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SO(3) ⊂ SU(3) REVISITED

Čestmír Burdík, Severin Pošta*, Erik Rapp

Czech Technical University in Prague, Faculty of Nuclear Sciences and Physical Engineering, Department of Mathematics, Trojanova 13, 120 00 Prague, Czech Republic

* corresponding author: severin.posta@fjfi.cvut.cz

Abstract. This paper reproduces the result of Elliot, namely that the irreducible finite dimensional representation of the Lie algebra su(3) of highest weight (m, n) is decomposed according to the embedding $\mathrm{so}(3) \subset \mathrm{su}(3)$. First, a realisation (a representation in terms of vector fields) of the Lie algebra su(3) is constructed on a space of polynomials of three variables. The special polynomial basis of the representation space is given. In this basis, we find the highest weight vectors of the representation of the Lie subalgebra so(3) and in this way the representation space is decomposed to the direct sum of invariant subspaces. The process is illustrated by the example of the decomposition of the representation of highest weight $(2, 2)$. As an additional result, the generating function of the decomposition is given.

Keywords: Lie algebra, realisation, representation, decomposition, embedding.

1. Introduction

In the article by Elliot [1], the plethsym method is used to decompose the irreducible finite dimensional representation of Lie algebra su(3), the Lie algebra of anti-Hermitian 3×3-matrices with vanishing trace, with the highest weight (m, n), where m, n are non-negative integers, into the direct sum of irreducible representations of so(3), the Lie algebra of 3×3 skew-symmetric matrices, according to the embedding $\mathrm{so}(3) \subset \mathrm{su}(3)$. The embedding $\mathrm{SO}(3) \subset \mathrm{SU}(3)$ is widely used in theoretical physics and corresponding irreducible bases, both non-orthogonal and orthogonal ones, have been intensively studied (see [2–5]). The formula (14) in [1] states that the representations (λ) of so(3) with the highest weight λ, which occur in the representation (m, n) of su(3), are given by:

$$\lambda = K, K+1, K+2, ..., K + \max\{m, n\},$$

where the integer:

$$K = \min\{m, n\}, \min\{m, n\} - 2, ..., 1 \text{ or } 0, \quad (1)$$

with the exception that if $K = 0$:

$$\lambda = \max\{m, n\}, \max\{m, n\} - 2, ..., 1 \text{ or } 0.$$

In this paper, we reproduce this result using differential operator realisations, which act on a space of polynomials of three independent variables. This tool is used in various contexts, namely in a modern group analysis of differential equations [6–9], in classification of gravity fields [10], in geometric control theory [11], in difference schemes for numerical solutions of differential equations [12], etc.

2. Realisation of su(3)

We make use of the realisation of the Lie algebra $\mathrm{gl}(3, \mathbb{C})$, which is a complex Lie algebra of elements E_{ij}, $i, j = 1, 2, 3$ satisfying commutation relations:

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj},$$

on a space of polynomials of three variables $\mathbb{C}[x_1, x_2, x_3]$. (For an extensive list of realisations of low dimensional Lie algebras see [13]. See [14] how to obtain all irreducible representations of classical Lie algebras in terms of polynomial vector fields.) The realisation ρ is given by the formulas [15]:

$$\begin{aligned}
\rho(E_{11}) &= x_1\partial_1 + x_3\partial_3 + i\alpha_1 + 1, \\
\rho(E_{12}) &= x_1\partial_2 + x_3^2\partial_3 + (1 + i(\alpha_1 - \alpha_2))x_3, \\
\rho(E_{13}) &= x_1^2\partial_1 + x_1x_2\partial_2 + x_3(x_1 + x_2x_3)\partial_3 + \\
&\quad (2 + i(\alpha_1 - \alpha_3))x_1 + x_2x_3(1 + i(\alpha_1 - \alpha_2)), \\
\rho(E_{21}) &= x_2\partial_1 - \partial_3, \\
\rho(E_{22}) &= x_2\partial_2 - x_3\partial_3 + i\alpha_2, \\
\rho(E_{23}) &= x_1x_2\partial_1 + x_2^2\partial_2 - (x_1 + x_2x_3)\partial_3 \\
&\quad + (1 + i(\alpha_2 - \alpha_3))x_2, \\
\rho(E_{31}) &= -\partial_1, \\
\rho(E_{32}) &= -\partial_2, \\
\rho(E_{33}) &= -x_1\partial_1 - x_2\partial_2 + i\alpha_3 - 1,
\end{aligned} \quad (2)$$

where $\alpha_1, \alpha_2, \alpha_3$ are arbitrary complex parameters. Introducing the generators:

$$\begin{aligned}
H_1 &= E_{11} - E_{22}, \\
H_2 &= E_{22} - E_{33},
\end{aligned}$$

and the operators:

$$\begin{aligned}
\rho(H_1) &= \rho(E_{11}) - \rho(E_{22}) = x_1\partial_1 - x_2\partial_2 + 2x_3\partial_3 + \\
&\quad i(\alpha_1 - \alpha_2) + 1, \\
\rho(H_2) &= \rho(E_{22}) - \rho(E_{33}) = x_1\partial_1 + 2x_2\partial_2 - x_3\partial_3 + \\
&\quad i(\alpha_2 - \alpha_3) + 1,
\end{aligned}$$

we obtain a realisation of the Lie algebra $\mathrm{sl}(3, \mathbb{C}) \simeq \mathrm{su}(3)_{\mathbb{C}}$, given by the generators E_{12}, E_{13},

E_{23}, E_{21}, E_{31}, E_{32}, H_1, and H_2 and commutation relations:

$$\begin{array}{ll}
[E_{12},E_{23}]=E_{13}, & [E_{12},E_{21}]=H_1,\\
[E_{31},E_{12}]=E_{32}, & [H_1,E_{12}]=2E_{12},\\
[E_{12},H_2]=E_{12}, & [E_{21},E_{13}]=E_{23},\\
[E_{13},E_{31}]=H_1+H_2, & [E_{13},E_{32}]=E_{12},\\
[H_1,E_{13}]=E_{13}, & [H_2,E_{13}]=E_{13},\\
[E_{23},E_{31}]=E_{21}, & [E_{23},E_{32}]=H_2,\\
[E_{23},H_1]=E_{23}, & [H_2,E_{23}]=2E_{23},\\
[E_{32},E_{21}]=E_{31}, & [E_{21},H_1]=2E_{21},\\
[H_2,E_{21}]=E_{21}, & [E_{31},H_1]=E_{31},\\
[E_{31},H_2]=E_{31}, & [H_1,E_{32}]=E_{32},\\
[E_{32},H_2]=2E_{32}, &
\end{array}$$

(other commutation relations are zero).

For any m,n non-negative integers, let us take:

$$\alpha_1=0,\quad \alpha_2=-i(n+1),\quad \alpha_3=-i(m+n+2).$$

This way (2) becomes a realisation of $\mathrm{su}(3)_{\mathbb{C}}$, and also of the real form su(3), which turns out to be reducible (see Theorem 1). (We denote this realisation by the same symbol ρ.)

From now on, we suppose, for technical reasons, m,n to be even integers such that $m\geq n$. (The process would differ slightly for the case of m,n being odd, we omit details here for brevity.)

Let us now denote:

$$y=x_1+x_2x_3,$$

and let us take the polynomials from the representation space $\mathbb{C}[x_1,x_2,x_3]$ which are given in Table 1. Let us define the subspace $P\subset\mathbb{C}[x_1,x_2,x_3]$ as a linear span of all these polynomials. This turns out to be an invariant subspace of ρ (see Lemma 2). To show this, we start with a finding of a suitable set of $\mathrm{su}(3)_{\mathbb{C}}$ algebra generators.

Lemma 1. *$\mathrm{su}(3)_{\mathbb{C}}$ is generated (as a Lie algebra) by the generators:*

$$E_{12},E_{23},E_{31}.$$

Proof. First, using E_{12} and E_{23}, we obtain the generator E_{13}, because:

$$[E_{12},E_{23}]=E_{13}.$$

Then, using E_{23} and E_{31}, we obtain E_{21}:

$$[E_{23},E_{31}]=E_{21}.$$

Similarly, we get H_1 by:

$$[E_{12},E_{21}]=H_1,$$

and E_{32} by:

$$[E_{31},E_{12}]=E_{32}.$$

Finally, we obtain H_2 using:

$$[E_{23},E_{32}]=H_2.$$

■

1)	$x_1^{m-j}x_3^{n-b-j}y^bx_1^k(x_2x_3)^{j-k}$, $0\leq b\leq n, 0\leq j\leq n-b, 0\leq k\leq j$,
2)	$x_1^{m-j}x_2^{j-(n-b)}y^bx_1^k(x_2x_3)^{n-b-k}$, $0\leq b\leq n, n-b+1\leq j\leq m, 0\leq k\leq n-b$,
3)	$x_2^{j-(n-b)}y^bx_1^k(x_2x_3)^{m+n-b-j-k}$, $0\leq b\leq n, m+1\leq j\leq m+n-b$, $0\leq k\leq m+(n-b)-j$,
4)	$x_1^{m-j}x_3^{n-k}x_1^l(x_2x_3)^{k-l}$, $1\leq j\leq n, 0\leq k\leq j-1, 0\leq l\leq k$,
5)	$x_1^{m-j}x_3^{n-k}x_1^l(x_2x_3)^{k-l}$, $n+1\leq j\leq m, 0\leq k\leq n, 0\leq l\leq k$,
6)	$x_1^{m-j}x_2^{k-n}x_1^l(x_2x_3)^{n-l}$, $n+2\leq j\leq m, n+1\leq k\leq j-1, 0\leq l\leq n$,
7)	$x_2^{m-j}x_1^l(x_2x_3)^{k-l}$, $1\leq j\leq n, 0\leq k\leq j-1, 0\leq l\leq k$,
8)	$x_2^{m-j}x_1^l(x_2x_3)^{k-l}$, $n+1\leq j\leq m, 0\leq k\leq n-1, 0\leq l\leq k$,
9)	$x_3^{n-j}x_1^l(x_2x_3)^{k-l}$, $1\leq j\leq n-1, 0\leq k\leq j-1, 0\leq l\leq k$.

TABLE 1. P.

Note 1. Similarly, one can show that $\mathrm{su}(3)_{\mathbb{C}}$ is generated by the triplet:

$$E_{13},E_{21},E_{32}.$$

Lemma 2. *P is an invariant subspace of the realisation ρ of $\mathrm{su}(3)_{\mathbb{C}}$ given by (2).*

Proof. Due to Lemma 1, it is sufficient to show that P is invariant with respect to $\rho(E_{12})$, $\rho(E_{23})$, and $\rho(E_{31})$.

Let us start with $\rho(E_{31})=-\partial_1$.

First, let us apply $\rho(E_{31})$ to polynomial of type 1) from Table 1. We obtain:

$$\begin{aligned}\rho(E_{31})(x_1^{m-j}x_3^{n-b-j}y^bx_1^k(x_2x_3)^{j-k})=\\ -bx_1^{m-j}x_3^{n-b-j}x_1^k(x_2x_3)^{j-k}y^{b-1}-\\ (m-j+k)x_1^{m-j-1}x_3^{n-b-j}x_1^k(x_2x_3)^{j-k}y^b.\end{aligned}$$

When $b>0$, this result falls into the group 1) in Table 1.

When $b>0$ and $j<n$, this result falls into the group 4).

Finally, when $b>0$ and $j=n$, this result falls into the group 5).

For other types 2)–9), we get similar results.

For the operators $\rho(E_{23})$ and $\rho(E_{31})$, we proceed similarly. ■

We will denote the restriction of ρ on the subspace P by the same symbol ρ. In this way, ρ becomes finite dimensional representation on P.

Theorem 1. *The polynomials in Table 1 form a basis of P.*

Proof. The group 1) in Table 1 contains a polynomial:

$$v = x_1^m y^n. \tag{3}$$

This vector v satisfies:

$$\rho(H_1)v = mv, \qquad \rho(E_{12})v = 0,$$
$$\rho(H_2)v = nv, \qquad \rho(E_{23})v = 0,$$

i. e. it is the highest weight vector of the representation ρ. Therefore, ρ contains, as a subrepresentation, the representation of the highest weight (m,n) and the dimension of P has to be greater or equal to:

$$\frac{1}{2}(m+1)(n+1)(m+n+2), \tag{4}$$

(for what is a well-known formula for the dimension of the representation of the highest weight (m,n), see [16], § 24.3). The numbers of vectors given in Table 1 are given in Table 2. Because their total count agrees with (4), the dimension of P is equal to (4) and the vectors in Table 1 are linearly independent and form a basis of P. ∎

Example 1. An example of the space P together with the weights of the 27 basis vectors for the case of the highest weight $(2,2)$ is shown in Figure 1.

Let us denote:

$$I_0 = i(E_{21} - E_{12}),$$
$$I_\pm = i(E_{31} - E_{13}) \pm (E_{23} - E_{32}).$$

Then:

$$[I_0, I_\pm] = \pm I_\pm, \quad [I_+, I_-] = 2I_0,$$

i. e. the generators I_0, I_+, and I_- form a su(2) $\simeq$ so(3) subalgebra of su(3).
Let us now denote:

$$z = 1 + ix_3,$$
$$\chi = x_2 - ix_1,$$
$$r = 2z - z^2 + y^2,$$
$$w = z + iy\chi,$$
$$\xi = 1 + x_1^2 + x_2^2,$$

and consider the polynomials given in the Table 3 (we call them "maximal vectors").

Lemma 3. *Maximal vectors s_{ab} and t_{ka} belong to P.*

Proof. This fact can be directly verified from the expanded form of the maximal vectors. ∎

Lemma 4. *Maximal vectors s_{ab} and t_{ka} are the highest weight vectors for the* so(3) *triple $(I_0, I_\pm)$, i. e.*

$$\rho(I_0)s_{ab} = (b+m)s_{ab}, \qquad \rho(I_+)s_{ab} = 0,$$
$$\rho(I_0)t_{ka} = (m - 2k + a \bmod 2)t_{ak}, \quad \rho(I_+)t_{ka} = 0.$$

1)	$\frac{1}{6}(n+1)(n^2+5n+6)$,
2)	$\frac{1}{6}(n+1)(n+2)(3m-2n)$,
3)	$\frac{1}{6}n(n+1)(n+2)$,
4)	$\frac{1}{6}n(n+1)(n+2)$,
5)	$\frac{1}{2}(n+1)(n+2)(m-n)$,
6)	$\frac{1}{2}(n+1)(m-n-1)(m-n)$,
7)	$\frac{1}{6}n(n+1)(n+2)$,
8)	$\frac{1}{2}n(n+1)(m-n)$,
9)	$\frac{1}{6}(n-1)n(n+1)$,
Total	$\frac{1}{2}(m+1)(n+1)(m+n+2)$.

Table 2. Vector counts.

Proof. This is verified by the direct computation. ∎

Lemma 5. *Maximal vectors s_{ab} and t_{ka} are linearly independent.*

Proof. As the linear independence of maximal vectors having different eigenvalues of $\rho(I_0)$ is clear, it remans to check the linear independence of maximal vectors having the same eigenvalues. But this is clear from the form of the maximal vectors. ∎

We are now ready to formulate the main theorem.

Theorem 2. *The representation space P is a direct sum of linear spans of mutually linearly independent vectors*

$$\rho(I_-)^j s_{ab}, \quad 0 \le j \le 2(b+m), \tag{5}$$

and

$$\rho(I_-)^j t_{ka}, \quad 0 \le j \le 2(m - 2k + a \bmod 2), \tag{6}$$

where s_{ab} and t_{ka} and indices a, b, k are given in the Table 3. The linear spans (5) *and* (6) *are minimal invariant subspaces of representation ρ of* su(3) *of highest weight (m,n) viewed as a (completely reducible) representation of* so(3)*, having highest weights $(b+n)$ resp. $(n - 2k + a \bmod 2)$.*

Proof. The linear independence of vectors (5) resp. (6) is clear from the linear independence of vectors s_{ab} and t_{ka}, because we can make use of the operator $\rho(I_+)^j$ to "come back" from $\rho(I_-)^j s_{ab}$ to the scalar multiple of the highest weight vector s_{ab}. The proof is thus reduced to verifying the fact that the number of vectors is equal to the dimension of P, namely (4). ∎

Corollary 1. *Elliot's result* (1) *is a direct consequence of the Theorem 2.*

Example 2. The list of maximal vectors for the case of highest weigth $(2,2)$ (i. e. $m = 2$, $n = 2$, see the Figure 1 contains the vectors:

$$z^2\chi^2, wz\chi, w^2, r\chi^2, r\xi,$$

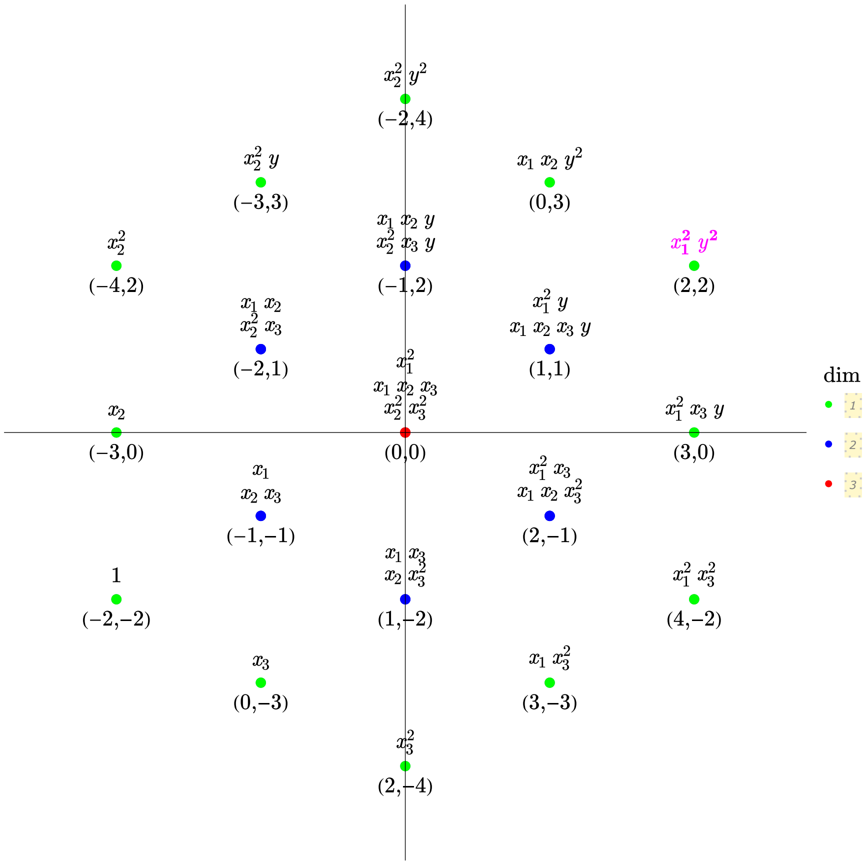

Figure 1. An example of P.

of highest weights

$$4, 3, 2, 2, 0.$$

This indicates the following decomposition of the representation $(2, 2)$:

$$(2, 2) \simeq (4) \oplus (3) \oplus 2(2) \oplus (0), \qquad (7)$$

or, in the dimensions of individual representations:

$$27 = 9 + 7 + 2 \times 5 + 1.$$

Note 2. Using the explicit decomposition formula (1), it is easy to obtain the generating function $F(P, Q, x)$ for the $\mathrm{so}(3) \subset \mathrm{su}(3)$ decomposition. It reads

$$\frac{1 + PQx}{(1 - P^2)(1 - Q^2)(1 - Px)(1 - Qx)}. \qquad (8)$$

For example, the result (7) can be quickly rediscovered using the generating function (8) by computing:

$$\frac{1}{2!}\frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}P^2}\frac{\mathrm{d}^2}{\mathrm{d}Q^2}F(P, Q, x)\bigg|_{\substack{P=0\\Q=0}} = x^4 + x^3 + 2x^2 + 1.$$

1)	$s_{ab} = w^a z^b \chi^{m-a} r^{\frac{n-(b+a)}{2}}$, $0 \le a \le m, 0 \le b \le n - a, a + b$ even,
2)	$t_{ka} = r^{[\frac{n-a}{2}]} \xi^{[\frac{a}{2}]+k} w^{a \bmod 2} z^a \chi^{m-2k-a}$, $1 \le k \le \frac{m}{2}, 0 \le a \le m - 2k, a \le n.$

Table 3. Maximal vectors.

3. Conclusion

The differential realisation method for obtaining the decomposition of a finite dimensional representation of the Lie algebra $\mathrm{su}(3)$ according to the embedding $\mathrm{so}(3) \subset \mathrm{su}(3)$ was presented. The parametrisation of the submodules is a convenient for the application in particle physics and in general for systems with the appropriate symmetries. The realisation way is shown to be convenient tool for constructing such decompositions and leads to the decomposition result using only basic, appropriate classical tools from representation theory. This makes the method approachable for a broad audience within mathematics and physics and it can be useful for obtaining similar decompositions based on other subalgebra–algebra pairs. The generating function provides a quick algorithm to determine the multiplicities for $\mathrm{so}(3)$ irreducible representations occurring in the decomposition of $\mathrm{su}(3)$ representations.

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