Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Higher-order derivative gravity

Gravitace s vyššími derivacemi

Bachelor's Degree Project

Author: **Aneta Pjatkanová**

Supervisor: **doc. Ing. Mgr. Petr Jizba, Ph.D.**

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ČVUT
ČESKÉ VYSOKÉ UČENÍ TECHNICKÉ V PRAZE

ZADÁNÍ BAKALÁŘSKÉ PRÁCE

I. OSOBNÍ A STUDIJNÍ ÚDAJE

Příjmení: **Pjatkanová** Jméno: **Aneta** Osobní číslo: **509330**

Fakulta/ústav: **Fakulta jaderná a fyzikálně inženýrská**

Zadávající katedra/ústav: **Katedra fyziky**

Studijní program: **Matematické inženýrství**

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II. ÚDAJE K BAKALÁŘSKÉ PRÁCI

Název bakalářské práce:

Gravitace s vyššími derivacemi

Název bakalářské práce anglicky:

Higher-order derivative gravity

Pokyny pro vypracování:

1) Seznámení se s problematikou gravitačních teorií s vyššími derivacemi.
2) Seznámení se s existujícími nestandardními řešeními vybraných gravitačních teorií.
3) Formulace analogu Schwarzschildova a Kerrova řešení pro vybrané gravitační teorie.
4) Diskuse získaných řešení.

Seznam doporučené literatury:

[1] S. Capozziello and V. Faraoni, Beyond Einstein Gravity; A Survey of Gravitational Theories for Cosmology and Astrophysics (Springer, New York, 2011).
[2] S. Carroll, Spacetime and Geometry: An Introduction to General Relativity, (Cambridge University Press, Cambridge, 2019).
[3] S. Capozziello and M. Francaviglia, Extended theories of gravity and their cosmological andastrophysical applications, General Relativity and Gravitation volume 40, pages 357–420 (2008).
[4] A. Idrissov, Higher-order derivative gravity and Black Holes, MSc Thesis, Imparial College London, 2017; https://inspirehep.net/files/43e23aae76425e0081d35dad2ebb454c

Jméno a pracoviště vedoucí(ho) bakalářské práce:

doc. Ing. Mgr. Petr Jizba, Ph.D. katedra fyziky FJFI

Jméno a pracoviště druhé(ho) vedoucí(ho) nebo konzultanta(ky) bakalářské práce:

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doc. Ing. Mgr. Petr Jizba, Ph.D.
podpis vedoucí(ho) práce

podpis vedoucí(ho) ústavu/katedry

doc. Ing. Václav Čuba, Ph.D.
podpis děkana(ky)

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Studentka bere na vědomí, že je povinna vypracovat bakalářskou práci samostatně, bez cizí pomoci, s výjimkou poskytnutých konzultací.
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ČESKÉ VYSOKÉ UČENÍ TECHNICKÉ V PRAZE

Fakulta jaderná a fyzikálně inženýrská

Břehová 7
115 19 Praha 1

PROHLÁŠENÍ

Já, níže podepsaná

Jméno a příjmení studenta:	Aneta Pjatkanová
Osobní číslo:	509330
Studijní program:	Matematické inženýrství
Specializace:	Matematická fyzika

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Gravitace s vyššími derivacemi

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Author's declaration:
I declare that this Bachelor's Degree Project is entirely my own work and I have listed all the used sources in the bibliography.

Prague, August 5, 2024 Aneta Pjatkanová

Název práce:

Gravitace s vyššími derivacemi

Autor: Aneta Pjatkanová

Studijní program: Matematické inženýrství

Specializace: Matematiká fyzika

Druh práce: Bakalářská práce

Vedoucí práce: doc. Ing. Mgr. Petr Jizba, Ph.D.,katedra fyziky, Fakulta jaderná a fyzikálně inženýrská ČVUT v Praze

Abstrakt: Gravitace s vyššími derivacemi odpovídá specifické modifikaci obecné torie relativity. V této práci jsou shrnuty poznatky o pěti základních typech gravitací s vyššími derivacemi a zdůrazněny jejich výhody. Pro $f(R)$ gravitaci jsou studovány možné způsoby řešení pro prostoročasy Schwarzschildova a Kerrova typu. Pro konformní gravitaci jsou analyzována řešení Mannheima a Kazanase pro sférický a osově symetrický problém.

Klíčová slova: $f(R)$ gravitace, Kerrovo řešení, Konformní gravitace, modifikovaná gravitace, sférické řešení

Title:

Higher-order derivative gravity

Author: Aneta Pjatkanová

Abstract: Higher-derivative theories of gravity are modifications of general relativity. In this work are reviewed five main categories of higher-derivative theories and their advantages. For $f(R)$ gravity there are studied possible ways to solve Schwarzschild-like and Kerr-like problems. For Conformal gravity, there are reiterated solutions to spherically symmetric and axially symmetric problem derived by Mannheim and Kazanas.

Key words: Conformal gravity, $f(R)$ gravity, Kerr solution, modified gravity, spherical solution

Contents

Introduction

So far, the General theory of relativity (GR) has been considered the best possible theory of gravity. However, with the increasing amount of data from cosmology and particle physics, and with the advancement of quantum theory, some shortcomings of GR became apparent.Since GR is not able to solve them sufficiently, this may indicate that GR is not the most fundamental theory, and that it needs to be adjusted.

One could take a radical path and abandon GR altogether and propose a new theory of gravity. Or one could keep the principles of GR and just modify them. There are also myriads of possible mathematical approaches to GR that may be advantageous in different circumstances. In this thesis we consider only a pure metric formalism in four dimensions.

In a purely metric four-dimensional approach, Lovelock's theorem dictates that in order to modify GR, new terms must be added to the Einstein-Hilbert action. The most natural terms that could be introduced into the Einstein–Hilbert action are higher-derivative (HD) terms, since similar contributions arise in the quantization of GR [31].

The thesis is organized as follows: First, we briefly review five basic categories of HD theories and point out their potential to solve problems that GR struggles with. For the $f(R)$ theory, we study possible methods to solve the Schwarzschild-like problem and the Kerr-like problem. For Conformal gravity, we will review the methods for solving the Schwarzschild-like and Kerr-like problems developed by Mannheim and Kazanas.

Throughout the work we consider geometrized units $G = 1 = c$, a metric of signature $(- + ++)$, and the notation for the covariant derivative is ";".

Chapter 1

Overview of higher derivative theories

1.1 Problems of General Relativity

Some problems of GR arising from Cosmology are the horizon problem, cosmological constant problem and problem with Dark energy and Dark matter.

The Horizon problem suggests that there might be a different theory of gravity, which describes the early universe. The Horizon problem briefly stated is following. The present universe is well described by the Robertson-Walker (R-W) metric, which is conformally flat. If we assume that the Universe is described by the R-W metric for its whole history, then it is concluded that the particle horizons have to be finite. Therefore some parts of the Universe can not be causally connected and the observed homogenity and isotropy could not be achieved. There are two possible solutions, the Universe was born homogeneous or at its early stages it was not described by the R-W metric but by other theory which gradually changed into the R-W model. The problem is in detail discussed in [32].

The cosmological constant problem dates back to Einstein himself. As is discussed in [33] Einstein included the cosmological term in his equations to obtain a static solution, since it was believed that the Universe is static. But equations of this form have another static solution, found by deSitter, which violates Mach's principle. Moreover, it was observed that the Universe undergoes expansion, which then eliminates the need for a static solution. One would then like to cross out the cosmological term, however, this is not possible, because any term that contributes to the energy density of the vacuum manifests as a contribution to the cosmological constant. So the Einstein equations have to include the cosmological term, so where is the problem? From the observations of the expansion of the Universe, it is possible to estimate the value of the energy density of an empty space. Comparison of this observational estimate with the prediction of quantum field theory reveals the problem, the prediction of the Quantum field theory for the energy density of vacuum is about 120 magnitudes larger [33]. Even after decades of research the problem remains and shows how challenging it is, to formulate such a theory, which would predict such a small value of vacuum energy density.

Other observational propositions of GR are Dark Energy and Dark Matter. For GR to adequately describe big structures of the Universe, it is necessary to introduce a new kind of matter, so-called Dark matter, which does not interact in any other way than gravitationally [7]. But, curiously, no Dark matter is necessary on the smaller scales as it has been pointed out in [22]. To explain the inflationary phase of the universe by GR, it is needed to introduce Dark energy [5].

Because GR is a classical theory, it does not have to work in the quantum domain and it certainly does not. Attempts to quantize GR have been met with failure, but once there are introduced some quantum corrections in the quantized GR, the theory becomes viable [7].

Besides the deviation from experimental data, GR seems to have a few fundamental defects, such as a violation of Mach's principle or incompatibility with other theories [7].

Mach's principle states that the inertia of an object is a result of the interaction of the object with the rest of the Universe. This principle is not incorporated fully in the GR, since there exists the deSitter solution, which is nontrivial and the mass of the universe is zero [33].

Any pursuit to link GR with other fundamental theories such as Superstring theory or Grand Unified Theories (GUT) has not been met with success yet [7]. The effective actions of these theories always include some modified terms, higher-order curvature invariants or non-minimally coupled scalar field [7]. Therefore, to unify GR with these possible fundamental theories, we do have to introduce some kind of modification into Hilbert-Einstein action.

Lastly, if we analyze closely the action of GR, we can easily see that it is the simplest possible form. However, there is no physical reason for the action to only involve Ricci scalar[7]. It is true, that with a more intricate term in the action, namely higher-derivative terms, it is inevitable to discuss the stability of such systems since they can suffer from Ostrogradsky's instability [34]. But some theories are perfectly stable, or there are promising methods to deal with the instability [29, 30, 12, 1].

1.2 Higher derivative gravity

In a search for a viable theory of gravity, it is possible to follow different paths. The most radical approach is to completely abandon GR and formulate a new theory of gravity. Creating new theories has its advantages like addressing directly structural problems of GR or using the same mathematical formulation as in other theories, which are considered more fundamental [7].

Or one can preserve the metric formalism of GR and its axioms and only modify the action of GR. These theories are called Extended Theories of Gravity (ETGs) and they venture to modify the standard Hilbert-Einstein action in two different ways

- they include non-minimally coupled scalar fields into action
- they include higher-curvature invariants into action.

Theories of the first type are called Scalar-tensor theories, theories of the second type are named higher-order theories and it is also possible to combine these two methods to formulate higher-order-scalar-tensor theories [8].

All three of these modified theories are conformally equivalent to each other and moreover, they are even equivalent to classical GR also via conformal transformation [8]. For classical theory, the mathematical conformal equivalence is also a physical one, unfortunately, incorrect interpretation of scaling in different frames leads to wrong judgements [14]. However, the meaning of conformal transformation is still unclear in the quantum domain [15].

There are several subcategories of higher-order theories, some of which are more successful than others. Individual theories are discussed in the following order:

1. $f(R)$ gravity
2. $f(R,G)$ gravity
3. Conformal gravity
4. Non-local gravity
5. $f(R^{\mu}{}_{\nu\sigma\rho})$ gravity

1.2.1 $f(R)$ gravity

The simplest modification of GR is so-called $f(R)$ gravity. This gravitational theory allows the action to have a nonlinear structure because $f(R)$ can be any analytical function. $f(R)$ gravity is characterized by Lagrangian density

$$\mathcal{L}_f = f(R)\,. \tag{1.1}$$

To derive equations of motion (EoM), calculus of variations is used on action

$$\mathcal{S}_{f(R)} = \frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\mathcal{L}_f + \mathcal{S}^{(m)}\,, \tag{1.2}$$

where g is determinant of the metric and $\mathcal{S}^{(m)}$ is action of the matter. Final EoM take form (detailed derivation can be found in (A.1)

$$\kappa T_{\mu\nu} = R_{\mu\nu}f'(R) - \frac{1}{2}g_{\mu\nu}f(R) + g_{\mu\nu}\Box f'(R) - f'(R)_{;\nu\mu}\,, \tag{1.3}$$

where $f'(R)$ is the derivative of $f(R)$ with respect to R.

The term $f'(R)_{;\mu\nu}$ suggests that the EoM are partial differential equations of fourth order with respect to the metric, since R depends on the second partial derivative of the metric.

Theories with $f(R)$ corrections (prominently polynomial or logarithmic) predict the galaxy rotation curves even without the introduction of Dark matter and some models also successfully describe inflation.

The simplest of $f(R)$ gravity is the Starobinsky gravity, having Lagrangian

$$\mathcal{L}_{Starobinsky} = R + \alpha R^2\,, \tag{1.4}$$

where α is a positive parameter. Similar contributions arise when higher-order quantum corrections in quantized GR are allowed [5]. This model can describe inflation, which could solve the horizon and flatness problem.

The generalization of Starobinsky gravity is described by Lagrangian [25]

$$\mathcal{L}_{polynom} = R + \frac{\alpha}{R^n} + \beta R^m\,, \tag{1.5}$$

where α, β are constants and n, m are positive numbers (can be fractions). These models unify the early cosmic inflation - terms with positive powers of R - and late-time cosmic acceleration - terms with negative powers of R. If the model includes the positive powers of R, then the instabilities of $\frac{1}{R^n}$ are improved. These polynomial corrections are interesting, because they might have an origin in the string theory and they could solve the Dark matter and cosmological constant problem without the assumption of Dark matter and Dark energy. [25]

In ref [26] has been suggested model with Lagrangian

$$\mathcal{L}_{\ln R} = R + \alpha \ln R + \beta R^m\,, \tag{1.6}$$

where α, β are constants and m is a positive real number. Authors were motivated to include lnR terms because similar contributions appear in the quantum theory in curved spacetime. Terms lnR support the late-time inflation and positive powers of R describe the inflation phase and also combat the instabilities posed by the lnR term even better than in the polynomial model above [26].

1.2.2 $f(R, G)$ gravity

More advanced modification of GR is $f(R, G)$ gravity, where the G is so called Gauss-Bonnet term (GB)

$$G = R^{\mu\nu\sigma\rho}R_{\mu\nu\sigma\rho} - 4R^{\mu\nu}R_{\mu\nu} + R^2 . \tag{1.7}$$

Integral over a manifold of (1.7) is the Euler characteristics of the manifold. The integral of the Gauss=Bonnet term is in four dimensions identically zero, on the other hand in quantum theory or in higher dimensions is the Gauss-Bonnet term nontrivial and therefore its contributions to the EoM do change the theory while having the same dynamics in classical four-dimensional case [7].

General Lagrangian of the $f(R, G)$ gravity is

$$\mathcal{L}_G = f(R, G) . \tag{1.8}$$

The EoM can be derived in the same manner as these of $f(R)$ gravity from action (derivation can be found in B)

$$\mathcal{S} = \frac{1}{2\kappa}\int d^4x\,\sqrt{-g}f(R, G) + \mathcal{S}^{(m)} \tag{1.9}$$

and they take form

$$\begin{aligned} T_{\kappa\lambda} =& -\frac{1}{2}g_{\kappa\lambda}f + \partial_R f R_{\kappa\lambda} + \Box\left(\partial_R f\right)g_{\kappa\lambda} - \partial_R f_{;\lambda\kappa} + 2\partial_G f R R_{\kappa\lambda} + 2\Box\left(\partial_G f R\right)g_{\kappa\lambda} \\ & - 2\left(\partial_G f R\right)_{;\lambda\kappa} - 4\Box\left(\partial_G f R_{\kappa\lambda}\right) - 4\left(\partial_G f R^{\mu\nu}\right)_{;\nu\mu}g_{\kappa\lambda} + 8\left(\partial_G f R^{\mu}{}_{\lambda}\right)_{;\kappa\mu} - 8\partial_G f R_{\kappa\nu}R^{\nu}{}_{\lambda} \\ & + 4\left(\partial_G f R_{\lambda\nu\kappa\rho}\right)^{;\rho\nu} + 4\left(\partial_G f R_{\lambda\kappa\nu\rho}\right)^{;\rho\nu} + 2\partial_G f R_{\mu\nu\rho\lambda}R^{\mu\nu\rho}{}_{\kappa} . \end{aligned} \tag{1.10}$$

Outstanding example of $f(R, G)$ is Einstein-Gauss-Bonnet gravity which is described by action

$$I_{GB} = \int d^4x\,\sqrt{-g}(R + \alpha G). \tag{1.11}$$

Corresponding EoM are similar to Einstein's equations, therefore this theory describes on the classical level universe in the same way as GR does.

The motivation to include Gauss-Bonnet term in the gravitational action rises from QFT. GB term appears in theories such as Yang Mills string theory [27]. In cosmology, the GB term describes well the late time development of the Universe. Also in ref. [27] has been shown that this theory passes the Solar system tests.

1.2.3 Conformal gravity

Conformal theory of gravity is a theory that instead of Riemann tensor $R_{\mu\nu\tau\sigma}$ utilizes only its anti-symmetric part, so-called Weyl tensor $C_{\mu\nu\tau\sigma}$ defined in four dimensions as [16]

$$C_{\mu\nu\kappa\lambda} = R_{\mu\nu\kappa\lambda} + \frac{1}{2}\left(R_{\mu\lambda}g_{\nu\kappa} - R_{\mu\kappa}g_{\nu\lambda} + R_{\nu\kappa}g_{\mu\lambda} - R_{\nu\lambda}g_{\mu\kappa}\right) + \frac{1}{6}R\left(g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa}\right) . \tag{1.12}$$

The conformal theory does not demand the equations of motion to be maximal of second order in derivatives of the metric. Rather it assumes that the fundamental gauge principle has to be preserved [20], i. e. the theory is invariant under certain transformations. The role of the gauge principle in this theory is the principle of local conformal invariance. The only conformally invariant gravitational action is [18]

$$\begin{aligned} \mathcal{S}_w &= -\alpha \int d^4x \sqrt{-g} C_{\mu\nu\sigma\rho} C^{\mu\nu\sigma\rho} = -\alpha \int d^4x \sqrt{-g} \left(R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho} - 2R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2 \right) = \\ &= -2\alpha \int d^4x \sqrt{-g} \left(R_{\mu\nu} R^{\mu\nu} - \frac{R^2}{3} \right) , \end{aligned} \tag{1.13}$$

where α is a dimensionless universal constant and in the second equation, the fact that the term $R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$ can be expressed via Gauss-Bonnet term is used.

The non-vacuum EoM can be easily derived via variational principle [16] (see C.1)

$$4\alpha W_{\mu\nu} = 4\alpha \left[{W^{(2)}}_{\mu\nu} - \frac{1}{3} {W^{(1)}}_{\mu\nu} \right] = 8\alpha B_{\mu\nu} = T_{\mu\nu} , \tag{1.14}$$

where ${W^{(1)}}_{\mu\nu}, {W^{(2)}}_{\mu\nu}$ are respectively variations of

$$\int d^4x \sqrt{-g} R^2 , \qquad \int d^4x \sqrt{-g} R^{\mu\nu} R_{\mu\nu} . \tag{1.15}$$

Tensor $B_{\mu\nu}$ is so-called Bach tensor, which can be expressed in terms of Weyl tensor $C_{\mu\nu\sigma\rho}$ as

$$B_{\mu\nu} = {C_{\mu\kappa\nu\lambda}}^{;\kappa\lambda} - \frac{1}{2} C_{\mu\kappa\nu\lambda} R^{\kappa\lambda} . \tag{1.16}$$

Since the EoM include a second covariant derivative of the Ricci tensor and the Ricci scalar the EoM are partial differential equations of fourth order with respect to metric tensors.

For theory to have conformal symmetry, it has to be massless theory, so Conformal gravity does not suffer from the cosmological constant problem. Another advantageous characteristic of Conformal gravity is that it naturally involves inflation [19]. It also passes the Solar system tests and describes well the large structures of the Universe [20].

1.2.4 Non-local theories

All theories of gravity mentioned above are local theories in nature [5]. Quantum theory on the other hand is inherently non-local theory, therefore a possible approach to merge QFT and GR is to reformulate gravity as a non-local theory as well. The non-locality can be introduced into action by the addition of non-local operators.

There are two types of Non-local theories

1. Infinite derivative gravity (IDG)
2. Integral kernel gravity (IKG)

The most general ghost-free action of IDG is [3]

$$I_{IDG} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[R + \alpha \left(R F_1(\Box_s) R + R_{\mu\nu} F_2(\Box_s) R^{\mu\nu} + R_{\mu\nu\sigma\rho} F_3(\Box_s) R^{\mu\nu\sigma\rho} \right) \right] , \tag{1.17}$$

where α is dimensionful coupling constant, $\Box_s = \frac{\Box}{M_s^2}$, with M_s being a scale of non-locality and F_i being analytical functions.

The EoM of the IDG include of course infinite number of derivatives of the metric. Solving the Cauchy problem in the framework of IDG requires an infinite number of initial conditions. However, there is no physical explanation of these initial conditions.

Integral kernel gravity incorporates in the action term $\Box^{-1}$, which can be expressed by its Green function, there comes the name of these theories. Any term with d'Alembert operator to negative power causes late-time cosmic expansion, hence they might be an alternative to Dark matter and Dark energy. Non-local theories have another advantage over $f(R)$ gravity and scalar theories and that is that they do not need any fine-tuning to correspond to experimental data [13].

1.2.5 $f(R^{\mu}{}_{\nu\sigma\rho})$ gravity

The most general higher derivative extension of GR is $f(R^{\mu}{}_{\nu\sigma\rho})$ theory. There are two subcategories of this type of gravity, one includes only curvature invariants of the Riemann tensor i.e. any contractions and any powers, and the second type includes also arbitrary covariant derivatives of the Riemann tensor.

The most general Lagrangian for diffeomorphism-invariant action is a scalar function of contractions of the Riemann tensor and its covariant derivatives [4]

$$\mathcal{L} = \mathcal{L}(g^{\mu\nu}, R_{\mu\nu\sigma\rho}, R_{\mu\nu\sigma\rho;\alpha}, R_{\mu\nu\sigma\rho;\alpha\beta}, \dots)\,. \tag{1.18}$$

Coupling the first type of $f(R^{\mu}{}_{\nu\sigma\rho})$ theories (without the covariant derivatives) to matter via variational principle the EoM are obtained in the form

$$P_{\mu}{}^{\alpha\beta\sigma}R_{\nu\alpha\beta\sigma} - \frac{1}{2}g_{\mu\nu}\mathcal{L} + 2P_{\mu\alpha\nu\beta}{}^{;\beta\alpha} = \frac{1}{2}T_{\mu\nu}\,, \tag{1.19}$$

where

$$P^{\mu\nu\rho\sigma} = \left[\frac{\partial\mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}\right]_{g^{\alpha\beta}}\,. \tag{1.20}$$

Thanks to the assumption that Lagrangian $\mathcal{L}$ does not involve any covariant derivative Lagrangian is a function of maximally second partial derivatives of the metric. Then the EoM (1.19)include a second covariant derivative of tensor $P_{\mu\alpha\nu\beta}$, therefore the EoM are of fourth order partial differential equations with respect to metric.

The EoM of the theories with additional terms with covariant derivatives of the Riemann tensor will be at least sixth-order partial differential equations for metric.

Quantization of GR produces quadratic corrections so the theory is renormalizable [31]. Therefore the most straightforward generalization of GR is to include said quadratic terms in the Lagrangian. The most general quadratic Lagrangian constructed from curvature invariants is

$$\mathcal{L} = R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}\,. \tag{1.21}$$

An example of the quadratic theory is Stelle gravity with Lagrangian [31]

$$\mathcal{L}_{Stelle} = R^{\mu\nu}R_{\mu\nu} + R^2\,. \tag{1.22}$$

In ref. [2] is formulated the most general theory of gravity involving cubic curvature invariants, which authors called Einstein Cubic gravity (ECG). In four dimensions it is described by Lagrangian

$$\mathcal{L}_{ECG} = R + \alpha G + \lambda\mathcal{P}\,, \tag{1.23}$$

where α and λ are dimensionless constants, G is Gauss-Bonnet term and $\mathcal{P}$ is

$$\mathcal{P} = 12R_{\mu\ \nu}^{\ \rho\ \sigma}R_{\rho\ \sigma}^{\ \gamma\ \delta}R_{\gamma\ \delta}^{\ \mu\ \nu} + R^{\rho\sigma}{}_{\mu\nu}R^{\gamma\delta}{}_{\rho\sigma}R^{\mu\nu}{}_{\gamma\delta} - 12R_{\mu\nu\rho\sigma}R^{\mu\nu}R^{\nu\sigma} + 8R^{\nu}{}_{\mu}R^{\rho}{}_{\nu}R^{\mu}{}_{\rho}\,. \tag{1.24}$$

The Gauss-Bonnet term does not contribute to the EoM, so the additional terms in EoM come from the variation of $\mathcal{P}$, which is a curvature invariant, therefore the EoM are of fourth order.

An interesting feature of ECG is, that it possesses dimensionless coupling constants, which is a necessary feature for theory to be renormalizable, and it shares its spectrum with GR [2].

The conformal theory discussed in 1.2.3 also belongs in the category of $f(R^{\mu}{}_{\nu\sigma\rho})$ theories, but since its main feature is the conformal invariance it is discussed separately.

Chapter 2

Schwarzschild type solution for HD theories

In this work as a Schwarzschild-type solution is considered any solution of EoM for spherically symmetric metric. For classical GR the spherically symmetric solution is

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega\,. \tag{2.1}$$

The only physical singularity appears at $r = 0$ and the event horizon is at $r = 2M$. The metric is static above the event horizon, but the roles of time t and areal radius r switch below the event horizon. Meaning that for r smaller than $2M$ it is impossible to be a static observer.

Spherical solutions can be classified using the Ricci scalar R into 4 categories [7]

1. solutions with $R = 0$
2. solutions with $R = constant$
3. solutions with $R = R(r)$
4. solutions with $R = R(r, t)$

2.1 $f(R)$ gravity

In general, for any analytical function $f(R)$ there exists a solution of second and third type [7].

Following the procedure in ref. [7] the derivation of Schwarzschild-like solution for $f(R)$ starts with the most general spherically symmetric metric

$$ds^2 = -A(r,t)dt^2 + B(r,t)dr^2 + r^2 d\Omega\,, \tag{2.2}$$

where A, B are arbitrary functions of r, t. Ricci scalar in this metric takes the form

$$R = \frac{1}{2r^2A^2B^2}\Big[r^2B\left(-\dot{A}\dot{B} + A'^2\right) + 4A^2\left(-B + B^2 + rB'\right) \\ +rA\left(-r\dot{B}^2 + rA'B' + 2B\left(r\ddot{B} - 2A' - rA''\right)\right)\Big]\,, \tag{2.3}$$

where $\dot{A}, \dot{B}$ are partial derivatives with respect to t and A', B' are partial derivatives with respect to r.

Assuming the metric (2.2) is static, i.e. $A(r,t) = A(r)$ and $B(r,t) = B(r)$, the expression for Ricci scalar simplifies to

$$R = \frac{1}{2r^2A^2B^2}\left[r^2BA'^2 + 4A^2\left(-B + B^2 + rB'\right) + rA\left(rA'B' - 2B\left(2A' + rA''\right)\right)\right] . \tag{2.4}$$

The EoM (1.3) when taken $R = R(r,t)$ can be rewritten as

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + \mathcal{H}_{\mu\nu} = \kappa T_{\mu\nu} , \tag{2.5}$$

where

$$\mathcal{H}_{\mu\nu} = f'''(R)\left(g_{\mu\nu}g^{\alpha\beta}R_{,\alpha}R_{,\beta} - R_{,\mu\nu}\right) + f''(R)\left(g_{\mu\nu}g^{\alpha\beta}\left(R_{,\alpha\beta} - \Gamma^{\sigma}{}_{\alpha\beta}R_{,\sigma}\right) - R_{,\mu\nu} + \Gamma^{\sigma}{}_{\mu\nu}R_{,\sigma}\right) . \tag{2.6}$$

It is clear from this form of EoM (2.5) that derivatives of $f(R)$ are distinct from the spatial and time derivatives of R.

2.1.1 Solutions with $R = const$

In ref. [7] is in detail discussed the derivation of Schwarzschild type solution for $R = const$. The method described therein is the following. Assuming that $R = R_0$ is constant the $\mathcal{H}_{\mu\nu}$ becomes zero and the EoM (2.5) reduce to

$$R_{\mu\nu} + \lambda g_{\mu\nu} = q\kappa T_{\mu\nu} , \tag{2.7}$$

where $\lambda = \frac{-f(R_0)}{2f'(R_0)}$ and $q^{-1} = f'(R_0)$.

Vacuum solution can be obtained when $T_{\mu\nu} = 0$. Only nontrivial EoM of form (2.7) when spherically symmetric metric (2.2) is substituted are components $(t,t), (r,r), (t,r) = (r,t), (\theta,\theta)$ and (ϕ,ϕ) (their full form can be found in A.2.1). The off-diagonal component determines that the $B(r,t) = B(r)$ and from component (ϕ,ϕ) follows that $A(r,t)$ is separable. The separability of $A(r,t)$ allows for a coordinate transformation, which eliminates the time dependence of $A(r,t)$. Hence the spherically symmetric solution for case $R = R_0$ in $f(R)$ gravity will be static. The EoM (2.7) with static metric can be easily solved and the most general solution to the spherical problem with $R = R_0$ in a vacuum is then

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2d\Omega , \tag{2.8}$$

where

$$B(r) = \frac{1}{1 + \frac{C}{r} + \frac{\lambda}{3}r^2} , \qquad A(r) = \frac{9}{B(R)} , \tag{2.9}$$

where C is a constant of integration. The structure of the solution for $R = const$ is very similar to the Schwarzschild solution in the framework of GR, the only difference is the quadratic correction $\frac{\lambda}{3}r^2$.

The metric is not defined for $r = 0$ and for

$$r = \frac{-2^{4/3}\lambda + \left(-6C\lambda^2 + \sqrt{\lambda^3(4 + 9C^2\lambda)}\right)^{2/3}}{2\lambda\left(-3C\lambda^2 + \sqrt{\lambda^3(4 + 9C^2\lambda)}\right)^{1/3}} . \tag{2.10}$$

To determine which of the values r are true singularities the curvature invariants are computed. The Kretschmann invariant in metric (2.8) has denominator equal to $36r^6$. Therefore the true singularity is $r = 0$. The second singular point of the metric might be an analogue of the event horizon.

2.1.2 Solutions with $R = R(r)$

In ref. [7] is also discussed the case of $R = R(r)$. For $R = R(r)$ the term $\mathcal{H}_{\mu\nu}$ simplifies to

$$\mathcal{H}_{\mu\nu} = f'''(R)\left(g_{\mu\nu}g^{rr}R_{,r}^2\right) + f''(R)\left(g_{\mu\nu}\left(g^{rr}R_{,rr} - g^{rr}\Gamma^r{}_{rr}R_{,r}\right) - R_{,\mu\nu} + \Gamma^r{}_{\mu\nu}R_{,r}\right) . \tag{2.11}$$

When spherical metric (2.2) is substituted in EoM (2.5) with $\mathcal{H}_{\mu\nu}$ of form (2.11) the off diagonal component (t, r) takes form

$$\frac{d}{dr}\left(r^2 f'(R)\right)\dot{B}(t,r) = 0 . \tag{2.12}$$

There are two solutions to this equation. First assuming that $\dot{B}(t,r) \neq 0$ implies that $f'(R) \sim \frac{1}{r^2}$, but in this case, the remaining field equations are not satisfied therefore the only other possible solution is when $\dot{B}(t,r) = 0$ and so $B(r,t) = B(r)$. Nontrivial EoM with $B(r,t) = B(r)$ can be found in A.2.2.

It was recovered just like in the previous case that $A(r,t)$ has separable structure of form $A(r,t) = a(t)B(r)\exp(...)$, and through coordinate transformation, it is possible to eliminate factor $a(t)$. Therefore the spherical symmetric solution in $f(R)$ gravity with $R = R(r)$ is static.

Trying to solve the EoM for static metric using the same approach as in the $R = const$ case is too complicated. Therefore in ref. [7, 6] is introduced a method which utilizes the Noether symmetry of the system. Instead of solving EoM with infinite degrees of freedom, the Lagrangian is modified to a point-like Lagrangian with a suitable Lagrange multiplier for Ricci scalar. The method from ref. [7, 6] has the following steps.

First the point-like Lagrangian for $f(R)$ gravity is formulated (in detail here A.2.3). The point-like Lagrangian takes the form

$$\begin{aligned} L = \sqrt{-g}\mathcal{L} = &-\frac{\sqrt{A}\partial_R f}{2M\sqrt{B}}M'^2 - \frac{\partial_R f}{\sqrt{AB}}A'M' - \frac{M\partial_{RR}f}{\sqrt{AB}}A'R' \\ &- \frac{2\sqrt{A}\partial_{RR}f}{\sqrt{B}}R'M' - \sqrt{AB}\left((2+MR)\,\partial_R f - Mf\right) , \end{aligned} \tag{2.13}$$

which can be rewritten in the matrix form when a general coordinate $q = (A, B, M, R)$ is defined

$$L = q'^T\left(\frac{\partial^2 L}{\partial q_i'\partial q_j'}\right)q' + V(q) = q'^T T q' + V(q) , \tag{2.14}$$

where T is kinetic term and $V(q)$ is potential. The Lagrangian (2.13) depends only on the first derivatives of the variables R, A, B, M, therefore the EoM are the Euler-Lagrange equations, which take form

$$\frac{d}{dr}\left(\nabla_{q'}L\right) - \nabla_q L = 2Tq'' + 2\left(q'\cdot\nabla_q T\right)q' - \nabla_q V - q'^T\left(\nabla_q T\right)q' = 0 , \tag{2.15}$$

where ∇_q is four divergence with respect to q. Lagrangian (2.13) does not depend on B', hence the B does not change the dynamics of the system, but equations of motion for B should be considered as a constraint on the system. From equation of motion for B which reads

$$\begin{aligned} 0 = &\frac{d}{dr}\left(\partial_{B'}L\right) - \partial_B L = \\ &-\frac{1}{2B^{3/2}}\left(\frac{\sqrt{A}\partial_R f}{2M}M'^2 + \frac{\partial_R f}{\sqrt{A}}A'M' + \frac{M\partial_{RR}f}{\sqrt{A}}A'R' + 2\sqrt{A}\partial_{RR}fR'M'\right) \\ &+ \frac{1}{2\sqrt{B}}\left(\sqrt{A}\left((2+MR)\,\partial_R f - Mf\right)\right) . \end{aligned} \tag{2.16}$$

it is possible to determine the formula for $B(r)$ that is

$$B(r) = \frac{A\partial_R f M'^2 + 2M\partial_R f A'M' + 2M^2\partial_{RR} f A'R' + 4AMM'\partial_{RR} f R'}{2MA\left((2+MR)\,\partial_R f - MR\right)}\,. \tag{2.17}$$

By substituting the formula for $B(r)$ into the Lagrangian (2.13) the new Lagrangian of three dynamical functions is obtained. This procedure of eliminating the variables in Lagrangian is indeed correct, thanks to the fact that $B(r)$ is non-dynamical. New Lagrangian with only three dynamical variables takes form

$$\begin{aligned} L^* =& q'^T \hat{L} q' = \\ & \frac{(2+MR)\partial_R f - fM}{M}\left(2M^2\partial_{RR} f A'R' + 2MM'\left(\partial_R f A' + 2A\partial_{RR} f R'\right) + A\partial_R f M'^2\right). \end{aligned} \tag{2.18}$$

Since the problem is spherically symmetric, Noether's theorem states that there has to exist a conserved quantity. The conserved quantity can be determined if the Lie derivative of Lagrangian disappears [7], i.e.

$$\mathcal{L}_X L^* = \alpha \cdot \nabla_q L^* + \alpha' \cdot \nabla_{q'} L^* = q'^T \left(\alpha \cdot \nabla_q L + 2\left(\nabla_q \alpha\right)^T L\right) q'\,, \tag{2.19}$$

where X is the Noether vector and α is a vector of coefficients which are to be solved. If α satisfies the condition

$$\alpha_i \frac{\partial \hat{L}_{km}}{\partial q_i} + 2\frac{\partial \alpha_i}{\partial q_k}\hat{L}_{im} = 0 \tag{2.20}$$

the Lie derivative of Lagrangian $L*$ disappears. To continue further, one has to specify the function $f(R)$, because the equation (2.20) depends on it implicitly. Still, once the form of the function $f(R)$ is specified and the vector α is computed, then the conserved current can be calculated. This invariant quantity helps to solve the equations of motion.

In ref. [23] is presented different approach. Authors consider the derivative of $f(R)$ as an independent function, denoted $F(R)$. Then the EoM (1.3) can be rewritten as

$$F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + g_{\mu\nu}\Box F(R) - F(R)_{;\mu\nu} = 0 \tag{2.21}$$

plus a condition on the function $F(R)$

$$F(R) = f'(R)\,. \tag{2.22}$$

Contracting (2.21) leads to the equation

$$F(R)R - 2f(R) + 3\Box F(R) = 0\,, \tag{2.23}$$

thanks to which $f(R)$ can be eliminated from equations (2.21)

$$E_{\mu\nu} \equiv F(R)R_{\mu\nu} - F(R)_{;\mu\nu} - \frac{1}{4}g_{\mu\nu}\left(F(R)R - \Box F(R)\right) = 0\,. \tag{2.24}$$

Since $F(R)$ is independent of $f(R)$ and the problem is spherically symmetric, the function $F(R)$ can be perceived as a function of r, then the unknown functions in equations (2.24) are $F(r), A(r), B(r)$, where $A(r), B(r)$ originate from the metric.

From equation (2.24) it is apparent that quantity

$$Q_\mu = \frac{FR_{\mu\mu} - F_{;\mu\mu}}{g_{\mu\mu}} \tag{2.25}$$

is scalar and it holds $Q_\mu = Q_\nu$ for any indices μ, ν. Combinations of indices $\mu = 0, \nu = 2$ and $\mu = 0, \nu = 2$ lead to following equations

$$2\frac{X'}{X}F + r\frac{X'}{X}F' - 2rF'' = 0\,, \tag{2.26}$$

$$\frac{F'}{F}\left(4rA - 2r^2A\right) + r^2A'\frac{X'}{X} - 2r\frac{X'}{X} - 4X + 4A - 2r^2A'' = 0\,, \tag{2.27}$$

where $X = A(r)B(r)$. From the first equation (2.26) quantity X'/X can be expressed and then substituted into the second equation. Arising formula for X then is

$$X(r) = A\left(1 + r\frac{F'}{F} - \frac{r^2F''}{2 + rF'}\right) + \frac{1}{2}r^2A'\left(\frac{rF''}{2 + rF'} - 1\right) - \frac{1}{2}r^2A''\,. \tag{2.28}$$

One can now compute the EoM (2.24) and express the function $B(r)$ using the formula (2.28), as it was done in ref. [23]. One obtains a set of equations for $A(r), F(r)$ with a common factor. So any pair of functions $A(r), F(r)$ for which the factor is zero satisfy the EoM (2.24). Then the $B(r)$ is computed via formula (2.28) and $f(R)$ is recovered from $F(r)$. However, this strategy is not suitable for computing the metric from $f(R)$.

In ref. [24] is suggested yet another approach, instead of using the quantity Q_μ the authors compute the EoM directly, i.e. they substitute spherically symmetric metric (2.2) and an expression for R (2.4). Subsequently, they eliminate one of the unknown functions $A(r)$, $B(r)$ through the introduction of a new function $N(r) = A(r)B(r)$ obtaining two independent EoM for three unknown functions $B(r)$, $N(r)$ and $F(r)$ (derivation can be found in (A.2.4)). Therefore it is not possible to solve this problem without any prior assumption about any function $A(r), N(r), F(r)$. However, this approach might be interesting, because the function $N(r)$ provides information about the relation of (t,t) component and (r,r) component of the metric tensor. So, for example, for solutions similar to the Schwarzschild solution in GR, the function $N(r) = 1$ and subsequently the equations can be solved.

Since in the section (2.1.1) has been shown that for a spherically symmetric solution will hold that $A(r)B(r) = 1$ (the factor 9 can be eliminated by rescaling the t coordinate). Then we might hypothesise, that this might hold even for the case $R = R(r)$.

For the spherically symmetric solution, which has structure $A(r) = \frac{1}{B(r)}$ only functions $F(r)$ of form $F(r) = kr + k_0$ are possible. For choice k_0 the function $B(r)$ becomes

$$B(r) = \frac{1}{1 - \frac{C_1}{r} + C_2r^2} \tag{2.29}$$

and for choice $k = 0$ function $B(r)$ obtains form

$$B(r) = \frac{1}{\frac{1}{2} - \frac{D_1}{r^2} + D_2r^2}\,. \tag{2.30}$$

One might also take inspiration in the derivation of a Schwarzschild-like solution for Conformal gravity (see section 2.2, where is used a theorem (2.34) for computation of diagonal components of tensor $E_{\mu\nu}$ from Lagrangian. However, the arising equations are much more complicated than any equations in previous methods. Still, there is a possibility that for some particular theories, this method could work.

2.1.3 Solutions with $R = R(r, t)$

Solutions with condition $R = R(r,t)$ can not be solved in all generality. The EoM also have the structure of a diagonal matrix with one off-diagonal element, but this element does not provide any constraints on the functions $A(r,t), B(r,t)$, which would allow for solving the problem.

2.2 Conformal theory

The exact exterior vacuum Schwarzschild-type solution for conformal gravity was derived in [20]. The EoM of Conformal gravity in a vacuum are

$$W^{\mu\nu} = 0\,, \tag{2.31}$$

where $W^{\mu\nu}$ is defined most precisely in (C.7). First authors consider spherically symmetric static metric of form

$$ds^2 = -b(\rho)dt^2 + a(\rho)d\rho^2 + \rho^2 d\Omega. \tag{2.32}$$

It is sufficient to consider only static metric since in ref. [28] has been proven that any spherically symmetric solution in Weyl gravity has to be static. Via certain transformations and thanks to conformal symmetry the general metric (2.32) is equivalent to metric (details can be found in (C.2))

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega\,. \tag{2.33}$$

The EoM (2.31) with metric (2.33) are still too complicated to solve directly. Because the metric (2.33) is static, there holds the following theorem.

Theorem. *For any action of form $I = \int d^4x \sqrt{-g}L$ with Lagrangian density L and static line element $ds^2 = -B(r)dt^2 + A(r)dr^2 + C(r)d\theta^2 + D(r,\theta)d\phi^2$ it is possible to calculate*

$$\frac{\delta I}{\delta A} = \partial_A(\sqrt{-g}L) - \partial_r\left(\sqrt{-g}\partial_{A'}L\right) + \partial_{rr}\left(\sqrt{-g}\partial_{A''}L\right)\,. \tag{2.34}$$

And other components can be computed analogically.

Specifically for W^{rr}

$$-\sqrt{-g}B^{-2}W^{rr} = \frac{\delta S_w}{\delta B} = \partial_B(\sqrt{-g}\mathcal{L}) - \partial_r\left(\sqrt{-g}\partial_{B'}\mathcal{L}\right) + \partial_{rr}\left(\sqrt{-g}\partial_{B''}\mathcal{L}\right)\,. \tag{2.35}$$

Authors then argue that component W^{rr} is the only independent component, the other non-zero components can be obtained using Bianchi identity and trace equation. The equation for W^{rr} derived using (2.35) can be transformed via certain substitutions into a differential equation, which can be solved analytically (for details see (C.2)).

The spherically symmetric metric takes the form

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega\,, \tag{2.36}$$

$$B(r) = 1 - \frac{\beta(2-3\beta\gamma)}{r} - 3\beta\gamma + \gamma r - kr^2\,, \tag{2.37}$$

where β, γ, k are constants.

Another possible approach to derive an exterior and also an interior solution for the spherically symmetric problem in a vacuum is to rewrite the equations for $B(r)$ into an easier one. This method was introduced in Ref. [18] and is following by combining the expressions derived for components W^{rr} and W^{00} using fact (2.34) the EoM can be rewritten as

$$B(r)_{;\alpha}{}^{;\alpha}{}_{;\beta}{}^{;\beta} = \frac{3(T^0{}_0 - T^r{}_r)}{4\alpha B(r)} \equiv f(r)\,, \tag{2.38}$$

where $f(r)$ is function describing the source. Integration of (2.38) leads to a new exact exterior and interior solution in the form

$$B(r > R) = w - \frac{1}{6}\int_0^R ds f(s)\left[3s^2 r + \frac{s^4}{r}\right] - kr^2\,, \tag{2.39}$$

$$B(r < R) = w - \frac{1}{6}\int_r^R ds f(s)\left[3s^3 + r^2 s\right] - \frac{1}{6}\int_0^r ds f(s)\left[3s^2 r + \frac{s^4}{r}\right] - kr^2\,. \tag{2.40}$$

The two exterior solutions are indeed equivalent if parameters in (2.36) are defined as

$$\beta(2 - 3\beta\gamma) = \frac{1}{6}\int_0^R ds f(s) s^4\,, \tag{2.41}$$

$$\gamma = -\frac{1}{2}\int_0^R ds f(s) s^2\,. \tag{2.42}$$

From the structure of the exterior solution (2.37) ensues that the Newton-like term $\frac{1}{r}$ dominates the dynamics on the small distances, terms γr and kr^2 dominate at the larger distances [20].

Equation (2.38) is the fourth-Poisson equation rather than the second-Poisson equation as it is in GR. Still, the Newton-like contribution is recovered prompting the possible needlessness of the Poisson equation as a condition to achieve $\frac{1}{r}$ term [22]. The fourth-order Poisson equation also determines that the sources in Conformal gravity have to be extended since the four-momentum of a delta function would disappear, which is unlike in classical mechanics, bit more in alignment with the reality [18].

The exterior solution depends on three constants of integration, two of which can be linked with the interior structure of the source. In ref. [20] it is argued that when $\gamma = 0$, the solution is similar to the Schwarzschild solution in the deSitter background with scalar curvature $R = -12k$. This solution in GR would only be possible if the cosmological constant is nonzero. Hence conformal gravity does not need cosmological constant at all [20].

Thanks to the linear term, this metric well describes the galactic rotation curves with higher precision than GR [20].

Schwarzschild solution in GR has only one true singularity for $r = 0$. Singularities of metric (2.36) can be determined from the form of the metric or they can be calculated from the curvature invariants $R^{\mu\nu\sigma\rho}R_{\mu\nu\sigma\rho}$. From the form of the metric (2.36) can be established that possible singularities can occur for $r = 0$ and for r that satisfies $B(r) = 0$. Calculating the curvature invariant $R^{\mu\nu\sigma\rho}R_{\mu\nu\sigma\rho}$ its denominator is

$$2r^8(-r + r^3 + 2\beta - r^2\gamma + 3r\beta\gamma - 3\beta^2\gamma)^4(-r + 2r^3 + 2\beta - r^2\gamma + 3r\beta\gamma - 3\beta^2\gamma)^2\,. \tag{2.43}$$

The true singularity does occur for $r = 0$ and there are three other values of r for which the curvature invariants are not well defined. These values of r are determined by equations

$$B(r) = 1 - \frac{\beta(2 - 3\beta\gamma)}{r} - 3\beta\gamma + \gamma r - kr^2 = 0\,, \tag{2.44}$$

$$-r + r^3 + 2\beta - r^2\gamma + 3r\beta\gamma - 3\beta^2\gamma = 0\,, \tag{2.45}$$

$$-r + 2r^3 + 2\beta - r^2\gamma + 3r\beta\gamma - 3\beta^2\gamma = 0\,. \tag{2.46}$$

Chapter 3

Kerr type solutions for HD theories

To obtain the Kerr solution even in GR is a fairly complicated task. Kerr-type solution is a solution to an axially symmetric problem for which the angular momentum is conserved, in other words, it is a case of a spinning black hole. In classical GR Kerr metric in Boyer-Lindquist coordinates tales form [10]

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{-2Mar\sin^2\theta}{\rho^2}\,(dtd\phi + d\phi dt) \\ &+ \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left(\left(r^2 + a^2\right)^2 - a^2\Delta\sin^2\theta\right)d\phi^2\,, \end{aligned} \tag{3.1}$$

where

$$\Delta(r) = r^2 - 2Mr + a^2, \qquad \rho^2(r,\theta) = r^2 + a^2\cos^2\theta\,. \tag{3.2}$$

Kerr solution has two parameters M and a respectively mass and angular momentum per unit of mass. Kerr's solution is particularly interesting since unusual behaviour occurs in the vicinity of horizons. Unlike Schwarzschild black hole Kerr's rotating black hole has two event horizons and an ergosphere. An ergosphere is a region of spacetime, where the particle has to move in the direction of rotation, i.e. in the direction of ϕ, but the particle still can travel towards or away from the event horizon [10].

3.1 $f(R)$ gravity

To derive a solution to an axially symmetric problem in $f(R)$ gravity one may follow several different paths. One may generalise Carter's procedure for the Kerr solution in GR. This method was successfully applied to Conformal gravity (see (3.2)), however when applied to $f(R)$ gravity the arising equations are quite complicated and one might prefer to find another way to solve the problem.

Probably the most general approach is to generalise the Chandrasekhar method, its generalization for $f(R)$ has been done in [17]. This method uses formalism of differential forms, therefore it will not be discussed further in this work.

In GR there exists a procedure called the Newman-Janis algorithm, which through complex coordinate transformation generalises the spherically symmetric solution to an axially symmetric one (in A.3 is summarized the algorithm for GR). This procedure has been generalised to $f(R)$ gravity in article [6]. This method only uses the spherically symmetric metric and only transforms it into different coordinates, therefore it does not have to correspond to the EoM of the axial symmetric problem. Hence any axially symmetric metric obtained by this algorithm has to be checked if it does satisfy the EoM.

3.2 Conformal theory

In ref. [21] is derived exact solution to Kerr type problem for conformal gravity. Authors use a similar method as Carter in [11] to derive the Kerr solution in GR. They start with the general Carter's metric

$$ds^2 = (bf - ce)\left[\frac{dx^2}{a} + \frac{dy^2}{d}\right] + \frac{1}{bf - ce}\left[d(bd\phi - cdt)^2 - a(ed\phi - fdt)^2\right], \tag{3.3}$$

where a, b, c are functions of x and d, e, f are functions of y. Subsequently, they extend the crucial characteristic of the metric - separability of the Klein-Gordon equation - into the framework of Conformal gravity. For Conformal gravity, it is however the separability of the vacuum Klein-Gordon equation which has to be preserved instead of the Klein-Gordon equation coupled to matter (Conformal theory is massless). Requiring the separability leads to similar kinematical conditions.

$$b = j^2 + x^2, \qquad e = j(1 - y^2), \qquad c = j, \qquad f = 1\,, \tag{3.4}$$

where j is angular momentum. Imposing the separability conditions simplifies the metric (3.3) so that it is determined only by two unknown functions $a(x), d(y)$. The EoM of the problem are of the form (1.14) with $T_{\mu\nu} = 0$. Thanks to Bianchi and trace identities

$$W^{\mu\nu}{}_{;\nu} = 0\,, \qquad W^{\mu}{}_{\mu} = 0\,, \tag{3.5}$$

which follow directly from the EoM, are only three components of $W^{\mu\nu}$ independent – W^{11}, W^{22}, W^{12}. Since the metric is static, to derive the EoM one can use the same method as in the derivation of EoM for Schwarzschild type solution for Conformal gravity, i.e. using theorem (2.34). However this time the resulting equations are too complicated to be solved in all generality.

Yet assuming that the unknown functions are polynomials of fourth order suffices to satisfy the EoM. Substituting (3.8) into the expressions for components $W^1{}_1, W^2{}_2$ following formula is obtained

$$W^1{}_1 = -W^2{}_2 = \frac{uv - rs}{(x^2 + j^2y^2)^2}\,, \tag{3.6}$$

which leads to the solution

$$ds^2 = (x^2 + y^2)\left[\frac{dx^2}{a(x)} + \frac{dy^2}{d(y)}\right] + \frac{1}{x^2 + y^2}\left[d(y)[(j^2 + x^2)d\phi - jdt]^2 - a(x)[j(1 - y^2)d\phi - dt]^2\right], \tag{3.7}$$

where the functions $a(x)$ and $d(y)$ are

$$\begin{aligned} a(x) &= j^2 + ux + px^2 + vx^3 - kx^4\,, \\ d(y) &= 1 + ry - py^2 + sy^3 - j^2ky^4\,, \end{aligned} \tag{3.8}$$

with the condition

$$uv - rs = 0. \tag{3.9}$$

This solution is conformal to the Kerr solution in GR [21] and therefore it does encompass the classical dynamics of GR.

Conclusion

The purpose of this work is to get familiar with the problematic of higher derivative theories of gravity as possible modifications of general relativity.

General relativity is so far the best description of gravitational interaction, however, the ongoing advancement of Cosmology, Particle physics and Quantum theory, exposes several problems which GR is not able to satisfactorily clarify. Some of these problems are Cosmological constant problems, the need for Dark matter and Dark Energy and nonrenormalizability of GR.

In a purely metric four-dimensional approach, the most natural way to generalise gravity and perhaps explain some of the problems of GR is to introduce into the Einstein-Hilbert action terms with higher than second order derivatives of the metric tensor.

The most straightforward generalization of GR is $f(R)$ gravity, which can explain the early inflation of the Universe and its late time expansion without the need for Dark Energy, however for good results, it is still necessary to fine-tune the constant of the theory.

HD theories which in certain settings might have EoM of lesser order than fourth, as it is common among HD theories, are $f(R,G)$ theories. These theories incorporate into action the Gauss-Bonnet term G, which is a topological term, meaning its value is determined by the topological structure of spacetime. In four dimensions the Gauss-Bonnet term is identically zero, which allows the EoM to be possibly simpler than of other HD theories.

For theories describing other physical interactions is typical gauge invariance. Insisting that the metric is invariant under conformal transformation gives rise to other modifications of GR, the Conformal theory of gravity. This theory predicts well the galactic rotation curves without the need for Dark matter.

As an attempt to unify the formalism of GR and quantum mechanics the Non-local modification of gravity was devised. The non-locality is introduced through the appearance of infinite derivatives or some non-local operators such as $\Box^{-1}$. With infinite derivatives comes hand in hand with the problem of formulating the initial conditions of EoM, there is still no physical explanation for this phenomenon.

The most general modification of the GR is $f(R_{\mu\nu\sigma\rho})$, which introduces into action any contraction and any covariant derivative of Riemann tensor. Quantum theory does predict the appearance of similar corrections in gravitational action.

In the second chapter were presented some attempts to solve the spherically symmetric problem in the framework of $f(R)$ gravity and Conformal gravity.

For $f(R)$ was recovered that for constant Ricci scalar and Ricci scalar being a function of radial coordinate, the spherical symmetric solution is static regardless of the function $f(R)$. The case of constant Ricci scalar was solved entirely for any function $f(R)$. For the case of Ricci scalar being a function of radial coordinate, there were suggested three feasible procedures to derive a solution in the most generality possible. First, the Noether symmetry approach permits solving the EoM by finding the conserved current, which does exist thanks to the Noether's theorem. Another two methods transfigured the EoM and regarded the derivative of function $f(R)$ as an independent function of the radial component. Then one of the unknown functions of metric was eliminated from the EoM by the introduction of a new function. One method arrived at an equation relating the derivative of the function $f(R)$ and one of the unknown functions of the metric and the second method arrived at equations for the derivative of the $f(R)$, the function of the metric and the newly introduced function relating the two unknown functions of the metric. All three of the above methods are highly dependent on the particular form of the function $f(R)$.

For Conformal gravity, there was reiterated the derivation of the Schwarzschild-like solution derived by Mannheim in [20].

In the last chapter, the possible ways to derive an axially symmetric solution in the framework of $f(R)$ gravity and conformal gravity were presented.

For $f(R)$ gravity, there are many approaches, the Generalization of Carter's derivation of Kerr solution, the generalization of the Chandrasekhar method or the generalization of the Newman-Janis algorithm.

For Conformal gravity, the derivation of Mannheim and Kazanas which generalizes Carter's approach to the axial problem was presented.

Higher derivative theories of gravity have the potential to solve some of the problems, which GR still fails to address. However, with more derivatives in the action come new problems, which have to be resolved before the HD theories might be considered as a new theory of gravity.

Any Lagrangian with a higher than the first derivative does suffer from Ostrogradsky instability, which can manifest in many ways, like ghosts, or complete collapse of the system.

More derivatives also mean the need for more initial conditions. The physical meaning of these additional conditions is still unclear. In some particular higher-derivative theories it is feasible to interpret the additional initial conditions as new healthy degrees of freedom. But for an infinite number of derivatives, nothing alike has not been proven yet.

Further research in this field might be interested in determining the stability of certain types of HD theories, interpretation of the initial conditions or to quantize some interesting HD models.

It might be interesting to thoroughly examine the singularities and horizons in the framework of HD theories, to see if there is any interesting behaviour.

Appendix A

$f\,(R)$ gravity

A.1 EoM

In this section is presented a detailed derivation of equations of motion for $f\,(R)$ gravity.

Standard approach is to vary the action $\mathcal{S}_{f(R)}$ defined in (1.2) with respect to $g^{\mu\nu}$. The variation can be then written as

$$\delta\mathcal{S}_{f(R)} = \int d^4x \left(-\frac{1}{2\sqrt{-g}} f(R)\delta g + \sqrt{-g}\partial_R f(R)\delta R \right) + \mathcal{S}^{(m)} \,. \tag{A.1}$$

In [10] is proved that variation of the determinant of metric tensor g with respect to $g^{\mu\nu}$ is

$$\delta g = -g g_{\mu\nu} \delta g^{\mu\nu} \,. \tag{A.2}$$

Variation of Ricci scalar is

$$\delta R = \delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu} \,. \tag{A.3}$$

The second element can be rewritten as

$$g^{\mu\nu}\delta R_{\mu\nu} = \left(g_{\mu\beta} \left(\delta g^{\mu\beta}\right)^{;\lambda} - \left(\delta g^{\mu\lambda}\right)_{;\mu} \right)_{;\lambda} \,, \tag{A.4}$$

where is used the relation $g_{\mu\nu;\lambda} = 0$ and the variation of Riemann tensor

$$\delta R^{\alpha}{}_{\beta\gamma\omega} \overset{O(\delta^2\Gamma)}{=} \left(\delta\Gamma^{\alpha}{}_{\omega\beta}\right)_{;\gamma} - \left(\delta\Gamma^{\alpha}{}_{\gamma\beta}\right)_{;\omega} \,. \tag{A.5}$$

Since the variation of affine connection is

$$\delta\Gamma^{\alpha}{}_{\beta\gamma} = -\frac{1}{2}[g_{\mu\beta}\left(\delta g^{\mu\alpha}\right)_{;\gamma} + g_{\mu\gamma}\left(\delta g^{\mu\alpha}\right)_{;\beta} - g_{\beta\mu}g_{\gamma\nu}\left(\delta g^{\mu\nu}\right)^{\alpha}] \,, \tag{A.6}$$

formula (A.4) follows immediately.

Then it is necessary to reformulate expression $f'\,(R)\, g^{\mu\nu}\delta R_{\mu\nu}$. Exploiting the Leibnitz rule and the fact that 4-divergence of a metric is zero we obtain

$$\begin{aligned} f'\,(R)\, g^{\mu\nu} R_{\mu\nu} = &\left(f'\,(R)\, g_{\mu\beta} g^{\lambda\omega} \delta g^{\mu\nu} - f'\,(R)\,\delta g^{\omega\lambda} \right)_{;\omega\lambda} + 2\left(f'\,(R)_{;\mu}\,\delta g^{\mu\lambda} \right)_{;\lambda} \\ &+ \left(f'\,(R)_{;\lambda}{}^{\lambda}\, g_{\mu\beta} - f'\,(R)_{;\beta\mu} \right)\delta g^{\mu\beta} \,. \end{aligned} \tag{A.7}$$

Then $\delta\mathcal{S}_{f(R)}$ is

$$\delta\mathcal{S}_{f(R)} = \int \sqrt{-g}d^4x\left(-\frac{1}{2}g_{\mu\nu}f(R) + f'(R)R_{\mu\nu} + \left(\Box f'(R)g_{\mu\nu} - f'(R)_{;\nu\mu}\right)\right)\delta g^{\mu\nu} + \tag{A.8}$$

$$+\int d^4x\sqrt{-g}\left(f'(R)g_{\mu\beta}g^{\lambda\omega} - f'(R)\delta g^{\omega\lambda}\right)_{;\omega\lambda} + \tag{A.9}$$

$$+\int d^4x\sqrt{-g}\left(2f'(R)_{;\mu}\delta g^{\mu\lambda}\right)_{;\lambda} + \delta\mathcal{S}^{(m)}, \tag{A.10}$$

the second integral is zero because it is possible to use twice the Stokes theorem and because the boundary of a boundary is zero and the third integral is zero because we assume variations with static ends. The EoM follow from

$$\delta\mathcal{S}_{f(R)} = 0, \tag{A.11}$$

where the variation of $\mathcal{S}^{(m)}$ is

$$\delta\mathcal{S}^{(m)} = -\frac{1}{2}\sqrt{-g}T_{\mu\nu}. \tag{A.12}$$

A.2 Schwarzschild Solution - $f(R)$ gravity

A.2.1 $R = R_0$

The $(t,t), (t,r), (r,r), (\phi,\phi) = (\theta,\theta)$ components of EoM (2.7) with a constraint that $R = R_0$ is constant are

$$-4r\lambda A^2B^2 + rB\dot{A}\dot{B} + rA\dot{B}^2 - 2rAB\ddot{B} + 4ABA' - rBA'^2 - rAA'B' + 2rABA'' = 0, \tag{A.13}$$

$$\frac{\dot{B}}{rB} = 0, \tag{A.14}$$

$$4r\lambda A^2B^2 - rB\dot{A}\dot{B} - rA\dot{B}^2 + 2rAB\ddot{B} + rBA'^2 + 4A^2B' + rAA'B' - 2rABA'' = 0, \tag{A.15}$$

$$-2AB + 2AB^2 + 2r^2\lambda AB^2 - rBA' + rAB' = 0. \tag{A.16}$$

Instantly from equation (A.14) follows $B = B(r)$. Substituting this fact into equations (A.13), (A.15), (A.16) leads to system of equations

$$-4r\lambda A^2B^2 + (4BA - rAB')A' - rBA'^2 + 2rABA'' = 0, \tag{A.17}$$

$$4r\lambda A^2B^2 + (4A^2 + rAA')B' + rBA'^2 - 2rABA'' = 0, \tag{A.18}$$

$$A\left(-2B + 2B^2 + 2\lambda r^2B^2 + rB'\right) - rBA' = 0. \tag{A.19}$$

Last equation (A.19) can be rewritten as

$$\frac{A'}{A} = \frac{B'}{B} - \frac{2}{r} + 2\lambda Br + \frac{2B}{r} = f(r), \tag{A.20}$$

which can be easily integrated

$$A(r,t) = a(t)\exp\left(\int dr f(r)\right) = a(t)B(r)\exp\left(2\int dr\frac{B(r) + \lambda B(r)r^2 - 1}{r}\right) = a(t)B(r)e(r). \tag{A.21}$$

The function $A(r,t)$ is therefore separable and the factor $a(t)$ can be eliminated rescaling the coordinate t. The metric is then static in these rescaled coordinates.

Substituting expression (A.21) into remaining equations (A.17) and (A.18) leads to differential equations of second order for the function $B(r)$.

$$-2B^3 + B^4\left(2 + 4r^2\lambda + 2r^4\lambda^2\right) + B'\left(rB + 3rB^2 + 3r^2\lambda B^2\right) - r^2B'^2 + r^2BB'' = 0\,, \tag{A.22}$$

$$-4B^2 + B^3\left(6 + 4r^2\lambda\right) + B^4\left(-2 - 4r^2\lambda - 2r^4\lambda^2\right) + B'\left(3rB - 3rB^2 - 3r^3\lambda B^2\right) + r^2B'^2 - r^2BB'' = 0\,. \tag{A.23}$$

Expressing r^2BB'' from the first equation

$$r^2BB'' = 2B^3 - B^4\left(2 + 4r^2\lambda + 2r^4\lambda^2\right) - B'\left(rB + 3rB^2 + 3r^2\lambda B^2\right) + r^2B'^2 \tag{A.24}$$

and substituting into the second equation it is possible to reduce the order of the differential equation

$$B' - \frac{B}{r} + B^2\frac{1 + \lambda r^2}{r} = 0\,, \tag{A.25}$$

whose solution is

$$B(r) = \frac{1}{1 + \frac{C}{r} + \frac{\lambda}{3}r^2}\,, \tag{A.26}$$

where C is an integration constant.

Function $A(r)$ then takes form

$$A(r,t) = 3\left(3 + \lambda r^2 + \frac{3C}{r}\right) = \frac{9}{B(r)}\,. \tag{A.27}$$

A.2.2 $R = R(r)$

The EoM (2.21) for choice $B(r,t) = B(r)$ take form

$$f'(R)\left(-\frac{B'A'}{4B^2} + \frac{4AA' - rA'^2 + 2rAA''}{4rBA}\right) + \frac{A}{2B}f(R)R'^2 - \frac{A}{B}f'''(R)R'^2 + f''(R)\left(-\frac{A}{B}\left(R'' + \frac{B'}{2B}R'\right) + \frac{A'R'}{2B}\right) = 0\,, \tag{A.28}$$

$$\frac{1}{4B}f'(R)\left(B'\left(\frac{4}{r} + \frac{A'}{A}\right) + \frac{1}{A^2}\left(BA'^2 - 2ABA''\right)\right) + \frac{B}{2}f'(R) + f'''(R)\left(R'^2 - R''\right) = 0\,, \tag{A.29}$$

$$\frac{1}{2}f'(R)\left(2 + \frac{rB'}{B^2} - \frac{2A + rA'}{BA}\right) - \frac{r^2}{2}f(R) + f'''(R)\left(\frac{r^2}{B}R'^2\right) + f''(R)\left(\frac{r^2}{B}\left(R'' - \frac{B'R'}{2B}\right) - \frac{rR'}{B}\right) = 0\,, \tag{A.30}$$

$$f'(R)\frac{1}{AB^2}\left(-2AB + 2B^2A + rB'A - rBA'\right) - \frac{r^2}{2}f(R) + f'''(R)\frac{r^2R'^2}{B} + f''(R)\left(r^2\frac{1}{B}\left(R'' - \frac{B'R'}{2B}\right) - \frac{rR'}{B}\right) = 0\,. \tag{A.31}$$

From equation (A.30) is again possible to express A'/A as a function of only r and by integrating the formula $A(r,t)$ can be rewritten as

$$A(r,t) = a(t)B(r)\exp\left(\int dr g(r)\right), \tag{A.32}$$

where

$$g(r) = -\frac{2}{r} + \frac{2B}{r} - \frac{Brf(R)}{f'(R)} + 2rR^2 f'''(R) + 2\frac{f''(R)r}{f'(R)}\left(R'' - \frac{B'R}{2B}\right) - \frac{2f''(R)R'}{f'(R)}. \tag{A.33}$$

Transforming the coordinate system so that the factor $a(t)$ in the definition of $A(r,t)$ disappears one obtains a static solution of similar structure as in the previous case for $R = const$.

A.2.3 The Point-like Lagragian

In ref. [9] is devised the point-like Lagrangian in the following manner. First, the static spherically symmetric metric is generalised like so

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + M(r)d\Omega, \tag{A.34}$$

where $A(r), B(r), M(r)$ are functions of areal radius. The action then takes the form

$$\mathcal{S} = \int d^4x L(A, A', B, B', R, R', M, M'). \tag{A.35}$$

Thanks to spherical symmetry the action of the theory has a finite number of degrees of freedom, which are Ricci scalar and functions $A(r), B(r), M(r)$.

The point-like Lagrangian is constructed by action

$$\mathcal{S} = \int d^4x \sqrt{-g}\left(f(R) - \lambda\left(R - \bar{R}\right)\right), \tag{A.36}$$

where $\bar{R}$ is Ricci scalar constructed from the metric (A.34) and λ is Lagrangian multiplier. The $\bar{R}$ takes form

$$\bar{R} = \frac{A''}{AB} + 2\frac{M''}{BM} + \frac{A'M'}{ABM} - \frac{A'^2}{2A^2B} - \frac{M'^2}{2BM^2} - \frac{A'B'}{2AB^2} - \frac{B'M'}{B^2M} - \frac{2}{M} = \tilde{R} + \frac{A''}{AB} + 2\frac{M''}{BM}. \tag{A.37}$$

Substituting the formula for $\bar{R}$ into action (A.36) and instead of determinant of the metric is substituted $-ABM^2$ the action can be rewritten as

$$\mathcal{S} = \int dr\left(\sqrt{AB}M\left[f - \partial_R f\left(R - \bar{R}\right)\right] - \left(\frac{\partial_R f M}{\sqrt{AB}}\right)' A' - 2\left(\frac{\sqrt{A}\partial_R f}{\sqrt{B}}\right)' M'\right) \tag{A.38}$$

where the last two terms arise from the per partes of the second derivative terms in the formula for $\bar{R}$. Explicit manipulation of the expression in the integral leads to the point-like Lagrangian L of a form (2.13).

A.2.4 Nashed-Nojiri approach

In this section is in detail performed the computation of method introduced in [24].

The full set of nontrivial EoM (2.24), when $F(R)$ is taken as function of r, with static spherical metric and R of form (2.4) take form

$$E_{tt} = \frac{1}{8r^2AB^2}\left(-r^2BFA'^2s + rA\left(-rFA'B' + B\left(4FA' + 3rA'F' + 2rFA''\right)\right) + A^2\left(4B^2F + rB'\left(4F + rF'\right) - 2B\left(2F + 2rF' + r^2F''\right)\right)\right) = 0\,, \quad \text{(A.39)}$$

$$E_{rr} = \frac{1}{8r^2A^2B}\left(r^2BFA'^2 + rA\left(rFA'B' + B\left(4fA' + rFA''\right)\right) + A^2\left(-4B^2F + rB'\left(4F + 3rF'\right) + B\left(4F + 4rF' - 6r^2F''\right)\right)\right) = 0\,, \quad \text{(A.40)}$$

$$E_{\theta\theta} = \frac{1}{\sin^2\theta}E_{\phi\phi} = \frac{1}{8A^2B^2}\left(-r^2BFA'^2 + r^2A\left(BA'F' + F\left(-A'B' + 2BA''\right)\right) + A^2\left(4B^2F - r^2B'F' + B\left(-4F - 4rF' + 2r^2F''\right)\right)\right) = 0\,. \quad \text{(A.41)}$$

Equations (A.39), (A.40), (A.41) can be rewritten using function $N(r) = A(r)B(r)$ and omitting the prefactors as

$$-4B^3FN^2 - 4r^2FN^2B'^2 + r^2BN\left(3FB'N' + 2N\left(B'F' + FB''\right)\right) + B^2\left(rN\left(-3rF'N' + 2N\left(2F' + rF''\right)\right) + F\left(4N^2 + r^2N'^2 - 2rN\left(2N' + rN''\right)\right)\right) = 0\,, \quad \text{(A.42)}$$

$$-4B^3FN^2 - 4r^2FN^2B'^2 + r^2BN\left(3FB'N' + 2N\left(B'F' + FB''\right)\right) + B^2\left(rN\left(rF'N' + N\left(4F' - 6rF''\right)\right) + F\left(4N^2 + r^2N'^2 - 2rN\left(2N' + rN''\right)\right)\right) = 0\,, \quad \text{(A.43)}$$

$$4B^3FN^2 + 4r^2FN^2B'^2 - r^2BN\left(3fB'N' + 2N\left(B'F' + FB''\right)\right) + B^2\left(rN\left(-4NF' + rF'N' + 2rNF''\right) - F\left(4N^2 + r^2N'^2 - 2r^2NN''\right)\right) = 0\,. \quad \text{(A.44)}$$

Only two of the equations (A.42)-(A.44) are independent, because subtracting equation (A.2.4) from (A.42) yields same quantity as adding equations (A.42) and (A.44). Therefore only two independent equations for three dynamical variables $F(r), B(r), N(r)$ were obtained. Hence the system can not be solved in all generality, but if one of the functions $F(r), N(r), B(r)$ is fixed beforehand, the system (A.42)-(A.44) can be solved.

Assuming that $N(r) = 1$ the equations (A.42) and (A.2.4) take form

$$-4B^3F - 4r^2F'B'^2 + 2r^2B\left(B'F' + FB''\right) + 2B^2\left(2F + 2rF' + r^2F''\right) = 0\,, \quad \text{(A.45)}$$

$$-4B^3F - 4r^2FB'^2 + 2r^2B\left(B'F' + FB''\right) + B^2\left(4F + 4rF' - 6r^2F''\right) = 0\,. \quad \text{(A.46)}$$

Subtracting equations (A.45) and (A.46) leads to

$$4B^2r^2F'' = 0\,, \tag{A.47}$$

which has two solutions, $B(r) = 0$ or

$$F(r) = kr + k_0\,. \tag{A.48}$$

Substituting $F(r)$ back into equation (A.45) yields

$$-4B^3(kr+k_0) - 4r^2(kr+k_0)B'^2 + r^2B(kr+k_0)2B'' + 4B^2(kr+k_0) = 0\,, \tag{A.49}$$

which has solution for $k_0 = 0$ of form

$$B(r) = \frac{1}{1 - \frac{C_1}{r} + C_2r^2}\,, \tag{A.50}$$

and for $k = 0$

$$B(r) = \frac{1}{\frac{1}{2} - \frac{D_1}{r^2} + D_2r^2}\,, \tag{A.51}$$

where C_1, C_2, D_1, D_2 are constants of integration.

A.3 Newmann-Janis algorithm for GR

In Ref. [6, 9] is summarized the Newman-Janis algorithm in GR and subsequently applied to specific $f(R)$ gravity. The procedure is following. Let us consider static spherically symmetric metric of form

$$ds^2 = -\exp^{2\phi(r)} dt^2 + \exp^{2\lambda(r)} dr^2 + r^2 d\Omega\,. \tag{A.52}$$

The metric (A.52) can be transformed into new coordinates (u, r, θ, ϕ) by defining $dt = du + F(r)dr$, where $F(r) = \pm\exp^{\lambda(r)-\phi(r)}$. The metric in these new coordinates is

$$ds^2 = -\exp^{2\phi(r)} du^2 \mp 2\exp^{\lambda(r)+\phi(r)} dudr - r^2 d\Omega\,. \tag{A.53}$$

Then the metric (A.53) can be rewritten in the matrix form

$$g_{\mu\nu} = \begin{pmatrix} -\exp^{2\phi(r)} & \mp\exp^{\lambda(r)+\phi(r)} & 0 & 0 \\ \mp\exp^{\lambda(r)+\phi(r)} & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \tag{A.54}$$

and the contravariant metric in the matrix form then takes the form

$$g^{\mu\nu} = \begin{pmatrix} 0 & \mp\exp^{-\lambda(r)-\phi(r)} & 0 & 0 \\ \mp\exp^{-\lambda(r)-\phi(r)} & \exp^{-2\lambda(r)} & 0 & 0 \\ 0 & 0 & 1/r^2 & 0 \\ 0 & 0 & 0 & 1/r^2\sin^2\theta \end{pmatrix}. \tag{A.55}$$

This metric can be rewritten in terms of a null tetrad (four basis vectors from which two are related to the t coordinate)

$$g^{\mu\nu} = l^\mu n^\nu + l^\nu n^\mu - m^\mu\bar{m}^\nu - m^\nu\bar{m}^\mu\,, \tag{A.56}$$

where the bar signifies the complex conjugate and the individual vectors l^μ, n^μ, m^ν have to satisfy

$$l_\mu l^\mu = n_\mu n^\mu = m_\mu m^\mu = 0\,, \qquad l_\mu n^\mu = -m_\mu \bar{m}^\mu = -1\,, \qquad l_\mu m^\mu = n_\mu m^\mu = 0\,. \tag{A.57}$$

In Ref. [6] it is stated that at any point of space-time it is possible to choose null tetrad satisfying conditions (A.57) so that vector l^μ is an outward null vector to the light cone, n^μ is the inward null vector pointed to the origin and $m^\mu, \bar{m}^\mu$ are tangent vectors to the two-dimensional sphere of constant radius r and constant u. For the metric (A.55) this null tetrad has been chosen as

$$\begin{aligned}
l^\mu &= \delta^\mu_1 \\
n^\mu &= \frac{1}{2}\exp^{-2\lambda(r)}\,\delta^\mu_1 + \exp^{-\lambda(r)-\phi(r)}\,\delta^\mu_0 \\
m^\mu &= \frac{1}{\sqrt{2}r}\left(\delta^\mu_2 + \frac{i}{\sin\theta}\delta^\mu_3\right) \\
\bar{m}^\mu &= \frac{1}{\sqrt{2}r}\left(\delta^\mu_2 - \frac{i}{\sin\theta}\delta^\mu_3\right)\,.
\end{aligned} \tag{A.58}$$

Now the set of coordinates $x^\mu = (u, r, \theta, \phi)$ has to be extended by a complex radial coordinate in a way

$$\begin{aligned}
l^\mu &= \delta^\mu_1 \\
n^\mu &= \frac{1}{2}\exp^{-2\lambda(r,\bar{r})}\,\delta^\mu_1 + \exp^{-\lambda(r,\bar{r})-\phi(r,\bar{r})}\,\delta^\mu_0 \\
m^\mu &= \frac{1}{\sqrt{2}\bar{r}}\left(\delta^\mu_2 + \frac{i}{\sin\theta}\delta^\mu_3\right) \\
\bar{m}^\mu &= \frac{1}{\sqrt{2}r}\left(\delta^\mu_2 - \frac{i}{\sin\theta}\delta^\mu_3\right)\,.
\end{aligned} \tag{A.59}$$

To obtain a real metric from the metric of the form (A.56) the following transformation is performed. Let coordinates x^μ undergo a transformation

$$x^\mu \to \tilde{x}^\mu = x^\mu + i y^\mu(x^\sigma)\,, \tag{A.60}$$

assuming that y^μ are analytical functions of the real coordinates x^μ and at the same time let the null tetrad vectors $Z^\mu_a = (l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$, where $a = 1, 2, 3, 4$, transform as

$$Z^\mu_a \to Z^\rho_a \frac{\partial \tilde{x}^\mu}{\partial x^\rho}\,. \tag{A.61}$$

This last transformation serves to make the metric (A.56) real again.

For the purposes of the paper [6] the transformation $\tilde{x}^\mu$ has been chosen as

$$\tilde{x}^\mu = x^\mu + ia\left(\delta^\mu_1 - \delta^\mu_0\right)\cos\theta\,, \tag{A.62}$$

the transformed coordinates read

$$\tilde{u} = u + ia\cos\theta \tag{A.63}$$

$$\tilde{r} = r - ia\cos\theta \tag{A.64}$$

$$\tilde{\theta} = \theta \tag{A.65}$$

$$\tilde{\phi} = \phi\,. \tag{A.66}$$

For the choice $\tilde{r} = \bar{\tilde{r}}$ the null tetrad vectors (A.59) becomes

$$
\begin{aligned}
\tilde{l}^{\mu} &= \delta_1^{\mu} \\
\tilde{n}^{\mu} &= \frac{1}{2} \exp^{-2\lambda(\bar{r},\theta)} \delta_1^{\mu} + \exp^{-\lambda(\bar{r},\theta)-\phi(\bar{r},\theta)} \delta_0^{\mu} \\
\tilde{m}^{\mu} &= \frac{1}{\sqrt{2}\,(\bar{r} - ia\cos\theta)} \left[ia\left(\delta_0^{\mu} - \delta_1^{\mu}\right)\sin\theta + \delta_2^{\mu} + \frac{i}{\sin\theta}\delta_3^{\mu} \right] \\
\bar{\tilde{m}}^{\mu} &= \frac{1}{\sqrt{2}\,(\bar{r} + ia\cos\theta)} \left[-ia\left(\delta_0^{\mu} - \delta_1^{\mu}\right)\sin\theta + \delta_2^{\mu} - \frac{i}{\sin\theta}\delta_3^{\mu} \right] .
\end{aligned}
\tag{A.67}
$$

The new transformed metric is then recovered by using (A.56) and it takes form in transformed coordinates $\tilde{x}^{\mu} = (\tilde{u}, \tilde{r}, \theta, \phi)$

$$
\tilde{g}^{\mu\nu} = \begin{pmatrix}
-\frac{a^2 \sin\theta}{\Sigma} & \exp^{-\lambda(\bar{r},\theta)-\phi(\bar{r},\theta)} + \frac{a^2 \sin^2\theta}{\Sigma} & 0 & \frac{a}{\Sigma} \\
\cdot & \exp^{-2\lambda(\bar{r},\theta)} - \frac{a^2 \sin^2\theta}{\Sigma} & 0 & \frac{a}{\Sigma} \\
\cdot & \cdot & \frac{1}{\Sigma} & 0 \\
\cdot & \cdot & \cdot & \frac{1}{\Sigma\sin^2\theta}
\end{pmatrix} ,
\tag{A.68}
$$

where $\Sigma = \tilde{r}^2 + a^2\cos^2\theta$ and the dots in the matrix symbolize the symmetric component.

Appendix B

$f(R,G)$ gravity

EoM of $f(R,G)$ gravity can be easily derived similarly as in (A). We will vary the action (1.9) with respect to $g^{\mu\nu}$

$$\delta\mathcal{S}_G = \delta\mathcal{S}^{(G)} + \delta\mathcal{S}^{(m)} = \frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\left(-\frac{1}{2}fg_{\mu\nu} + \partial_R f\delta R\right)\delta g^{\mu\nu} + \frac{1}{2\kappa}\int d^4\,\sqrt{-g}\partial_G f\delta G + \delta\mathcal{S}^{(m)}\,. \tag{B.1}$$

The first integral is similar to the variation action of $f(R)$ gravity (A.1), hence the only unknown term remaining is term $\partial_G f(R,G)\delta G$. Variation of the Gauss-Bonnet term is

$$\delta G = 2R\delta R - 4\delta R_{\mu\nu}R^{\mu\nu} - 4R_{\mu\nu}\delta R^{\mu\nu} + \delta R^{\mu\nu\sigma\rho}R_{\mu\nu\sigma\rho} + R^{\mu\nu\sigma\rho}\delta R_{\mu\nu\sigma\rho} \tag{B.2}$$

Variations of the Ricci scalar, Ricci tensor and covariant Riemann tensor are

$$\delta R = \delta g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} = \delta g^{\mu\nu}R_{\mu\nu} + \left(g_{\mu\beta}\left(\delta g^{\mu\beta}\right)^{;\lambda} - \left(\delta g^{\lambda\mu}\right)_{;\mu}\right)_{;\lambda}\,, \tag{B.3}$$

$$\delta R_{\mu\nu} = \frac{1}{2}\left(g_{\alpha\lambda}(\delta g^{\alpha\lambda})_{;\mu\nu} - g_{\lambda\nu}(\delta g^{\lambda\alpha})_{;\mu\alpha} - g_{\lambda\mu}(\delta g^{\lambda\alpha})_{;\nu\alpha} + g_{\kappa\nu}g_{\lambda\mu}(\delta g^{\kappa\lambda})^{;\alpha}_{;\alpha}\right), \tag{B.4}$$

$$\begin{aligned}\delta R_{\mu\nu\sigma\rho} =& - g_{\mu\kappa}R_{\lambda\nu\sigma\rho}\delta g^{\kappa\lambda} + \frac{1}{2}(g_{\rho\kappa}g_{\nu\kappa}\delta g^{\kappa\lambda})_{;\mu\sigma} - \frac{1}{2}(g_{\sigma\kappa}g_{\nu\lambda}\delta g^{\lambda\kappa})_{;\mu\rho}\\ &+ \frac{1}{2}g_{\mu\alpha}\left[(-g_{\lambda\rho}\delta g^{\lambda\alpha})_{;\nu\sigma} - (g_{\lambda\nu}\delta g^{\lambda\alpha})_{;\rho\sigma} + (g_{\lambda\sigma}\delta g^{\lambda\alpha})_{;\nu\rho} + (g_{\lambda\nu}\delta g^{\lambda\alpha})_{;\sigma\rho}\right].\end{aligned} \tag{B.5}$$

Using same procedure as in deriving formula (A.7), individual terms of $\partial_G f\delta G$ can be rewritten as

$$\begin{aligned}\partial_G fR\delta R =& \left(\partial_G fRR_{\kappa\lambda} + \Box(\partial_G fR)g_{\kappa\lambda} - (\partial_G fR)_{;\lambda\kappa}\right)\delta g^{\kappa\lambda} + \\ &+ 2\left[(\partial_G fR)_{;\kappa}\delta g^{\kappa\lambda} - (\partial_G fRg_{\kappa\beta})^{;\lambda}\delta g^{\kappa\beta}\right]_{;\lambda} + \Box(\partial_G fRg_{\mu\beta}\delta g^{\mu\beta}) - (\partial_G fR\delta g^{\mu\lambda})_{;\lambda\mu}\,,\end{aligned} \tag{B.6}$$

$$\begin{aligned}\partial_G fR^{\mu\nu}\delta R_{\mu\nu} =& \frac{1}{2}\left[\Box(\partial_G fR_{\kappa\lambda}) + (\partial_G fR^{\mu\nu})_{;\nu\mu}\,g_{\kappa\lambda} - 2\left(\partial_G fR^{\mu}{}_{\lambda}\right)_{;\kappa\mu}\right]\delta g^{\kappa\lambda}\\ &+ \left[\left(\partial_G fR_{\kappa\lambda}\right)^{;\alpha}\delta g^{\kappa\lambda} + \left(\partial_G fR^{\mu}{}_{\lambda}\right)_{;\mu}\delta g^{\lambda\alpha} + \left(\partial_G fR^{\alpha}{}_{\lambda}\right)_{;\mu}\delta g^{\mu\lambda} - (\partial_G fR^{\mu\alpha})_{;\mu}\,g_{\kappa\lambda}\delta g^{\kappa\lambda}\right]_{;\alpha}\\ &+ \frac{1}{2}\left(\partial_G fR^{\mu\nu}g_{\kappa\lambda}\delta g^{\kappa\lambda}\right)_{;\mu\nu} + \frac{1}{2}\Box\left(\partial_G fR_{\kappa\lambda}\delta g^{\kappa\lambda}\right) - \left(\partial_G fR^{\mu}{}_{\lambda}g^{\lambda\alpha}\right)_{;\mu\alpha}\end{aligned} \tag{B.7}$$

$$
\begin{aligned}
\partial_G f R_{\mu\nu}\delta R^{\mu\nu} = &\left[2\partial_G f R_{\kappa\nu} R^{\nu}{}_{\lambda} + \frac{1}{2}\Box\left(\partial_G f R_{\kappa\lambda}\right) + \frac{1}{2}\left(\partial_G f R^{\mu\nu}\right)_{;\nu\mu} g_{\kappa\lambda} - \left(\partial_G f R^{\mu}{}_{\lambda}\right)_{;\kappa\mu}\right]\delta g^{\kappa\lambda} \\
&+ \left[\left(\partial_G f R_{\kappa\lambda}\right)^{;\alpha}\delta g^{\kappa\lambda} + \left(\partial_G f R^{\mu}{}_{\lambda}\right)_{;\mu}\delta g^{\lambda\alpha} + \left(\partial_G f R^{\alpha}{}_{\lambda}\right)_{;\mu}\delta g^{\mu\lambda} - \left(\partial_G f R^{\mu\alpha}\right)_{;\mu} g_{\kappa\lambda}\delta g^{\kappa\lambda}\right]_{;\alpha} \\
&+ \frac{1}{2}\left(\partial_G f R^{\mu\nu} g_{\kappa\lambda}\delta g^{\kappa\lambda}\right)_{;\mu\nu} + \frac{1}{2}\Box\left(\partial_G f R_{\kappa\lambda}\delta g^{\kappa\lambda}\right) - \left(\partial_G f R^{\mu}{}_{\lambda} g^{\lambda\alpha}\right)_{;\mu\alpha}
\end{aligned} \tag{B.8}
$$

$$
\begin{aligned}
\partial_G f R^{\mu\nu\sigma\rho}\delta R_{\mu\nu\sigma\rho} = &\left[-\partial_G f R_{\kappa}{}^{\nu\sigma\rho} R_{\lambda\nu\sigma\rho} + 2\left(\partial_G f R_{\lambda\nu\kappa\rho}\right)^{;\rho\nu} + \left(\partial_G f R_{\lambda\kappa\nu\rho}\right)^{;\rho\nu}\right]\delta g^{\kappa\lambda} \\
&- \left[\left(\partial_G f R_{\alpha\rho\kappa\nu}\right)^{;\nu}\delta g^{\alpha\kappa}\right]^{;\rho} + \left(\partial_G f R_{\lambda\kappa\sigma\rho}\delta g^{\kappa\lambda}\right)_{;\sigma\rho} - \left(\partial_G f R_{\sigma\lambda\kappa\rho}\delta g^{\kappa\lambda}\right)^{;\sigma\rho} ,
\end{aligned} \tag{B.9}
$$

$$
\begin{aligned}
\partial_G f R_{\mu\nu\sigma\rho}\delta R^{\mu\nu\sigma\rho} = &\left[\partial_G f R_{\lambda\mu\nu\rho} R_{\kappa}{}^{\mu\nu\rho} + 2\partial_G f R_{\mu\nu\rho\lambda} R^{\mu\nu\rho}{}_{\kappa} + 2\left(\partial_G f R_{\lambda\nu\kappa\rho}\right)^{;\rho\sigma} + \left(\partial_G f R_{\lambda\kappa\nu\rho}\right)^{;\rho\nu}\right]\delta g^{\kappa\lambda} \\
&- \left[\left(\partial_G f R_{\alpha\rho\kappa\nu}\right)^{;\nu}\delta g^{\alpha\kappa}\right]^{;\rho} + \left(\partial_G f R_{\lambda\kappa\sigma\rho}\delta g^{\kappa\lambda}\right)_{;\sigma\rho} - \left(\partial_G f R_{\sigma\lambda\kappa\rho}\delta g^{\kappa\lambda}\right)^{;\sigma\rho} ,
\end{aligned} \tag{B.10}
$$

Assuming variations with fixed ends four-divergence terms in equations (B.6)-(B.10) do not contribute to the EoM. Terms rewritten as second covariant derivatives also do not contribute to the EoM thanks to the fact that the boundary of a boundary is zero. Omitting four-divergence terms and second covariant derivatives yields

$$
\begin{aligned}
\frac{\partial_G f \delta G}{\delta g^{\kappa\lambda}} = &\, 2\partial_G f R R_{\kappa\lambda} + 2\Box\left(\partial_G f R\right) g_{\kappa\lambda} - 2\left(\partial_G f R\right)_{;\lambda\kappa} - 4\Box\left(\partial_G f R_{\kappa\lambda}\right) - 4\left(\partial_G f R^{\mu\nu}\right)_{;\nu\mu} g_{\kappa\lambda} \\
&+ 8\left(\partial_G f R^{\mu}{}_{\lambda}\right)_{;\kappa\mu} - 8\partial_G f R_{\kappa\nu} R^{\nu}{}_{\lambda} + 4\left(\partial_G f R_{\lambda\nu\kappa\rho}\right)^{;\rho\nu} + 4\left(\partial_G f R_{\lambda\kappa\nu\rho}\right)^{;\rho\nu} \\
&+ 2\partial_G f R_{\mu\nu\rho\lambda} R^{\mu\nu\rho}{}_{\kappa}
\end{aligned} \tag{B.11}
$$

The EoM then take the form

$$
\begin{aligned}
-\frac{1}{2} g_{\kappa\lambda} f + \partial_R f R_{\kappa\lambda} + \Box\left(\partial_R f\right) g_{\kappa\lambda} - \partial_R f_{;\lambda\kappa} + 2\partial_G f R R_{\kappa\lambda} + 2\Box\left(\partial_G f R\right) g_{\kappa\lambda} \\
- 2\left(\partial_G f R\right)_{;\lambda\kappa} - 4\Box\left(\partial_G f R_{\kappa\lambda}\right) - 4\left(\partial_G f R^{\mu\nu}\right)_{;\nu\mu} g_{\kappa\lambda} + 8\left(\partial_G f R^{\mu}{}_{\lambda}\right)_{;\kappa\mu} \\
- 8\partial_G f R_{\kappa\nu} R^{\nu}{}_{\lambda} + 4\left(\partial_G f R_{\lambda\nu\kappa\rho}\right)^{;\rho\nu} + 4\left(\partial_G f R_{\lambda\kappa\nu\rho}\right)^{;\rho\nu} + 2\partial_G f R_{\mu\nu\rho\lambda} R^{\mu\nu\rho}{}_{\kappa} = \kappa T_{\kappa\lambda}
\end{aligned} \tag{B.12}
$$

Appendix C

Conformal gravity

C.1 EoM

The EoM of Conformal gravity can be derived by varying action (1.13). The arising EoM can be then rewritten as

$$W_{\mu\nu} = W^{(2)}{}_{\mu\nu} - \frac{1}{3}W^{(1)}{}_{\mu\nu}, \tag{C.1}$$

where $W^{(2)}{}_{\mu\nu}$ and $W^{(1)}{}_{\mu\nu}$ are respectively variations of

$$\int d^4x\,\sqrt{-g}R_{\kappa\lambda}R^{\kappa\lambda}\,, \qquad \int d^4x\,\sqrt{-g}R^2\,. \tag{C.2}$$

Variations of these integrals can be readily done using the formulas (B.6), (B.7), (B.8), when $\partial_G f = 1$

$$\begin{aligned}\delta\int d^4x\,\sqrt{-g}R_{\kappa\lambda}R^{\kappa\lambda} = &\int d^4x\,\sqrt{-g}\left(-\frac{1}{2}R^{\kappa\lambda}R_{\kappa\lambda}g_{\mu\nu} + 2R_{\mu\kappa}R^{\kappa\nu} + \Box R_{\mu\nu} + R^{\kappa\lambda}{}_{;\lambda\kappa}g_{\mu\nu} - 2R^{\kappa}{}_{\nu;\mu\kappa}\right)\delta g^{\mu\nu}\\ &+ 2\int d^4x\,\sqrt{-g}\left(R_{\kappa\lambda}{}^{;\alpha}\delta g^{\kappa\lambda} + R^{\mu}{}_{\lambda;\mu}\delta g^{\lambda\alpha} + R^{\alpha}{}_{\lambda;\mu}\delta g^{\mu\lambda} - R^{\mu\alpha}{}_{;\mu}g_{\kappa\lambda}\delta g^{\kappa\lambda}\right)_{;\alpha}\\ &+ \int d^4x\,\sqrt{-g}\left\{\left(R^{\mu\nu}g_{\kappa\lambda}\delta g^{\kappa\lambda}\right)_{;\mu\nu} + \Box\left(R_{\kappa\lambda}\delta g^{\kappa\lambda}\right) - 2\left(R^{\mu}{}_{\lambda}\delta g^{\lambda\alpha}\right)_{;\mu\alpha}\right\}\,,\end{aligned} \tag{C.3}$$

$$\begin{aligned}\delta\int d^4x\,\sqrt{-g}R^2 = &\int d^4x\,\sqrt{-g}\left(-\frac{1}{2}R^2 g_{\mu\nu} + 2RR_{\mu\nu} + 2\Box R g_{\mu\nu} - 2R_{;\nu\mu}\right)\delta g^{\mu\nu}\\ &+ 2\int d^4x\,\sqrt{-g}\left(R_{;\kappa}\delta g^{\kappa\lambda} - \left(Rg_{\kappa\beta}\right)^{;\lambda}\delta g^{\kappa\beta}\right)_{;\lambda}\\ &+ \int d^4x\,\sqrt{-g}\left\{\Box\left(Rg_{\mu\beta}\delta g^{\mu\beta}\right) - \left(R\delta g^{\mu\lambda}\right)_{;\lambda\mu}\right\}\,.\end{aligned} \tag{C.4}$$

The second and third terms are again zero. Therefore the $W^{(2)}{}_{\mu\nu}$ and $W^{(1)}{}_{\mu\nu}$ are

$$W^{(2)}{}_{\mu\nu} = -\frac{1}{2}R^{\kappa\lambda}R_{\kappa\lambda}g_{\mu\nu} + 2R_{\mu\kappa}R^{\kappa\nu} + \Box R_{\mu\nu} + R^{\kappa\lambda}{}_{;\lambda\kappa}g_{\mu\nu} - 2R^{\kappa}{}_{\nu;\mu\kappa}\,, \tag{C.5}$$

$$W^{(1)}{}_{\mu\nu} = -\frac{1}{2}R^2 g_{\mu\nu} + 2RR_{\mu\nu} + 2\Box R g_{\mu\nu} - 2R_{;\nu\mu}\,, \tag{C.6}$$

and $W_{\mu\nu}$ is

$$W_{\mu\nu} = \Box R_{\mu\nu} - \frac{2}{3}\Box R g_{\mu\nu} - 2R^{\kappa}{}_{\nu;\mu\kappa} + \frac{2}{3}R_{;\nu\mu} + R^{\kappa\lambda}{}_{;\lambda\kappa}g_{\mu\nu} + \frac{1}{6}R^2 g_{\mu\nu} - \frac{2}{3}RR_{\mu\nu} + 2R_{\mu\kappa}R^{\kappa}{}_{\nu} - \frac{1}{2}R^{\kappa\lambda}R_{\kappa\lambda}\,. \tag{C.7}$$

Expressing Bach's tensor in terms of Riemann tensor leads to

$$2B_{\mu\nu} = 2R_{\mu\kappa\nu\lambda}{}^{;\kappa\lambda} + R_{\mu}{}^{\lambda}{}_{;\nu\lambda} + R_{\nu}{}^{\lambda}{}_{;\lambda\mu} - \Box R_{\mu\nu} - R_{\kappa\lambda}{}^{;\kappa\lambda} g_{\mu\nu} + \frac{1}{3}\Box R g_{\mu\nu} - \frac{1}{3}R_{;\mu\nu} - R^{\kappa\lambda}R_{\mu\kappa\nu\lambda} - R_{\mu\kappa}R^{\kappa}{}_{\nu}$$
$$+ \frac{2}{3}RR_{\mu\nu} - \frac{1}{2}R_{\kappa\lambda}R^{\kappa\lambda}g_{\mu\nu} - \frac{1}{6}R^2 g_{\mu\nu} \,, \tag{C.8}$$

commuting the covariant derivatives

$$R^{\lambda}{}_{\nu;\lambda\mu} = R^{\lambda}{}_{\nu;\mu\lambda} - R_{\mu\alpha}R^{\alpha}{}_{\nu} + R_{\kappa\nu\lambda\mu}R^{\kappa\lambda} \,, \tag{C.9}$$

and rewriting second covariant derivative of Riemann tensor via Ricci tensor as

$$R_{\mu\kappa\nu\lambda}{}^{;\kappa\lambda} = g_{\mu\nu}R_{\kappa\lambda}{}^{;\kappa\lambda} \,, \tag{C.10}$$

leads to relation for Bach's tensor

$$2B_{\mu\nu} = R_{\kappa\lambda}{}^{;\kappa\lambda} + \frac{2}{3}R_{;\mu\nu} + R^{\lambda}{}_{\nu;\mu\lambda} - 2R_{\mu\kappa}R^{\kappa}{}_{\nu} - \Box(R_{\mu\nu}) + \frac{1}{3}\Box R g_{\mu\nu} + \frac{2}{3}RR_{\mu\nu} - \frac{1}{2}R_{\kappa\lambda}R^{\kappa\lambda}g_{\mu\nu} - \frac{1}{6}R^2 g_{\mu\nu} \,. \tag{C.11}$$

This formula has the same terms as (**??**) but some are with different factors. However, these might be eliminated by adding a suitable four-divergence term, which will be zero under the integral. Hence the EoM can be rewritten using Bach's tensor.

C.2 Schwarzschild type solution

In Ref. [20] taken advantage of the conformal symmetry to transform general spherical metric (2.32) into coordinates such that the metric will have a special form. The static spherically symmetric metric (2.32) can be transformed into a coordinate system

$$\rho = p(r), \qquad B(r) = \frac{r^2 b(r)}{p^2(r)}, \qquad A(r) = \frac{r^2 a(r) p'^2(r)}{p^2(r)} \,. \tag{C.12}$$

The metric (2.32) in coordinates (C.12) takes form

$$ds^2 = \frac{p^2(r)}{r^2}\left(-B(r)dt^2 + A(r)dr^2 + r^2 d\Omega\right) \,. \tag{C.13}$$

Thanks to conformal symmetry the function $p(r)$ can be chosen as it satisfied

$$-\frac{1}{p(r)} = \int \frac{dr}{r^2 (a(r)b(r))^{\frac{1}{2}}} \,. \tag{C.14}$$

This condition (C.14) ensures that $A^{-1}(r) = B(r)$.

Then the metric (C.13) takes form

$$ds^2 = \frac{p^2(r)}{r^2}\left(-B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega\right) \,. \tag{C.15}$$

The prefactor $\frac{p^2(r)}{r^2}$ can be omitted since it is conformal scaling. Therefore the most general static spherical metric (2.32) is conformally equivalent to metric

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega \,. \tag{C.16}$$

Using theorem (2.34) the component W^{rr} and other diagonal components of $W^{\mu\nu}$ are derived. Thanks to the fact that the metric (2.33) is static all elements W^{r0} are identically zero and components $W^{00}, W^{\theta\theta}$ can be calculated from Bianchi and trace identity.

$$W^{\mu\nu}{}_{;\nu} = 0\,, \qquad W^{\mu}{}_{\nu} = 0\,, \tag{C.17}$$

which follow directly from the EoM (2.31). Therefore all information about the system is contained in the component W^{rr}.

The theorem (2.35) applied to the action (1.13) using metric (2.33) leads to equation

$$B^{-1}W^{rr} = \frac{1}{6}B'B''' - \frac{1}{12}B''^2 - \frac{1}{3r}(BB''' - B'B'') - \frac{1}{3r^2}(BB'' + B'^2) + \frac{2}{3r^3}BB' - \frac{B^2}{3r^4} + \frac{1}{3r^4}\,. \tag{C.18}$$

When substituting first $B(r) = r^3 f(r)$ and then $f'(r) = y^2(r)r^{-4}$ the following equation is obtained

$$0 = B^{-1}W^{rr} = \frac{1}{3r^4}(1 + y^3 y'') \tag{C.19}$$

which can be integrated to obtain metric (2.36).

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