Czech Technical University in Prague
Faculty of Electrical Engineering
Department of Mathematics

Habilitation Thesis

Norm-attainment, Smoothness and Lineability

Sheldon Dantas

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Contents

Chapter 1

Introduction

This thesis is a collection of six selected research publications. Essentially, they are part of three different subareas of Banach space theory: norm-attaining theory, which is the study of functions defined in infinite-dimensional spaces and when they attain their maxima; smoothness in Banach spaces, which is the study of when one can replace a norm of a normed space by another one with some desired property; and lineability, which is the study of finding finite or infinite linear vector spaces inside non-linear subsets.

Let us describe the relevance of these subareas in the field.

It is well-known that in a Banach space X, the set of all continuous linear functionals on X determines almost totally the structure of X as a Banach space. As a matter of fact, by the Hahn-Banach theorem, the norm of any element $x \in X$ can be calculated as the supremum over all continuous functionals in the unit ball B_{X^*} of the topological dual space X^* of X. On the other hand, James theorem [56] states that a Banach space is reflexive if and only if every functional in the dual attains its norm. Moreover, Bishop and Phelps [11] proved that every functional can be approximated by functionals which attain their norms. For these reasons, since James and Bishop-Phelps, the theory of norm-attaining functionals was intensively studied. In fact, it has been widely considered and extended to different contexts besides linear functionals. For instance, there several relevant results on the topic for bounded linear operators, homogeneous polynomials, holomorphic functions, Lipschitz functions, among others.

On the other hand, the existence of a smooth norm (by this we mean an equivalent norm) has deep structural consequences for a given Banach space X. For example, let us suppose that X has a C^1-smooth renorming (or just a bump function). In this case, it must be an Asplund space, that is, the dual of every separable subspace of X is also separable [38]. Furthermore, if X has a C^2-smooth renorming then X either contains a copy of c_0 or it is superreflexive [41]. Also, it is known that if X has a C^∞-smooth renorming then it contains a copy of c_0 or ℓ_p, where p is an even integer [31].

Finally, finding a linear structure inside a non-linear subset might bring relevant consequences for the space itself. In the last decade, there have been a large amount of papers, books and monographs (see, for instance, [4]) related to lineability properties. In fact, this problem was considered within many different areas as Real

Analysis, Operator Theory, Linear Dynamics, classical Functional Analysis, Linear and Multilinear Algebra, or even Probability Theory.

Thesis outline

Let us briefly describe the contents of the dissertation. Chapter 2 provides the necessary background and notation so that the reader can follow this thesis without having to jump into many different references too often. Chapter 3 describes the results of three different papers (papers 1, 2 and 3) on norm-attaining theory. We start by considering the problem on the existence of bounded linear operators which do *not* attain their norms. As the reader will realize, this problem is related to the question on when the Banach space of all bounded linear operators is reflexive and also when all the operators are compact. We also describe the Schur property in terms of norm-attaining operators. Afterwards, we move to the study of norm-attaining tensors and nuclear operators. As one can immediately realize, there exists a deep connection between nuclear operators and tensor products, which allows us to study both concepts (almost) simultaneously. We provide several examples of norm-attaining tensors and nuclear operators as well as tensors and nuclear operators which never attain their projective tensor norms and nuclear norms, respectively. This opens the gate to study a Bishop-Phelp type theorem in this context. It turns out that, for most of the classical Banach spaces, the set of all norm-attaining tensors in dense in the projective tensor product. The same holds true for nuclear operators . The delicate problem here is then to find Banach spaces X and Y such that no tensor can be approximated by norm-attaining ones. We conclude this chapter by considering the symmetric projective tensor products and its counterpart results. In Chapter 4 we describe our main result in this line of research. We draw a complete picture concerning smoothness in the sense that it implies the existence of smooth norms, norm approximation by smooth norms, C^1-smooth LUR norms and the existence of partitions of unity. Finally, we expose our results concerning lineability in Chapter 5. We study several different problems in Functional Analysis concerning results that hold true for sequences but are not for nets and we wonder about lineability properties in this context. We finally consider the problem of embedding c_0 isometrically inside the space of all Lipschitz functions which attain their norms strongly. Appendix A contains all the attached publications we have considered for this habilitation dissertation.

Chapter 2

Preliminaries

Let X, Y, and Z be Banach spaces over the field $\mathbb{K}$, which can be either $\mathbb{R}$ or $\mathbb{C}$. We denote by B_X and S_X the closed unit ball and the unit sphere, respectively, of the Banach space X. We denote by $\mathcal{L}(X,Y)$ the set of all bounded linear operators from X into Y. If $Y = \mathbb{K}$, then $\mathcal{L}(X,\mathbb{K})$ is denoted by X^*, the topological dual space of X. We denote by $\mathcal{B}(X \times Y, Z)$ the Banach space of bounded bilinear mappings from $X \times Y$ into Z. When $Z = \mathbb{K}$, we denote this space by $\mathcal{B}(X \times Y)$. It is well-known that the space $\mathcal{B}(X \times Y)$ and $\mathcal{L}(X, Y^*)$ are isometrically isomorphic as Banach spaces. We denote by $\mathcal{K}(X,Y)$ the set of all compact operators and by $\mathcal{F}(X,Y)$ the space of all operators of finite-rank from X into Y.

2.1 Basic notation

In this section, we provide the main tools we will be using throughout this thesis. We slip it into different subsections so that the reader can go direct to topic they consider more convenient.

2.1.1 Norm-attainment

We say that a bounded linear functional $x^* \in X^*$ *attains its norm* or it is *norm-attaining* if there exists $x_0 \in S_X$ such that $\|x^*\| = |x^*(x_0)|$. Analogously, we define a norm-attaining operator and a norm-attaining bilinear mapping. The sets of all norm-attaining functionals, norm-attaining operators and norm-attaining bilinear mappings are defined, respectively, by $\operatorname{NA}(X)$, $\operatorname{NA}(X,Y)$ and $\operatorname{NA}(X \times Y, Z)$. Here, we understand that the functionals are of the form $x^* : X \to \mathbb{K}$, the operators are the form $T : X \to Y$ and the bilinear mappings of the form $A : X \times Y \to Z$, where X, Y and Z are normed spaces (we will be clear when they are not necessarily complete). The symbol τ_c denotes the topology of compact convergence and $\|\cdot\|$ denotes the norm topology in $\mathcal{L}(X,Y)$.

2.1.2 Lipschitz functions

When we are considering Lipschitz functions in this thesis, all the vector spaces will be considered to be *real*. Let (M,d) be a pointed metric space (that is, a metric space with a distinguished point 0). We denote by $\operatorname{Lip}_0(M)$ the Banach space of all Lipschitz functions $f : M \to \mathbb{R}$ such that $f(0) = 0$, endowed with the Lipschitz norm

$$\|f\|_{\mathrm{Lip}_0} := \sup \left\{ \frac{|f(y) - f(x)|}{d(x, y)} : x, y \in M, \, x \neq y \right\}.$$

We say that a Lipschitz function $f \in \mathrm{Lip}_0(M)$ *strongly attains its norm*, or that it is *strongly norm-attaining*, if there exist two different points $p, q \in M$ such that

$$\|f\|_{\mathrm{Lip}_0} = \frac{|f(p) - f(q)|}{d(p, q)}.$$

The set of strongly norm-attaining Lipschitz functions on M will be denoted by $\mathrm{SNA}(M)$. Let X be a separable Banach space with a Schauder basis denoted by $\{x_n\}_{n=1}^{\infty}$. We say that a sequence $\{y_n\}_{n=1}^{\infty}$ in a Banach space Y is *(isometrically) equivalent* to the basis $\{x_n\}_{n=1}^{\infty}$ if there exists a linear (isometric) isomorphism $T\colon \overline{\mathrm{span}}\{y_n\colon n \in \mathbb{N}\} \to X$ such that $T(y_n) = x_n$ for all $n \in \mathbb{N}$. The following facts will be used throughout the text without any explicit reference.

(i) A sequence $\{x_n\}_{n=1}^{\infty}$ is isometrically equivalent to the canonical basis of c_0 if and only if the equality $\left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\| = \max_n |\lambda_n|$ holds for every sequence $\{\lambda_n\}_{n=1}^{\infty} \in c_0$.

(ii) If a sequence $\{x_n\}_{n=1}^{\infty}$ is isometrically equivalent to the canonical basis of c_0, then so is the sequence $\{\varepsilon_n x_n\}_{n=1}^{\infty}$, where $\varepsilon_n \in \{-1, 1\}$ for every $n \in \mathbb{N}$.

(iii) Any subsequence of a sequence which is isometrically equivalent to the canonical basis of c_0 is once again isometrically equivalent to the same basis.

2.1.3 Tensor products

We use essentially the notation from [83]. The projective tensor product of X and Y, denoted by $X \widehat{\otimes}_\pi Y$, is the completion of the space $X \otimes Y$ endowed with the norm given by

$$\begin{aligned} \|z\|_\pi &= \inf \left\{ \sum_{n=1}^{\infty} \|x_n\| \|y_n\| : \sum_{n=1}^{\infty} \|x_n\| \|y_n\| < \infty, z = \sum_{n=1}^{\infty} x_n \otimes y_n \right\} \\ &= \inf \left\{ \sum_{n=1}^{\infty} |\lambda_n| : z = \sum_{n=1}^{\infty} \lambda_n x_n \otimes y_n, \sum_{n=1}^{\infty} |\lambda_n| < \infty, \|x_n\| = \|y_n\| = 1 \right\}, \end{aligned}$$

where the infimum is taken over all such representations of z. It is well-known that $\|x \otimes y\|_\pi = \|x\| \|y\|$ for every $x \in X$, $y \in Y$, and the closed unit ball of $X \widehat{\otimes}_\pi Y$ is the closed convex hull of the set $B_X \otimes B_Y = \{x \otimes y : x \in B_X, y \in B_Y\}$. Let us recall also that there is a canonical operator $J : X^* \widehat{\otimes}_\pi Y \longrightarrow \mathcal{L}(X, Y)$ with $\|J\| = 1$ defined by $z = \sum_{n=1}^{\infty} \varphi_n \otimes y_n \mapsto L_z$, where $L_z : X \longrightarrow Y$ is given by

$$L_z(x) = \sum_{n=1}^{\infty} \varphi_n(x) y_n \quad (x \in X). \tag{2.1}$$

The operators that arise in this way are called *nuclear operators*. We denote the set of such operators by $\mathcal{N}(X,Y)$ endowed with the *nuclear norm*

$$\|T\|_N = \inf\left\{\sum_{n=1}^{\infty}\|x_n^*\|\|y_n\| : T(x) = \sum_{n=1}^{\infty} x_n^*(x)y_n\right\},$$

where the infimum is taken over all representations of T of the form $T(x) = \sum_{n=1}^{\infty} x_n^*(x)y_n$ for bounded sequences $(x_n^*) \subseteq X^*$ and $(y_n) \subseteq Y$ such that $\sum_{n=1}^{\infty}\|x_n^*\|\|y_n\| < \infty$. Notice that every nuclear operator is compact since it is the limit in the operator norm of a sequence of finite-rank operators. Using the function J, we can identify the space $\mathcal{N}(X,Y)$ with $X^*\widehat{\otimes}_\pi Y/\ker J$ isometrically.

A Banach space is said to have the *approximation property* if for every compact subset K of X and every $\varepsilon > 0$, there exists a finite-rank operator $T : X \longrightarrow X$ such that $\|T(x) - x\| \leq \varepsilon$ for every $x \in K$. Let us take into account that if X^* or Y has the approximation property, then $X^*\widehat{\otimes}_\pi Y = \mathcal{N}(X,Y)$ (see, for instance, [83, Corollary 4.8]). For a complete background on tensor products in Banach spaces, we refer the reader to the books [30, 83].

The *N-fold projective symmetric tensor product of* X, denoted by $\widehat{\otimes}_{\pi,s,N}X$, is the completion of the linear space $\otimes_{\pi,s,N}X$, generated by $\{z^N : z \in X\}$, under the norm given by

$$\|z\|_{\pi,s,N} := \inf\left\{\sum_{k=1}^{n}|\lambda_k| : z := \sum_{k=1}^{n}\lambda_k x_k^N, n \in \mathbb{N}, x_k \in S_X, \lambda_k \in \mathbb{K}\right\}$$

where the infimum is taken over all the possible representations of z. Its topological dual $\left(\widehat{\otimes}_{\pi,s,N}X\right)^*$ can be identified (there exists an isometric isomorphism) with $\mathcal{P}(^N X)$. Indeed, every polynomial $P \in \mathcal{P}(^N X)$ acts as a linear functional on $\widehat{\otimes}_{\pi,s,N}X$ through its associated symmetric N-linear form $\overline{P}$ and satisfies

$$P(x) = \overline{P}(x,\ldots,x) = \langle P, x^N\rangle$$

for every $x \in X$. We also have that $B_{\widehat{\otimes}_{\pi,s,N}X} = \overline{\text{aco}}(\{x^N : x \in S_X\})$. To save notation, by a symmetric tensor we will refer to a generic element of $\widehat{\otimes}_{\pi,s,N}X$. For more information about symmetric tensor products, we suggest [43] and also [14, 16, 17].

2.1.4 Smoothness

Let X, Y be normed linear spaces. We say that the norm $\|\cdot\|$ of X is *C^k-smooth* if its kth Fréchet derivative exists and is continuous at every point of $X \setminus \{0\}$. The norm is said to be *C^∞-smooth* if this holds for every $k \in \mathbb{N}$. We denote by $\mathcal{P}(^n X; Y)$ the normed linear space of all n-homogeneous continuous polynomials from X into Y. If $U \subset X$ is an open subset, then we say that a function $f : U \to Y$ is *analytic* if, for every $a \in U$, there exist $P_n \in \mathcal{P}(^n X; Y)$ $(n \in \mathbb{N} \cup \{0\})$ and $\delta > 0$ such that, for all $x \in U(a,\delta)$,

$$f(x) = \sum_{n=0}^{\infty} P_n(x-a).$$

2.1.5 Lineability

We will be using basic concepts and notations from Set Theory found, for instance, in [15, 58]. Ordinal numbers will be identified with the set of their predecessors and cardinal numbers with the initial ordinals. Given a set A, the cardinality of A will be denoted by card(A). We denote by $\aleph_0$, $\aleph_1$ and co the first infinite cardinal, the second infinite cardinal and the cardinality of the continuum, respectively. The *cofinality* $\mathrm{cof}(\alpha)$ of an ordinal α is the smallest ordinal β such that $\alpha = \sup_{\gamma<\beta} \alpha_\gamma$, where $\{\alpha_\gamma\}_{\gamma<\beta}$ is an ordinal sequence of length β with $\alpha_\gamma < \alpha$ for all $\gamma < \beta$. We say that a cardinal number κ is regular if $\mathrm{cof}(\kappa) = \kappa$ (see, for instance, [68]).

A set $\mathcal{A}$ is a *directed set* (also known as an index set) if $\mathcal{A}$ is a nonempty set that is endowed with a preorder $\leq$ (a reflexive and transitive relation) such that every pair of elements of $\mathcal{A}$ has an upper bound. A *net* in a set X is a function from a directed set $\mathcal{A}$ to X which will be denoted by $(x_a)_{a\in\mathcal{A}}$. We denote the set of nets in X indexed by $\mathcal{A}$ as $X^{\mathcal{A}}$.

Given a topological space X and $(x_a)_{a\in\mathcal{A}}$ a net in X, we say that $(x_a)_{a\in\mathcal{A}}$ converges to $x \in X$ if for every neighborhood U^x of x, there exists an element $a_0 \in \mathcal{A}$ such that $x_a \in U^x$ for every $a \geq a_0$. Recall that if X is a topological vector space, then a net $(x_a)_{a\in\mathcal{A}}$ in X weakly converges to $x \in X$ (denoted by $x_a \xrightarrow{w} x$) if and only if $(x^*(x_a))_{a\in\mathcal{A}}$ converges to $x^*(x)$ for every $x^* \in X^*$.

If a directed set $\mathcal{A}$ is in particular an ordinal number α, then we have the so-called *α-sequences* instead of nets defined in a set. It is known that the convergence of α-sequences in a topological space can be reduced to the convergence of $\mathrm{cof}(\alpha)$-sequences (see, for instance, [81, Propositions 3.1 and 3.2]). Therefore, we simply consider κ-sequences, where κ is a regular cardinal number. This notion of κ-sequence was introduced in 1907 by J. Mollerup [76] and has been studied throughout the 20th and 21st centuries by many mathematicians in several contexts (see [67, 87, 78, 82] and the references therein). Given a κ-sequence $(x_\alpha)_{\alpha<\kappa}$ in X, we say that $(x_{\beta_\alpha})_{\alpha<\kappa}$ is a κ-subsequence of $(x_\alpha)_{\alpha<\kappa}$ if there exists an increasing injective function $\varphi : \kappa \to \kappa$ such that $x_{\beta_\alpha} = x_{\varphi(\alpha)}$ for every $\alpha < \kappa$.

Recall that a *topological vector space* (TVS, for short) is a vector space endowed with a topology such that vector addition and scalar multiplication are both continuous. In this case, we denote by X^* its topological dual and $\sigma(X, X^*)$ the weak topology on X.

Our main definition in this section is the following. We say that a subset M of a vector space X is *lineable* (respectively, *κ-lineable*, for a cardinal κ) if $M \cup \{0\}$ contains a vector space of infinite-dimension (respectively, of dimension κ).

Chapter 3

Norm-attainment

One of the most classical topics in the theory of Banach spaces is the study of norm-attaining functions. In fact, one of the most famous characterizations of reflexivity, due to R. James, is described in terms of linear functionals which attain their norms (see, for instance, [40, Corollary 3.56]). In the same line of research, E. Bishop and R. Phelps proved that the set of all norm-attaining linear functionals is dense in X^* (see [11]). This motivated J. Lindenstrauss to study the analogous problem for bounded linear operators in his seminal paper [71], where it was obtained for the first time an example of a Banach space such that the Bishop-Phelps theorem is no longer true for this class of functions. Consequently, this opened the gate for a crucial and vast research on the topic during the past fifty years in many different directions. Indeed, J. Bourgain, R.E. Huff, J. Johnson, W. Schachermayer, J.J. Uhl, J. Wolfe and V. Zizler continued the study about the set of all linear operators which attain their norms ([13, 54, 61, 86, 90, 91]); M. Acosta, R. Aron, F.J. Aguirre, Y.S. Choi, R. Payá ([1, 6, 22] tackled problems in the same line involving bilinear mappings; D. García and M. Maestre considered it for homogeneous polynomials (see [2, 7]); and more recently several problems on norm-attainment of Lipschitz maps were considered (see [20, 21, 45, 65]).

In this section, we consider basically two problems on norm-attaining theory. We will start by trying to reply the question on when there exists a non-norm-attaining operator between Banach spaces. Our results improve some other results in the literature and bring back some relevant questions in the theory. We then move to the problem on finding norm-attaining and non-norm-attaining nuclear operators as an attempt of giving one step further in the theory and trying to help to reply to the question on whether every finite-rank operator can be approximated by norm-attaining operators. The results described here can be found in Papers 1, 2 and 3.

3.1 Non-norm-attaining operators

One can check that if every bounded linear operator from a Banach space X into a Banach space Y is norm-attaining, then X must be reflexive, whereas the range space Y is not forced to be reflexive in general. Indeed, every bounded linear operator from a reflexive space into a Banach space which satisfies the Schur property is compact (by the Šmulyan theorem) and any compact operator from a reflexive space into an arbitrary Banach space is norm-attaining. Therefore, it seems natural to wonder whether it is possible to guarantee the existence of a non-norm-attaining operator from the existence of a non-compact operator. This brings us back to the 70's when J.R. Holub proved that this is, in fact, true under approximation property assumptions (see [54, Theorem 2]). Almost thirty years later, J. Mujica improved Holub's result by using the compact approximation property (see [77, Theorem 2.1]), which is a weaker assumption than the approximation property. However, both results require the reflexivity on both domain and range spaces, so the following question arises naturally.

Given a reflexive space X and any Banach space Y, under which assumptions we may guarantee the existence of non-norm-attaining operators in $\mathcal{L}(X,Y)$?

In connection with the previous question, Holub and Mujica (in fact, Mujica's result is an improvement of Holub's) proved the following result.

Theorem 3.1.1 ([53, Theorem 2] and [77, Theorem 2.1])**.** *Let X and Y be both reflexive spaces.*

(a) If $\mathcal{L}(X,Y)$ is non-reflexive, there is a non-norm-attaining operator $S \in \mathcal{L}(X,Y)$.

(b) If X or Y has the (compact) approximation property, then the following statements are equivalent.

- *(i) There exists a non-norm-attaining operator $S \in \mathcal{L}(X,Y)$;*
- *(ii) $\mathcal{L}(X,Y) \neq \mathcal{K}(X,Y)$;*
- *(iii) $\mathcal{L}(X,Y)$ is non-reflexive;*
- *(iv) $(\mathcal{L}(X,Y), \|\cdot\|)^* \neq (\mathcal{L}(X,Y), \tau_c)^*$.*

The proof of the above result relies on the fact that if Y is a reflexive space, then $\mathcal{L}(X,Y)$ is the dual space of the projective tensor product $X\widehat{\otimes}_\pi Y^*$. However, if the range space Y is non-reflexive, then $\mathcal{L}(X,Y)$ is always non-reflexive (see, for instance, [84]). As a way of extending the above results to the case of non-reflexive range spaces, we borrow some of the techniques used by R.C. James (see [56, 57]) and we consider the following notion. We will prove that the James property is a sufficient condition to guarantee the existence of an operator which never attains its norm.

Definition 3.1.2. We say that a pair (X,Y) of Banach spaces has the *James property* if there exists a relatively WOT-compact set $K \subseteq \mathcal{L}(X,Y)$ such that $0 \in \overline{K}^{WOT}$ and $0 \notin \overline{\text{co}}^{\|\cdot\|}(K)$.

We have the following existence result.

Theorem 3.1.3. *Let X and Y be Banach spaces. If the pair (X,Y) has the James property, then there exists a non-norm-attaining operator in $\mathcal{L}(X,Y)$.*

In order to expose the new result, we need the following definitions.

Definition 3.1.4. The pair (E, F) of Banach spaces is said to have the *compact approximation property* (CAP, for short) if every operator $T \in \mathcal{L}(E, F)$ belongs to $\overline{\mathcal{K}(E, F)}^{\tau_c}$. If $\lambda \geq 1$ and every operator $T \in \mathcal{L}(E, F)$ belongs to $\lambda \|T\| \overline{B_{\mathcal{K}(E,F)}}^{\tau_c}$, then (E, F) is said to have the *λ-BCAP* or simply the *BCAP*. In the case when $\lambda = 1$, we say that the pair (E, F) has the metric compact approximation property (MCAP, for short). Moreover, we say that the pair (E, F) has the *pointwise-BCAP* if for every operator $T \in \mathcal{L}(E, F)$ there is a constant $\lambda_T \geq 1$ such that $T \in \lambda_T \overline{B_{\mathcal{K}(E,F)}}^{\tau_c}$.

We observe, for a reflexive space E and an arbitrary Banach space F, that (1) the unit ball of $\mathcal{K}(E, F)$ is closed in the strong operator topology if and only if it is sequentially closed in this topology and (2) $\mathcal{K}(E, F) = \mathcal{L}(E, F)$ implies that $(\mathcal{L}(E, F), \|\cdot\|)^* = (\mathcal{L}(E, F), \tau_c)^*$ by using the result [42, Theorem 1] due to M. Feder and P. Saphar. Besides that, we consider the concept of the (bounded) compact approximation property for a pair of Banach spaces in the way as it is done in [12] (see Definition 3.1.4) and prove that $\mathcal{K}(E, F) = \mathcal{L}(E, F)$ when either (3) the norm-closed unit ball of $\mathcal{K}(E, F)$ is closed in the strong operator topology or (4) $(\mathcal{L}(E, F), \|\cdot\|)^* = (\mathcal{L}(E, F), \tau_c)^*$ under the just mentioned approximation property assumption. Combining (1)-(4) together with Theorem 3.1.3, we get a generalization of Holub and Mujica's results, where we no longer need reflexivity on the target space F.

Theorem 3.1.5. *Let E be a reflexive space and F be an arbitrary Banach space. Consider the following conditions.*

(a) $\mathcal{K}(E, F) = \mathcal{L}(E, F)$.

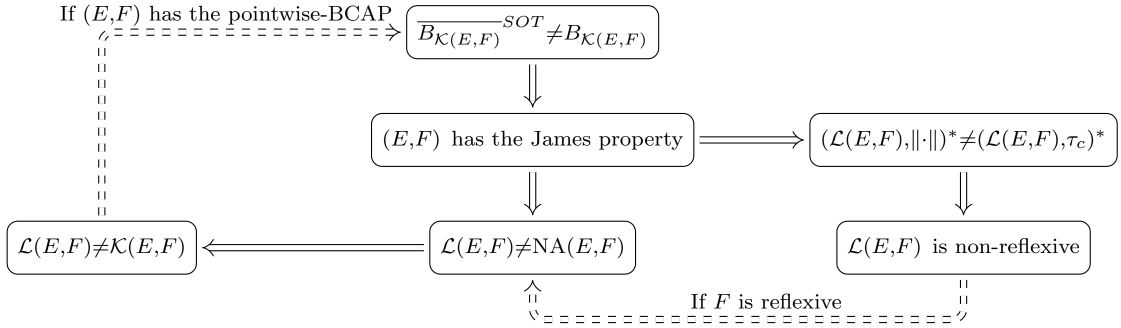
(b) Every operator from E into F attains its norm.

(c) The unit ball $B_{\mathcal{K}(E,F)}$ is closed in the strong operator topology.

(d) $(\mathcal{L}(E, F), \tau_c)^* = (\mathcal{L}(E, F), \|\cdot\|)^*$.

Then, we always have (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) and (a) $\Longrightarrow$ (d) $\Longrightarrow$ (c). Additionally, if the pair (E, F) has the bounded compact approximation property, then (c) $\Longrightarrow$ (a) and therefore all the statements are equivalent.

The following diagram summarizes most of the results included in this work. In what follows, E is supposed to be any reflexive space and F is any arbitrary Banach space. See Definition 3.1.4 in Section 2 below for the definition of the pointwise-BCAP.



As an application of Theorem 3.1.5, we provide a characterization of the Schur property in terms of norm-attaining operators.

Theorem 3.1.6. *Let Y be a Banach space. The following statements are equivalent.*

(a) Y *has the Schur property.*

(b) $\mathcal{K}(X,Y) = \mathcal{L}(X,Y)$ *for every reflexive space* E.

(c) $\mathrm{NA}(X,Y) = \mathcal{L}(X,Y)$ *for every reflexive space* X.

(d) $\mathcal{K}(Z,Y) = \mathcal{L}(Z,Y)$ *for every reflexive space* Z *with basis.*

(e) $\mathrm{NA}(Z,Y) = \mathcal{L}(Z,Y)$ *for every reflexive space* Z *with basis.*

3.2 Tensor products and nuclear operators

M. Martín solved negatively a problem from the 1970s (posed explicitly by J. Diestel and J. Uhl in [35] and J. Johnson and J. Wolfe in [61]) on whether or not every compact operator can be approximated by norm-attaining operators (see [74, Theorem 1]). On the other hand, the main open problem in the theory of norm-attaining operators nowadays seems to be if every finite-rank operator can be approximated by norm-attaining operators (see [74, Question 9]). Since every nuclear operator is a limit of a sequence of finite-rank operators, we were motivated to try to take one step further in the theory by studying the set of all nuclear operators which attain their (nuclear) norms systematically. On account of clear relations between nuclear operators and projective tensor products, we focus also on a concept of norm-attainment in projective tensor products. We also consider the problem for symmetric tensor products and homogeneous polynomials.

3.2.1 Projective tensor products

Definition 3.2.1. We say that

(1) $z \in X \widehat{\otimes}_\pi Y$ *attains its projective norm* if there is a bounded sequence $(x_n, y_n) \subseteq X \times Y$ with $\sum_{n=1}^\infty \|x_n\|\|y_n\| < \infty$ such that $z = \sum_{n=1}^\infty x_n \otimes y_n$ and that $\|z\|_\pi = \sum_{n=1}^\infty \|x_n\|\|y_n\|$. In this case, we say that z is a *norm-attaining tensor.*

(2) $T \in \mathcal{N}(X,Y)$ *attains its nuclear norm* if there is a bounded sequence $(x_n^*, y_n) \subseteq X^* \times Y$ with $\sum_{n=1}^\infty \|x_n^*\|\|y_n\| < \infty$ such that $T = \sum_{n=1}^\infty x_n^* \otimes y_n$ and that $\|T\|_N = \sum_{n=1}^\infty \|x_n^*\|\|y_n\|$. In this case, we say that T is a *norm-attaining nuclear operator.*

We will be using the classical notation $\mathrm{NA}(X,Y)$ for the subset $\{T \in \mathcal{L}(X,Y) : T \text{ attains its norm}\}$ and $\mathrm{NA}(X \times Y, Z)$ for the subset of all $B \in \mathcal{B}(X \times Y, Z)$ such that B attains its norm. If $Z = \mathbb{K}$, then we simply denote it as $\mathrm{NA}(X \times Y)$. For our new concepts, we use the following notations:

(1) $\mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y) = \{z \in X \widehat{\otimes}_\pi Y : z \text{ attains its projective norm}\}$.

(2) $\mathrm{NA}_{\mathcal{N}}(X,Y) = \{T \in \mathcal{N}(X,Y) : T \text{ attains its nuclear norm}\}$.

Theorem 3.2.2. *Let X, Y be Banach spaces. Let $z \in X\widehat{\otimes}_\pi Y$ with*

$$z = \sum_{n=1}^{\infty} \lambda_n x_n \otimes y_n,$$

where $\lambda_n \in \mathbb{R}^+$, $x_n \in S_X$, and $y_n \in S_Y$ for every $n \in \mathbb{N}$. Then, the following assertions are equivalent:

1. *$\|z\|_\pi = \sum_{n=1}^{\infty} \lambda_n$; in other words, $z \in \mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$.*
2. *There is $G \in \mathcal{L}(X, Y^*)$ with $\|G\| = 1$ such that $G(x_n)(y_n) = 1$ for every $n \in \mathbb{N}$.*
3. *Every norm one $G \in \mathcal{L}(X, Y^*)$ such that $G(z) = \|z\|_\pi$ satisfies that $G(x_n)(y_n) = 1$ for every $n \in \mathbb{N}$.*

Remark 3.2.3. *Taking into account the explanation just before (2.1), we call that we have an isometric isomorphism between $\mathcal{N}(X, Y)$ and $X^*\widehat{\otimes}_\pi Y/\ker(J)$. Therefore, it is natural to think that there is also a nuclear operator version of Theorem 3.2.2. We will not highlight it in this thesis. In fact, we will be focusing on the tensor products versions of our results rather than the nuclear operators versions. Let us observe that every time we consider the nuclear operator versions of these results, the subset $\ker J)^\perp$ plays a relevant role there.*

Let us see a simple example on how to apply Theorem 3.2.2.

Example 3.2.4. Let X, Y be two reflexive Banach spaces such that X^* or Y has the approximation property (in this case, we have $X^*\widehat{\otimes}_\pi Y = \mathcal{N}(X, Y)$). Assume further that X^* is isometrically isomorphic to a subspace of Y^*. Take $G : X^* \longrightarrow Y^*$ to be a linear isometry and pick $(x_n^*)_n \subseteq S_{X^*}$. Now, for any $n \in \mathbb{N}$, notice that $\|G(x_n^*)\| = \|x_n^*\| = 1$. Since Y is reflexive, by using the James Theorem, we have that $G(x_n^*) \in S_{Y^*}$ attains its norm, so there exists $y_n \in S_Y$ so that $G(x_n^*)(y_n) = 1$. Now, Theorem 3.2.2 implies that, given any sequence $(\lambda_n)_n \subseteq (0, 1]$ with $\sum_{n=1}^{\infty} \lambda_n < \infty$, the nuclear operator

$$T := \sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n \in \mathcal{N}(X, Y)$$

attains its nuclear norm.

In finite-dimensional spaces, *every* tensor is norm-attaining. This is so because of the compactness of the unit ball as, in this case, we have $\overline{\mathrm{co}}(B_X \otimes B_Y) = \mathrm{co}\,(B_X \otimes B_Y)$, which is a consequence of Minkowski-Carathéodory theorem (see, for instance, [40, Exercises 1.57 and 1.58]).

Proposition 3.2.5. *Let X, Y be finite dimensional Banach spaces. Then, we have that $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) = X\widehat{\otimes}_\pi Y$.*

It turns out that the classical norm-attainment concept for operators has nothing to do with the norm-attainment for tensor products and nuclear operators. This can be justified by using the next result. Indeed, let us observe that one can construct a Banach space Y and operator $T : c_0 \to Y$ such that T does not attain its (classical) norm (see [74, Lemma 2.2] or the proof of [71, Proposition 4]). Nevertheless, we have the following result.

Proposition 3.2.6. *Let Y be a Banach space. Then,*

1. *every $T \in \mathcal{N}(c_0, Y)$ attains its nuclear norm. Equivalently,*
2. *every element in $\ell_1 \widehat{\otimes}_\pi Y$ attains its projective norm.*

Complex Hilbert spaces also satisfy a similar statement as in Proposition 3.2.6 (as Hilbert space satisfy the approximation property, we can write the result in terms of nuclear operators).

Proposition 3.2.7. *Let H be a complex Hilbert space. Then, every nuclear operator $T \in \mathcal{N}(H, H)$ attains its nuclear norm.*

At this point, it is natural to ask whether the equality $\mathrm{NA}_{\mathcal{N}}(X, Y) = \mathcal{N}(X, Y)$ (or $\mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y) = X \widehat{\otimes}_\pi Y$) holds for all (infinite-dimensional) Banach spaces X and Y. It turns out that this is not the case as we can see in the next result.

Proposition 3.2.8. *Let X, Y be Banach spaces. If every element in $X \widehat{\otimes}_\pi Y$ attains its projective norm, then the set of all bilinear forms on $X \times Y$ which attain their norms is dense in $\mathcal{B}(X \times Y)$. Equivalently, under the same hypothesis, the set of norm-attaining operators from X into Y^* is dense in $\mathcal{L}(X, Y^*)$. In particular, there exist elements $z \in X \widehat{\otimes}_\pi Y$ such that $z \notin \mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y)$ in the following cases.*

1. *When $X = L_1(\mathbb{T})$, where the unit circle $\mathbb{T}$ is equipped with the Haar measure m, and Y is the two-dimensional Hilbert space [45, Remark 5.7.(2)].*

2. *When X is $L_1[0, 1]$ and Y^* is a strictly convex Banach space without the Radon-Nikodým property [90, Theorem 3].*

3. *When $Y = \ell_p$ for $1 < p < \infty$ and X is the Banach space constructed by Gowers (see [47, Theorem, page 149]).*

4. *When X and Y are both $L_1[0, 1]$ [22, Theorem 3].*

So far we have seen that there are many norm-attaining tensors, that there is a direct connection between this new theory and the classical norm-attaining theory and finally that there are non-norm-attaining tensors (and nuclear operators). Thus, the natural question we would like to tackle is whether one can find a Bishop-Phelps type theorem for this class. It turns out that this is true for all the classical Banach spaces. For this, we take advantage of the finite dimensional case (where we know that every tensor is norm-attaining) to obtain a general result on denseness of norm-attaining tensors. The only problem here is the fact that in general the projective norm does not respect subspaces, but it does respect 1-complemented subspaces. For this reason, we need a property that guarantees the existence of many 1-complemented subspaces. This is the case for the metric π-property (see. for instance, [19, Definition 5.1]).

Definition 3.2.9. Let X be a Banach space. We will say that X has the *metric π-property* if given $\varepsilon > 0$ and $\{x_1, \ldots, x_n\} \subseteq S_X$ a finite collection in the sphere, then we can find a finite dimensional 1-complemented subspace $M \subseteq X$ such that for each $i \in \{1, \ldots, n\}$ there exists $x_i' \in M$ with $\|x_i - x_i'\| < \varepsilon$.

The following Banach spaces satisfy the metric π-property.

(a) Banach spaces with a finite dimensional decomposition with the decomposition constant 1 (consequently, every Banach space with Schauder basis can be renormed to have the metric π-property).

(b) $L_p(\mu)$-spaces for any $1 \leq p < \infty$ and any measure μ.

(c) L_1-predual spaces.

(d) $X \oplus_a Y$, whenever X, Y satisfy the metric π-property and $|\cdot|_a$ is an absolute norm.

(e) $X = \left[\bigoplus_{n \in \mathbb{N}} X_n\right]_{c_0}$ or $\left[\bigoplus_{n \in \mathbb{N}} X_n\right]_{\ell_p}$, $\forall\ 1 \leq p < \infty$, X_n satisfying the metric π-property, $\forall n$.

(f) $X \widehat{\otimes}_\pi Y$, whenever X, Y satisfy the metric π-property.

Coming back to norm-attaining tensors, we have the following density result. For the second part of the result we need a property defined in [27] (see also [29]).

Theorem 3.2.10. *Let X be a Banach space satisfying the metric π-property.*

(a) If Y satisfies the metric π-property, then $\overline{\operatorname{NA}_\pi(X \widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X \widehat{\otimes}_\pi Y$.

(b) If Y is uniformly convex, then $\overline{\operatorname{NA}_\pi(X \widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X \widehat{\otimes}_\pi Y$.

Consequently, we also have the following result.

Corollary 3.2.11. *Let $N \in \mathbb{N}$ be given. Let $X_1, \ldots, X_N$ be Banach spaces with the metric π-property, and Y be a Banach space. Then,*

$$\overline{\operatorname{NA}_\pi(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_N \widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_N \widehat{\otimes}_\pi Y.$$

Recall from Theorem 3.2.8 that when $X = L_1(\mathbb{T})$, where the unit circle $\mathbb{T}$ is equipped with the Haar measure, and Y is the two-dimensional Hilbert space ℓ_2^2, we have that $\operatorname{NA}_\pi(X \widehat{\otimes}_\pi Y) \neq X \widehat{\otimes}_\pi Y$. This shows that finite dimensionality on just one of the factors is not enough to guarantee that every tensor in $X \widehat{\otimes}_\pi Y$ is norm-attaining. Nevertheless, we have the following result.

Theorem 3.2.12. *Let X be a Banach space with $B_X = \operatorname{co}(\{x_1, \ldots, x_n\})$ for some $x_1, \ldots, x_n \in S_X$ and assume that Y is a dual space. Then, every tensor in $X \widehat{\otimes}_\pi Y$ attains its projective tensor norm.*

In fact, we can use Theorem 3.2.12 to get the following denseness result. Let us recall a Banach space X is said to be polyhedral if the unit ball of every finite-dimensional subspace is a polytope, that is, the convex hull of a finite set.

Theorem 3.2.13. *Let X be a Banach which is polyhedral and satisfies the metric π-property. Assume that Y is a dual space. Then, every tensor in $X \widehat{\otimes}_\pi Y$ can be approximated by tensors that attain their norms.*

We have that $(c_0, \|\cdot\|_\infty)$ is the canonical example of a polyhedral space. We, have the following immediate consequence of Theorem 3.2.13.

Corollary 3.2.14. *Let Y be a dual space. Then,* $\mathrm{NA}_\pi(c_0\widehat{\otimes}_\pi Y)$ *is dense in* $c_0\widehat{\otimes}_\pi Y$*.*

Next, we can prove the following result on the denseness of nuclear operators which attain their nuclear norms under the RNP assumption.

Theorem 3.2.15. *Let X, Y be Banach spaces such that X^* and Y^* have the RNP. Then, every nuclear operator from X into Y^* can be approximated by norm-attaining nuclear operators.*

After seeing Theorems 3.2.10, 3.2.12 and 3.2.13, we can think that the Bishop-Phelps always holds true in the setting of norm-attaining tensors. This is not true in general as we can see in the next result. For this, we will use Read's space $\mathcal{R}$ (see [62, 63, 81] for all the details on this space). Read's space is a renorming of the Banach space c_0, $\mathcal{R} = (c_0, |||\cdot|||)$, which has bidual $\mathcal{R}^{**}$ strictly convex (see [62, Theorem 4]).

Theorem 3.2.16. *Let $\mathcal{R}$ be Read's space. There exist subspaces X of c_0 and Y of $\mathcal{R}$ such that the set of tensors in $X\widehat{\otimes}_\pi Y^*$ which attain their projective norms is not dense in $X\widehat{\otimes}_\pi Y^*$.*

Considering X and Y^* as in Theorem 3.2.16, we can see that there exist $\alpha > 0$ and $z \in X\widehat{\otimes}_\pi Y^*$ such that $\operatorname{dist}(z, \mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y^*)) \geq \alpha$. If one takes u a finite-rank tensor such that $\|z-u\|_\pi < \frac{\alpha}{2}$. Then, this element cannot attain its projective norm. In other words, being finite-rank is not enough for being norm-attaining.

Corollary 3.2.17. *There are tensors of finite-rank which do not attain their projective norm.*

We do not know whether Theorem 3.2.16 holds for nuclear operators.

3.2.2 Symmetric tensor products

Analogously to what we have done in the previous section, we have the following definition. We say that $z \in \widehat{\otimes}_{\pi,s,N}X$ *attains its projective symmetric norm* if there are bounded sequences $(\lambda_n)_{n=1}^\infty \subset \mathbb{K}$ and $(x_n)_{n=1}^\infty \subseteq B_X$ such that $\|z\|_{\pi,s,N} = \sum_{n=1}^\infty |\lambda_n|$ for $z = \sum_{n=1}^\infty \lambda_n x_n^N$. In this case, we say that z is a *norm-attaining symmetric tensor.* We then denote

$$\mathrm{NA}_{\pi,s,N}(X) := \Big\{ z \in \widehat{\otimes}_{\pi,s,N}X : z \text{ attains its symmetric norm} \Big\}.$$

As a counterpart of Proposition 3.2.8, we have the following result.

Proposition 3.2.18. *Let X be a Banach space and suppose that every element in $\widehat{\otimes}_{\pi,s,N}X$ attains its norm. Then the set of all N-homogeneous polynomials that attain their norms is dense in the space of all N-homogeneous polynomials.*

With Proposition 3.2.18 in mind, we are able to present some examples where there exist symmetric tensors z which do not attain their norms. It is known (see [3, 60]) that if $X = d_*(w, 1)$ with $w \in \ell_2 \setminus \ell_1$, the predual of the Lorentz sequence space, then the set $\mathcal{P}(^N X)$, for $N \geq 2$, of all norm-attaining N-homogeneous polynomials on X, is *not* dense in $\mathcal{P}(^N X)$. Thus, Theorem 3.2.18 implies that there exists an element z in $\widehat{\otimes}_{\pi,s,N} X$ which does not attain its norm.

At the same way as Proposition 3.2.5, we have that if X is a finite dimensional Banach space, then every symmetric tensor attains its projective symmetric tensor norm. And at the same way as Theorem 3.2.10, we have the following.

Theorem 3.2.19. *Let X be a Banach space with the metric π-property. Then, every symmetric tensor can be approximated by symmetric tensors which attain their norms.*

We also have the following result on the denseness of norm-attaining elements in $\widehat{\otimes}_{\pi,s,N} X^*$ under the hypothesis of Radon-Nikodým property (for short, RNP), which is not covered by Theorem 3.2.19.

Theorem 3.2.20. *Let X be a Banach space. Suppose that X^* has the RNP and the AP. Then, every symmetric tensor in $\widehat{\otimes}_{\pi,s,N} X^*$ can be approximated by symmetric tensors that attain their norms.*

As one can see (after Theorems 3.2.19 and 3.2.20)we only have positive results about the denseness of norn-attaining symmetric tensors. In fact, we do not know whether Theorem 3.2.16 holds for symmetric projective tensor products. In other words, we do not know if the set of norm-attaining symmetric tensors is always dense for every Banach space X.

Chapter 4

Smoothness

In this chapter, we present the result of paper 4.

It is well-known that the existence of a smooth norm on a Banach space X has several deep structural consequences. For instance, the presence of a C^1-smooth bump implies that the space is Asplund [38]; the presence of an LFC bump yields that the space is a c_0-saturated Asplund space [39, 80]. If X admits a C^2-smooth bump, then either it contains a copy of c_0, or it is super-reflexive with type 2 [41]. Finally, if X admits a C^∞-smooth bump and it contains no copy of c_0, then it has exact cotype $2k$, for some $k \in \mathbb{N}$, and it contains ℓ_{2k} [34]. Each of these results involves at some point the *completeness* of the space X, most frequently via the appeal to some form of variational principles, such as the Ekeland variational principle [37], Stegall's variational principle [88], the Borwein–Preiss smooth variational principle [11], or the compact variational principle [30]. It is therefore unclear whether any, possibly weaker, form of the above results could be valid for general *normed* spaces. In this direction, it was pointed out in [10, p. 96] that it is not known whether X is an Asplund space provided the set where its norm fails to be Fréchet differentiable is 'small' in some sense (also see [48, Problem 148]). For example, it is unknown if there is a norm on ℓ_1 that is Fréchet differentiable outside a countable union of hyperplanes.

Nevertheless, some scattered results concerning normed spaces are present in the literature. Vanderwerff [89] proved that every normed space with a countable algebraic basis admits a C^1-smooth norm; this result was later improved to obtain a C^∞-smooth norm [49], a polyhedral norm [32], and an analytic one [28]. These results and the previous discussion motivated [48, Problem 149], [51], and recent research of our paper [28], where the following problem was posed.

Problem 4.0.1. *Let X be a Banach space and $k \in \mathbb{N} \cup \{\infty, \omega\}$. Is there a dense subspace Y of X that admits a C^k-smooth norm?*

This problem seems to be very general and ambitious. However, [28] answers it in the positive for X separable and $k = \omega$. Moreover, also in [28] it was solved in the positive for ℓ_∞ and $k = \omega$, $\ell_1(\mathfrak{c})$ and $k = \omega$, and spaces with long unconditional bases and $k = \infty$.

4.1 Main Result

The main result we have in this line of research can be summed up in only one theorem. It is a vast generalization of the previous results mentioned before.

Theorem 4.1.1. *Let X be a Banach space with a fundamental biorthogonal system $\{e_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma}$. Consider the dense subspace Y of X given by $Y := span\{e_\alpha\}_{\alpha\in\Gamma}$. Then:*

1. *Y admits a polyhedral and LFC norm,*
2. *Y admits a C^∞-smooth and LFC norm,*
3. *Y admits a C^∞-smooth and LFC bump,*
4. *Y admits locally finite, σ-uniformly discrete C^∞-smooth and LFC partitions of unity,*
5. *Y admits a C^1-smooth LUR norm.*

Moreover, norms as in (1), (2), and (5) are dense in the set of all equivalent norms on Y.

We would like to point out that Theorem 4.1.1 draws a complete picture concerning smoothness in the sense that it implies the existence of smooth norms, norm approximation by smooth norms, C^1-smooth LUR norms, and the existence of partitions of unity, which are instrumental for the smooth approximation of continuous or Lipschitz functions (see, for instance, [33, §VIII.3] or [50, Chapter 7]).

Chapter 5

Lineability

In this section, we present the results of papers 6 and 5. The first one concerns a contribution about the study of "pathological" nets in Functional Analysis in the sense of lineability. These families of nets arise from well-known results and we will detail our results in a minute. On the other hand, in the second section, as the main result, we construct an infinite metric space M such that the set $\mathrm{SNA}(M)$ of strongly norm-attaining Lipschitz functions on M does not contain a subspace which is linearly isometric to c_0.

5.1 Lineability properties in Functional Analysis

In Functional Analysis, certain results hold only when one considers sequences. In other words, they are not longer true when we consider the context of nets. In this section, we study lineability properties of families of

1. nets that are weakly convergent and unbounded,
2. nets that fail the Banach-Steinhaus theorem,
3. nets indexed by a regular cardinal κ that are weakly dense and norm-unbounded and finally
4. convergent series which have associated nets that are divergent.

We start with the following result. Theorem 5.1.1 improves and generalizes [18]Theorem 2.1 by considering arbitrary infinite-dimensional TVS over $\mathbb{R}$ or $\mathbb{C}$ and decreasing the size of the index set to make it $\leq \mathfrak{c}$ while still having the property of being co-lineable.

Theorem 5.1.1. *Let $\aleph_0 \leq \kappa \leq \mathfrak{c}$ be a cardinal number. Let X be a real or complex infinite-dimensional TVS. There exists a directed set $\mathcal{A}$ of cardinality κ such that the family of nets in X indexed by $\mathcal{A}$ that are unbounded and weakly convergent is 2^κ-lineable.*

In order to prove Theorem 5.1.1, we apply the Fichtenholz-Kantorovich-Hausdorff theorem [44, 52], which says that, for Γ a set of infinite cardinality κ, there exists always a family of independent subsets $\mathcal{Y}$ of Γ of cardinality 2^κ. Let us recall that a family $\mathcal{Y}$ of subsets of Γ is *independent* if for any pairwise distinct sets $Y_1, \dots, Y_n \in \mathcal{Y}$ and any $\varepsilon_1, \dots, \varepsilon_n \in \{0,1\}$ we have that $Y_1^{\varepsilon_1} \cap \cdots \cap Y_n^{\varepsilon_n} \neq \varnothing$, where Y^1 and Y^0 denote Y and $\Gamma \setminus Y$, respectively.

Recall that the Banach-Steinhaus theorem states the following (see, for instance, [26, Chapter 3, Theorem 14.6]): let X be Banach space and Y be a normed space; denote by $\mathcal{L}(X,Y)$ the Banach space of all continuous linear operators from X into Y; if a sequence $(T_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}(X,Y)$ strongly converges pointwise, then there is a $T \in \mathcal{L}(X,Y)$ such that $(T_n)_{n\in\mathbb{N}}$ strongly converges pointwise to T and $\{\|T_n\| \colon n \in \mathbb{N}\}$ is uniformly bounded. Let us recall that this theorem is a result only about sequences, not nets (for an easy example, see [26, page 97] just after its proof as a consequence of the Principle of Uniform Boundedness). In terms of lineability of the nets which do not satisfy the Banach-Steinhaus theorem, we have the following result.

Proposition 5.1.2. *Let $\aleph_0 \leq \kappa \leq \mathfrak{c}$ be a cardinal number. If X and Y are nonzero real or complex normed spaces, then there exists a directed set $\mathcal{A}$ with* $\operatorname{card}(\mathcal{A}) = \kappa$ *such that the set of nets of continuous linear operators $(T_a)_{a\in\mathcal{A}}$ in $\mathcal{L}(X,Y)$ that converge pointwise and also strongly converge pointwise to an operator but $\{\|T_a\| \colon a \in \mathcal{A}\}$ is not bounded is 2^κ-lineable.*

In [5, Corollary 5] (see also [9, 64, 66] for more general results in this line), the authors show that, in a separable Banach space X, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $\{x_n : n \in \mathbb{N}\}$ is weakly dense in X and $\|x_n\| \longrightarrow \infty$ as $n \to \infty$. As an immediate consequence of Theorem 5.1.4 below, we obtain Corollary 5.1.3 below which is related to the existence of norm divergent sequences that are weakly dense.

Corollary 5.1.3. *Let X be real or complex separable Banach space. The set of all sequences $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that*

(a) $\{\|x_n\| : n \in \mathbb{N}\}$ *is unbounded and*

(b) $\overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,X^*)} = X$

is $\mathfrak{c}$-lineable.

Theorem 5.1.4 reads as follows.

Theorem 5.1.4. *Let X be a real or complex normed space with dens$(X) = \kappa \geq \aleph_0$, where κ is a regular cardinal. Denote by* UWD_κ *the set of all κ-sequences $(x_\alpha)_{\alpha<\kappa}$ such that*

(i) $\{\|x_\alpha\| : \alpha < \kappa\}$ *is unbounded and*

(ii) $\overline{\{x_\alpha : \alpha < \kappa\}}^{\sigma(X,X^*)} = X$.

Then, the set UWD_κ *is κ^+-lineable. Moreover, if $2^{<\kappa} = \kappa$, then* UWD_κ *is 2^κ-lineable.*

Let X be a normed space and $\mathcal{I}$ an infinite set. We can give meaning to the convergence of the (possibly) "uncountable sum" in X, denoted by $\sum_{i\in I} x_i$, where each x_i belongs to X, as follows: consider $\mathcal{F}$ to be the set of all finite subsets of I

endowed with the inclusion $\subseteq$. Bearing this in mind, we have that $\mathcal{F}$ is a directed set. Now, for every $F \in \mathcal{F}$, we define

$$x_F := \sum_{i \in F} x_i.$$

Each x_F is then a well-defined vector of X (since F is finite) and $(x_F)_{F\in\mathcal{F}}$ is a net. In the same line, we have the following definition. Given $x_i \in X$ for all $i \in I$, we say that $\sum_{i\in I} x_i$ converges to $x \in X$ whenever $\lim_{F\in\mathcal{F}} x_F = x$. Recall that in Hilbert spaces, the latter definition can be used to obtain some relevant characterization in the non-separable case (see, for instance, [26, Chapter 1, Theorem 4.13]). We have the following result.

Theorem 5.1.5. *Let X be a normed space defined over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $\mathcal{F}$ the family of finite subsets of $\mathbb{N}$ endowed with the order $\subseteq$. The set of all sequences $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $\sum_{n=1}^{\infty} x_n$ is convergent and $(x_F)_{F\in\mathcal{F}}$ diverges is $\mathfrak{c}$-lineable.*

5.2 Embeddings into the sets of Lipschitz functions

In this short section, we will discuss the possibility of finding a linear space isometrically isomorphic to c_0 in the set $\mathrm{SNA}(M)$. This situation can be reduced only to infinite metric spaces. Note also that the choice of the distinguished point 0 in the pointed metric space M is irrelevant in our context. Indeed, if 0 and $0'$ are two distinguished points in M, then the mapping from $\mathrm{Lip}_0(M)$ to $\mathrm{Lip}_{0'}(M)$ defined as $f \mapsto f - f(0')$ is a linear isometry that completely preserves the strong norm-attainment behaviour of the mappings, so we do not need to worry about the choice of the distinguished point.

We start by guarantying the existence of an infinite complete metric space M such that the set $\mathrm{SNA}(M)$ of strongly norm-attaining Lipschitz functions does not contain a linearly isometric copy of c_0, answering a question posed in [8, Remark 3.6]. See also [24, 25] for more relevant and important results/references regarding spaceability in certain sets of Lipschitz functions.

Theorem 5.2.1. *There exists an infinite bounded uniformly discrete metric space M such that c_0 is not isometrically contained in $\mathrm{SNA}(M)$ and for which no point in M attains its separation radius.*

Going in the opposite direction of Theorem 5.2.1, next we show that we can always embed c_0 isometrically in $\mathrm{SNA}(M)$ whenever M is infinite but not uniformly discrete by using Ramsey's theorem.

Theorem 5.2.2. *Let M be an infinite non uniformly discrete metric space. Then, the set $\mathrm{SNA}(M)$ contains a linearly isometric copy of c_0.*

From [8, Theorem 3.3] we have that if M is any infinite compact metric space, then c_0 is isomorphically embedded into $\mathrm{SNA}(M)$ (for countable compact metric spaces this was achieved non constructively using the little Lipschitz space). Our previous theorem provides a constructive proof that for any infinite compact metric space M with a finite amount of cluster points, $\mathrm{SNA}(M)$ actually contains c_0 isometrically.

Corollary 5.2.3. *Let M be an infinite compact metric space. Then, the subset $\mathrm{SNA}(M)$ contains a linearly isometric copy of c_0.*

It turns out that Corollary 5.2.3 cannot be improved to include all proper metric spaces. Indeed, we have the following result.

Theorem 5.2.4. *There exists an infinite proper uniformly discrete metric space M such that c_0 is not isometrically contained in $\mathrm{SNA}(M)$ and for which every point in M attains its separation radius.*

We tackle the problem of embedding $c_0(\Gamma)$ in $\mathrm{SNA}(M)$ isometrically, where Γ is an arbitrary set of large cardinality. The next result is essentially based on the proof of [55, Proposition 3].

Theorem 5.2.5. *Let M be a metric space with $dens(M) = \Gamma$ for some cardinal Γ. Then, there exists a discrete set $L \subseteq M$ with $\mathrm{card}(L) = \Gamma$. Moreover, if $cof(\Gamma) > \aleph_0$, then L can be chosen to be uniformly discrete.*

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Appendix A

Attached publications

The following publications are included (listed in the order they appear):

1. S. Dantas, M. Jung and G. Martínez-Cervantes, *On the existence of non-norm-attaining operators*, J. Inst. Math. Jussieu 22 (2023), no. 3, 1023–1035.

2. S. Dantas, M. Jung, O. Roldán and A. Rueda Zoca, *Norm-attaining nuclear operators*. Mediterr. J. Math. 19 (2022), no.1, Paper No. 38, 27 pp.

3. S. Dantas, L.C. García-Lirola, M. Jung and A. Rueda Zoca, *On norm-attainment in (symmetric) tensor products*. Quaest. Math. 46 (2023), no. 2, 393–409.

4. Dantas, P. Hájek and T. Russo, *Smooth and polyhedral norms via fundamental biorthogonal systems*. Int. Math. Res. Not. IMRN(2023), no. 16, 13909–13939.

5. S. Dantas and D.L. Rodríguez-Vidanes, *Algebraic genericity of certain families of nets in Functional Analysis*. Accepted by Israel Journal of Mathematics (2024).

6. S. Dantas, R. Medina, A. Quilis and Ó. Roldán, *On isometric embeddings into the set of strongly norm-attaining Lipschitz functions*. Nonlinear Anal. 232 (2023), Paper No. 113287, 15 pp.

J. Inst. Math. Jussieu (2021), 1–13
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ON THE EXISTENCE OF NON-NORM-ATTAINING OPERATORS

SHELDON DANTAS [1], MINGU JUNG [2] AND GONZALO MARTÍNEZ-CERVANTES [3]

[1] *Departament de Matemàtiques and Institut Universitari de Matemàtiques i Aplicacions de Castelló (IMAC), Universitat Jaume I, Campus del Riu Sec. s/n, 12071 Castelló Spain*
(dantas@uji.es)

[2] *Basic Science Research Institute and Department of Mathematics, POSTECH, Pohang 790-784, Republic of Korea*
(jmingoo@postech.ac.kr)

[3] *Universidad de Murcia, Departamento de Matemáticas, Campus de Espinardo 30100 Murcia, Spain*
(gonzalo.martinez2@um.es)

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Abstract In this article, we provide necessary and sufficient conditions for the existence of non-norm-attaining operators in $\mathcal{L}(E,F)$. By using a theorem due to Pfitzner on James boundaries, we show that if there exists a relatively compact set K of $\mathcal{L}(E,F)$ (in the weak operator topology) such that 0 is an element of its closure (in the weak operator topology) but it is not in its norm-closed convex hull, then we can guarantee the existence of an operator that does not attain its norm. This allows us to provide the following generalisation of results due to Holub and Mujica. If E is a reflexive space, F is an arbitrary Banach space and the pair (E,F) has the (pointwise-)bounded compact approximation property, then the following are equivalent:

(i) $\mathcal{K}(E,F) = \mathcal{L}(E,F)$;
(ii) Every operator from E into F attains its norm;
(iii) $(\mathcal{L}(E,F),\tau_c)^* = (\mathcal{L}(E,F),\|\cdot\|)^*$,

where τ_c denotes the topology of compact convergence. We conclude the article by presenting a characterisation of the Schur property in terms of norm-attaining operators.

Keywords and phrases: James theorem, norm-attaining operators, compact approximation property

2020 Mathematics Subject Classification: 46B20, 46B10, 46B28

1. Introduction

The famous James theorem states that a Banach space E is reflexive if and only if every bounded linear functional on E attains its norm. By using this characterisation, one can check that if every bounded linear operator from a Banach space E into a Banach space

F is norm-attaining, then E must be reflexive, whereas the range space F is not forced to be reflexive in general. Indeed, every bounded linear operator from a reflexive space into a Banach space that satisfies the Schur property is compact (by the Eberlein-Šmulian theorem) and any compact operator from a reflexive space into an arbitrary Banach space is norm-attaining. Therefore, it seems natural to wonder whether it is possible to guarantee the existence of a non-norm-attaining operator from the existence of a noncompact operator. This brings us back to the 1970s when J.R. Holub proved that this is, in fact, true under approximation property assumptions (see [16, Theorem 2]). Almost 30 years later, J. Mujica improved Holub's result by using the compact approximation property (see [25, Theorem 2.1]), which is a weaker assumption than the approximation property. However, both results require the reflexivity on both domain and range spaces, so the following question arises naturally:

Given a reflexive space E and an arbitrary Banach space F, under which assumptions may we guarantee the existence of non-norm-attaining operators in $\mathcal{L}(E,F)$?

Coming back to Holub and Mujica's results, we would like to highlight what they proved in the direction of the above question. For a background on necessary definitions and notations, we refer the reader to Section 2. In what follows, τ_c denotes the topology of compact convergence and $\|\cdot\|$ denotes the norm topology in $\mathcal{L}(E,F)$.

Theorem ([16, Theorem 2] and [25, Theorem 2.1]). *Let E and F be both reflexive spaces.*

(a) *If $\mathcal{L}(E,F)$ is nonreflexive, there is a non-norm-attaining operator $S \in \mathcal{L}(E,F)$.*

(b) *If E or F has the (compact) approximation property, then the following statements are equivalent:*
   - (i) *There exists a non-norm-attaining operator $S \in \mathcal{L}(E,F)$;*
   - (ii) $\mathcal{L}(E,F) \neq \mathcal{K}(E,F)$;
   - (iii) $\mathcal{L}(E,F)$ *is nonreflexive;*
   - (iv) $(\mathcal{L}(E,F),\|\cdot\|)^* \neq (\mathcal{L}(E,F),\tau_c)^*$.

The proof of the above result relies on the fact that if F is a reflexive space, then $\mathcal{L}(E,F)$ is the dual space of the projective tensor product $E\widehat{\otimes}_\pi F^*$. However, if the range space F is nonreflexive, then $\mathcal{L}(E,F)$ is always nonreflexive (see, for instance, [30]).

As a way of extending the above results to the case of nonreflexive range spaces, we borrow some of the techniques used by R.C. James (see [17, 18]). As a matter of fact, one of his results [18, Theorem 1] implies that a separable Banach space E is nonreflexive if and only if given $0 < \theta < 1$, there exists a sequence (x_n^*) in B_{E^*} such that $x_n^* \xrightarrow{w^*} 0$ and dist $(0, \operatorname{co}\{x_n^* : n \in \mathbb{N}\}) > \theta$, which in turn is equivalent to the existence of a relatively weak-star compact set $K \subseteq B_{E^*}$ such that $0 \in \overline{K}^{w^*}$ and $0 \notin \overline{\operatorname{co}}^{\|\cdot\|}(K)$. This motivates us to define the following property.

Definition 1.1. We say that a pair (E,F) of Banach spaces has the *James property* if there exists a relatively weak operator topology (WOT)-compact set $K \subseteq \mathcal{L}(E,F)$ such that $0 \in \overline{K}^{WOT}$ and $0 \notin \overline{\operatorname{co}}^{\|\cdot\|}(K)$.

We will prove that the James property is a sufficient condition to guarantee the existence of an operator that does not attain its norm, which is our first aim in the present article.

Theorem A. *Let E and F be Banach spaces. If the pair (E,F) has the James property, then there exists a non-norm-attaining operator in $\mathcal{L}(E,F)$.*

Next, we prove that $(\mathcal{L}(E,F),\|\cdot\|)^*$ does not coincide with $(\mathcal{L}(E,F),\tau_c)^*$ whenever a pair (E,F) satisfies the James property (see Proposition 3.1). From this, we can see that whenever the pair (E,F) has the James property, the Banach space $\mathcal{L}(E,F)$ cannot be reflexive due to [25, Lemma 2.3].

We observe, for a reflexive space E and an arbitrary Banach space F, that (1) the unit ball of $\mathcal{K}(E,F)$ is closed in the strong operator topology if and only if it is sequentially closed in this topology (see Lemma 3.4) and (2) $\mathcal{K}(E,F) = \mathcal{L}(E,F)$ implies that $(\mathcal{L}(E,F),\|\cdot\|)^* = (\mathcal{L}(E,F),\tau_c)^*$ by using the result [12, Theorem 1] due to M. Feder and P. Saphar. In addition, we consider the concept of the pointwise-bounded compact approximation property for a pair of Banach spaces in the way it is done in [3] (see Definition 2.1) and prove that $\mathcal{K}(E,F) = \mathcal{L}(E,F)$ when either (3) the norm-closed unit ball of $\mathcal{K}(E,F)$ is closed in the strong operator topology or (4) $(\mathcal{L}(E,F),\|\cdot\|)^* = (\mathcal{L}(E,F),\tau_c)^*$ under the just mentioned approximation property assumption (see Lemma 3.7). Combining (1)–(4) together with Theorem A, we get a generalisation of Holub and Mujica's results, where we no longer need reflexivity on the target space F and E and F might not have the bounded compact approximation property (CAP) (see Example 2.2).

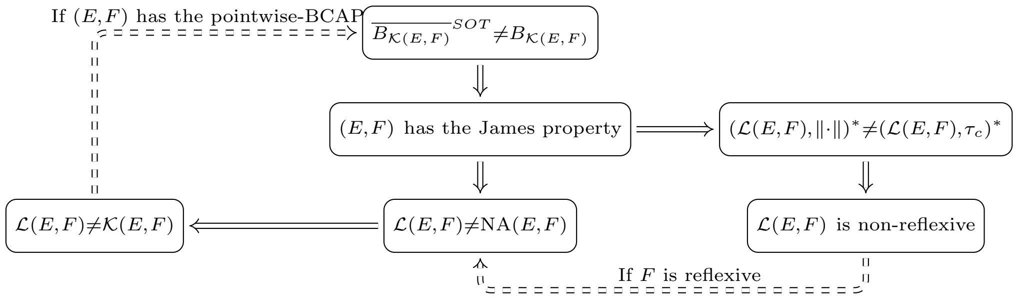
Theorem B. *Let E be a reflexive space and F be an arbitrary Banach space. Consider the following conditions:*

(a) $\mathcal{K}(E,F) = \mathcal{L}(E,F)$.
(b) *Every operator from E into F attains its norm.*
(c) *The unit ball $B_{\mathcal{K}(E,F)}$ is closed in the strong operator topology.*
(d) $(\mathcal{L}(E,F),\tau_c)^* = (\mathcal{L}(E,F),\|\cdot\|)^*$.

Then, we always have (a) $\implies$ (b) $\implies$ (c) and (a) $\implies$ (d) $\implies$ (c). Additionally, if the pair (E,F) has the bounded compact approximation property, then (c) $\implies$ (a) and therefore all of the statements are equivalent.

The following diagram summarises most of the results included in this article. In what follows, E is supposed to be any reflexive space and F is any arbitrary Banach space. See Definition 2.1 in Section 2 for the definition of the pointwise-bounded compact approximation property (BCAP).

Finally, as an application of Theorem B, we connect the Schur property with the case where every operator attains its norm and obtain the following characterisation, which follows from Theorem 3.10.



Theorem C. *Let F be a Banach space. The following statements are equivalent:*

(a) *F has the Schur property.*

(b) *$\mathcal{K}(E,F) = \mathcal{L}(E,F)$ for every reflexive space E.*

(c) *$\mathrm{NA}(E,F) = \mathcal{L}(E,F)$ for every reflexive space E.*

(d) *$\mathcal{K}(G,F) = \mathcal{L}(G,F)$ for every reflexive space G with basis.*

(e) *$\mathrm{NA}(G,F) = \mathcal{L}(G,F)$ for every reflexive space G with basis.*

2. Preliminaries

Throughout the article, E and F will be Banach spaces over a field $\mathbb{K}$, which can be either real or complex numbers. We denote by B_E and S_E the closed unit ball and the unit sphere of the Banach space E, respectively. For a subset K of E, $\mathrm{co}(K)$ (respectively, $\overline{\mathrm{co}}(K)$) denotes the convex hull (respectively, closed convex hull) of K. The space of all bounded linear operators from E into F is denoted by $\mathcal{L}(E,F)$. The symbol $\mathcal{K}(E,F)$ (respectively, $\mathcal{W}(E,F)$) stands for the space of all compact operators (respectively, weakly compact operators) from E into F, whereas the symbol $\mathcal{F}(E,F)$ is used to denote the space of all finite-rank operators. Finally, let us recall that an operator $T \in \mathcal{L}(E,F)$ *attains its norm* or is *norm-attaining* if there exists $x \in B_E$ such that $\|T(x)\| = \|T\|$. By $\mathrm{NA}(E,F)$ we mean the set of all norm-attaining operators from E into F. If $E = F$, then we simply write $\mathrm{NA}(E)$ instead of $\mathrm{NA}(E,E)$ and we do the same with the above classes of operators.

We will be using different topologies in $\mathcal{L}(E,F)$. We denote by τ_c the *topology of compact convergence*; that is, the topology of uniform convergence on compact subsets of E. The WOT is defined by the basic neighbourhoods

$$N(T;A,B,\varepsilon) := \big\{S \in \mathcal{L}(E,F) : |y^*(T-S)(x)| < \varepsilon, \text{ for every } y^* \in B, x \in A\big\},$$

where A and B are arbitrary finite sets in E and F^*, respectively, and $\varepsilon > 0$. Thus, in the WOT, a net (T_α) converges to T if and only if $(y^*(T_\alpha(x)))$ converges to $y^*(T(x))$ for every $x \in E$ and $y^* \in F^*$. Analogously, the *strong operator topology* (SOT) is defined by the basic neighbourhoods

$$N(T;A,\varepsilon) := \big\{S \in \mathcal{L}(E,F) : \|(T-S)(x)\| < \varepsilon, \text{ for every } x \in A\big\},$$

where A is an arbitrary finite set in E and $\varepsilon > 0$. Thus, a net (T_α) converges in the SOT to T if and only if $(T_\alpha(x))$ converges in norm to $T(x)$ for every $x \in E$. We will deal with SOT and WOT closures of bounded sets in $\mathcal{L}(E,F)$. It is worth mentioning that for a bounded convex set in $\mathcal{L}(E,F)$, the WOT closure and the SOT closure coincide [10, Corollary VI.1.5]. Thus, the SOT and the WOT in some results in this article can be interchanged. For a more detailed exposition on topologies in $\mathcal{L}(E,F)$, we refer the reader to [8, 10].

Let us present now the necessary definitions on approximation properties we will need. A Banach space E is said to have the *approximation property* (AP) if the identity operator Id_E in $\mathcal{L}(E)$ belongs to $\overline{\mathcal{F}(E)}^{\tau_c}$. Given $\lambda \geqslant 1$, E is said to have the λ-*bounded approximation property* (λ-BAP) when Id_E belongs to $\lambda\overline{B_{\mathcal{F}(E)}}^{\tau_c}$. A Banach space is said to have the *bounded approximation property* (BAP) if it has the λ-BAP for some $\lambda \geqslant 1$. In the special case when $\lambda = 1$, we say that E has the *metric approximation property* (MAP). Also, recall that E is said to have the CAP if the identity operator Id_E in $\mathcal{L}(E)$ belongs to $\overline{\mathcal{K}(E)}^{\tau_c}$. The λ-*bounded compact approximation property* (λ-BCAP), BCAP and *metric compact approximation property* (MCAP) for a Banach space E can be defined in an analogous way. It is known that a reflexive space has the AP if and only if it has the MAP (see [14]). Analogously, every reflexive space with the CAP also has the MCAP (see [5, Proposition 1 and Remark 1]). We refer the reader to [22, 23] and [4] for background.

On the other hand, E. Bonde introduced in [3] the AP and λ-BAP for pairs of Banach spaces in the following way: a pair (E,F) of Banach spaces is said to have the *AP* if any operator $T \in \mathcal{L}(E,F)$ belongs to $\overline{\mathcal{F}(E,F)}^{\tau_c}$. If $\lambda \geqslant 1$ and every operator $T \in \mathcal{L}(E,F)$ belongs to $\lambda\|T\|\overline{B_{\mathcal{F}(E,F)}}^{\tau_c}$, then (E,F) is said to have the λ-*BAP* or simply the *BAP*.

It is clear that if E or F has the AP (respectively, BAP), then the pair (E,F) has the AP (respectively, BAP). It is observed in [3, Section 4] that there are pairs of Banach spaces (E,F) with the BAP such that E and F do not have the BAP. Similarly, we have the following.

Definition 2.1. The pair (E,F) of Banach spaces is said to have the *compact approximation property* (CAP) if every operator $T \in \mathcal{L}(E,F)$ belongs to $\overline{\mathcal{K}(E,F)}^{\tau_c}$. If $\lambda \geqslant 1$ and every operator $T \in \mathcal{L}(E,F)$ belongs to $\lambda\|T\|\overline{B_{\mathcal{K}(E,F)}}^{\tau_c}$, then (E,F) is said to have the λ-*BCAP* or simply the *BCAP*. In the case when $\lambda = 1$, we say that the pair (E,F) has the MCAP.

Moreover, as one of the anonymous referees and Miguel Martín suggested, we say that the pair (E,F) has the *pointwise-BCAP* if for every operator $T \in \mathcal{L}(E,F)$ there is a constant $\lambda_T \geqslant 1$ such that $T \in \lambda_T\overline{B_{\mathcal{K}(E,F)}}^{\tau_c}$.

Let us note that the BCAP implies the pointwise-BCAP and the pointwise-BCAP implies the CAP. In addition, for a Banach space E, the pair (E,E) has the BCAP if and only if it has the pointwise-BCAP (just take $\lambda := \lambda_I$ with I the identity on E from the definition of the pointwise-BCAP and note that $T = T \circ I \subseteq \overline{\{T \circ K : K \in \lambda_I B_{\mathcal{K}(E,E)}\}}^{\tau_c} \subseteq \lambda\|T\|\overline{B_{\mathcal{K}(E,E)}}^{\tau_c}$ for every $T \in \mathcal{L}(E,E)$). However, we do not know whether the pointwise-BCAP of a pair (E,F) is equivalent to the BCAP or the CAP for an arbitrary Banach space F even if E is assumed to be reflexive. The next example shows that a pair (E,F) might have the BCAP even if E and F do not have the CAP.

Example 2.2. In [3, Example 4.2], it is shown that whenever E is a subspace of ℓ_{p_1} and F is a subspace of ℓ_{p_2} with $1 \leqslant p_2 < 2 < p_1 < \infty$, the pair (E,F) has the BAP and therefore the BCAP. Nevertheless, for every $1 < p < \infty$ with $p \neq 2$ there is a subspace of ℓ_p failing the CAP (see [6] and [22, Theorem 1.g.4 and Remark 2 in pg. 111]). In particular, there are Banach spaces E and F such that (E,F) has the BCAP and E and F do not have the CAP. Therefore, assuming that a pair (E,F) of Banach spaces has the BCAP is more general than E or F having the CAP.

3. The Results

In this section, we shall prove Theorems A, B, C and their consequences. We start by proving Theorem A. To do so, let us recall that a set $B \subseteq B_{E^*}$ is called a *James boundary* of a Banach space E if for every $x \in S_E$ there exists $f \in B$ such that $f(x) = 1$. For a subset G of E^*, we shall denote by $w(E,G)$ the weak topology of X induced by G.

Proof of Theorem A. Let us assume by contradiction that every operator from E into F attains its norm. Then, the family

$$B := \Big\{ x \otimes y^* : x \in S_E, y^* \in S_{F^*} \Big\}$$

is a James boundary of $\mathcal{L}(E,F)$. Indeed, for an arbitrary operator $T \in \mathcal{L}(E,F) = \mathrm{NA}(E,F)$, take $x \in S_E$ to be such that $\|T(x)\| = \|T\|$ and then $y^* \in S_{F^*}$ to be such that $|y^*(T(x))| = \|T(x)\| = \|T\|$. Now, because (E,F) has the James property, there exists a relatively WOT-compact set $K \subseteq \mathcal{L}(E,F)$ such that $0 \in \overline{K}^{WOT}$ and $0 \notin \overline{\mathrm{co}}^{\|\cdot\|}(K)$. By the uniform boundedness principle, the set $\overline{K}^{WOT}$ is norm-bounded. Note that the WOT-topology is the topology of pointwise convergence on B, so it coincides with $w(\mathcal{L}(E,F),B)$. By hypothesis, $\overline{K}^{WOT}$ is WOT-compact or, equivalently, $w(\mathcal{L}(E,F),B)$-compact. By a theorem of Pfitzner (see [27] or [11, Theorem 3.121]), we have that $\overline{K}^{WOT}$ is weakly compact. Therefore, $0 \in \overline{K}^{WOT} = \overline{K}^{w}$, which in particular gives that $0 \in \overline{\mathrm{co}}^{w}(K) = \overline{\mathrm{co}}^{\|\cdot\|}(K)$. This contradiction yields a non-norm-attaining operator $T \in \mathcal{L}(E,F)$ as desired. □

Let us observe that if a pair (E,F) of Banach spaces has the James property, then the dual of $\mathcal{L}(E,F)$ endowed with the norm topology does not coincide with the dual of $\mathcal{L}(E,F)$ endowed with the topology τ_c of compact convergence. As a matter of fact, if K is a subset of E given as in Definition 1.1, then there exists $\varphi \in (\mathcal{L}(E,F),\|\cdot\|)^*$ such that $0 = \mathrm{Re}\,\varphi(0) > \sup\{\mathrm{Re}\,\varphi(T) : T \in \overline{\mathrm{co}}(K)\}$ thanks to the Hahn-Banach separation theorem. This implies that φ cannot be in $(\mathcal{L}(E,F),\tau_c)^*$ because $0 \in \overline{\mathrm{co}}^{WOT}(K) = \overline{\mathrm{co}}^{\tau_c}(K)$. Moreover, using [25, Lemma 2.3], we see that if $(\mathcal{L}(E,F),\|\cdot\|)^* \neq (\mathcal{L}(E,F),\tau_c)^*$, then the space $\mathcal{L}(E,F)$ cannot be reflexive. Summarising, we obtain the following result.

Proposition 3.1. *Let E and F be Banach spaces. If the pair (E,F) has the James property, then*

(i) $(\mathcal{L}(E,F),\|\cdot\|)^* \neq (\mathcal{L}(E,F),\tau_c)^*$.

(ii) $\mathcal{L}(E,F)$ *is nonreflexive.*

One easy consequence of Theorem A is that if E is reflexive and a pair (E,F) has the James property, then $\mathcal{K}(E,F)$ cannot be equal to the whole space $\mathcal{L}(E,F)$. As a matter of fact, the following result gives us a rather general observation.

Proposition 3.2. *Let E be a reflexive space and F be an arbitrary Banach space. If $\mathcal{K}(E,F) = \mathcal{L}(E,F)$, then $(\mathcal{L}(E,F),\|\cdot\|)^* = (\mathcal{L}(E,F),\tau_c)^*$.*

Proof. Let $D : E\widehat{\otimes}_\pi F^* \longrightarrow (\mathcal{L}(E,F),\tau_c)^*$ be defined by $D(z)(T) := \sum_{n=1}^\infty y_n^*(T(x_n))$ for every $z \in E\widehat{\otimes}_\pi F^*$ with $z = \sum_{n=1}^\infty x_n \otimes y_n^*$ and $T \in \mathcal{L}(E,F)$. It is well known that D is a surjective map (see, for example, [8, Section 5.5, pg. 62]). Therefore, we have that $(\mathcal{L}(E,F),\tau_c)^* = (E\widehat{\otimes}_\pi F^*)/\ker D$. On the other hand, from the result [12, Theorem 1], we have that the map $V : E\widehat{\otimes}_\pi F^* \longrightarrow (\mathcal{K}(E,F),\|\cdot\|)^*$ defined by $V(z)(T) := \sum_{n=1}^\infty y_n^*(T(x_n))$ for $z = \sum_{n=1}^\infty x_n \otimes y_n^*$ and $T \in \mathcal{K}(E,F)$ satisfies the following: for every $\varphi \in (\mathcal{K}(E,F),\|\cdot\|)^*$, there exists $v \in E\widehat{\otimes}_\pi F^*$ such that $\varphi = V(v)$ and $\|\varphi\| = \|v\|$. In particular, we have that $(\mathcal{K}(E,F),\|\cdot\|)^* = (E\widehat{\otimes}_\pi F^*)/\ker V$. Thus, if $\mathcal{K}(E,F) = \mathcal{L}(E,F)$, then $D(z)(T) = V(z)(T)$ for every $z \in E\widehat{\otimes}_\pi F^*$ and every $T \in \mathcal{K}(E,F)$; hence, $\ker D = \ker V$ and $(\mathcal{L}(E,F),\|\cdot\|)^* = (\mathcal{L}(E,F),\tau_c)^*$. □

Let us now go towards the proof of Theorem B. We show the following result, which will help us to prove that if (E,F) does not satisfy the James property, then $\overline{B_{\mathcal{K}(E,F)}}^{SOT}$ coincides with $B_{\mathcal{K}(E,F)}$. Recall that the sequential closure of a set in a topological space is the family of all limit points of sequences on the set in consideration.

Lemma 3.3. *Let E and F be Banach spaces. Suppose that there exists a norm-closed convex set $C \subseteq \mathcal{L}(E,F)$ that is not sequentially closed in the strong operator topology. Then (E,F) has the James property.*

Proof. Suppose that $C \subseteq \mathcal{L}(E,F)$ is norm-closed but not SOT-sequentially closed. This implies that there exists a sequence of operators $(R_n) \subseteq C$ such that (R_n) converges in the SOT (and therefore in the WOT) to an operator $R \notin C$. We may (and we do) suppose that $R = 0$. Set $K := \{R_n : n \in \mathbb{N}\} \subseteq \mathcal{L}(E,F)$. Therefore, K is relatively WOT-compact, $0 \in \overline{K}^{WOT}$ but 0 cannot be in $\overline{\text{co}}(K)$ by hypothesis. Therefore, (E,F) has the James property. □

It is not difficult to check that for a bounded subset C of $\mathcal{L}(E,F)$, with E separable, the SOT-closure of C coincides with the SOT-sequential closure of C. Namely, if $D = \{x_n : n \in \mathbb{N}\}$ is a countable norm-dense subset of E and C is a bounded subset of $\mathcal{L}(E,F)$ with $T \in \overline{C}^{SOT}$, then for each $n \in \mathbb{N}$ we can pick $T_n \in C$ such that $\|T_n(x_m) - T(x_m)\| \leqslant \frac{1}{n}$ for every $m \leqslant n$. Because (T_n) is a uniformly bounded sequence of operators converging in norm on a dense set, it follows from a routine computation that $T_n(x)$ converges in norm to $T(x)$ for every $x \in E$, so T is in the SOT-sequential closure of C.

Furthermore, the following result shows that the unit ball of $\mathcal{K}(E,F)$ is SOT-closed if it is SOT-sequentially closed under the assumption that E has the separable complementation property. Recall that a Banach space E is said to have the *separable complementation property* if for every separable subspace Y in E there is a separable subspace Z with $Y \subseteq Z \subseteq E$ and Z is complemented in E. It is worth mentioning that D. Amir and J. Lindenstrauss proved in [2] that every weakly compactly generated Banach space (and therefore every reflexive space) has the separable complementation property.

Lemma 3.4. *Let E be a Banach space with the separable complementation property and F be an arbitrary Banach space. Then, the unit ball $B_{\mathcal{K}(E,F)}$ is SOT-closed if and only if it is SOT-sequentially closed.*

Proof. It is enough to check that if $B_{\mathcal{K}(E,F)}$ is not SOT-closed then it is not SOT-sequentially closed. Suppose that T is an operator that belongs to the SOT-closure of $B_{\mathcal{K}(E,F)}$ but not to $B_{\mathcal{K}(E,F)}$. Note that T is noncompact; hence, there exists a separable subspace E_0 of E such that $T|_{E_0}$ is noncompact. Choose a separable subspace Z of E such that $E_0 \subseteq Z \subseteq E$ and Z is complemented in E. Note that $T|_Z$ is noncompact and belongs to the SOT-closure of $B_{\mathcal{K}(Z,F)}$. Because Z is separable, we have that $T|_Z$ is indeed in the SOT-sequential closure of $B_{\mathcal{K}(Z,F)}$. Let (K_n) be a sequence in $B_{\mathcal{K}(Z,F)}$ converging to $T|_Z$ in the SOT. Letting P be a projection from E onto Z, it is immediate that $T|_Z \circ P$ is noncompact and $K_n \circ P$ is SOT-convergent to $T|_Z \circ P$. This proves that $B_{\mathcal{K}(E,F)}$ is not SOT-sequentially closed. □

It is worth mentioning, however, that the SOT-closure and SOT-sequential closure are different in general, as the following remark shows.

Remark 3.5. *In general, it is not true that the SOT-sequential closure of a bounded convex set C in $\mathcal{L}(E,F)$ coincides with the SOT-closure of C. An example is given by*

$$C := \Big\{ T \in B_{\mathcal{L}(\ell_2(\omega_1))} : \text{there is } \alpha < \omega_1 \text{ such that } (T(x))_\beta = 0 \text{ for every } \beta > \alpha,\ x \in \ell_2(\omega_1) \Big\}.$$

It is immediate that C is SOT-sequentially closed. Nevertheless, because the canonical projections $P_\alpha \in \mathcal{L}(\ell_2(\omega_1))$ with $\alpha < \omega_1$, defined by $(P_\alpha(x))_\beta = x_\beta$ if $\beta \leqslant \alpha$ and 0 otherwise are in C and satisfy that $\{P_\alpha\}_{\alpha<\omega_1}$ SOT-converges to the identity, which is not in C, it follows that C is not SOT-closed.

Notice that if E is reflexive, then it has the separable complementation property. By Lemma 3.4, $B_{\mathcal{K}(E,F)}$ is SOT-closed if and only if it is SOT-sequentially closed. Therefore, if we assume that $\overline{B_{\mathcal{K}(E,F)}}^{SOT} \neq B_{\mathcal{K}(E,F)}$, then (E,F) has the James property by Lemma 3.3. Therefore, we have the following result.

Proposition 3.6. *Let E and F be Banach spaces. If $\overline{B_{\mathcal{K}(E,F)}}^{SOT} \neq B_{\mathcal{K}(E,F)}$, then (E,F) has the James property.*

In order to prove Theorem B, we also need the following lemma. We thank Miguel Martín and one of the anonymous referees for suggesting the use of the pointwise-BCAP instead of the BCAP in the following lemma.

Lemma 3.7. *Let E and F be Banach spaces. Suppose that the pair (E,F) has the pointwise-BCAP. Then $\mathcal{K}(E,F) = \mathcal{L}(E,F)$ if and only if the unit ball $B_{\mathcal{K}(E,F)}$ is SOT-closed.*

Proof. First, note that because the pair (E,F) has the pointwise-BCAP, we have $\mathcal{L}(E,F) = \bigcup_{\lambda>0} \lambda \overline{B_{\mathcal{K}(E,F)}}^{\tau_c}$. Because $\overline{B_{\mathcal{K}(E,F)}}^{\tau_c} \subseteq \overline{B_{\mathcal{K}(E,F)}}^{SOT}$, we have that $\mathcal{L}(E,F) = \bigcup_{\lambda>0} \lambda \overline{B_{\mathcal{K}(E,F)}}^{SOT}$. So, if we assume $B_{\mathcal{K}(E,F)}$ to be SOT-closed, then

$$\mathcal{L}(E,F) = \bigcup_{\lambda>0} \lambda \overline{B_{\mathcal{K}(E,F)}}^{SOT} = \bigcup_{\lambda>0} \lambda B_{\mathcal{K}(E,F)} = \mathcal{K}(E,F).$$

The other implication is immediate. □

Let us finally recall the following conditions and prove Theorem B as a consequence of Theorem A, Proposition 3.1, Proposition 3.2, Proposition 3.6 and Lemma 3.7.

(a) $\mathcal{K}(E,F) = \mathcal{L}(E,F)$.

(b) Every operator from E into F attains its norm.

(c) The unit ball $B_{\mathcal{K}(E,F)}$ is closed in the strong operator topology.

(d) $(\mathcal{L}(E,F),\tau_c)^* = (\mathcal{L}(E,F),\|\cdot\|)^*$.

Proof of Theorem B. Let E be reflexive and F be an arbitrary Banach space. It was noted in the Introduction that (a) $\Longrightarrow$ (b) holds. Moreover, (b) implies that (E,F) does not have the James property (by applying Theorem A), which in turn implies (c) (by applying Proposition 3.6). On the other hand, Proposition 3.2 shows (a) $\Longrightarrow$ (d). By Proposition 3.1, (d) implies that (E,F) does not have the James Property and, therefore, it implies (c) (by applying Proposition 3.6). Finally, if the pair (E,F) has the BCAP, then the implication (c) $\Longrightarrow$ (a) follows from Lemma 3.7 and all statements (a)–(d) are equivalent. □

M.I. Ostrovskii asked in [24, §12, pg. 65] whether there exist infinite-dimensional Banach spaces on which every operator attains its norm (this question is also asked in [20, Problem 8] and [15, Problem 217]). By Holub's theorem [16], if such an infinite-dimensional Banach space exists, it cannot have the AP. Theorem 3.8 is a generalisation of this fact. Let us recall that given a (norm-closed) operator ideal $\mathcal{A}$ and $\lambda \geqslant 1$, a Banach space E is said to have the *λ-$\mathcal{A}$-approximation property* (λ-$\mathcal{A}$-AP) if the identity operator Id_E belongs to $\overline{\{T \in \mathcal{A}(E,E) : \|T\| \leqslant \lambda\}}^{\tau_c}$. We say that E has the bounded-$\mathcal{A}$-AP if it has the λ-$\mathcal{A}$-AP for some $\lambda \geqslant 1$. This general approximation property has been studied, for instance, in [13, 21, 26, 29].

Theorem 3.8. *If there is an infinite-dimensional Banach space E such that every operator on $\mathcal{L}(E)$ attains its norm, then E does not have the bounded $\mathcal{A}$-approximation property for any nontrivial ideal $\mathcal{A}$ (i.e., for any ideal $\mathcal{A} \neq \mathcal{L}(E)$).*

Proof. As highlighted in [24, §12, pg. 66], due to a result of N.J. Kalton, if such a Banach space E exists, then it must be separable. Therefore, the SOT-closure of the set $B := \{T \in \mathcal{A}(E,E) : \|T\| \leqslant 1\}$ in $\mathcal{L}(E)$ coincides with its SOT-sequential closure. Thus, if every operator on $\mathcal{L}(E)$ attains its norm, then B is SOT-closed by Lemma 3.3. Suppose that E has the bounded $\mathcal{A}$-approximation property. Then, because $\overline{B}^{\tau_c} \subseteq \overline{B}^{SOT} = B$, we have that B contains a multiple of the identity and therefore $\mathcal{A}$ contains the identity on E, so $\mathcal{A} = \mathcal{L}(E)$. $\square$

We finally present the proof of Theorem C as a direct consequence of Theorem B and Proposition 3.9. Recall that a Banach space E has the *Schur property* if every weakly convergent sequence is norm convergent. It is known that a Banach space F has the Schur property if and only if every weakly compact operator from E into F is compact for any Banach space E (see, for example, [28, Section 3.2, pg. 61]). Also, it is proved in [9, Theorem 1] that a Banach space F has the Schur property if and only if the weak Grothendieck compactness principle holds in F; that is, every weakly compact subset of F is contained in the closed convex hull of a weakly null sequence. Then W.B. Johnson et al. gave an alternative proof in [19, Theorem 1.1] for this result by using the Davis-Figiel-Johnson-Pełczyński factorisation theorem [7]. Moreover, it was observed in [19, Theorem 3.3] that a Banach space F has the Schur property if and only if $\mathcal{W}_\infty(E,F) \subseteq \mathcal{W}(E,F)$ for every Banach space E (see the precise definition of these sets just after Proposition 3.9).

The following result will be used as an important tool in the proof of Theorem 3.10.

Proposition 3.9. *Let F be a Banach space. If F fails to have the Schur property, then there exists a reflexive space with basis E such that $\mathcal{K}(E,F) \neq \mathcal{L}(E,F)$.*

Proof. Take $(x_n) \subseteq S_F$ to be a weakly null sequence in F, which is not norm null. Because the absolute closed convex hull of $\{x_n : n \in \mathbb{N}\}$ is weakly compact, the operator $T \in \mathcal{L}(\ell_1, F)$ given by $T(e_n) := x_n$ for each $n \in \mathbb{N}$ defines a weakly compact operator (which is not compact). By the Davis-Figiel-Johnson-Pełczyński factorisation theorem [7], there exists a reflexive space E_0 such that $T = S \circ R$, where $R \in \mathcal{L}(\ell_1, E_0)$ and $S \in \mathcal{L}(E_0, F)$. In particular, note that S cannot be a compact operator. Now, pick a weakly null sequence $(v_n) \subseteq E_0$ so that $S(v_n)$ does not admit a convergent subsequence. Because (v_n) is weakly null, consider a subsequence that is a basic sequence of E_0 (see [1, Proposition 1.5.4]) and denote it again by (v_n). Let $E := \overline{\text{span}}\{v_n\}_{n \in \mathbb{N}}$. Then, E is a closed reflexive space with basis and $S(v_n)$ does not admit a convergent subsequence. Therefore, we conclude that $\mathcal{K}(E,F) \neq \mathcal{L}(E,F)$. $\square$

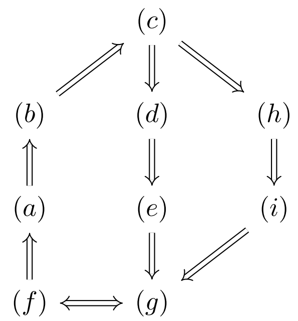
Compared to the previously known results in [19], Theorem 3.10 not only provides a new characterisation of the Schur property in terms of norm-attaining operators but also shows that we can restrict the possible candidates for a domain space as in the below items (f)–(i) by considering only reflexive Banach spaces *with basis*. Recall that $T \in \mathcal{L}(E,F)$ is *completely continuous* if T sends weakly null sequences in E to norm null sequences in F. We denote by $\mathcal{V}(E,F)$ the space of all completely continuous operators from E into F. Let us denote by $\mathcal{W}_\infty(E,F)$ the space of all weakly ∞-compact operators from E into F, which are introduced in [31]. A subset C of a Banach space E is called *relatively weakly ∞-compact* if there exists a weakly null sequence (x_n) in E such that $C \subseteq \{\sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{\ell_1}\}$ and an operator $T \in \mathcal{L}(E,F)$ is said to be *weakly ∞-compact* if $T(B_E)$ is a relatively weakly ∞-compact subset of F.

It is immediate to notice that Theorem C follows from Theorem 3.10.

Theorem 3.10. *Let F be an arbitrary Banach space. The following are equivalent:*

(a) *F has the Schur property.*
(b) *$\mathcal{K}(E,F) = \mathcal{L}(E,F)$ for every reflexive space E.*
(c) *$\mathcal{W}_\infty(E,F) = \mathcal{L}(E,F)$ for every reflexive space E.*
(d) *$\mathcal{V}(E,F) = \mathcal{L}(E,F)$ for every reflexive space E.*
(e) *$\mathrm{NA}(E,F) = \mathcal{L}(E,F)$ for every reflexive space E.*
(f) *$\mathcal{K}(G,F) = \mathcal{L}(G,F)$ for every reflexive space G with basis.*
(g) *$\mathrm{NA}(G,F) = \mathcal{L}(G,F)$ for every reflexive space G with basis.*
(h) *$\mathcal{W}_\infty(G,F) = \mathcal{L}(G,F)$ for every reflexive space G with basis.*
(i) *$\mathcal{V}(G,F) = \mathcal{L}(G,F)$ for every reflexive space G with basis.*

Proof. The following diagram holds.



Indeed, by definition we have that $\mathcal{K}(E,F) \subseteq \mathcal{W}_\infty(E,F) \subseteq \mathcal{W}(E,F)$ for any Banach space E and F, and it is also known that $\mathcal{K}(E,F) \subseteq \mathcal{W}_\infty(E,F) \subseteq \mathcal{V}(E,F)$ (see [19, Proposition 3.1]). Moreover, if T is an element of $\mathcal{V}(E,F)$ with E reflexive, then $T \in \mathrm{NA}(E,F)$ thanks to the weak sequential compactness of B_E. Thus, it is immediate that (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) $\Longrightarrow$ (e) $\Longrightarrow$ (g) and (c) $\Longrightarrow$ (h) $\Longrightarrow$ (i) $\Longrightarrow$ (g) hold. Because a reflexive Banach space with basis has the MAP, (f) $\Longleftrightarrow$ (g) follows from Theorem B. Finally, (f) $\Longrightarrow$ (a) is already obtained by Proposition 3.9. □

Questions and comments. Let us conclude the article by recalling some open problems. In [16], Holub conjectured that if E and F are both reflexive, then $\mathcal{L}(E,F)$ is reflexive if and only if $\mathcal{L}(E,F) = \mathcal{K}(E,F) = NA(E,F)$. We do not know whether, in general, when E and F are both reflexive spaces all of the implications in the diagram of the Introduction are indeed equivalences. A similar open question posed by Miguel Martín and one of the anonymous referees asks whether Theorem A is in general an equivalence; that is, whether a pair (E,F) has the James property whenever there exists a non-norm-attaining operator in $\mathcal{L}(E,F)$. Note that James proved that this implication holds when E is separable and $F = \mathbb{R}$ (recall the paragraph preceding Definition 1.1).

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Norm-Attaining Tensors and Nuclear Operators

Sheldon Dantas, Mingu Jung, Óscar Roldán and Abraham Rueda Zoca

Abstract. Given two Banach spaces X and Y, we introduce and study a concept of norm-attainment in the space of nuclear operators $\mathcal{N}(X,Y)$ and in the projective tensor product space $X\widehat{\otimes}_\pi Y$. We exhibit positive and negative examples where both previous norm-attainment hold. We also study the problem of whether the class of elements which attain their norms in $\mathcal{N}(X,Y)$ and in $X\widehat{\otimes}_\pi Y$ is dense or not. We prove that, for both concepts, the density of norm-attaining elements holds for a large class of Banach spaces X and Y which, in particular, covers all classical Banach spaces. Nevertheless, we present Banach spaces X and Y failing the approximation property in such a way that the class of elements in $X\widehat{\otimes}_\pi Y$ which attain their projective norms is not dense. We also discuss some relations and applications of our work to the classical theory of norm-attaining operators throughout the paper.

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1. Introduction

One of the most classical topics in the theory of Banach spaces is the study of norm-attaining functions. As a matter of fact, one of the most famous characterizations of reflexivity, due to James, is described in terms of linear functionals which attain their norms (see, for instance, [17, Corollary 3.56]). In the same direction, Bishop and Phelps proved that the set of all norm-attaining linear functionals is dense in X^* (see [5]). This motivated Lindenstrauss to study the analogous problem for bounded linear operators in his seminal paper [29], where it was obtained for the first time an example of a Banach space such that the Bishop–Phelps theorem is no longer true for this class of functions. Consequently, this opened the gate for a crucial and vast research on the topic during the past 50 years in many different directions. Indeed, just to name a few, J. Bourgain, R.E. Huff, J. Johnson, W. Schachermayer, J.J. Uhl, J. Wolfe, and V. Zizler continued the study about the set of all linear operators which attain their norms ([6,21,23,38–40]); M. Acosta, R. Aron, F.J. Aguirre, Y.S. Choi, R. Payá ([1,3,10] tackled problems in the same line involving bilinear mappings; García and Maestre considered it for homogeneous polynomials (see [2,4]); and more recently several problems on norm-attainment of Lipschitz maps were considered (see [8,9,18,26]).

Six years ago, Martín solved negatively a problem from the 1970s (posed explicitly by Diestel and Uhl in [15] and Johnson and Wolfe in [23]) on whether every compact operator can be approximated or not by norm-attaining operators (see [32, Theorem 1]). On the other hand, the main open problem in the theory of norm-attaining operators nowadays seems to be if every finite-rank operator can be approximated by norm-attaining operators (see [32, Question 9]). Since every nuclear operator is a limit of a sequence of finite-rank operators, we were motivated to try to take one step further in the theory by studying the set of all nuclear operators which attain their (nuclear) norms systematically.

On account of clear relations between nuclear operators and projective tensor products, we focus also on a concept of norm-attainment in projective tensor products (see Definition 2.1). This is justifiable, since it has strong and deep connections with different open problems coming from the study of norm-attaining operators. To mention one of them, let Y be a finite-dimensional Banach space. Then, for an arbitrary Banach space Z, every operator from Y into Z attains its norm by using the compactness of the unit ball of Y. If we suppose that the same happens with the nuclear operators, since Y is finite dimensional, we would have that the set of all norm-attaining tensors in $Y^* \widehat{\otimes}_\pi Z$ is the whole set $Y^* \widehat{\otimes}_\pi Z$ for every Banach space Z. By Corollary 3.11, the set of all norm-attaining operators from Z into Y would be dense in $\mathcal{L}(Z, Y)$ for every Banach space Z and this would mean finally that every finite-rank operator can be approximated by norm-attaining operators.

We proceed now to describe the content of the paper. In Sect. 2, we give the necessary background material to help the reader to follow the track of ideas from the text without having to jump into references so often. In particular, we give the precise definitions of norm-attainment in the context of nuclear operators and tensor products (see Sect. 2.3) as well as the concepts of approximations. Section 3 is devoted to the first examples of nuclear operators and tensors which attain their norms. We give a characterization for these kind of elements, which will be very helpful during the entire paper. We prove that if every element in the projective tensor product between two Banach spaces X, Y attains its projective norm, then the set of all norm-attaining operators from X into Y^* is dense. Since there exist operators which cannot be approximated by norm-attaining operators, this result gives the first examples of nuclear operators that do not attain their nuclear norms, meaning that the study of norm-attaining nuclear operators is not a trivial problem. In Sect. 4, we show that the set of all norm-attaining tensors (in particular, we get the analogous result for nuclear operators) is dense in the projective tensor product whenever both factors are finite-dimensional Banach spaces (actually, our result is more general than this). By using this result and the fact that the projective norm respects 1-complemented subspaces, we prove that the density problem holds in a much more general scenario. Indeed, we prove that if the involved Banach spaces satisfy a property which guarantees the existence of many 1-complemented subspaces (see Definition 4.7), then every tensor can be approximated (in the projective norm) by norm-attaining tensors (and the result for nuclear operators follows as a particular case). Since this property is satisfied by Banach spaces with finite-dimensional decompositions of constant 1, L_p-spaces, and L_1-predual spaces, the problem of denseness for nuclear operators and tensors is covered by all classical Banach spaces. Moreover, we prove that such a property is stable by finite absolute sums, countable c_0- and ℓ_p-sums, projective tensor products, and injective tensor products. In Sect. 5, we present an example of two Banach spaces X and Y, both failing the approximation property, which shows that the set of norm-attaining tensors is not always dense in the projective tensor product space based on the counterexample given in [32] with the existence of an equivalent renorming of c_0 which has bidual strictly convex (see [24,25,35]). Finally, we finish the paper with a discussion on some open problems.

2. Background, Notation, and Concepts

2.1. Basic Notation

We use essentially the notation from [36]. Let X, Y, and Z be Banach spaces over the field $\mathbb{K}$, which can be either $\mathbb{R}$ or $\mathbb{C}$. We denote by B_X and S_X the closed unit ball and the unit sphere, respectively, of the Banach space X. We denote by $\mathcal{L}(X, Y)$ the set of all bounded linear operators from X into Y. If $Y = \mathbb{K}$, then $\mathcal{L}(X, \mathbb{K})$ is denoted by X^*, the topological dual space of X. We denote by $\mathcal{B}(X \times Y, Z)$ the Banach space of bounded bilinear mappings from $X \times Y$ into Z. When $Z = \mathbb{K}$, we denote this space by $\mathcal{B}(X \times Y)$. It is well known that the space $\mathcal{B}(X \times Y)$ and $\mathcal{L}(X, Y^*)$ are isometrically isomorphic as Banach spaces. We denote by $\mathcal{K}(X, Y)$ the set of all compact operators and by $\mathcal{F}(X, Y)$ the space of all operators of finite-rank from X into Y. Given an absolute norm $|\cdot|_a$ defined on $\mathbb{R}^2$, let us denote by $X \oplus_a Y$ the *absolute sum* of X and Y with respect to $|\cdot|_a$, which is a Banach space $X \times Y$ endowed with the norm $\|(x, y)\|_a = |(\|x\|, \|y\|)|_a$ for every $x \in X$ and $y \in Y$.

2.2. Tensor Products and Nuclear Operators

The projective tensor product of X and Y, denoted by $X \widehat{\otimes}_\pi Y$, is the completion of the space $X \otimes Y$ endowed with the norm given by

$$\begin{aligned}\|z\|_\pi &= \inf\left\{\sum_{n=1}^{\infty} \|x_n\|\|y_n\| : \sum_{n=1}^{\infty} \|x_n\|\|y_n\| < \infty, z = \sum_{n=1}^{\infty} x_n \otimes y_n\right\} \\ &= \inf\left\{\sum_{n=1}^{\infty} |\lambda_n| : z = \sum_{n=1}^{\infty} \lambda_n x_n \otimes y_n, \sum_{n=1}^{\infty} |\lambda_n| < \infty, \|x_n\| = \|y_n\| = 1\right\},\end{aligned}$$

where the infinum is taken over all such representations of z. It is well known that $\|x \otimes y\|_\pi = \|x\|\|y\|$ for every $x \in X$, $y \in Y$, and the closed unit ball of $X \widehat{\otimes}_\pi Y$ is the closed convex hull of the set $B_X \otimes B_Y = \{x \otimes y : x \in B_X, y \in B_Y\}$. Throughout the paper, we will be using both formulas indistinctly, without any explicit reference. The canonical identification $\mathcal{B}(X \times Y, Z) = \mathcal{L}(X \widehat{\otimes}_\pi Y, Z)$ allows us to obtain the canonical identification $\mathcal{B}(X \times Y) = (X \widehat{\otimes}_\pi Y)^*$. Using the fact that the spaces $\mathcal{B}(X \times Y)$ and $\mathcal{L}(X, Y^*)$ are isometrically isomorphic, we also have the identification $(X \widehat{\otimes}_\pi Y)^* = \mathcal{L}(X, Y^*)$, where the action of an operator $G : X \longrightarrow Y^*$ as a linear functional on $X \widehat{\otimes}_\pi Y$ is given by

$$G\left(\sum_{n=1}^{\infty} x_n \otimes y_n\right) = \sum_{n=1}^{\infty} G(x_n)(y_n),$$

for every $\sum_{n=1}^{\infty} x_n \otimes y_n \in X \widehat{\otimes}_\pi Y$. Let us recall also that there is a canonical operator $J : X^* \widehat{\otimes}_\pi Y \longrightarrow \mathcal{L}(X, Y)$ with $\|J\| = 1$ defined by $z = \sum_{n=1}^{\infty} \varphi_n \otimes y_n \mapsto L_z$, where $L_z : X \longrightarrow Y$ is given by

$$L_z(x) = \sum_{n=1}^{\infty} \varphi_n(x) y_n \quad (x \in X).$$

The operators that arise in this way are called *nuclear operators*. We denote the set of such operators by $\mathcal{N}(X, Y)$ endowed with the *nuclear norm*

$$\|T\|_N = \inf\left\{\sum_{n=1}^{\infty} \|x_n^*\|\|y_n\| : T(x) = \sum_{n=1}^{\infty} x_n^*(x) y_n\right\},$$

where the infimum is taken over all representations of T of the form $T(x) = \sum_{n=1}^{\infty} x_n^*(x) y_n$ for bounded sequences $(x_n^*) \subseteq X^*$ and $(y_n) \subseteq Y$ such that $\sum_{n=1}^{\infty} \|x_n^*\|\|y_n\| < \infty$. Notice that every nuclear operator is compact since it is the limit in the operator norm of a sequence of finite-rank operators. Using the function J, we can identify the space $\mathcal{N}(X, Y)$ with $X^* \widehat{\otimes}_\pi Y / \ker J$ isometrically. In order to clarify the relations between the set of nuclear operators, the quotient space of the projective tensor product and their respective duals, we consider the following diagram:

$$\begin{array}{ccccc} (\ker J)^{\perp} & \xrightarrow{\ \delta\ } & \left(X^*\widehat{\otimes}_\pi Y/\ker J\right)^* & \xleftarrow{\ \widetilde{J}^*\ } & \mathcal{N}(X,Y)^* \\ & & \vdots & & \vdots \\ & & X^*\widehat{\otimes}_\pi Y/\ker J & \xrightarrow{\ \widetilde{J}\ } & \mathcal{N}(X,Y) \end{array},$$

where $\widetilde{J}$ and δ are isometric isomorphisms between $X^*\widehat{\otimes}_\pi Y/\ker J$ and $\mathcal{N}(X,Y)$, and $(\ker J)^{\perp}$ and $\left(X^*\widehat{\otimes}_\pi Y/\ker J\right)^*$, respectively. If we consider a nuclear operator $T \in \mathcal{N}(X,Y)$ given by $T = \sum_{n=1}^{\infty} x_n^* \otimes y_n$ for some $(x_n^*)_{n\in\mathbb{N}} \subset X^*$ and $(y_n)_{n\in\mathbb{N}} \subset Y$ bounded with $\sum_{n=1}^{\infty} \|x_n^*\|\|y_n\| < \infty$, then for every $H \in \mathcal{N}(X,Y)^*$, we have

$$H(T) = \sum_{n=1}^{\infty} G(x_n^*)(y_n),$$

where $G = (\delta^{-1} \circ \widetilde{J}^*)(H) \in (\ker J)^{\perp}$.

Recall that a Banach space is said to have the *approximation property* if for every compact subset K of X and every $\varepsilon > 0$, there exists a finite-rank operator $T : X \longrightarrow X$ such that $\|T(x) - x\| \leqslant \varepsilon$ for every $x \in K$. Let us take into account that if X^* or Y has the approximation property, then $X^*\widehat{\otimes}_\pi Y = \mathcal{N}(X,Y)$ (see, for instance, [36, Corollary 4.8]). Recall also that the injective norm of $z \in X \otimes Y$ is defined by

$$\|z\|_\varepsilon = \sup\left\{\left|\sum_{i=1}^{n} x^*(x_i)y^*(y_i)\right| : x^* \in B_{X^*}, y^* \in B_{Y^*}\right\},$$

where $\sum_{i=1}^{n} x_i \otimes y_i$ is any representation of z. We denote by $X \otimes_\varepsilon Y$ the tensor product $X \otimes Y$ with the injective norm and its completion, denoted by $X\widehat{\otimes}_\varepsilon Y$, is called the *injective tensor product* of X and Y.

For a complete background on tensor products in Banach spaces, we refer the reader to the books [14,36].

2.3. Norm-Attaining Concepts

Recall that $T \in \mathcal{L}(X,Y)$ *attains its norm* (in the classical way) if there is $x_0 \in S_X$ such that $\|T(x_0)\| = \|T\| = \sup_{x\in S_X} \|T(x)\|$. In this case, we say that T is a *norm-attaining* operator. Recall also that $B \in \mathcal{B}(X \times Y, Z)$ attains its norm if there is $(x_0, y_0) \in S_X \times S_Y$ such that $\|B(x_0, y_0)\| = \|B\| = \sup_{(x,y)\in S_X\times S_Y} \|B(x,y)\|$. In this case, we say that B is a norm-attaining bilinear mapping. In the next sections, we will be considering the concepts of attainment on the Banach spaces $X\widehat{\otimes}_\pi Y$ and $\mathcal{N}(X,Y)$. For us, the most natural approach is the following one.

Definition 2.1. Let X, Y be Banach spaces. We say that

1. $z \in X\widehat{\otimes}_\pi Y$ *attains its projective norm* if there is a bounded sequence $(x_n, y_n) \subseteq X \times Y$ with $\sum_{n=1}^{\infty} \|x_n\|\|y_n\| < \infty$ such that $z = \sum_{n=1}^{\infty} x_n \otimes y_n$

and that $\|z\|_\pi = \sum_{n=1}^\infty \|x_n\|\|y_n\|$. In this case, we say that z is a *norm-attaining tensor.*

2. $T \in \mathcal{N}(X,Y)$ *attains its nuclear norm* if there is a bounded sequence $(x_n^*, y_n) \subseteq X^* \times Y$ with $\sum_{n=1}^\infty \|x_n^*\|\|y_n\| < \infty$ such that $T = \sum_{n=1}^\infty x_n^* \otimes y_n$ and that $\|T\|_N = \sum_{n=1}^\infty \|x_n^*\|\|y_n\|$. In this case, we say that T is a *norm-attaining nuclear operator.*

If (1) (respectively, (2)) holds, then we say that $\sum_{n=1}^\infty x_n \otimes y_n$ (respectively, $\sum_{n=1}^\infty x_n^* \otimes y_n$) is a *norm-attaining representation.* Let us fix the notation for the set of norm-attaining operators, bilinear mappings, tensors, and nuclear operators. For the first two, we continue using the classical notation $\operatorname{NA}(X,Y) = \{T \in \mathcal{L}(X,Y) : T \text{ attains its norm}\}$ and $\operatorname{NA}(X \times Y, Z) = \{B \in \mathcal{B}(X \times Y, Z) : B \text{ attains its norm}\}$, respectively; if $Z = \mathbb{K}$, then we simply denote it as $\operatorname{NA}(X \times Y)$. For the last two, we shall use the following notations:

(1') $\operatorname{NA}_\pi(X\widehat{\otimes}_\pi Y) = \{z \in X\widehat{\otimes}_\pi Y : z \text{ attains its projective norm}\}$.
(2') $\operatorname{NA}_\mathcal{N}(X,Y) = \{T \in \mathcal{N}(X,Y) : T \text{ attains its nuclear norm}\}$.

Notice that, as we have pointed out before, when X^* or Y has the approximation property then $X^*\widehat{\otimes}_\pi Y$ is isometrically isomorphic to $\mathcal{N}(X,Y)$. In such case, it is clear that both concepts of norm-attainment agree. Due to the connection between projective tensor products, bilinear mappings, and operators, we are forced to observe also that the denseness of the sets $\operatorname{NA}(X \times Y)$ and $\operatorname{NA}(X, Y^*)$ are *not* equivalent in general, but the first implies the later.

Let us finish this introduction by clarifying what we mean by approximating elements from $X\widehat{\otimes}_\pi Y$ or $\mathcal{N}(X,Y)$ by norm-attaining ones. Evidently, when working with $X\widehat{\otimes}_\pi Y$, it is natural to make the approximation of an element $z \in X\widehat{\otimes}_\pi Y$ by an element $z' \in \operatorname{NA}_\pi(X\widehat{\otimes}_\pi Y)$ using the tensor norm $\|\cdot\|_\pi$. Similarly, we shall be dealing with the nuclear operator norm $\|\cdot\|_N$ whenever we approximate a given nuclear operator T by a norm-attaining nuclear operator T'.

3. Nuclear Operators and Tensors Which Attain Their Norms

In this section, we provide the first examples of elements in $X\widehat{\otimes}_\pi Y$ and $\mathcal{N}(X,Y)$ which attain their norms. The first result gives us an important characterization used abundantly in the rest of the paper.

Theorem 3.1. *Let X, Y be Banach spaces. Let $z \in X\widehat{\otimes}_\pi Y$ with*

$$z = \sum_{n=1}^\infty \lambda_n x_n \otimes y_n,$$

where $\lambda_n \in \mathbb{R}^+$, $x_n \in S_X$, and $y_n \in S_Y$ for every $n \in \mathbb{N}$. Then, the following assertions are equivalent:

1. $\|z\|_\pi = \sum_{n=1}^\infty \lambda_n$*; in other words,* $z \in \operatorname{NA}_\pi(X\widehat{\otimes}_\pi Y)$.
2. *There is $G \in \mathcal{L}(X, Y^*)$ with $\|G\| = 1$ such that $G(x_n)(y_n) = 1$ for every $n \in \mathbb{N}$.*
3. *Every norm one $G \in \mathcal{L}(X, Y^*)$ such that $G(z) = \|z\|_\pi$ satisfies that $G(x_n)(y_n) = 1$ for every $n \in \mathbb{N}$.*

Proof. Suppose that $\|z\|_\pi = \sum_{n=1}^{\infty} \lambda_n$ with $z = \sum_{n=1}^{\infty} \lambda_n x_n \otimes y_n$ with $(\lambda_n) \subseteq \mathbb{R}^+$, $(x_n) \subseteq S_X$, and $(y_n) \subseteq S_Y$. Pick any $G \in (X \widehat{\otimes}_\pi Y)^* = \mathcal{L}(X, Y^*)$ such that $\|G\| = 1$ and $G(z) = \|z\|_\pi$. Since we have

$$\sum_{n=1}^{\infty} \lambda_n = \|z\|_\pi = G(z) = \sum_{n=1}^{\infty} \lambda_n G(x_n)(y_n),$$

it follows that $G(x_n)(y_n) = 1$ for each $n \in \mathbb{N}$, which proves that (1) implies (3). It is obvious that (3) implies (2). Finally, assume that there exists $G \in \mathcal{L}(X, Y^*)$ with $\|G\| = 1$ such that $G(x_n)(y_n) = 1$ for every $n \in \mathbb{N}$. Then,

$$\sum_{n=1}^{\infty} \lambda_n = \sum_{n=1}^{\infty} \lambda_n G(x_n)(y_n) = G(z) \leqslant \|z\|_\pi \leqslant \sum_{n=1}^{\infty} \lambda_n.$$

This completes the proof. □

Taking into account the isometric isomorphism between $\mathcal{N}(X, Y)$ and $X^* \widehat{\otimes}_\pi Y / \ker(J)$, we can take advantage of the previous estimates to prove a nuclear operator version of Theorem 3.1 as follows.

Theorem 3.2. *Let X, Y be Banach spaces. Let $T \in \mathcal{N}(X, Y)$ with*

$$T = \sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n,$$

where $\lambda_n \in \mathbb{R}^+$, $x_n \in S_X$, and $y_n \in S_Y$ for every $n \in \mathbb{N}$. Then, the following assertions are equivalent:

1. *$\|T\|_N = \sum_{n=1}^{\infty} \lambda_n$; in other words, $T \in \mathrm{NA}_{\mathcal{N}}(X, Y)$.*
2. *There is $G \in (\ker J)^\perp$ with $\|G\| = 1$ such that $G(x_n^*)(y_n) = 1$ for every $n \in \mathbb{N}$.*
3. *For any $G \in (\ker J)^\perp$ with $\|G\| = 1$ and $G(T) = \|T\|_N$ we get that $G(x_n^*)(y_n) = 1$ holds for every $n \in \mathbb{N}$.*

Proof. Let $\widetilde{J} : X^* \widehat{\otimes}_\pi Y / \ker J \longrightarrow \mathcal{N}(X, Y)$ be an isometric isomorphism which maps, according to the notation of Sect. 2.2, $z + \ker J$ to L_z. If we let $z_0 := \sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n \in X^* \widehat{\otimes}_\pi Y$, then $J(z_0) = T$ and $\|T\|_N = \|z_0 + \ker J\|$. Now assume (1) and let us prove (3). To this end, pick any $G \in (\ker J)^\perp$ with $\|G\| = 1$ and $G(z_0 + \ker J) = \|z_0 + \ker J\|$. Then,

$$\begin{aligned}\sum_{n=1}^{\infty} \lambda_n = \|z_0 + \ker J\| = |G(z_0)| &= \left| G\left(\sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n \right) \right| \\ &\leqslant \sum_{n=1}^{\infty} \lambda_n |G(x_n^*)(y_n)| \\ &\leqslant \sum_{n=1}^{\infty} \lambda_n.\end{aligned}$$

Then, we have $|G(x_n^*)(y_n)| = 1$ for each $n \in \mathbb{N}$. Using a convexity argument, we get that $G(x_n^*)(y_n) = 1$ for every $n \in \mathbb{N}$. The other implications can be proved as in Theorem 3.1. □

With Theorems 3.1 and 3.2 in mind, we can now exhibit examples of nuclear operators which attain their nuclear norms.

Example 3.3. Let X, Y be two reflexive Banach spaces such that X^* or Y has the approximation property (recall that, in this case, we have $X^*\widehat{\otimes}_\pi Y = \mathcal{N}(X,Y)$). Assume further that X^* is isometrically isomorphic to a subspace of Y^*. Take $G : X^* \longrightarrow Y^*$ to be a linear isometry and pick $(x_n^*)_n \subseteq S_{X^*}$. Now, for any $n \in \mathbb{N}$, notice that $\|G(x_n^*)\| = \|x_n^*\| = 1$. Since Y is reflexive, by using the James Theorem, we have that $G(x_n^*) \in S_{Y^*}$ attains its norm, so there exists $y_n \in S_Y$ so that $G(x_n^*)(y_n) = 1$. Now, Theorem 3.1 (or Theorem 3.2) implies that, given any sequence $(\lambda_n)_n \subseteq (0,1]$ with $\sum_{n=1}^\infty \lambda_n < \infty$, the nuclear operator

$$T := \sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n \in \mathcal{N}(X,Y)$$

attains its nuclear norm.

One may think that a norm-attaining nuclear operator should attain its norm (in the classical way). This is not true in general as observed below.

Remark 3.4. Let Y be an infinite-dimensional strictly convex Banach space. Then, there is $T \in \mathrm{NA}_{\mathcal{N}}(c_0,Y)$ such that $T \notin \mathrm{NA}(c_0,Y)$. Indeed, let $(y_n)_n \subseteq S_Y$ be linearly independent. For every $n \in \mathbb{N}$, find $y_n^* \in S_{Y^*}$ such that $y_n^*(y_n) = 1$. Define $\phi : Y \longrightarrow \ell_\infty$ by $\phi(y) := (y_j^*(y))_{j=1}^\infty \in \ell_\infty$ for every $y \in Y$. Given $n \in \mathbb{N}$ we get that $|y_n^*(y)| \leqslant \|y\|$ since $\|y_n^*\| = 1$ holds for every $n \in \mathbb{N}$. This implies that $\sup_{n\in\mathbb{N}} |y_n^*(y)| \leqslant \|y\|$, which proves that $\phi(y) \in \ell_\infty$ for every y (i.e., ϕ is well defined) . In view of the linearity, we have that ϕ is continuous and $\|\phi\| \leqslant 1$. Furthermore, notice that $\phi(y_n)(e_n) = 1$ holds for every $n \in \mathbb{N}$, where $(e_n)_n$ is the basis of ℓ_1. This proves that the nuclear operator $T : c_0 \longrightarrow Y$ defined by

$$T = \sum_{n=1}^{\infty} \frac{1}{2^n} e_n \otimes y_n \in \ell_1 \widehat{\otimes}_\pi Y$$

attains its nuclear norm by Theorem 3.2. Nevertheless, notice that T is not a finite-rank operator and, consequently, T does not belong to $\mathrm{NA}(c_0,Y)$ (see [32, Lemma 2.2] or the proof of [29, Proposition 4]).

We prove next that on the the finite-dimensional setting, *every* tensor is norm-attaining. Before presenting a proof of it, let us notice that since the convex hull of a compact set is compact when X and Y are both finite-dimensional spaces, we have that $\overline{\mathrm{co}}(B_X \otimes B_Y) = \mathrm{co}\,(B_X \otimes B_Y)$, which is a consequence of Minkowski–Carathéodory theorem (see, for instance, [17, Exercises 1.57 and 1.58]).

Proposition 3.5. *Let X, Y be finite-dimensional Banach spaces. Then, every tensor attains its projective tensor norm. In other words,* $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) = X\widehat{\otimes}_\pi Y$.

Proof. Let $z \in X \widehat{\otimes}_\pi Y$ with $\|z\|_\pi = 1$ be given and let us prove that $z \in \mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y)$. As we have mentioned before, since X and Y are finite-dimensional Banach spaces, $B_X \otimes B_Y$ is compact in $X \widehat{\otimes}_\pi Y$ and this implies that $B_{X \widehat{\otimes}_\pi Y} = \overline{\mathrm{co}}(B_X \otimes B_Y) = \mathrm{co}(B_X \otimes B_Y)$. Therefore, z can be written as a finite convex combination of elements in $B_X \otimes B_Y$, i.e.,

$$z = \sum_{j=1}^{n} \lambda_j x_j \otimes y_j \quad \text{with} \quad \sum_{j=1}^{n} \lambda_j = 1,$$

where $\lambda_j \in \mathbb{R}^+$, $x_j \in B_X$, and $y_j \in B_Y$ for $j = 1, \ldots, n$, that is, z is norm-attaining. □

Let us notice that in Remark 3.4, we have constructed by hand a nuclear operator from c_0 into a particular Y which attains its nuclear norm. It turns out that *every* nuclear operator from c_0 into *any* Banach space Y attains its nuclear norm. This should be compared to the fact that, in the classical theory, whenever X is a Banach space such that $\mathrm{NA}(X, Y) = \mathcal{L}(X, Y)$ for some $Y \neq \{0\}$, X must be reflexive (this is an application of James theorem). In other words, this result is no longer true in the context of nuclear operators.

Proposition 3.6. *Let Y be a Banach space. Then,*

(a) *every nuclear operator $T \in \mathcal{N}(c_0, Y)$ attains its nuclear norm. Equivalently,*

(b) *every element in $\ell_1 \widehat{\otimes}_\pi Y$ attains its projective norm.*

Proof. Indeed, in the last part of [36, Lemma 2.6], it is proved that $\Phi : \ell_1(Y) \longrightarrow \ell_1 \widehat{\otimes}_\pi Y$ given by

$$\Phi((x_n)_n) = \sum_{n=1}^{\infty} e_n \otimes x_n$$

is an onto linear isometry, where $(e_n)_n$ is the basis of ℓ_1 (in fact, $\Phi = J^{-1}$ in the proof given there). Let $T \in \mathcal{N}(c_0, Y) = \ell_1 \widehat{\otimes}_\pi Y$ be given. By the surjectivity of Φ, we can find an element $(x_n)_n \in \ell_1(Y)$ such that $\Phi((x_n)_n) = T$. Consequently, $T = \sum_{n=1}^{\infty} e_n \otimes x_n$. Then,

$$\|T\|_N = \|\Phi((x_n)_n)\| = \|(x_n)_n\| = \sum_{n=1}^{\infty} \|x_n\| = \sum_{n=1}^{\infty} \|e_n\| \|x_n\|.$$

This proves that T attains its nuclear norm, as desired. □

Remark 3.7. Notice that Proposition 3.6 is also true for $c_0(I)$ and $\ell_1(I)$ for any arbitrary index set I (see [36, Example 2.6]).

In the infinite-dimensional case, besides the nuclear operators from c_0 into an arbitrary Banach space Y, we have that every nuclear operator on a complex Hilbert space attains its nuclear norm. Although we prove this result for nuclear operators (justified by the fact that we will be dealing with eigenvalues and Schatten classes), we also get that every tensor in $H \widehat{\otimes}_\pi H$ attains its projective norm as every Hilbert space H has the approximation property.

Proposition 3.8. *Let H be a complex Hilbert space. Then, every nuclear operator $T \in \mathcal{N}(H,H)$ attains its nuclear norm.*

Proof. Note that $T \in \mathcal{N}(H,H)$ can be written as

$$T = \sum_{j=1}^{n_0} \lambda_j \langle \cdot, x_j \rangle y_j,$$

where $n_0 \in \mathbb{N} \cup \{\infty\}$, $(\lambda_j)_j$ is the sequence of nonzero eigenvalues of $|T| = (T^*T)^{\frac{1}{2}}$, and $(x_j)_j$ and $(y_j)_j$ are orthonormal systems in H (see [19, Theorem 2.1]). On the other hand, it is well known that $\|T\|_N = \sigma_1(T) = \sum_{j=1}^{n_0} \lambda_j$, where $\sigma_1(\cdot)$ is the Schatten 1st norm (see, for example, [19, pages 96-97]). This completes the proof. □

Taking into account Propositions 3.5, 3.6 and 3.8, it is natural to ask whether or not the equality $\mathrm{NA}_{\mathcal{N}}(X,Y) = \mathcal{N}(X,Y)$ (or $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) = X\widehat{\otimes}_\pi Y$) holds for all Banach spaces X and Y. We will give a negative answer for this problem by proving that if this happens, then the set of norm-attaining bilinear forms which attain their norms is dense in $\mathcal{B}(X \times Y)$. From our point of view, this shows that the study of norm-attaining nuclear operators is not a trivial task.

Lemma 3.9. *Let X, Y be Banach spaces. If $B \in \mathcal{B}(X \times Y) = (X\widehat{\otimes}_\pi Y)^*$ attains its norm (as a functional) at an element of $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$, then $B \in \mathrm{NA}(X \times Y)$.*

Proof. Let $B \in \mathcal{B}(X \times Y) = (X\widehat{\otimes}_\pi Y)^*$ and $z \in S_{X\widehat{\otimes}_\pi Y}$ with $z = \sum_{n=1}^{\infty} \lambda_n x_n \otimes y_n \in \mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$ be such that $B(z) = 1$, where $\lambda_n \in \mathbb{R}^+$, $x_n \in S_X$, and $y_n \in S_Y$. By Theorem 3.1, $B(x_n, y_n) = 1$ for every $n \in \mathbb{N}$. In particular, $B \in \mathrm{NA}(X \times Y)$. □

Proposition 3.10. *Let X, Y be Banach spaces. If every element in $X\widehat{\otimes}_\pi Y$ attains its projective norm, then the set of all bilinear forms on $X \times Y$ which attain their norms is dense in $\mathcal{B}(X \times Y)$. In other words, if $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) = X\widehat{\otimes}_\pi Y$, then*

$$\overline{\mathrm{NA}(X \times Y)}^{\|\cdot\|} = \mathcal{B}(X \times Y).$$

Proof. Let $\varepsilon > 0$. Let $B \in \mathcal{B}(X \times Y) = (X\widehat{\otimes}_\pi Y)^*$ with $\|B\| = 1$. By the Bishop–Phelps theorem, for $X\widehat{\otimes}_\pi Y$, there are $B_0 \in (X\widehat{\otimes}_\pi Y)^*$ with $\|B_0\| = 1$ and $z_0 \in S_{X\widehat{\otimes}_\pi Y}$ such that $B_0(z_0) = 1$ and $\|B_0 - B\| < \varepsilon$. By hypothesis, $z_0 \in \mathrm{NA}_\pi(X, Y)$ attains its projective norm and by Lemma 3.9 we have that $B_0 \in \mathrm{NA}(X \times Y)$. Since $\|B_0 - B\| < \varepsilon$, we are done. □

Proposition 3.10 yields the following consequence:

Corollary 3.11. *Let X, Y be Banach spaces. Suppose that every element in $X\widehat{\otimes}_\pi Y$ attains its projective norm. Then, the set of norm-attaining operators from X into Y^* is dense in $\mathcal{L}(X, Y^*)$. In other words, if $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) = X\widehat{\otimes}_\pi Y$, then*

$$\overline{\mathrm{NA}(X, Y^*)}^{\|\cdot\|} = \mathcal{L}(X, Y^*).$$

Now, by using Lemma 3.9, Proposition 3.10, and Corollary 3.11, we can get examples of pairs of Banach spaces (X, Y) such that there are elements in the projective tensor product $X\widehat{\otimes}_\pi Y$ which *do not* attain their projective norms.

Example 3.12. There are elements $z \in X\widehat{\otimes}_\pi Y$ such that $z \notin \mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$ in the following cases:

(a) [1] When $X = L_1(\mathbb{T})$, where the unit circle $\mathbb{T}$ is equipped with the Haar measure m, and Y is the two-dimensional Hilbert space. Indeed, it is shown in [18, Remark 5.7.(2)] that there is $T \in \mathcal{B}(X \times Y)$ which attains its norm as a linear functional on $X\widehat{\otimes}_\pi Y$ but not as an operator from X into Y^* (nor the more as a bilinear form on $X \times Y$). By Lemma 3.9, it follows that $\mathrm{NA}_\pi(X \times Y) \neq X\widehat{\otimes}_\pi Y$.

(b) When X is $L_1[0,1]$ and Y^* is a strictly convex Banach space without the Radon–Nikodým property. Indeed, by [39, Theorem 3], the set $\mathrm{NA}(L_1[0,1], Y^*)$ is not dense in $\mathcal{L}(L_1[0,1], Y^*)$. Let us notice that this also shows that Proposition 3.6 is no longer true if we consider an $L_1(\mu)$-space for a non-purely atomic measure μ.

(c) When $Y = \ell_p$ for $1 < p < \infty$ and X is the Banach space constructed by Gowers. Indeed, there is a Banach space G such that $\mathrm{NA}(G \times \ell_p)$ is not dense in $\mathcal{B}(G \times \ell_p)$ (see [20, Theorem, page 149]). We should notice that the unit ball of G lacks extreme points. This result should be compared to the fact that, if X is reflexive and Y is any Banach space, then $\mathcal{K}(X, Y) \subseteq \mathrm{NA}(X, Y)$.

(d) When X and Y are both $L_1[0,1]$. Indeed, [10, Theorem 3] shows that the set $\mathrm{NA}(L_1[0,1] \times L_1[0,1])$ is not dense in $\mathcal{B}(L_1[0,1] \times L_1[0,1])$.

Let us finish this section by highlighting two observations.

Remark 3.13. Notice that if we weaken the hypothesis in Proposition 3.10 by assuming that $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$ is dense in $X\widehat{\otimes}_\pi Y$, the result does *not* remain true. Indeed, by using Example 3.12.(c), we know that $\mathrm{NA}(L_1[0,1] \times L_1[0,1])$ is not dense in $\mathcal{B}(L_1[0,1] \times L_1[0,1])$, but we will see in Section 4 that the set of all tensors which attain their projective norm on $L_1[0,1]\widehat{\otimes}_\pi L_1[0,1]$ is dense in $L_1[0,1]\widehat{\otimes}_\pi L_1[0,1]$ (see Theorem 4.8 and Example 4.12.(b)). Nevertheless, we will always have that $\mathrm{NA}(X, Y^*) \cap B_{\mathcal{L}(X,Y^*)}$ is w^*-dense in $B_{\mathcal{L}(X,Y^*)}$ under this hypothesis (see Remark 5.4).

Remark 3.14. Let Y be a finite-dimensional Banach space. Then, $\mathrm{NA}(Y, Z) = \mathcal{L}(Y, Z)$ for every Banach space Z by using the compactness of the unit ball of Y. Let us suppose for a second that the same holds for nuclear operators. Then, $\mathrm{NA}_{\mathcal{N}}(Y, Z) = \mathcal{N}(Y, Z)$ for every Banach space Z. Since Y is finite-dimensional, it has the approximation property and then we would have that $\mathrm{NA}_\pi(Z\widehat{\otimes}_\pi Y^*) = Z\widehat{\otimes}_\pi Y^*$ for every Banach space Z. By Corollary 3.11, we would have that the set $\mathrm{NA}(Z, Y)$ is dense in $\mathcal{L}(Z, Y)$ for every Banach space Z, which would imply that Y has property B of Lindenstrauss (solving positively [32, Question 9]). Therefore, it is natural to wonder whether every

[1] The authors are thankful to the referee who provided us this example.

nuclear operator $T : Y \longrightarrow Z$ attain its nuclear norm for every Banach space Z whenever Y is finite dimensional. This is *not* the case due to Example 3.12.(a)[2] by taking $Z = Ł_1(\mathbb{T})$ and $Y = \ell_2^2$, the Euclidean plane (see [18, Remark 5.7.(2)]).

4. Denseness of Nuclear Operators and Tensors Which Attain Their Norms

Here we will be focusing on examples of Banach spaces X and Y such that the sets $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$ and $\mathrm{NA}_{\mathcal{N}}(X,Y)$ are dense in norm in $X\widehat{\otimes}_\pi Y$ and $\mathcal{N}(X,Y)$, respectively. As we have seen in the previous section, there are many examples of projective tensor products where we can guarantee the existence of elements which do not attain their projective norms even when one of the factors is reflexive (see Example 3.12(b)). In spite of the existence of such non-norm-attaining tensors, it is natural to ask if the set of elements in a tensor product space which attain their projective norms is dense in the whole space.

Let us start by explaining where the difficulty comes from when one tries to get such a property. Assume that $z \in \mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$ is a norm-attaining tensor in $X\widehat{\otimes}_\pi Y$. This implies that there are bounded sequences $(x_n)_n \subseteq X$ and $(y_n)_n \subseteq Y$ such that $z = \sum_{n=1}^\infty x_n \otimes y_n$ with $\|z\|_\pi = \sum_{n=1}^\infty \|x_n\|\|y_n\|$. It is clear that the task of choosing the optimal representation for z as a series of basic tensors is the most difficult part. In order to avoid this inconvenience, let us make use of Theorem 3.1. By applying it, for any bilinear mapping $B \in S_{\mathcal{B}(X\times Y)} = S_{(X\widehat{\otimes}_\pi Y)^*}$ such that $B(z) = \|z\|_\pi$, we have that $B(x_n)(y_n) = \|x_n\|\|y_n\|$ for every $n \in \mathbb{N}$. In other words, B attains its bilinear norm at the pair $\left(\frac{x_n}{\|x_n\|}, \frac{y_n}{\|y_n\|}\right)$ for every $n \in \mathbb{N}$. Because of this, in order to get examples of Banach spaces X and Y where the set $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$ is dense in $X\widehat{\otimes}_\pi Y$, we need somehow that the space $\mathcal{B}(X \times Y)$ contains many bilinear forms which attain their bilinear norm at many elements of $S_X \times S_Y$. This motivates us to make use of the following definitions, which can be found in [12] and [13].

Definition 4.1. Let X, Y and Z be Banach spaces.

(a) We say that (X,Y) has the $\mathbf{L}_{o,o}$ for operators if given $\varepsilon > 0$ and $T \in \mathcal{L}(X,Y)$ with $\|T\| = 1$, there is $\eta(\varepsilon,T) > 0$ such that whenever $x \in S_X$ satisfies $\|T(x)\| > 1 - \eta(\varepsilon,T)$, there is $x_0 \in S_X$ such that $\|T(x_0)\| = 1$ and $\|x_0 - x\| < \varepsilon$.

(b) We say that $(X \times Y, Z)$ satisfies the $\mathbf{L}_{o,o}$ for bilinear mappings if given $\varepsilon > 0$ and $B \in \mathcal{B}(X \times Y, Z)$ with $\|B\| = 1$, there exists $\eta(\varepsilon,B) > 0$ such that whenever $(x,y) \in S_X \times S_Y$ satisfies $\|B(x,y)\| > 1 - \eta(\varepsilon,B)$, there is $(x_0,y_0) \in S_X \times S_Y$ such that $\|B(x_0,y_0)\| = 1$, $\|x - x_0\| < \varepsilon$, and $\|y - y_0\| < \varepsilon$.

[2] It is worth mentioning that this question was posed by the authors in a preliminary version of this manuscript; they thank the anonymous referee who answered it negatively.

Example 4.2. Let us highlight some examples and results related to the properties just defined.

(a) If $\dim(X), \dim(Y) < \infty$, then $(X \times Y, Z)$ has the $\mathbf{L}_{o,o}$ for every Banach space Z (see [13, Proposition 2.2]).

(b) $(X \times Y, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ for bilinear mappings if and only if (X, Y^*) has the $\mathbf{L}_{o,o}$ for operators, whenever Y is uniformly convex (see [13, Lemma 2.6]). In particular, if X is finite dimensional and Y is uniformly convex, then $(X \times Y, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ for bilinear forms (see [12, Theorem 2.4]).

(c) If $1 < p, q < \infty$, then $(\ell_p \times \ell_q, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ if and only if $p > q'$, where q' is the conjugate of q (see [13, Theorem 2.7.(b)]).

(d) There are reflexive spaces X and Y such that $(X \times Y, \mathbb{K})$ fails the $\mathbf{L}_{o,o}$ (see [12, Theorem 2.21.(ii)]).

Our next aim is to prove that every nuclear operator between finite-dimensional Banach spaces can be approximated by nuclear operators which attain their nuclear norm. This will follow from a more general result.

Proposition 4.3. *Let X, Y be Banach spaces. Suppose that $(X^* \times Y, \mathbb{K})$ has $\mathbf{L}_{o,o}$ for bilinear forms. Then, every nuclear operator from X into Y can be approximated (in the nuclear norm) by nuclear operators which attain their nuclear norm. In other words,*

$$\overline{\mathrm{NA}_{\mathcal{N}}(X, Y)}^{\|\cdot\|_N} = \mathcal{N}(X, Y).$$

We get the following particular case by combining Proposition 4.3 with Example 4.2.

Corollary 4.4. *Let X be finite-dimensional Banach space. If Y is uniformly convex, then*

$$\overline{\mathrm{NA}_{\mathcal{N}}(X, Y)}^{\|\cdot\|_N} = \mathcal{N}(X, Y).$$

Now, we prove Proposition 4.3.

Proof of Proposition 4.3. Let $T \in \mathcal{N}(X, Y)$ and $\varepsilon > 0$ be given. There exists $H \in \mathcal{N}(X, Y)^*$ with $\|H\| = 1$ such that $H(T) = \|T\|_N$. Consider $G := (\delta^{-1} \circ \widetilde{J}^*)(H) \in (\ker J)^\perp$ (see Subsection 2.1). Let A_G be the bilinear form on $X^* \times Y$ defined by $A_G(x^*, y) = G(x^*)(y)$ for every $x^* \in X^*$ and $y \in Y$. Then $\|A_G\| = \|G\| = 1$. Consider the positive value $\eta(\varepsilon, A_G) > 0$ from the assumption that $(X^* \times Y, \mathbb{K})$ has $\mathbf{L}_{o,o}$ for bilinear forms. Now, choose $(\lambda_n)_n \subseteq \mathbb{R}^+$, $(x_n^*)_n \subseteq S_{X^*}$, and $(y_n)_n \subseteq S_Y$ so that $T = \sum_{n=1}^\infty \lambda_n x_n^* \otimes y_n$ with

$$\sum_{n=1}^\infty \lambda_n < \|T\|_N + \eta(\varepsilon, A_G)^2.$$

We get that

$$\begin{aligned}\|T\|_N = H(T) = \operatorname{Re} H(T) &= \sum_{n=1}^\infty \lambda_n \operatorname{Re}\left(G(x_n^*)(y_n)\right) \\ &\leqslant \sum_{n=1}^\infty \lambda_n |G(x_n^*)(y_n)|\end{aligned}$$

$$\leqslant \sum_{n=1}^{\infty} \lambda_n < \|T\|_N + \eta(\varepsilon, A_G)^2.$$

In particular,

$$\sum_{n\in\mathbb{N}} \lambda_n \left(1 - \operatorname{Re}\left(G(x_n^*)(y_n)\right)\right) < \eta(\varepsilon, A_G)^2 \tag{4.1}$$

Consider the following set

$$I = \{n \in \mathbb{N} : \operatorname{Re}\left(G(x_n^*)(y_n)\right) > 1 - \eta(\varepsilon, A_G)\}.$$

From (4.1), notice that

$$\eta(\varepsilon, A_G) \sum_{n\in I^c} \lambda_n \leqslant \sum_{n\in I^c} \lambda_n \left(1 - \operatorname{Re}\left(G(x_n^*)(y_n)\right)\right) < \eta(\varepsilon, A_G)^2,$$

which implies that $\sum_{n\in I^c} \lambda_n < \eta(\varepsilon, A_G)$. On the other hand, for each $n \in I$,

$$\operatorname{Re} A_G(x_n^*, y_n) = \operatorname{Re}\left(G(x_n^*)(y_n)\right) > 1 - \eta(\varepsilon, A_G).$$

Thus, there exist norm one vectors $(\widetilde{x}_n^*)_{n\in I}$ in X^* and $(\widetilde{y}_n)_{n\in I}$ in Y such that

$$|A_G(\widetilde{x}_n^*, \widetilde{y}_n)| = |G(\widetilde{x}_n^*)(\widetilde{y}_n)| = 1, \quad \|\widetilde{x}_n^* - x_n^*\| < \varepsilon, \quad \text{and} \quad \|\widetilde{y}_n - y_n\| < \varepsilon$$

for every $n \in I$. Let us write $G(\widetilde{x}_n^*)(\widetilde{y}_n) = e^{i\theta_n}$ with some $\theta_n \in \mathbb{R}$ for every $n \in I$. Notice that $|1 - e^{i\theta_n}| < \sqrt{2\eta(\varepsilon, A_G)}$ for every $n \in I$. Let us define

$$T' := \sum_{n\in I} \lambda_n e^{-i\theta_n} \widetilde{x}_n^* \otimes \widetilde{y}_n.$$

Then,

$$\begin{aligned}
\|T' - T\|_N &\leqslant \left\| \sum_{n\in I} \lambda_n (e^{-i\theta_n} \widetilde{x}_n^* \otimes \widetilde{y}_n - x_n^* \otimes y_n) \right\|_N + \sum_{n\in I^c} \lambda_n \\
&< \sum_{n\in I} \lambda_n |1 - e^{i\theta_n}| + \left\| \sum_{n\in I} \lambda_n (\widetilde{x}_n^* \otimes \widetilde{y}_n - x_n^* \otimes y_n) \right\|_N + \eta(\varepsilon, A_G) \\
&< \sqrt{2\eta(\varepsilon, A_G)} (\|T\|_N + \eta(\varepsilon, A_G)^2) \\
&\quad + 2\varepsilon (\|T\|_N + \eta(\varepsilon, A_G)^2) + \eta(\varepsilon, A_G) \\
&= (\sqrt{2\eta(\varepsilon, A_G)} + 2\varepsilon)(\|T\|_N + \eta(\varepsilon, A_G)^2) + \eta(\varepsilon, A_G).
\end{aligned}$$

Finally, it is clear by definition that $\|T'\|_N \leqslant \sum_{i\in I} \lambda_n$. On the other hand,

$$\|T'\|_N \geqslant |H(T')| = \left| \sum_{n\in I} \lambda_n e^{-i\theta_n} G(\widetilde{x}_n^*)(\widetilde{y}_n) \right| = \sum_{n\in I} \lambda_n.$$

This shows that T' attains its nuclear norm and completes the proof. □

Using very similar arguments to Proposition 4.3 and Corollary 4.4, we can obtain the following results.

Proposition 4.5. *Let X, Y be Banach spaces. Suppose that $(X \times Y, \mathbb{K})$ has $\mathbf{L}_{o,o}$ for bilinear forms. Then, every tensor in $X \widehat{\otimes}_\pi Y$ can be approximated by tensors which attain their projective norm. In other words,*

$$\overline{\mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X \widehat{\otimes}_\pi Y.$$

Corollary 4.6. *Let X be a finite-dimensional Banach space. If Y is uniformly convex, then*

$$\overline{\mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X \widehat{\otimes}_\pi Y.$$

Let us notice that, although we have the first examples of denseness by using Propositions 4.3 and 4.5, property $\mathbf{L}_{o,o}$ seems to be very restrictive. Indeed, when a pair of Banach spaces satisfies this property, both of them must be reflexive since every bilinear mapping attains its norm. Moreover, there are reflexive spaces X and Y such that $(X \times Y, \mathbb{K})$ fails this property (see Example 4.2.(d)). On the other hand, we could have used the previous results together with Example 4.2.(c) in order to get examples where the denseness holds for ℓ_p-spaces: for instance, if $1 < p, q < \infty$ and $p > q'$, then the set $\mathrm{NA}_\pi(\ell_p \widehat{\otimes}_\pi \ell_q)$ is dense in $\ell_p \widehat{\otimes}_\pi \ell_q$ by Proposition 4.5. Nevertheless, in what follows we will take advantage of the finite-dimensional case to obtain more general examples of Banach spaces where the density follows. The only problem here is the fact that in general the projective norm does not respect subspaces, but it does respect 1-complemented subspaces. For this reason, intuitively, we need a property of Banach spaces which guarantees the existence of many 1-complemented subspaces. Motivated by this, we consider the following definition.

Definition 4.7. Let X be a Banach space. We will say that X has the *metric π-property* if given $\varepsilon > 0$ and $\{x_1, \ldots, x_n\} \subseteq S_X$ a finite collection in the sphere, then we can find a finite-dimensional 1-complemented subspace $M \subseteq X$ such that for each $i \in \{1, \ldots, n\}$ there exists $x_i' \in M$ with $\|x_i - x_i'\| < \varepsilon$.

Before proceeding, let us make a small observation. Let $\varepsilon > 0$ and $F = \{x_1, \ldots, x_n\} \subseteq S_X$ be given. Suppose that X has metric π-property as defined above and let M be a finite-dimensional subspace of X with $\|x_i' - x_i\| < \varepsilon$ for $x_i' \in M$ and $i = 1, \ldots, n$. Let $P_{\varepsilon,F}$ be the norm one projection onto M. Then, for each $i = 1, \ldots, n$, we have

$$\|P_{\varepsilon,F}(x_i) - x_i\| \leqslant \|P_{\varepsilon,F}(x_i) - P_{\varepsilon,F}(x_i')\| + \|P_{\varepsilon,F}(x_i') - x_i\| < 2\varepsilon.$$

Consider now the net $\{P_{\varepsilon,F} : \varepsilon > 0,\ F \subset S_X \text{ a finite set}\}$ with $(\varepsilon_1, F_1) \leqslant (\varepsilon_2, F_2)$ if and only if $\varepsilon_2 < \varepsilon_1$ and $F_1 \subseteq F_2$. Then, $(P_{\varepsilon,F})_{(\varepsilon,F)}$ strongly converges to the identity on S_X and hence on X with $\|P_{\varepsilon,F}\| \leqslant 1$ for every ε and F. This shows that Definition 4.7 is in fact equivalent to [7, Definition 5.1] as the classical way of defining the metric π-property (we also send the reader to [22] and [30] for more information on the π-property).

We have the following general result, which confirms that our intuition of finding a property of Banach spaces, which guarantees the existence of many 1-complemented subspaces, was in the right direction. This result will give us many positive examples of denseness in both norm-attaining tensor and nuclear operator cases (see Examples 4.12).

Theorem 4.8. *Let X be a Banach space satisfying the metric π-property.*

(a) *If Y satisfies the metric π-property, then $\overline{\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X\widehat{\otimes}_\pi Y$.*

(b) *If Y is uniformly convex, then $\overline{\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X\widehat{\otimes}_\pi Y$.*

Proof. (a). Let $u \in S_{X\widehat{\otimes}_\pi Y}$ and $\varepsilon > 0$ be given. By [36, Proposition 2.8], there are bounded sequences $(\lambda_n)_n \subseteq \mathbb{R}^+$, $(x_n)_n \subseteq S_X$, and $(y_n)_n \subseteq S_Y$ with $u = \sum_{n=1}^{\infty} \lambda_n x_n \otimes y_n$ and

$$\sum_{n=1}^{\infty} \lambda_n < 1 + \varepsilon. \tag{4.2}$$

Find $k \in \mathbb{N}$ large enough so that $\|u - z\|_{\pi_{X\widehat{\otimes}_\pi Y}} < \frac{\varepsilon}{2}$ for $z := \sum_{n=1}^{k} \lambda_n x_n \otimes y_n$. Since X and Y have the metric π-property, we can find finite-dimensional subspaces X_0 of X and Y_0 of Y which are 1-complemented and such that, for every $n \in \{1, \ldots, k\}$, there are $x_n' \in X_0$ and $y_n' \in Y_0$ such that

$$\max\{\|x_n - x_n'\|, \|y_n - y_n'\|\} < \frac{\varepsilon}{4k\lambda_n}.$$

Define $z' = \sum_{n=1}^{k} \lambda_n x_n' \otimes y_n'$ and notice that $\|z' - z\|_{\pi_{X\widehat{\otimes}_\pi Y}} < \frac{\varepsilon}{2}$. Moreover, note that $z' \in X_0 \otimes Y_0$. We have that X_0 is 1-complemented in X and Y_0 is 1-complemented in Y. Consequently, by [36, Proposition 2.4] we get that norm of $X\widehat{\otimes}_\pi Y$ agrees on $X_0 \otimes Y_0$ with the norm of $X_0\widehat{\otimes}_\pi Y_0$. In particular,

$$\|z'\|_{\pi_{X_0\widehat{\otimes}_\pi Y_0}} = \|z'\|_{\pi_{X\widehat{\otimes}_\pi Y}}. \tag{4.3}$$

Finally, since X_0, Y_0 are finite-dimensional spaces, we use Proposition 3.5 to show that z' attains its projective norm in $X_0\widehat{\otimes}_\pi Y_0$. Since (4.3) holds, z' attains its norm in $X\widehat{\otimes}_\pi Y$ and we are done.

(b). Let $u \in S_{X\widehat{\otimes}_\pi Y}$ and $\varepsilon > 0$ be given. There are bounded sequences $(\lambda_n)_n \subseteq \mathbb{R}^+$, $(x_n)_n \subseteq S_X$, and $(y_n)_n \subseteq S_Y$ with $u = \sum_{n=1}^{\infty} \lambda_n x_n \otimes y_n$ and (4.2) holds. We can find k large enough such that $\|u - z\|_{\pi_{X\widehat{\otimes}_\pi Y}} < \frac{\varepsilon}{3}$ for $z := \sum_{n=1}^{k} \lambda_n x_n \otimes y_n$. Since X satisfies the metric π-property, we can find a finite-dimensional subspace X_0 which is 1-complemented and such that for every $n \in \{1, \ldots, k\}$, there is $x_n' \in X_0$ such that $\|x_n - x_n'\| < \frac{\varepsilon}{6k\lambda_n}$. Define $z' = \sum_{n=1}^{k} \lambda_n x_n' \otimes y_n$. Notice that $\|z' - z\|_{\pi_{X\widehat{\otimes}_\pi Y}} < \frac{\varepsilon}{3}$ and that $z' \in X_0 \otimes Y$. Since X_0 is finite dimensional and Y is uniformly convex, by Corollary 4.6, we can find $z'' \in X_0\widehat{\otimes}_\pi Y$ such that

$$\|z' - z''\|_{\pi_{X_0\widehat{\otimes}_\pi Y}} < \frac{\varepsilon}{3} \quad \text{with} \quad z''$$
$$= \sum_{n=1}^{\infty} a_n \otimes b_n \quad \text{and} \quad \|z''\|_{\pi_{X_0\widehat{\otimes}_\pi Y}} = \sum_{n=1}^{\infty} \|a_n\|\|b_n\|.$$

Since the norm of $X\widehat{\otimes}_\pi Y$ agrees on $X_0 \otimes Y$ with the norm of $X_0\widehat{\otimes}_\pi Y$, the result follows as in the previous item. □

Let us notice that if a Banach space Z satisfies the metric π-property, then it has the metric approximation property and then the analogous result for nuclear operators follows immediately from Theorem 4.8 and [36, Corollary 4.8].

Corollary 4.9. *Let X be Banach space such that X^* satisfies the metric π-property.*

(a) *If Y satisfies the metric π-property, then $\overline{\mathrm{NA}_{\mathcal{N}}(X,Y)}^{\|\cdot\|_N} = \mathcal{N}(X,Y)$.*
(b) *If Y is uniformly convex, then $\overline{\mathrm{NA}_{\mathcal{N}}(X,Y)}^{\|\cdot\|_N} = \mathcal{N}(X,Y)$.*

To finish this section, let us see particular cases where Theorem 4.8 and Corollary 4.9 can be applied. This shows that we always have denseness in all classical Banach spaces. Note that item (a) tells us that the metric π-property happens very often. Also, the stability results, (d), (e), (f), and (g), allow us to get more positive examples on denseness. We will first recall the following definition:

Definition 4.10. Let X be a Banach space. A sequence $\{X_n\}_{n\in\mathbb{N}}$ of finite-dimensional subspaces of X is called a *finite-dimensional decomposition* of X (F.D.D. for short) if every $x \in X$ has a unique representation of the form $x = \sum_{n=1}^{+\infty} x_n$ with $x_n \in X_n$ for every $n \in \mathbb{N}$.

Remark 4.11. A F.D.D. on a Banach space X determines a sequence $\{P_n\}_{n\in\mathbb{N}}$ of projections (called the *partial sum projections* of the decomposition) such that if $x = \sum_{n=1}^{\infty} x_n \in X$, then $P_j(x) = \sum_{n=1}^{j} x_n$ for all $j \in \mathbb{N}$. These projections are commuting, have increasing range, and converge strongly to the identity operator on X. The supremum of the norms of those projections is finite and is called the *decomposition constant.*

Example 4.12. The following Banach spaces satisfy the metric π-property (which might be well known for some readers, but we could not find proper references and we include the proof for completeness).

(a) Banach spaces with a finite-dimensional decomposition with the decomposition constant 1 (consequently, every Banach space with Schauder basis can be renormed to have the metric π-property)as follows;
(b) $L_p(\mu)$-spaces for any $1 \leqslant p < \infty$ and any measure μ;
(c) L_1-predual spaces;
(d) $X \oplus_a Y$, whenever X, Y satisfy the metric π-property and $|\cdot|_a$ is an absolute norm;
(e) $X = \left[\bigoplus_{n\in\mathbb{N}} X_n\right]_{c_0}$ or $\left[\bigoplus_{n\in\mathbb{N}} X_n\right]_{\ell_p}$, $\forall\ 1 \leqslant p < \infty$, X_n satisfying the metric π-property, $\forall n$;
(f) $X\widehat{\otimes}_\pi Y$, whenever X, Y satisfy the metric π-property;
(g) $X\widehat{\otimes}_\varepsilon Y$, whenever X, Y satisfy the metric π-property.

Proof. (a). Given a Banach space X, if there exists a sequence of finite-dimensional Banach spaces and 1-complemented subspaces $\{E_n\}_{n\in\mathbb{N}}$ such that $E_n \subseteq E_{n+1}$ holds for every n and such that $\bigcup_{n\in\mathbb{N}} E_n$ is dense in X, then X has the metric π-property. In particular, it applies whenever X is a Banach space with an F.D.D. with the decomposition constant 1 (if $P_n : X \longrightarrow X$ are the associated norm-one projections, take $E_n := P_n(X)$).
(b). Let $1 \leqslant p < \infty$ be given. Then, $L_p(\mu)$ has the metric π-property regardless the measure μ. Let us write $X = L_p(\mu)$, for short. Consider

$x_1, \ldots, x_n \in S_X, \varepsilon > 0$. For every $i \in \{1, \ldots, n\}$, we can find a simple function $x_i' \in S_X$ such that

$$\|x_i - x_i'\| < \varepsilon, \tag{4.4}$$

where $x_i' = \sum_{j=1}^m a_{ij}\chi_{A_j}$ for suitable $m \in \mathbb{N}$, $a_{ij} \in \mathbb{R}$ and pairwise disjoint $A_j \in \Sigma$. Now, in order to prove that X has the metric π-property, define $M := \operatorname{span}\{\chi_{A_j} : 1 \leqslant j \leqslant m\}$ and let us construct $P : X \longrightarrow X$ by the equation

$$T(f) := \sum_{j=1}^m \frac{1}{\mu(A_j)} \int_{A_j} f \, d\mu \chi_{A_j}.$$

It is clear from the disjointedness of $A_1, \ldots, A_m$ and the fact that $\|P(f)\| \leqslant \|f\|$ holds for every $f \in X$. Furthermore, it is clear from the definition that $P(f) = f$ holds for every $f \in M$, so P is a norm-one projection onto M. The result follows since $x_i' \in M$ and by the arbitrariness of $\varepsilon > 0$. This proves (b).
(c). If X is a Banach space with $X^* = L_1$, then X has the metric π-property. Indeed, let $\varepsilon > 0$ and $\{x_1, \ldots, x_n\} \subset S_X$ be given. Define $F_1 = \{0\}$ and $F_2 = \operatorname{span}\{x_1, \ldots, x_n\}$. By [28, Theorem 3.1] and [34, Theorem 1.3], we may find a subspace E of X such that E is isometric to ℓ_∞^m for some $m \in \mathbb{N}$ and $d(x, E) < \varepsilon$ for all $x \in F_2$. For each $1 \leqslant i \leqslant n$, pick $x_i' \in E$ so that $\|x_i - x_i'\| < \varepsilon$. By [33, Lemma 2.1], there exists a norm one projection P from X to E; hence E is indeed an 1-complemented finite-dimensional subspace of X.
(d). To prove that the metric π-property is stable by absolute sums, let us first notice that S_X, in its definition, can be replaced by B_X (indeed, let $\varepsilon > 0$ and $\{x_1, \ldots, x_n\} \subset B_X$ be given; without loss of generality, we may assume that $x_i \neq 0$ for all $1 \leqslant i \leqslant n$; from the metric π-property, we may find a 1-complemented finite-dimensional space M of X with $x_i' \in M$ such that $\|x_i/\|x_i\| - x_i'\| < \varepsilon$ for every $1 \leqslant i \leqslant n$; thus, $\|x_i - \|x_i\|x_i'\| < \varepsilon$ and $\{\|x_1\|x_1', \ldots, \|x_n\|x_n'\} \subset M$). Set $Z = X \oplus_a Y$. Let $\varepsilon > 0$ and $\{z_1, \ldots, z_n\} \subset S_Z$ be given. If we write $z_i = (x_i, y_i)$ for each $1 \leqslant i \leqslant n$, then $\max\{\|x_i\|, \|y_i\|\} \leqslant \|z_i\|_a = 1$ for every $1 \leqslant i \leqslant n$. As X has the metric π-property and $\{x_1, \ldots, x_n\} \subset B_X$, there exist a 1-complemented finite-dimensional subspace M of X and $\{x_1', \ldots, x_n'\} \subseteq M$ such that $\|x_i - x_i'\| < \varepsilon$. Similarly, there exist a 1-complemented finite-dimensional subspace N of Y and $\{y_1', \ldots, y_n'\} \subset N$ such that $\|y_i - y_i'\| < \varepsilon$. If we let $z_i' = (x_i', y_i')$ for each $1 \leqslant i \leqslant n$, then for every $1 \leqslant i \leqslant n$, we have

$$\|z_i - z_i'\|_a \leqslant \|x_i - x_i'\| + \|y_i - y_i'\| < 2\varepsilon.$$

Let P and Q be norm one projections from X onto M and Y onto N, respectively. Consider the map (P, Q) defined on $X \oplus_a Y$ as $(x, y) \mapsto (Px, Qy)$ for every $(x, y) \in X \oplus_a Y$. Note that (P, Q) is a projection with (closed) range $M \oplus_a N$. Moreover,

$$\|(Px, Qy)\|_a = |(\|Px\|, \|Qy\|)|_a \leqslant |(\|x\|, \|y\|)|_a = \|(x, y)\|_a$$

for every $(x, y) \in X \oplus_a Y$; hence $M \oplus_a N$ is a 1-complemented finite-dimensional subspace of Z with $\{z_1', \ldots, z_n'\} \subset M \oplus_a N$ satisfying $\|z_i - z_i'\| < 2\varepsilon$ for each $1 \leqslant i \leqslant n$.

(e). This can be obtained by extending the proof of (d). Let $\{x_1, \ldots, x_n\} \subseteq S_X$ be given. First, approximate x_i by x'_i of finite support. Now, say $x'_i = (x_{i1}, \ldots, x_{ik}, 0, 0, \ldots)$ with some common $k \in \mathbb{N}$. Find a 1-complemented subspace M_j in X_j containing $x_{1j}, \ldots, x_{nj}$ from the assumption that X_j enjoys the metric π-property for each $1 \leqslant j \leqslant k$. Then, $M = \{(z_1, z_2, \ldots, z_k, 0, 0, \ldots) : z_i \in M_i, 1 \leqslant i \leqslant k\}$ is a finite-dimensional subspace of X which is 1-complemented by the projection $(P_1, P_2, \ldots, P_k, 0, 0, \ldots)$ (defined similarly as in the item (d)) and M contains the set $\{x'_1, \ldots, x'_n\}$.

(f). Let $\varepsilon > 0$ and $z_1, \ldots, z_n \in S_{X \widehat{\otimes}_\pi Y}$ be given. For each $1 \leqslant i \leqslant n$, consider $\{x_j^{(i)}, y_j^{(i)}\} \subseteq B_X \times B_Y$ to be such that

$$z_i = \sum_{j=1}^{\infty} x_j^{(i)} \otimes y_j^{(i)} \quad \text{with} \quad \|z_i\|_\pi > \sum_{j=1}^{\infty} \|x_j^{(i)}\| \|y_j^{(i)}\| - \varepsilon.$$

For each $i = 1, \ldots, n$, let $N_i \in \mathbb{N}$ be such that

$$\sum_{j=N_i+1}^{\infty} \|x_j^{(i)}\| \|y_j^{(i)}\| < \frac{\varepsilon}{2}.$$

Now, since X has the metric π-property, there exists a 1-complemented finite-dimensional subspace M of X with

$$\left\{ \widetilde{x_j}^{(i)} : 1 \leqslant j \leqslant N_i, 1 \leqslant i \leqslant n \right\}$$
$$\subseteq M \text{ such that } \|\widetilde{x_j}^{(i)} - x_j^{(i)}\| < \min \left\{ \frac{\varepsilon}{4N_i} : 1 \leqslant i \leqslant n \right\}$$

and, analogously, there exists a 1-complemented finite-dimensional subspace N of Y with

$$\left\{ \widetilde{y_j}^{(i)} : 1 \leqslant j \leqslant N_i, 1 \leqslant i \leqslant n \right\}$$
$$\subseteq N \text{ such that } \|\widetilde{y_j}^{(i)} - y_j^{(i)}\| < \min \left\{ \frac{\varepsilon}{4N_i} : 1 \leqslant i \leqslant n \right\}$$

for each $1 \leqslant j \leqslant N_i$ with $i = 1, \ldots, n$. By [36, Proposition 2.4], $M \widehat{\otimes}_\pi N$ is an 1-complemented space. Let $\widetilde{z_i} := \sum_{j=1}^{N_i} \widetilde{x_j}^{(i)} \otimes \widetilde{y_j}^{(i)}$. Then,

$$\left\| \widetilde{z_i} - \sum_{j=1}^{N_i} x_j^{(i)} \otimes y_j^{(i)} \right\|_\pi \leqslant 2N_i \min \left\{ \frac{\varepsilon}{4N_i} : 1 \leqslant i \leqslant n \right\} \leqslant \frac{\varepsilon}{2}$$

for every $i = 1, \ldots, n$. Then, $X \widehat{\otimes}_\pi Y$ has the metric π-property, as desired.

(g). Let $z_1, \ldots, z_n \in S_{X \widehat{\otimes}_\varepsilon Y}$ and $\delta > 0$ be given. For each $i \in \{1, \ldots, n\}$, let $\widetilde{z_i} \in X \otimes Y$ be such that $\|z_i - \widetilde{z_i}\|_\varepsilon < \frac{\delta}{2}$. Let $\sum_{j=1}^{N_i} x_j^{(i)} \otimes y_j^{(i)}$ be a representation of $\widetilde{z_i}$ for each $i = 1, \ldots, n$. Since

$$\{x_j^{(i)} : 1 \leqslant j \leqslant N_i, 1 \leqslant i \leqslant n\} \subseteq X \quad \text{and} \quad \{y_j^{(i)} : 1 \leqslant j \leqslant N_i, 1 \leqslant i \leqslant n\} \subseteq Y,$$

there are 1-complemented finite-dimensional subspaces $M \leqslant X$ and $N \leqslant Y$ with $\{\widetilde{x_j}^{(i)} : 1 \leqslant j \leqslant N_i, 1 \leqslant i \leqslant n\} \subseteq M$ and $\{\widetilde{y_j}^{(i)} : 1 \leqslant j \leqslant N_i, 1 \leqslant i \leqslant n\} \subseteq N$ such that

$$\|x_j^{(i)} - \widetilde{x_j}^{(i)}\| < \min\left\{\frac{\varepsilon}{4N_i} : 1 \leqslant i \leqslant n\right\}$$
$$\text{and } \|y_j^{(i)} - \widetilde{y_j}^{(i)}\| < \min\left\{\frac{\varepsilon}{4N_i} : 1 \leqslant i \leqslant n\right\}.$$

As $M\widehat{\otimes}_\varepsilon N$ is a 1-complemented subspace of $X\widehat{\otimes}_\varepsilon Y$ (see, for instance, [36, Proposition 3.2]),

$$\widetilde{v_i} = \sum_{j=1}^{N_i} \widetilde{x_j}^{(i)} \otimes \widetilde{y_j}^{(i)} \in M\widehat{\otimes}_\varepsilon N \quad \text{and} \quad \|\widetilde{z_i} - \widetilde{v_i}\|_\varepsilon \leqslant \|\widetilde{z_i} - \widetilde{v_i}\|_\pi \leqslant \frac{\delta}{2},$$

which implies that $\|z_i - \widetilde{v_i}\|_\varepsilon < \delta$, we have that $X\widehat{\otimes}_\varepsilon Y$ satisfies the metric π-property. □

Remark 4.13. From the estimates of case (g) above it follows that $X\widehat{\otimes}_\alpha Y$ has the metric π-property whenever X and Y enjoy the metric π-property and α is a *uniform cross norm* (see [36, Section 6.1] for background and details).

Example 4.12.(g) allows us to extend Theorem 4.8 for larger projective tensor products.

Corollary 4.14. *Let $N \in \mathbb{N}$ be given. Let $X_1, \ldots, X_N$ be Banach spaces with the metric π-property, and Y be a Banach space. Then,*

$$\overline{\mathrm{NA}_\pi(X_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_N \widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_N \widehat{\otimes}_\pi Y.$$

5. There are Tensors Which Cannot be Approximated by Norm-Attaining Tensors

By the results from previous section, one may think that the denseness for norm-attaining tensors always holds true. In this section, we will see that this is not the case. We show that there are Banach spaces X and Y such that the set of all tensors in $X\widehat{\otimes}_\pi Y^*$ which attain their projective norms *is not* dense in $X\widehat{\otimes}_\pi Y^*$. In order to do that, let us notice that, by Theorem 3.1, it would be enough to show that $\mathrm{NA}(X, Y^{**}) \cap B_{L(X,Y^{**})}$ is not norming for $X\widehat{\otimes}_\pi Y^*$ (and in fact that is what we do; see Remark 5.4). On the other hand, in view of the proof of [27, Proposition 2.3], note that if either X or Y^{**} satisfies the metric approximation property (respectively, bounded approximation property), then $\mathcal{F}(X, Y^{**})$ is norming (respectively, K-norming) for $X\widehat{\otimes}_\pi Y^*$, and this implies that $\mathcal{F}(X, Y^{**})$ is w^*-dense in $\mathcal{L}(X, Y^{**})$. This suggests us to look for our counterexample in the context of Banach spaces failing the approximation property and trying to guarantee that the set of operators which attain their norms is not bigger than the set of finite-rank operators. This is the reason why we will adapt [32, Theorem 1] taking into account all the previous considerations.

For this, we will use Read's space $\mathcal{R}$ (see [24,25,35] for all the details on this space). Read's space is a renorming of the Banach space c_0, $\mathcal{R} = (c_0, |||\cdot|||)$, which has bidual $\mathcal{R}^{**}$ strictly convex (see [24, Theorem 4]). This implies that $\mathrm{NA}(X, \mathcal{R}^{**}) \subseteq \mathcal{F}(X, \mathcal{R}^{**})$ whenever X is a closed subspace of c_0 (see

[32, Lemma 2]). It is worth mentioning that we are not using here the deep properties of $\mathcal{R}$ (that it contains no two-codimensional proximal subspaces) but only the fact that its bidual is strictly convex for the bidual norm and that it contains c_0 (this is in fact well known; the existence of such norms can be justified, for instance, by using [25, Lemma 2.1] and taking R as a one-to-one operator from c_0 into ℓ_2).

Theorem 5.1. *Let $\mathcal{R}$ be Read's space. There exist subspaces X of c_0 and Y of $\mathcal{R}$ such that the set of tensors in $X\widehat{\otimes}_\pi Y^*$ which attain their projective norms is not dense in $X\widehat{\otimes}_\pi Y^*$.*

In order to prove Theorem 5.1, we would like to present several results, which, from our point of view, have their own interest.

Lemma 5.2. *Let X, Y be a Banach spaces such that Y^* is separable. If $\mathcal{F}(X, Y^{**})$ is viewed as a subspace of $(X\widehat{\otimes}_\pi Y^*)^* = \mathcal{L}(X, Y^{**})$, we have*

$$B_{\mathcal{F}(X,Y^{**})} \subset \overline{B_{\mathcal{F}(X,Y)}}^{w^*}.$$

Proof. Let $T \in \mathcal{F}(X, Y^{**})$ with $\|T\| < 1$. Choose a countable dense subset $(y_n^*)_{n\in\mathbb{N}}$ of Y^* and let $F_n = \operatorname{span}\{y_1^*, \ldots, y_n^*\}$ for each $n \in \mathbb{N}$. By the Principle of Local Reflexivity, for each $n \in \mathbb{N}$, there exists an operator $\phi_n : T(X) \to Y$ such that

1. $\left(1 - \frac{1}{n}\right)\|T(x)\| \leqslant \|\phi_n(T(x))\| \leqslant \left(1 + \frac{1}{n}\right)\|T(x)\|$ for every $x \in X$,
2. $y^*(\phi_n(T(x))) = y^*(T(x))$ for every $y^* \in F_n$ and $x \in X$.

Choose $n_0 \in \mathbb{N}$ so that $\frac{1}{n} < \frac{1}{\|T\|} - 1$ whenever $n \geqslant n_0$. Let us define $K_n = \phi_n \circ T \in \mathcal{F}(X, Y)$ for each $n \geqslant n_0$. Then $\|K_n\| \leqslant \|\phi_n\|\|T\| < 1$ for each $n \geqslant n_0$. We claim that $K_n \xrightarrow{w^*} T$. First, observe that given $x \in X$ and $m \in \mathbb{N}$, we have

$$y_m^*(K_n(x)) = y_m^*(\phi_n(T(x))) = y_m^*(T(x)) \text{ for every } n \geqslant m. \tag{5.1}$$

Now, let $x \in X \setminus \{0\}$, $y^* \in Y^*$ and $\varepsilon > 0$ be given. Pick $n_0 \in \mathbb{N}$ so that $\|y_{n_0}^* - y^*\| < \frac{\varepsilon}{2\|x\|}$. By (5.1), we have for $n \geqslant n_0$,

$$\begin{aligned}|y^*(K_n(x)) - y^*(T(x))| &\leqslant |y^*(K_n(x)) - y_{n_0}^*(K_n(x))| + |y_{n_0}^*(K_n(x)) - y_{n_0}^*(T(x))| \\ &\quad + |y_{n_0}^*(T(x)) - y^*(T(x))| \\ &\leqslant \|y^* - y_{n_0}^*\|\|K_n\|\|x\| + \|y_{n_0}^* - y^*\|\|T\|\|x\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

By a linearity argument we get that $K_n(z) \to T(z)$ for every $z \in X \otimes Y$. Finally, since the sequence K_n is bounded we get that $K_n \to T$ in the w^*-topology.

This implies that $\{T \in \mathcal{F}(X, Y^{**}) : \|T\| < 1\} \subset \overline{B_{\mathcal{F}(X,Y)}}^{w^*}$. As a w^*-closed set in $\mathcal{L}(X, Y^{**})$ is $\|\cdot\|$-closed, we conclude that $B_{\mathcal{F}(X,Y^{**})} \subset \overline{B_{\mathcal{F}(X,Y)}}^{w^*}$. □

In what follows, we will be using the *strong operator topology* (*SOT*, for short) and the *weak operator topology* (*WOT*, for short). Recall that the strong operator topology in $\mathcal{L}(X,Y)$ is the topology defined by the basic neighborhoods

$$N(T;A,\varepsilon)=\{S\in\mathcal{L}(X,Y):\|(T-S)(x)\|<\varepsilon, x\in A\},$$

where A is an arbitrary finite subset of X and $\varepsilon>0$. Thus, in the SOT, a net (T_α) converges to T if and only if $(T_\alpha(x))$ converges to $T(x)$ for every $x\in X$. On the other hand, the weak operator topology is defined by the basic neighborhoods

$$N(T;A,A^*,\varepsilon)=\{S\in\mathcal{L}(X,Y),|y^*(T-S)(x)|<\varepsilon, y^*\in A^*, x\in A\},$$

where A and A^* are arbitrary finite sets in X and Y^*, respectively, and $\varepsilon>0$. Thus, in the WOT, a net T_α converges to T if and only if $(y^*(T_\alpha(x)))$ converges to $y^*(T(x))$ for every $x\in X$ and $y^*\in Y^*$.

Let us notice that a convex set in $\mathcal{L}(X,Y)$ has the same closure in the WOT as it does in the SOT (see, for instance, [16, Corollary 5, page 477]). We will use this fact in the proof of Theorem 5.1 below.

Lemma 5.3. *Let X be a Banach space failing the approximation property. Then, the identity map on X does not belong to $\overline{RB_{\mathcal{F}(X,X)}}^{WOT}$ for any $R>0$.*

Proof. Let X be a Banach space which fails the approximation property and let us denote the identity map on X by Id_X. Then, by definition, $\mathrm{Id}_X\notin\overline{\mathcal{F}(X,X)}^{\tau}$, where τ is the topology of uniform convergence on compact sets. For given $R>0$, let us prove that $\mathrm{Id}_X\notin\overline{RB_{\mathcal{F}(X,X)}}^{SOT}$. In order to get a contradiction, let us assume $\mathrm{Id}_X\in\overline{RB_{\mathcal{F}(X,X)}}^{SOT}$. Then there exists a net $(T_\alpha)_{\alpha\in\Lambda}\subset RB_{\mathcal{F}(X,X)}$ such that $T_\alpha\xrightarrow{SOT}\mathrm{Id}_X$. Now, let K be a compact set in X and $\varepsilon>0$ be given. Choose a $\left(\min\left\{\frac{\varepsilon}{3R},\frac{\varepsilon}{3}\right\}\right)$-net $\{x_1,\dots,x_k\}$ for K. Pick $\alpha_0\in\Lambda$ such that for every $\alpha\geqslant\alpha_0$

$$\max_{1\leqslant i\leqslant k}\|T_\alpha(x_i)-\mathrm{Id}_X(x_i)\|=\max_{1\leqslant i\leqslant k}\|T_\alpha(x_i)-x_i\|<\frac{\varepsilon}{3}.$$

Given $x\in K$, take $i\in\{1,\dots,k\}$ so that $\|x-x_i\|<\min\left\{\frac{\varepsilon}{3R},\frac{\varepsilon}{3}\right\}$. Then,

$$\begin{aligned}\|T_\alpha(x)-\mathrm{Id}_X(x)\|&\leqslant\|T_\alpha(x)-T_\alpha(x_i)\|+\|T_\alpha(x_i)-x_i\|+\|x_i-x\|\\&\leqslant\|T_\alpha\|\|x-x_i\|+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}\\&<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon\end{aligned}$$

for every $\alpha\geqslant\alpha_0$. This implies that $\mathrm{Id}_X\in\overline{\mathcal{F}(X,X)}^{\tau}$, a contradiction. So, $\mathrm{Id}_X\notin\overline{RB_{\mathcal{F}(X,X)}}^{SOT}=\overline{RB_{\mathcal{F}(X,X)}}^{WOT}$. □

Now we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. Let X be a closed subspace of c_0 which fails the approximation property (see, for instance, [31, Theorem 2.d.6]). Then, by Lemma 5.3, the identity map on X does not belong to $\overline{RB_{\mathcal{F}(X,X)}}^{WOT}$ for any $R>0$. Let $Y=(X,|||\cdot|||)$, where $|||\cdot|||$ is the norm that defines Read's space. Let us denote by $\iota\in\mathcal{L}(X,Y)$ the formal identity map from X to Y. Then

$T = \iota/\|\iota\|$ does not belong to $R\overline{B_{\mathcal{F}(X,Y)}}^{WOT}$ for any $R > 0$. It follows that T does not belong to $R\overline{B_{\mathcal{F}(X,Y)}}^{w^*}$, where the previous weak-star topology refers to $\sigma(\mathcal{L}(X, Y^{**}), X\widehat{\otimes}_\pi Y^*)$, for any $R > 0$. Indeed, if $T \in R\overline{B_{\mathcal{F}(X,Y)}}^{w^*}$ for some $R > 0$, given $x \in X$, $y^* \in Y^*$ and $\varepsilon > 0$, there exists $T_0 \in RB_{\mathcal{F}(X,Y)}$ such that

$$|y^*(T(x) - T_0(x))| = |(T - T_0)(x \otimes y^*)| < \varepsilon,$$

which implies that $T \in R\overline{B_{\mathcal{F}(X,Y)}}^{WOT}$, a contradiction. In particular, T does not belong to $\overline{B_{\mathcal{F}(X,Y)}}^{w^*}$. As Y^* is separable, by Lemma 5.2, T does not belong to $\overline{B_{\mathcal{F}(X,Y^{**})}}^{w^*}$. Thus, by the Hahn–Banach theorem we have that the unit ball $B_{\mathcal{F}(X,Y^{**})}$ is not norming for $X\widehat{\otimes}_\pi Y^*$. Take $z \in X\widehat{\otimes}_\pi Y^*$ with $\|z\|_\pi = 1$ and $\alpha > 0$ such that

$$\sup\{|G(z)| : G \in B_{\mathcal{F}(X,Y^{**})}\} < 1 - \alpha. \tag{5.2}$$

Claim: dist $\left(z, \mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y^*)\right) > \dfrac{\alpha}{2}$.

If this is not the case, there exists $z' \in \mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y^*)$ such that $\|z - z'\|_\pi \leqslant \frac{\alpha}{2}$. This implies that $\|z'\|_\pi \geqslant 1 - \frac{\alpha}{2}$. Let $G \in \mathcal{L}(X, Y^{**})$ with $\|G\| = 1$ such that $|G(z')| = \|z'\|_\pi$. In particular, $G \in \mathrm{NA}(X, Y^{**})$ by Theorem 3.1. Notice that $Y^{**} = Y^{\perp\perp}$ is a closed subspace of $\mathcal{R}^{**}$, so Y^{**} is strictly convex. Thus, we have that $G \in \mathcal{F}(X, Y^{**})$ by [32, Lemma 2], which implies by (5.2) that $|G(z)| < 1 - \alpha$. Nevertheless,

$$|G(z)| \geqslant |G(z')| - \|z - z'\|_\pi \geqslant 1 - \frac{\alpha}{2} - \frac{\alpha}{2} = 1 - \alpha,$$

which is a contradiction. □

Remark 5.4. Notice that from the above proof it follows that, given two Banach spaces X and Y, if $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$ is dense in $X\widehat{\otimes}_\pi Y$, then $\mathrm{NA}(X, Y^*) \cap B_{\mathcal{L}(X,Y^*)}$ is norming for $X\widehat{\otimes}_\pi Y$.

In fact, from the proof of Theorem 5.1 (and its lemmas) we extract more information. Recall that for every non-zero tensor $u \in X \otimes Y$, there is a smallest $N \in \mathbb{N}$ for which there is a representation for z containing N terms. The number N is known as the rank of u. Because of this, we will say that u is a *finite-rank tensor* if $u \in X \otimes Y$. Although it is not known whether every finite-rank operator can be approximated by norm-attaining operators, the case for tensors does not hold in general.

Proposition 5.5. *There are tensors of finite rank which do not attain their projective norm.*

Proof. Consider X and Y^* as in Theorem 5.1. Then, there exist $\alpha > 0$ and $z \in X\widehat{\otimes}_\pi Y^*$ such that dist$\left(z, \mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y^*)\right) \geqslant \alpha$. Now, take u of finite-rank such that $\|z - u\|_\pi < \frac{\alpha}{2}$. Then, this element cannot attain its projective norm. □

As we have commented at the beginning of this section, let us notice that from the proof of Theorem 5.1, there exist some Banach spaces X and Y such that $\mathrm{NA}(X, Y^{**}) \cap B_{\mathcal{L}(X,Y^{**})}$ is not w^*-dense in $B_{\mathcal{L}(X,Y^{**})}$. Actually, we have the following result:

Corollary 5.6. *There are Banach spaces X and Y such that*

$$\overline{\mathrm{co}(\mathrm{NA}(X, Y^{**}) \cap B_{\mathcal{L}(X,Y^{**})})}^{w^*} \neq B_{\mathcal{L}(X,Y^{**})}.$$

6. Open Questions

In this section, we would like to discuss and propose some open questions.

We have proved that if H is a complex Hilbert space, every tensor in $H \widehat{\otimes}_\pi H$ attains its projective norm (see Proposition 3.8) and that the set $\mathrm{NA}_\pi(L_p(\mu) \widehat{\otimes}_\pi L_q(\nu))$ is dense in $L_p(\mu) \widehat{\otimes}_\pi L_p(\nu)$ for $1 < p, q < \infty$ and measures μ and ν (see Example 4.12.(b)). However, we do not know what happens in general when both factors are reflexive spaces.

Question 6.1. *Let X, Y be reflexive Banach spaces. Is it true that the set of all norm-attaining tensors is dense in $X \widehat{\otimes}_\pi Y$?*

Let us notice that, by trying to mimic the proof of Theorem 5.1 (and its lemmas) for the nuclear operator case, one would realize that

$$(\ker J)^\perp \neq \overline{(\ker J)^\perp \cap F(Y, X^{**})}^{w^*}$$

needs to be one the hypothesis (which we cannot guarantee that it holds). We do not know if there is a version of Theorem 5.1 for nuclear operators.

Question 6.2. *Are there Banach spaces X and Y so that $\mathrm{NA}_{\mathcal{N}}(X, Y)$ is not dense in $\mathcal{N}(X, Y)$?*

We say that a Banach space X has *property quasi-α* if, for an index set Γ, there are $A = \{x_\gamma \in S_X : \gamma \in \Gamma\}$, $A^* = \{x^*_\gamma \in S_{X^*} : \gamma \in \Gamma\}$, and $\lambda : A \longrightarrow \mathbb{R}$ such that $x^*_\gamma(x_\gamma) = 1$ for every $\gamma \in \Gamma$; $|x^*_\gamma(x_\eta)| \leqslant \lambda(x_\gamma) < 1$ for $\gamma \neq \eta$; and for every $e \in \mathrm{Ext}(B_{X^{**}})$, there is a subset $A_e \subseteq A$ and a scalar t with $|t| = 1$ such that $te \in \overline{Q(A_e)}^{w^*}$ and $r_e = \sup\{\lambda(x) : x \in A_e\} < 1$, where Q is the canonical embedding on X^{**} (see [11]). Let us notice that property quasi-α is weaker than property α introduced by Schachermayer in [37]. We have proved that $\mathrm{NA}_\pi(\ell_1 \widehat{\otimes}_\pi Y) = \ell_1 \widehat{\otimes}_\pi Y$ for every Banach space Y (see Proposition 3.6). Consequently, by using Proposition 3.10, we get that

$$\overline{\mathrm{NA}(\ell_1 \times Y)}^{\|\cdot\|} = \mathcal{B}(\ell_1 \times Y)$$

for every Banach space Y. This is a particular case of [11, Theorem 2.17], which we wonder if it could be extended in the following sense.

Question 6.3. *Let X be a Banach space with property α (or quasi-α). Is it true that the equality $\mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y) = X \widehat{\otimes}_\pi Y$ holds for every Banach space Y?*

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Sheldon Dantas
Departament de Matemàtiques and
Institut Universitari de Matemàtiques i Aplicacions de Castelló (IMAC)
Universitat Jaume I
Campus del Riu Sec. s/n
12071 Castelló
Spain
e-mail: dantas@uji.es
URL: https://www.sheldondantas.com

Mingu Jung
Basic Science Research Institute and Department of Mathematics
POSTECH
Pohang 790-784
Republic of Korea
e-mail: jmingoo@postech.ac.kr

Óscar Roldán
Departamento de Análisis Matemático
Universitat de València
Valencia
Spain
e-mail: oscar.roldan@uv.es

Abraham Rueda Zoca
Universidad de Murcia, Departamento de Matemáticas
Campus de Espinardo
30100 Murcia
Spain
e-mail: abraham.rueda@um.es
URL: https://arzenglish.wordpress.com

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ON NORM-ATTAINMENT IN (SYMMETRIC) TENSOR PRODUCTS

Sheldon Dantas

Departament de Matemàtiques and Institut Universitari de Matemàtiques i Aplicacions de Castelló (IMAC), Universitat Jaume I, Campus del Riu Sec. s/n, 12071 Castelló, Spain.
E-Mail dantas@uji.es

Luis C. García-Lirola

Departamento de Matemáticas, Universidad de Zaragoza, 50009, Zaragoza, Spain.
E-Mail luiscarlos@unizar.es

Mingu Jung[1]

Basic Science Research Institute and Department of Mathematics, POSTECH, Pohang 790-784, Republic of Korea.
E-Mail jmingoo@postech.ac.kr

Abraham Rueda Zoca

Universidad de Murcia, Departamento de Matemáticas, Campus de Espinardo 30100 Murcia, Spain.
E-Mail abraham.rueda@um.es

Abstract. In this paper, we introduce a concept of norm-attainment in the projective symmetric tensor product $\widehat{\otimes}_{\pi,s,N} X$ of a Banach space X, which turns out to be naturally related to the classical norm-attainment of N-homogeneous polynomials on X. Due to this relation, we can prove that there exist symmetric tensors that do not attain their norms, which allows us to study the problem of when the set of norm-attaining elements in $\widehat{\otimes}_{\pi,s,N} X$ is dense. We show that the set of all norm-attaining symmetric tensors is dense in $\widehat{\otimes}_{\pi,s,N} X$ for a large set of Banach spaces such as L_p-spaces, isometric L_1-predual spaces or Banach spaces with monotone Schauder basis, among others. Next, we prove that if X^* satisfies the Radon-Nikodým and approximation properties, then the set of all norm-attaining symmetric tensors in $\widehat{\otimes}_{\pi,s,N} X^*$ is dense. From these techniques, we can present new examples of Banach spaces X and Y such that the set of all norm-attaining tensors in the projective tensor product $X \widehat{\otimes}_\pi Y$ is dense, answering positively an open question from the paper [10].

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Key words: Bishop-Phelps theorem, symmetric tensor product, projective tensor product, Radon-Nikodým property, norm attaining polynomials.

1. Introduction. Recently, it was studied the set of the nuclear operators $T\colon X \longrightarrow Y$ between two Banach spaces X, Y that attains their nuclear norm in the sense that

$$T = \sum_{n=1}^{\infty} x_n^* \otimes y_n, \text{ and } \|T\|_{\mathcal{N}} = \sum_{n=1}^{\infty} \|x_n^*\| \, \|y_n\|$$

for some $(x_n^*)_{n=1}^{\infty} \subseteq X^*$ and $(y_n)_{n=1}^{\infty} \subseteq Y$ [10]. From a practical point of view, it has been shown that this new concept has great connections with different norm-attainment concepts like the norm-attainment of bounded functionals, the norm-attainment of operators and the norm-attainment of bilinear forms coming from the identification $(X \widehat{\otimes}_\pi Y)^* = \mathcal{L}(X, Y^*) = \mathcal{B}(X \times Y)$ (see [10, Proposition 3.10 and Corollary 3.11]). In this paper, we focus on a related norm-attaining notion on the N-fold symmetric tensor product of a Banach space X: we say that an element $z \in \widehat{\otimes}_{\pi,s,N} X$ attains its norm if

$$z = \sum_{n=1}^{\infty} \lambda_n x_n^N, \text{ and } \|z\| = \sum_{n=1}^{\infty} |\lambda_n|$$

for some $(\lambda_n)_{n=1}^{\infty} \subseteq \mathbb{K}$ and $(x_n)_{n=1}^{\infty} \subseteq X$ (given a natural number $N \in \mathbb{N}$, we denote by z^N the element $z \otimes \overset{N}{\cdots} \otimes z \in X \otimes \overset{N}{\cdots} \otimes X$ for every $z \in X$). Due to the fact that the dual $(\widehat{\otimes}_{\pi,s,N} X)^*$ of the N-fold symmetric tensor product of a Banach space X is identified with the space $\mathcal{P}(^N X)$ of all N-homogeneous polynomials on X, this norm-attainment notion turns out to be closely related to the theory of norm-attaining homogeneous polynomials for which the reader is referred to [1, 7, 9, 21].

We present a characterization for the set of all norm-attaining elements in $\widehat{\otimes}_{\pi,s,N} X$, denoted by $\mathrm{NA}_{\pi,s,N}(X)$, and we use it to prove that if every element in $\widehat{\otimes}_{\pi,s,N} X$ attains its norm, then the set $\mathrm{NA}(^N X)$ of all N-homogeneous polynomials which attain their norms in dense in $\mathcal{P}(^N X)$. This, together with the fact that there are Banach spaces X such that the set $\mathrm{NA}(^N X)$ is not dense in $\mathcal{P}(^N X)$ (see [1, 18]), allows us to get our first examples of spaces X so that we can guarantee the existence of non-norm-attaining elements in $\widehat{\otimes}_{\pi,s,N} X$. As there exist elements in $\widehat{\otimes}_{\pi,s,N} X$ which do not attain their norms, it is natural to ask when the set of norm-attaining elements in $\widehat{\otimes}_{\pi,s,N} X$ forms a dense subset. We prove that, under the metric π-property (see [8, 19]) on X, the denseness of $\mathrm{NA}_{\pi,s,N}(X)$ holds from the fact that every tensor in $\widehat{\otimes}_{\pi,s,N} Z$ attains its norm whenever Z is a finite dimensional space. This shows that, for a large class of Banach spaces, as for instance L_p-spaces, L_1-predual spaces, and Banach spaces with monotone Schauder basis, the set $\mathrm{NA}_{\pi,s,N}(X)$ is dense in $\widehat{\otimes}_{\pi,s,N} X$. We also present a result not covered by the previous ones which holds under the Radon-Nikodým property assumption. More precisely, we show that if X^* has the Radon-Nikodým property and the approximation property, then the set of tensors in $\widehat{\otimes}_{\pi,s,N} X^*$ which attain their norms is dense. Moreover, we observe that the problem whether the set $\mathrm{NA}_{\pi,s,N}(X)$ is dense in $\widehat{\otimes}_{\pi,s,N} X$ for every Banach space, is separably determined.

[1]Corresponding author.

We finish the paper by considering the set $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$ of all norm-attaining tensors in $X\widehat{\otimes}_\pi Y$ and obtaining some positive results on the denseness of the set $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$. For instance, we prove that if X is the convex hull of a finite set and Y is a dual space, then *every* element in $X\widehat{\otimes}_\pi Y$ attains its norm, which seems to be surprising somehow since there exists a Banach space X so that $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi \ell_2^2) \neq X\widehat{\otimes}_\pi \ell_2^2$ (see [10, Example 3.12.(a)]). This result allows us to show that if X is a polyhedral Banach space with the metric π-property, then the set $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$ is dense in $X\widehat{\otimes}_\pi Y$ whenever Y is a dual space. Moreover, in the same line as the symmetric tensor product case, we give a positive answer to an open question from [10] by proving that $\mathrm{NA}(X^*\widehat{\otimes}_\pi Y^*)$ is dense in $X^*\widehat{\otimes}_\pi Y^*$, provided that X^* and Y^* both have the Radon-Nikodým property, and at least one of them has the approximation property.

2. Notation and preliminary results. In this section, we give the necessary notation and some preliminary results we will be using throughout the paper.

The letters X, Y, and Z stand for Banach spaces over the field $\mathbb{K}$ which will be $\mathbb{R}$ or $\mathbb{C}$. We denote by B_X and S_X the closed unit ball and unit sphere of X, respectively. Given a subset $B \subseteq X$, we denote the convex hull of B by $\mathrm{co}(B)$. The symbol $\mathrm{aco}(B)$ stands for the absolutely convex (i.e., the convex and balanced) hull of the set B. If A, B are subsets of X, Y, respectively, we denote by $A \otimes B$ the set $\{x \otimes y \in X \otimes Y : x \in A, y \in B\}$. Given a subset C of X, a point $x \in C$ is said to be an extreme point of C if x cannot be written as a convex combination of points in C which are different from x itself. We denote by $\mathrm{ext}\,(C)$ the set of all extreme points of C.

For two Banach spaces X and Y, the symbol $\mathcal{L}(X,Y)$ stands for the space of all bounded linear operators from X into Y. By $\mathcal{B}(X \times Y)$ we mean the space of all bilinear forms on $X \times Y$ taking values in $\mathbb{K}$. The Banach space of all scalar-valued N-homogeneous polynomials on X is denoted by $\mathcal{P}(^N X)$, which is endowed with the norm $\|P\| = \sup_{x \in B_X} |P(x)|$ for every $P \in \mathcal{P}(^N X)$. In this case, P is said to attain its norm when this supremum becomes a maximum. We denote by $\mathrm{NA}(^N X)$ the set of all N-homogeneous polynomials which attain their norms on X. For background on homogeneous polynomials we refer the refer to [13, 16, 23].

The projective and injective tensor product between X and Y, denoted by $X\widehat{\otimes}_\pi Y$ and $X\widehat{\otimes}_\varepsilon Y$, respectively, are the completion of the algebraic tensor product $X \otimes Y$ endowed with the norms

$$\|z\|_\pi := \inf\left\{\sum_{i=1}^n \|x_i\|\|y_i\| : z = \sum_{i=1}^n x_i \otimes y_i\right\}, \tag{2.1}$$

where the infimum is taken over all such representations of z, and

$$\left\|\sum_{i=1}^n x_i \otimes y_i\right\|_\varepsilon = \sup\left\{\left|\sum_{i=1}^n x^*(x_i) y^*(y_i)\right| : x^* \in B_{X^*}, y^* \in B_{Y^*}\right\}.$$

It is well known that $B_{X\widehat{\otimes}_\pi Y} = \overline{\mathrm{co}}(B_X \otimes B_Y)$ and that $(X\widehat{\otimes}_\pi Y)^* = \mathcal{B}(X \times Y) = \mathcal{L}(X, Y^*)$. There is a canonical operator $J\colon X^*\widehat{\otimes}_\pi Y \longrightarrow \mathcal{L}(X,Y)$ with

$\|J\| = 1$ defined by $z = \sum_{n=1}^{\infty} x_n^* \otimes y_n \mapsto L_z$, where $L_z : X \longrightarrow Y$ is given by $L_z(x) = \sum_{n=1}^{\infty} x_n^*(x) y_n$. The operators that arise in this way are called nuclear operators and we denote them by $\mathcal{N}(X,Y)$ endowed with the nuclear norm

$$\|T\|_{\mathcal{N}} := \inf \left\{ \sum_{n=1}^{\infty} \|x_n^*\| \|y_n\| : T(x) = \sum_{n=1}^{\infty} x_n^*(x) y_n \right\}, \tag{2.2}$$

where the infimum is taken over all representations of T of that form. Let us notice that every nuclear operator is the limit (in the operator norm) of a sequence of finite-rank operators, so every nuclear operator is compact.

Recall that a Banach space X satisfies the approximation property (AP, for short) if for every compact subset K of X and for every $\varepsilon > 0$, there exists a finite-rank operator $T \colon X \longrightarrow X$ such that $\|T(x) - x\| \leqslant \varepsilon$ for every $x \in K$. It turns out that whenever X^* or Y has the approximation property, then $X^* \widehat{\otimes}_\pi Y = \mathcal{N}(X,Y)$. For a detailed account on tensor products and nuclear operators, we refer the reader to [11, 26].

The (N-fold) projective symmetric tensor product of X, denoted by $\widehat{\otimes}_{\pi,s,N} X$, is the completion of the linear space $\otimes_{\pi,s,N} X$, generated by $\{z^N : z \in X\}$, under the norm given by

$$\|z\|_{\pi,s,N} := \inf \left\{ \sum_{k=1}^{n} |\lambda_k| : z := \sum_{k=1}^{n} \lambda_k x_k^N, n \in \mathbb{N}, x_k \in S_X, \lambda_k \in \mathbb{K} \right\}, \tag{2.3}$$

where the infimum is taken over all the possible representations of z. Its topological dual $\left(\widehat{\otimes}_{\pi,s,N} X\right)^*$ can be identified (there exists an isometric isomorphism) with $\mathcal{P}(^N X)$. Indeed, every polynomial $P \in \mathcal{P}(^N X)$ acts as a linear functional on $\widehat{\otimes}_{\pi,s,N} X$ through its associated symmetric N-linear form $\overline{P}$ and satisfies

$$P(x) = \overline{P}(x, \ldots, x) = \langle P, x^N \rangle$$

for every $x \in X$. We also have that $B_{\widehat{\otimes}_{\pi,s,N} X} = \overline{\text{aco}}(\{x^N : x \in S_X\})$. To save notation, by a symmetric tensor we will refer to a generic element of $\widehat{\otimes}_{\pi,s,N} X$. For more information about symmetric tensor products, we send the reader to [14] and also to recent papers as [4, 5, 6].

Throughout the paper, we will be interested in studying the concepts of norm-attainment on $X \widehat{\otimes}_\pi Y$, $\mathcal{N}(X,Y)$, and $\widehat{\otimes}_{\pi,s,N} X$, meaning that their norms (2.1), (2.2), and (2.3) are respectively attained. More precisely, we have the following definitions, which will be our main notions in this paper:

(1) $z \in X \widehat{\otimes}_\pi Y$ *attains its projective norm* if there is a bounded sequence $(x_n, y_n) \subseteq X \times Y$ with $\sum_{n=1}^{\infty} \|x_n\| \|y_n\| < \infty$ such that $z = \sum_{n=1}^{\infty} x_n \otimes y_n$ and that $\|z\|_\pi = \sum_{n=1}^{\infty} \|x_n\| \|y_n\|$. In this case, we say that z is a *norm-attaining tensor.*

(2) $T \in \mathcal{N}(X,Y)$ *attains its nuclear norm* if there is a bounded sequence $(x_n^*, y_n) \subseteq X^* \times Y$ with $\sum_{n=1}^{\infty} \|x_n^*\| \|y_n\| < \infty$ such that $T = \sum_{n=1}^{\infty} x_n^* \otimes y_n$ and that

$\|T\|_{\mathcal{N}} = \sum_{n=1}^{\infty} \|x_n^*\|\|y_n\|$. In this case, we say that T is a *norm-attaining nuclear operator*.

(3) $z \in \widehat{\otimes}_{\pi,s,N} X$ *attains its projective symmetric norm* if there are bounded sequences $(\lambda_n)_{n=1}^{\infty} \subset \mathbb{K}$ and $(x_n)_{n=1}^{\infty} \subseteq B_X$ such that $\|z\|_{\pi,s,N} = \sum_{n=1}^{\infty} |\lambda_n|$ for $z = \sum_{n=1}^{\infty} \lambda_n x_n^N$. In this case, we say that z is a *norm-attaining symmetric tensor*.

When there is no confusion of misunderstanding and it is clear on what spaces we are working with, we denote the norms $\|\cdot\|_{\pi}$, $\|\cdot\|_{\mathcal{N}}$, and $\|\cdot\|_{\pi,s,N}$ simply by $\|\cdot\|$. Therefore, we set

(i) $\mathrm{NA}_{\pi}(X \widehat{\otimes}_{\pi} Y) = \Big\{ z \in X \widehat{\otimes}_{\pi} Y : z \text{ attains its projective norm} \Big\}$,

(ii) $\mathrm{NA}_{\mathcal{N}}(X, Y) = \Big\{ T \in \mathcal{N}(X, Y) : T \text{ attains its nuclear norm} \Big\}$,

(iii) $\mathrm{NA}_{\pi,s,N}(X) = \Big\{ z \in \widehat{\otimes}_{\pi,s,N} X : z \text{ attains its symmetric norm} \Big\}$.

Recall that a subspace Y of a Banach space X is said to be an ideal of X if for every finite-dimensional subspace E of X and every $\varepsilon > 0$, there is a linear operator $T \in \mathcal{L}(E, Y)$ such that $T(e) = e$ for every $e \in E \cap Y$ and $\|T\| \leqslant 1 + \varepsilon$. Let us notice that 1-complemented subspaces are ideals and that the concept of being an ideal of X coincides with the one of locally complemented subspace of X (see [20]). The following result is motivated by [24, Theorem 1.(i)], where the author proves that if X and Z are Banach spaces and Y is an ideal of Z, then $X \widehat{\otimes}_{\pi} Y$ is a subspace of $X \widehat{\otimes}_{\pi} Z$ and it is an ideal. In what follows, $\widehat{\otimes}_{\pi,s,N} Y$ being an isometric subspace means that if we consider the natural embedding of it into $\widehat{\otimes}_{\pi,s,N} X$, then the norms in $\widehat{\otimes}_{\pi,s,N} Y$ and $\widehat{\otimes}_{\pi,s,N} X$ coincide on $\widehat{\otimes}_{\pi,s,N} Y$.

Theorem 2.1. *Let X be a Banach space and Y an ideal of X. Then, $\widehat{\otimes}_{\pi,s,N} Y$ is an isometric subspace of $\widehat{\otimes}_{\pi,s,N} X$.*

Proof. Notice first that, by a denseness argument, it is enough to prove the theorem for $z = \sum_{i=1}^{n} \lambda_i y_i \in \otimes_{\pi,s,N} Y \subseteq \otimes_{\pi,s,N} X$. By the definition of the norm (see (2.3)), we have that $\|z\|_{\widehat{\otimes}_{\pi,s,N} X} \leqslant \|z\|_{\widehat{\otimes}_{\pi,s,N} Y}$. Now, let us prove the other inequality.

Let $\varepsilon > 0$ be given. Since the norm on a symmetric tensor product is finitely generated (see [14, Subsection 2.2]), there exists a finite-dimensional subspace F of X containing $\{y_1, \ldots, y_n\}$ such that $\|z\|_{\widehat{\otimes}_{\pi,s,N} F} < \|z\|_{\widehat{\otimes}_{\pi,s,N} X} + \varepsilon$. Since Y is an ideal in X, there exists a linear operator $T \in \mathcal{L}(F, Y)$ such that $\|T\| \leqslant \sqrt[N]{1+\varepsilon}$ and $T(y_i) = y_i$ for every $i = 1, \ldots, n$. Let us define $T^N \in \mathcal{L}(\widehat{\otimes}_{\pi,s,N} F, \widehat{\otimes}_{\pi,s,N} Y)$ by $T^N(m^N) := T(m)^N$ for every $m \in F$. This operator is well-defined and satisfies $\|T^N\| = \|T\|^N \leqslant 1 + \varepsilon$ (see [14, Subsection 2.2]). Therefore, we have that

$$\sum_{i=1}^{n} \lambda_i y_i^N = \sum_{i=1}^{n} \lambda_i T(y_i)^N = \sum_{i=1}^{n} \lambda_i T^N(y_i^N) = T^N \left(\sum_{i=1}^{n} \lambda_i y_i^N \right)$$

and then

$$\begin{aligned}\left\|\sum_{i=1}^{n}\lambda_i y_i^N\right\|_{\widehat{\otimes}_{\pi,s,N}Y} = \left\|T^N\left(\sum_{i=1}^{n}\lambda_i y_i^N\right)\right\|_{\widehat{\otimes}_{\pi,s,N}Y} &\leqslant \|T^N\|\left\|\sum_{i=1}^{n}\lambda_i y_i^N\right\|_{\widehat{\otimes}_{\pi,s,N}F}\\ &\leqslant (1+\varepsilon)\|z\|_{\widehat{\otimes}_{\pi,s,N}F}\\ &< (1+\varepsilon)(\|z\|_{\widehat{\otimes}_{\pi,s,N}X}+\varepsilon).\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $\|z\|_{\widehat{\otimes}_{\pi,s,N}Y} \leqslant \|z\|_{\widehat{\otimes}_{\pi,s,N}X}$ and we are done. □

We will be using also the following straightforward fact.

LEMMA 2.2. *Let X be a Banach space. Let $\varepsilon > 0$ and $x, y \in S_X$. Then,*

$$\|x^N - y^N\|_{\widehat{\otimes}_{\pi,s,N}X} \leqslant \frac{N^{N+1}}{N!}\|x-y\|.$$

Proof. By the polarization constant (see [14, Subsection 2.3]), we have that

$$\|x^N - y^N\|_{\widehat{\otimes}_{\pi,s,N}X} \leqslant \frac{N^N}{N!}\|x^N - y^N\|_{X\widehat{\otimes}_\pi X\ldots\widehat{\otimes}_\pi X}.$$

Now, let us notice that

$$x^N - y^N = \sum_{k=1}^{N} x^{N-k}\otimes(x-y)\otimes y^{k-1}.$$

This proves the statement since

$$\|x^N - y^N\|_{X\widehat{\otimes}_\pi\ldots\widehat{\otimes}_\pi X} \leqslant \sum_{k=1}^{N}\|x\|^{N-k}\|x-y\|\|y\|^{k-1} = N\|x-y\|.$$ □

3. Results for symmetric tensor products. In this section we show that there are symmetric tensors that *do not* attain their norms and study the denseness problem for norm-attaining elements in $\widehat{\otimes}_{\pi,s,N}X$. We start by giving a relation between the concepts of norm-attainment for symmetric tensors and N-homogeneous polynomials.

THEOREM 3.1. *Let X be a Banach space and suppose that every element in $\widehat{\otimes}_{\pi,s,N}X$ attains its norm. Then the set of all N-homogeneous polynomials that attain their norms is dense in the space of all N-homogeneous polynomials. In other words,*

$$\overline{\mathrm{NA}(^N X)} = \mathcal{P}(^N X).$$

In order to prove Theorem 3.1, we present a characterization for elements of $\widehat{\otimes}_{\pi,s,N}X$ to attain their norms. We have the following result, which is the counterpart of [10, Theorem 3.1] for symmetric tensors and homogeneous polynomials. We denote by $\mathrm{sign}(\lambda)$ the complex number $\frac{\overline{\lambda}}{|\lambda|}$ for each $\lambda \in \mathbb{C}\setminus\{0\}$.

LEMMA 3.2. *Let X be a Banach space and let*

$$z = \sum_{n=1}^{\infty} \lambda_n x_n^N \in \widehat{\otimes}_{\pi,s,N} X$$

where $\lambda_n \in \mathbb{C} \setminus \{0\}$ and $(x_n)_{n=1}^{\infty} \subseteq S_X$. Then, the following statements are equivalent.

(1) *$\|z\| = \sum_{n=1}^{\infty} |\lambda_n|$; in other words, $z \in \mathrm{NA}_{\pi,s,N}(X)$.*

(2) *There exists $P \in S_{\mathcal{P}(^N X)}$ such that $P(x_n) = \mathrm{sign}(\lambda_n), \forall n \in \mathbb{N}$.*

(3) *Every $P \in S_{\mathcal{P}(^N X)}$ such that $P(z) = \|z\|$ satisfies $P(x_n) = \mathrm{sign}(\lambda_n), \forall n \in \mathbb{N}$.*

Proof. Let us suppose that (1) holds. Pick any $P \in \left(\widehat{\otimes}_{\pi,s,N} X\right)^* = \mathcal{P}(^N X)$ with $\|P\| = 1$ and $P(z) = \|z\|$. We have that

$$\sum_{n=1}^{\infty} |\lambda_n| = \|z\| = P(z) = \sum_{n=1}^{\infty} \lambda_n \mathrm{Re} P(x_n) \leqslant \sum_{n=1}^{\infty} |\lambda_n|,$$

which implies that $P(x_n) = \mathrm{sign}(\lambda_n)$ for every $n \in \mathbb{N}$. This shows that (3) holds. The implication (3) $\Rightarrow$ (2) is immediate. Assume now that (2) holds. Then, there exists $P \in \mathcal{P}(^N X)$ with $\|P\| = 1$ such that $P(x_n) = \mathrm{sign}(\lambda_n)$ for every $n \in \mathbb{N}$. So,

$$\sum_{n=1}^{\infty} |\lambda_n| \geqslant \|z\| \geqslant P(z) = \sum_{n=1}^{\infty} \lambda_n P(x_n) = \sum_{n=1}^{\infty} |\lambda_n|$$

and this implies $\|z\| = \sum_{n=1}^{\infty} |\lambda_n|$. Therefore, (2) implies (1). □

By using Lemma 3.2 above, we can now prove Theorem 3.1.

Proof of Theorem 3.1. Let $\varepsilon > 0$ and $P \in \mathcal{P}(^N X) = (\widehat{\otimes}_{\pi,s,N} X)^*$ with $\|P\| = 1$ be given. By the Bishop-Phelps theorem for the Banach space $\widehat{\otimes}_{\pi,s,N} X$, there are $P_0 \in \mathcal{P}(^N X)$ with $\|P_0\| = 1$ and $z_0 \in S_{\widehat{\otimes}_{\pi,s,N} X}$ such that

$$P(z_0) = 1 \quad \text{and} \quad \|P_0 - P\| < \varepsilon.$$

By hypothesis we have that $z_0 \in \mathrm{NA}_{\pi,s,N}(X)$. So, there are $(\lambda_n)_{n=1}^{\infty} \subseteq \mathbb{R} \setminus \{0\}$ and $(x_n)_{n=1}^{\infty} \subseteq S_X$ such that $\|z_0\| = \sum_{n=1}^{\infty} |\lambda_n|$ for $z_0 = \sum_{n=1}^{\infty} \lambda_n x_n^N$. By Lemma 3.2, $P_0(x_n) = \mathrm{sign}(\lambda_n)$ for every $n \in \mathbb{N}$. In particular, $P_0 \in \mathrm{NA}(^N X)$ and we are done. □

Now we are able to present some examples where there exist symmetric tensors z which do not attain their norms.

REMARK 3.3. It is known (see [1, 18]) that if $X = d_*(w, 1)$ with $w \in \ell_2 \setminus \ell_1$, the predual of the Lorentz sequence space, then the set $\mathcal{P}(^N X)$, for $N \geqslant 2$, of all norm-attaining N-homogeneous polynomials on X, is *not* dense in $\mathcal{P}(^N X)$. Thus, Theorem 3.1 implies that there exists an element z in $\widehat{\otimes}_{\pi,s,N} X$ which does not attain its norm.

In contrast to Remark 3.3, when X is finite-dimensional, we do have that *every* symmetric tensor is norm-attaining (we send the reader also to Theorem 4.1 for an analogous phenomenon on projective tensor products). Its proof can be obtained by arguing as in [10, Proposition 3.5] with the aid of the fact that a convex hull of a compact set in a finite dimensional space is again compact and that $B_{\widehat{\otimes}_{\pi,s,N}X} = \overline{\mathrm{aco}}(\{x^N : x \in S_X\})$.

PROPOSITION 3.4. *Let X be a finite dimensional Banach space. Then, every symmetric tensor attains its projective symmetric tensor norm. In other words,*

$$\mathrm{NA}_{\pi,s,N}(X) = \widehat{\otimes}_{\pi,s,N}X.$$

As promised, we shall investigate when it is possible to approximate an arbitrary element $z \in \widehat{\otimes}_{\pi,s,N}X$ by a norm-attaining symmetric tensor. Similarly to what it is done in [10], this is achieved under the assumption that X contains "many" 1-complemented subspaces.

DEFINITION 3.5. Let X be a Banach space. We say that X has the metric π-property if given $\varepsilon > 0$ and $\{x_1, \ldots, x_n\} \subseteq S_X$, we can find a finite dimensional 1-complemented subspace $M \subseteq X$ and $x_i' \in M$ with $\|x_i - x_i'\| < \varepsilon$ for every $i = 1, \ldots, n$.

We invite the reader to [8] (and also to [19, 22]) for more information about π-properties. Moreover, [10, Example 4.12] sums up known examples of Banach spaces satisfying the metric π-property. Just to name a few, it is known that L_p-spaces, L_1-predual spaces, and Banach spaces with a finite dimensional decomposition with decomposition constant 1 satisfy such a property. Now, we present the following result analogous to [10, Theorem 4.8].

THEOREM 3.6. *Let X be a Banach space with the metric π-property. Then, every symmetric tensor can be approximated by symmetric tensors which attain their norms. In other words,*

$$\overline{\mathrm{NA}_{\pi,s,N}(X)}^{\|\cdot\|_{\pi,s,N}} = \widehat{\otimes}_{\pi,s,N}X.$$

Proof. Let $u \in S_{\widehat{\otimes}_{\pi,s,N}X}$ and $\varepsilon > 0$ be given. There are $(\lambda_n)_{n=1}^{\infty} \subseteq \mathbb{R} \setminus \{0\}$ and $(x_n)_{n=1}^{\infty} \subseteq S_X$ such that

$$u = \sum_{n=1}^{\infty} \lambda_n x_n^N \quad \text{and} \quad \sum_{n=1}^{\infty} |\lambda_n| < 1 + \varepsilon.$$

Find $k \in \mathbb{N}$ large enough such that

$$\|u - z\| < \frac{\varepsilon}{2} \quad \text{for} \quad z = \sum_{n=1}^{k} \lambda_n x_n^N \in \otimes_{\pi,s,N}X.$$

Since X has the metric π-property, we can find a finite dimensional space M of X which is 1-complemented and such that, for every $n \in \{1, \ldots, k\}$, there exists $x_n' \in M$ such that

$$\|x_n - x_n'\| < \frac{N!}{N^{N+1}} \cdot \frac{\varepsilon}{4}.$$

Define $z' := \sum_{n=1}^{k} \lambda_n (x_n')^N \in \otimes_{\pi,s,N} M$. Since M is finite dimensional, by Proposition 3.4 we have $z' \in \mathrm{NA}_{\pi,s,N}(M)$, and, since $\|z'\|_{\widehat{\otimes}_{\pi,s,N} M} = \|z'\|_{\widehat{\otimes}_{\pi,s,N} X}$, we also have $z' \in \mathrm{NA}_{\pi,s,N}(X)$. Finally, by using Lemma 2.2, we have that

$$\|z' - z\| = \left\| \sum_{n=1}^{k} \lambda_n \Big((x_n')^N - x_n^N \Big) \right\| \leqslant \sum_{n=1}^{k} |\lambda_n| \|(x_n')^N - x_n^N\| < \frac{\varepsilon}{2}.$$

Therefore, $\|z' - u\| < \varepsilon$ and we are done. □

Before proceeding, let us use Theorem 3.6 to point out the following observation on the hypothesis of Theorem 3.1.

REMARK 3.7. In Theorem 3.1, the assumption that every element of $\widehat{\otimes}_{\pi,s,N} X$ attains its norm *cannot* be relaxed to the case that $\mathrm{NA}_{\pi,s,N}(X)$ is dense in $\widehat{\otimes}_{\pi,s,N} X$. Indeed, if $X = d_*(w, 1)$ with $w \in \ell_2 \setminus \ell_1$, then X has monotone symmetric basis (see, for instance, [29, Proposition 2.2] and [18, Lemma 2.2]) and, therefore, satisfies the metric π-property (see, for instance, [10, Example 4.12]), which implies that $\mathrm{NA}_{\pi,s,N}(X)$ is dense in $\widehat{\otimes}_{\pi,s,N} X$ by Theorem 3.6. On the other hand, as we already have mentioned in Remark 3.3, the set of all norm-attaining N-homogeneous polynomials is not dense in $\mathcal{P}(^N X)$ for $N \geqslant 2$.

Our next goal will be obtaining the following result on the denseness of norm-attaining elements in $\widehat{\otimes}_{\pi,s,N} X^*$ under the hypothesis of Radon-Nikodým property (for short, RNP), see Theorem 4.5 and Corollary 4.6 for its counterpart for nuclear operators and projective tensor products, respectively.

THEOREM 3.8. *Let X be a Banach space. Suppose that X^* has the RNP and the AP. Then, every symmetric tensor in $\widehat{\otimes}_{\pi,s,N} X^*$ can be approximated by symmetric tensors that attain their norms. In other words,*

$$\overline{\mathrm{NA}_{\pi,s,N}(X^*)}^{\|\cdot\|_{\pi,s,N}} = \widehat{\otimes}_{\pi,s,N} X^*.$$

In order to prove Theorem 3.8, we need two preliminary results. Let us start with the following general lemma for spaces satisfying the RNP, which will also be used to prove Theorem 4.5.

LEMMA 3.9. *Let X be a Banach space with the RNP. Then*

$$A := \left\{ x = \sum_{i=1}^{n} \lambda_i x_i \in X : \lambda_1, \ldots, \lambda_n > 0, x_1, \ldots, x_n \in \operatorname{ext}(B_X), \|x\| = \sum_{i=1}^{n} \lambda_i \right\}$$

is dense in X.

Proof. Let $x_0 \in S_X$. Pick $x^* \in S_{X^*}$ to be such that $x^*(x_0) = 1$. Now, let us consider the closed convex set $C := \{x \in B_X : x^*(x) = 1\}$. Since X has the RNP, we have that $C = \overline{\text{co}}\,\text{ext}\,(C)$. Moreover, C is a face of B_X and so $\text{ext}\,(C) \subseteq \text{ext}\,(B_X)$. Thus, $x_0 \in \overline{\text{co}}\{x \in \text{ext}\,(B_X) : x^*(x) = 1\}$. To conclude, it suffices to check that

$$\text{co}\{x \in \text{ext}\,(B_X) : x^*(x) = 1\} \subseteq A.$$

To this end, take $v = \sum_{i=1}^n \lambda_i x_i$, where $x_i \in \text{ext}\,(B_X)$, $x^*(x_i) = 1$ and $\lambda_i > 0$ for all $i = 1, \ldots, n$, and $\sum_{i=1}^n \lambda_i = 1$. Then,

$$1 \geqslant \|v\| \geqslant \langle x^*, v\rangle = \sum_{i=1}^n \lambda_i = 1$$

and so $v \in A$. A straightforward homogeneity argument allows us to restrict the assumption $\|x_0\| = 1$, and the lemma is proved. □

We also need the following result, which is a consequence of Lemma 3.2 and Lemma 3.9.

LEMMA 3.10. *Let X be a Banach space. Assume that $\widehat{\otimes}_{\pi,s,N} X$ has the RNP and that $\text{ext}\left(B_{\widehat{\otimes}_{\pi,s,N} X}\right) \subseteq \{\pm x^N : x \in B_X\}$. Then, every symmetric tensor can be approximated by symmetric tensors which attain their norms. In other words,*

$$\overline{\text{NA}_{\pi,s,N}(X)}^{\|\cdot\|_{\pi,s,N}} = \widehat{\otimes}_{\pi,s,N} X.$$

Proof. By Lemma 3.9, the set

$$\begin{aligned} A &= \left\{ z = \sum_{i=1}^n \varepsilon_i \lambda_i x_i^N \in \widehat{\otimes}_{\pi,s,N} X : \varepsilon_i \in \{1,-1\}, \lambda_i > 0, x_i \in S_X, \|z\| = \sum_{i=1}^n \lambda_i \right\} \\ &= \left\{ z = \sum_{i=1}^n \lambda_i x_i^N \in \widehat{\otimes}_{\pi,s,N} X : \lambda_i \in \mathbb{R}, x_i \in S_X, \|z\| = \sum_{i=1}^n |\lambda_i| \right\} \end{aligned}$$

is dense in X. Clearly, $A \subseteq \text{NA}_{\pi,s,N}(X)$. □

Now we are ready to prove Theorem 3.8.

Proof of Theorem 3.8. Let us observe first that if X^* has the RNP and the AP, then $\widehat{\otimes}_{\pi,s,N} X^*$ has the RNP. Indeed, by using [14, Subsection 2.3], we have that $\widehat{\otimes}_{\pi,s,N} X^*$ is isomorphic to a subspace of $X^* \widehat{\otimes}_\pi \ldots \widehat{\otimes}_\pi X^*$. Since X^* has the RNP and AP, we have that $X^* \widehat{\otimes}_\pi \ldots \widehat{\otimes}_\pi X^*$ has the RNP (see [12, Theorem VIII.4.7]) and then we can conclude that $\widehat{\otimes}_{\pi,s,N} X^*$ has the RNP. Now by using [4, Proposition 1], we have that

$$\text{ext}\left(B_{\widehat{\otimes}_{\pi,s,N} X^*}\right) = \text{ext}\left(B_{(\widehat{\otimes}_{\varepsilon,s,N} X)^*}\right) \subseteq \{\pm \varphi^N : \varphi \in X^*, \|\varphi\| = 1\}$$

and by Lemma 3.10, the set $\mathrm{NA}_{\pi,s,N}(X^*)$ is dense in $\widehat{\otimes}_{\pi,s,N}X^*$, as desired. $\square$

Notice that if X^* has the RNP, then $\mathcal{P}_I(^N X)$, the Banach space of all N-homogeneous integral polynomials on X, has the RNP. Consequently, $\widehat{\otimes}_{\varepsilon,s,N}X$ cannot contain an isomorphic copy of ℓ_1, which in turn implies that $\mathcal{P}_I(^N X)$ is isometrically isomorphic to the Banach space $\mathcal{P}_{\mathrm{nu}}(^N X)$ of all N-homogeneous nuclear polynomials on X (see [4, Theorem 2]).

Theorem 3.11. *Let X be a Banach space. Suppose that $\widehat{\otimes}_{\varepsilon,s,N}X$ does not contain a copy of ℓ_1. Then the set of norm-attaining elements in $\mathcal{P}_{nu}(^N X)$ is w^*-dense in $\mathcal{P}_{nu}(^N X) = (\widehat{\otimes}_{\varepsilon,s,N}X)^*$.*

Proof. Let $P \in S_{\mathcal{P}_{\mathrm{nu}}(^N X)}$ be given. By the Bishop-Phelps theorem [3], given $\varepsilon > 0$, we can find $P_0 \in S_{(\widehat{\otimes}_{\varepsilon,s,N}X)^*}$ so that P_0 attains its norm at some $u_0 \in S_{\widehat{\otimes}_{\varepsilon,s,N}X}$ and $\|P_0 - P\| < \varepsilon$. We will prove that P_0 can be approximated by norm-attaining elements in $\mathcal{P}_{\mathrm{nu}}(^N X)$ in the w^*-topology. For this, let us consider the set

$$C := \Big\{Q \in B_{(\widehat{\otimes}_{\varepsilon,s,N}X)^*} : \langle Q, u_0\rangle = 1\Big\}.$$

Notice that C is a w^*-compact and convex set. It follows from Krein-Milman theorem (see, for instance, [2, Theorem 7.68]) that $C = \overline{\mathrm{co}}^{w^*}(\mathrm{ext}\,(C))$. As C is a face of $B_{(\widehat{\otimes}_{\varepsilon,s,N}X)^*}$, due to [4, Proposition 1], we have that

$$C \subseteq \overline{\mathrm{co}}^{w^*}\Big(\Big\{ \pm (x^*)^N : x^* \in S_{X^*},\ \langle \pm(x^*)^N, u_0\rangle = 1\Big\}\Big).$$

It follows from Lemma 3.2 that P_0 can be approximated by norm-attaining elements in $\mathcal{P}_{\mathrm{nu}}(^N X)$ in the w^*-topology and we are done. $\square$

Recall that $\mathcal{P}_{\mathrm{nu}}(^N X)$ coincides with $\widehat{\otimes}_{\pi,s,N}X^*$ isometrically whenever X^* has the AP. It is known that the James-Hagler space JH is an example of a Banach space whose dual does not have the RNP while the symmetric injective tensor product $\widehat{\otimes}_{\varepsilon,s,N}JH$ does not contain a copy of ℓ_1 (see [15]). Thus, the assumption in Corollary 3.12 below is strictly weaker than that of Theorem 3.8.

Corollary 3.12. *Let X be a Banach space such that X^* has the AP. If $\widehat{\otimes}_{\varepsilon,s,N}X$ does not contain a copy of ℓ_1, then the set $\mathrm{NA}_{\pi,s,N}(X^*)$ is w^*-dense in $\widehat{\otimes}_{\pi,s,N}X^*$.*

Let us observe that so far we have presented only positive results on the (w^*-) denseness of symmetric tensors which attain their norms in $\widehat{\otimes}_{\pi,s,N}X$. In fact, we do not know whether the set $\mathrm{NA}_{\pi,s,N}(X)$ is dense in $\widehat{\otimes}_{\pi,s,N}X$ for *every* Banach space X. The first candidate that would pop up in our minds would be a Banach space X such that the set $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi X)$ is not dense in $X\widehat{\otimes}_\pi X$. Nevertheless, the techniques from [10, Section 5] (where the authors show that there exist subspaces X of c_0 and Y of the Read's space $\mathcal{R}$ such that the set $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y^*)$ is not dense) do not seem to work. Indeed, the idea behind was requiring that every element of $\mathrm{NA}(X, Y^*)$

has finite rank, and then working with a bounded operator $T : X \longrightarrow Y$ which can not be approximated by finite rank operators from the failure of the approximation property. This construction is doable since X and Y are isomorphic and then T can be taken as a formal identity thanks to classical results on AP. However, for one such example of the form $X\widehat{\otimes}_\pi X$, we would need to work with an operator $T : X \longrightarrow X^*$, for a certain subspace X of c_0, which is not approximable by finite rank operators and, to the best of our knowledge, the existence of such a space X and such a T is unknown. Despite that, we shall conclude this section by showing that this open problem is separably determined.

THEOREM 3.13. *Let $N \in \mathbb{N}$ be fixed. If $\mathrm{NA}_{\pi,s,N}(Y)$ is dense in $\widehat{\otimes}_{\pi,s,N}Y$ for every separable Banach space Y, then $\mathrm{NA}_{\pi,s,N}(X)$ is dense in $\widehat{\otimes}_{\pi,s,N}X$ for every Banach space X.*

Proof. Let X be a Banach space, $z \in \widehat{\otimes}_{\pi,s,N}X$, and let $\varepsilon > 0$ be given. Choose a representation $z = \sum_{n=1}^{\infty} \lambda_n x_n^N$ with $(\lambda_n)_{n=1}^{\infty} \subseteq \mathbb{R} \setminus \{0\}$ and $(x_n)_{n=1}^{\infty} \subseteq S_X$ satisfying that $\sum_{n=1}^{\infty} |\lambda_n| < \|z\|_{\widehat{\otimes}_{\pi,s,N}X} + \varepsilon$. Let $Z := \overline{\operatorname{span}}\{x_n : n \in \mathbb{N}\}$. Thus, Z is a separable Banach space. By [28, Proposition 2] (see also [17, Lemma 4.3]), there exists a separable ideal Y of X such that $Z \subseteq Y$. As Y is an ideal, by Theorem 2.1, we have that $\|z\|_{\widehat{\otimes}_{\pi,s,N}X} = \|z\|_{\widehat{\otimes}_{\pi,s,N}Y}$. By the hypothesis, there exists $z' = \sum_{n=1}^{\infty} \mu_n y_n^N \in \mathrm{NA}_{\pi,s,N}(Y)$ with $\|z'\|_{\widehat{\otimes}_{\pi,s,N}Y} = \sum_{n=1}^{\infty} |\mu_n|$ which satisfies that $\|z - z'\|_{\widehat{\otimes}_{\pi,s,N}Y} < \varepsilon$. Considering z' as an element of $\widehat{\otimes}_{\pi,s,N}X$, we notice that

$$\sum_{n=1}^{\infty} |\mu_n| = \|z'\|_{\widehat{\otimes}_{\pi,s,N}Y} = \|z'\|_{\widehat{\otimes}_{\pi,s,N}X} \leqslant \sum_{n=1}^{\infty} |\mu_n|,$$

which implies that $z' \in \mathrm{NA}_{\pi,s,N}(X)$. Finally, $\|z-z'\|_{\widehat{\otimes}_{\pi,s,N}X} = \|z-z'\|_{\widehat{\otimes}_{\pi,s,N}Y} < \varepsilon$. □

4. Results for projective tensor products. In this section, we present some results on the denseness of tensors in projective tensor products of Banach spaces.

Let us first notice that when $X = L_1(\mathbb{T})$, where the unit circle $\mathbb{T}$ is equipped with the Haar measure, and Y is the two-dimensional Hilbert space ℓ_2^2, we have that $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) \neq X\widehat{\otimes}_\pi Y$ [10, Example 3.12 (a)]. This shows that finite dimensionality on just one of the factors is not enough to guarantee that every tensor in $X\widehat{\otimes}_\pi Y$ is norm-attaining. Nevertheless, we have the following result.

THEOREM 4.1. *Let X be a Banach space with $B_X = \operatorname{co}(\{x_1, \ldots, x_n\})$ for some $x_1, \ldots, x_n \in S_X$ and assume that Y is a dual space. Then, every tensor in $X\widehat{\otimes}_\pi Y$ attains its projective tensor norm. In other words,*

$$\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) = X\widehat{\otimes}_\pi Y.$$

Proof. Let us assume that $Y = Z^*$. Notice first that since X is finite dimensional, we have that $\mathcal{L}(X, Z) = \mathcal{K}(X, Z) = X^*\widehat{\otimes}_\varepsilon Z$ and then $\mathcal{L}(X, Z)^* = X\widehat{\otimes}_\pi Z^*$. We will use this fact to prove the following statement.

Claim: The set $\{x_i\} \otimes B_{Z^*}$ is a w^*-compact convex subset of $X\widehat{\otimes}_\pi Z^*$ for each $i = 1, \ldots, n$.

Indeed, for each $i = 1, \ldots, n$, let us take $T_i\colon \mathcal{L}(X, Z) \longrightarrow Z$ to be defined by $T_i(T) := T(x_i)$ for every $T \in \mathcal{L}(X, Z)$. Therefore, its adjoint operator $T_i^*\colon Z^* \longrightarrow \mathcal{L}(X, Z)^* = X\widehat{\otimes}_\pi Z^*$ satisfies $T_i^*(z^*) = x_i \otimes z^*$ for every $z^* \in Z^*$ and $i = 1, \ldots, n$. This implies that $T_i^*(B_{Z^*}) = x_i \otimes B_{Z^*}$ and since T^* is w^*-w^* continuous, we can conclude that $\{x_i\} \otimes B_{Z^*}$ is w^*-compact convex in $X\widehat{\otimes}_\pi Z^*$.

Thus, $A := \operatorname{co}\left(\bigcup_{i=1}^n x_i \otimes B_{Z^*}\right)$ is w^*-compact as being the convex hull of a finite number of w^*-compact convex sets (see, for instance, [2, Lemma 5.29]). So, in particular, A is w^*-closed and then norm-closed. Finally, if $z \in B_X \otimes B_{Z^*}$, then there are $\lambda_i > 0$ with $\sum_{i=1}^n \lambda_i = 1$ such that

$$z = \left(\sum_{i=1}^n \lambda_i x_i\right) \otimes y = \sum_{i=1}^n \lambda_i x_i \otimes y.$$

Therefore, $B_X \otimes B_{Z^*} \subseteq A$ and by taking convex hulls, we get $A = B_{X\widehat{\otimes}_\pi Z^*} = B_{X\widehat{\otimes}_\pi Y}$. In particular, every element of $S_{X\widehat{\otimes}_\pi Z^*}$ can be written as a finite convex combination of basic tensors in $B_X \otimes B_{Z^*}$, so it is norm attaining. □

Recall a Banach space X is said to be polyhedral if the unit ball of every finite-dimensional subspace is a polytope, that is, the convex hull of a finite set. We can use Theorem 4.1 to get the following denseness result.

Theorem 4.2. *Let X be a Banach which is polyhedral and satisfies the metric π-property. Assume that Y is a dual space. Then, every tensor in $X\widehat{\otimes}_\pi Y$ can be approximated by tensors that attain their norms. In other words,*

$$\overline{\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X\widehat{\otimes}_\pi Y.$$

Proof. Let $u \in S_{X\widehat{\otimes}_\pi Y}$ and $\varepsilon \in (0, 1)$ be given. Then, there exist sequences $(\lambda_n) \subseteq \mathbb{R}^+$, $(x_n) \subseteq S_X$, and $(y_n) \subseteq S_Y$ with $u = \sum_{n=1}^\infty \lambda_n x_n \otimes y_n$ and $\sum_{n=1}^\infty \lambda_n < 1 + \varepsilon$. We may find $k \in \mathbb{N}$ so that $\|u - z\| < \frac{\varepsilon}{2}$, where $z := \sum_{n=1}^k \lambda_n x_n \otimes y_n$. Since X satisfies the metric π-property, we can find a finite-dimensional subspace M of X which is 1-complemented and such that for every $n \in \{1, \ldots, k\}$, there exists $x_n' \in M$ such that $\|x_n - x_n'\| < \frac{\varepsilon}{4}$. Define $z' := \sum_{n=1}^k \lambda_n x_n' \otimes y_n$ and notice that

$$\begin{aligned} \|z' - z\| = \left\|\sum_{n=1}^k \lambda_n x_n' \otimes y_n - \sum_{n=1}^k \lambda_n x_n \otimes y_n\right\| &= \left\|\sum_{n=1}^k \lambda_n (x_n' - x_n) \otimes y_n\right\| \\ &\leqslant \sum_{n=1}^\infty \lambda_n \|x_n' - x_n\| \|y_n\| \\ &< \frac{\varepsilon}{4} \sum_{n=1}^k \lambda_n \\ &< \frac{\varepsilon}{2}. \end{aligned}$$

Notice now that $z' \in M\widehat{\otimes}_\pi Y$ and since M is 1-complemented in X, we have that $\|z'\|_{M\widehat{\otimes}_\pi Y} = \|z'\|_{X\widehat{\otimes}_\pi Y}$. Moreover, since X is a polyhedral, we have that B_M is equal to $\operatorname{co}\{x_1, \ldots, x_m\}$ for some $x_1, \ldots, x_m \in S_M$. Theorem 4.1 shows then that $z' \in \mathrm{NA}_\pi(M\widehat{\otimes}_\pi Y)$ and, therefore, $z' \in \mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$. Finally, $\|z' - u\| \leqslant \|z' - z\| + \|z - u\| < \varepsilon$. □

The Banach space c_0 endowed with $\|\cdot\|_\infty$ is the canonical example of a polyhedral space. So, we have the following immediate consequence of Theorem 4.2, which is not covered by [10, Theorem 4.8].

COROLLARY 4.3. *Let Y be a dual space. Then,* $\mathrm{NA}_\pi(c_0\widehat{\otimes}_\pi Y)$ *is dense in* $c_0\widehat{\otimes}_\pi Y$.

REMARK 4.4. Notice that in Theorem 4.2 the hypothesis of X having the metric π-property is essential. Indeed, in [10, Section 5] the authors show that if X is a closed subspace of c_0 (and hence polyhedral since this property is hereditary) failing the approximation property and $Y := (X, |||\cdot|||)$ is a renorming of X, where $|||\cdot|||$ is the norm that defines Read's space, then $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y^*)$ is not dense in $X\widehat{\otimes}_\pi Y^*$.

Next, we can prove the following result on the denseness of nuclear operators which attain their nuclear norms under the RNP assumption.

THEOREM 4.5. *Let X, Y be Banach spaces such that X^* and Y^* have the RNP. Then, every nuclear operator from X into Y^* can be approximated by norm-attaining nuclear operators. In other words,*

$$\overline{\mathrm{NA}_{\mathcal{N}}(X, Y^*)}^{\|\cdot\|_{\mathcal{N}}} = \mathcal{N}(X, Y^*).$$

Proof. Suppose that X^* and Y^* have the RNP. Then, $\mathcal{N}(X, Y^*) = (X\widehat{\otimes}_\varepsilon Y)^*$ also has the RNP (this is shown in [12, Theorem VIII.4.7, pg. 249] under the additional assumption that X^* or Y^* have the AP, which is only used to get $\mathcal{N}(X, Y^*) = X^*\widehat{\otimes}_\pi Y^*$). Also, $\operatorname{ext}\left(B_{\mathcal{N}(X,Y^*)}\right) = \operatorname{ext}\left(B_{(X\widehat{\otimes}_\varepsilon Y)^*}\right) \subseteq S_{X^*} \otimes S_{Y^*}$ (cf. [27]). By Lemma 3.9, the set

$$A = \left\{ T = \sum_{i=1}^{n} \lambda_i x_i^* \otimes y_i^* : \lambda_i > 0, x_i^* \in S_{X^*}, y_i^* \in S_{Y^*} \text{ for } i = 1, \ldots, n, \|T\| = \sum_{i=1}^{n} \lambda_i \right\}$$

is dense in $\mathcal{N}(X, Y^*)$. Clearly, $A \subseteq \mathrm{NA}_{\mathcal{N}}(X, Y^*)$. □

If we are under the hypotheses of Theorem 4.5 together with the extra assumption that one of the spaces has the approximation property, then the equality $X^*\widehat{\otimes}_\pi Y^* = \mathcal{N}(X, Y^*)$ holds (see, for instance, [25, Corollary 4.8]). By using Theorem 4.5, we get the following counterpart of Theorem 3.8 for non-symmetric tensors, which provides a positive answer for [10, Question 6.1] in the case that one of the spaces has the approximation property.

Corollary 4.6. *Let X, Y be Banach spaces such that X^* and Y^* have the RNP and at least one of them has the AP. Then, every tensor in $X^*\widehat{\otimes}_\pi Y^*$ can be approximated by tensors that attain their norms. In other words,*

$$\overline{\mathrm{NA}_\pi(X^*\widehat{\otimes}_\pi Y^*)}^{\|\cdot\|_\pi} = X^*\widehat{\otimes}_\pi Y^*.$$

Even if we weaken the assumption in Theorem 4.5 so that only X^* has the RNP, we are still able to obtain the denseness result, but in the w^*-topology, of the set of norm-attaining nuclear operators as in Theorem 3.11. Notice that $\mathcal{N}(X, Y^*)$ is identified with $(X\widehat{\otimes}_\varepsilon Y)^*$, the dual of injective tensor space, under the assumption that X^* has the RNP. Arguing in the same way as in the proof of Theorem 3.11 but using the fact that $\operatorname{ext}\left(B_{(X\widehat{\otimes}_\varepsilon Y)^*}\right) \subseteq S_{X^*} \otimes S_{Y^*}$ and [10, Theorem 3.2] instead of [4, Proposition 1] and Lemma 3.2, respectively, the following result can be obtained.

Theorem 4.7. *Let X be a Banach space such that X^* has the RNP. Then, the set $\mathrm{NA}_{\mathcal{N}}(X, Y^*)$ is w^*-dense in $\mathcal{N}(X, Y^*) = (X\widehat{\otimes}_\varepsilon Y)^*$ for any Banach space Y.*

Using the equality $X^*\widehat{\otimes}_\pi Y^* = \mathcal{N}(X, Y^*)$ provided that one of X^* or Y^* has the AP, we get the following immediate consequence of Theorem 4.7.

Corollary 4.8. *Let X, Y be Banach spaces such that X^* has the RNP and at least one of X^* or Y^* has the AP. Then, the set $\mathrm{NA}_\pi(X^*\widehat{\otimes}_\pi Y^*)$ is w^*-dense in $X^*\widehat{\otimes}_\pi Y^* = (X\widehat{\otimes}_\varepsilon Y)^*$.*

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ORCID

S. Dantas https://orcid.org/0000-0001-8117-3760

L.C. García-Lirola https://orcid.org/0000-0001-9211-4475

M. Jung https://orcid.org/0000-0003-2240-2855

A. Rueda Zoca https://orcid.org/0000-0003-0718-1353

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Smooth and Polyhedral Norms via Fundamental Biorthogonal Systems

Sheldon Dantas[1,*], Petr Hájek[2], and Tommaso Russo[2,3]

[1]Departament de Matemàtiques and Institut Universitari de Matemàtiques i Aplicacions de Castelló (IMAC), Universitat Jaume I, Campus del Riu Sec. s/n, 12071 Castelló, Spain, [2]Department of Mathematics, Faculty of Electrical Engineering, Czech Technical University in Prague, Technická 2, 166 27 Prague 6, Czech Republic, and [3]Institute of Mathematics, Czech Academy of Sciences, Žitná 25, 115 67 Prague 1, Czech Republic

[*]*Correspondence to be sent to: e-mail: dantas@uji.es*

Let $\mathcal{X}$ be a Banach space with a fundamental biorthogonal system, and let $\mathcal{Y}$ be the dense subspace spanned by the vectors of the system. We prove that $\mathcal{Y}$ admits a C^∞-smooth norm that locally depends on finitely many coordinates (LFC, for short), as well as a polyhedral norm that locally depends on finitely many coordinates. As a consequence, we also prove that $\mathcal{Y}$ admits locally finite, σ-uniformly discrete C^∞-smooth and LFC partitions of unity and a C^1-smooth locally uniformly rotund norm. This theorem substantially generalises several results present in the literature and gives a complete picture concerning smoothness in such dense subspaces. Our result covers, for instance, every weakly Lindelöf determined Banach space (hence, all reflexive ones), $L_1(\mu)$ for every measure μ, $\ell_\infty(\Gamma)$ spaces for every set Γ, $C(K)$ spaces where K is a Valdivia compactum or a compact Abelian group, duals of Asplund spaces, or preduals of Von Neumann algebras. Additionally, under Martin Maximum MM, all Banach spaces of density ω_1 are covered by our result.

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1 Introduction

The topic of smooth approximation is one of the classical themes in analysis. In the setting of Banach spaces the problem has several aspects, among which the existence of smooth partitions of unity, smooth extensions, approximation of continuous or Lipschitz functions by smooth ones, smooth renormings, approximation with smooth norms, study of polynomials and spaces of polynomials, and so on. For an introduction to these directions of research we refer to the monographs [15, 18, 28, 49].

It is by now a well-known fact that the existence of a smooth norm (or more generally a smooth bump) on a Banach space $\mathcal{X}$ has several deep structural consequences for the space. For example, the presence of a C^1-smooth bump implies that the space is Asplund [17]; the presence of an LFC bump yields that the space is a c_0-saturated Asplund space [22, 48]. If $\mathcal{X}$ admits a C^2-smooth bump, then either it contains a copy of c_0, or it is super-reflexive with type 2 [21]. Finally, if $\mathcal{X}$ admits a C^∞-smooth bump and it contains no copy of c_0, then it has exact cotype $2k$, for some $k \in \mathbb{N}$, and it contains ℓ_{2k} [12]. Each of these results involves at some point the completeness of the space $\mathcal{X}$, most frequently via the appeal to some form of variational principles, such as the Ekeland variational principle [16], Stegall's variational principle [56], the Borwein–Preiss smooth variational principle [8], or the compact variational principle [13]. It is therefore unclear whether any, possibly weaker, form of the above results could be valid for general normed spaces. In this direction, it was pointed out in [3, p. 96] that it is not known whether $\mathcal{X}$ is an Asplund space provided the set where its norm fails to be Fréchet differentiable is "small" in some sense (also see [25, Problem 148]). For example, it is unknown if there is a norm on ℓ_1 that is Fréchet differentiable outside a countable union of hyperplanes.

Nevertheless, some scattered results concerning normed spaces are present in the literature. Vanderwerff [60] proved that every normed space with a countable algebraic basis admits a C^1-smooth norm; this result was later improved to obtain a C^∞-smooth norm [27], a polyhedral norm [14], and an analytic one [9]. These results and the previous discussion motivated [25, Problem 149], [33], and recent research of the present authors [9], where the following problem was posed.

Problem 1.1. Let $\mathcal{X}$ be a Banach space and $k \in \mathbb{N} \cup \{\infty, \omega\}$. Is there a dense subspace $\mathcal{Y}$ of $\mathcal{X}$ that admits a C^k-smooth norm?

Although the problem is seemingly very general and ambitious, note that [14] answers it in the positive for $\mathcal{X}$ separable and $k = \omega$. Moreover, in [9] it was solved in the positive for ℓ_∞ and $k = \omega$, $\ell_1(\mathfrak{c})$ and $k = \omega$, and spaces with long unconditional bases and $k = \infty$. It is worth pointing out that in these results the density of smooth norms cannot be guaranteed in general. The main contribution of the present paper is a vast generalisation of the previous results by means of the following theorem.

Theorem A. Let $\mathcal{X}$ be a Banach space with a fundamental biorthogonal system $\{e_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma}$. Consider the dense subspace $\mathcal{Y}$ of $\mathcal{X}$ given by $\mathcal{Y} := \operatorname{span}\{e_\alpha\}_{\alpha\in\Gamma}$. Then,

(i) $\mathcal{Y}$ admits a polyhedral and LFC norm,
(ii) $\mathcal{Y}$ admits a C^∞-smooth and LFC norm,
(iii) $\mathcal{Y}$ admits a C^∞-smooth and LFC bump,
(vi) $\mathcal{Y}$ admits locally finite, σ-uniformly discrete C^∞-smooth and LFC partitions of unity,
(v) $\mathcal{Y}$ admits a C^1-smooth locally uniformly rotund (LUR) norm.

Moreover, norms as in (i), (ii), and (v) are dense in the set of all equivalent norms on $\mathcal{Y}$.

The main novel parts of Theorem A are claims (i) and (ii), whose proof constitutes the core of the paper and is presented in Section 3. Clauses (iii)–(v) follow from the former ones via known results or adaptations of known techniques. More precisely, (iii) is an obvious consequence of (ii), while the existence of locally finite, σ-uniformly discrete C^∞-smooth partitions of unity follows from (ii) via [36, Corollary 6] or [34, Theorem 2]. The fact that functions in the partition of unity can be chosen to be LFC follows by inspection of the proof of [34] as we will briefly discuss in Section 4. Finally, (v) follows from (ii) by using ideas from [31], and we shall explain this in Section 5.

Let us point out that Theorem A draws a complete picture concerning smoothness in the sense that it implies the existence of smooth norms, norm approximation by smooth norms, C^1-smooth LUR norms, and the existence of partitions of unity, which are instrumental for the smooth approximation of continuous or Lipschitz functions (see, e.g., [15, §VIII.3] or [28, Chapter 7]). The unique part of the result where one might ponder possible improvements is (v), where it is natural to ask whether $\mathcal{Y}$ admits C^k-smooth LUR norms for some $k \geqslant 2$. Nonetheless, this is not the case in general, since a normed space with a C^2-smooth LUR norm has super-reflexive completion (Theorem 5.2). Hence, in general it is not possible to replace C^1-smoothness with higher-order smoothness in (v) (even in the separable case). Although Theorem 5.2 is a more or less formal consequence of [21, Theorem 3.3(ii)], it is of notable importance in our context since it is one of the few instances where the existence of a smooth norm on an incomplete normed space bears structural consequences for the space.

As a simple consequence of Theorem A, we can obtain one further such instance. Indeed, if a Banach space $\mathcal{X}$ admits a fundamental biorthogonal system, then densely many norms on $\mathcal{X}$ are C^∞-smooth and LFC on a dense and open subset of $\mathcal{X}$ (Corollary 3.5). On the one hand, this result should be compared to the classical characterisation of Asplund spaces, that a Banach space $\mathcal{X}$ is Asplund if and only if every norm on $\mathcal{X}$ is Fréchet differentiable on a dense G_δ set. On the other hand, in the particular case of Banach spaces with fundamental biorthogonal systems, it generalises Moreno's result that every Banach space admits a norm that is Fréchet differentiable on a dense open set [44] (with Moreno's argument it doesn't seem possible to obtain the density of such norms).

We now discuss how general our results are and compare them to the literature. If we restrict our attention to separable Banach spaces, a classical result due to Markuševič [42] asserts that every separable Banach space admits an M-basis, hence Theorem A applies to every separable Banach space. Therefore, our result generalises simultaneously [27], where a C^∞-smooth LFC norm is constructed in every normed space with a countable algebraic basis, and [14], where a polyhedral norm is constructed in such spaces. Here we should observe that, for a normed space, admitting a countable algebraic basis is equivalent to being the linear span of the vectors of an M-basis, again by [42]. On the other hand, in [9] an analytic norm is also constructed in normed spaces with a countable algebraic basis, while in our result it is not possible in general to obtain analytic norms [9, Theorem 3.10].

For non-separable Banach spaces the problem has only been faced in [9, Theorem B], where a C^∞-smooth norm is constructed in the linear span of every long unconditional Schauder basis (and in [9, Theorem A], concerning the concrete spaces ℓ_∞ and $\ell_1(\mathfrak{c})$, as mentioned above). Once more, Theorem A is substantially stronger, since we additionally obtain an approximation result, the LFC condition, polyhedral norms, partitions of unity, and C^1-smooth LUR norms. Moreover, the assumption on the space is much more general, for the existence of an unconditional basis is a rather strong assumption, while the existence of a fundamental biorthogonal system is a much weaker one, as we now discuss.

A large class of Banach spaces that admit a fundamental biorthogonal system (even an M-basis) is the class of Plichko spaces [30, 37, 38]. Such a class of Banach spaces contains all weakly Lindelöf determined (WLD, for short) Banach spaces, hence all weakly compactly generated (hereinafter, WCG) spaces and in particular all reflexive ones; besides, every $L_1(\mu)$ space and every $C(K)$ space, where K is a Valdivia compactum or an Abelian compact group, is a Plichko space (see, e.g., [38, §6.2 and §5.1], [39]).

More generally, Kalenda [40] recently proved that every Banach space with a projectional skeleton admits a (strong) M-basis. Among the Banach spaces that admit a projectional skeleton we could additionally mention duals of Asplund spaces [41], preduals of Von Neumann algebras [6], or preduals of JBW*-triples [7]. Additionally, there are several examples of concrete Banach spaces where a fundamental biorthogonal system can be constructed, for example, $\ell_\infty(\Gamma)$ for every set Γ [11], $\ell_\infty^c(\Gamma)$ when $|\Gamma| \leqslant \mathfrak{c}$ [23, 51], or $C([0,\eta])$ for every ordinal η (this is standard; see, e.g., [40, Proposition 5.11]). More generally, $C(T)$ has an M-basis, for every tree T, [40, §5.3]. Moreover, it is proved in [11] that a Banach space $\mathcal{X}$ with $\operatorname{dens}\mathcal{X} = \kappa$ admits a fundamental biorthogonal system provided that $\mathcal{X}$ has a WCG quotient of density κ. Similarly, Plichko [50] proved that a Banach space $\mathcal{X}$ with $\operatorname{dens}\mathcal{X} = \kappa$ admits a fundamental biorthogonal system if and only if $\mathcal{X}$ has a quotient of density κ with a long Schauder basis. Finally, it is consistent with ZFC, and in particular true under Martin Maximum MM, that every Banach space of density ω_1 admits a fundamental biorthogonal system [57].

Let us add one more comment concerning the space ℓ_∞. On the one hand, in [9, Theorem 3.1], an analytic norm is constructed in the dense subspace of ℓ_∞ comprising all sequences that attain finitely many values; while Theorem A only gives a C^∞-smooth norm in a subspace with a less explicit description. On the other hand, Theorem A also yields the LFC condition, a polyhedral norm, a C^1-smooth LUR norm, and partitions of unity; additionally, it holds for $\ell_\infty(\Gamma)$ for every set Γ.

Several months after the present research was completed, the authors obtained the following result [10], related to Theorem A. If $1 \leqslant p < \infty$, the dense subspace $\mathcal{Y}_p := \bigcup_{0<q<p} \ell_q(\Gamma)$ of $\ell_p(\Gamma)$ admits a C^∞-smooth and LFC norm. The interest of the result is that $\mathcal{Y}_p$ is not the linear span of a biorthogonal system and it has linear dimension equal to that of $\ell_p(\Gamma)$.

Finally, recall that it is in general unknown if, in a non-separable Banach space with a C^k-smooth norm, C^k-smooth norms are dense in the set of all equivalent norms. Among the few results available in the literature let us mention [1, 5, 54, 55], where the problem is solved for spaces with a small boundary such as $c_0(\Gamma)$. In particular, the C^k-smooth approximation of norms is open in $\ell_2(\omega_1)$, or $C([0,\omega_1])$, while our Theorem A gives the C^∞-smooth approximation in some dense subspace of the said spaces.

Our paper is organised as follows: Section 2 contains the definitions of some notions that we will need and some auxiliary (known or folklore) results. Section 3 is devoted to the main part of the proof of Theorem A and we prove clauses (i) and (ii). A brief discussion of (iv) is given in Section 4. Finally, in Section 5 we discuss the existence of C^k-smooth LUR norms: we give the proof of Theorem A(v) borrowing our methods from [31] and we prove Theorem 5.2.

2 Preliminary Material

Our notation is standard as in, for example, [2, 15, 19]. Throughout the paper we always consider normed spaces over the reals. We use the calligraphic font $\mathcal{X}, \mathcal{Y}$ for infinite-dimensional normed spaces and we denote by $F, G, H, \ldots$ their finite-dimensional subspaces. For an infinite-dimensional normed space $\mathcal{X}$, we denote by $\mathcal{X}^*$, $\mathcal{S}_{\mathcal{X}}$, and $\mathcal{B}_{\mathcal{X}}$ the dual space, the unit sphere, and the closed unit ball respectively; accordingly, the unit sphere and ball of the finite-dimensional normed space F are $\mathcal{S}_F$ and $\mathcal{B}_F$, respectively. We use the calligraphic notation for the unit sphere since we keep the letter S for a generic slice as we will make extensive use of slices in our arguments. We write $\langle \varphi, x \rangle$ to denote the action of a functional $\varphi \in \mathcal{X}^*$ at a point $x \in \mathcal{X}$. When talking about norm approximations, hence in particular density of norms, we always refer to uniform approximation on bounded sets. More precisely, the assertion that a norm $\|\cdot\|$ on $\mathcal{X}$ can be approximated by norms with property P means that, for every $\varepsilon > 0$, there is a norm $|||\cdot|||$ on $\mathcal{X}$ with property P and such that $(1-\varepsilon)|||\cdot||| \leqslant \|\cdot\| \leqslant (1+\varepsilon)|||\cdot|||$.

2.1 Convexity and slices

A finite-dimensional normed space F is *polyhedral* if its unit ball is a polyhedron, that is, it is a finite intersection of closed half-spaces; an infinite-dimensional normed space is polyhedral if every its finite-dimensional subspace is so. A normed space $(\mathcal{X}, \|\cdot\|)$ is *locally uniformly rotund* (LUR, for short) if, for every $x \in \mathcal{S}_{\mathcal{X}}$ and every sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{S}_{\mathcal{X}}$ with $\|x_n + x\| \to 2$ one has $x_n \to x$. A *norming functional* for $x \in \mathcal{X}$ is a functional $\psi \in \mathcal{S}_{\mathcal{X}^*}$ such that $\langle \psi, x \rangle = \|x\|$. A *slice* of $\mathcal{B}_{\mathcal{X}}$ is a set of the form $S(\psi, \delta) := \{y \in \mathcal{B}_{\mathcal{X}} : \langle \psi, y \rangle > 1 - \delta\}$, for some $\psi \in \mathcal{S}_{\mathcal{X}^*}$ and $\delta > 0$. For us it will be convenient to consider slices $S(\psi, \delta)$ where the functional ψ attains its norm. Therefore, we write $S(x, \psi, \delta)$ to indicate the slice $S(\psi, \delta)$ whenever ψ attains its norm at x. A point $x \in \mathcal{S}_{\mathcal{X}}$ is *strongly exposed* if there is a norming functional ψ for x such that $\operatorname{diam}(S(x, \psi, \delta)) \to 0$, as $\delta \to 0^+$. It is a standard fact that if $\mathcal{X}$ is LUR then every point of $\mathcal{S}_{\mathcal{X}}$ is strongly exposed (see, for instance, [19, Problem 8.27]), namely we have the following:

Fact 2.1. Let $\mathcal{X}$ be an LUR normed space, $x \in \mathcal{S}_{\mathcal{X}}$, and $\psi \in \mathcal{S}_{\mathcal{X}^*}$ be a norming functional for x. Then $\operatorname{diam}(S(x, \psi, \delta)) \to 0$, as $\delta \to 0^+$.

We also collect here for future reference the following standard observation concerning slices; the proof is so simple that we include it here for the sake of completeness.

Fact 2.2. Let $\mathcal{X}$ be a normed space and let $\psi \in \mathcal{S}_{\mathcal{X}^*}$ attain its norm at $x \in \mathcal{S}_{\mathcal{X}}$. Then $\delta \mapsto \operatorname{diam}(S(x, \psi, \delta))$ is a continuous function on $(0, \infty)$.

Proof. Fix arbitrarily $\varepsilon > 0$ and take $y, z \in S(x, \psi, \delta + \varepsilon)$. Consider the points $y_\lambda, z_\lambda \in \mathcal{B}_{\mathcal{X}}$ defined by $y_\lambda := \lambda x + (1 - \lambda)y$ and $z_\lambda := \lambda x + (1 - \lambda)z$, where $\lambda := \frac{\varepsilon}{\delta+\varepsilon} \in (0, 1)$. Then $y_\lambda, z_\lambda \in S(x, \psi, \delta)$, since

$$\langle \psi, y_\lambda \rangle = \langle \psi, \lambda x + (1 - \lambda)y \rangle > \lambda + (1 - \lambda)(1 - \delta - \varepsilon) = 1 - \delta$$

(and analogously for z_λ). Hence,

$$\|y - z\| = \frac{1}{1 - \lambda}\|y_\lambda - z_\lambda\| \leqslant \frac{1}{1 - \lambda}\operatorname{diam}(S(x, \psi, \delta)) = \left(1 + \frac{\varepsilon}{\delta}\right)\operatorname{diam}(S(x, \psi, \delta)).$$

This yields $0 \leqslant \operatorname{diam}(S(x, \psi, \delta + \varepsilon)) - \operatorname{diam}(S(x, \psi, \delta)) \leqslant 2\varepsilon/\delta$ and we are done. ■

2.2 Fundamental biorthogonal systems

A *biorthogonal system* in a normed space $\mathcal{X}$ is a system $\{e_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma}$, with $e_\alpha \in \mathcal{X}$ and $\varphi_\alpha \in \mathcal{X}^*$, such that $\langle \varphi_\alpha, e_\beta \rangle = \delta_{\alpha,\beta}$ $(\alpha, \beta \in \Gamma)$. A biorthogonal system is *fundamental* if $\operatorname{span}\{e_\alpha\}_{\alpha\in\Gamma}$ is dense in $\mathcal{X}$; it is *total* when $\operatorname{span}\{\varphi_\alpha\}_{\alpha\in\Gamma}$ is w^*-dense in $\mathcal{X}^*$. A *Markuševič basis* (M-basis, for short) is a fundamental and total biorthogonal system. A biorthogonal system $\{e_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma}$ is *bounded* if there is $M < \infty$ with $\|x_\alpha\| \cdot \|\varphi_\alpha\| \leqslant M$ $(\alpha \in \Gamma)$.

The following standard lemma, concerning distances of vectors from finite-dimensional subspaces in presence of a bounded biorthogonal system, will be used frequently in our argument. We refer to [26, §1.2] for a more general treatment of such types of results.

Lemma 2.3. Let $(\mathcal{X}, \|\cdot\|)$ be a normed space and $\{e_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma}$ be a bounded biorthogonal system in $\mathcal{X}$ with $\|e_\alpha\| = 1$, $\|\varphi_\alpha\| \leqslant M$ $(\alpha \in \Gamma)$. Let $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m \in \Gamma$ be such that $\alpha_i \neq \alpha_j$ and $\beta_i \neq \beta_j$ for $i \neq j$; set $F := \operatorname{span}\{e_{\alpha_1}, \dots, e_{\alpha_n}\}$ and $G := \operatorname{span}\{e_{\beta_1}, \dots, e_{\beta_m}\}$.

(i) If $\{\alpha_1, \dots, \alpha_n\} \cap \{\beta_1, \dots, \beta_m\} = \emptyset$ and $x \in \mathcal{S}_F$, then

$$\operatorname{dist}(x, G) \geqslant \frac{1}{nM}.$$

(ii) More generally, for every $x \in F$,

$$\operatorname{dist}(x, F \cap G) \leqslant nM \cdot \operatorname{dist}(x, G).$$

Proof. (i) is a particular case of (ii) since the assumption of (i) gives $F \cap G = \{0\}$, so $\operatorname{dist}(x, F \cap G) = \|x\|$. For the proof of (ii), assume that, for some $k \geqslant 0$, $\alpha_1 = \beta_1, \dots, \alpha_k = \beta_k$ and $\{\alpha_{k+1}, \dots, \alpha_n\} \cap \{\beta_{k+1}, \dots, \beta_m\} = \emptyset$. Fix $x \in F$ and let

$$\tilde{x} := x - \sum_{j=k+1}^{n} \langle \varphi_{\alpha_j}, x \rangle e_{\alpha_j} = \sum_{j=1}^{k} \langle \varphi_{\alpha_j}, x \rangle e_{\alpha_j} \in F \cap G.$$

Moreover, taking $y \in G$ with $\operatorname{dist}(x, G) = \|x - y\|$, we can estimate

$$\begin{aligned} \operatorname{dist}(x, F \cap G) \leqslant \|x - \tilde{x}\| = \left\| \sum_{j=k+1}^{n} \langle \varphi_{\alpha_j}, x \rangle e_{\alpha_j} \right\| \leqslant \sum_{j=k+1}^{n} |\langle \varphi_{\alpha_j}, x \rangle| \\ = \sum_{j=k+1}^{n} |\langle \varphi_{\alpha_j}, x - y \rangle| \leqslant nM \cdot \|x - y\| = nM \cdot \operatorname{dist}(x, G). \end{aligned}$$

■

In the proof of Theorem A we shall need two important results concerning fundamental biorthogonal systems in Banach spaces, which we collect below.

Theorem 2.4. Let $\mathcal{X}$ be a Banach space with a fundamental biorthogonal system $\{e_\alpha; \varphi_\alpha\}_{\alpha \in \Gamma}$ and let $\mathcal{Y} := \operatorname{span}\{e_\alpha\}_{\alpha \in \Gamma}$. Then,

(i) there exists a bounded fundamental biorthogonal system $\{e'_\alpha; \varphi'_\alpha\}_{\alpha \in \Gamma}$ such that $\mathcal{Y} = \operatorname{span}\{e'_\alpha\}_{\alpha \in \Gamma}$ [29].

(ii) $\mathcal{Y}$ admits an LUR norm (which approximates the original norm of $\mathcal{Y}$) [45, 58].

The claim in (i) is stated and proved in [29] for M-bases only; however, an inspection of the argument shows that it holds true for every fundamental biorthogonal system. Perhaps the simplest way to see this is to note that the argument in [29] never uses the completeness of the space, hence if $\{e_\alpha; \varphi_\alpha\}_{\alpha \in \Gamma}$ is a fundamental biorthogonal system in a Banach space $\mathcal{X}$, it is an M-basis for the normed space $\mathcal{Y} := \operatorname{span}\{e_\alpha\}_{\alpha \in \Gamma}$ and we can apply [29] to the space $\mathcal{Y}$. Earlier partial results due to Plichko can be found in [50, 52]. Part (ii) is essentially Troyanski's renorming technique [58] (also see [15, Chapter VII], or [30, Theorem 3.48]). The same type of results can also be found in the literature under the name *Deville's master lemma*, for example, in [46, 47]. The formulation given here is stated explicitly, for example, in [45, Lemma 2.1], and some variants of it are also used in [24, 35, 62]. The assertion in parentheses concerning the density of LUR norms is standard [15, p. 52]. Finally, note that Theorem A(v) constitutes an improvement of Theorem 2.4(ii).

2.3 Smooth norms via Minkowski functionals

A norm $\|\cdot\|$ on $\mathcal{X}$ is *C^k-smooth* if its k-th Fréchet derivative exists and it is continuous at every point of $\mathcal{X}\setminus\{0\}$ (equivalently, of $\mathcal{S}_{\mathcal{X}}$). The norm $\|\cdot\|$ *locally depends on finitely many coordinates* (is LFC, for short) on $\mathcal{X}$ if for each $x \in \mathcal{S}_{\mathcal{X}}$ there exist an open neighbourhood $\mathcal{U}$ of x and functionals $\varphi_1, \dots, \varphi_k \in \mathcal{X}^*$ such that $\|y\| = \|z\|$ for every $y, z \in \mathcal{U}$ with $\langle \varphi_j, y\rangle = \langle \varphi_j, z\rangle$ for every $j = 1, \dots, k$.

A *convex body* is a convex set with nonempty interior. A canonical way to build an equivalent norm on a normed space $(\mathcal{X}, \|\cdot\|)$ consists in building a bounded, symmetric convex body $\mathcal{D}$. Then $\mathcal{D}$ induces an equivalent norm on $\mathcal{X}$ via its Minkowski functional $\mu_{\mathcal{D}}$, defined by $\mu_{\mathcal{D}}(x) := \inf\{t > 0\colon x \in t\mathcal{D}\}$. If $\mathcal{D}$ is additionally closed, then the unit ball of $(\mathcal{X}, \mu_{\mathcal{D}})$ coincides with $\mathcal{D}$ itself. Moreover, if $(1-\delta)\mathcal{B}_{\mathcal{X}} \subseteq \mathcal{D} \subseteq \mathcal{B}_{\mathcal{X}}$ for some $\delta > 0$, then $\|\cdot\| \leqslant \mu_{\mathcal{D}} \leqslant (1-\delta)^{-1}\|\cdot\|$. This approach is ubiquitous in smooth renorming, combined with the following standard lemma, a version of the Implicit Function theorem; the formulation given here follows from [28, Lemma 5.23]. We refer, for example, to [4, 14, 20, 21, 32, 48] for some instances of uses of this technique.

Lemma 2.5. Let $(\mathcal{X}, \|\cdot\|)$ be a normed space, $\mathcal{D} \neq \emptyset$ be an open, convex, and symmetric subset of $\mathcal{X}$ and $f\colon \mathcal{D} \to \mathbb{R}$ be even, convex, and continuous. Assume that there is $a > f(0)$ such that $\mathcal{B} := \{f \leqslant a\}$ is bounded and closed in $\mathcal{X}$. If f is C^k-smooth for some $k \in \mathbb{N} \cup \{\infty, \omega\}$ (resp. LFC) on $\mathcal{D}$, then the Minkowski functional $\mu_{\mathcal{B}}$ of $\mathcal{B}$ is a C^k-smooth (resp. LFC) norm on $\mathcal{X}$.

We will also need the following folklore lemma, a variation of the above result.

Lemma 2.6. Let $\mathcal{D}$ be a bounded, symmetric, convex body in a normed space $\mathcal{X}$. Assume that for every $x \in \partial\mathcal{D}$ there are a neighbourhood $\mathcal{U}$ of x and functionals $\varphi_1, \dots, \varphi_n \in \mathcal{X}^*$ such that

$$\forall y \in \mathcal{U}\colon \quad y \in \mathcal{D} \iff \langle \varphi_i, x\rangle \leqslant 1,\ \forall i = 1, \dots, n. \tag{2.1}$$

Then, $\mu_{\mathcal{D}}$ is an LFC norm.

Proof. Pick any $x \in \mathcal{X}$ with $\mu_{\mathcal{D}}(x) = 1$, namely $x \in \partial\mathcal{D}$. Let a neighbourhood $\mathcal{U}$ of x and functionals $\varphi_1, \ldots, \varphi_n \in \mathcal{X}^*$ be as in the statement of the lemma. There are an open neighbourhood $\mathcal{V}$ of x and $\varepsilon > 0$ such that $s^{-1} \cdot y \in \mathcal{U}$ for every $y \in \mathcal{V}$ and every $s \in \mathbb{R}$ with $1-\varepsilon \leqslant s \leqslant 1+\varepsilon$. Moreover, the continuity of $\mu_{\mathcal{D}}$ yields the existence of a neighbourhood $\mathcal{W}$ of x, $\mathcal{W} \subseteq \mathcal{V}$, such that $1-\varepsilon \leqslant \mu_{\mathcal{D}}(y) \leqslant 1+\varepsilon$ for every $y \in \mathcal{W}$.

We claim that the neighbourhood $\mathcal{W}$ of x and the functionals $\varphi_1, \ldots, \varphi_n$ witness that $\mu_{\mathcal{D}}$ satisfies the LFC condition at x. Indeed, towards a contradiction, assume that there are $y, z \in \mathcal{W}$ such that $\langle \varphi_i, y \rangle = \langle \varphi_i, z \rangle$ for $i = 1, \ldots, n$, but $\mu_{\mathcal{D}}(y) < \mu_{\mathcal{D}}(z)$. Pick $s \in \mathbb{R}$ with $\mu_{\mathcal{D}}(y) < s < \mu_{\mathcal{D}}(z)$; then $1-\varepsilon \leqslant s \leqslant 1+\varepsilon$, by definition of $\mathcal{W}$. The definition of $\mathcal{V}$ now yields that $s^{-1} \cdot y, s^{-1} \cdot z \in \mathcal{U}$. Moreover, the definition of $\mu_{\mathcal{D}}$ gives $s^{-1} \cdot y \in \mathcal{D}$ and $s^{-1} \cdot z \notin \mathcal{D}$. Hence, by (2.1), we derive that there is $i_0 \in \{1, \ldots, n\}$ such that $\langle \varphi_{i_0}, s^{-1} \cdot z \rangle > 1$, while $\langle \varphi_i, s^{-1} \cdot y \rangle \leqslant 1$ for each $i = 1, \ldots, n$. However, this contradicts the fact that $\langle \varphi_{i_0}, y \rangle = \langle \varphi_{i_0}, z \rangle$ and concludes the proof. ■

3 Proof of the Main Result

The goal of this section is the proof of the core parts of our main result, items (i) and (ii) of Theorem A. Before diving into the details of the proof, let us present here some of the main ideas involved in the argument.

Let $\mathcal{Y} := \operatorname{span}\{e_\alpha\}_{\alpha\in\Gamma}$ be the linear span of the fundamental biorthogonal system and $F := \operatorname{span}\{e_{\alpha_1}, \ldots, e_{\alpha_n}\}$ be a finite-dimensional subspace. By Theorem 2.4(ii), we can assume that the norm on $\mathcal{Y}$ is LUR, so we can cover the unit sphere of F with finitely many open slices of $\mathcal{B}_{\mathcal{Y}}$ with arbitrary small diameters. By removing such slices from $\mathcal{B}_{\mathcal{Y}}$ we end up with a convex body $\mathcal{P}_F$ whose intersection with F is a polyhedron. After having performed such a construction in the single subspace F, one would like to "glue together" all such convex sets $\mathcal{P}_F$ in order to get the desired polyhedral norm on $\mathcal{Y}$. When trying to implement this idea, we face two main difficulties. The first one is that, if $G := \operatorname{span}\{e_{\beta_1}, \ldots, e_{\beta_k}\}$ is a subspace of F, the polyhedron $\mathcal{P}_F \cap F$ corresponding to F intersected with G should coincide with $\mathcal{P}_G \cap G$. In order to solve this, when choosing the slices in F, we only add those that have small intersection with the slices coming from proper subspaces of F. Moreover, the diameter of these "new" slices is much smaller than the one of the slices constructed before. This inductive construction will be carried out in Step 1. The second, and main, difficulty is that the slices corresponding to some other subspace $G := \operatorname{span}\{e_{\beta_1}, \ldots, e_{\beta_k}\}$ (now, not necessarily contained in F) could intersect F. Since this could happen for infinitely many of such subspaces G, there would be no way to assure that the ball of F remains a polyhedron. The rather delicate choice of the slices in Step 1 is justified by the need to circumvent such a problem. Indeed, in Step 2 we show that there even exists a neighbourhood of $\mathcal{B}_F$, that we will denote by $T(F, \theta_F/2)$, that "protects" F, in the sense that the other slices do not intersect such a neighbourhood. The existence of this neighbourhood will be the crucial ingredient in the proof of the LFC and for smoothing the norm in Steps 3 and 4, respectively.

Proof of Theorem A(i) and (ii). Let $\mathcal{X}$ be a Banach space with a fundamental biorthogonal system $\{e_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma}$, set $\mathcal{Y} := \operatorname{span}\{e_\alpha\}_{\alpha\in\Gamma}$, and choose any equivalent norm $\|\cdot\|$ on $\mathcal{Y}$. Theorem 2.4(i) yields that $\mathcal{Y}$ is the linear span of a bounded (fundamental) biorthogonal system, hence it allows us to assume that $\{e_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma}$ is bounded. Our task consists in building two norms, the former polyhedral and LFC, the latter C^∞-smooth and LFC, that approximate $\|\cdot\|$. By Theorem 2.4(ii) there exists an LUR norm on $\mathcal{Y}$ that approximates $\|\cdot\|$, hence we can (and do) assume that $\|\cdot\|$ is already LUR on $\mathcal{Y}$. Finally, up to rescaling we can also assume that $\|e_\alpha\| = 1$, $\|\varphi_\alpha\| \leqslant M$ ($\alpha \in \Gamma$), for some $M \geqslant 1$.

We start by fixing one piece of notation. Denote, for $n \in \mathbb{N}$,

$$\mathcal{F}^n := \left\{\operatorname{span}\{e_{\alpha_1}, \ldots, e_{\alpha_n}\}\colon \alpha_1, \ldots, \alpha_n \in \Gamma,\ \alpha_i \neq \alpha_j \text{ for } i \neq j\right\}$$

$$\mathcal{F}^{<\omega} := \bigcup_{n\in\mathbb{N}} \mathcal{F}^n.$$

Note that $\mathcal{F}^{<\omega}$ is a directed set by inclusion. Moreover, for $F \in \mathcal{F}^{<\omega}$ and $\theta > 0$, we consider the open "tubular" neighbourhood of F in $\mathcal{B}_\mathcal{Y}$ defined by

$$T(F, \theta) := \{y \in \mathcal{B}_\mathcal{Y}\colon \operatorname{dist}(y, F) < \theta\}.$$

Step 1. Construction of the slices.

We start by building a collection of slices parametrised by $\mathcal{F}^{<\omega}$ that satisfy the conditions in the following claim.

Claim 3.1. Let $\varepsilon > 0$ be fixed. Then there are nets $(\varepsilon_F)_{F\in\mathcal{F}^{<\omega}}$ and $(\theta_F)_{F\in\mathcal{F}^{<\omega}}$ of positive reals and sets $(\Omega_F)_{F\in\mathcal{F}^{<\omega}}$, where Ω_F is a finite set of slices of $\mathcal{B}_\mathcal{Y}$, such that

(i) $\theta_F \leqslant \varepsilon_F \leqslant \varepsilon$ for every $F \in \mathcal{F}^{<\omega}$;

(ii) $\varepsilon_F \leqslant \frac{1}{4nM}$ for every $F \in \mathcal{F}^n$;

(iii) $\varepsilon_F \leqslant \frac{1}{4nM}\theta_G$ for every $F \in \mathcal{F}^n$ and every $G \in \mathcal{F}^{<\omega}$ with $G \subsetneq F$;

(iv) if $S \in \Omega_F$, then $-S \in \Omega_F$ as well;

(v) $\operatorname{diam}(S) < \varepsilon_F$ for every $S \in \Omega_F$;

(vi) if $S \in \Omega_F$, then S is of the form $S = S(x, \psi, \delta)$, for some $\delta > 0$, some functional $\psi \in \mathcal{S}_{\mathcal{Y}^*}$ that is norming for x, where

$$x \in \mathcal{S}_F \setminus \bigcup_{\substack{G \in \mathcal{F}^{<\omega} \\ G \subsetneq F}} T(G, \theta_G);$$

(vii) Setting, for $F \in \mathcal{F}^{<\omega}$,

$$\mathcal{U}_F := \bigcup_{\substack{G \in \mathcal{F}^{<\omega} \\ G \subseteq F}} \bigcup_{S \in \Omega_G} S,$$

we have $2\theta_F \leqslant \operatorname{dist}(\mathcal{S}_F, \mathcal{B}_{\mathcal{Y}} \setminus \mathcal{U}_F)$. In particular, $\mathcal{S}_{\mathcal{Y}} \cap T(F, \theta_F) \subseteq \mathcal{U}_F$.

Proof of Claim 3.1. Without loss of generality, we assume that $\varepsilon < 1$. We start by proving that the second clause in (vii) indeed follows from the first part. Pick any $x \in \mathcal{S}_{\mathcal{Y}} \cap T(F, \theta_F)$ and find $w \in F$ with $\|x - w\| < \theta_F$. Since $\|x\| = 1$, we have $|\|w\| - 1| < \theta_F$, hence the vector $\tilde{w} := \frac{w}{\|w\|}$ satisfies $\|w - \tilde{w}\| < \theta_F$. Thus, $\|\tilde{w} - x\| < 2\theta_F \leqslant \operatorname{dist}(\mathcal{S}_F, \mathcal{B}_{\mathcal{Y}} \setminus \mathcal{U}_F)$. This inequality and $\tilde{w} \in \mathcal{S}_F$ imply $x \in \mathcal{U}_F$, as desired.

We now build $(\varepsilon_F)_{F \in \mathcal{F}^{<\omega}}$, $(\theta_F)_{F \in \mathcal{F}^{<\omega}}$, and $(\Omega_F)_{F \in \mathcal{F}^{<\omega}}$ with the above properties and we argue by induction on $n := \dim F$ (where $F \in \mathcal{F}^{<\omega}$). To check the statement for $n = 1$, pick any $F \in \mathcal{F}^1$, namely $F = \operatorname{span}\{e_\alpha\}$, for some $\alpha \in \Gamma$. Set $\varepsilon_F := \min\{\varepsilon, \frac{1}{4M}\}$ and let $\psi_\alpha \in \mathcal{S}_{\mathcal{Y}^*}$ be a norming functional for e_α. By Fact 2.1 there is δ_α such that the slice $S(e_\alpha, \psi_\alpha, \delta_\alpha)$ has diameter smaller than ε_F. Let $\Omega_F := \{\pm S(e_\alpha, \psi_\alpha, \delta_\alpha)\}$ and $\mathcal{U}_F := S(e_\alpha, \psi_\alpha, \delta_\alpha) \cup -S(e_\alpha, \psi_\alpha, \delta_\alpha)$. Then $\mathcal{S}_F \subseteq \mathcal{U}_F$, so $\operatorname{dist}(\mathcal{S}_F, \mathcal{B}_{\mathcal{Y}} \setminus \mathcal{U}_F) > 0$ and we can finally choose $\theta_F \leqslant \min\{\varepsilon_F, \frac{1}{2}\operatorname{dist}(\mathcal{S}_F, \mathcal{B}_{\mathcal{Y}} \setminus \mathcal{U}_F)\}$. With this construction, all conditions (i)–(vii) are clearly satisfied.

Now fix $n \geqslant 2$ and assume inductively that ε_F, θ_F, and Ω_F have already been defined for every $F \in \mathcal{F}^{<\omega}$ with $\dim F \leqslant n - 1$ and satisfy (i)–(vii). Fix $F \in \mathcal{F}^n$ arbitrarily. There are only finitely many $G \in \mathcal{F}^{<\omega}$ with $G \subsetneq F$ and θ_G has already been defined for each such G, hence we can choose $\varepsilon_F > 0$ such that

$$\varepsilon_F \leqslant \frac{1}{4nM} \min\{\theta_G \colon G \in \mathcal{F}^{<\omega}, G \subsetneq F\}.$$

This gives conditions (iii) and (ii), since $\theta_G \leqslant \varepsilon_G \leqslant \varepsilon < 1$. Consider now the set

$$\mathcal{V} := \bigcup_{\substack{G\in\mathcal{F}^{<\omega}\\ G\subsetneq F}} \bigcup_{S\in\Omega_G} S,$$

pick $x \in \mathcal{S}_F \setminus \mathcal{V}$ and a norming functional ψ_x for x. By Fact 2.1 there exists $\delta_x > 0$ such that $\operatorname{diam}(S(x,\psi_x,\delta_x)) < \varepsilon_F$. The collection $\{S(x,\psi_x,\delta_x) \colon x \in \mathcal{S}_F \setminus \mathcal{V}\}$ is an open cover of the compact set $\mathcal{S}_F \setminus \mathcal{V}$, so we can extract a finite subcover $\{S(x_j,\psi_{x_j},\delta_{x_j})\}_{j=1}^k$ of $\{S(x,\psi_x,\delta_x) \colon x \in \mathcal{S}_F \setminus \mathcal{V}\}$. We set $\Omega_F := \{\pm S(x_j,\psi_{x_j},\delta_{x_j})\}_{j=1}^k$. Then conditions (iv) and (v) are satisfied. (vi) is satisfied as well because

$$\mathcal{S}_{\mathcal{Y}} \cap \bigcup_{\substack{G\in\mathcal{F}^{<\omega}\\ G\subsetneq F}} T(G,\theta_G) \subseteq \bigcup_{\substack{G\in\mathcal{F}^{<\omega}\\ G\subsetneq F}} \bigcup_{S\in\Omega_G} S = \mathcal{V}$$

by (vii) of the inductive assumption.

Finally, we define $\mathcal{U}_F$ as in (vii); by construction $\mathcal{U}_F$ is an open subset of $\mathcal{B}_{\mathcal{Y}}$ that contains $\mathcal{S}_F$. Hence, the closed sets $\mathcal{S}_F$ and $\mathcal{B}_{\mathcal{Y}} \setminus \mathcal{U}_F$ are disjoint and, thus, they have positive distance ($\mathcal{S}_F$ being compact). So, we can choose $\theta_F > 0$ such that

$$\theta_F \leqslant \min\left\{\varepsilon_F, \frac{1}{2}\operatorname{dist}(\mathcal{S}_F, \mathcal{B}_{\mathcal{Y}} \setminus \mathcal{U}_F)\right\}.$$

This yields conditions (i) and (vii) and concludes the inductive step. ∎

Step 2. The convex bodies $\mathcal{P}_F$ and the compatibility condition.

Next, we use the family of slices from the previous step to build, for every $F \in \mathcal{F}^{<\omega}$, a convex body $\mathcal{P}_F$. We prove the crucial fact (see Fact 3.2 below) that this construction is compatible, in the sense that the construction of $\mathcal{P}_F$ does not interfere with the one of $\mathcal{P}_G$, for any $F \neq G \in \mathcal{F}^{<\omega}$.

For every $F \in \mathcal{F}^{<\omega}$ and every $n \in \mathbb{N}$ we define

$$\mathcal{P}_F := \mathcal{B}_{\mathcal{Y}} \setminus \bigcup_{\substack{G\in\mathcal{F}^{<\omega}\\ G\subseteq F}} \bigcup_{S\in\Omega_G} S \tag{3.1}$$

$$\mathcal{P}_n := \bigcap_{F\in\mathcal{F}^n} \mathcal{P}_F = \bigcap_{\substack{G\in\mathcal{F}^{<\omega}\\ \dim G\leqslant n}} \bigcap_{S\in\Omega_G} \mathcal{B}_{\mathcal{Y}} \setminus S. \tag{3.2}$$

Let us note here the following monotonicity properties: $\mathcal{P}_F \subseteq \mathcal{P}_G$ whenever F, $G \in \mathcal{F}^{<\omega}$, $G \subseteq F$. Hence, $\mathcal{P}_{n+1} \subseteq \mathcal{P}_n$ for $n \geqslant 1$. We shall discuss further properties of such sets at the beginning of Step 3, while we now turn to the main property of the construction.

Fact 3.2. (Compatibility). For every $F \in \mathcal{F}^{<\omega}$ and every $n \in \mathbb{N}$ with $\dim F \leqslant n$ we have

$$\mathcal{P}_n \cap T(F, \theta_F/2) = \mathcal{P}_F \cap T(F, \theta_F/2). \tag{†}$$

Proof of Fact 3.2. The "$\subseteq$" inclusion is clear since $\mathcal{P}_n \subseteq \mathcal{P}_F$ if $\dim F \leqslant n$, by the monotonicity properties mentioned above. So, we only need to prove that, for every $F \in \mathcal{F}^{<\omega}$ and every $n \in \mathbb{N}$, one has

$$\mathcal{P}_F \cap T(F, \theta_F/2) \subseteq \mathcal{P}_n.$$

(Notice that checking the inclusion $\mathcal{P}_F \cap T(F, \theta_F/2) \subseteq \mathcal{P}_n$ for every $n \geqslant \dim F$ is equivalent to checking it for every $n \in \mathbb{N}$, since $(\mathcal{P}_n)_{n\in\mathbb{N}}$ is a decreasing sequence.) By the definition (3.2) of $\mathcal{P}_n$ it is thus sufficient to prove that, for every $F, G \in \mathcal{F}^{<\omega}$ and every $S \in \Omega_G$,

$$\mathcal{P}_F \cap T(F, \theta_F/2) \subseteq \mathcal{B}_{\mathcal{Y}} \setminus S. \tag{*}$$

We prove this by induction on $n := \max\{\dim F, \dim G\}$. Throughout the argument, we assume $F \neq G$ as (∗) is trivially true when $F = G$.

For the case $n = 1$, assume by contradiction that there are $F \neq G \in \mathcal{F}^1$ and $S \in \Omega_G$ with $\mathcal{P}_F \cap T(F, \theta_F/2) \cap S \neq \emptyset$. Pick $x \in \mathcal{P}_F \cap T(F, \theta_F/2) \cap S$. Then $\operatorname{dist}(x, F) < \theta_F/2$. Moreover, by condition (vi) of Claim 3.1, $S \in \Omega_G$ implies the existence of $z \in \mathcal{S}_G \cap S$, hence by (v), $\|x - z\| < \varepsilon_G$. Thus, we get $\operatorname{dist}(z, F) < \varepsilon_G + \theta_F/2$. Therefore, Lemma 2.3(i) and conditions (i) and (ii) give the following contradiction

$$\frac{1}{M} \leqslant \operatorname{dist}(z, F) < \varepsilon_G + \theta_F/2 \leqslant \frac{1}{4M} + \frac{1}{8M} \leqslant \frac{1}{2M}.$$

Now fix $n \geqslant 2$ and assume by induction that (∗) holds for every $F, G \in \mathcal{F}^{<\omega}$ with $\max\{\dim F, \dim G\} \leqslant n-1$ and every $S \in \Omega_G$. Take $F, G \in \mathcal{F}^{<\omega}$ with $\max\{\dim F, \dim G\} = n$ and $S \in \Omega_G$. First of all, since $S \in \Omega_G$, (v) and (vi) respectively imply that $\operatorname{diam}(S) < \varepsilon_G$ and $S = S(z, \psi, \delta)$ for some

$$z \in \mathcal{S}_G \setminus \bigcup_{\substack{H\in\mathcal{F}^{<\omega} \\ H\subsetneq G}} T(H, \theta_H). \tag{3.3}$$

Let us stress that, in particular, $z \in S \cap G$, fact that we shall use several times below. We distinguish two cases: $\dim F = n$, or $\dim G = n$ and $\dim F \leqslant n-1$.

Case 1. $\dim F = n$.

Set $k := \dim G \leqslant n$ and, towards a contradiction, assume that there is $x \in \mathcal{P}_F \cap T(F, \theta_F/2)$ such that $x \in S$. Since $x, z \in S$, by (v), $\|x - z\| < \varepsilon_G$. Moreover, $x \in T(F, \theta_F/2)$ implies $\operatorname{dist}(x, F) < \theta_F/2$, so $\operatorname{dist}(z, F) < \varepsilon_G + \theta_F/2$. Now there are three sub-cases.

- If $F \cap G = \{0\}$, then we readily have a contradiction. Indeed, applying Lemma 2.3(i) to $z \in \mathcal{S}_G$ and F gives the absurdity that

$$\frac{1}{kM} \leqslant \operatorname{dist}(z, F) < \varepsilon_G + \theta_F/2 \overset{(i),(ii)}{\leqslant} \frac{1}{4kM} + \frac{1}{8nM} \overset{k\leqslant n}{\leqslant} \frac{1}{2kM}.$$

- If $G \subseteq F$, then $\mathcal{P}_F \subseteq \mathcal{P}_G$. However, this contradicts $x \in \mathcal{P}_F \setminus \mathcal{P}_G$ ($x \in S$ with $S \in \Omega_G$ implies $x \notin \mathcal{P}_G$ by using item (vi)).
- The last sub-case is that $F \cap G \neq \{0\}$ and $G \not\subseteq F$. In particular, these conditions give $F \cap G \subsetneq G$ (and $F \cap G \subsetneq F$ as well, since $\dim(F \cap G) \leqslant k-1$). We can now apply Lemma 2.3(ii) to get

$$\begin{aligned}\operatorname{dist}(z, F \cap G) \leqslant kM \cdot \operatorname{dist}(z, F) < kM(\varepsilon_G + \theta_F/2) &\overset{(i)}{\leqslant} kM(\varepsilon_G + \varepsilon_F/2) \\ &\overset{(iii)}{\leqslant} kM\left(\frac{1}{4kM}\theta_{F\cap G} + \frac{1}{8nM}\theta_{F\cap G}\right) \overset{k\leqslant n}{\leqslant} \frac{1}{2}\theta_{F\cap G}.\end{aligned}$$

  Notice that, when applying condition (iii) we are using the assumptions that $F \cap G \subsetneq G$ and $F \cap G \subsetneq F$. This estimate implies that $z \in T(F \cap G, \theta_{F\cap G})$; however, this is a contradiction with (3.3), since $F \cap G \subsetneq G$.

Case 2. $\dim G = n$ and $\dim F \leqslant n-1$.

Here we set $k := \dim F \leqslant n-1$ and, as in Case 1, we assume towards a contradiction that there is $x \in \mathcal{P}_F \cap T(F, \theta_F/2)$ with $x \in S$. As above, $x, z \in S$ and (v) imply $\|x - z\| < \varepsilon_G$; we again distinguish the same three sub-cases.

- If $F \cap G = \{0\}$, we use the assumption that $x \in T(F, \theta_F/2)$ to find $w \in F$ with $\|x - w\| < \theta_F/2$. Since $x \in S$, $1 - \varepsilon_G \leqslant \|x\| \leqslant 1$, so $1 - \varepsilon_G - \theta_F/2 \leqslant \|w\| \leqslant 1 + \theta_F/2$. Then the vector $\tilde{w} := \frac{w}{\|w\|}$ satisfies $\|w - \tilde{w}\| \leqslant \varepsilon_G + \theta_F/2$, hence $\|x - \tilde{w}\| \leqslant \varepsilon_G + \theta_F$. Moreover, $\operatorname{dist}(x, G) \leqslant \|x - z\| < \varepsilon_G$. This finally yields $\operatorname{dist}(\tilde{w}, G) \leqslant 2\varepsilon_G + \theta_F$. We can now apply Lemma 2.3(i) to $\tilde{w} \in \mathcal{S}_F$ and G to get the following contradiction:

$$\frac{1}{kM} \leqslant \operatorname{dist}(\tilde{w}, G) \leqslant 2\varepsilon_G + \theta_F \leqslant \frac{2}{4nM} + \frac{1}{4kM} \leqslant \frac{3}{4kM},$$

where we are using again (i), (ii), and that $k \leqslant n$.

- If $F \subseteq G$, then actually $F \subsetneq G$, as $\dim F = k \leqslant n-1$. The assumption $x \in T(F, \theta_F/2)$ gives $\operatorname{dist}(x, F) < \theta_F/2$, hence (using (iii) and that $F \subsetneq G$)

$$\operatorname{dist}(z, F) < \varepsilon_G + \theta_F/2 \overset{(iii)}{\leqslant} \frac{1}{4nM}\theta_F + \theta_F/2 < \theta_F.$$

  Hence, $z \in T(F, \theta_F)$, which is in contradiction with (3.3) since $F \subsetneq G$.
- If $F \cap G \neq \{0\}$ and $F \not\subseteq G$, then $F \cap G \subsetneq F$ (and $F \cap G \subsetneq G$ as well since $\dim(F \cap G) \leqslant k \leqslant n-1$). As before, we can pick $w \in F$ with $\|x - w\| < \theta_F/2$. Since $\|x - z\| < \varepsilon_G$, such a w satisfies $\operatorname{dist}(w, G) \leqslant \|w - z\| \leqslant \varepsilon_G + \theta_F/2$. Combining the estimate $\operatorname{dist}(w, G) \leqslant \varepsilon_G + \theta_F/2$ with Lemma 2.3(ii) applied to $w \in F$ (the lemma is used in the second inequality) we obtain

$$\begin{aligned}\operatorname{dist}(z, F \cap G) &\leqslant \operatorname{dist}(w, F \cap G) + (\varepsilon_G + \theta_F/2)\\ &\leqslant kM \cdot \operatorname{dist}(w, G) + (\varepsilon_G + \theta_F/2)\\ &\leqslant (kM+1) \cdot (\varepsilon_G + \theta_F/2) \overset{(i)}{\leqslant} 2kM \cdot (\varepsilon_G + \varepsilon_F/2)\\ &\overset{(iii)}{\leqslant} 2kM\left(\frac{1}{4nM}\theta_{F\cap G} + \frac{1}{8kM}\theta_{F\cap G}\right) \overset{k\leqslant n}{\leqslant} \frac{3}{4}\theta_{F\cap G}.\end{aligned}$$

  Again, this implies $z \in T(F \cap G, \theta_{F\cap G})$, a contradiction with (3.3).

This concludes the induction step, hence the proof of Fact 3.2. ∎

Step 3. Construction of a polyhedral LFC norm and proof of Theorem A(i).

In this step we shall use the sets $(\mathcal{P}_F)_{F\in\mathcal{F}^{<\omega}}$ and $(\mathcal{P}_n)_{n\in\mathbb{N}}$ to build a polyhedral and LFC norm on $\mathcal{Y}$ that approximates $\|\cdot\|$. The desired norm will be the Minkowski functional of the set $\mathcal{P}$ defined by

$$\mathcal{P} := \bigcap_{n\in\mathbb{N}} \mathcal{P}_n. \tag{3.4}$$

To begin with, every set $\mathcal{P}_F$ ($F \in \mathcal{F}^{<\omega}$) is a closed, convex, and symmetric set (the symmetry is consequence of condition (iv)); plainly, we also have $\mathcal{P}_F \subseteq \mathcal{B}_{\mathcal{Y}}$. Moreover, by (v), $\mathcal{P}_F$ is obtained from $\mathcal{B}_{\mathcal{Y}}$ by removing slices of diameter at most $\varepsilon_F \leqslant \varepsilon$, so $(1-\varepsilon)\mathcal{B}_{\mathcal{Y}} \subseteq \mathcal{P}_F$. As a consequence of these remarks, the set $\mathcal{P}$ is a closed, convex and symmetric set with

$$(1-\varepsilon)\mathcal{B}_{\mathcal{Y}} \subseteq \mathcal{P} \subseteq \mathcal{B}_{\mathcal{Y}}.$$

Thus, the Minkowski functional $\mu_{\mathcal{P}}$ of $\mathcal{P}$ is an equivalent norm on $\mathcal{Y}$, whose unit ball is precisely $\mathcal{P}$. Further, the previous chain of inclusions gives $\|\cdot\| \leqslant \mu_{\mathcal{P}} \leqslant (1-\varepsilon)^{-1}\|\cdot\|$, hence $\mu_{\mathcal{P}}$ approximates $\|\cdot\|$. We now show that $\mu_{\mathcal{P}}$ is a polyhedral and LFC norm on $\mathcal{Y}$, thereby concluding the proof of Theorem A(i).

In order to check that $(\mathcal{Y}, \mu_{\mathcal{P}})$ is polyhedral, take a finite-dimensional subspace E of $\mathcal{Y}$. Then there is $F \in \mathcal{F}^{<\omega}$ with $E \subseteq F$. Hence, it suffices to prove that the unit ball $\mathcal{P} \cap F$ of F is a polyhedron. From Fact 3.2 we obtain in particular $\mathcal{P}_n \cap F = \mathcal{P}_F \cap F$ for every $n \in \mathbb{N}$, $n \geqslant \dim F$; thus, $\mathcal{P} \cap F = \mathcal{P}_F \cap F$. We enumerate all the slices that appear in the definition of $\mathcal{P}_F$:

$$\bigcup_{\substack{G \in \mathcal{F}^{<\omega} \\ G \subseteq F}} \Omega_G = \{S_j\}_{j=1}^N, \tag{3.5}$$

where $S_j = S(x_j, \psi_j, \delta_j)$. Then we can write

$$\begin{aligned} \mathcal{P} \cap F = \mathcal{P}_F \cap F &= \{y \in \mathcal{B}_F \colon \langle \psi_j, y\rangle \leqslant 1 - \delta_j \text{ for all } j = 1, \dots, N\} \\ &= \{y \in F \colon \langle \psi_j, y\rangle \leqslant 1 - \delta_j \text{ for all } j = 1, \dots, N\} =: \mathcal{C}. \end{aligned}$$

Indeed, the second equality is just the definition of $\mathcal{P}_F$, while the "$\subseteq$" inclusion of the third equality is obvious. In order to prove that $\mathcal{C} \subseteq \mathcal{P}_F \cap F$, it is sufficient to prove that $\mathcal{C} \subseteq \mathcal{B}_F$. Towards a contradiction, assume that there is $x \in \mathcal{C}$ with $\|x\| > 1$. Then $x/\|x\| \in \mathcal{C}$ as well. Notice that the second clause of (vii) gives in particular $\mathcal{S}_F \subseteq \cup_{j=1}^N S_j$. Hence, there is $j \in \{1, \dots, N\}$ such that $x/\|x\| \in S_j$, which contradicts $x/\|x\| \in \mathcal{C}$.

Consequently, we obtained that

$$\mathcal{P} \cap F = \mathcal{P}_F \cap F = \{y \in F \colon \langle \psi_j, y\rangle \leqslant 1 - \delta_j \text{ for all } j = 1, \dots, N\}$$

is a polyhedron, hence $\mu_{\mathcal{P}}$ is a polyhedral norm.

Finally, we use Lemma 2.6 to show that the norm is LFC. First, in the notation of (vii), we have $\mathcal{P}_F = \mathcal{B}_\mathcal{Y} \setminus \mathcal{U}_F$, so $\operatorname{dist}(\mathcal{S}_F, \mathcal{P}_F) \geqslant 2\theta_F$ for every $F \in \mathcal{F}^{<\omega}$. Thus,

$$\|x\| \leqslant 1 - 2\theta_F \text{ for all } x \in \mathcal{P}_F \cap F. \tag{3.6}$$

Moreover, as before, Fact 3.2 gives $\mathcal{P} \cap T(F, \theta_F/2) = \mathcal{P}_F \cap T(F, \theta_F/2)$ for every $F \in \mathcal{F}^{<\omega}$.

Now fix arbitrarily $x \in \partial\mathcal{P}$. Then there is $F \in \mathcal{F}^{<\omega}$ with $x \in F$, hence $x \in \mathcal{P}_F \cap F$ and $\|x\| \leqslant 1 - 2\theta_F$ by (3.6). Consider the open ball $\mathcal{U} := \mathcal{B}^o_{\theta_F/2}(x) := \{y \in \mathcal{Y} \colon \|y - x\| < \theta_F/2\}$; then $\mathcal{U} \subseteq T(F, \theta_F/2)$. Indeed, for every $y \in \mathcal{U}$, $\operatorname{dist}(y, F) \leqslant \|y - x\| < \theta_F/2$ and $\|y\| \leqslant \|x\| + \theta_F/2 \leqslant 1 - 2\theta_F + \theta_F/2 < 1$. The inclusion $\mathcal{U} \subseteq T(F, \theta_F/2)$ then allows us to "localise" the compatibility condition and deduce from Fact 3.2 that $\mathcal{P} \cap \mathcal{U} = \mathcal{P}_F \cap \mathcal{U}$.

Moreover, using (3.5) and the definition of $\mathcal{P}_F$, we can write

$$\mathcal{P}_F = \left\{ y \in \mathcal{B}_\mathcal{Y} \colon \left\langle \frac{\psi_j}{1 - \delta_j}, y \right\rangle \leqslant 1 \text{ for all } j = 1, \dots, N \right\}.$$

Consequently, for every $y \in \mathcal{U}$, $y \in \mathcal{P}$ if and only if $\left\langle \frac{\psi_j}{1-\delta_j}, y \right\rangle \leqslant 1$ for every $j = 1, \dots, N$ (here we are using the equality $\mathcal{P} \cap \mathcal{U} = \mathcal{P}_F \cap \mathcal{U}$). Therefore, Lemma 2.6 implies that $\mu_\mathcal{P}$ is LFC, as desired.

Step 4. Smoothing and proof of Theorem A(ii).

In this step we glue together in a smooth way the functionals corresponding to the slices from Step 1 and obtain a C^∞-smooth and LFC norm on $\mathcal{Y}$. The smoothness of the resulting norm will crucially depend again on the compatibility condition (†). In order to leave room for the smoothing, the main technical point consists in actually applying Fact 3.2 to some suitably enlarged slices and not to the slices from Step 1.

Fix $F \in \mathcal{F}^{<\omega}$ and write (note that, differently from (3.5), here we do not enumerate the slices in Ω_G with $G \subsetneq F$) $\Omega_F = \left\{ \pm S(x_j, \psi_{x_j}, \delta_{x_j}) \right\}_{j=1}^k$. We first define

$$\Theta_F := \left\{ \frac{\psi_{x_j}}{1 - \delta_{x_j}} \right\}_{j=1}^k.$$

Observe that this notation allows us to write

$$\mathcal{P}_F := \left\{ y \in \mathcal{B}_\mathcal{Y} \colon |\langle \psi, y \rangle| \leqslant 1 \text{ for all } \psi \in \bigcup_{\substack{G \in \mathcal{F}^{<\omega} \\ G \subseteq F}} \Theta_G \right\}. \tag{3.7}$$

Moreover, since all the finitely many slices in Ω_F have diameter less than ε_F, by Fact 2.2 there is $\delta_F > 0$ such that $\operatorname{diam}(S(x_j, \psi_{x_j}, \delta_{x_j} + \delta_F)) < \varepsilon_F$ for every $j = 1, \dots, k$. We then pick, for every $F \in \mathcal{F}^{<\omega}$, an even, convex, and C^∞-smooth function $\varrho_F \colon \mathbb{R} \to [0, \infty)$ such that $\varrho_F(1) = 1$ and $\varrho_F(s) = 0$ if and only if $|s| \leqslant 1 - \delta_F$. Note that every such a function is strictly increasing on $[1 - \delta_F, \infty)$. We are now in position to define $\Phi \colon \mathcal{Y} \to [0, \infty]$ by

$$\Phi(y) := \sum_{F \in \mathcal{F}^{<\omega}} \sum_{\psi \in \Theta_F} \varrho_F\big(\langle \psi, y \rangle\big).$$

For $F \in \mathcal{F}^{<\omega}$, we also define $\Phi_F \colon \mathcal{Y} \to [0, \infty)$ by

$$\Phi_F(y) := \sum_{\substack{G \in \mathcal{F}^{<\omega} \\ G \subseteq F}} \sum_{\psi \in \Theta_G} \varrho_G\big(\langle \psi, y \rangle\big).$$

We wish to apply Lemma 2.5 to the function Φ and sets $\mathcal{D} := \{\Phi < 1\}$ and $\mathcal{B} := \{\Phi \leqslant 1 - \varepsilon\}$. To this aim we first observe that (recall the definition of $\mathcal{P}$ from (3.4))

$$(1 - \varepsilon)\mathcal{B}_{\mathcal{Y}} \subseteq \{\Phi = 0\} \subseteq \mathcal{B} \subseteq \mathcal{D} \subseteq \mathcal{P} \subseteq \mathcal{B}_{\mathcal{Y}}. \tag{3.8}$$

Indeed, in order to check the first inclusion note that, for $S = S(x_j, \psi_{x_j}, \delta_{x_j}) \in \Omega_F$, the condition $\operatorname{diam}(S(x_j, \psi_{x_j}, \delta_{x_j} + \delta_F)) < \varepsilon_F$, together with (i) of Claim 3.1, implies $\delta_{x_j} + \delta_F < \varepsilon_F \leqslant \varepsilon$. Hence, if $y \in \mathcal{Y}$ satisfies $\|y\| \leqslant 1 - \varepsilon$, then for every $F \in \mathcal{F}^{<\omega}$ and every $\psi = \frac{\psi_{x_j}}{1 - \delta_{x_j}} \in \Theta_F$ we have

$$|\langle \psi, y \rangle| = \left| \left\langle \frac{\psi_{x_j}}{1 - \delta_{x_j}}, y \right\rangle \right| \leqslant \frac{1 - \varepsilon}{1 - \delta_{x_j}} \leqslant \frac{1 - \delta_F - \delta_{x_j}}{1 - \delta_{x_j}} \leqslant 1 - \delta_F.$$

This yields $\Phi(y) = 0$ and shows the first inclusion. Next, if $y \in \mathcal{D}$, then for every $\psi \in \Theta_F$ and every $F \in \mathcal{F}^{<\omega}$ we have $\varrho_F\big(\langle \psi, y \rangle\big) < 1$, so $|\langle \psi, y \rangle| < 1$ by the properties of the functions ϱ_F. Hence, $y \in \mathcal{P}_F$ for every $F \in \mathcal{F}^{<\omega}$, thus $\mathcal{D} \subseteq \mathcal{P}$. The other inclusions being trivial, (3.8) is proved.

In particular, the set $\mathcal{B}$ is bounded and closed in $\mathcal{Y}$ (since Φ is lower semi-continuous on $\mathcal{Y}$ by Fatou's lemma). Moreover, Φ is even and convex, hence $\mathcal{D}$ is convex and symmetric. In order to be able to apply Lemma 2.5 we need to show that $\mathcal{D}$ is open and that Φ is C^∞-smooth and LFC on $\mathcal{D}$. All these properties follow rather easily from the next claim, asserting that Φ is locally a finite sum on $\mathcal{D}$.

Claim 3.3. For every $y \in \mathcal{D}$ there are an open subset $\mathcal{U}$ of $\mathcal{Y}$ with $y \in \mathcal{U}$ and $F \in \mathcal{F}^{<\omega}$ such that $\Phi = \Phi_F$ on $\mathcal{U}$.

Assuming the validity of the claim, note that the function Φ_F is clearly C^∞-smooth and LFC on $\mathcal{Y}$ (it is defined via a finite sum). Thus, Claim 3.3 immediately yields that Φ is C^∞-smooth and LFC on some open neighbourhood of $\mathcal{D}$ in $\mathcal{Y}$. Hence, it also follows that $\mathcal{D}$ is open. We are then in position to apply Lemma 2.5, which leads us to the conclusion that $\mu_{\mathcal{B}}$ is a C^∞-smooth and LFC norm. Finally, (3.8) implies $\|\cdot\| \leqslant \mu_{\mathcal{B}} \leqslant (1-\varepsilon)^{-1}\|\cdot\|$, hence $\mu_{\mathcal{B}}$ approximates the norm $\|\cdot\|$. Consequently, in order to conclude the proof of Theorem A(ii), we only need to prove Claim 3.3.

Proof of Claim 3.3. Take $F \in \mathcal{F}^{<\omega}$ with $y \in F$ and consider the set $\mathcal{U}$ defined by

$$\mathcal{U} := T(F, \theta_F/2) \cap \left\{ z \in \mathcal{Y} \colon \|z\| < 1,\ |\langle \psi, z\rangle| < 1 \text{ for every } \psi \in \bigcup_{\substack{G \in \mathcal{F}^{<\omega} \\ G \subseteq F}} \Theta_G \right\}.$$

Observe that $\mathcal{U}$ is indeed open in $\mathcal{Y}$. In fact, $T(F, \theta_F/2)$ is open in $\mathcal{B}_{\mathcal{Y}}$, so $T(F, \theta_F/2) \cap \{z \in \mathcal{Y} \colon \|z\| < 1\}$ is open in $\mathcal{Y}$; this and the fact that $\cup_{\substack{G \in \mathcal{F}^{<\omega} \\ G \subseteq F}} \Theta_G$ is a finite set yield that $\mathcal{U}$ is open in $\mathcal{Y}$. Moreover, $y \in \mathcal{U}$. Indeed, the assumption that $y \in \mathcal{D}$ yields $|\langle \psi, y\rangle| < 1$ for every $\psi \in \Theta_G$ and every $G \in \mathcal{F}^{<\omega}$, $G \subseteq F$. By the same reason (or from (3.8)) we also get that $y \in \mathcal{P}_F \cap F$. Hence, $\|y\| < 1$, in light of (3.6). Finally, since $y \in \mathcal{B}_F \subseteq T(F, \theta_F/2)$, it follows that $y \in \mathcal{U}$.

We shall now prove that $\Phi = \Phi_F$ on the set $\mathcal{U}$. This amounts to proving that for every $z \in \mathcal{U}$, every $G \in \mathcal{F}^{<\omega}$, with $G \not\subseteq F$, and every $\psi \in \Theta_G$ one has $\varrho_G(\langle \psi, z\rangle) = 0$; equivalently, that $|\langle \psi, z\rangle| \leqslant 1 - \delta_G$.

Now the main idea comes. Fix $G \in \mathcal{F}^{<\omega}$ with $G \not\subseteq F$ and write $\Omega_G = \left\{ \pm S(x_j, \psi_{x_j}, \delta_{x_j}) \right\}_{j=1}^k$. Set $\tilde{\Omega}_G := \left\{ \pm S(x_j, \psi_{x_j}, \delta_{x_j} + \delta_G) \right\}_{j=1}^k$ and $\tilde{\Omega}_H = \Omega_H$ for $H \in \mathcal{F}^{<\omega}$, $H \neq G$. Then define sets $\tilde{\mathcal{P}}_H$ and $\tilde{\mathcal{P}}_n$ as in (3.1) and (3.2), but replacing the slices $(\Omega_H)_{H \in \mathcal{F}^{<\omega}}$ with the slices $(\tilde{\Omega}_H)_{H \in \mathcal{F}^{<\omega}}$. Since $G \not\subseteq F$, we have $\tilde{\Omega}_H = \Omega_H$ for every $H \in \mathcal{F}^{<\omega}$, $H \subseteq F$, hence $\tilde{\mathcal{P}}_F = \mathcal{P}_F$. Moreover, the system of slices $(\tilde{\Omega}_H)_{H \in \mathcal{F}^{<\omega}}$ satisfies all conditions (i)–(vii) in Claim 3.1 with the same parameters $(\varepsilon_H)_{H \in \mathcal{F}^{<\omega}}$ and $(\theta_H)_{H \in \mathcal{F}^{<\omega}}$. Consequently, the sets $\tilde{\mathcal{P}}_H$ and $\tilde{\mathcal{P}}_n$ also satisfy the compatibility condition (†) from Fact 3.2. Hence, given $n \in \mathbb{N}$ with $n \geqslant \dim F, \dim G$, we have

$$\mathcal{P}_F \cap T(F, \theta_F/2) = \tilde{\mathcal{P}}_F \cap T(F, \theta_F/2) \overset{(\dagger)}{\subseteq} \tilde{\mathcal{P}}_n \subseteq \tilde{\mathcal{P}}_G.$$

Besides, $\mathcal{U} \subseteq \mathcal{P}_F \cap T(F, \theta_F/2)$ by (3.7). Therefore, for every $z \in \mathcal{U}$ and every slice $\pm S(x_j, \psi_{x_j}, \delta_{x_j} + \delta_G) \in \tilde{\Omega}_G$, we have $z \notin \pm S(x_j, \psi_{x_j}, \delta_{x_j} + \delta_G)$; in other words, $|\langle \psi_{x_j}, z\rangle| \leqslant 1 - \delta_{x_j} - \delta_G$, for every $j = 1, \dots, k$.

Finally, if $\psi \in \Theta_G$, there is $j \in \{1, \dots, k\}$ with $\psi = \frac{\psi_{x_j}}{1-\delta_{x_j}}$, hence

$$|\langle \psi, z\rangle| = \left|\left\langle \frac{\psi_{x_j}}{1 - \delta_{x_j}}, z\right\rangle\right| \leqslant \frac{1 - \delta_{x_j} - \delta_G}{1 - \delta_{x_j}} \leqslant 1 - \delta_G.$$

This yields $\varrho_G(\langle \psi, z\rangle) = 0$, hence $\Phi(z) = \Phi_F(z)$, as desired. ■

As we explained before, this concludes Step 4 and the proof of Theorem A(ii). ■

Remark 3.4. It is apparent from the above proof that we only used the LUR condition via Fact 2.1, namely we only used that the fact that if $\mathcal{Y}$ is LUR, then every point of $\mathcal{S}_{\mathcal{Y}}$ is strongly exposed. One could wonder whether the argument could be modified as to only require every point of $\mathcal{S}_{\mathcal{Y}}$ being denting for $\mathcal{B}_{\mathcal{Y}}$. However, this is not a more general assumption, since Raja [53] proved that every normed space $\mathcal{Y}$ such that every point of $\mathcal{S}_{\mathcal{Y}}$ is a denting point for $\mathcal{B}_{\mathcal{Y}}$ actually has a LUR renorming. Indeed, although [53, Theorem 1] is stated for Banach spaces, it is readily seen that the proof of the implication (iii)⇒(i) there is valid for every normed space. The same result appeared earlier in [59], but the (probabilistic) proof there seems to really depend on completeness.

As we already mentioned in the Introduction, the following corollary of Theorem A generalises a result from [44].

Corollary 3.5. Let $\mathcal{X}$ be a Banach space with a fundamental biorthogonal system. Then densely many norms on $\mathcal{X}$ are C^∞-smooth and LFC on a dense open subset of $\mathcal{X}$.

Proof. Let $\{e_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma}$ be a fundamental biorthogonal system in $\mathcal{X}$ and let $\mathcal{Y} := \operatorname{span}\{e_\alpha\}_{\alpha\in\Gamma}$. According to Theorem A(ii), we can take a C^∞-smooth and LFC norm $\|\cdot\|$ on $\mathcal{Y}$. Then for every $y \in \mathcal{Y}$, $y \neq 0$, there are $\delta_y > 0$, functionals $\psi_1, \dots, \psi_n \in \mathcal{X}^*$, and a C^∞-smooth function $G\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\|z\| = G\big(\langle \psi_1, z\rangle, \dots, \langle \psi_n, z\rangle\big) \qquad \text{for every } z \in \mathcal{B}^{\mathrm{o}}_{\mathcal{Y}}(y, \delta_y), \tag{3.9}$$

where $\mathcal{B}^{\mathrm{o}}_{\mathcal{Y}}(y, \delta_y) := \{z \in \mathcal{Y}\colon \|y - z\| < \delta_y\}$ (see, e.g., [28, Fact 5.79]). Since obviously $\mathcal{B}^{\mathrm{o}}_{\mathcal{Y}}(y, \delta_y)$ is dense in $\mathcal{B}^{\mathrm{o}}_{\mathcal{X}}(y, \delta_y)$, the formula (3.9) is actually valid for every $z \in \mathcal{B}^{\mathrm{o}}_{\mathcal{X}}(y, \delta_y)$.

Consequently, setting $\mathcal{U} := \cup_{y\in\mathcal{Y}\setminus\{0\}}\mathcal{B}^{\circ}_{\mathcal{X}}(y,\delta_y)$, $\mathcal{U}$ is a dense open set in $\mathcal{X}$, and $\|\cdot\|$ is C^∞-smooth and LFC in $\mathcal{U}$. Finally, C^∞-smooth and LFC norms are dense in $\mathcal{Y}$, so we are done. ■

4 Partitions of Unity

In this short section we explain how Theorem A(iv) follows from [34] and we begin by recalling Haydon's result from [34] that we need (also see [28, Theorem 7.53]). We denote by $\{e^*_\gamma\}_{\gamma\in\Gamma}$ the coordinate functionals on $c_0(\Gamma)$ and we refer to [28, §7.5] for basic definitions concerning partitions of unity.

Theorem 4.1. (Haydon, [34]). Let $\mathcal{Y}$ be a normed space with a C^k-smooth bump function. Assume the following:

- (i) there is a continuous function $\Phi\colon \mathcal{Y} \to c_0(\Gamma)$ such that $e^*_\gamma \circ \Phi$ is C^k-smooth where non-zero, for every $\gamma \in \Gamma$;
- (ii) for every finite set $F \subseteq \Gamma$ there is a C^k-smooth map $P_F\colon \mathcal{Y} \to \mathcal{Y}$ such that $\operatorname{span}(P_F(\mathcal{Y}))$ has locally finite C^k-smooth partitions of unity;
- (iii) for every $x \in \mathcal{Y}$ and $\varepsilon > 0$ there is $\delta > 0$ such that $\|x - P_F(x)\| < \varepsilon$, where $F := \{\gamma \in \Gamma\colon |\Phi(x)(\gamma)| \geqslant \delta\}$.

Then $\mathcal{Y}$ admits locally finite and σ-uniformly discrete C^k-smooth partitions of unity.

In the case when $\mathcal{Y} := \operatorname{span}\{e_\alpha\}_{\alpha\in\Gamma}$ is the linear span of some fundamental biorthogonal system $\{e_\alpha;\varphi_\alpha\}_{\alpha\in\Gamma}$ we can apply Haydon's result as follows. The map Φ is defined by

$$\Phi(x) := \left(\left\langle \frac{\varphi_\alpha}{\|\varphi_\alpha\|}, x\right\rangle\right)_{\alpha\in\Gamma},$$

so that $e^*_\gamma \circ \Phi$ is evidently C^∞-smooth and LFC on $\mathcal{Y}$. The map P_F is just the canonical projection from $\mathcal{Y}$ onto $\operatorname{span}\{e_\alpha\}_{\alpha\in F}$; surely, the finite-dimensional normed space $\operatorname{span}\{e_\alpha\}_{\alpha\in F}$ admits locally finite C^∞-smooth and LFC partitions of unity. The approximation condition is also easily satisfied, since for every $x \in \mathcal{Y}$ there is a finite set $F \subseteq \Gamma$ with $x = P_F(x)$. Hence, Theorem 4.1 yields that $\mathcal{Y}$ admits locally finite and σ-uniformly discrete C^∞-smooth partitions of unity.

In order to explain why such partitions of unity are also LFC, we follow the proof and notation in [28, Theorem 7.53]. Since the functions $\varphi_{F,q,r}$ defined there are LFC and Φ is linear, every map $\varphi_{F,q,r} \circ \Phi$ is LFC (see, e.g., [28, Fact 5.80]). Next, instead of using the partition ring $C^\infty(\mathcal{Z})$, where $\mathcal{Z}$ is a normed space, we replace it with the partition ring of C^∞-smooth LFC functions on $\mathcal{Z}$ (we refer to [28, Definition 7.47] for the definition of partition ring). Since the partition ring of C^∞-smooth LFC functions on $\mathcal{Z}$ is determined locally, in the sense of [28, Definition 7.48], all the statements in [28, Lemma 7.49] are equivalent. Also notice that the partition ring of C^∞-smooth LFC functions on $\mathcal{Y}$ contains a bump function by Theorem A(iii). After these remarks, we can return to the argument in [28, Theorem 7.53]. The unique additional difference is that we need to show that each of the sets $\Phi^{-1}(W_{F,q,r})$, $P_F^{-1}(V)$, and $(Id - P_F)^{-1}(U_m)$ is of the form $\{f \neq 0\}$, for some C^∞-smooth and LFC function $f\colon \mathcal{Y} \to \mathbb{R}$. We explain this for sets $P_F^{-1}(V)$, the other two cases being analogous. By assumption $V = \{g \neq 0\}$, for some C^∞-smooth and LFC function $g\colon \mathrm{span}\{e_\alpha\}_{\alpha\in F} \to \mathbb{R}$. Then $P_F^{-1}(V) = \{g \circ P_F \neq 0\}$ and the function $g \circ P_F\colon \mathcal{Y} \to \mathbb{R}$ is C^∞-smooth and LFC (again, the LFC property follows from the linearity of P_F (this argument does not work in the general setting of Theorem 4.1, since there the maps Φ and P_F are not necessarily linear)). All the remaining steps of the proof being identical to the argument in [28], we conclude the validity of Theorem A(iv).

5 C^1-Smooth LUR Norms

In this section we discuss the proof of Theorem A(v), thereby concluding the proof of our main result. As we said already, the argument is an adaptation of [31], therefore we shall restrict ourselves to defining the desired C^1-smooth LUR norm and refer to [31] for the verification of the various properties. Very roughly speaking, the main idea in [31] is to smoothen up the formula for the norm from Troyanski's renorming technique. Since LUR renormings typically involve countably many contributions around each point, it is crucial to have uniform Lipschitz estimates in order to obtain C^1-smoothness in the limit. Two earlier results, based on the same strategy but substantially less technical, are [43] and [48] (also see [15, §V.1]), where a C^1-smooth LUR norm is constructed in every separable Banach space and in $c_0(\Gamma)$, respectively.

If $\mathcal{Y} := \mathrm{span}\{e_\alpha\}_{\alpha\in\Gamma}$ is the linear span of a fundamental biorthogonal system $\{e_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma}$, then, to begin with, we can assume that $\{e_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma}$ is bounded, thanks to Theorem 2.4(i). Secondly, in light of Theorem A(ii), C^∞-smooth norms are dense in the set of all equivalent norms on $\mathcal{Y}$. Hence, Theorem A(v) is a consequence of Theorem 5.1 below.

Theorem 5.1. Let $\mathcal{X}$ be a Banach space with a bounded fundamental biorthogonal system $\{e_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma}$. Let $\mathcal{Y} := \mathrm{span}\{e_\alpha\}_{\alpha\in\Gamma}$ and let $\|\cdot\|$ be a C^1-smooth norm on $\mathcal{Y}$. Then $\|\cdot\|$ can be approximated by C^1-smooth LUR norms.

Before the proof we need to recall a couple of definitions. Let $\mathcal{Y}$ be a normed space and $\mathcal{D}$ be a convex subset of $\mathcal{Y}$. A convex function $f\colon \mathcal{D} \to \mathbb{R}$ is *strictly convex* if its graph contains no non-trivial segments. Given a set Λ, a function $g = (g_\gamma)_{\gamma\in\Lambda}\colon \mathcal{Y} \to \ell_\infty(\Lambda)$ is *coordinate-wise convex* (resp. *coordinate-wise C^k-smooth*) if, for every $\gamma \in \Lambda$, the function g_γ is convex (resp. C^k-smooth).

Proof of Theorem 5.1. Without loss of generality, we can assume that $\|e_\alpha\| = 1$ and $\|\varphi_\alpha\| \leqslant M$, for some $M \geqslant 1$ (M corresponds to the constant $2C$ in [31, §4.2]). We begin by fixing some notation. Let, for every $n \in \mathbb{N}$, $\xi_n\colon [0,\infty) \to [0,\infty)$ be a C^∞-smooth, 1-Lipschitz convex function such that $\xi_n(t) = 0$ when $t \in [0, 1/n]$ and $\xi_n(t) = t - \frac{2}{n}$ when $t \geqslant 3/n$ (take the convolution of $t \mapsto \max\{0, t - \frac{2}{n}\}$ with a suitable bump). Let $\mathcal{B}^{\mathrm{o}}_{\mathcal{Y}}$ be the open unit ball of $\mathcal{Y}$. Given a set Λ, on $\ell_\infty(\Lambda)$ we consider the seminorm $\lceil\cdot\rceil$ defined by

$$\lceil z\rceil := \inf\left\{t > 0\colon \{\gamma \in \Lambda\colon |z(\gamma)| > t\} \text{ is finite}\right\} = \|q(z)\|_{\ell_\infty(\Lambda)/c_0(\Lambda)},$$

where $q\colon \ell_\infty(\Lambda) \to \ell_\infty(\Lambda)/c_0(\Lambda)$ is the quotient map. For $\eta \in (0,1)$ we consider the set

$$A_\eta(\Lambda) := \left\{z \in \ell_\infty(\Lambda)\colon \lceil z\rceil < (1-\eta)\|z\|_\infty\right\}.$$

Note that $z \in A_\eta(\Lambda)$ is a "strong maximum" condition, in the sense that finitely many coordinates of z are quantitatively larger than the others. The construction of the norm is then performed in three steps.

First, consider the system of functions $\left\{g_{n,m,l}\colon n, m, l \in \mathbb{N}, l \leqslant n\right\}$ on $\mathbb{R}^2$ obtained by shifting and scaling a certain function g. Set $g\colon \mathbb{R}^2 \to [0,\infty)$ to be $g(t,s) = 0$ for $t \leqslant 0$ and, for $t > 0$,

$$g(t,s) := \exp\left(-10/t\right) \cdot \left(\frac{s^2}{100} + \frac{s}{10} + 1\right).$$

For $n, m, l \in \mathbb{N}$ with $l \leqslant n$ define

$$g_{n,m,l}(t,s) := g\left(\frac{t - l/n}{1 + nM}, \theta_{n,m}\frac{s}{1 + nM}\right).$$

The main properties of the system $\left\{g_{n,m,l}\right\}$ are listed in [31, Lemma 4.5], where in particular the parameters $\rho_n \in (0, 1/2)$, $\theta_{n,m} \in (0,1)$, and $\kappa_{n,m} \in (0, \rho_n)$ are fixed.

Let $\eta_{n,m} := \rho_n - \kappa_{n,m} \in (0, 1/2)$. Consider the set

$$\Lambda_n := \{(A, B) : \emptyset \neq B \subseteq A \subseteq \Gamma, |A| \leqslant n\}$$

and the functions $H_{n,m} : \mathcal{B}^{\text{o}}_{\mathcal{Y}} \to \ell_\infty(\Lambda_n)$ defined by

$$H_{n,m}y(A, B) := g_{n,m,|A|}\left(\sum_{\alpha \in A} \xi_n\big(|\langle \varphi_\alpha, y\rangle|\big), \xi_n\left(\left\| y - \sum_{\alpha \in B} \langle \varphi_\alpha, y\rangle e_\alpha \right\|\right)\right).$$

It is easily seen that $H_{n,m}$ is 1-Lipschitz, coordinate-wise convex and coordinate-wise C^1-smooth. Additionally, [31, Lemma 4.7] asserts that $H_{n,m}y \in A_{\eta_{n,m}}(\Lambda_n)$ for every $y \in \mathcal{B}^{\text{o}}_{\mathcal{Y}}$ with $H_{n,m}y \neq 0$.

The second step consists in building the norms for gluing together the functions $H_{n,m}$ in the standard way. For $\eta \in (0, 1/2)$, take a C^∞-smooth convex function $\psi_\eta : [0, \infty) \to [0, \infty)$ such that $\psi_\eta(t) = 0$ for $t \in [0, 1 - \eta]$, ψ_η is strictly convex on $[1 - \eta, \infty)$, and $\psi_\eta(1) = 1$. We also require that $\psi_{\eta_1}(t) \leqslant \psi_{\eta_2}(t)$ for $t \in [0, 1]$ and $\eta_1 < \eta_2$. Set $\Phi_\eta : \ell_\infty(\Lambda) \to [0, \infty]$ by

$$\Phi_\eta(z) := \sum_{\gamma \in \Lambda} \psi_\eta(|z(\gamma)|)$$

and let Z_η be the Minkowski functional of the set $\{\Phi_\eta \leqslant 1\}$. One readily sees that $Z_{\eta_1} \leqslant Z_{\eta_2}$ if $\eta_1 \leqslant \eta_2$, that Z_η is a lattice norm, and that $(1 - \eta)Z_\eta \leqslant \|\cdot\|_\infty \leqslant Z_\eta$. In particular, Z_η is 2-Lipschitz on $\ell_\infty(\Lambda)$. Moreover, Z_η is C^∞-smooth and LFC on the set $A_\eta(\Lambda)$ and $(1 - \eta)Z_\eta(z) < \|z\|_\infty$ for every $z \in A_\eta(\Lambda)$, [31, Lemma 4.1]. Additionally, Z_η satisfies a LUR condition for "large coordinates" (namely for those coordinates γ for which $z(\gamma) > (1 - \gamma)Z_\eta(z)$), [31, Lemma 4.3 and Lemma 4.4].

Finally, in the last step we glue all the ingredients together. Consider the norm Z_η on $\ell_\infty(\Lambda)$ with $\eta = \eta_{n,m}$ and $\Lambda = \Lambda_n$ as defined above. For $j, n, m \in \mathbb{N}$ define $J_{j,n,m} : \mathcal{B}^{\text{o}}_{\mathcal{Y}} \to [0, \infty)$ by

$$J_{j,n,m} := \xi_j \circ Z_{\eta_{n,m}} \circ H_{n,m}.$$

Clearly, $J_{j,n,m}$ is 2-Lipschitz on $\mathcal{B}^{\text{o}}_{\mathcal{Y}}$ and $J_{j,n,m}(0) = 0$. Moreover, $Z_{\eta_{n,m}} \circ H_{n,m}$ is C^1-smooth on the set $\{H_{n,m} \neq 0\}$; this follows from the facts that $H_{n,m}$ is coordinate-wise C^1-smooth, that $H_{n,m}y \in A_{\eta_{n,m}}(\Lambda_n)$ when $H_{n,m}y \neq 0$, and that $Z_{\eta_{n,m}}$ is C^1-smooth on $A_{\eta_{n,m}}(\Lambda_n)$.

Hence, $J_{j,n,m}$ is C^1-smooth on $\mathcal{B}^{\circ}_{\mathcal{Y}}$. Finally, fix $\varepsilon > 0$ and define $J\colon \mathcal{B}^{\circ}_{\mathcal{Y}} \to [0,\infty)$ by

$$J(y)^2 := \|y\|^2 + \varepsilon \sum_{j,n,m\in\mathbb{N}} 2^{-(j+n+m)} J_{j,n,m}(y)^2.$$

Since each $J_{j,n,m}$ is 2-Lipschitz, the series of the derivatives converges and J is C^1-smooth on $\mathcal{B}^{\circ}_{\mathcal{Y}}$. Also, $\|y\| \leqslant J(y) \leqslant \sqrt{1+4\varepsilon}\|y\|$, hence

$$\frac{1-\varepsilon}{\sqrt{1+4\varepsilon}}\mathcal{B}_{\mathcal{Y}} \subseteq \{J \leqslant 1-\varepsilon\} \subseteq \mathcal{B}_{\mathcal{Y}}.$$

Therefore, the Minkowski functional $|||\cdot|||$ of the set $\{J \leqslant 1-\varepsilon\}$ is a norm that approximates $\|\cdot\|$. Moreover, J is C^1-smooth on $\mathcal{B}^{\circ}_{\mathcal{Y}} \supseteq \{J < 1\}$, so Lemma 2.5 yields that $|||\cdot|||$ is C^1-smooth. What remains to be proved is that $|||\cdot|||$ is LUR, which follows the argument in [31, Proposition 4.10]. As a matter of fact the argument is even simpler in our context, since instead of finding a finite set A with

$$\left\| y - \sum_{\gamma\in A} Q_\gamma y \right\| < \varepsilon,$$

we can find a finite set $A = \operatorname{supp}(y)$ with $y = \sum_{\alpha\in A}\langle\varphi_\alpha, y\rangle e_\alpha$ (Q_γ in [31] are projections that correspond to the rank-one projections $\langle\varphi_\alpha,\cdot\rangle e_\alpha$). ■

5.1 Higher-order smoothness and super-reflexivity

We shall conclude the section observing that the C^1-smooth LUR norms that we constructed above can't in general admit any higher-order smoothness. This essentially follows from results in [21] (also see [28, §5.2]) and we sketch the explanation below.

We write that a norm on $\mathcal{X}$ is $C^{1,+}_{\mathrm{loc}}$-smooth (resp. $C^{1,+}$-smooth) if it is differentiable with locally uniformly continuous (resp. uniformly continuous) derivative on $\mathcal{S}_{\mathcal{X}}$. Moreover, we also recall that the *modulus of smoothness* of a norm $\|\cdot\|$ on a Banach space $\mathcal{X}$ is the function $\rho_{\mathcal{X}}\colon (0,\infty) \to (0,\infty)$ defined by

$$\rho_{\mathcal{X}}(\tau) := \sup\left\{\frac{\|x+\tau h\| + \|x-\tau h\| - 2}{2} \colon \|x\| = \|h\| = 1\right\}.$$

A Banach space $\mathcal{X}$ is *uniformly smooth* if $\frac{\rho_{\mathcal{X}}(\tau)}{\tau} \to 0$ as $\tau \to 0^+$; it is a classical result that uniformly smooth Banach spaces are super-reflexive.

Theorem 5.2. Let $\mathcal{Y}$ be a normed space with a $C^{1,+}_{\mathrm{loc}}$-smooth LUR norm $\|\cdot\|$. Then the completion $\hat{\mathcal{Y}}$ of $\mathcal{Y}$ is super-reflexive.

Proof. Since $\|\cdot\|$ is LUR, every point of its unit sphere is strongly exposed. Hence, [28, Theorem 5.46] (which comes from [21]) implies that $\mathcal{Y}$ has a $C^{1,+}$-smooth norm $|||\cdot|||$. Let $g\colon \mathcal{Y}\setminus\{0\}\to\mathbb{R}$ be the derivative of $|||\cdot|||$; by homogeneity, g is uniformly continuous on $\{y\in\mathcal{Y}\colon |||y|||\geqslant 1/2\}$ with modulus of continuity, say ω_g. Fix $y,h\in\mathcal{Y}$ with $|||y|||=|||h|||=1$ and $\tau\in(0,1/2)$. By Lagrange's theorem, there are $\theta^{\pm}\in(0,1)$ with

$$|||y\pm\tau h|||-|||y|||=\langle g(y\pm\theta^{\pm}\tau h),\pm\tau h\rangle$$

(here we are using the facts that $|||\cdot|||$ is differentiable on $\mathcal{Y}$ and $y,h\in\mathcal{Y}$). Note that $\left|\left|\left|y\pm\theta^{\pm}\tau h\right|\right|\right|\geqslant 1/2$. Then we have

$$\begin{aligned}|||y+\tau h|||+|||y-\tau h|||-2|||y||| &= \left\langle g(y+\theta^{+}\tau h)-g(y-\theta^{-}\tau h),\tau h\right\rangle\\ &\leqslant \left|\left|\left|g(y+\theta^{+}\tau h)-g(y-\theta^{-}\tau h)\right|\right|\right|\cdot\tau\\ &\leqslant \omega_g\left(\left|\left|\left|\theta^{+}\tau h+\theta^{-}\tau h\right|\right|\right|\right)\cdot\tau\leqslant\omega_g(2\tau)\cdot\tau.\end{aligned}$$

Since $\mathcal{Y}$ is dense in $\hat{\mathcal{Y}}$, the previous inequality is also valid for every $y,h\in\hat{\mathcal{Y}}$ with $|||y|||=|||h|||=1$. Hence, (dividing by 2τ and) passing to the supremum over such y,h yields

$$\frac{\rho_{\hat{\mathcal{Y}}}(\tau)}{\tau}\leqslant\frac{\omega_g(2\tau)}{2}\to 0,\qquad \text{as } \tau\to 0^{+}.$$

Consequently, $(\hat{\mathcal{Y}},|||\cdot|||)$ is uniformly smooth, as desired. (This argument showing that $C^{1,+}$-smooth norm are uniformly smooth certainly is a classical one. We presented it just because of the passage to the completion that we needed in the midst of it.) ■

Remark 5.3. In particular, no dense subspace $\mathcal{Y}$ of $c_0(\Gamma)$ admits a $C^{1,+}_{\mathrm{loc}}$-smooth LUR norm. This particular case could also be proved by combining [28, Theorem 5.46] with [61]. More precisely, a standard "small perturbations" argument shows that it is sufficient to consider $\Gamma=\mathbb{N}$ and $\mathcal{Y}=c_{00}$ (see, e.g., [33, Theorem 2.1]). Then [28, Theorem 5.46] would imply that c_{00} has a $C^{1,+}$-smooth norm, which is false [61].

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Conflict of interest

The authors have no conflict of interest to declare that are relevant to the content of this article.

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On isometric embeddings into the set of strongly norm-attaining Lipschitz functions

Sheldon Dantas [a,*], Rubén Medina [b,c], Andrés Quilis [d,c], Óscar Roldán [a]

[a] *Departamento de Análisis Matemático, Facultad de Ciencias Matemáticas, Universidad de Valencia, Doctor Moliner 50, 46100 Burjasot (Valencia), Spain*
[b] *Universidad de Granada, Facultad de Ciencias. Departamento de Análisis Matemático, 18071 Granada, Spain*
[c] *Czech Technical University in Prague, Faculty of Electrical Engineering. Department of Mathematics, Technická 2, 166 27, Praha 6, Czech Republic*
[d] *Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, Camíde Vera, s/n, Spain*

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In this paper, we provide an infinite metric space M such that the set $\mathrm{SNA}(M)$ of strongly norm-attaining Lipschitz functions on M does not contain a subspace which is linearly isometric to c_0. This answers a question posed by Antonio Avilés, Gonzalo Martínez-Cervantes, Abraham Rueda Zoca, and Pedro Tradacete. On the other hand, we prove that $\mathrm{SNA}(M)$ contains an isometric copy of c_0 whenever M is an infinite metric space which is not uniformly discrete. In particular, the latter holds true for all infinite compact metric spaces while it does not hold true for all proper metric spaces. We also provide some positive results in the non-separable setting.

1. Introduction

Let M be a pointed metric space. We consider the subset $\mathrm{SNA}(M)$ of $\mathrm{Lip}_0(M)$ of all strongly norm-attaining Lipschitz functions on M. In 2016, Marek Cúth, Michal Doucha, and Przemysław Wojtaszczyk [8] proved that ℓ_∞ (and hence c_0) embeds isomorphically in $\mathrm{Lip}_0(M)$ for any infinite metric space M. One year later, this result was improved by Marek Cúth and Michal Johanis by showing that ℓ_∞ (and hence c_0) embeds isometrically in $\mathrm{Lip}_0(M)$ [9]. Motivated by the papers [1,13,19], we turn our attention to the study

* Corresponding author.
E-mail addresses: sheldon.dantas@uv.es (S. Dantas), rubenmedina@ugr.es (R. Medina), anquisan@posgrado.upv.es (A. Quilis), oscar.roldan@uv.es (Ó. Roldán).

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of the analogous problems for the subset $\mathrm{SNA}(M)$. In big contrast to what happens in the classical norm-attainment theory, where Martin Rmoutil [21] proved that the set of all norm-attaining functionals needs not contain 2-dimensional spaces, Antonio Avilés, Gonzalo Martínez Cervantes, Abraham Rueda Zoca, and Pedro Tradacete [1] provided a beautiful and interesting construction, and showed that $\mathrm{SNA}(M)$ contains an isomorphic copy of c_0 for every infinite complete metric space M (answering [19, Question 2] in the positive). At the very ending of that paper, the authors wondered whether an isometric version of this result holds true (see [1, Remark 3.6]).

In this paper, we answer the latter question by providing an example of an infinite uniformly discrete metric space M such that $\mathrm{SNA}(M)$ does *not* contain any subspace which is linearly isometric to c_0 (see Theorem 4.1). On the other hand, we prove that $\mathrm{SNA}(M)$ does contain a linearly isometric copy of c_0 whenever M is infinite but not uniformly discrete (see Theorem 4.2). It turns out that this is no longer true even for proper metric spaces (see Theorem 4.4). We conclude the paper by tackling the problem in the non-separable setting; we prove that whenever $\mathrm{dens}(M') = \varGamma > \aleph_0$, then $\mathrm{SNA}(M)$ contains a linearly isometric copy of $c_0(\varGamma)$ (see Theorem 5.2).

2. Preliminaries and notation

Throughout the text, all the vector spaces will be considered to be *real*. Let (M, d) be a pointed metric space (that is, a metric space with a distinguished point 0). We denote by $\mathrm{Lip}_0(M)$ the Banach space of all Lipschitz functions $f : M \to \mathbb{R}$ such that $f(0) = 0$, endowed with the Lipschitz norm

$$\|f\|_{\mathrm{Lip}} := \sup\left\{\frac{|f(y) - f(x)|}{d(x, y)} : x, y \in M,\ x \neq y\right\}.$$

We say that a Lipschitz function $f \in \mathrm{Lip}_0(M)$ *strongly attains its norm*, or that it is *strongly norm-attaining*, if there exist two different points $p, q \in M$ such that

$$\|f\|_{\mathrm{Lip}} = \frac{|f(p) - f(q)|}{d(p, q)}.$$

The set of strongly norm-attaining Lipschitz functions on M will be denoted by $\mathrm{SNA}(M)$. In the last few years, this topic has been intensively studied. We send the reader to [1–7,11,12,14,15,17–19] and the references therein.

In this paper, we will discuss the possibility of finding a linear space isometrically isomorphic to c_0 inside the set $\mathrm{SNA}(M)$. Clearly, this situation is reduced only to infinite metric spaces. Note also that, for our purposes, the choice of the distinguished point 0 in the pointed metric space M is irrelevant in our context. Indeed, if 0 and $0'$ are two distinguished points in M, then the mapping from $\mathrm{Lip}_0(M)$ to $\mathrm{Lip}_{0'}(M)$ defined as $f \mapsto f - f(0')$ is a linear isometry that completely preserves the strong norm-attainment behaviour of the mappings, so we do not need to worry about the choice of the distinguished point.

In this document, the expression *linear subspaces of* $\mathrm{SNA}(M)$ should be understood as linear subspaces of $\mathrm{Lip}_0(M)$ consisting of strongly norm-attaining Lipschitz functions. Also, if Y is a Banach space, then we say that Y isometrically embeds in $\mathrm{SNA}(M)$ (or, equivalently, $\mathrm{SNA}(M)$ contains a linearly isometric copy of Y), whenever there exists a linear isometric embedding $U : Y \to \mathrm{Lip}_0(M)$ such that $U(Y) \subseteq \mathrm{SNA}(M)$.

The *separation radius* of a point $x \in M$ is defined by

$$R(x) := \inf\Big\{d(x, y) : y \in M \setminus \{x\}\Big\},$$

and it will play an important role in some of the upcoming results. We will say that a point x from a metric space M *attains its separation radius* whenever there is some $y \in M$ such that $R(x) = d(x, y)$. The symbol M' stands for the set of all cluster points of M. Recall that a metric space M is said to be *discrete* if $M' = \emptyset$, *uniformly discrete* if $\inf\{R(x) : x \in M\} > 0$, and *proper* if every closed and bounded subset of M is compact. The notation $B(x, R)$ stands for the closed ball of centre $x \in M$ and radius $R > 0$.

Throughout the entire note, we will denote by $c_0(\Gamma)$ the space of all real valued functions over a set Γ satisfying that for every element $x \in c_0(\Gamma)$, the set $\{\gamma \in \Gamma : |x(\gamma)| \geqslant \varepsilon\}$ is finite for every $\varepsilon > 0$. If Γ is countable, we will refer to $c_0(\Gamma)$ simply as c_0.

Let X be a separable Banach space with a Schauder basis denoted by $\{x_n\}_{n=1}^{\infty}$. We say that a sequence $\{y_n\}_{n=1}^{\infty}$ in a Banach space Y is *(isometrically) equivalent* to the basis $\{x_n\}_{n=1}^{\infty}$ if there exists a linear (isometric) isomorphism $T : \overline{\text{span}}\{y_n : n \in \mathbb{N}\} \to X$ such that $T(y_n) = x_n$ for all $n \in \mathbb{N}$. The following straightforward facts will be used throughout the text without any explicit reference.

(i) A sequence $\{x_n\}_{n=1}^{\infty}$ is isometrically equivalent to the canonical basis of c_0 if and only if the equality $\left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\| = \max_n |\lambda_n|$ holds for every sequence $\{\lambda_n\}_{n=1}^{\infty} \in c_0$.

(ii) If a sequence $\{x_n\}_{n=1}^{\infty}$ is isometrically equivalent to the canonical basis of c_0, then so is the sequence $\{\varepsilon_n x_n\}_{n=1}^{\infty}$, where $\varepsilon_n \in \{-1, 1\}$ for every $n \in \mathbb{N}$.

(iii) Any subsequence of a sequence which is isometrically equivalent to the canonical basis of c_0 is once again isometrically equivalent to the same basis.

Given any set A and a natural number $k \in \mathbb{N}$, we denote by $A^{[k]}$ the set of all subsets of A with exactly k elements. We will use Ramsey's Theorem intensively throughout the text, which ensures that given any infinite set A and any finite partition of the set $A^{[k]}$, $\{B_1, \ldots, B_n\}$ for some $n \in \mathbb{N}$, there exists an infinite subset S of A and a number $i \in \{1, \ldots n\}$ such that $S^{[k]}$ is contained in B_i (see, for instance, [10, Proposition 6.4]).

Finally, let us note the following. If M is any metric space and M_c is its completion, then, as Lipschitz mappings extend uniquely by uniform continuity, one clearly has $\text{Lip}_0(M) = \text{Lip}_0(M_c)$. However, it can happen that a Lipschitz mapping strongly attains its norm on M_c but not on M, so $\text{SNA}(M)$ can be a strict subset of $\text{SNA}(M_c)$ sometimes. For this reason, we will state our main positive results for general metric spaces (complete or not) whenever possible.

3. Some useful tools

In this section, we state and prove some auxiliary results that will be crucial for the rest of the note. The following are three essential yet straightforward lemmas that hold in any metric space. We provide their proofs for the sake of completeness.

Lemma 3.1. *Let M be a metric space. Suppose that $\{f_n\}_{n=1}^{\infty} \subseteq \text{Lip}_0(M)$ is a sequence isometrically equivalent to the canonical basis of c_0. If for every $n \in \mathbb{N}$, the function f_n strongly attains its Lipschitz norm at a pair of points $x_n, y_n \in M$, then $f_m(x_n) = f_m(y_n)$ for every $m \in \mathbb{N} \setminus \{n\}$.*

Proof. Let $n \in \mathbb{N}$ be fixed. Suppose that the function f_n strongly attains its Lipschitz norm at a pair of points $x_n, y_n \in M$. Without loss of generality, we may (and we do) assume that $|f_n(x_n) - f_n(y_n)| = f_n(x_n) - f_n(y_n) = d(x_n, y_n)$. Let us suppose by contradiction that there exist natural numbers $m \neq n$ such that $f_m(x_n) \neq f_m(y_n)$. We may again suppose without loss of generality that $f_m(x_n) > f_m(y_n)$ (otherwise we may consider the sequence $\{g_k\}_{k=1}^{\infty}$ defined as $g_m = -f_m$ and $g_k = f_k$ for $k \neq m$, which is still equivalent to the c_0 basis). Set $f := f_n + f_m$. Then, we have that

$$\begin{aligned}|f(x_n) - f(y_n)| &\geqslant (f_n + f_m)(x_n) - (f_n + f_m)(y_n)\\ &= f_n(x_n) - f_n(y_n) + f_m(x_n) - f_m(y_n)\\ &> d(x_n, y_n),\end{aligned}$$

which yields a contradiction with the fact that f is 1-Lipschitz. ■

Lemma 3.2. *Let M be a metric space. Suppose that $\{f_n\}_{n=1}^{\infty} \subseteq \mathrm{Lip}_0(M)$ is a sequence equivalent to the canonical basis of c_0. Then, for all $p \in M$, $\lim_{n\to\infty} |f_n(p)| = 0$.*

Proof. Let $T: c_0 \to \overline{\mathrm{span}}\{f_n : n \in \mathbb{N}\}$ be a linear isomorphism with $T(e_n) = f_n$ for all $n \in \mathbb{N}$ and set $C = \|T\|$. Suppose that for some $p \in M$, the sequence $\{f_n(p)\}_{n=1}^{\infty}$ does not converge to 0. Then, there exists $N \in \mathbb{N}$ such that $\sum_{n=1}^{N} |f_n(p)| > C \cdot d(p, 0)$. However, this implies that there exist $\{\varepsilon_n\}_{n=1}^{N} \subseteq \{-1, 1\}^N$ such that the function $\sum_{k=1}^{N} \varepsilon_n f_n$ is not C-Lipschitz, contradicting the fact that the operator norm of T is C. ■

Finally, for the upcoming positive results of the paper, we need the following generalization of [1, Lemma 3.1].

Lemma 3.3. *Let Γ be a nonempty index set. Let M be a pointed metric space such that there exist $\{(x_\gamma, y_\gamma)\}_{\gamma\in\Gamma} \subseteq M \times M$ with $x_\gamma \neq y_\gamma$ for every $\gamma \in \Gamma$ satisfying that*

$$d(x_\alpha, x_\beta) \geqslant d(x_\alpha, y_\alpha) + d(x_\beta, y_\beta) \tag{3.1}$$

for every $\alpha \neq \beta \in \Gamma$. Then there is a linear subspace of $\mathrm{SNA}(M)$ linearly isometric to $c_0(\Gamma)$.

Proof. Since the choice of the point 0 is not relevant as noted before, we may assume that $0 \in \{y_\gamma\}_{\gamma\in\Gamma}$. Pick $\gamma' \in \Gamma$ such that $y_{\gamma'} := 0$. For each $\gamma \in \Gamma$, define $f_\gamma : M \to \mathbb{R}$ by

$$f_\gamma(x) := \max\{0, d(x_\gamma, y_\gamma) - d(x, x_\gamma)\} \quad (x \in M).$$

Clearly, $f_{\gamma'}(0) = 0$. Also, if $\gamma \neq \gamma'$ we have by means of the triangle inequality,

$$d(x_\gamma, y_\gamma) \overset{(3.1)}{\leqslant} d(x_\gamma, x_{\gamma'}) - d(x_{\gamma'}, y_{\gamma'}) \leq d(x_\gamma, y_{\gamma'}).$$

Therefore, $f_\gamma(0) = \max\{0, d(x_\gamma, y_\gamma) - d(x_\gamma, y_{\gamma'})\} = 0$ for every $\gamma \in \Gamma$.

Notice that for every $\gamma \in \Gamma$, $f_\gamma(x) \neq 0$ if and only if x is such that $d(x_\gamma, x) < d(x_\gamma, y_\gamma)$. Also, notice that by (3.1), for every $\alpha \neq \beta$ in Γ, $B(x_\alpha, d(x_\alpha, y_\alpha)) \cap B(x_\beta, d(x_\beta, y_\beta))$ has empty interior.

Let $\lambda := (\lambda_\gamma)_{\gamma\in\Gamma} \in c_0(\Gamma)$ and let $\gamma_0 \in \Gamma$ be such that $|\lambda_{\gamma_0}| = \|\lambda\|_\infty$. Set $f = \sum_{\gamma\in\Gamma} \lambda_\gamma f_\gamma$. Clearly,

$$|f(x_{\gamma_0}) - f(y_{\gamma_0})| = |\lambda_{\gamma_0}| d(x_{\gamma_0}, y_{\gamma_0}).$$

Therefore, we will be done if we show that $\|f\|_{\mathrm{Lip}} \leqslant |\lambda_{\gamma_0}|$:

Let $x \neq y$ be two points in M. We will distinguish several cases.

(a) If both x and y lie outside of $\bigcup_{\gamma\in\Gamma} B(x_\gamma, d(x_\gamma, y_\gamma))$, then clearly $|f(x) - f(y)| = 0$.
(b) Assume that $x \notin \bigcup_{\gamma\in\Gamma} B(x_\gamma, d(x_\gamma, y_\gamma))$ and that there exists some $\alpha \in \Gamma$ such that $y \in B(x_\alpha, d(x_\alpha, y_\alpha))$. Since $0 \leqslant d(x_\alpha, y_\alpha) - d(x_\alpha, y) \leqslant d(x_\alpha, x) - d(x_\alpha, y) \leqslant d(x, y)$, we have

$$|f(x) - f(y)| = |\lambda_\alpha|(d(x_\alpha, y_\alpha) - d(x_\alpha, y)) \leqslant |\lambda_{\gamma_0}| d(x, y).$$

(c) Assume now that there is some $\gamma \in \Gamma$ such that $x, y \in B(x_\gamma, d(x_\gamma, y_\gamma))$. Then, by the triangle inequality,

$$\begin{aligned} |f(x) - f(y)| &= |\lambda_\gamma||(d(x_\gamma, y_\gamma) - d(x_\gamma, x)) - (d(x_\gamma, y_\gamma) - d(x_\gamma, y))| \\ &= |\lambda_{\gamma_0}|\,|d(x_\gamma, x) - d(x_\gamma, y)| \\ &\leq |\lambda_{\gamma_0}|d(x, y). \end{aligned}$$

(d) Finally, if there are different $\alpha, \beta \in \Gamma$ such that $x \in B(x_\alpha, d(x_\alpha, y_\alpha))$ and $y \in B(x_\beta, d(x_\beta, y_\beta))$, thanks to (3.1), we know that $d(x_\alpha, y_\alpha) \leqslant d(x_\alpha, x_\beta) - d(x_\beta, y_\beta)$. Hence, by means of this last inequality and the triangle inequality, we have that

$$\begin{aligned} d(x_\alpha, y_\alpha) - d(x_\alpha, x) + d(x_\beta, y_\beta) - d(x_\beta, y) &\leqslant d(x_\alpha, x_\beta) - d(x_\alpha, x) - d(x_\beta, y) \\ &\leqslant d(x, y). \end{aligned} \tag{3.2}$$

Then,

$$\begin{aligned} |f(x) - f(y)| &= |\lambda_\alpha f_\alpha(x) - \lambda_\beta f_\beta(y)| \\ &\leqslant |\lambda_{\gamma_0}|(f_\alpha(x) + f_\beta(y)) \\ &= |\lambda_{\gamma_0}|(d(x_\alpha, y_\alpha) - d(x_\alpha, x) + d(x_\beta, y_\beta) - d(x_\beta, y)) \\ &\overset{(3.2)}{\leqslant} |\lambda_{\gamma_0}|d(x, y). \quad \blacksquare \end{aligned}$$

4. The isometric containment of c_0 in $\mathrm{SNA}(M)$

In this section we turn our attention to the main results of the paper.

4.1. A bounded and uniformly discrete counterexample

In this subsection, we construct an infinite complete metric space M such that the set $\mathrm{SNA}(M)$ of strongly norm-attaining Lipschitz functions does not contain a linearly isometric copy of c_0, answering a question posed in [1, Remark 3.6]. It is worth mentioning that no point of the constructed metric space attains its separation radius.

Theorem 4.1. *There exists an infinite bounded uniformly discrete metric space M such that c_0 is not isometrically contained in* $\mathrm{SNA}(M)$ *and for which no point in M attains its separation radius.*

Proof. Let $M = \{p_n\}_{n\in\mathbb{N}}$ be any countable set endowed with the metric d given by

$$d(p_n, p_m) = \begin{cases} 1 + \frac{1}{\max\{m,n\}} & \text{if } m \neq n, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the diameter of M is $3/2$.

For the sake of contradiction, let us suppose that there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of strongly norm-attaining functions which is isometrically equivalent to the canonical basis of c_0. For each $n \in \mathbb{N}$, let $x_n, y_n \in M$ be such that $x_n \neq y_n$ and $|f_n(x_n) - f_n(y_n)| = d(x_n, y_n)$. Our goal is to find two natural numbers $n_0 \neq m_0$ and $\delta \in \{-1, 1\}$ such that the Lipschitz function $f_{n_0} + \delta f_{m_0}$ has Lipschitz norm strictly greater than 1. This will lead to a contradiction.

Let us consider the sets

$$A := \left\{ \{n, m\} \in \mathbb{N}^{[2]} : \{x_n, y_n\} \cap \{x_m, y_m\} = \emptyset \right\},$$

$$B_1 := \left\{ \{n, m\} \in \mathbb{N}^{[2]} : x_n = x_m \right\},$$
$$B_2 := \left\{ \{n, m\} \in \mathbb{N}^{[2]} : y_n = y_m \right\}, \text{ and}$$
$$B_3 := \left\{ \{n, m\} \in \mathbb{N}^{[2]} : x_n = y_m \text{ or } x_m = y_n \right\}.$$

Note that, as an immediate consequence of Lemma 3.1, those sets form a partition of $\mathbb{N}^{[2]}$. By Ramsey's theorem, there exists $C \in \{A, B_1, B_2, B_3\}$ and an infinite set $S \subseteq \mathbb{N}$ such that $S^{[2]} \subseteq C$.

Case 1: $C = A$.

By restricting ourselves to S, we have that $\{x_n, y_n\} \cap \{x_m, y_m\} = \emptyset$ for every $n, m \in S$ with $n \neq m$. For each $n \in S$, let us set

$$\varepsilon_n := \frac{1}{2k(n)}, \quad \text{where} \quad k(n) := \max\{k \in \mathbb{N} : p_k = x_n \quad \text{or} \quad p_k = y_n\}.$$

Let us fix some $n_0 \in S$. Since $\{x_n, y_n\} \cap \{x_m, y_m\} = \emptyset$ for every $n, m \in S$ with $n \neq m$, by Lemma 3.2 and the definition of the metric d, there exists $m_0 \in S \setminus \{n_0\}$ such that

(i) $\max\{|f_{m_0}(x_{n_0})|, |f_{m_0}(y_{n_0})|\} \leqslant \frac{\varepsilon_{n_0}}{3}$ and

(ii) $\max\{d(x_{n_0}, x_{m_0}), d(x_{n_0}, y_{m_0}), d(y_{n_0}, x_{m_0}), d(y_{n_0}, y_{m_0})\} \leqslant 1 + \frac{\varepsilon_{n_0}}{3}$.

Now, by Lemma 3.1, there is a constant $C_{m_0} \in \mathbb{R}$ such that $f_{n_0}(x_{m_0}) = f_{n_0}(y_{m_0}) = C_{m_0}$. Since all the previous inequalities still hold if we relabel any of the pairs (x_{n_0}, y_{n_0}) or (x_{m_0}, y_{m_0}), we may assume that

$$|f_{n_0}(x_{n_0}) - C_{m_0}| \geqslant |f_{n_0}(y_{n_0}) - C_{m_0}| \qquad \text{and} \qquad |f_{m_0}(x_{m_0})| \geqslant |f_{m_0}(y_{m_0})|.$$

Note that the triangle inequality yields that

$$|f_{n_0}(x_{n_0}) - C_{m_0}| + |f_{n_0}(y_{n_0}) - C_{m_0}| \geqslant d(x_{n_0}, y_{n_0}) \;\; \text{and} \;\; |f_{m_0}(x_{m_0})| + |f_{m_0}(y_{m_0})| \geqslant d(x_{m_0}, y_{m_0}),$$

and so, under our previous assumption we get that

$$|f_{m_0}(x_{m_0})| \geqslant \frac{1}{2} + \varepsilon_{m_0} \qquad \text{and} \qquad |f_{n_0}(x_{n_0}) - C_{m_0}| \geqslant \frac{1}{2} + \varepsilon_{n_0}. \tag{4.1}$$

In particular, $f_{m_0}(x_{m_0}) \neq 0$. Set now $\delta := \frac{|f_{m_0}(x_{m_0})|}{f_{m_0}(x_{m_0})} \in \{-1, 1\}$. To finish the proof of this case, we distinguish two possibilities depending on the sign of $f_{n_0}(x_{n_0}) - C_{m_0}$:

If $f_{n_0}(x_{n_0}) < C_{m_0}$, consider the function $f = f_{n_0} + \delta f_{m_0}$, which is 1-Lipschitz by assumption. However, using properties (i) and (ii) and Eq. (4.1) we obtain that

$$\begin{aligned} |f(x_{n_0}) - f(x_{m_0})| &\geqslant -f_{n_0}(x_{n_0}) - \delta f_{m_0}(x_{n_0}) + f_{n_0}(x_{m_0}) + \delta f_{m_0}(x_{m_0}) \\ &> \frac{1}{2} + \varepsilon_{n_0} + \frac{1}{2} - |f_{m_0}(x_{n_0})| > d(x_{n_0}, x_{m_0}), \end{aligned}$$

a contradiction.

On the other hand, if $f_{n_0}(x_{n_0}) \geqslant C_{m_0}$, an analogous procedure shows that the function $g = f_{n_0} - \delta f_{m_0}$ has a Lipschitz norm greater than 1 witnessed by the same pair (x_{n_0}, x_{m_0}). This again yields a contradiction.

Case 2: $C = B_1$.

Restricting ourselves to S once more, note that there exists $k^* \in S$ such that $x_n = p_{k^*}$ for every $n \in S$. By Lemma 3.1, we have that $y_n \neq y_m$ for every $n, m \in S$ with $n \neq m$. Using Lemma 3.2, we may find $n_0, m_0 \in S$ with $n_0 \neq m_0$ such that

$$|f_{n_0}(x_{n_0})| \leqslant \frac{1}{10} \qquad \text{and} \qquad |f_{m_0}(x_{m_0})| \leqslant \frac{1}{10}, \tag{4.2}$$

and so, by the triangle inequality, we have that

$$|f_{n_0}(y_{n_0})| \geqslant \frac{9}{10} \qquad \text{and} \qquad |f_{m_0}(y_{m_0})| \geqslant \frac{9}{10}. \tag{4.3}$$

Since $x_{n_0} = x_{m_0}$, another consequence of Eq. (4.2) together with Lemma 3.1 is that

$$|f_{n_0}(y_{m_0})| \leqslant \frac{1}{10} \qquad \text{and} \qquad |f_{m_0}(y_{n_0})| \leqslant \frac{1}{10}. \tag{4.4}$$

Changing signs of f_{n_0} and f_{m_0} if necessary, we may assume that $f_{n_0}(y_{n_0}) > 0$ and $f_{m_0}(y_{m_0}) > 0$. Finally, consider the function $f := f_{n_0} - f_{m_0}$, which is 1-Lipschitz by the assumption on the sequence $\{f_n\}_{n\in\mathbb{N}}$. However, applying (4.3) and (4.4), and recalling that the diameter of M is $3/2$ we obtain that

$$\begin{aligned} |f(y_{n_0}) - f(y_{m_0})| &\geqslant f_{n_0}(y_{n_0}) + f_{m_0}(y_{m_0}) - f_{m_0}(y_{n_0}) - f_{n_0}(y_{m_0}) \\ &\geqslant \frac{9}{5} - (|f_{m_0}(y_{n_0})| + |f_{n_0}(y_{m_0})|) > d(y_{n_0}, y_{m_0}). \end{aligned}$$

We have then a contradiction and the proof of this case is over.

Case 3: $C = B_2$.

If $C = B_2 = \big\{\{n,m\} \in \mathbb{N}^{[2]} : y_n = y_m\big\}$ and $S^{[2]} \subset B_2$, defining $x'_n = y_n$ and $y'_n = x_n$ for all $n \in \mathbb{N}$ we observe that $S^{[2]} \subset \big\{\{n,m\} \in \mathbb{N}^{[2]} : x'_n = x'_m\big\}$, so this case can be solved in an analogous way to the case $C = B_1$.

Case 4: $C = B_3$.

We will show that this case cannot happen. Suppose for the sake of contradiction that $S^{[2]} \subset B_3 = \big\{\{n,m\} \in \mathbb{N}^{[2]} : x_n = y_m \text{ or } x_m = y_n\big\}$, and fix $n_0 \in S$. Defining $S_1 = \{m \in S \setminus \{n_0\} : x_m = y_{n_0}\}$ and $S_2 = \{m \in S \setminus \{n_0\} : y_m = x_{n_0}\}$ we obtain that

$$\begin{aligned} S_1^{[2]} &\subset \Big\{\{n,m\} \in \mathbb{N}^{[2]} : x_n = x_m = y_{n_0}\Big\} \subset B_1, \\ S_2^{[2]} &\subset \Big\{\{n,m\} \in \mathbb{N}^{[2]} : y_n = y_m = x_{n_0}\Big\} \subset B_2. \end{aligned}$$

The sets S_1 and S_2 must be singletons. Indeed, suppose first that there existed $n_1 \neq n_2 \in S_1$. Then $x_{n_1} = x_{n_2}$, and since $\{n_1, n_2\} \in B_3$, one of $x_{n_1} = y_{n_2}$ or $y_{n_1} = x_{n_2}$ holds. This implies that either $x_{n_1} = y_{n_1}$ or $x_{n_2} = y_{n_2}$, which is a contradiction with the fact that the functions f_{n_1} and f_{n_2} strongly attain their Lipschitz norm at the corresponding pairs of points. Similarly, we obtain that S_2 is a singleton as well.

Finally, since for each $m \in S \setminus \{n_0\}$ we have that either $x_{n_0} = y_m$ or $x_m = y_{n_0}$, it follows that $S = S_1 \cup S_2 \cup \{n_0\}$, and thus the cardinality of S is at most 3. This is a contradiction with the choice of S. ■

It is worth mentioning that there exist countable bounded uniformly discrete metric spaces M with the condition that no point x in M attains its separation radius, but such that c_0 embeds isometrically in $\text{SNA}(M)$. Indeed, it suffices to consider a countable collection $\{M_n\}_{n\in\mathbb{N}}$ of copies of the previous space in such a way that $d(M_n, M_m) = 3$ for all different $n, m \in \mathbb{N}$, and observe that, in this context, Lemma 3.3 applies. This means that the aforementioned property is not sufficient for c_0 not to be contained in $\text{SNA}(M)$ isometrically.

Likewise, one could be tempted to assume that the condition of not attaining the separation radii is at least necessary in negative results like Theorem 4.1. However, this is far from being true as well. In fact, later in this section we will exhibit a proper unbounded uniformly discrete metric space M such that c_0 cannot be embedded in $\text{SNA}(M)$ isometrically (see Theorem 4.4). In particular, every point of M attains its separation radius, since closed bounded sets in M are compact. On the other hand, we will see in the next subsection that the property of being uniformly discrete is indeed necessary in order to get such negative results (see Theorem 4.2).

4.2. Non uniformly discrete metric spaces

Let us move on to the main positive result of the paper. Going in the opposite direction of Theorem 4.1, here we show that we can *always* embed c_0 isometrically in $\mathrm{SNA}(M)$ whenever M is infinite but not uniformly discrete. Recall that M' is the set of cluster points of M. Let us remark that the complete metric spaces such that the set of cluster points is either empty or infinite were already covered in [1], and we do not address these cases at all in the proof.

Theorem 4.2. *Let M be an infinite non uniformly discrete metric space. Then, the set* $\mathrm{SNA}(M)$ *contains a linearly isometric copy of* c_0.

Proof. Assume first that M is complete. By [1, Theorems 3.2 and 3.4] it suffices to assume that M' is non-empty and finite. We can also assume without loss of generality that $0 \in M'$. Now we can find a sequence $\{x_n\}_{n\in\mathbb{N}}$ in M converging to 0 such that $R(x_n) > 0$ for all $n \in N$. It is clear that the sequence $\{R(x_n)\}_{n\in\mathbb{N}}$ converges to 0.

We are going to define a sequence $\{f_k\}_{k\in\mathbb{N}}$ of 1-Lipschitz functions in $\mathrm{SNA}(M)$ which will be isometrically equivalent to the canonical basis of c_0 and such that the subspace $\overline{\mathrm{span}}\{f_k : k \in \mathbb{N}\}$ (which is linearly isometric to c_0) is contained in $\mathrm{SNA}(M)$.

We define the following sets:

$$A := \big\{\{n,m\} \in \mathbb{N}^{[2]} : d(x_n,x_m) \geqslant R(x_n) + R(x_m)\big\},$$
$$B := \big\{\{n,m\} \in \mathbb{N}^{[2]} : d(x_n,x_m) < R(x_n) + R(x_m)\big\},$$

which form a partition of $\mathbb{N}^{[2]}$. By Ramsey's theorem, there is $C \in \{A,B\}$ and an infinite subset $S \subseteq \mathbb{N}$ such that $S^{[2]} \subseteq C$. These two possibilities give us two separate cases.

Case 1: $C = A$.

Consider the subset $\{x_n\}_{n\in S}$, which satisfies that $d(x_n,x_m) \geqslant R(x_n)+R(x_m)$ for all $n \neq m \in S$. Assume first that there is an infinite subset of S, which we denote by S_0, such that $R(x_n)$ is attained for every $n \in S_0$. Consider now for each $n \in S_0$ an element $y_n \in M$ such that $d(x_n,y_n) = R(x_n)$. It is straightforward to see that the sequences $\{x_n\}_{n\in S_0}, \{y_n\}_{n\in S_0}$ satisfy the assumptions of Lemma 3.3 and we are done.

Otherwise, by passing to a subsequence if necessary, we may assume that $R(x_n)$ is not attained for any $n \in S$. Let us then choose inductively a sequence $\{a_k\}_{k\in\mathbb{N}}$ among the elements of the sequence $\{x_n\}_{n\in S}$ satisfying that for every $k \in \mathbb{N}$,

$$d(a_k,0) \leqslant \frac{d(a_j,0) - R(a_j)}{3} \quad \forall j < k. \tag{4.5}$$

For the sake of clarity, let us denote $\Delta_k = \frac{d(a_k,0)-R(a_k)}{3}$ for every $k \in \mathbb{N}$. It is clear from (4.5) that $\{\Delta_k\}_{k=1}^{\infty}$ is a decreasing sequence. Now, by definition of $R(a_k)$, we deduce that for every $k \in \mathbb{N}$, there is some $b_k \in B(a_k, R(a_k) + \Delta_k)$. Finally, let us prove that the sequences $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ are under the assumptions of Lemma 3.3. Pick $n,m \in \mathbb{N}$ with $n < m$. Clearly, the following expressions hold

$$\begin{aligned} d(a_n,0) =&R(a_n) + 3\Delta_n, \quad d(a_m,0) \leqslant \Delta_n, \quad d(a_n,b_n) \leqslant R(a_n) + \Delta_n \\ &d(a_m,b_m) \leqslant R(a_m) + \Delta_m \leqslant R(a_m) + 3\Delta_m = d(a_m,0) \leqslant \Delta_n. \end{aligned} \tag{4.6}$$

Hence, by (4.6) we have that

$$d(a_n,a_m) \geqslant d(a_n,0) - d(a_m,0) \geqslant R(a_n) + 2\Delta_n \geq d(a_n,b_n) + d(a_m,b_m).$$

This finishes the first case.

Case 2: $C = B$.

Since the set S is infinite and the sequence $\{x_n\}_{n\in S}$ is convergent, we can inductively define a pair of sequences $\{a_k\}_{k\in\mathbb{N}} \subseteq \{x_n\}_{n\in S}$ and $\{b_k\}_{k\in\mathbb{N}} \subseteq \{x_n\}_{n\in S}$ satisfying the following properties:

(i) $R(a_k) < \varepsilon_j/2$, for $j, k \in \mathbb{N}$ with $j < k$, where $\varepsilon_j = R(a_j) + R(b_j) - d(a_j, b_j) > 0$.
(ii) $R(b_k) < R(a_k)/2$ for every $k \in \mathbb{N}$.

Fixed $k \in \mathbb{N}$, we define $f_k : M \to \mathbb{R}$ by

$$f_k(p) = \begin{cases} R(a_k) - \dfrac{\varepsilon_k}{2} & \text{if } p = a_k, \\ -R(b_k) + \dfrac{\varepsilon_k}{2} & \text{if } p = b_k, \\ 0 & \text{otherwise.} \end{cases}$$

Property (i) and the definition of ε_k ensure that $f_k(a_k) \geqslant 0$ and $f_k(b_k) \leqslant 0$ for every $k \in \mathbb{N}$. With this, we obtain that

$$|f_k(a_k)| = R(a_k) - \frac{\varepsilon_k}{2}, \qquad \text{and} \qquad |f_k(b_k)| = R(b_k) - \frac{\varepsilon_k}{2}, \qquad \text{for all } k \in \mathbb{N}. \tag{4.7}$$

Let $\{\lambda_k\}_{k\in\mathbb{N}} \in c_0$. Again we will show that $f := \sum_{k\in\mathbb{N}} \lambda_k f_k \in \mathrm{SNA}(M)$ and also that $\|f\|_{\mathrm{Lip}} = \max_{k\in\mathbb{N}}\{|\lambda_k|\}$. Choose $k_0 \in \mathbb{N}$ such that $|\lambda_{k_0}| = \max_{k\in\mathbb{N}}\{|\lambda_k|\}$.

We again start by proving that f is $|\lambda_{k_0}|$-Lipschitz. Take $p, q \in M$ with $p \neq q$. We will show that $|f(p) - f(q)| \leqslant |\lambda_{k_0}| d(p, q)$. If both p and q form a pair $\{a_k, b_k\}$ for some $k \in \mathbb{N}$, the previous inequality is clear. We need to study now the two remaining possibilities:

(a) Suppose that there exist $k_1, k_2 \in \mathbb{N}$ with $k_1 < k_2$ such that $p \in \{a_{k_1}, b_{k_1}\}$ and $q \in \{a_{k_2}, b_{k_2}\}$. Then, by (4.7) we have in particular that $|f(p)| = |\lambda_{k_1}| \left(R(p) - \frac{\varepsilon_{k_1}}{2}\right)$ and $|f(q)| \leqslant |\lambda_{k_2}| R(a_{k_2})$. Hence, we obtain that

$$\begin{aligned} |f(p) - f(q)| \leqslant& |\lambda_{k_0}| \cdot \left(R(p) - \frac{\varepsilon_{k_1}}{2} + R(a_{k_2})\right) \\ \leqslant& |\lambda_{k_0}| \cdot R(p) \leqslant |\lambda_{k_0}| \cdot d(p, q). \end{aligned}$$

(b) If $p \in M \setminus \{x_k\}_{k\in\mathbb{N}}$, then $f(p) = 0$ and, using (4.7) again, we have that

$$|f(p) - f(q)| = |f(q)| \leqslant |\lambda_{k_0}| \cdot R(q) \leqslant |\lambda_{k_0}| \cdot d(p, q).$$

We have proven then that the Lipschitz norm of f is smaller or equal than $|\lambda_{k_0}|$. Finally, considering the pair of points a_{k_0} and b_{k_0}, we quickly observe that $\|f\|_{\mathrm{Lip}} = |\lambda_{k_0}|$ and that f strongly attains its Lipschitz norm at this pair of points. This finishes the proof for complete metric spaces.

Finally, assume now that M is not complete and let M_c be its completion. Note that if M_c' is empty or infinite, we already have the result using again the constructions from [1, Theorems 3.2 and 3.4]. Otherwise, if M_c' is finite, it suffices to do the same procedure we described but on M_c instead of M, and also asking to our original sequence $\{x_n\}_{n\in\mathbb{N}}$ to be in M. Then, the constructed sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ from $\mathrm{SNA}(M_c)$ will still be strongly norm-attaining when restricted to M. Finally, it suffices to consider the functions $f - f(0_M)$ if needed (for instance, if our original limit point 0 was in $M_c \setminus M$, which means that it cannot be the distinguished point from the original space M). ■

It is an immediate consequence of [1, Theorem 3.3] that if M is any infinite compact metric space, then c_0 is isomorphically embedded into $\mathrm{SNA}(M)$ (for countable compact metric spaces this was achieved non constructively using the little Lipschitz space). Note that our previous theorem provides a constructive proof that for any infinite compact metric space M with a finite amount of cluster points, $\mathrm{SNA}(M)$ actually contains c_0 isometrically.

Corollary 4.3. *Let M be an infinite compact metric space. Then, the subset* $\mathrm{SNA}(M)$ *contains a linearly isometric copy of* c_0.

4.3. A proper and uniformly discrete counterexample

To finish this section, we show that Corollary 4.3 cannot be improved to include all proper metric spaces. Indeed, we have the following result.

Theorem 4.4. *There exists an infinite proper uniformly discrete metric space M such that c_0 is not isometrically contained in* $\mathrm{SNA}(M)$ *and for which every point in M attains its separation radius.*

Proof. Let $M = \{p_k\}_{k=0}^{\infty}$, with distinguished point $p_0 = 0$, be a countable set endowed with the metric $d: M \times M \to \mathbb{R}$ given by:

$$d(p_k, p_j) = \begin{cases} k + j - \varepsilon_{\max\{k,j\}} & \text{if } k \neq j \in \mathbb{N} \setminus \{0\}, \\ k & \text{if } j = 0, \\ j & \text{if } k = 0, \\ 0 & \text{if } j = k, \end{cases}$$

where $\{\varepsilon_k\}_{k\in\mathbb{N}}$ is a sequence of positive numbers such that $\varepsilon_{k+1} > \varepsilon_k$ and $\varepsilon_k < 1/2$ for all $k \in \mathbb{N}$. For convenience, write $\delta_k = \varepsilon_{k+1} - \varepsilon_k > 0$ for all $k \in \mathbb{N}$. It is clear that M is proper since every bounded set is finite.

As in the proof of Theorem 4.1, we start by assuming that there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions in $\mathrm{SNA}(M)$ isometrically equivalent to the canonical basis of c_0, and we are going to find two natural numbers $n_0 \neq m_0$ such that $f_{n_0} - f_{m_0}$ is not 1-Lipschitz, which will yield a contradiction. For each $n \in \mathbb{N}$, since f_n is strongly norm-attaining, we may consider two points $x_n \neq y_n$ such that $|f_n(x_n) - f_n(y_n)| = d(x_n, y_n)$.

We write $k(n)$ and $j(n)$ to denote the natural numbers such that $x_n = p_{k(n)}$ and $y_n = p_{j(n)}$ for every $n \in \mathbb{N}$. By relabelling the pairs (x_n, y_n) if needed, we may assume that $k(n) < j(n)$ for all $n \in \mathbb{N}$.

We now define the sets A, B_1, B_2, and B_3 as in the proof of Theorem 4.1. By Ramsey's theorem, there exists $C \in \{A, B_1, B_2, B_3\}$ and an infinite set $S \subseteq \mathbb{N}$ such that $S^{[2]} \subseteq C$.

Case 1: $C = A$.

In this case, choose an arbitrary $n_0 \in S$ such that $k(n_0), j(n_0) \neq 0$. By Lemma 3.2, and using that $\{x_n, y_n\} \cap \{x_m, y_m\} = \emptyset$ for all $n \neq m \in S$, we can find $m_0 \in S$ with $k(m_0) > j(n_0)$ such that

$$|f_{m_0}(x_{n_0})| < \frac{1}{2}\delta_{j(n_0)} \qquad \text{and} \qquad |f_{m_0}(y_{n_0})| < \frac{1}{2}\delta_{j(n_0)}. \tag{4.8}$$

Using Lemma 3.1 we can define $C_{m_0} \in \mathbb{R}$ such that $C_{m_0} = f_{n_0}(x_{m_0}) = f_{n_0}(y_{m_0})$. Using the triangle inequality, we know that

$$|f_{n_0}(x_{n_0}) - C_{m_0}| + |f_{n_0}(y_{n_0}) - C_{m_0}| \geqslant d(x_{n_0}, y_{n_0}),$$

and so, we obtain that either

(a_0) $|f_{n_0}(x_{n_0}) - C_{m_0}| \geqslant k(n_0) - \frac{1}{2}\varepsilon_{j(n_0)}$, or
(a_1) $|f_{n_0}(y_{n_0}) - C_{m_0}| \geqslant j(n_0) - \frac{1}{2}\varepsilon_{j(n_0)}$.

Similarly, since $|f_{m_0}(x_{m_0})| + |f_{m_0}(y_{m_0})| \geqslant d(x_{m_0}, y_{m_0})$ by the triangle inequality, we get that either

(b_0) $|f_{m_0}(x_{m_0})| \geqslant k(m_0) - \left(\frac{1}{2}\varepsilon_{j(n_0)} + \frac{1}{2}\delta_{j(n_0)}\right)$, or
(b_1) $|f_{m_0}(y_{m_0})| \geqslant j(m_0) - \left(\varepsilon_{j(m_0)} - \frac{1}{2}\varepsilon_{j(n_0)} - \frac{1}{2}\delta_{j(n_0)}\right)$.

In total, there are now 4 different possibilities that must be checked for contradiction. We will only expand on the two possibilities where (a_0) holds, since the two remaining possibilities (where (a_1) holds) are proven similarly. Hence, suppose first that (a_0) and (b_0) hold. By changing the signs of f_{n_0} and f_{m_0} if necessary, we may suppose that $f_{n_0}(x_{n_0})-C_{m_0} \geqslant k(n_0)-\frac{1}{2}\varepsilon_{j(n_0)}$ and $f_{m_0}(x_{m_0}) \geqslant k(m_0)-\big(\frac{1}{2}\varepsilon_{j(n_0)}+\frac{1}{2}\delta_{j(n_0)}\big)$. Consider the function $f = f_{n_0} - f_{m_0}$, which is 1-Lipschitz since we are assuming that $\{f_n\}_{n\in\mathbb{N}}$ is isometrically equivalent to the canonical basis of c_0. However, using (4.8), we have that

$$\begin{aligned}|f(x_{n_0}) - f(x_{m_0})| &\geqslant f_{n_0}(x_{n_0}) - f_{m_0}(x_{n_0}) - C_{m_0} + f_{m_0}(x_{m_0})\\ &> k(n_0) - \frac{1}{2}\varepsilon_{j(n_0)} - \frac{1}{2}\delta_{j(n_0)} + k(m_0) - \frac{1}{2}\varepsilon_{j(n_0)} - \frac{1}{2}\delta_{j(n_0)}\\ &\geqslant k(n_0) + k(m_0) - \varepsilon_{k(m_0)} = d(x_{n_0}, x_{m_0}),\end{aligned}$$

which yields a contradiction. Suppose now that (a_0) and (b_1) hold. Again we may suppose that $f_{n_0}(x_{n_0}) - C_{m_0} \geqslant k(n_0)-\frac{1}{2}\varepsilon_{j(n_0)}$ and $f_{m_0}(y_{m_0}) \geqslant j(m_0)-\big(\varepsilon_{j(m_0)}-\frac{1}{2}\varepsilon_{j(n_0)}-\frac{1}{2}\delta_{j(n_0)}\big)$. Using (4.8) again, the 1-Lipschitz function $f = f_{n_0} - f_{m_0}$ now tested at the pair (x_{n_0}, y_{m_0}) yields

$$\begin{aligned}|f(x_{n_0}) - f(y_{m_0})| &> k(n_0) - \frac{1}{2}\varepsilon_{j(n_0)} - \frac{1}{2}\delta_{j(n_0)} + j(m_0) - \varepsilon_{j(m_0)} + \frac{1}{2}\varepsilon_{j(n_0)} + \frac{1}{2}\delta_{j(n_0)}\\ &= k(n_0) + j(m_0) - \varepsilon_{j(m_0)} = d(x_{n_0}, y_{m_0}),\end{aligned}$$

which is again a contradiction. This finishes the proof for Case 1.

Case 2: $C = B_1$.

Write k^* to denote the non-negative integer (including 0) such that $p_{k^*} = x_n$ for all $n \in S$. Suppose first that $k^* = 0$. Then, choose any two different numbers $n_0 \neq m_0 \in S$. Since both f_{n_0} and f_{m_0} strongly attain their norm, at the pairs $(0, y_{n_0})$ and $(0, y_{m_0})$ respectively, and both f_{n_0} and f_{m_0} vanish at 0, we have that $|f_{n_0}(y_{n_0})| = j(n_0)$ and $|f_{m_0}(y_{m_0})| = j(m_0)$. With Lemma 3.1 we obtain that $f_{n_0}(y_{m_0}) = f_{m_0}(y_{n_0}) = 0$. By changing the signs of both functions if needed, we may suppose that $f_{n_0}(y_{n_0}) = j(n_0)$ and $f_{m_0}(y_{m_0}) = j(m_0)$, producing a contradiction directly by considering the mapping $f = f_{n_0} - f_{m_0}$, which is not 1-Lipschitz as witnessed by the pair (y_{n_0}, y_{m_0}). Indeed,

$$|f(y_{n_0}) - f(y_{m_0})| = j(n_0) + j(m_0) > d(y_{n_0}, y_{m_0}).$$

Suppose now that $k^* \neq 0$. Using Lemma 3.2, choose two different natural numbers $n_0 \neq m_0 \in S$ with $j(m_0) > j(n_0) > k^*$ such that

$$|f_{n_0}(p_{k^*})| < \frac{1}{4} \qquad \text{and} \qquad |f_{m_0}(p_{k^*})| < \frac{1}{4}.$$

On the one hand, these inequalities imply, by the triangle inequality, that $|f_{n_0}(y_{n_0})| > k^*+j(n_0)-\varepsilon_{j(n_0)}-\frac{1}{4}$ and $|f_{m_0}(y_{m_0})| > k^* + j(m_0) - \varepsilon_{j(m_0)} - \frac{1}{4}$, while, on the other hand, they imply by Lemma 3.1 that

$$|f_{n_0}(y_{m_0})| < \frac{1}{4} \qquad \text{and} \qquad |f_{m_0}(y_{n_0})| < \frac{1}{4}.$$

Finally, we may again suppose without loss of generality that f_{n_0} and f_{m_0} are both positive at the points y_{n_0} and y_{m_0} respectively, and consider the function $f = f_{n_0} - f_{m_0}$, which is assumed to be 1-Lipschitz. However, we have that

$$\begin{aligned}|f(y_{n_0}) - f(y_{m_0})| &\geqslant f_{n_0}(y_{n_0}) - f_{m_0}(y_{n_0}) - f_{n_0}(y_{m_0}) + f_{m_0}(y_{m_0})\\ &\geqslant j(n_0) + j(m_0) + 2k^* - 1 - \varepsilon_{j(n_0)} - \varepsilon_{j(m_0)}\\ &> j(n_0) + j(m_0) > d(y_{n_0}, y_{m_0}),\end{aligned}$$

a contradiction. This finishes the proof of Case 2.

Case 3: $C = B_2$.

We will show that this case is impossible. Suppose, for the sake of contradiction, that $S^{[2]} \subseteq B_2$. Then for all $m, n \in S$, $y_n = y_m = p_{j^*}$ for some fixed $j^* \in \mathbb{N}$. Since $k(n) < j(n)$ for all $n \in \mathbb{N}$, we obtain that $k(n) < j^*$ for all $n \in S$. The set S is infinite, so there exist $n \neq m \in S$ such that $x_n = x_m$ and $y_n = y_m$. It follows that the functions f_n and f_m from the basis strongly attain their norms at the same pair of points, contradicting Lemma 3.1.

Case 4: $C = B_3$.

Similarly to Case 4 of the proof of Theorem 4.1, if $S \subset \mathbb{N}$ satisfies $S^{[2]} \subset B_3$, the cardinality of S is at most 3. Hence, Ramsey's Theorem cannot apply to B_3 either and the proof is finished. ■

5. The non-separable case

In this section we tackle the problem of embedding $c_0(\Gamma)$ in $\mathrm{SNA}(M)$ isometrically, where Γ is an arbitrary set of large cardinality. Let us first introduce some basic concepts and results of set theory that will be heavily used in this section.

We denote by $\mathrm{dens}(M)$ the *density character* of a metric space M, defined as the smallest cardinal Γ such that there is a dense subset of M of cardinality Γ. The *cofinality* $\mathrm{cof}(\alpha)$ of an ordinal α is the smallest ordinal β such that $\alpha = \sup_{\gamma<\beta} \alpha_\gamma$, where $\{\alpha_\gamma\}_{\gamma<\beta}$ is an ordinal sequence with $\alpha_\gamma < \alpha$ for all $\gamma < \beta$. A cardinal Γ is *regular* if $\mathrm{cof}(\Gamma) = \Gamma$. Following the notation of [20], for an ordinal α, we denote by α^+ the least cardinal strictly bigger than α, which is always a regular cardinal (see [20, Lemma 10.37]). We again refer the reader to [20] for a comprehensive background on this topic. Finally, recall that a subset of a metric space $S \subseteq M$ is called *r-separated* for some $r > 0$ whenever $d(x, y) \geqslant r$ for all $x \neq y \in S$.

The next result is essentially based on the proof of [16, Proposition 3].

Proposition 5.1. *Let M be a metric space with $\mathrm{dens}(M) = \Gamma$ for some cardinal Γ. Then, there exists a discrete set $L \subseteq M$ with $\mathrm{card}(L) = \Gamma$. Moreover, if $\mathrm{cof}(\Gamma) > \aleph_0$, then L can be chosen to be uniformly discrete.*

Proof. Note that if Γ is finite, the result is trivial, so we will only prove the case where Γ is infinite. Note also that it is well known that if M is any infinite Hausdorff topological space (in particular if M is an infinite metric space), then it always contains an infinite discrete set, so the case where $\Gamma = \aleph_0$ is already finished. Hence, we will assume now that Γ is uncountable. For every $k \in \mathbb{N}$, let M_k be some maximal $\frac{1}{2^k}$-separated subset of M. Denote $\Gamma_k := |M_k|$ for all $k \in \mathbb{N}$, and note that $\overline{\bigcup_{k\in\mathbb{N}} M_k} = M$, and so, $\sup_{k\in\mathbb{N}} \Gamma_k = \Gamma$. If $\mathrm{cof}(\Gamma) > \aleph_0$, then, since $\overline{\bigcup_{k\in\mathbb{N}} M_k} = M$, we have that there is $k_0 \in \mathbb{N}$ such that $\Gamma_{k_0} = \Gamma$ and so we take $L := M_{k_0}$.

Now, let us assume that $\mathrm{cof}(\Gamma) = \aleph_0$. If there exists $k_0 \in \mathbb{N}$ such that $\Gamma_{k_0} = \Gamma$, we are done, since we can take once again $L := M_{k_0}$. On the other hand, if this is not the case, we have that $\Gamma_k < \Gamma$ for every $k \in \mathbb{N}$ and $\mathrm{cof}(\Gamma) = \aleph_0$. Since Γ is not regular, we know that $\Gamma_k^+ < \Gamma$, for every $k \in \mathbb{N}$. Using this, and the fact that $\sup_{k\in\mathbb{N}} \Gamma_k = \Gamma$, it is straightforward to inductively construct a subsequence $\{\Gamma_{k_n}\}_{n\in\mathbb{N}}$ of $\{\Gamma_k\}_{k\in\mathbb{N}}$ with Γ_{k_1} infinite and such that $\Gamma_{k_n}^+ < \Gamma_{k_{n+1}}$ for all $n \in \mathbb{N}$. Notice that since $\{\Gamma_k\}_{k\in\mathbb{N}}$ is an increasing sequence, we have that $\sup_{n\in\mathbb{N}} \Gamma_{k_n} = \Gamma$.

Now, for each $n \in \mathbb{N}$, let us consider a sequence of sets $\{\widetilde{M}_n\}_{n\in\mathbb{N}}$ such that $\widetilde{M}_n$ is a subset of $M_{k_{n+1}}$ with $|\widetilde{M}_n| = \Gamma_{k_n}^+$ for all $n \in \mathbb{N}$. Let us write $\widetilde{M}_n = \{x_\alpha^n : \alpha \in \Gamma_{k_n}^+\}$.

For each $n \in \mathbb{N}$, each $j \leqslant n$, and each $\alpha \in \Gamma_{k_j}^+$, we define

$$A_{j,\alpha}^n := \widetilde{M}_{n+1} \cap B\left(x_\alpha^j, \frac{1}{2^{k_{j+1}+1}}\right).$$

We will inductively construct, for every $n \in \mathbb{N}$, a set $L_n \subseteq \widetilde{M}_n$ with $|L_n| = \Gamma^+_{k_n}$ and a finite subset $N_n \subseteq M$ such that whenever $j < n$,

$$d(L_n,\, L_j \setminus N_n) \geqslant \frac{1}{2^{k_{j+1}+2}}. \tag{5.1}$$

Set $L_1 := \widetilde{M}_1$ and $N_1 := \emptyset$. Now, assuming that for some $n \in \mathbb{N}$ we have constructed L_j and N_j for all $j \leqslant n$, we can do the inductive step towards $n+1$.

(a) Suppose that $|A^n_{j,\,\alpha}| < \Gamma^+_{k_{n+1}}$ for every $j \leqslant n$ and $\alpha \in \Gamma^+_{k_n}$. Since $\Gamma^+_{k_n} < \Gamma^+_{k_{n+1}}$ and $\Gamma^+_{k_{n+1}}$ is regular, we have that

$$\left| \bigcup_{j\leqslant n,\, \alpha\in\Gamma^+_{k_j}} A^n_{j,\,\alpha} \right| < \Gamma^+_{k_{n+1}} = \operatorname{card}(\widetilde{M}_{n+1}).$$

Therefore, the set

$$L_{n+1} := \widetilde{M}_{n+1} \setminus \bigcup_{j\leqslant n,\, \alpha\in\Gamma^+_{k_j}} A^n_{j,\,\alpha}$$

satisfies $|L_{n+1}| = \Gamma^+_{k_{n+1}}$ and (5.1) holds by setting $N_{n+1} = \emptyset$. Indeed, for any $j \in \{1,\ldots,n\}$, every point in L_j is of the form x^j_α for some $\alpha \in \Gamma^+_{k_j}$. Hence, if there exists a point $p \in L_{n+1}$ such that $d(p, x^j_\alpha) < \frac{1}{2^{k_{j+1}+2}}$, then p belongs to the set $A^n_{j,\,\alpha}$, which leads to a contradiction with the definition of L_{n+1}.

(b) Suppose now that $|A^n_{j_0,\,\alpha_0}| = \Gamma^+_{k_{n+1}}$ for some $j_0 \leqslant n$ and some $\alpha_0 \in \Gamma^+_{k_{j_0}}$. Without loss of generality we consider $j_0 \in \{1,\ldots,n\}$ to be such that $|A^n_{j,\,\alpha}| < \Gamma^+_{k_{n+1}}$ for all $j_0 < j \leqslant n$ and all $\alpha \in \Gamma^+_{k_j}$. Define

$$L_{n+1} := A^n_{j_0,\,\alpha_0} \setminus \bigcup_{j_0<j\leqslant n,\, \alpha\in\Gamma^+_{k_j}} A^n_{j,\,\alpha}.$$

Arguing as in case (a), we obtain that $|L_{n+1}| = \Gamma^+_{k_{n+1}}$. Finally, define

$$N_{n+1} := \left\{ x \in M : \exists i \in \{1,\ldots,j_0\} \text{ such that } x \in L_i \text{ and } d(x, L_{n+1}) < \frac{1}{2^{k_{i+1}+2}} \right\}$$

which is finite since for each $i \in \{1,\ldots,j_0\}$, there can only be at most a single point x_i in L_i such that $d(x_i, L_{n+1}) < \frac{1}{2^{k_{i+1}+2}}$. Indeed, if $i = j_0$, the only point in L_{j_0} that can satisfy that property is $x^{j_0}_{\alpha_0}$, since for every $\beta \in \Gamma^+_{k_{j_0}} \setminus \{\alpha_0\}$, $d(x^{j_0}_\beta, A^n_{j_0,\,\alpha_0}) \geqslant \frac{1}{2^{k_{j_0+1}+1}}$. On the other hand, if $i < j_0$, if there were two points $x_i \neq y_i \in L_i$ with that property, we would have that

$$d(x_i, y_i) \leqslant d(x_i, A^n_{j_0,\,\alpha_0}) + d(y_i, A^n_{j_0,\,\alpha_0}) + \operatorname{diam}(A^n_{j_0,\,\alpha_0}) < \frac{1}{2^{k_{i+1}}},$$

a contradiction with the fact that L_i is $\frac{1}{2^{k_{i+1}}}$-separated.
Let us check that the sets L_{n+1} and N_{n+1} satisfy Eq. (5.1) for each $j \in \{1,\ldots,n\}$. Fix $j \in \{1,\ldots,n\}$. If $j \leqslant j_0$ then the inequality follows directly by definition of N_{n+1}. Otherwise, if $j > j_0$, then the inequality holds following the same argument as in case (a).

Having discussed both possibilities, the induction is finished. To finish the proof, set $L := \left(\bigcup_{n\in\mathbb{N}} L_n\right) \setminus \left(\bigcup_{n\in\mathbb{N}} N_n\right)$. It is clear that $|L| = \Gamma$, and using Eq. (5.1), it is straightforward to prove that all convergent sequences in L are eventually constant, and thus L is discrete. ∎

As an application of Lemma 3.3 and Proposition 5.1, we have the following isometric result.

Theorem 5.2. *Let M be a pointed metric space such that* $\operatorname{dens}(M') = \Gamma$ *for some infinite cardinal Γ. Then there is a linear subspace of* $\operatorname{SNA}(M)$ *that is isometrically isomorphic to* $c_0(\Gamma)$.

Proof. The case where $\Gamma = \aleph_0$ is already covered in [1, Theorem 3.2]. Assume now that $\Gamma > \aleph_0$. If we apply Proposition 5.1 to the set M', we find a discrete set $L \subseteq M'$ with $\operatorname{card}(L) = \operatorname{dens}(L) = \Gamma$ and such that all points of L are cluster points of M. Finally, Lemma 3.3 can be applied now if we consider $\{x_\gamma\}_{\gamma\in\Gamma}$ to be L itself and for each $\gamma \in \Gamma$, we set y_γ to be sufficiently close to x_γ. ■

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ALGEBRAIC GENERICITY OF CERTAIN FAMILIES OF NETS IN FUNCTIONAL ANALYSIS

SHELDON DANTAS AND DANIEL L. RODRÍGUEZ-VIDANES

Abstract. In Functional Analysis, certain conclusions apply to sequences, but they cannot be carried over when we consider nets. In fact, some nets, including sequences, can behave unexpectedly. In this paper we are interested in exploring the prevalence of these unusual nets in terms of linearity. Each problem is approached with different methods, which have their own interest. As our results are presented in the contexts of topological vector spaces and normed spaces, they generalize or improve a few ones in the literature. We study lineability properties of families of (1) nets that are weakly convergent and unbounded, (2) nets that fail the Banach-Steinhaus theorem, (3) nets indexed by a regular cardinal κ that are weakly dense and norm-unbounded, and finally (4) convergent series which have associated nets that are divergent.

Contents

1. Introduction

The present paper is about lineability: we say that a subset M of a vector space X is *lineable* (respectively, κ-*lineable*, for a cardinal κ) if $M \cup \{0\}$ contains a vector space of infinite-dimension (respectively, of dimension κ). These sort of problems have been studied intensively during the past few years since the term lineability was coined by V. I. Gurariy in the early 2000s and since then have appeared in the literature in different areas of Mathematics such as Functional Analysis, Real Analysis, Complex Analysis, Set Theory, Dynamical Systems, among others (we send the reader to [2, 8, 15, 16, 24, 25, 28, 30] and the references therein).

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Our main interest here is to provide a contribution about the study of "pathological" nets in Functional Analysis in the sense of lineability. Some of these families of nets that are studied in this work arise from two well-known results that hold true for sequences. They are the following ones: (a) every weakly convergent sequence is bounded and (b) the Banach-Steinhaus theorem, which states that every sequence of bounded linear operators that converges pointwise to a bounded linear operator is uniformly bounded. On the other hand, for the upcoming results it is also important to recall also the following. Some authors have thoroughly worked on the problem about finding conditions that $(\|x_n\|)_{n\in\mathbb{N}}$ has to satisfy in order that the set $\{x_n : n \in \mathbb{N}\}$ is weakly closed (see, for instance, [3, 6, 20, 21]). In particular, it is known that for every separable Banach space X, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $\|x_n\| \longrightarrow \infty$ as $n \to \infty$ and the set $\{x_n : n \in \mathbb{N}\}$ is weakly dense in X (see, for instance, [3, Corollary 5] and also [21, Section 2] for a refinement).

1.1. **Preliminaries and notation.** In what follows, we present briefly the notation we will be using throughout the paper and then we will describe our main results. All the spaces that we work with here are considered to be nonzero. We will be using basic concepts and notations from Set Theory found, for instance, in [10, 18]. Ordinal numbers will be identified with the set of their predecessors and cardinal numbers with the initial ordinals. Given a set A, the cardinality of A will be denoted by card(A). We denote by $\aleph_0$, $\aleph_1$ and $\mathfrak{c}$ the first infinite cardinal, the second infinite cardinal and the cardinality of the continuum, respectively. The *cofinality* cof(α) of an ordinal α is the smallest ordinal β such that $\alpha = \sup_{\gamma<\beta} \alpha_\gamma$, where $\{\alpha_\gamma\}_{\gamma<\beta}$ is an ordinal sequence of length β with $\alpha_\gamma < \alpha$ for all $\gamma < \beta$. We say that a cardinal number κ is regular if cof$(\kappa) = \kappa$ (see, for instance, [23]).

A set $\mathcal{A}$ is a *directed set* (also known as an index set) if $\mathcal{A}$ is a nonempty set that is endowed with a preorder $\leqslant$ (a reflexive and transitive relation) such that every pair of elements of $\mathcal{A}$ has an upper bound. A *net* in a set X is a function from a directed set $\mathcal{A}$ to X which will be denoted by $(x_a)_{a\in\mathcal{A}}$. We denote the set of nets in X indexed by $\mathcal{A}$ as $X^{\mathcal{A}}$.

Given a topological space X and $(x_a)_{a\in\mathcal{A}}$ a net in X, we say that $(x_a)_{a\in\mathcal{A}}$ converges to $x \in X$ if for every neighborhood U^x of x, there exists an element $a_0 \in \mathcal{A}$ such that $x_a \in U^x$ for every $a \geqslant a_0$. Recall that if X is a topological vector space, then a net $(x_a)_{a\in\mathcal{A}}$ in X weakly converges to $x \in X$ (denoted by $x_a \xrightarrow{w} x$) if and only if $(x^*(x_a))_{a\in\mathcal{A}}$ converges to $x^*(x)$ for every $x^* \in X^*$.

If a directed set $\mathcal{A}$ is in particular an ordinal number α, then we have the so-called α-*sequences* instead of nets defined in a set. It is known that the convergence of α-sequences in a topological space can be reduced to the convergence of cof(α)-sequences (see, for instance, [29, Propositions 3.1 and 3.2]). Therefore, we simply consider κ-sequences, where κ is a regular cardinal number. This notion of κ-sequence was introduced in 1907 by J. Mollerup [26] and has been studied throughout the 20th and 21st centuries by many mathematicians in several contexts (see [22, 27, 29, 31] and the references therein). Given a κ-sequence $(x_\alpha)_{\alpha<\kappa}$ in X, we say that $(x_{\beta_\alpha})_{\alpha<\kappa}$ is a κ-subsequence of $(x_\alpha)_{\alpha<\kappa}$ if there exists an increasing injective function $\varphi : \kappa \to \kappa$ such that $x_{\beta_\alpha} = x_{\varphi(\alpha)}$ for every $\alpha < \kappa$.

As stated earlier, in this work we are interested in studying the lineability of families of nets. This sort of approach was initiated by J. Carmona Tapia *et. al.* in [9] from several points of view and recently the second author in [29] analyzed several "monstrous" families of nets and κ-sequences related to Measure Theory. Still about reference [29], more specifically about

[29, Section 1.2], the author showed several details regarding the study of lineability involving families of nets. Let us dive into some of these ideas since they will be relevant in our context as well. Given a vector space V and a directed set $\mathcal{A}_0$, we assume that there exists $M_{\mathcal{A}_0} \subset V^{\mathcal{A}_0}$ satisfying a pathological property (P). It may be possible that we can extend $\mathcal{A}_0$ to a directed set $\mathcal{A}_1$ having greater cardinality such that we can find $M_{\mathcal{A}_1} \subseteq V^{\mathcal{A}_1}$ still satisfying property (P). With this mind, not only we are interested in studying the lineability properties of the family of nets satisfying property (P) indexed by an arbitrary directed set, but also we are looking for the smallest indexed set $\mathcal{A}$ in terms of cardinality for which there is a net indexed by $\mathcal{A}$ satisfying such a property and having the same lineability properties. Likewise, in terms of κ-sequences, we are looking for the smallest κ.

1.2. **The main results.** Now we are ready to describe briefly our main results. By using the Fichtenholz-Kantorovich-Hausdorff theorem (see [14, 17]), we show in Theorem 2.1 that, for an infinite-dimensional topological vector space, given a cardinal number κ between $\aleph_0$ and $\mathfrak{c}$, there exists a directed set $\mathcal{A}$ of cardinality κ such that the family of all nets indexed by $\mathcal{A}$ that are weakly convergent and unbounded is 2^κ-lineable. Next, in Proposition 2.8, we study "how big" is the set of nets which do not satisfy the Banach-Steinhaus theorem. More precisely, we show that, for every normed spaces X and Y, there exists a directed set $\mathcal{A}$ with cardinality κ between $\aleph_0$ and $\mathfrak{c}$ such that the set of all nets of continuous linear operators $(T_a)_{a\in\mathcal{A}}$ from X into Y that are pointwise convergent and $\{\|T_a\| : a \in \mathcal{A}\}$ is unbounded is 2^κ-lineable. In Section 2.3, we study lineability properties related to the following (already mentioned) property: in every (separable) Banach space, there is a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $\|x_n\| \longrightarrow \infty$ as $n \to \infty$ and $\{x_n : n \in \mathbb{N}\}$ is weakly dense in X. We are interested in both separable and non-separable (using κ-sequences) cases. To be more precise, we will study the lineability properties of the set of all κ-sequences $(x_\alpha)_{\alpha<\kappa}$ such that $\{\|x_\alpha\| : \alpha < \kappa\}$ is unbounded and $\{x_\alpha : \alpha < \kappa\}$ is weakly dense in X, where X is a normed space with density character $\text{dens}(X) = \kappa$. Finally we study lineability properties related to a family of nets that are divergent but its series is convergent (see Theorem 2.13). The last section is dedicated to remarks and open problems related to the topics of the paper.

2. Main Results

Since we will be dealing with different contexts as we have mentioned in the introduction, we split this section into four subsections.

2.1. **Weakly convergent and unbounded nets.** We will prove the following result in terms of lineability for a family of nets which are weakly convergent and unbounded. Recall that a *topological vector space* (TVS, for short) is a vector space endowed with a topology such that vector addition and scalar multiplication are both continuous. In this case, we denote by X^* its topological dual and $\sigma(X, X^*)$ the weak topology on X. The symbol $\mathbb{K}$ stands for the set of real or complex numbers.

Theorem 2.1. Let $\aleph_0 \leqslant \kappa \leqslant \mathfrak{c}$ be a cardinal number. Let X be a real or complex infinite-dimensional TVS. There exists a directed set $\mathcal{A}$ of cardinality κ such that the family of nets in X indexed by $\mathcal{A}$ that are unbounded and weakly convergent is 2^κ-lineable.

Note that Theorem 2.1 improves and generalizes [9, Theorem 2.1] by considering arbitrary infinite-dimensional TVS over $\mathbb{R}$ or $\mathbb{C}$ and decreasing the size of the index set to make it $\leqslant \mathfrak{c}$ while still having the property of being $\mathfrak{c}$-lineable.

In Remark 2.2 below we provide some remarks regarding Theorem 2.1, which depend on the model of ZFC that we consider.

Remark 2.2. Assume that X is an infinite-dimensional TVS. On the one hand, under ZFC+CH (where CH denotes the Continuum Hypothesis), it is clear that $\aleph_0 < \aleph_1 = \mathfrak{c}$ and $2^{\aleph_0} = \mathfrak{c} < 2^{\mathfrak{c}} = 2^{\aleph_1}$. Therefore, by Theorem 2.1, there exist directed sets $\mathcal{A}_0$ and $\mathcal{A}_1$ with cardinalities $\text{card}(\mathcal{A}_0) = \aleph_0$ and $\text{card}(\mathcal{A}_1) = \aleph_1$ such that the families of nets in X indexed by $\mathcal{A}_0$ and $\mathcal{A}_1$ that are unbounded and weakly convergent are $\mathfrak{c}$-lineable and $2^{\aleph_1}$-lineable, respectively. So, under ZFC+CH, the size of the directed set $\mathcal{A}$ can affect the dimension of the desired vector space based on Theorem 2.1.

On the other hand, under ZFC+$\neg$CH+MA (where MA denotes Martin's Axiom and $\neg$CH the negation of the CH), we have that $2^{\aleph_0} = 2^{\aleph_1}$ since $\aleph_0 < \aleph_1 < \mathfrak{c}$ (see [10, Theorem 9.5.9] and [19, Theorem 16.20]). Therefore, by Theorem 2.1 and taking $\kappa = \aleph_0$, there is a directed set $\mathcal{A}$ having cardinality $\aleph_0$ such that the set of nets in X indexed by $\mathcal{A}$ being unbounded and weakly convergent is $\mathfrak{c}$-lineable. If we took $\kappa = \aleph_1$, we would increase the size of our index set but the dimension of the desired vector space would still be $\mathfrak{c}$; therefore, we would not obtain a larger algebraic structure even though we are increasing the size of the index set.

In order to prove Theorem 2.1 we need to introduce some notation and remind some relevant results in our context. We start with the concept of independent families.

Definition 2.3. Let Γ be a nonempty set. We say that a family $\mathcal{Y}$ of subsets of Γ is *independent* if for any pairwise distinct sets $Y_1, \ldots, Y_n \in \mathcal{Y}$ and any $\varepsilon_1, \ldots, \varepsilon_n \in \{0, 1\}$ we have that

$$Y_1^{\varepsilon_1} \cap \cdots \cap Y_n^{\varepsilon_n} \neq \varnothing,$$

where Y^1 and Y^0 denote Y and $\Gamma \setminus Y$, respectively.

We will use the Fichtenholz-Kantorovich-Hausdorff theorem (FKH, for short) as stated below.

Theorem 2.4 (Fichtenholz-Kantorovich-Hausdorff theorem)**.** [14, 17] *Let Γ be a set of infinite cardinality κ. There is a family of independent subsets $\mathcal{Y}$ of Γ of cardinality 2^κ.*

It is worth mentioning the following observation below about Theorem 2.4.

Remark 2.5. When one applies FKH, one gets a family $\mathcal{Y}$ of 2^κ-many subsets of nonempty sets such that $Y_1^{\varepsilon_1} \cap \cdots \cap Y_n^{\varepsilon_n} \neq \varnothing$ whenever $Y_1, \ldots, Y_n \in \mathcal{Y}$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{0, 1\}$. As a matter of fact, the sets $Y_1^{\varepsilon_1} \cap \cdots \cap Y_n^{\varepsilon_n}$ besides being nonempty are in fact infinite. The reader can go to the observation right after [13, Definition 1.3] for a simple proof of this fact (alternatively the reader may consider the definition of independent subsets given in the paragraph below [18, Theorem 2.7]).

We are now ready to provide a proof for Theorem 2.1.

Proof of Theorem 2.1. Let $\mathcal{A}^- \subseteq (-\infty, 0)$ and $\mathcal{A}^+ \subseteq (0, \infty)$ be such that $\text{card}(\mathcal{A}^-) = \text{card}(\mathcal{A}^+) = \kappa$. Let $\mathcal{A} = \mathcal{A}^- \cup \mathcal{A}^+$ endowed with the standard order of $\mathbb{R}$ and note that $\text{card}(\mathcal{A}) = \kappa$. By

FKH, there is a family $\mathcal{K} \subseteq \mathcal{P}(\mathcal{A}^-)$ of independent subsets of $\mathcal{A}^-$ such that $\mathrm{card}(\mathcal{K}) = 2^\kappa$. Fix $x \in X \setminus \{0\}$. For every $K \in \mathcal{K}$, we define

$$(x_a^K)_{a\in\mathcal{A}} = \begin{cases} |a|x, & \text{if } a \in K, \\ \frac{1}{a}x, & \text{if } a \in \mathcal{A}^+, \\ 0, & \text{if } a \notin K \cup \mathcal{A}^+. \end{cases}$$

We will see that given nonzero scalars $\lambda_1, \dots, \lambda_m \in \mathbb{K}$ and $K_1, \dots, K_m \in \mathcal{K}$ distinct, the net $\sum_{j=1}^m \lambda_j \left(x_a^{K_j}\right)_{a\in\mathcal{A}}$ is unbounded, which also immediately implies the linear independence of $\left\{\left(x_a^K\right)_{a\in\mathcal{A}} : K \in \mathcal{K}\right\}$. Indeed, since $\mathcal{K}$ is a family of independent sets, by Remark 2.5, there exists a sequence of distinct terms $(a_l^1)_{l\in\mathbb{N}} \subseteq K_1 \setminus (K_2 \cup \dots \cup K_m)$, that is, unbounded in $\mathcal{A}^-$. Therefore, the set

$$\left\{\sum_{i=1}^m \lambda_i x_{a_l^1}^{K_i} : l \in \mathbb{N}\right\} = \left\{\lambda_1 \left|a_l^1\right| x : l \in \mathbb{N}\right\}$$

is unbounded.

Finally, it is enough to prove that each $\left(x_a^K\right)_{a\in\mathcal{A}}$ weakly converges to 0. Fix $x^* \in X^*$ and $K \in \mathcal{K}$. Since $\mathcal{A}$ is a linearly ordered set and $x^*(x)$ is fixed, for every $\varepsilon > 0$, there is an $a_0 \in \mathcal{A}^+ \subseteq \mathcal{A}$ such that for any $a \geqslant a_0$ we have $\frac{1}{a}|x^*(x)| < \varepsilon$. Hence, for any $a \geqslant a_0$, it yields

$$\left|x^*\left(x_a^K\right) - x^*(0)\right| = \left|x^*\left(x_a^K\right)\right| = \frac{1}{a}|x^*(x)| < \varepsilon.$$

This finishes the proof. □

Let us provide a proof of the following weaker version of Theorem 2.1 which is interesting on its own (this also motives a natural problem posed in Section 3). By weaker we mean that the size of the directed set may be larger than the one considered in Theorem 2.1. More specifically, we have the following result.

Proposition 2.6. Let X be a real or complex infinite-dimensional TVS. There exists a directed set $\mathcal{A}$ of cardinality $\mathrm{card}(X^*)$ such that the family of nets in X indexed by $\mathcal{A}$ that are unbounded and weakly convergent is $2^{\mathfrak{c}}$-lineable.

Let us introduce some notation and preliminary results which will help us in the proof of this latter result. On c_{00} over $\mathbb{R}$, we define the partial order $\preccurlyeq$ as follows: for any $P = (P_i)_{i\in\mathbb{N}}$, $Q = (Q_i)_{i\in\mathbb{N}} \in c_{00}$, we set

$$P \preccurlyeq Q \text{ if and only if } |P_i| \leqslant |Q_i| \text{ for every } i \in \mathbb{N}.$$

For every $n \in \mathbb{N}$, we denote $n^{\mathbb{N}} := \{0, 1, \dots, n-1\}^{\mathbb{N}}$. On $\bigcup_{n=1}^\infty n^{\mathbb{N}}$, we define the partial order $\leqslant^*$ in the following manner: for any $a = (a_j)_{j\in\mathbb{N}}$, $b = (b_j)_{j\in\mathbb{N}} \in \bigcup_{n=1}^\infty n^{\mathbb{N}}$,

$$a \leqslant^* b \text{ if and only if } a_j \geqslant b_j \text{ for every } j \in \mathbb{N}.$$

Let $\mathcal{P}_1^n := \mathbb{R}_1[x_1, \dots, x_n]$ be the set of all real polynomials of degree 1 in n variables. The map $\Phi\colon \bigcup_{n=1}^\infty \mathcal{P}_1^n \to c_{00} \setminus \{0\}$ defined for every $P = \sum_{k=1}^m P_k x_k \in \bigcup_{n=1}^\infty \mathcal{P}_1^n$ by

$$\Phi(P) := (P_1, \dots, P_m, 0, \dots)$$

is a bijection. We denote by $\beta\mathbb{N}$ the Stone-Čech compactification of $\mathbb{N}$ and we set

$$\mathcal{P} := \bigcup_{n\in\mathbb{N}} \mathcal{P}_1^n \times n^{\mathbb{N}}.$$

We also need the following result from [11].

Lemma 2.7. There exists a family of $2^{\mathfrak{c}}$-many real functions $\{g_{\mathcal{U}}\colon \mathcal{U} \in \beta\mathbb{N}\}$ and $\mathfrak{c}$-many distinct real numbers $\{x_{P,a}\colon \langle P, a\rangle \in \mathcal{P}\}$ such that for distinct $\mathcal{U}_1, \dots, \mathcal{U}_n \in \beta\mathbb{N}$ and $P \in \mathcal{P}_1^n$, there exists $a \in n^{\mathbb{N}}$ with

$$P(g_{\mathcal{U}_1}, \dots, g_{\mathcal{U}_n})(x_{P,a}) \neq 0.$$

We are now ready to present a proof for Proposition 2.6.

Proof of Proposition 2.6. Let X be an infinite-dimensional TVS and $\mathbb{F}(X^*)$ the family of all finite subsets of X^*. Let us recall that $\operatorname{card}(\mathbb{F}(X^*)) = \operatorname{card}(X^*)$ since X^* is infinite.

On $\mathcal{A} := \mathbb{F}(X^*) \times (c_{00} \setminus \{0\}) \times \bigcup_{n=1}^{\infty} n^{\mathbb{N}} \times (0, \infty)$, we define the partial order $\leqslant$ as follows: for any $\langle F, P, a, \varepsilon\rangle,\ \langle G, Q, b, \varepsilon'\rangle \in \mathcal{A}$,

$$\langle F, P, a, \varepsilon\rangle \leqslant \langle G, Q, b, \varepsilon'\rangle \text{ if and only if } F \subseteq G,\ P \preccurlyeq Q,\ a \leqslant^* b,\ \varepsilon \geqslant \varepsilon'.$$

Observe that $\mathcal{A}$ endowed with $\leqslant$ is a directed set and $\operatorname{card}(\mathcal{A}) = \operatorname{card}(X^*)$. For every $\langle F, P, a, \varepsilon\rangle \in \mathbb{F}(X^*) \times (c_{00} \setminus \{0\}) \times \bigcup_{n=1}^{\infty} n^{\mathbb{N}} \times (0, \infty)$, where $P = (P_i)_{i\in\mathbb{N}}$ and $a = (a_j)_{j\in\mathbb{N}}$, let us define

$$V_{\langle F,P,a,\varepsilon\rangle} := \left\{ x \in X\colon |x^*(x)| < \varepsilon \cdot \min\left\{1, \frac{1}{\sum_{i=1}^{\infty} |P_i|}, \sum_{j=1}^{\infty} 2^{-(j(a_j+1))}\right\},\ \forall x^* \in F \right\}.$$

Notice that

$$\mathcal{B}_0 := \left\{ V_{\langle F,P,a,\varepsilon\rangle}\colon \langle F, P, a, \varepsilon\rangle \in \mathbb{F}(X^*) \times (c_{00} \setminus \{0\}) \times \bigcup_{n=1}^{\infty} n^{\mathbb{N}} \times (0, \infty) \right\}$$

is a neighborhood basis of 0 for the weak topology $\sigma(X, X^*)$. Since X is infinite-dimensional, for every $F \in \mathbb{F}(X^*)$, there is a nonzero vector $x_F \in X$ such that $x^*(x_F) = 0$ for any $x^* \in F$. Now, for each $\mathcal{U} \in \beta\mathbb{N}$, let us define the net

$$\left(x^{\mathcal{U}}_{\langle F,P,a,\varepsilon\rangle}\right)_{\langle F,P,a,\varepsilon\rangle\in\mathcal{A}} := \left(\frac{g_{\mathcal{U}}(x_{\Phi^{-1}(P),a})}{\varepsilon} x_F\right)_{\langle F,P,a,\varepsilon\rangle\in\mathcal{A}}$$

where $x_{\Phi^{-1}(P),a}$ witnesses the conclusion of Lemma 2.7.

Let us prove that any nonzero linear combination of the nets in $\mathcal{B} := \left\{ \left(x^{\mathcal{U}}_{\langle F,P,a,\varepsilon\rangle}\right)_{\langle F,P,a,\varepsilon\rangle\in\mathcal{A}} \right\}$ weakly converges to 0. To do so, it is enough to show that $\left(x^{\mathcal{U}}_{\langle F,P,a,\varepsilon\rangle}\right)_{\langle F,P,a,\varepsilon\rangle\in\mathcal{A}}$ converges to 0 weakly for every $\mathcal{U} \in \beta\mathbb{N}$. Given W a weak neighborhood of 0, there exists $\langle F, P, a, \varepsilon\rangle \in \mathcal{A}$ such that $V_{\langle F,P,a,\varepsilon\rangle} \subseteq W$ since $\mathcal{B}_0$ is a neighborhood basis of 0 for the weak topology $\sigma(X, X^*)$. Notice that for any $\langle G, Q, b, \varepsilon'\rangle \geqslant \langle F, P, a, \varepsilon\rangle$ we have that $V_{\langle G,Q,b,\varepsilon'\rangle} \subseteq V_{\langle F,P,a,\varepsilon\rangle} \subseteq W$. By definition $F \subseteq G$, therefore $x^*(x_G) = 0$ for every $x^* \in F$. Hence,

$$\left|x^*\left(x^{\mathcal{U}}_{\langle G,Q,b,\varepsilon'\rangle}\right)\right| = \frac{\left|g_{\mathcal{U}}(x_{\Phi^{-1}(Q),b})\right|}{\varepsilon'} |x^*(x_G)| = 0 < \varepsilon,$$

for every $x^* \in F$. Therefore, $x^{\mathcal{U}}_{\langle G,Q,b,\varepsilon'\rangle} \in V_{\langle G,Q,b,\varepsilon'\rangle} \subseteq W$.

We will now distinguish between the real and complex cases.

Case 1. Assume that X is a real vector space. Let us show that the nets in $\mathcal{B}$ are linearly independent over $\mathbb{R}$ and any nonzero linear combination is unbounded. Let $P = \sum_{k=1}^{n} P_k x_k \in \mathcal{P}_1^n$ be arbitrary and $\mathcal{U}_1, \ldots, \mathcal{U}_n \in \beta\mathbb{N}$ pairwise distinct. Then there exists $a \in n^{\mathbb{N}}$ such that $P(g_{\mathcal{U}_1}, \ldots, g_{\mathcal{U}_n})(x_{P,a}) \neq 0$ by Lemma 2.7. Take any $F \in \mathbb{F}(X^*)$. Then

$$\left|\sum_{k=1}^{n} P_k \frac{g_{\mathcal{U}_k}(x_{P,a})}{\varepsilon}\right| x_F = \frac{1}{\varepsilon}\left|P(g_{\mathcal{U}_1}, \ldots, g_{\mathcal{U}_n})(x_{P,a})\right| x_F.$$

Since $P(g_{\mathcal{U}_1}, \ldots, g_{\mathcal{U}_n})(x_{P,a}) \neq 0$ and $x_F \neq 0$, by taking $\varepsilon \to 0$, we are done.

Case 2. Assume that X is a complex vector space. Let us show that the nets in $\mathcal{B}$ are linearly independent over $\mathbb{C}$ and any nonzero linear combination is unbounded. First of all, observe that $\mathcal{P}_1^n + i\mathcal{P}_1^n$ (where $i = \sqrt{-1}$) can be identified with the set $\mathbb{C}_1^n[x_1, \ldots, x_n]$, where $\mathbb{C}_1^n[x_1, \ldots, x_n]$ denotes the set of complex polynomials of degree 1 in n real variables. Let $P = \sum_{k=1}^{n} P_k x_k \in \mathcal{P}_1^n$ and $\mathcal{U}_1, \ldots, \mathcal{U}_n \in \beta\mathbb{N}$ pairwise distinct, then there is $a \in n^{\mathbb{N}}$ such that $P(g_{\mathcal{U}_1}, \ldots, g_{\mathcal{U}_n})(x_{P,a}) \neq 0$. Take any $Q = \sum_{k=1}^{n} Q_k x_k \in \mathcal{P}_1^n$ and $F \in \mathbb{F}(X^*)$. Then

$$\begin{aligned}\left|\sum_{j=1}^{n} (P_j + iQ_j) \frac{g_{\mathcal{U}_j}(x_{P,a})}{\varepsilon}\right| x_F &= \frac{1}{\varepsilon}\left|(P + iQ)(g_{\mathcal{U}_1}, \ldots, g_{\mathcal{U}_n})(x_{P,a})\right| x_F \\ &= \frac{1}{\varepsilon}\sqrt{\left(P(g_{\mathcal{U}_1}, \ldots, g_{\mathcal{U}_n})(x_{P,a})\right)^2 + \left(Q(g_{\mathcal{U}_1}, \ldots, g_{\mathcal{U}_n})(x_{P,a})\right)^2}\; x_{F_\alpha}\end{aligned}$$

Since $P(g_{\mathcal{U}_1}, \ldots, g_{\mathcal{U}_n})(x_{P,a}) \neq 0$ and $x_F \neq 0$, taking $\varepsilon \to 0$, we conclude the proof. □

2.2. Nets that fail the Banach-Steinhaus theorem.

The idea of the construction of the vector space in the proof of Theorem 2.1 can be carried out to show the existence of large vector spaces of nets that fail the Banach-Steinhaus theorem. Although its proof uses similar ideas from the proof of Theorem 2.1, we provide a detailed argument for the sake of completeness.

Recall that the well-known Banach-Steinhaus theorem states the following (see, for instance, [12, Chapter 3, Theorem 14.6]): let X be Banach space and Y be a normed space; denote by $\mathcal{L}(X, Y)$ the Banach space of all continuous linear operators from X into Y; if a sequence $(T_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}(X, Y)$ strongly converges pointwise, then there is a $T \in \mathcal{L}(X, Y)$ such that $(T_n)_{n\in\mathbb{N}}$ strongly converges pointwise to T and $\{\|T_n\| \colon n \in \mathbb{N}\}$ is uniformly bounded. Let us recall that this theorem is a result only about sequences, not nets (for an easy example, see [12, page 97] just after its proof as a consequence of the Principle of Uniform Boundedness).

In terms of lineability of the nets which do not satisfy the Banach-Steinhaus theorem, we have the following result.

Proposition 2.8. Let $\aleph_0 \leqslant \kappa \leqslant \mathfrak{c}$ be a cardinal number. If X and Y are nonzero real or complex normed spaces, then there exists a directed set $\mathcal{A}$ with $\mathrm{card}(\mathcal{A}) = \kappa$ such that the set of nets of continous linear operators $(T_a)_{a\in\mathcal{A}}$ in $\mathcal{L}(X, Y)$ that converge pointwise and also strongly converge pointwise to an operator but $\{\|T_a\| \colon a \in \mathcal{A}\}$ is not bounded is 2^κ-lineable.

Proof. Fix $I \in \mathcal{L}(X,Y)\setminus\{0\}$ (this can be done since $X \neq \{0\} \neq Y$). Let $\mathcal{A}^- \subset (-\infty,0)$ and $\mathcal{A}^+ \subset (0,\infty)$ be such that $\mathrm{card}(\mathcal{A}^-) = \mathrm{card}(\mathcal{A}^+) = \kappa$, and take $\mathcal{A} = \mathcal{A}^- \cup \mathcal{A}^+$ endowed with the standard order of $\mathbb{R}$. Note that $\mathrm{card}(\mathcal{A}) = \kappa$. By FKH, there exists $\mathcal{K} \subseteq \mathcal{P}(\mathcal{A}^-)$ a family of independent subsets of $\mathcal{A}^-$ having cardinality 2^κ. For any $K \in \mathcal{K}$, define

$$\left(T_a^K\right)_{a\in\mathcal{A}} = \begin{cases} |a|I, & \text{if } a \in K, \\ \frac{1}{a}I, & \text{if } a \in \mathcal{A}^+, \\ 0, & \text{otherwise.} \end{cases}$$

Given nonzero scalars $\lambda_1,\dots,\lambda_m \in \mathbb{K}$ and $K_1,\dots,K_m \in \mathcal{K}$ distinct, let us show that

$$\left\{ \left\| \sum_{j=1}^m \lambda_j T_a^{K_j} \right\| : a \in \mathcal{A} \right\}$$

is unbounded, proving also the linear independence of the subset $\left\{ \left\| \left(T_a^K\right)_{a\in\mathcal{A}} \right\| : K \in \mathcal{K} \right\}$. As $\mathcal{K}$ is a family of independent subsets of $\mathcal{A}^-$, we can take an unbounded sequence $(a_l^1)_{l\in\mathbb{N}} \subseteq K_1 \setminus (K_2 \cup \dots \cup K_m)$, which shows that

$$\left\{ \left\| \sum_{j=1}^m \lambda_j T_{a_l^1}^{K_j} \right\| : l \in \mathbb{N} \right\} = \left\{ |\lambda_1 a_l^1|\, \|I\| : l \in \mathbb{N} \right\}$$

is unbounded.

Finally, observe for any $K \in \mathcal{K}$ and for every $x \in X$, we have that $\left(T_a^K(x)\right)_{a\in\mathcal{A}}$ converges to 0 and also $\left(T_a^K\right)_{a\in\mathcal{A}}$ strongly converges pointwise to the null operator. Thus, the vector space generated by the family of nets $\left\{ \left\| \left(T_a^K\right)_{a\in\mathcal{A}} \right\| : K \in \mathcal{K} \right\}$ is as needed. □

2.3. **Weakly dense and norm-unbounded nets.** In this section, we will be interested in nets $(x_a)_{a\in\mathcal{A}}$ such that $\{x_a : a \in \mathcal{A}\}$ is weakly dense and the set $\{\|x_a\| : a \in \mathcal{A}\}$ is unbounded. In [3, Corollary 5] (see also [7,20,21] for more general results in this line), the authors show that, in a separable Banach space X, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $\{x_n : n \in \mathbb{N}\}$ is weakly dense in X and $\|x_n\| \longrightarrow \infty$ as $n \to \infty$. Our aim here is to prove Theorem 2.12 below, which deals with a family of κ-sequences satisfying that $\{x_\alpha : \alpha < \kappa\}$ is weakly dense in X and also that the set $\{\|x_\alpha\| : \alpha < \kappa\}$ is unbounded. We show that such a family is κ^+-lineable under some condicions on κ. As an immediate consequence of Theorem 2.12, we obtain Corollary 2.9 below which is related to the existence of norm divergent sequences that are weakly dense.

Corollary 2.9. Let X be real or complex separable Banach space. The set of all sequences $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that

(a) $\{\|x_n\| : n \in \mathbb{N}\}$ is unbounded and
(b) $\overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,X^*)} = X$

is $\mathfrak{c}$-lineable.

For the proof of Theorem 2.12, we need the definition of almost disjoint subsets of a regular cardinal number as well as Lemma 2.11, which is taken from [18]. We start with almost disjointness.

Definition 2.10. Let κ be a regular cardinal number. We say that $K_1, K_2 \subseteq \mathcal{P}(\kappa)$ are *almost disjoint* if $|K_1| = |K_2| = \kappa$ and $|K_1 \cap K_2| < \kappa$.

We will use the following lemma.

Lemma 2.11. [18, Lemma 9.23 and Exercise 9.12] Let κ be a regular cardinal.

(a) There exists an almost disjoint $\mathcal{K} \subseteq \mathcal{P}(\kappa)$ such that $|\mathcal{K}| = \kappa^+$.
(b) If $2^{<\kappa} = \kappa$, then there exists an almost disjoint $\mathcal{K} \subseteq \mathcal{P}(\kappa)$ such that $|\mathcal{K}| = 2^\kappa$.

Theorem 2.12. Let X be a real or complex normed space with $\operatorname{dens}(X) = \kappa \geqslant \aleph_0$, where κ is a regular cardinal. Denote by UWD_κ the set of all κ-sequences $(x_\alpha)_{\alpha<\kappa}$ such that

(i) $\{\|x_\alpha\| : \alpha < \kappa\}$ is unbounded and
(ii) $\overline{\{x_\alpha : \alpha < \kappa\}}^{\sigma(X,X^*)} = X$.

Then, the set UWD_κ is κ^+-lineable. Moreover, if $2^{<\kappa} = \kappa$, then UWD_κ is 2^κ-lineable.

Before we get into the proof of Theorem 2.12, let us provide a comment about properties (i) and (ii) above. Suppose that the net $(x_a)_{a\in\mathcal{A}}$ satisfies that $\{x_a : a \in \mathcal{A}\} = X$ or $\{x_a : a \in \mathcal{A}\}$ is norm-dense in X. Then, clearly we have that $\{\|x_a\| : a \in \mathcal{A}\}$ is unbounded and $\{x_a : a \in \mathcal{A}\}$ is weakly dense (i.e., conditions (i) and (ii) are both satisfied for the net $(x_a)_{a\in\mathcal{A}}$). Moreover, if $\operatorname{dens}(X) = \kappa$, we can assume that $|\mathcal{A}| = \kappa$ and consider a bijection $f : \mathcal{A} \longrightarrow \kappa$. Since (i) and (ii) are satisfied for $(x_a)_{a\in\mathcal{A}}$, conditions (i) and (ii) will be also satisfied for a κ-sequence thanks to the bijection f. This guarantees the existence of κ-sequences satisfying such properties.

Proof of Theorem 2.12. We start the proof by defining a family $\mathcal{K}$ of almost disjoint subsets (see Definition 2.10 above) of κ that witnesses either Lemma 2.11(a) or Lemma 2.11(b). For each $K \in \mathcal{K}$, we can consider an increasing bijection $\varphi_K : K \longrightarrow \kappa$. Now, let $(x_\alpha)_{\alpha<\kappa}$ be a κ-sequence satisfying properties (i) and (ii). For a fixed $K \in \mathcal{K}$ and every $\alpha < \kappa$, we define

$$x_\alpha^K := \begin{cases} x_{\varphi_K(\alpha)}, & \text{whenever } \alpha \in K, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Write $K = \{\alpha_\beta : \beta < \kappa\}$, where $\beta_1 < \beta_2 < \kappa$ implies that $\alpha_{\beta_1} < \alpha_{\beta_2}$. Since each $K \in \mathcal{K}$ is such that $|K| = \kappa$, we have that the set $\{\|x_\alpha^K\| : \alpha < \kappa\}$ is also unbounded. Moreover, since for every $K \in \mathcal{K}$ we have that

$$\{x_\alpha : \alpha < \kappa\} \subseteq \{x_\alpha^K : \alpha < \kappa\},$$

then we obtain

$$X = \overline{\{x_\alpha : \alpha < \kappa\}}^{\sigma(X,X^*)} \subseteq \overline{\{x_\alpha^K : \alpha < \kappa\}}^{\sigma(X,X^*)} \subseteq X.$$

This shows that $\overline{\{x_\alpha^K : \alpha < \kappa\}}^{\sigma(X,X^*)} = X$ for every $K \in \mathcal{K}$.

Now, let $\lambda_1, \dots, \lambda_n$ be nonzero scalars and $K_1, \dots, K_n \in \mathcal{K}$ distinct. Since $\operatorname{card}(K_1) = \kappa$ and $\operatorname{card}(K_1 \cap K_j) < \kappa$ for every $j = 2, \dots, n$, we have that the cardinality of the set

$$\mathcal{I}_1 := K_1 \setminus \left(\bigcup_{j=2}^n (K_1 \cap K_j) \right)$$

is also κ, that is, $\operatorname{card}(\mathcal{I}_1) = \kappa$. Consider $(x_{\alpha_\beta})_{\beta<\kappa}$ to be a κ-subsequence of $(x_\alpha)_{\alpha<\kappa}$ indexed by $\mathcal{I}_1$. Then, by the election of the index set $\mathcal{I}_1$ and by using (1), we get that

$$\left\{ \left\| \sum_{j=1}^{n} \lambda_j x_{\alpha_\beta}^{K_j} \right\| : \beta < \kappa \right\} = \left\{ |\lambda_1| \left\| x_{\varphi_{K_1}(\beta)} \right\| : \beta < \kappa \right\}.$$

This argument shows at once that $(x_\alpha^{K_1})_{\alpha<\kappa}, \ldots, (x_\alpha^{K_n})_{\alpha<\kappa}$ are linearly independent and also that the set

$$\left\{ \left\| \sum_{j=1}^{n} \lambda_j x_\alpha^{K} \right\| : \alpha < \kappa \right\}$$

is unbounded in X. It remains to prove that

$$\overline{\left\{ \sum_{j=1}^{n} \lambda_j x_\alpha^{K_j} : \alpha < \kappa \right\}}^{\sigma(X,X^*)} = X. \tag{2}$$

Before doing that, let us observe the following. Since $\operatorname{card}\left(\bigcup_{j=2}^{n} (K_1 \cap K_j) \right) < \kappa$ and κ is regular it yields

$$\overline{\alpha} := \sup \left\{ \alpha \in \bigcup_{j=2}^{n} (K_1 \cap K_j) \right\} < \kappa$$

by [18, Lemma 3.9]. Why we are considering such an $\overline{\alpha}$ will be clear below. In order to prove (2), let us fix $x \in X$. Since (ii) holds true, there exists $(z_\beta)_{\beta<\kappa} \subseteq \{x_\alpha : \alpha < \kappa\}$ such that $z_\beta \xrightarrow{w} \frac{1}{\lambda_1} x$. Considering once again the index set $\mathcal{I}_1$ and $\overline{\alpha}$, there exists $\overline{\alpha} \leqslant \alpha_0 < \kappa$ such that for every $\alpha_0 \leqslant \beta < \kappa$, we have that

$$\sum_{j=1}^{n} \lambda_j x_\beta^{K_j} = \lambda_1 x_{\varphi_{K_1}(\beta)}.$$

Since φ_{K_1} is an increasing bijection and $\operatorname{card}(\mathcal{I}_1) = \kappa$, we have that

$$\left\{ \lambda_1 x_{\varphi_{K_1}(\beta)} : \alpha_0 \leqslant \beta < \kappa \right\} = \{\lambda_1 x_\alpha : \alpha < \kappa\} \setminus \mathcal{F},$$

where $\mathcal{F}$ is a set with elements of the κ-sequence $(\lambda_1 x_\alpha)_{\alpha<\kappa}$ such that its cardinality is $< \kappa$. Therefore, there exists a κ-sequence $(\tilde{z}_\beta)_{\beta<\kappa} \subseteq \{\sum_{j=1}^{n} \lambda_j x_\alpha^{K_j} : \alpha < \kappa\}$ such that

$$\tilde{z}_\beta \xrightarrow{w} \lambda_1 \cdot \frac{1}{\lambda_1} x = x$$

and this proves (2) as desired. □

2.4. **Convergent series with associated divergent nets.** Let X be a normed space and $\mathcal{I}$ an infinite set. We can give meaning to the convergence of the (possibly) "uncountable sum" in X, denoted by $\sum_{i\in I} x_i$, where each x_i belongs to X, as follows: consider $\mathcal{F}$ to be the set of all finite subsets of I endowed with the inclusion $\subseteq$. Bearing this in mind, we have that $\mathcal{F}$ is a directed set. Now, for every $F \in \mathcal{F}$, we define

$$x_F := \sum_{i \in F} x_i.$$

Each x_F is then a well-defined vector of X (since F is finite) and $(x_F)_{F \in \mathcal{F}}$ is a net. In the same line, we have the following definition. Given $x_i \in X$ for all $i \in I$, we say that $\sum_{i \in I} x_i$ converges to $x \in X$ whenever $\lim_{F \in \mathcal{F}} x_F = x$. Recall that in Hilbert spaces, the latter definition can be used to obtain some relevant characterization in the non-separable case (see, for instance, [12, Chapter 1, Theorem 4.13]).

Our next result is motivated by the following fact. If $X = H$ is a Hilbert space and $I = \mathbb{N}$, it is known that if $\lim_{F \in \mathcal{F}} h_F = h \in \mathcal{H}$, then $\sum_{n=1}^{\infty} h_n = h$, but the converse is not true in general (it is important to mention that if $\sum_{n=1}^{\infty} h_n$ is absolutely convergent, then the converse implication is satisfied; see, for instance, [12, Chapter 1, Section 4, Exercises 10 and 11]). The following result shows that we can find (in terms of lineability) large sets of sequences $(x_n)_{n \in \mathbb{N}}$ in a normed space such that the series $\sum_{n=1}^{\infty} x_n$ is convergent but the net $(x_F)_{F \in \mathcal{F}}$ diverges. In what follows, we denote by $\mathrm{CS}(\mathbb{K}) \subseteq \mathbb{K}^{\mathbb{N}}$ the set of all sequences $(k_n)_{n \in \mathbb{N}}$ such that the series with general terms $(k_n)_{n \in \mathbb{N}}$ is conditionally convergent.

Theorem 2.13. Let X be a normed space defined over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $\mathcal{F}$ the family of finite subsets of $\mathbb{N}$ endowed with the order $\subseteq$. The set of all sequences $(x_n)_{n \in \mathbb{N}} \subseteq X$ such that $\sum_{n=1}^{\infty} x_n$ is convergent and $(x_F)_{F \in \mathcal{F}}$ diverges is $\mathfrak{c}$-lineable.

Proof. Since a series $\sum a_n$ of complex numbers is conditionally convergent if and only if $\sum \mathrm{Re}(a_n)$ or $\sum \mathrm{Im}(a_n)$ is conditionally convergent, we restrict ourselves to the real case, that is, $\mathbb{K} = \mathbb{R}$. Fix $x \in X \setminus \{0\}$. Let $V_1 \subseteq (\mathrm{CS}(\mathbb{K}) \cup \{0\})$ be a vector subspace of dimension $\mathfrak{c}$ (that we might consider thanks to [1, Theorem 2.1], which holds for both real and complex cases). Then, the set

$$xV_1 := \{(k_n x)_{n \in \mathbb{N}} \colon (k_n)_{n \in \mathbb{N}} \in V_1\} \subseteq X^{\mathbb{N}}$$

yields the desired result. Indeed, it is easy see that xV_1 is a vector subspace of $X^{\mathbb{N}}$ of dimension $\mathfrak{c}$ such that $\sum_{n=1}^{\infty} k_n x = x \sum_{n=1}^{\infty} k_n$ converges. Recall that given a conditionally convergent series $\sum_{n=1}^{\infty} k_n$, the series of positive terms $\sum_n k_n^+$ diverges. Now fix $M > 0$ and $F \in \mathcal{F}$ to be arbitrary. Let $F_M^+ \in \mathcal{F}$ be such that $F_M^+ \supseteq F$, $k_n = k_n^+$ provided that $n \in F_M^+ \setminus F$, and

$$\sum_{n \in F_M^+} k_n = \sum_{n \in F} k_n + \sum_{n \in F_M^+ \setminus F} k_n > M.$$

Therefore,

$$\left\| \sum_{n \in F_M^+} k_n x \right\| = \left\| x \sum_{n \in F_M^+} k_n \right\| = \left| \sum_{n \in F_M^+} k_n \right| \|x\| > M \|x\|.$$

□

3. Remarks and Open Questions

We conclude the paper by presenting a series of remarks and open problems related to the contents of the paper.

The proof of Theorem 2.6 uses Lemma 2.7, but in [11] there is a stronger version of this lemma which states the following.

Lemma 3.1. There exists a family of $2^{\mathfrak{c}}$-many real functions $\{g_{\mathcal{U}} : \mathcal{U} \in \beta\mathbb{N}\}$ and $\mathfrak{c}$-many distinct real numbers $\{x_{P,a} : \langle P, a\rangle \in \mathcal{P}\}$ such that for distinct $\mathcal{U}_1, \ldots, \mathcal{U}_n \in \beta\mathbb{N}$ and any real polynomial P of degree at least 1 in n variables, there exists $a \in n^{\mathbb{N}}$ with

$$P(g_{\mathcal{U}_1}, \ldots, g_{\mathcal{U}_n})(x_{P,a}) \neq 0.$$

In view of Lemma 3.1 and the usage of Lemma 2.7 in Theorem 2.6, the following problem seems to be natural. For the precise concepts of algebrability we send the reader to [2].

Problem 3.2. Let X be a real or complex topological algebra. What can be said about the algebrability of the family of nets that are weakly convergent and norm-unbounded? What is the smallest possible directed set that witnesses our desired algebrability property?

Given a regular cardinal κ, it is well-known that under ZFC we can only guarantee that all almost disjoint families have cardinality at most κ^+. Now, under additional assumptions without the Generalized Continuum Hypthesis such as $2^{<\kappa} = \kappa$, we can assure the existence of an almost disjoint family of the largest possible cardinality (recall that in ZFC we have that $2^{<\aleph_0} = \aleph_0$, so in this case no extra assumptions are needed). However, J. E. Baumgartner proved that it is consistent with ZFC that there is no almost disjoint family of $\aleph_1$ having cardinality $2^{\aleph_1}$ (see [6, Theorem 5.6]). The existence of large families of almost disjoint subsets of κ are used in the proof of Theorem 2.12 in order to construct the desired vector space, but they do not play a role in the definition of the set UWD_κ. For this, it is natural to ask the following question which we were not able to solve.

Problem 3.3. Let κ be a regular cardinal number. Can we guarantee the 2^κ-lineability of UWD_κ within ZFC?

Related to this question we have the following one, which we wonder whether Corollary 2.9 holds also when one considers $\|x_n\| \longrightarrow \infty$ as $n \to \infty$ instead of only assuming that the set $\{\|x_n\| : n \in \mathbb{N}\}$ is unbounded. This is relevant since in the references [3, 20, 21] the authors study these kind of sequences.

Problem 3.4. Is the set of all sequences $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that

(i) $\|x_n\| \to \infty$ and
(ii) $\overline{\{x_n : n \in \mathbb{N}\}}^{w(X,X^*)} = X$

$\mathfrak{c}$-lineable?

As a counterpart of Problem 3.4 for κ-sequences, we also wonder whether there exists κ-sequences which satisfy properties (i) and (ii) from Problem (3.4) above and, in this case, what are its lineability properties.

Let us remind that in the proof of Theorem 2.13 we use the fact that $\mathrm{CS}(\mathbb{K})$ is $\mathfrak{c}$-lineable. Now, it is also known that $\mathrm{CS}(\mathbb{R})$ is $(\aleph_0, 1)$-algebrable with respect to the Cauchy product [4] and $\mathrm{CS}(\mathbb{C})$ is $\mathfrak{c}$-algebrable with repect to the pointwise product [5] (once again, we send the reader to [2] for the algebrability properties). In view of these facts, one could wonder whether the set of conditionally convergent series can be used to construct algebras that satisfy the conditions of Theorem 2.13. We highlight this question as follows.

Problem 3.5. Under (possibly) additional assumptions on a normed algebra X, what are the algebrability properties of the set of sequences $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $\sum_{n=1}^{\infty} x_n$ converges but $(x_F)_{F\in\mathcal{F}}$ diverges?

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(Dantas) Departamento de Análisis Matemático, Facultad de Ciencias Matemáticas, Universidad de Valencia, Doctor Moliner 50, 46100 Burjasot (Valencia), Spain.
ORCID: 0000-0001-8117-3760
Email address: sheldon.dantas@uv.es

(Rodríguez-Vidanes) Departamento de Matemática Aplicada a la Ingeniería Industrial, Escuela Técnica Superior de Ingeniería y Diseño Industrial, Ronda de Valencia 3, Universidad Politécnica de Madrid, Madrid, 28012, Spain
ORCID: 0000-0003-2240-2855
Email address: dl.rodriguez.vidanes@upm.es