wilhem Weber [1] tried to explain natural diamagnetism (discovered by Faraday some years before) as a consequence of the excitation of induced currents in some hypothetical conducting loops that supposedly existed in diamagnetic materials. In order to enhance the magnetic properties of artificial media (or metamaterials in modern terminology) made from metallic conducting rings, Shelkunoff proposed in 1952 to introduce a capacitor [2], so that the rings become resonant. More recently Pendry *et al.* [3] proposed to replace the capacitively loaded rings by planar split-ring resonators (SRRs) which substitute the lumped capacitor with a distributed capacitance between the rings. Because Pendry's SRRs can be easily manufactured by using standard printed circuit technologies, this design opened the way to manufacturing true magnetic metamaterials made of many individual elements (SRRs) at many laboratories around the world. As a consequence of this resonant behavior, capacitively loaded rings and/or SRRs can produce metamaterials with negative magnetic permeability above resonance. It is also well known [4–6] that when a system of these elements is properly combined with another system of elements (metallic wires or plates, for instance) producing a negative electric permittivity [7], a metamaterial with simultaneously negative permittivity and permeability (or left-handed metamaterial [8]) arises in the frequency band where both subsystems present negative parameters. Remarkably, the electric and magnetic properties of such combinations are, quite approximately, the superposition of the electric and magnetic properties of each subsystem. This superposition hypothesis is not apparent at all (see, for instance, Refs. [9,10]) and, for the specific SRRs and wire configuration proposed in Ref. [4], it can be admitted that it is valid provided the elements of both subsystems are placed in such a way that their quasistatic fields do not interact or interact weakly [10,11]. Almost simultaneously, other analyses and experiments [12,13] did show that SRR-based metamaterials also support, in some frequency bands, slow waves based on short-range interactions between the SRRs: the so-called magnetoinductive (MI) waves, which cannot be deduced from the usually assumed local magnetic permeability of the metamaterial. Interestingly, many of the physical effects expected in negative permeability and left-handed metamaterials, such as frequency band gaps and frequency bands of backward-wave propagation, also come out from the analysis when the coupling between electromagnetic and MI waves in SRR systems is considered [14], thus providing an alternative explanation for such effects. Although the analysis in Ref. [14] has a great heuristic value, it cannot be considered as fully satisfactory because it only considers one-dimensional systems in the nearest-neighbor approximation. On the other hand, the presence of waves which cannot be deduced from a local time dispersive magnetic permeability in split-ring metamaterials can be expected from the fact that its periodicity is usually not smaller than one tenth of a wavelength. As is well known [15], when the periodicity of a given medium approaches the wavelength of the electromagnetic radiation, it becomes not only time dispersive but also spatially dispersive. Therefore, it can be expected that both electromagnetic and MI waves would come out from the analysis if spatial dispersion in split ring metamaterials were taken into account. In fact, the main purpose of this paper is to develop a spatially dispersive homogenization procedure able to describe both types of waves.

II. ANALYSIS

In order to simplify the analysis, we will consider an ideal metamaterial made of a cubic arrangement of LC circuits supporting current loops as sketched in Fig. 1(a) for a unit cell. We define the current vector $\mathbf{I}^{\mathbf{n}}$ on each unit cell as $\mathbf{I}^{\mathbf{n}} = (I_x^{\mathbf{n}}, I_y^{\mathbf{n}}, I_z^{\mathbf{n}})$, where $\mathbf{n} = (n_x, n_y, n_z)$ specifies the location of each unit cell in the lattice and $I_i^{\mathbf{n}}$ denotes the current along the loop located in the face normal to the i direction of the unit cell of index $\mathbf{n}$. The time dependence is assumed to be of

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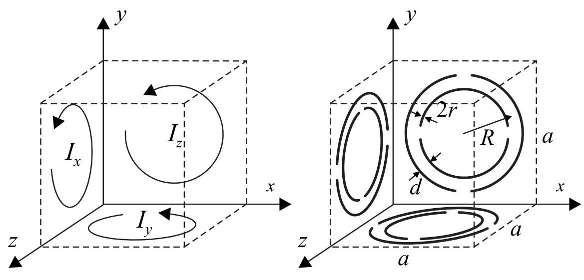

FIG. 1. (a) Unit cell of a material formed by a cubic array of current loops. Each unit cell has three current loops centered at the faces of the cube. (b) Similar to (a) but for a realistic metamaterial formed by edge coupled SRRs with two splits. The SRRs are formed by circular wires with radius r. The distance between the inner and outer rings is d and the average radius of the particle is R.

the form $\mathbf{I}^{\mathbf{n}} \propto \exp(j\omega t)$. Each component of the current vector is governed by the equation

$$\left(j\omega L + \frac{1}{j\omega C}\right) I_i^{\mathbf{n}} = -j\omega \Phi_i^{\mathbf{n}}, \tag{1}$$

where L, C are self-inductance and self-capacitance of the circuit (losses are neglected by simplicity, although they can be easily introduced in the analysis through a ring resistance). In Eq. (1) $\Phi_i^{\mathbf{n}}$ is the total magnetic flux through the considered loop which, using Lorentz local field approximation, can be calculated as

$$\Phi_i^{\mathbf{n}} = A\mu_0\left(H_i + \frac{M_i}{3}\right) + \sum_{(\mathbf{m}\neq\mathbf{n})\wedge(j\neq i)}^{r<R} M_{ij}^{\mathbf{nm}} I_j^{\mathbf{m}}, \tag{2}$$

where A is the area of the loop, $\mathbf{H}$, $\mathbf{M}$ are the macroscopic magnetic field and magnetization on the ring, and $M_{ij}^{\mathbf{nm}}$ are the mutual inductances between the loops oriented along the i and j axes in unit cells with indexes $\mathbf{n}$ and $\mathbf{m}$. The summation is extended to the cells inside a sphere centered around the nth unit cell and with radius R sufficiently large so that the region outside can be approximated by a continuous material but such that R is smaller than the wavelength to guarantee that the Lorentz approximation can be used. Therefore, in Eq. (2) the first term accounts for the contribution of all rings outside the sphere which, according to the standard Lorentz local field theory, are considered as a "continuous medium," whereas the summation accounts for the detailed contribution of each ring inside the sphere.

In the following we will assume a spatial field dependence of the kind $\{\mathbf{H},\mathbf{M}\}=\{\mathbf{H}_0,\mathbf{M}_0\}\exp(-j\mathbf{k}\cdot\mathbf{r})$ and $\mathbf{I}^{\mathbf{n}}=\mathbf{I}_0\exp(-ja\mathbf{k}\cdot\mathbf{n})$, where a is the lattice periodicity and $\mathbf{I}_0$, $\mathbf{H}_0$, $\mathbf{M}_0$ are constant vectors. With the assumed time and space dependence, the macroscopic Maxwell equations lead to

$$(k_m^2 - k^2)\mathbf{H}_0 + (k_m^2 - \mathbf{k}\mathbf{k}\cdot)\mathbf{M}_0 = 0, \tag{3}$$

with $k_m = k_0\sqrt{\epsilon_r} = \omega\sqrt{\epsilon_r\epsilon_0\mu_0}$, where ϵ_r is the macroscopic relative dielectric constant of the metamaterial. By combining Eqs. (1) and (2) the following equation for $\mathbf{I}_0$ is obtained:

$$\bar{\bar{Z}}(\mathbf{k},\omega)\cdot\mathbf{I}_0 = -j\omega A\mu_0\left(\mathbf{H}_0 + \frac{\mathbf{M}_0}{3}\right), \tag{4}$$

where $\bar{\bar{Z}}(\mathbf{k},\omega)$ is an impedance matrix which incorporates all the magnetoinductive effects between the neighboring rings. Explicit expressions for the diagonal and the off-diagonal terms of $\bar{\bar{Z}}$ are

$$Z_{ii} = j\omega L\left\{1 - \frac{\omega_0^2}{\omega^2} + \sum_{\mathbf{n}\neq\mathbf{0}}^{r<R} \frac{M_{ii}^{\mathbf{0n}}}{L} e^{-ja\mathbf{k}\cdot\mathbf{n}}\right\}, \tag{5}$$

$$Z_{ij} = Z_{ji} = j\omega L \sum_{\mathbf{n}}^{r<R} \frac{M_{ij}^{\mathbf{0n}}}{L} e^{-ja\mathbf{k}\cdot\mathbf{n}}, \quad i \neq j, \tag{6}$$

where $\omega_0 = 1/\sqrt{LC}$ is the frequency of resonance of the rings. Taking into account that $\mathbf{M}_0 = A\mathbf{I}_0/a^3$, it is possible to combine Eq. (3) with Eq. (4) which gives

$$\left\{\bar{\bar{Z}}(\mathbf{k},\omega) - \frac{j\omega\mu_0 A^2}{3a^3}\frac{2k_m^2 + k^2 - 3\mathbf{k}\mathbf{k}}{k_m^2 - k^2}\right\}\cdot\mathbf{I}_0 = 0. \tag{7}$$

The dispersion relation for plane waves in the metamaterial is obtained by equating the determinant of Eq. (7) to zero. In the most general case this equation can be only solved numerically. However, in some cases, it is possible to give analytical solutions. In particular, for propagation along one of the coordinate axes (for $\mathbf{k}=k\hat{\mathbf{x}}$, for instance) the summation in Eq. (6) vanishes because all unit cells in planes perpendicular to the x axis are in phase and mutual inductances cancel out couple by couple (for instance, the mutual inductances between the ring marked I_x and the rings placed on the top and lower faces of the cube of Fig. 1(a) cancel each other, and so on). Therefore, the matrix $\bar{\bar{Z}}(\mathbf{k},\omega)$ becomes diagonal, and Eq. (7) can be easily solved. This gives two branches: a longitudinal wave with $\mathbf{I}_0 = I_{0x}\hat{\mathbf{x}}$ given by

$$Z_{xx} - \frac{2}{3}\frac{j\omega\mu_0 A^2}{a^3} = 0 \tag{8}$$

and a transverse wave with $\mathbf{I}_0 = I_{0y}\hat{\mathbf{y}} + I_{0z}\hat{\mathbf{z}}$ given by

$$Z_{yy} - \frac{j\omega\mu_0 A^2}{3a^3}\frac{2k_m^2 + k_x^2}{k_m^2 - k_x^2} = 0. \tag{9}$$

If only interactions with the closest rings are considered for the computation of the summations in Eqs. (5) and (6) the dispersion equation for the longitudinal wave becomes

$$\frac{\omega_0^2}{\omega^2} = 1 + 2\frac{M_a}{L}\cos(ak_x) + 4\frac{M_c}{L} - \frac{2}{3}\frac{\alpha_0}{a^3}, \tag{10}$$

where $\alpha_0 = \mu_0 A^2/L$ and M_a and M_c are the mutual inductances between closest rings of the same orientation, placed in the axial and the coplanar directions, respectively. It can be easily recognized that Eq. (10) corresponds to the dispersion relation for longitudinal magnetoinductive waves [12], with some small corrections, which take into account the effects of the rings other than the nearest neighbors in the axial direction. In the same approximation the dispersion equation for transverse waves can be written as

$$\frac{k_x^2}{k_m^2} - 1 = \frac{\frac{\alpha_0}{a^3}}{\frac{\omega_0^2}{\omega^2} - 1 - \frac{2M_c}{L}\cos(ak_x) - \frac{2(M_a + M_c)}{L} - \frac{\alpha_0}{3a^3}}. \tag{11}$$

For high values of k_x ($k_x \gg k_m$) this equation reduces to

$$\frac{\omega_0^2}{\omega^2} = 1 + \frac{2M_c}{L}\cos(ak_x) + \frac{2(M_a + M_c)}{L} + \frac{\alpha_0}{3a^3}, \tag{12}$$

which corresponds to the dispersion relation for transverse magnetoinductive waves [12]. On the other hand, in the long wavelength limit ($ak_x \ll 1$) Eq. (11) reduces to

$$\chi(\omega) = \frac{k_x^2}{k_m^2} - 1 = \frac{\frac{\alpha_0}{a^3}}{\frac{\omega_0^2}{\omega^2} - 1 - \frac{2M_a}{L} - \frac{4M_c}{L} - \frac{\alpha_0}{3a^3}}. \tag{13}$$

This equation gives the value for the magnetic susceptibility that is obtained when the Lorentz homogenization procedure is applied to the metamaterial with the ring magnetic polarizabilities $\alpha = \alpha_0(\omega_0^2/\omega^2 - 1)^{-1}$ already proposed in Ref. [16], except for a small correction term $2M_a/L + 4M_c/L$ accounting for the effect of the closest rings. Actually, if such a correction term is calculated by assuming a magnetic dipole approximation for the rings, it can be easily shown that it vanishes, thus giving exactly the Clausius-Mossotti formula for the susceptibility. Therefore, we can conclude that in the long wavelength limit, the transverse waves (9) correspond to the electromagnetic waves that are obtained from the local time-dispersive permeability $\mu = \mu_0\{1 + \chi(\omega)\}$. Conversely, in the short wavelength limit ($k_x \gg k_m$), they converge to the transverse magnetoinductive waves (12). Furthermore, from $\mathbf{M}_0 = A\mathbf{I}_0/a^3$ and Eq. (4)

$$\mathbf{M}_0 = \left\{ \frac{ja^3}{\mu_0 \omega A^2} \bar{\bar{Z}}(\mathbf{k},\omega) - \frac{1}{3} \right\}^{-1} \cdot \mathbf{H}_0 = \bar{\bar{\chi}}(\mathbf{k},\omega) \cdot \mathbf{H}_0 \tag{14}$$

can be obtained. Now in Eq. (3) $\mathbf{M}_0$ can be replaced by Eq. (14) leading to

$$\{(-k^2 + k_m^2) + (-\mathbf{k}\mathbf{k} + k_m^2) \cdot \bar{\bar{\chi}}(\mathbf{k},\omega)\} \cdot \mathbf{H}_0 = 0, \tag{15}$$

which gives the same dispersion equation as Eq. (7). Therefore we can conclude that the nonlocal (i.e., time and spatially dispersive) magnetic permeability

$$\bar{\bar{\mu}}(\mathbf{k},\omega) = \mu_0 \left(1 + \left\{ \frac{ja^3}{\mu_0 \omega A^2} \bar{\bar{Z}}(\mathbf{k},\omega) - \frac{1}{3} \right\}^{-1} \right) \tag{16}$$

provides a complete characterization of the metamaterial, accounting for all kinds of waves propagating through it. In the long wavelength limit ($a|\mathbf{k}| \ll 1$) all the exponential terms in $\bar{\bar{Z}}(\mathbf{k},\omega)$ can be equated to unity and the magnetic permeability (14) reduces to the scalar time-dispersive permeability $\mu = \mu_0[1 + \chi(\omega)]$ with $\chi(\omega)$ given by Eq. (13). Alternatively, following Landau's description [15], the metamaterial can be also described by an equivalent spatially dispersive permittivity $\bar{\bar{\epsilon}}(\mathbf{k},\omega)$. The relation between this dielectric permittivity and the proposed magnetic permeability Eq. (16) is given by Eq. (43) in Ref. [17].

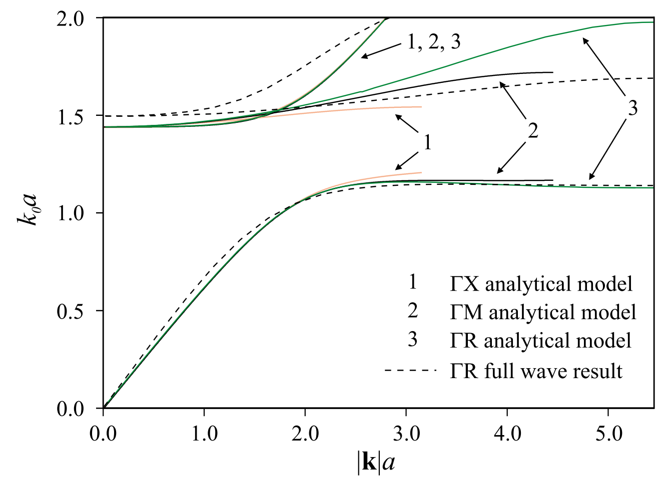

FIG. 2. (Color online) Dispersion diagram along Γ-X, Γ-M, Γ-R directions obtained from the analytical model (lines 1, 2, 3) and from a full-wave simulation (dashed line). The coordinates of selected points are $\Gamma = (0,0,0)$, $X = (\pi/a, 0, 0)$, $M = (\pi/a, \pi/a, 0)$, $R = (\pi/a, \pi/a, \pi/a)$.

III. NUMERICAL EXAMPLE

As a numerical example, we have studied the propagation of electromagnetic waves in a metamaterial formed by the simple cubic lattice of split-ring resonators whose unit cell is depicted in Fig. 1(b). The capacitance C and self-inductance L were calculated following the ideas of Ref. [18] but for the case of a SRR made of wires instead of planar strips. This calculation was carried out by using well-known formulas [19] as explained in Ref. [20]. The mutual inductances $M_{ij}^{\mathbf{nm}}$ were calculated numerically using Neumann's formula including time retardation. The macroscopic permittivity was evaluated by substituting the SRRs by planar conducting disks of the same external radius, and by using the static Lorentz homogenization theory [21]. This approach yielded the approximate value $\epsilon_r = 2.5$. Using Eq. (7) and the first neighbor approximation, the dispersion characteristic of the electromagnetic modes supported by the metamaterial along different directions of the first Brillouin zone was calculated. The result for the geometry associated with $R = 0.44a$, $r = 0.005a$, and $d = 0.03a$ [see Fig. 1(b)] is depicted in Fig. 2. It can be seen that the band structure is formed by three branches and contains a band gap for all the depicted directions of $\mathbf{k}$. For $\mathbf{k}$ along Γ-X, the first and third branches correspond to the transversal mode described by Eq. (9) and the second branch corresponds to the longitudinal mode described by Eq. (8). Figure 2 also shows the high isotropy of the transversal mode even for moderate values of $\mathbf{k}$, a fact that is expected from the tetrahedral symmetry of the system [22].

To assess the accuracy of the proposed analytical model, we have also numerically computed the exact band structure of the aforementioned periodic material using the hybrid-

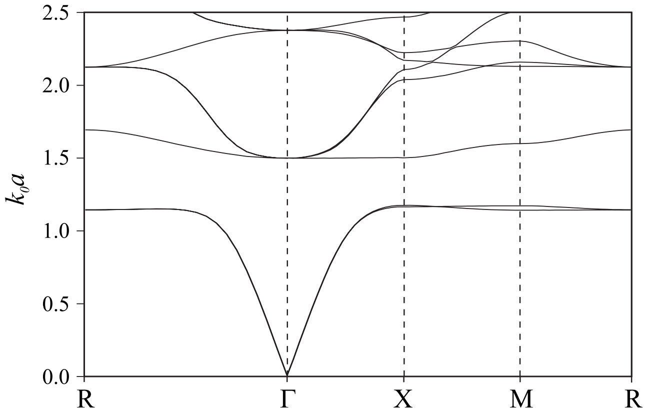

FIG. 3. Dispersion diagram along the closed path R-Γ-X-M-R obtained from full-wave simulation. The horizontal axis shows a projection of **k** on the corresponding line on the boundary of the Brillouin zone.

plane-wave-integral-equation formalism introduced in Ref. [23]. The result of the numerical simulation is presented in Fig. 2 for the specific direction Γ-R, and in Fig. 3 for the closed path R-Γ-X-M-R. Good qualitative agreement between theory and simulation can be seen from Fig. 2. We think that the quantitative disagreement for high values of k in the second branch can be attributed to the specific local field approximation considered in Eq. (2), which is strictly valid only for small values of k. In the case of the third branch, the disagreement is due to the proximity of the second resonance of the SRRs, which is not taken into account in the model. This effect is more visible in Fig. 3, where higher frequency branches are included. This figure also shows a complete electromagnetic band gap in the range $1.18 < k_0 a < 1.50$, in agreement with the hypothesis of a negative permeability in such a frequency band. It is worth noting that the effect of the substitution $k_m^2 \to -k_m^2$ (or equivalently $\epsilon_r \to -\epsilon_r$) into Eq. (11) is the onset of a backward wave pass-band in the frequency range of the stop-band of Fig. 3, as well as the conversion of the pass-bands of Fig. 2 into stop-bands. Therefore, the proposed model will be also useful for the analysis of isotropic left-handed media made of SRRs and wires or any other elements providing a macroscopic negative permittivity (provided the conditions for the validity of the superposition hypothesis previously discussed in Ref. [10] are fulfilled). Work in this direction is in progress.

IV. CONCLUSIONS

A homogenization procedure for split-ring metamaterials taking into account spatial dispersion has been developed. The spatially dispersive permeability arising from this homogenization accounts for all the electromagnetic spectra observed in these composites, including electromagnetic and magnetoinductive waves. It has been also shown that this spatially dispersive permeability continuously approaches to the Lorentz local permeability in the long wavelength limit. From this analysis follows that transverse magnetoinductive waves are the continuation, at short wavelengths, of the well known transverse electromagnetic waves that can be found in the long wavelength limit. However, longitudinal magnetoinductive waves are not related to electromagnetic waves but to the collective oscillations of the metamaterial arising at $\mu=0$ in the long wavelength limit. It has been also observed that, when a macroscopic negative permittivity is imposed to the metamaterial, a typical left-handed pass-band appears at those frequencies where the magnetic permeability becomes negative. Therefore, we feel that the proposed homogenization procedure provides a complete macroscopic characterization of negative-μ and left-handed split-ring metamaterials.

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