Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering
Department of Physics

The Wave Equation with Dirac Damping

Vlnová rovnice s dirakovským tlumením

Bachelor's Degree Project

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Academic year: 2023/2024

ZADÁNÍ BAKALÁŘSKÉ PRÁCE

I. OSOBNÍ A STUDIJNÍ ÚDAJE

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Studijní program: **Matematické inženýrství**

Specializace: **Matematická fyzika**

II. ÚDAJE K BAKALÁŘSKÉ PRÁCI

Název bakalářské práce:

Vlnová rovnice s dirakovským tlumením

Název bakalářské práce anglicky:

The wave equation with Dirac damping

Pokyny pro vypracování:

1. Vlnová rovnice s útlumem: Fyzikální motivace.
2. Matematické uchopení: Neomezený nesamosdružený generátor na Hilbertově prostoru.
3. Základní spektrální vlastnosti regulárních a distributivních útlumů.
4. Bazické vlastnosti dirakovského útlumu na omezeném intervalu.

Seznam doporučené literatury:

A. Bamberger, J. Rauch, and M. Taylor, A model for harmonics on stringed instruments, Arch. Rat. Mech. Anal. 79 (1982), 267–290.

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Datum zadání bakalářské práce: **31.10.2023** Termín odevzdání bakalářské práce: **05.08.2024**

Platnost zadání bakalářské práce: **31.10.2025**

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Vlnová rovnice s dirakovským tlumením

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Acknowledgment:
I would like to express gratitude to my supervisor prof. David Krejčiřík for his expert guidance, limitless patience, and for introducing me to operator theory.

Author's declaration:
I declare that this Bachelor's Degree Project is entirely my own work and I have listed all the used sources in the bibliography.

Prague, April 26, 2024 Mikuláš Kučera

Název práce:
Vlnová rovnice s dirakovským tlumením

Autor: Mikuláš Kučera

Studijní program: Matematické inženýrství

Specializace: Matematická fyzika

Druh práce: Bakalářská práce

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Abstrakt: Spektrální teorie je účinným nástrojem při vyšetřování vlnové rovnice s tlumením. Nejprve představíme základy teorie neomezených operátorů a pojem Rieszovy báze v Hilbertově prostoru. Vlnovou rovnici s dirakovským tlumením přeformulujeme pomocí neomezeného operátoru na konkrétním Hilbertově prostoru. Ukážeme, že jeho čistě diskrétní spektrum je určeno kořeny jisté celé funkce. Tohoto poznatku využijeme k popisu vlastností vlastních čísel a k řešení otázky optimálního tlumení pro jeho jednoduché umístění. Najdeme explicitní vztahy pro vlastní funkce a ukážeme symetrický vztah pro sdružený operátor. Nalezneme stopu reálné části inverzního operátoru a související spektrální charakteristiku. S využitím Livšicova kritéria ukážeme, že pro jistou kritickou hodnotu tlumení zobecněné vlastní vektory netvoří Rieszovu bázi. Nakonec určíme, zda zobecněné vlastní vektory tvoří Rieszovu bázi pro libovolné umístění dirakovského tlumení s jakoukoli reálnou hodnotou síly.

Klíčová slova: dirakovské tlumení, neomezený operátor, Rieszova báze, Sobolevovy prostory, spektrální teorie, vlnová rovnice

Title:
The Wave Equation with Dirac Damping

Author: Mikuláš Kučera

Abstract: Spectral theory is a potent tool in investigating the damped wave equation. First, we introduce the basics of the theory of unbounded operators and the notion of a Riesz basis in a Hilbert space. We reformulate the wave equation with Dirac damping in terms of an unbounded operator in a particular Hilbert space. We show that its purely discrete spectrum is determined by the roots of an entire function. We use this knowledge to describe properties of the eigenvalues and to solve the problem of the optimal damping for its simple placements. We find explicit formulas for the generalised eigenfunctions and show a symmetry result for the adjoint operator. We calculate the trace of the real part of the inverse and a related spectral characteristic. Invoking the Livšic trace criterion, we disprove the Riesz basis property of the root vectors for a certain critical value of the damping parameter. Finally, we determine the Riesz basis property for a general placement of any real-valued Dirac damping.

Key words: Dirac damping, Riesz basis, Sobolev spaces, spectral theory, unbounded operator, wave equation

Contents

Introduction

When modelling the evolution of a string of length L, e.g. to describe playing on stringed instruments, one typically considers the damped wave equation

$$u_{tt} - v^2 u_{xx} + b(x) u_t = 0, \tag{1}$$

where $u : [0, L] \times [0, \infty) \to \mathbb{C}$ is the wave evolution function, v is the speed of wave propagation and $b : [0, L] \to \mathbb{C}$ is the (regular) damping. By the choice of proper coordinates, we can set $v = 1$. To represent a string with fixed ends, we impose the Dirichlet boundary condition on the evolution function:

$$u(0, t) = u(L, t) = 0, \quad t \geq 0. \tag{2}$$

Finally, to provide a solution of (1) with (2), we must specify the initial condition (Cauchy data):

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x). \tag{3}$$

The wave equation, however, has far more applications than just to model strings. Its higher-dimension form is used to describe mechanical waves spreading through a medium. In electromagnetism, the Maxwell equations can be written as wave equations with sources and in vacuum, a solution in the form of a plane wave can be obtained. Additionally, the Dirac equation is a relativistic wave equation that models the propagation of relativistic quantum particles expressed via operators. [4]

We will mainly study spectral properties of a particular wave equation when reformulated in terms of a first-order operator differential equation. To vindicate the spectral approach, we will present a simple example. First, note that setting $\psi := \begin{pmatrix} u \\ u_t \end{pmatrix}$, we can rewrite equation (1) with initial data (3) as

$$A(b)\psi = \psi_t, \quad \psi(x, 0) = \psi_0(x),$$

where

$$A(b) := \begin{pmatrix} 0 & I \\ \partial_{xx} & -b \end{pmatrix}, \quad \psi_0(x) := \begin{pmatrix} u_0(x) \\ v_0(x) \end{pmatrix}.$$

As will be further discussed in the thesis, the natural setting for this problem is the Hilbert space $\mathcal{H} := H_0^1(0, L) \times L^2(0, L)$ and the natural domain of $A(b)$ is

$$\operatorname{dom} A(b) = \left(H_0^1(0, L) \cap L^2(0, L)\right) \times H_0^1(0, L).$$

Note that the Dirichlet boundary conditions are included in the domain. (The Sobolev spaces H^k are discussed in more detail in section 1.3.)

The solution is given by the theory of semigroups (see [10]) as

$$\psi(x, t) = \exp(tA)\psi_0(x),$$

given that the semigroup $\exp(tA)$ exists, which is ensured for regular dampings as well as for the distributional Dirac damping. Its behaviour is strongly dependent on spectral properties of operator $A(b)$, which is our motivation to study them.

Consider now the simple case when $b(x) = b > 0$ is constant. This model corresponds to constant dissipation of energy (and it is simple matter to confirm that operator $A(b)$ is indeed dissipative). An interesting problem is to find the optimal damping b for which the decay of any initial condition will be the fastest. It will clearly not be the case $b = 0$ which corresponds to the undamped wave equation where the amplitudes of all modes are conserved. However, with b large, the initial modes will be preserved over time as the solution tends to constant. In a vast simplification, one can imagine a mathematical pendulum (in the approximation of small oscillations) in some homogeneous viscous medium and ask what is the right resistance of the medium to stabilize the pendulum in the equilibrium position over the shortest period of time.

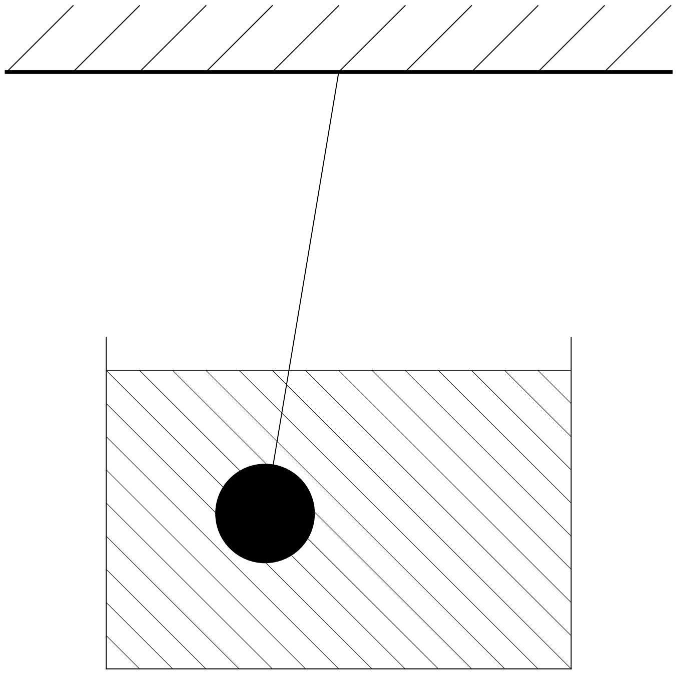


Figure 1: An illustration of a pendulum in a viscous medium.

The beauty of this question lies in the fact that the solution is purely spectral. As was shown in [1], given that the root vectors of A form a Riesz basis in $\mathcal{H}$, the spectral abscissa

$$\mu(A(b)) = \max\left\{\operatorname{Re}\lambda_j \mid \lambda_j \in \sigma_p(A(b))\right\}$$

is the optimal damping constant c such that there exists $A > 0$ satisfying $\left\|\exp(tA)\psi\right\| \leq A e^{tc} \|\psi\|$ for any $\psi \in \mathcal{H}$. In other words, the goal is to find the value of b such that $\mu(A(b))$ is minimal. For simplicity, let us from now on consider $L = \pi$. A calculation provides the solution illustrated by Figure 2.

We can see that the eigenvalues follow circular paths before hitting the real axis, where a double eigenvalue occurs. From that point, one of the collided eigenvalues exits to infinity, while the other returns to the imaginary axis. It can be easily deduced that the optimal damping with $\mu(A(b)) = -1$ is $b = 2$. In Chapter 2, the wave equation with Dirac damping will be observed to exhibit similar behaviour.

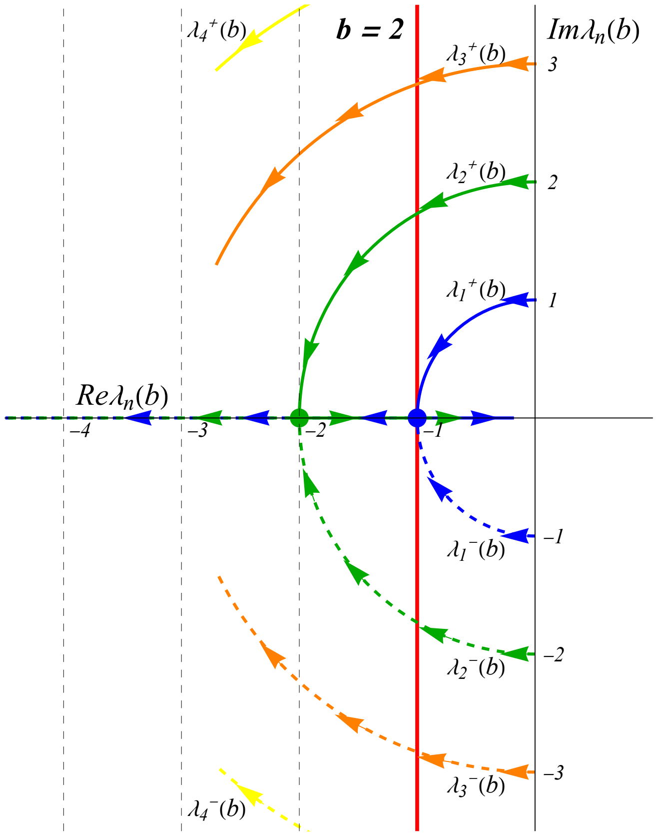


Figure 2: The spectrum of $A(b)$ with constant damping. Arrows indicate the evolution of b.

The Dirac damping is introduced to the wave equation by Bamberger, Rauch, and Taylor in [2] to attack the problem of playing harmonics on a string. Their results are outlined in Chapter 2. The operator in question is

$$A(a,b) = \begin{pmatrix} 0 & I \\ \partial_{xx} & 0 \end{pmatrix}$$

$$\operatorname{dom} A(a,b) = \left\{ \psi \in \left(H_0^1(0,\pi) \cap H^2(0,a) \cap H^2(a,\pi) \right) \times H_0^1(0,\pi) \mid \psi_1'(a+) - \psi_1'(a-) = b\psi_2(a) \right\},$$

with $b > 0$ and $a \in (0,\pi)$. It is shown that for $a = \pi/2$, $b = 2$ appears to be the optimal damping. At the same time, the spectrum shows singular behaviour at $b = 2$.

In [3], it is observed that the eigenvalues of $A(a,b)$ are the non-zero roots of

$$S(\lambda; a, b) := \sinh(\lambda\pi) + b\sinh(\lambda a)\sinh(\lambda(\pi - a)) = 0.$$

This knowledge is then used to further analyse the spectrum and the root vectors of $A(a,b)$. For the special case $a = p\pi/q$ and $b \neq 2$, the root vectors are shown to form a Riesz basis in $\mathcal{H}$. However, analysis of the case $b = 2$ is omitted.

In this thesis, we first build the apparatus needed to attack the problem in Chapter 1. For readers' convenience, we structure the theory from the basics, proving vast majority of the claims along the way. We state results considering bounded and unbounded operators and their spectrum. Great attention is paid to the study of bases in a Hilbert space, especially the Riesz bases. We also introduce the Sobolev spaces that play a pivotal role in our mathematical setting and show their connection to the Dirichlet Laplacian. We particularly investigate the properties of the Sobolev space H^1 on a subset of the real line.

Equipped with the theory, we analyse spectral properties of the operator $A(a, b)$. In addition to the results of [2] and [3], we consider a general $b \in \mathbb{R}$ or even $b \in \mathbb{C}$ where plausible, since the complex damping has applications in relativistic quantum mechanics – see [4]. It is shown that harmonic eigenfunctions exist if and only if a is a rational multiple of π. An original result is that the function $S(\lambda; a, b)$ has the properties of the characteristic polynomial known from finite-dimensional spaces. We study its properties and use it to describe the disposition of the spectrum. In particular, we derive a criterion for $A(a, b)$ to have algebraically double multiplicities and show the solution of the optimal damping problem for $a = \pi/3$.

In Chapter 3, we concern ourselves with the root vectors of $A(a, b)$. We use a trace criterion to disprove the Riesz basis property for $b = 2$ with rational placement of the damping. Along the way, we provide an explicit calculation of the sum $\sum_{\sigma(A)} \operatorname{Re} \frac{1}{\lambda}$ which, compared with the trace of the real part of the inverse $\operatorname{tr}(\operatorname{Re} A^{-1}(a, b))$, gives some quantitative insight on the singular behaviour of our model at $b = 2$. Subsequently, we extend both our result and that of [3] for a general $a \in (0, \pi)$ and $b \in \mathbb{R}$ yielding a definitive condition of the Riesz basis property for these values.

The thesis is finished with an appendix that provides proofs of several results needed for our approach.

Notation

Throughout the text, $\mathcal{X}$ and $\mathcal{Y}$ are used to denote Banach spaces, while $\mathcal{H}$ denotes a Hilbert space. The inner product of $x, y \in \mathcal{H}$ is denoted by $\langle x, y \rangle$. Inner product is always thought of as linear in the *second* argument. Unless stated otherwise, all vector spaces are considered over $\mathbb{C}$.

For the vector space X, $V \subset\subset X$ reads 'V is a subspace of X'. For vector spaces X, Y, symbol $\mathcal{L}(X, Y)$ denotes the vector space of linear mappings from X to Y (defined everywhere), with $\mathcal{L}(X) := \mathcal{L}(X, X)$.

The set of smooth, compactly supported functions on $\Omega = \Omega^{\circ} \subset \mathbb{R}^n$ is denoted by $C_0^\infty(\Omega)$. For partial derivatives, we use the multi-index notation, i.e. for $\alpha \equiv (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, we have

$$D^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_n}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.$$

To simplify enumeration, if $n \in \mathbb{N}$, we denote $\hat{n} := \{1, 2, \ldots, n\}$.

Chapter 1

Basic notions

1.1 Unbounded linear operator in a Hilbert space

By unbounded linear operator we mean 'not necessarily bounded'. This section provides a short overview of basic properties of unbounded operators in Banach and Hilbert spaces and their spectra. The theory is drawn mainly from [5] and [6].

1.1.1 Bounded linear operators

In this subsection, we briefly summarise fundamental knowledge about bounded linear operators in normed, Banach, and Hilbert spaces.

Definition 1.1. Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be normed spaces. $A \in \mathcal{L}(X, Y)$ is said to be *bounded* if there exists $M \geq 0$ such that for all $x \in X$ holds $\|Ax\|_Y \leq M \|x\|_X$. The set of bounded linear mappings from $\mathcal{L}(X, Y)$ is denoted by $\mathcal{B}(X, Y)$. The *dual space* to X over field $\mathbb{C}$ is $X^* := \mathcal{B}(X, \mathbb{C})$.

Definition 1.2. Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be normed spaces. The mapping

$$\|\cdot\| : \mathcal{B}(X, Y) \to [0, \infty) : \|A\| := \sup_{x \in X,\, \|x\| \leq 1} \|Ax\|_Y$$

is called the *operator norm* on $\mathcal{B}(X, Y)$.

In further text, X, Y, Z are automatically considered normed spaces. The norms on X, Y, and Z are no more distinguished by notation.

Lemma 1.3. *Let $A \in \mathcal{B}(X, Y)$, $X \neq \{0\}$. Then*

$$\|A\| = \min \{M \geq 0 \mid (\forall x \in X)\, (\|Ax\| \leq M \|x\|)\} = \sup_{x \in X,\, \|x\| = 1} \|Ax\|_Y = \sup_{x \in X \setminus \{0\}} \frac{\|Ax\|_Y}{\|x\|_X} = \sup_{x \in X,\, \|x\| < 1} \|Ax\|_Y .$$

Proposition 1.4. *$\mathcal{B}(X, Y)$ with the operator norm is a normed vector space.*

Proposition 1.5. *Let $A \in \mathcal{L}(X, Y)$. The following statements are equivalent*

1. *A is bounded,*

2. *A is uniformly continuous,*

3. *A is continuous,*

4. *A is continuous in* 0.

Remark. Let $A \in \mathcal{B}(X, Y)$, $B \in \mathcal{B}(Y, Z)$. Then $BA \in \mathcal{B}(X, Z)$ and $\|BA\| \leq \|B\| \, \|A\|$.

Proposition 1.6. *Let $\mathcal{Y}$ be a Banach space. Then $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ is Banach. In particular, if $\mathcal{X}$ is Banach, then $\mathcal{B}(\mathcal{X})$ is Banach.*

Theorem 1.7. *Let $V \subset\subset X$, $\overline{V} = X$. Let $\mathcal{Y}$ be a Banach space, $A \in \mathcal{B}(V, \mathcal{Y})$. Then there exists precisely one $\hat{A} \in \mathcal{B}(X, \mathcal{Y})$ such that $\hat{A}|_V = A$. Additionally, $\left\|\hat{A}\right\| = \|A\|$.*

Proposition 1.8. *Let $\{x_n\}_{n=1}^{\infty} \subset \mathcal{X}$. If $\sum\limits_{n=1}^{+\infty} \|x_n\|$ converges in $\mathbb{R}$, then $\sum\limits_{n=1}^{+\infty} x_n$ converges in $\mathcal{X}$. Additionally, $\left\|\sum\limits_{n=1}^{+\infty} x_n\right\| \leq \sum\limits_{n=1}^{+\infty} \|x_n\|$.*

Proposition 1.9. *Let $A \in \mathcal{B}(\mathcal{X})$, $\|A\| < 1$. Then there exists the inverse $(I - A)^{-1} = \sum_{n=0}^{\infty} A^n \in \mathcal{B}(\mathcal{X})$. Additionally, $\left\|(I - A)^{-1}\right\| \leq (1 - \|A\|)^{-1}$.*

Definition 1.10. Let $\{A_n\}_{n=1}^{\infty} \subset \mathcal{B}(X, Y)$, $A \in \mathcal{B}(X, Y)$. We say that

(i) $\{A_n\}_{n=1}^{\infty}$ *converges to A uniformly* (or *in operator norm*) $\iff \|A_n - A\| \to 0$,

(ii) $\{A_n\}_{n=1}^{\infty}$ *converges to A strongly* $\iff (\forall x \in X)(A_n x \to Ax)$,

(iii) $\{A_n\}_{n=1}^{\infty}$ *converges to A weakly* $\iff (\forall x \in X)(\forall \varphi \in Y^*)(\varphi(A_n x) \to \varphi(Ax))$.

Theorem 1.11. *Let $A \in \mathcal{B}(\mathcal{H})$. Then there exists exactly one $A^* \in \mathcal{B}(\mathcal{H})$ such that $(\forall x, y \in \mathcal{H})(\langle x, Ay \rangle = \langle A^* x, y \rangle)$.*

Definition 1.12. Let $A \in \mathcal{B}(\mathcal{H})$. Operator A^* from the previous theorem is called the *adjoint operator* to A.

Proposition 1.13. *Let $A, B \in \mathcal{B}(\mathcal{H})$, $\alpha \in \mathbb{C}$. The following holds:*

1. $(\alpha A + B)^* = \overline{\alpha} A^* + B^*$,
2. $(AB)^* = B^* A^*$,
3. *if A is an isomorphism, then* $(A^{-1})^* = (A^*)^{-1}$,
4. $A^{**} = A$,
5. $\|A^*\| = \|A\|$,
6. $\ker A^* = (\operatorname{ran} A)^{\perp}$.

Remark. It is established by Corollary 1.18 that the inverse of a bounded isomorphism between two Banach spaces is also bounded.

1.1.2 Basics of unbounded operators

Let us first state several fundamental results of functional analysis for further use and reference.

Theorem 1.14. *(Baire Category Theorem). Let X be a complete metric space and $\{A_n\}_{n=1}^{\infty}$ be a system of open dense subsets of X. Then $\cap_{n=1}^{\infty} A_n$ is dense in X.*

Corollary 1.15. *Let X be a complete metric space. Then X cannot be written as a countable union of nowhere dense subsets.*

Theorem 1.16. *(Banach-Steinhaus; Uniform Boundedness Principle). Let $\mathcal{X}$ be a Banach space, Y be a normed space, $\mathcal{A} \subset \mathcal{B}(\mathcal{X}, Y)$. Then exactly one of the following is true*

1. $\sup\limits_{A \in \mathcal{A}} \|A\| < +\infty$,
2. $(\exists G \subset \mathcal{X}, \overline{G} = \mathcal{X})(\forall x \in G)(\sup\limits_{A \in \mathcal{A}} \|Ax\| = +\infty)$.

Theorem 1.17. *(Open Mapping Theorem). Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces, $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. If A is surjective, then A is an open mapping.*

Corollary 1.18. *Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces, $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. If A is an isomorphism, then $A^{-1} \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$.*

Now we introduce unbounded operators.

Definition 1.19. *Linear operator* on $\mathcal{X}$ is a linear map $T : \operatorname{dom} T \subset\subset \mathcal{X} \to \mathcal{X}$. The subspace $\operatorname{dom} T$ is called the *domain* of T. If $\overline{\operatorname{dom} T} = \mathcal{X}$, we say that T is a *densely defined* linear operator in $\mathcal{X}$. The set of densely defined linear operators in $\mathcal{X}$ is denoted by $\mathscr{L}(\mathcal{X})$.

Definition 1.20. Let $T, S \in \mathscr{L}(\mathcal{X})$. If $\operatorname{dom} T \subset \operatorname{dom} S$ and $S|_{\operatorname{dom} T} = T$, then S is said to be the *extension* of T; we denote $T \subset S$.

On a Hilbert space, the adjoint operator can be defined analogously as in the bounded case. To do so, we will make use of the following lemmas.

Lemma 1.21. *Let $a \in \mathcal{H}$, $M \subset \mathcal{H}$, $\overline{M} = \mathcal{H}$. If $(\forall x \in M)(\langle a, x \rangle = 0)$, then $a = 0$.*

Proof. Since $a \in \mathcal{H} = \overline{M}$, there exists $\{a_n\}_{n=1}^{\infty} \subset M$ with $a_n \to a$. From the continuity of inner product in its arguments, we have

$$\|a\|^2 = \langle a, a \rangle = \langle a - a_n, a \rangle + \langle a_n, a \rangle = \langle a - a_n, a \rangle \to 0. \qquad \square$$

Lemma 1.22. *Let $T \in \mathscr{L}(\mathcal{H})$, $y \in \mathcal{H}$. Then there exists at most one $z \in \mathcal{H}$ such that $(\forall x \in \operatorname{dom} T)(\langle y, Tx \rangle = \langle z, x \rangle)$.*

Proof. Suppose $z_1, z_2 \in \mathcal{H}$ both satisfy $(\forall x \in \operatorname{dom} T)(\langle y, Tx \rangle = \langle z_i, x \rangle)$. Therefore,

$$(\forall x \in \operatorname{dom} T)(\langle z_1, x \rangle = \langle z_2, x \rangle).$$

Using the fact that T is densely defined and Lemma 1.21, we get $z_1 = z_2$. $\square$

We are now equipped to define the adjoint operator.

Definition 1.23. Let $T \in \mathscr{L}(\mathcal{H})$. The adjoint operator of T is the operator T^*, whose domain is

$$\operatorname{dom} T^* = \{y \in \mathcal{H} \mid (\exists z \in \mathcal{H})(\forall x \in \operatorname{dom} T)(\langle y, Tx\rangle = \langle z, x\rangle)\}.$$

For $y \in \operatorname{dom} T^*$, we put $T^* y := z$.

Remark. Linearity of the adjoint operator is evident. Furthermore, if $T \in \mathcal{B}(\mathcal{H})$, Definition 1.23 coincides with the standard definition.

The following definition presents some fundamental types of unbounded operators with respect to their adjoint.

Definition 1.24. Let T be a linear operator in $\mathcal{H}$. We say that T is

(i) *symmetric* $\iff$ $(\forall x, y \in \operatorname{dom} T)(\langle x, Ty\rangle = \langle Tx, y\rangle)$,

(ii) *skew-symmetric* $\iff$ $(\forall x, y \in \operatorname{dom} T)(\langle x, Ty\rangle = -\langle Tx, y\rangle)$

(iii) *self-adjoint* $\iff$ T is symmetric and $\operatorname{dom} T = \operatorname{dom} T^*$.

(iv) *skew-adjoint* $\iff$ T is skew-symmetric and $\operatorname{dom} T = \operatorname{dom} T^*$.

Remark. In other words, T is symmetric if and only if $T \subset T^*$ and self-adjoint if and only if $T = T^*$. Additionally, T is skew-symmetric (skew-adjoint) if and only if $\mathrm{i}T$ is symmetric (self-adjoint).

Proposition 1.25. *Linear operator T on $\mathcal{H}$ is symmetric if and only if* $(\forall x \in \operatorname{dom} T)(\langle x, Tx\rangle \in \mathbb{R})$.

Proof. Let T be symmetric and $x \in \operatorname{dom} T$. Then $\langle x, Tx\rangle = \langle Tx, x\rangle = \overline{\langle x, Tx\rangle}$.

Conversely, suppose $\langle x, Tx\rangle \in \mathbb{R}$ for all $x \in \operatorname{dom} T$. Applying this to $x + y$ for any $x, y \in \operatorname{dom} T$, we have

$$\langle x, Ty\rangle + \langle y, Tx\rangle = \langle Tx, y\rangle + \langle Ty, x\rangle \implies \langle x, Ty\rangle - \overline{\langle x, Ty\rangle} = \langle Tx, y\rangle - \overline{\langle Tx, y\rangle}.$$

Therefore, $\operatorname{Im}\langle x, Ty\rangle = \operatorname{Im}\langle Tx, y\rangle$. Substituting $y \to \mathrm{i}y$, we obtain equality of the real parts. □

Definition 1.26. A symmetric operator T on $\mathcal{H}$ is said to be *bounded from below* if $(\exists c \in \mathbb{R})(\forall x \in \operatorname{dom} T)(\langle x, Tx\rangle \geq c\,\|x\|^2)$.

1.1.3 Spectrum of a closed linear operator

In this section, we expand the concept of spectrum of an operator to infinite-dimensional Banach spaces.

Definition 1.27. Let T be a linear mapping from X to Y. *Graph of linear mapping T* is the set

$$\Gamma(T) := \{(x, Tx) \mid x \in \operatorname{dom} T\}$$

.

Definition 1.28. A *closed linear mapping* is a linear mapping T from X to Y such that $\Gamma(T)$ is closed in $X \times Y$ endowed with the product topology. The set of closed linear operator in $\mathcal{X}$ is denoted by $\mathcal{C}(\mathcal{X})$.

Proposition 1.29. *Let $T \in \mathcal{C}(\mathcal{X})$ be injective. Then T^{-1} is closed.*

The following is a direct consequence of the definition of product topology and provides us with a characterisation of closed operators.

Lemma 1.30. *Let T be a linear mapping from $\mathcal{X}$ to $\mathcal{Y}$. Then T is closed if and only if for every sequence $\{x_n\}_{n=1}^{\infty}$ in* $\operatorname{dom} T$ *holds:*

$$x_n \to x \in \mathcal{X} \text{ and } Tx_n \to y \in \mathcal{Y} \implies x \in \operatorname{dom} T \text{ and } Tx = y.$$

The lemma has a simple consequence.

Corollary 1.31. *Let $T \in \mathcal{B}(\mathcal{X})$. Then T is closed.*

Proof. Consider a convergent sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{X}$. Since T is defined on $\mathcal{X}$ it suffices to realise that, thanks to T being continuous, $Tx_n \to Tx$. □

Thanks to the following fundamental result, the previous implication can be reversed under the condition that T is defined everywhere.

Theorem 1.32. *(Closed Graph Theorem). Let $T \in \mathcal{C}(\mathcal{X}, \mathcal{Y})$. If* $\operatorname{dom} T = \mathcal{X}$*, then $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$.*

Now we are fully equipped to define the spectrum.

Definition 1.33. Let $T \in \mathscr{L}(\mathcal{X})$. The *resolvent set* of T is $\rho(T) := \{\lambda \in \mathbb{C} \mid (T - \lambda)^{-1} \in \mathcal{B}(\mathcal{X})\}$. The *spectrum* of T is the complement to the resolvent set, i.e. $\sigma(T) := \mathbb{C} \setminus \rho(T)$.

In general, it is practical to only deal with spectra of closed linear operators thanks to the following lemma. The density of the domain is supposed automatically from now on.

Lemma 1.34. *Let $T \in \mathscr{L}(\mathcal{X}) \setminus \mathcal{C}(\mathcal{X})$. Then $\sigma(T) = \mathbb{C}$.*

Proof. We will prove the lemma by contradiction. Suppose there exists some $\lambda \in \rho(T)$, meaning that $(T-\lambda)^{-1} \in \mathcal{B}(\mathcal{X})$. By Corollary 1.31 we have $(T-\lambda)^{-1} \in \mathcal{C}(\mathcal{X})$ and by Proposition 1.29 also $T-\lambda \in \mathcal{C}(\mathcal{X})$. It follows that T is closed - a contradiction. □

Proposition 1.35. *Let $T \in \mathcal{C}(\mathcal{X})$. Then $\sigma(T)$ is closed.*

Remark. It is clear from the definition that $\lambda \in \mathbb{C}$ is in the spectrum of $T \in \mathcal{C}(\mathcal{X})$ if and only if one of the following three *mutually exclusive* conditions is satisfied:

(i) $T - \lambda$ is not injective,

(ii) $T - \lambda$ is injective but it is not surjective, $\overline{\operatorname{ran} T} = \mathcal{X}$,

(iii) $T - \lambda$ is injective but it is not surjective, $\overline{\operatorname{ran} T} \neq \mathcal{X}$,

This allows us to define three disjoint parts of the spectrum.

Definition 1.36. Let $T \in \mathcal{C}(\mathcal{X})$.

(i) Let $\lambda \in \mathbb{C}$. If $T - \lambda$ is not injective, λ is called an *eigenvalue* of T. Non-zero vector $x \in \mathcal{X}$ such that $(T - \lambda)x = 0$ is called an *eigenvector* of T associated with (or corresponding to) the eigenvalue λ. The set of eigenvalues of T is called the *point spectrum* of T and denoted by $\sigma_p(T)$.

(ii) The set of $\lambda \in \mathbb{C}$ such that $T - \lambda$ is injective but it is not surjective, and $\overline{\operatorname{ran} T} = \mathcal{X}$ is called the *continuous spectrum* of T and denoted by $\sigma_c(T)$.

(iii) The set of $\lambda \in \mathbb{C}$ such that $T - \lambda$ is injective but it is not surjective, and $\overline{\operatorname{ran} T} \neq \mathcal{X}$ is called the *residual spectrum* of T and denoted by $\sigma_r(T)$.

Remark. We can write the spectrum of $T \in C(X)$ as the disjoint union

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T).$$

Throughout this thesis, we will be mainly concerned with the first of the three sets, that is, with eigenvalues.

We define multiplicities of eigenvalues similarly to the finite-dimensional case. In fact, the equivalence of definitions of algebraic multiplicities is analysed in detail in the appendix (see Section A.5).

Definition 1.37. Let $T \in C(X)$ and $\lambda \in \sigma_p(T)$.

(i) The *geometric multiplicity* of λ is $\nu_g(\lambda) := \dim \ker(T - \lambda)$. The subspace $\ker(T - \lambda)$ is called the *eigenspace* corresponding to λ.

(ii) The *algebraic multiplicity* of λ is $\nu_a(\lambda) := \dim \cup_{n=1}^{\infty} \ker(A - \lambda I)^n$. The subspace $\cup_{n=1}^{\infty} \ker(A - \lambda I)^n$ is called the *generalised eigenspace* corresponding to λ. Its non-zero elements are called *generalised eigenvectors* or *root vectors* associated with λ.

Remark. Note that by definition it always holds that $\nu_g(\lambda) \le \nu_a(\lambda)$.

The relationship between eigenvalues and eigenvectors of an operator and its inverse remain valid after transition from finite- to infinite-dimensional spaces.

Proposition 1.38. *Let $T \in C(X)$ be injective. Then $\lambda \in \sigma_p(T) \iff 1/\lambda \in \sigma_p\left(T^{-1}\right)$. Moreover, $x \in X$ is a (generalised) eigenvector of T if and only if it is a (generalised) eigenvector of T^{-1}. Consequently, for all $\lambda \in \sigma_p(T)$ holds*

$$\nu_g^T(\lambda) = \nu_g^{T^{-1}}\left(\frac{1}{\lambda}\right), \quad \nu_a^T(\lambda) = \nu_a^{T^{-1}}\left(\frac{1}{\lambda}\right).$$

Proof. Note that since T is injective, $0 \notin \sigma_p(T)$. Let $\lambda \in \sigma_p(T)$ and $x \in \mathcal{H}$, $x \neq 0$ such that $Tx = \lambda x$. Then clearly $x \in \operatorname{ran} T = \operatorname{dom} T^{-1}$. Applying T^{-1} gives us $x = \lambda T^{-1} x$. This proves the correspondence of the point spectra and of eigenvectors.

Let $\tilde{x} \in \operatorname{dom} T$ be a generalised eigenvector of T corresponding to λ. We will consider the case

$$(T - \lambda I)\tilde{x} = x, \tag{1.1}$$

where $Tx = \lambda x$. For generalised eigenvectors of higher order one can proceed by induction.

Since

$$\tilde{x} = T\left(\frac{1}{\lambda}\tilde{x} - \frac{1}{\lambda^2}x\right) \in \operatorname{ran} T = \operatorname{dom} T^{-1},$$

we can apply $-\frac{1}{\lambda}T^{-1}$ to (1.1) and find

$$\left(T^{-1} - \frac{1}{\lambda}\right)\tilde{x} = -\frac{1}{\lambda^2}x.$$

It follows that $\tilde{x}$ is a generalised eigenvector of T^{-1} corresponding to the eigenvalue $1/\lambda$. □

At the end of this section regarding spectrum, we state a potent theorem that will help us analyse the Sobolev spaces in Section 1.3.

Theorem 1.39. *(Mini-max Principle; [13], Theorem 4.10). Let T be a self-adjoint operator in $\mathcal{H}$ bounded from below. Let $\sigma_{ess}(T) = \emptyset$. Then $\sigma(T) = \sigma_{disc}(T) \subset \mathbb{R}$ has a minimal point, let us denote it by λ_1. Additionally,*

$$\lambda_1 = \inf_{x \in \operatorname{dom} T \setminus \{0\}} \frac{\langle x, Tx \rangle}{\|x\|^2}.$$

1.1.4 Compact operators

For the sake of reference and later use, here we briefly define compact operators and state several fundamental properties they possess.

Definition 1.40. Let X be a topological space, $A \subset X$. A is called *precompact* if $\overline{A}$ is compact.

Definition 1.41. Operator $T \in \mathcal{B}(\mathcal{X})$ is called *compact* if it maps bounded sets to precompact sets, i.e. for every $V \subset \mathcal{X}$ bounded, the set $T(V)$ is precompact.

Lemma 1.42. *Let T be a compact operator in $\mathcal{X}$, $\dim \mathcal{X} = \infty$. The following hold*

1. $0 \in \sigma(T)$,
2. $\sigma(T) \setminus \{0\} \subset \sigma_p(T)$,
3. *$\sigma(T)$ is countable,*
4. $(\forall \lambda \in \sigma_p(T) \setminus \{0\})(\nu_a(\lambda) < \infty)$,
5. *$\sigma_p(T)$ has no accumulation points except for zero, i.e. $(\sigma_p(T))' \subset \{0\}$.*

Theorem 1.43. *(Fredholm Alternative). Let T be a compact operator on $\mathcal{H}$, $\lambda \in \mathbb{C} \setminus \{0\}$. The following statements hold*

1. *$T - \lambda I$ is surjective $\iff$ $T - \lambda I$ is injective.*
2. $\operatorname{ran}(T - \lambda I) = \ker\left(T^* - \overline{\lambda} I\right)^{\perp}$.
3. $\dim\ker(T - \lambda I) = \dim\ker\left(T^* - \overline{\lambda} I\right)$.

1.1.5 Dissipative operators

The main object of this thesis is a particular dissipative operator in a Hilbert space. Hence, here we provide proofs of some of the properties of dissipative operators that will be needed later.

Definition 1.44. Linear operator $T \in \mathscr{L}(\mathcal{X})$ is called *dissipative* if $(\forall \lambda > 0)(\forall x \in \mathcal{X})(\|(T - \lambda I)x\| \geq \lambda \|x\|)$. T is called *accretive* if $-T$ is dissipative.

Lemma 1.45. *Let $T \in \mathscr{L}(\mathcal{X})$ be dissipative, $\lambda > 0$. Then $T - \lambda I$ is injective.*

Proof. Let $(T - \lambda I)x = 0$. Then $\lambda \|x\| \leq \|(T - \lambda I)x\| = 0$, so $\|x\| = 0$. □

On Hilbert spaces, dissipativness can be expressed by the following simple condition.

Proposition 1.46. *Linear operator $T \in \mathscr{L}(\mathcal{H})$ is dissipative if and only if* $(\forall x \in \operatorname{dom} T)(\operatorname{Re}\langle x, Tx\rangle \leq 0)$.

Proof. Suppose that T is dissipative. For any $\lambda > 0$ and $x \in \mathcal{H}$ we have $\langle (T - \lambda I)x, (T - \lambda I)x\rangle \geq \lambda^2 \|x\|^2$. Distributing the left-hand side, this becomes

$$-2\lambda \operatorname{Re}\langle x, Tx\rangle + \|Tx\|^2 \geq 0. \tag{1.2}$$

If $\operatorname{Re}\langle x, Tx\rangle > 0$, choosing λ large enough would contradict the inequality.

Conversely, if $(\forall x \in \operatorname{dom} T)(\operatorname{Re}\langle x, Tx\rangle \leq 0)$, then (1.2) holds for any $\lambda > 0$ which implies that T is dissipative. □

Remark. It is clear that any accretive symmetric operator in $\mathcal{H}$ is bounded from below.

Definition 1.47. Dissipative linear operator $T \in \mathscr{L}(\mathcal{X})$ is said to be *maximally dissipative* if

$$(\forall \lambda > 0)(\operatorname{ran}(T - \lambda I) = \mathcal{X}),$$

i.e. $T - \lambda I$ is surjective for all $\lambda > 0$.

Theorem 1.48. *([10], Theorem 4.3). Let $T \in \mathscr{L}(\mathcal{X})$ be a dissipative operator. Then T is maximally dissipative if and only if*

$$(\exists \lambda_0 > 0)(\operatorname{ran}(T - \lambda_0 I) = \mathcal{X}).$$

Proof. The implication T maximally dissipative $\implies$ $(\exists \lambda_0 > 0)(\operatorname{ran}(T - \lambda_0 I) = \mathcal{X})$ is obvious.

Suppose $\lambda_0 > 0$ such that $T - \lambda_0 I$ is surjective. By Lemma 1.45, there exists $(T - \lambda_0 I)^{-1} \in \mathcal{B}(\mathcal{X})$. Consequently, $T - \lambda_0 I$ is closed and thus T is closed.

Put $M := \{\lambda > 0 \mid \operatorname{ran}(T - \lambda I) = \mathcal{X}\}$. By Lemma 1.45, we have $M \subset \rho(T)$. Since $\rho(T)$ is open in $\mathbb{C}$, clearly also M must be open in the topological space $(0, \infty)$ with topology induced from $\mathbb{R}$.

Let $\{\lambda_n\}_{n=1}^{\infty} \subset M$, $\lambda_n \to \lambda \in (0, \infty)$. Choose an arbitrary $y \in \mathcal{X}$. By definition of M, we can construct $\{x_n\}_{n=1}^{\infty} \subset \operatorname{dom} T$ such that $(\forall n \in \mathbb{N})(Tx_n - \lambda_n x_n = y)$. From the dissipativness of T, we have

$$\lambda_n \|x_n\| \leq \|y\| \implies \|x_n\| \leq \frac{1}{\lambda_n} \|y\| \leq C,$$

where such $C > 0$ exists since $\{\lambda_n\}_{n=1}^{\infty}$ is convergent in $(0, \infty)$. For $m, n \in \mathbb{N}$, we can write

$$\begin{aligned} \lambda_m \|x_n - x_m\| &\leq \|T(x_n - x_m) - \lambda_m(x_n - x_m) + \lambda_n x_n - \lambda_n x_n\| = \|y - y + (\lambda_n - \lambda_m)x_n\| \\ &= |\lambda_n - \lambda_m| \, \|x_n\| \leq C|\lambda_n - \lambda_m|. \end{aligned}$$

Therefore, $\{x_n\}_{n=1}^{\infty}$ is Cauchy and thanks to completeness, it converges to $x \in \mathcal{X}$. By construction, we also have $Tx_n \to \lambda x + y$. Since T is closed, necessarily $x \in \operatorname{dom} T$ and $Tx = \lambda x + y$, i.e. $y = (T - \lambda I)x$. Consequently, $\operatorname{ran}(T - \lambda I) = \mathcal{X}$ and $\lambda \in M$; so M is also closed in $(0, \infty)$. As a clopen non-empty $(\lambda_0 \in M)$ subset of a connected space, it is equal to $(0, \infty)$. □

In a criterion that will be pivotal later, we will make use of the notion of the trace of a symmetric dissipative operator.

Definition 1.49. Let $T \in \mathcal{B}(\mathcal{H})$ be a symmetric dissipative operator and $\{u_n\}_{n=1}^{\infty}$ be an ON basis in $\mathcal{H}$. The *trace* of T is

$$\operatorname{tr} T := \sum_{n=1}^{\infty} \langle u_n, Tu_n \rangle .$$

Remark. The trace of a bounded symmetric dissipative operator T on $\mathcal{H}$ is well defined (possibly infinite). Thanks to T being dissipative and symmetric, it follows from Proposition 1.46 and Proposition 1.25 that the terms in the sum are all real and non-positive. Moreover, the trace does not depend on the choice of ON basis (see [11], Lemma 8.1).

The same definition can be used if we consider an accretive operator instead of a dissipative one.

1.1.6 The Friedrichs extension

We will now present a tool for extending symmetric operators to self-adjoint ones.

Definition 1.50. Let $\operatorname{dom} t \subset\subset \mathcal{H}$. A *sesquilinear form* in $\mathcal{H}$ is a mapping $t : \operatorname{dom} t \times \operatorname{dom} t \to \mathbb{C}$ that is linear in the second argument and antilinear in the first argument. The mapping $t[\cdot] : \operatorname{dom} t \to \mathbb{C} : t[\psi] := t(\psi, \psi)$ is called the *quadratic form* associated with t. $\operatorname{dom} t$ is the *domain of the sesquilinear form* t.

The sesquilinear form t is said to be

(i) *densely defined* if $\operatorname{dom} t$ is dense in $\mathcal{H}$;

(ii) *symmetric* if $(\forall \psi, \phi \in \operatorname{dom} t)(t(\psi, \phi) = \overline{t(\phi, \psi)})$.

If t is a symmetric form, we say that t is *bounded from below* if $(\exists c \in \mathbb{R})(\forall \psi \in \operatorname{dom} t)(t[\psi] \geq c\,\|\psi\|^2)$. A symmetric form bounded from below is said to be *closed* if the space $(\operatorname{dom} t, \|\cdot\|_t)$, where $\|\psi\|_t := \sqrt{\|a\|^2 + t[a]}$, is complete;

Remark. Note that every symmetric sesquilinear form (in a complex space) is uniquely determined by its diagonal.

Definition 1.51. Let t be a symmetric form bounded from below on $\mathcal{H}$. Then t is said to be *closable* if there exists a closed symmetric form c with $\operatorname{dom} t \subset \operatorname{dom} c$ such that $(\forall \phi, \psi \in \operatorname{dom} t)(t(\phi, \psi) = c(\phi, \psi))$. The smallest closed extension of a closable form t (in the sense of inclusion of domains) is called the *closure* of t.

In our construction, we will make use of the fact that a form generated by a symmetric operator bounded from below is closable. A proof can be found in [5], Theorem 7.5.7.

Proposition 1.52. *Let T be a symmetric operator in $\mathcal{H}$ bounded from below. Let* $\operatorname{dom} t := \operatorname{dom} T$ *and* $t(\phi, \psi) := \langle \phi, T\psi \rangle$. *Then t is a closable symmetric sesquilinear form bounded from below.*

The following theorem ([5], Theorem 7.5.8) provides a mutually unambiguous relationship between closed symmetric forms bounded from below and self-adjoint operators bounded from below.

Theorem 1.53. *(Representation Theorem). Let t be a densely defined, closed, symmetric form on $\mathcal{H}$ bounded from below. Then there exists a unique self-adjoint operator T on $\mathcal{H}$ such that* $\operatorname{dom} T \subset \operatorname{dom} t$ *and* $(\forall \psi \in \operatorname{dom} T)(\forall \phi \in \operatorname{dom} t)(t(\phi, \psi) = \langle \phi, T\psi \rangle)$.

Conversely, let T be a self-adjoint operator in $\mathcal{H}$ bounded from below by constant $c \in \mathbb{R}$. Then the sesquilinear form $\tilde{t}$, where $\operatorname{dom} \tilde{t} = \operatorname{dom} T$ *and* $\tilde{t}(\phi, \psi) := \langle \phi, T\psi \rangle$, *is closable. Let t denote its closure. Then t is densely defined, closed, symmetric and bounded from below. The operator associated with t by the first part of the theorem is T.*

Now suppose we have an operator $\tilde{H}$ in $\mathcal{H}$ that is symmetric and bounded from below. We can define the corresponding form $\tilde{h}$ with $\operatorname{dom} \tilde{h} := \operatorname{dom} \tilde{H}$, $\tilde{h}(\phi, \psi) := \left\langle \phi, \tilde{H}\psi \right\rangle$. By Proposition 1.52, $\tilde{h}$ is closable. Let h denote its closure. By Theorem 1.53, there exists a unique self-adjoint operator H bounded from below that is associated to h. By [5], Theorem 7.5.11, H is an extension of $\tilde{H}$, i.e. $\tilde{H} \subset H$. We have

$$\operatorname{dom} H = \{\psi \in \operatorname{dom} h \mid (\exists \eta \in \mathcal{H})(\forall \phi \in \mathcal{H})(h(\phi, \psi) = \langle \phi, \eta \rangle)\}\,, \quad H\psi = \eta. \tag{1.3}$$

1.2 Bases in a Hilbert space

To study the root vectors of the operator introduced in Chapter 3, we need to establish a class of total subsets in a Hilbert space – Riesz bases. This chapter covers in detail the basics of their properties. The following trivial proposition will be a neat tool.

Proposition 1.54. *Let $x, y \in \mathcal{H}$. Then $\|x + y\|^2 = \|x\|^2 + \left\|y^2\right\| + 2\,\mathrm{Re}\,\langle x, y\rangle$.*
In particular, if $x \perp y$, they satisfy the Pythagorean equality $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

1.2.1 Orthonormal basis

Definition 1.55. Let $U \subset \mathcal{H}$. U is said to be an *orthonormal (ON) subset* in $\mathcal{H}$ if $(\forall x, y \in U, x \neq y)(\langle x, y\rangle = 0)$ and $(\forall x \in U)(\|x\| = 1)$.

Theorem 1.56. *(Bessel inequality). Let U be an ON subset in $\mathcal{H}$, $x \in \mathcal{H}$. Then*

$$\sum_{u \in U} |\langle u, x\rangle|^2 \leq \|x\|^2 . \tag{1.4}$$

Proof.

(i) Let us first assume that U is finite, denote $U =: \{u_1, \ldots, u_n\}$, $n \in \mathbb{N}$. Let P denote the projection on $\mathrm{span}\{u_1, \ldots, u_n\}$. Then we have

$$\|x\|^2 = \|x - Px + Px\|^2 = \|x - Px\|^2 + \|Px\|^2 \geq \|Px\|^2 = \sum_{j=1}^{n} \left|\left\langle u_j, x\right\rangle\right|^2 ,$$

where the second equality is Pythagorean.

(ii) Suppose now that U is infinite. By definition

$$\sum_{u \in U} |\langle u, x\rangle|^2 = \sup_{B \subset U, |B| < \infty} \sum_{u \in B} |\langle u, x\rangle|^2 .$$

Thanks to the first part of the proof, inequality holds for every finite subset B of A; therefore, it must hold for the supremum as well. □

It is useful to state the following definition in a general Banach space.

Definition 1.57. Let $U \subset \mathcal{X}$. U is called a *total subset* in $\mathcal{X}$ if $\overline{\mathrm{span}\, U} = \mathcal{X}$.

Definition 1.58. A total orthonormal subset in $\mathcal{H}$ is called an *orthonormal (ON) basis*.

Remark. It is easy to see that an ON subset U in $\mathcal{H}$ is an ON basis if and only if U is a maximal ON subset in $\mathcal{H}$.

Theorem 1.59. *Let $U \subset \mathcal{H}$ be an ON subset. The following statements are equivalent.*

1. *U is an ON basis.*
2. *$U^{\perp} = \{0\}$.*
3. *For all $x \in \mathcal{H}$ holds the* Parseval equality*:*

$$\sum_{u \in U} |\langle u, x\rangle|^2 = \|x\|^2 . \tag{1.5}$$

Proof. The equivalence of 1. and 2. is obvious from linearity of inner product and the fact that for any subspace V holds $V^{\perp\perp} = \overline{V}$.

Let us now show the implication 1. $\implies$ 3. Take an arbitrary $x \in \mathcal{H}$. We already know that every ON subset satisfies the Bessel inequality, so $\sum_{u\in U} \| \langle u, x\rangle \|^2 \leq \|x\|^2$. It remains to prove the opposite inequality. For any $\varepsilon > 0$, we know by assumption 1. that there exist $n \in \mathbb{N}$, $\lambda_1, \dots, \lambda_n \in T$, and $u_1, \dots, u_n \in U$ such that $\left\| x - \sum_{j=1}^{n} \lambda_j u_j \right\| < \varepsilon$. Denote P the projection on $\operatorname{span}\{u_1, \dots, u_n\}$. Then we have

$$\|x\|^2 = \|Px\|^2 + \|x - Px\|^2 \leq \|Px\|^2 + \left\| x - \sum_{j=1}^{n} \lambda_j u_j \right\|^2 < \sum_{j=1}^{n} \left| \left\langle u_j, x \right\rangle \right|^2 + \varepsilon^2.$$

Therefore

$$\|x\|^2 - \varepsilon^2 \leq \sum_{j=1}^{n} \left| \left\langle u_j, x \right\rangle \right|^2.$$

Taking the limit $\varepsilon \to 0$ gives us the desired inequality.

The implication 3. $\implies$ 2. is simple. Consider $x \in U^\perp$. By Parseval equality, we have $\|x\|^2 = \sum_{u\in U} |\langle u, x\rangle|^2$. The right-hand side is zero since $x \in U^\perp$; thus $x = 0$. □

Theorem 1.60. *Let $U = \{u_n\}_{n=1}^\infty$ be a countable ON basis in $\mathcal{H}$. Then every $x \in \mathcal{H}$ can be written as*

$$x = \sum_{n=1}^{\infty} \langle u_n, x\rangle\, u_n. \tag{1.6}$$

Proof. Let $x \in \mathcal{H}$. For arbitrary $N \in \mathbb{N}$ we have

$$\|x\|^2 = \sum_{n=1}^{N} |\langle u_n, x\rangle|^2 + \left\| x - \sum_{n=1}^{N} \langle u_n, x\rangle\, u_n \right\|^2.$$

Taking the limit $N \to \infty$, Parseval equality gives us the statement. □

Definition 1.61. Coefficients $\langle u_n, x\rangle$ from the theorem above are called the *Fourier coefficients* of x in basis U. The series is called the *Fourier series (expansion)* of x in ON basis U.

Remark. Fourier series of a vector in an ON basis is unique. Suppose $x = \sum_{n=1}^\infty \lambda_n u_n$. Applying $\langle u_m, \cdot\rangle$ to the equality and using continuity of the inner product yields $(\forall n \in \mathbb{N})(\lambda_n = \langle u_n, x\rangle)$.

1.2.2 Riesz basis

In this subsection, we will introduce the concept of a Riesz basis, i.e. a basis isomorphic to an ON basis. We will study several equivalent characterisations and derive a sufficient criterion for later use. The results are drawn from [9] with occasional deviations and simplifications.

Let us start with a simple convergence criterion for series in $\mathcal{H}$.

Lemma 1.62. *Let $\{x_n\}_{n=1}^\infty$ be an orthogonal subset in $\mathcal{H}$. Then series $\sum_{n=1}^\infty x_n$ converges in $\mathcal{H}$ if and only if $\sum_{n=1}^\infty \|x_n\|^2$ converges in $\mathbb{R}$.*

Proof. Thanks to the completeness of both spaces, we can only examine whether the sequences of partial sums are Cauchy. Therefore, $\sum_{n=1}^{\infty} x_n$ converges if and only if for all $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for all $N \in \mathbb{N}$, $N \geq N_0$ and all $p \in \mathbb{N}$ holds $\left\|\sum_{n=N}^{N+p} x_n\right\| < \varepsilon$. Thanks to the Pythagorean identity, this is equivalent to

$$\varepsilon^2 > \left\|\sum_{n=N}^{N+p} x_n\right\| = \sum_{n=N}^{N+p} \|x_n\|^2 ,$$

that is, the convergence of $\sum_{n=1}^{\infty} \|x_n\|^2$ in $\mathbb{R}$. □

Corollary 1.63. *Let $\{u_n\}_{n=1}^{\infty}$ be an ON subset in $\mathcal{H}$ and $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{C}$. Then $\sum_{n=1}^{\infty} \lambda_n u_n$ converges in $\mathcal{H}$ if and only if $\{\lambda_n\}_{n=1}^{\infty} \in \ell^2$.*

Definition 1.64. Sequence $\{e_n\}_{n=1}^{\infty} \subset \mathcal{H}$ is called a *Riesz basis* if there exist isomorphism $T \in \mathcal{B}(\mathcal{H})$ and ON basis $\{u_n\}_{n=1}^{\infty}$ such that $(\forall n \in \mathbb{N})(e_n = Tu_n)$.

Remark. It follows from the definition that a Riesz basis is isomorphic to any ON basis in $\mathcal{H}$ with the underlying isomorphism being bounded. Indeed, let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be two ON bases in $\mathcal{H}$; then linear operator J defined as $Ju_n := v_n$ and extended to $\mathcal{H}$ using Theorem 1.7 is clearly a bounded isomorphism.

Remark. A simple consequence of the definition is that any ON basis in $\mathcal{H}$ is also a Riesz basis.

Definition 1.65. A *Riesz sequence* in $\mathcal{H}$ is any $\{e_n\}_{n=1}^{\infty} \subset \mathcal{H}$ for which there exist constants $0 < c \leq C$ such that

$$c \sum_{n=1}^{\infty} |\lambda_n|^2 \leq \left\|\sum_{n=1}^{\infty} \lambda_n e_n\right\|^2 \leq C \sum_{n=1}^{\infty} |\lambda_n|^2 \tag{1.7}$$

for any $\{\lambda_n\}_{n=1}^{\infty} \in \ell^2$.

Remark. The definition implicitly contains the assumption that for any $\{\lambda_n\}_{n=1}^{\infty} \in \ell^2$ the series $\sum_{n=1}^{\infty} \lambda_n e_n$ converges.

Theorem 1.66. *Let $\{e_n\}_{n=1}^{\infty} \subset \mathcal{H}$. The following statements are equivalent.*

1. *$\{e_n\}_{n=1}^{\infty}$ is a Riesz basis.*
2. *$\{e_n\}_{n=1}^{\infty}$ is a total Riesz sequence.*
3. *There exists a topologically equivalent inner product $\langle\cdot,\cdot\rangle_1$ on $\mathcal{H}$ such that $\{e_n\}_{n=1}^{\infty}$ is an ON basis in $(\mathcal{H}, \langle\cdot,\cdot\rangle_1)$.*

Proof.
1. $\Longrightarrow$ 3.: Suppose $\{e_n\}_{n=1}^{\infty}$ is the image of the ON basis $\{u_n\}_{n=1}^{\infty}$ under the isomorphism $T \in \mathcal{B}(\mathcal{H})$. Let us define

$$\langle x, y\rangle_1 := \left\langle T^{-1}x, T^{-1}y\right\rangle$$

for $x, y \in \mathcal{H}$. It is simple matter to confirm that $\langle\cdot,\cdot\rangle_1$ is linear in the second argument and Hermitian. Furthermore, if $\|x\|_1^2 := \langle x, x\rangle_1 = \left\|T^{-1}x\right\|^2 = 0$, we have $T^{-1}x = 0$ and from injectivity of T^{-1} clearly $x = 0$. To conclude, $\langle\cdot,\cdot\rangle_1$ is an inner product on $\mathcal{H}$.

Since $\langle e_n, e_m\rangle_1 = \left\langle T^{-1}Tu_n, T^{-1}Tu_m\right\rangle = \delta_{n,m}$, sequence $\{e_n\}_{n=1}^{\infty}$ is orthonormal. Consider $x \in \left(\{e_n\}_{n=1}^{\infty}\right)^{\perp_1}$, where $\perp_1$ denotes the orthogonal complement with respect to $\langle\cdot,\cdot\rangle_1$. Then $\left\langle u_n, T^{-1}x\right\rangle = \langle Tu_n, x\rangle_1 = \langle e_n, x\rangle_1 = 0$ for all $n \in \mathbb{N}$. Since $\{u_n\}_{n=1}^{\infty}$ is an ON basis (in particular, it is total in $\mathcal{H}$), we have $T^{-1}x = 0$. It follows that $x = 0$. We have shown that $\{e_n\}_{n=1}^{\infty}$ is total and thus an ON basis in $(\mathcal{H}, \langle\cdot,\cdot\rangle_1)$.

To prove the equivalence, suppose $x \in \mathcal{H}$. Then by the Cauchy-Schwarz inequality

$$\|x\|_1^2 = \langle x, x\rangle_1 = \left\langle T^{-1}x, T^{-1}x\right\rangle \leq \left\|T^{-1}\right\|^2 \|x\|^2$$

and

$$\|x\|^2 = \langle x, x\rangle = \left\langle TT^{-1}x, TT^{-1}x\right\rangle \leq \|T\|^2 \left\|T^{-1}x\right\|^2 = \|T\|^2 \|x\|_1^2 .$$

3. $\Longrightarrow$ 2.: Suppose we have an equivalent inner product $\langle\cdot,\cdot\rangle_1$ on $\mathcal{H}$ such that $\{e_n\}_{n=1}^{\infty}$ is an ON basis of $\mathcal{H}$. The first property means that there exist $0 < c \leq C$ such that for all $x \in \mathcal{H}$ holds $c\|x\|_1^2 \leq \|x\|^2 \leq C\|x\|_1^2$.

It is immediate that $\{e_n\}_{n=1}^{\infty}$ is total in $\mathcal{H}$.

Let us now prove the Riesz sequence property. Let $\{\lambda_n\}_{n=1}^{\infty} \in \ell^2$. Then by Corollary 1.63, $\sum_{n=1}^{\infty} \lambda_n e_n$ converges in $(\mathcal{H}, \langle\cdot,\cdot\rangle_1)$ and thus also in $\mathcal{H}$. Let x denote its limit. Using the equivalence of the norms and Parseval equality, we have

$$c\sum_{n=1}^{\infty} |\lambda_n|^2 = c\|x\|_1^2 \leq \|x\|^2 \leq C\|x\|_1^2 = C\sum_{n=1}^{\infty} |\lambda_n|^2 .$$

2. $\Longrightarrow$ 1.: Suppose that $\{e_n\}_{n=1}^{\infty}$ is total in $\mathcal{H}$ and satisfies the Riesz sequence property. Let $\{u_n\}_{n=1}^{\infty}$ be an arbitrary ON basis in $\mathcal{H}$. Let us define $Tu_n := e_n$. It suffices to show the boundedness of T on $\operatorname{span}\{u_n\}_{n=1}^{\infty}$. Then by Theorem 1.7, T can be extended to a unique bounded linear operator in $\mathcal{H}$.

Let $x = \sum_{j=1}^{k} \lambda_j u_j \in \operatorname{span}\{u_n\}_{n=1}^{\infty}$. Then

$$\|Tx\|^2 = \left\|\sum_{j=1}^{k} \lambda_j e_j\right\|^2 \leq C\sum_{j=1}^{k} \left|\lambda_j\right|^2 = C\|x\|^2 ,$$

where we defined $\lambda_j := 0$ for $j > k$ and used (1.7). Therefore, $T \in \mathcal{B}(\mathcal{H})$ and $\|B\| \leq C$.

We can now define $Se_n := u_n$. Again, recalling Theorem 1.7, it is enough to prove the boundedness of S on $\operatorname{span}\{e_n\}_{n=1}^{\infty}$. Additionally, since $ST = I$ on $\operatorname{span}\{u_n\}_{n=1}^{\infty}$ and $TS = I$ on $\operatorname{span}\{e_n\}_{n=1}^{\infty}$, it will be proven that $S = T^{-1}$ on $\mathcal{H}$.

Let $y = \sum_{j=1}^{k} \alpha_j e_j \in \operatorname{span}\{e_n\}_{n=1}^{\infty}$. Analogously as above, we have

$$\|Sy\|^2 = \left\|\sum_{j=1}^{k} \alpha_j u_j\right\|^2 = \sum_{j=1}^{k} \left|\alpha_j\right|^2 \leq \frac{1}{c}\left\|\sum_{j=1}^{k} \alpha_j e_j\right\|^2 = \frac{1}{c}\|y\|^2 . \qquad \square$$

Definition 1.67. Sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ are said to be *biorthogonal* if for all $n, m \in \mathbb{N}$ holds $\langle x_n, y_m\rangle = \delta_{n,m}$.

Theorem 1.68. *Let $\{e_n\}_{n=1}^{\infty}$ be a Riesz basis in $\mathcal{H}$. Then there exists a unique sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{H}$ biorthogonal to $\{e_n\}_{n=1}^{\infty}$. Additionally, $\{f_n\}_{n=1}^{\infty}$ is also a Riesz basis.*

Proof. $\{e_n\}_{n=1}^{\infty}$ is the image of some ON basis $\{u_n\}_{n=1}^{\infty}$ under the bounded isomorphism T, whose inverse T^{-1} is also bounded. Therefore, we can define $f_n := (T^{-1})^* u_n$. This gives us

$$\langle f_n, e_m\rangle = \left\langle (T^{-1})^* u_n, Tu_m\right\rangle = \left\langle u_n, T^{-1}Tu_m\right\rangle = \langle u_n, u_m\rangle = \delta_{n,m}.$$

Suppose that $\{g_n\}_{n=1}^{\infty}$ is also a sequence biorthogonal to $\{e_n\}_{n=1}^{\infty}$. Then we have

$$\delta_{n,m} = \langle e_n, g_m\rangle = \langle Tu_n, g_m\rangle = \langle u_n, T^* g_m\rangle .$$

By Theorem 1.60, $T^* g_m = u_m$, i.e. $g_m = (T^*)^{-1} u_m = (T^{-1})^* u_m = f_m$. The sequence is thus unique.

To prove that $\{f_n\}_{n=1}^\infty$ is also a Riesz basis, we must only realise that $(T^{-1})^*$ is itself a bounded isomorphism. □

As we will see, the implication that Riesz basis always possesses a biorthogonal sequence can be reversed under some additional conditions. First, we will prove an analogy of Theorem 1.60 for Riesz bases.

Theorem 1.69. *Let $\{e_n\}_{n=1}^\infty$ be a Riesz basis in $\mathcal{H}$ and $\{f_n\}_{n=1}^\infty$ its biorthogonal sequence. Let $x \in \mathcal{H}$. Then*

$$x = \sum_{n=1}^{\infty} \langle f_n, x \rangle e_n. \tag{1.8}$$

The decomposition is unique in the following sense: if $\{\lambda_n\}_{n=1}^\infty \subset \mathbb{C}$ such that $x = \sum_{n=1}^\infty \lambda_n e_n$, then $(\forall n \in \mathbb{N})(\lambda_n = \langle f_n, x \rangle)$.

Proof. Let $\{u_n\}_{n=1}^\infty$ denote an ON basis isomorphic to $\{e_n\}_{n=1}^\infty$ under $T \in \mathcal{B}(\mathcal{H})$. From Theorem 1.60 we have the unique decomposition

$$T^{-1} x = \sum_{n=1}^{\infty} \left\langle u_n, T^{-1} x \right\rangle u_n.$$

Applying T to the equality and using its continuity, we have

$$x = \sum_{n=1}^{\infty} \left\langle u_n, T^{-1} x \right\rangle e_n = \sum_{n=1}^{\infty} \left\langle (T^{-1})^* u_n, x \right\rangle e_n = \sum_{n=1}^{\infty} \langle f_n, x \rangle e_n,$$

where we used knowledge from the proof of Theorem 1.68 that $f_n = (T^{-1})^* u_n$. Uniqueness follows from the uniqueness of decomposition (1.6): if $x = \sum_{n=1}^\infty \lambda_n e_n$, we can apply T^{-1} and see that $\lambda_n = \left\langle u_n, T^{-1} x \right\rangle = \langle f_n, x \rangle$. □

The following property is a generalisation of Theorem 1.56.

Definition 1.70. Sequence $\{x_n\}_{n=1}^\infty \subset \mathcal{H}$ is said to be a *Bessel sequence* if there exists $B > 0$ such that for all $x \in \mathcal{H}$ holds

$$\sum_{n=1}^{\infty} |\langle x_n, x \rangle|^2 \leq B \|x\|^2 .$$

It shows that the Bessel property is equivalent to a seemingly weaker property.

Lemma 1.71. *Let $\{x_n\}_{n=1}^\infty \subset \mathcal{H}$. Then $\{x_n\}_{n=1}^\infty$ is Bessel if and only if*

$$\sum_{n=1}^{\infty} |\langle x_n, x \rangle|^2 < \infty \tag{1.9}$$

for all $x \in \mathcal{H}$.

Proof. Implication $\{x_n\}_{n=1}^\infty$ is Bessel $\implies$ (1.9) is trivial.

Suppose that (1.9) holds for all $x \in \mathcal{H}$. Let $e_j := \{\delta_{j,n}\}_{n=1}^\infty \in \ell^2$ and define the linear mappings $T_n : \mathcal{H} \to \ell^2$:

$$T_n x := \sum_{j=1}^{n} \left\langle x_j, x \right\rangle e_j = \{\langle x_1, x \rangle, \langle x_2, x \rangle \ldots, \langle x_n, x \rangle, 0, 0, \ldots\}.$$

Every T_n is bounded which follows from the Cauchy-Schwarz inequality. Thanks to (1.9) we have for any $x \in \mathcal{H}$: $\sup\limits_{n\in\mathbb{N}} \|T_n x\| < \infty$. By Banach-Steinhaus Theorem (Theorem 1.16), $\{T_n\}_{n=1}^{\infty}$ is uniformly bounded, i.e. there exists $B > 0$ such that $(\forall n \in \mathbb{N})(\|T_n\| \le B)$. Therefore, $\sum_{n=1}^{\infty} |\langle x_n, x\rangle|^2 = \lim \|T_n x\|^2 \le B\,\|x\|^2$. □

Lemma 1.72. *Let $\{e_n\}_{n=1}^{\infty}$ be a Riesz basis in $\mathcal{H}$. Then $\{e_n\}_{n=1}^{\infty}$ is Bessel.*

Proof. Take an arbitrary $x \in \mathcal{H}$. We only need to realise that

$$\langle e_n, x\rangle = \langle T u_n, x\rangle = \langle u_n, T^* x\rangle\,,$$

where $\{u_n\}_{n=1}^{\infty}$ is an ON basis. Parseval equality reads $\sum_{n=1}^{\infty} |\langle e_n, x\rangle|^2 = \|T^* x\|^2 \le \|T^*\|^2\,\|x\|^2$. □

Remark. It follows from the lemma and Theorem 1.66 that any Riesz sequence is Bessel.

The following two lemmas prepare us for the main result of this chapter.

Lemma 1.73. *Let T be a closed injective operator in $\mathcal{X}$. Suppose there exists $C > 0$ such that for all $x \in \mathcal{X}$ holds $C\,\|x\| \le \|T x\|$. Then* $\operatorname{ran} T$ *is closed.*

Proof. Let $\{y_n\}_{n=1}^{\infty} \subset \operatorname{ran} T$ with $y_n \to y \in \mathcal{X}$. Since T is injective, there exists a unique sequence $\{x_n\}_{n=1}^{\infty} \subset \operatorname{dom} T$ such that $(\forall n \in \mathbb{N})(y_n = T x_n)$. We have

$$\|x_n - x_m\| \le \frac{1}{C}\,\|T x_n - T x_m\|\,.$$

Since $\{T x_n\}_{n=1}^{\infty}$ is Cauchy, $\{x_n\}_{n=1}^{\infty}$ is also Cauchy, so $x_n \to x \in \mathcal{X}$ by completeness. Closedness of T and Lemma 1.30 give $x \in \operatorname{dom} T$ and $T x = y$; therefore, $\operatorname{ran} T$ is closed. □

Lemma 1.74. *Let $T \in \mathcal{B}(\mathcal{H})$ such that T^* is surjective. Then there exists $C > 0$ such that for all $x \in \mathcal{H}$ holds*

$$C\,\|x\| \le \|T x\|\,.$$

Proof. Denote $M := \{x \in \mathcal{H} \mid \|T x\| = 1\}$. For all $x \in M$ define the Riesz functional $\iota_x := \langle x, \cdot\rangle$. Since T^* is surjective, we have

$$\sup_{x\in M} |\,\langle x, y\rangle\,| = \sup_{x\in M} |\,\langle x, T^* z\rangle\,| \le \sup_{x\in M} \|T x\|\,\|z\| = \|z\| < \infty$$

for any $y \in \mathcal{H}$ with $T^* z = y$. Invoking the Banach-Steinhaus Theorem (Theorem 1.16), there exists $B > 0$ such that for all $x \in M$ holds

$$\|x\| = \|\iota_x\| \le B.$$

By homogeneity of the norm, we have $(\forall x \in \mathcal{H})(C\,\|x\| \le \|T x\|)$, where we set $C := 1/B$. □

Finally, we can formulate the desired criterion which will be of great importance later.

Theorem 1.75. *Let $\{e_n\}_{n=1}^{\infty} \subset \mathcal{H}$. Then $\{e_n\}_{n=1}^{\infty}$ is a Riesz basis if and only if it is total in $\mathcal{H}$ and Bessel and there exists $\{f_n\}_{n=1}^{\infty} \subset \mathcal{H}$ biorthogonal to $\{e_n\}_{n=1}^{\infty}$ that is also total and Bessel.*

Proof. Let us first suppose that $\{e_n\}_{n=1}^{\infty}$ is a Riesz basis. By Theorem 1.66 it is total and by Lemma 1.72 it is also Bessel. The same is true for sequence $\{f_n\}_{n=1}^{\infty}$ biorthogonal to $\{e_n\}_{n=1}^{\infty}$ which exists by Theorem 1.68.

On the contrary, suppose we have a pair of biorthogonal sequences $\{e_n\}_{n=1}^{\infty}$, $\{f_n\}_{n=1}^{\infty}$ that are total and Bessel. We will proceed in several steps.

(i) Let us define $A : \mathcal{H} \to \ell^2 : Ax := \{\langle e_n, x\rangle\}_{n=1}^\infty$. Then A is bounded. Indeed, let $\{x_k\}_{k=1}^\infty \subset \mathcal{H}$ such that $x_k \to x \in \mathcal{H}$ and $Ax_k \to y \in \ell^2$. That means $\sum_{n=1}^\infty |\langle e_n, x_k\rangle - y_n|^2 \to 0$ for $k \to \infty$. Consequently, $|\langle e_n, x_k\rangle - y_n| \to 0$ for all $n \in \mathbb{N}$. Since $x_k \to x$, by continuity we have $\langle e_n, x\rangle = y_n$, yielding $Ax = y$. A is closed according to Lemma 1.30 and by the Closed Graph Theorem (Theorem 1.32), A is bounded. Analogously, $B : \mathcal{H} \to \ell^2 : Bx := \{\langle f_n, x\rangle\}_{n=1}^\infty$ is also bounded. In conclusion, for all $x \in \mathcal{H}$ holds

$$\sum_{n=1}^\infty |\langle e_n, x\rangle|^2 \le \|A\|^2 \|x\|^2 \text{ and } \sum_{n=1}^\infty |\langle f_n, x\rangle|^2 \le \|B\|^2 \|x\|^2 .$$

(ii) Let $\{u_n\}_{n=1}^\infty$ be an arbitrary ON basis in $\mathcal{H}$. We define $T : \operatorname{span}\{e_n\}_{n=1}^\infty \to \mathcal{H}$ by $Te_n := u_n$. Let $x = \sum_{j=1}^k c_j e_j \in \operatorname{span}\{e_n\}_{n=1}^\infty$. Then applying $\langle f_l, \cdot\rangle$ to the equality gives $c_j = \left\langle f_j, x\right\rangle$. Therefore,

$$\|Tx\|^2 = \left\|\sum_{j=1}^k \left\langle f_j, x\right\rangle u_j\right\|^2 = \sum_{j=1}^k \left|\left\langle f_j, x\right\rangle\right|^2 \le \|B\|^2 \|x\|^2 .$$

Analogously, if we define $S : \operatorname{span}\{f_n\}_{n=1}^\infty \to \mathcal{H} : Sf_n := u_n$, then

$$\|Sx\|^2 \le \|A\|^2 \|x\|^2 .$$

Hence both T and S are bounded operators and they can be extended to bounded operators in $\mathcal{H}$ by virtue of Theorem 1.7 (using the supposition that $\{e_n\}_{n=1}^\infty$ and $\{f_n\}_{n=1}^\infty$ are total).

(iii) Let $x, y \in \mathcal{H}$. Then we can write them as $x = \sum_{n=1}^\infty c_n e_n$ and $y = \sum_{m=1}^\infty d_m f_m$ for some sequences $\{c_n\}_{n=1}^\infty, \{d_m\}_{m=1}^\infty \subset \mathbb{C}$. By linearity and continuity of the inner product, we have

$$\langle Tx, Sy\rangle = \left\langle \sum_{n=1}^\infty c_n u_n, \sum_{m=1}^\infty d_m u_m \right\rangle = \sum_{n=1}^\infty \overline{c_n} d_n = \left\langle \sum_{n=1}^\infty c_n e_n, \sum_{m=1}^\infty d_m f_m \right\rangle = \langle x, y\rangle .$$

Therefore, for all $x, y \in \mathcal{H}$ holds

$$0 = \langle x, (T^*S - I)y\rangle = \langle (S^*T - I)x, y\rangle ,$$

giving us $T^*S = I = S^*T$. We can immediately see that T is injective because it has a left inverse. To see that T is also surjective, note that $\operatorname{ran} T$ is dense in $\mathcal{H}$ since $\{u_n\}_{n=1}^\infty$ is total. Thanks to the surjectivity of T^* (provided by the existence of a right inverse), Lemma 1.74 and Lemma 1.73 then give $\operatorname{ran} T = \overline{\operatorname{ran} T} = \mathcal{H}$.

We have shown that $\{e_n\}_{n=1}^\infty$ is isomorphic to $\{u_n\}_{n=1}^\infty$ with a bounded isomorphism. (For concreteness $T^{-1}u_n = e_n$.) □

Later we will find ourselves in need of using Theorem 1.66. The following proposition will simplify checking whether its assumptions hold.

Proposition 1.76. *Let T be a compact invertible operator in $\mathcal{H}$. Then the generalised eigenvectors of T and T^* can be ordered to form biorthogonal sequences.*

Proof. We will prove the proposition for geometrically simple and algebraically at most double eigenvalues. By the Fredholm Alternative (Theorem 1.43), the number of generalised eigenvectors of T and T^* of a given rank is identical. Let us introduce the following notation

$$(T-\lambda I)x_\lambda = 0, \quad (T-\lambda I)\tilde{x}_\lambda = x_\lambda, \quad \left(T^* - \overline{\lambda} I\right) y_\lambda = 0, \quad \left(T^* - \overline{\lambda} I\right)\tilde{y}_\lambda = y_\lambda.$$

Clearly, if $\lambda_1 \neq \lambda_2$ are two distinct eigenvalues, then

$$\lambda_1 \left\langle y_{\lambda_2}, x_{\lambda_1}\right\rangle = \left\langle y_{\lambda_2}, T x_{\lambda_1}\right\rangle = \left\langle T^* y_{\lambda_2}, x_{\lambda_1}\right\rangle = \lambda_2 \left\langle y_{\lambda_2}, x_{\lambda_1}\right\rangle,$$

so $\left\langle y_{\lambda_2}, x_{\lambda_1}\right\rangle = 0$.

Analogously,

$$\lambda_1 \left\langle y_{\lambda_2}, \tilde{x}_{\lambda_1}\right\rangle = \left\langle y_{\lambda_2}, T\tilde{x}_{\lambda_1} - x_{\lambda_1}\right\rangle = \left\langle T^* y_{\lambda_2}, \tilde{x}_{\lambda_1}\right\rangle = \lambda_2 \left\langle y_{\lambda_2}, \tilde{x}_{\lambda_1}\right\rangle,$$

so $\left\langle y_{\lambda_2}, \tilde{x}_{\lambda_1}\right\rangle = 0$ and similarly $\left\langle \tilde{y}_{\lambda_2}, x_{\lambda_1}\right\rangle = 0$. Finally,

$$\lambda_1 \left\langle \tilde{y}_{\lambda_2}, \tilde{x}_{\lambda_1}\right\rangle = \left\langle \tilde{y}_{\lambda_2}, T\tilde{x}_{\lambda_1} - x_{\lambda_1}\right\rangle = \left\langle T^* \tilde{y}_{\lambda_2}, \tilde{x}_{\lambda_1}\right\rangle = \left\langle \overline{\lambda_2}\tilde{y}_{\lambda_2} + y_{\lambda_2}, \tilde{x}_{\lambda_1}\right\rangle = \lambda_2 \left\langle \tilde{y}_{\lambda_2}, \tilde{x}_{\lambda_1}\right\rangle$$

and thus also $\left\langle \tilde{y}_{\lambda_2}, \tilde{x}_{\lambda_1}\right\rangle = 0$.

Now we will find the non-zero inner products. First, suppose that $\nu_a(\lambda) = 1$. If $\langle y_\lambda, x_\lambda\rangle = 0$, then by Theorem 1.43 $x_\lambda \in \ker\left(T^* - \overline{\lambda} I\right)^\perp = \operatorname{ran}(T-\lambda I)$ – a contradiction with $\nu_a(\lambda) = 1$. The normalisation only depends on λ and can be chosen to satisfy $\langle y_\lambda, x_\lambda\rangle = 1$.

Let $\nu_a(\lambda) = 2$. Then

$$\langle y_\lambda, x_\lambda\rangle = \langle y_\lambda, (T-\lambda I)\tilde{x}_\lambda\rangle = \left\langle \left(T^* - \overline{\lambda} I\right) y_\lambda, \tilde{x}_\lambda\right\rangle = 0.$$

Note that $\tilde{x}_\lambda$ is determined uniquely up to addition of a multiple of x_λ and similarly for $\tilde{y}_\lambda$ and y_λ. Furthermore,

$$\langle y_\lambda, \tilde{x}_\lambda\rangle = \left\langle \left(T^* - \overline{\lambda} I\right)\tilde{y}_\lambda, \tilde{x}_\lambda\right\rangle = \langle \tilde{y}_\lambda, (T-\lambda I)\tilde{x}_\lambda\rangle = \langle \tilde{y}_\lambda, x_\lambda\rangle.$$

Once again, if $\langle y_\lambda, \tilde{x}_\lambda\rangle = 0$, then using Theorem 1.43 we reach a contradiction with $\nu_a(\lambda) = 2$.

Therefore, normalisation can be chosen so that

$$\langle y_\lambda, \tilde{x}_\lambda\rangle = \langle \tilde{y}_\lambda, x_\lambda\rangle = 1. \tag{1.10}$$

To finish the proof, we must show that it is possible to choose $\tilde{x}_\lambda$ and $\tilde{y}_\lambda$ in a way that preserves (1.10) and ensures $\langle \tilde{y}_\lambda, \tilde{x}_\lambda\rangle = 0$. Denote $\tilde{x}_\lambda^{(0)}$ and $\tilde{y}_\lambda^{(0)}$ particular solutions to the generalised eigenvector problems. Then we can choose $\tilde{x}_\lambda := \tilde{x}_\lambda^{(0)} + \alpha x_\lambda$, $\tilde{y}_\lambda := \tilde{y}_\lambda^{(0)} + \beta y_\lambda$; therefore,

$$\langle \tilde{y}_\lambda, \tilde{x}_\lambda\rangle = \left\langle \tilde{y}_\lambda^{(0)}, \tilde{x}_\lambda^{(0)}\right\rangle + \alpha + \overline{\beta},$$

since the choice does not affect (1.10). Choosing arbitrary α, β satisfying $\alpha + \overline{\beta} = -\left\langle \tilde{y}_\lambda^{(0)}, \tilde{x}_\lambda^{(0)}\right\rangle$ completes the proof. □

1.2.3 Bases in a Banach space

In this subsection, we briefly introduce the concept of a basis in a general Banach space.

Definition 1.77. Let $\{e_n\}_{n=1}^\infty \subset \mathcal{X}$. $\{e_n\}_{n=1}^\infty$ is called a *Schauder basis* in the space $\mathcal{X}$ if for every $x \in \mathcal{X}$ there exists a *unique* sequence $\{c_n\}_{n=1}^\infty \subset \mathbb{C}$ such that $x = \sum_{n=1}^\infty c_n e_n$.

Remark. It follows from the definition that if $\{e_n\}_{n=1}^\infty$ is a Schauder basis in $\mathcal{X}$, it is total in $\mathcal{X}$. Converse statement is obviously not true - to see that it suffices to consider any total sequence that contains a linearly dependent finite subset.

Proposition 1.78. *Let $\mathcal{X}$ contain a Schauder basis. Then $\mathcal{X}$ is separable.*

Proof. Let $\{e_n\}_{n=1}^\infty$ denote the basis. It is simple matter to confirm that the set

$$S := \left\{ \sum_{n=1}^{\infty} q_n e_n \mid \{q_n\}_{n=1}^\infty \subset \mathbb{Q} + \mathrm{i}\mathbb{Q} \right\}$$

is countable and dense in $\mathcal{X}$. □

Remark. While every Hilbert space has an ON basis (by Zorn's lemma every ON subset of $\mathcal{H}$ is contained in some maximal ON subset), there exist even separable Banach spaces without a Schauder basis. [7]

Remark. Theorem 1.69 shows that any Riesz basis in Hilbert space $\mathcal{H}$ is also a Schauder basis. We have the following hierarchy of embedding:

$$\text{ON basis} \prec \text{Riesz basis} \prec \text{Schauder basis} \prec \text{total set}.$$

Definition 1.79. Let X be a vector space. *Hamel basis* (or *algebraic basis*) of X is any set of linearly independent vectors in X that spans the whole X.

Remark. Note that a Schauder basis in $\mathcal{X}$ is not necessarily a Hamel basis. In fact, equivalence holds if and only if $\mathcal{X}$ is finite-dimensional. A reason behind this is that every Hamel basis in an infinite-dimensional Banach space is uncountable (while Schauder bases are countable by definition). Indeed, suppose $\{e_n\}_{n=1}^\infty$ is a Hamel basis of $\mathcal{X}$. Denote $X_n := \operatorname{span}\{e_1, \dots, e_n\}$. Then $\mathcal{X} = \cup_{n=1}^\infty X_n$. Since X_n is a finite-dimensional subspace of a complete space, it is closed. Additionally, $(X_n)^\circ = \emptyset$ because X_n is a proper subspace of $\mathcal{X}$. We have managed to write $\mathcal{X}$ as a countable union of nowhere dense sets which is a contradiction with the Baire Category Theorem (specifically Corollary 1.15).

1.3 Sobolev spaces and the Dirichlet Laplacian

When dealing with differential operators, one has to be especially cautious concerning their domains. Natural structures that often arise are L^2 functions whose several first derivatives are also in L^2. These spaces are commonly called *Sobolev spaces*. Here we will show their natural construction and prove their basic properties on bounded intervals. Throughout this section, Ω denotes an open subset of $\mathbb{R}^d$, $d \in \mathbb{N}$.

It is worth noting that Sobolev spaces work with weak derivatives in the following sense.

Definition 1.80. Let ψ, $\eta \in L^1_{\mathrm{loc}}(\Omega)$. Let $\alpha \in \mathbb{N}_0^n$. η is said to be the α-th *weak partial derivative* of ψ if for all $\varphi \in C_0^\infty(\Omega)$ holds

$$\langle \psi, D^\alpha \varphi \rangle = \int_\Omega \overline{\psi} D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega \overline{\eta} \varphi = (-1)^{|\alpha|} \langle \eta, \varphi \rangle .$$

We denote the weak derivative by $\eta =: D^\alpha \psi$.

Remark. For convenience, we use the L^2 inner product notation $\langle \cdot, \cdot \rangle$ in the definition although ψ and η do not necessarily have to be in $L^2(\Omega)$. The opposite inclusion, however, holds.

Theorem 1.81. *Let Ω be bounded and $1 \leq p \leq q \leq \infty$. Then $L^q(\Omega) \subset L^p(\Omega)$.*

Proof. The statement is trivial for $p = q$ or $q = \infty$. For $1 \leq p < q < \infty$ let $\psi \in L^q(\Omega)$. Hölder inequality for functions $|\psi|^p$ and 1 with dual exponents $\frac{q}{p}$ and $\frac{q}{q-p}$ reads

$$\int_\Omega |\psi|^p \leq \left(\int_\Omega |\psi|^q\right)^{p/q} \left(\int_\Omega 1\right)^{1-p/q} .$$

Therefore, $\|\psi\|_p < +\infty$ and $\psi \in L^p(\Omega)$. □

Corollary 1.82. $L^2(\Omega) \subset L^1_{loc}(\Omega)$.

Proof. Since any L^2 function is square integrable also locally, we can assume without loss of generality that Ω is bounded. Then the statement follows from Theorem 1.81. □

Definition 1.83. We define the *Sobolev space*

$$W^{k,p}(\Omega) := \left\{\psi \in L^p(\Omega) \mid (\forall \alpha \in \mathbb{N}_0^n, |\alpha| \leq k)(\exists D^\alpha \psi \in L^p(\Omega)\right\} .$$

Remark. It follows from the theory of distributions that if there exists a weak derivative of a function, it is defined uniquely almost everywhere, i.e. it is defined uniquely as an element of L^1_{loc}.

Furthermore, the space $W^{k,p}(\Omega)$ can be equipped with norm $\|\psi\|_{k,p} := \sum_{|\alpha|\leq k} \|D^\alpha \psi\|_p$ under which it becomes a Banach space.

Remark. In the special case $p = 2$, we denote $W^{k,2}(\Omega) =: H^k(\Omega)$. It is thought of as a Hilbert space endowed with the inner product $\langle \phi, \psi \rangle := \sum_{|\alpha|\leq k} \langle D^\alpha \phi, D^\alpha \psi \rangle$. In the case $k = 1$, the corresponding norm is called the *energy norm* on $H^1(\Omega)$.

An important subspaces of $H^k(\Omega)$ can be derived by completing C_0^∞ in the Sobolev space norm.

Definition 1.84. Let $k \in \mathbb{N}$. We define the following inner product on C_0^∞:

$$\langle \phi, \psi \rangle_1 := \sum_{|\alpha|\leq k} \langle D^\alpha \phi, D^\alpha \psi \rangle .$$

Then we define the space $H_0^k(\Omega)$ as the completion of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|_1$ induced by this inner product.

Proposition 1.85. $H_0^k(\Omega) \subset H^k(\Omega)$.

Proof. Let $\psi \in H_0^k(\Omega)$. That means there exists a corresponding sequence $\{\psi_n\}_{n=1}^\infty \subset C_0^\infty$ that is Cauchy in the norm $\|\cdot\|_1$. Therefore, $\{\psi_n\}_{n=1}^\infty$ is Cauchy in L^2 norm with limit function ψ. Additionally, $\{D^\alpha \psi_n\}_{n=1}^\infty$ is also Cauchy in $L^2(\Omega)$ for any $\alpha \in \mathbb{N}_0^d$, $|\alpha| \leq k$. Let $\eta_\alpha \in L^2(\Omega)$ denote its limit. Take any $\varphi \in C_0^\infty(\Omega)$. By continuity of the inner product, we have

$$\langle \eta_\alpha, \varphi \rangle = \lim \langle D^\alpha \psi_n, \varphi \rangle = \lim (-1)^{|\alpha|} \langle \psi_n, D^\alpha \varphi \rangle = (-1)^{|\alpha|} \langle \psi, D^\alpha \varphi \rangle ,$$

meaning η_α is the weak α-th derivative of ψ. To conclude, $D^\alpha \psi \in L^2(\Omega)$. □

Remark. For $\Omega = \mathbb{R}^d$ the spaces can be shown to be identical.

We will now proceed to find a natural domain for the Dirichlet Laplace operator and show its expression using Sobolev spaces. Note that the approach can be replicated for the Laplace operator on R^d. Let Ω be a region (i.e. non-empty connected open set).

We start with the operator $\tilde{H} := -\Delta \equiv \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$ with $\operatorname{dom} \tilde{H} := C_0^\infty(\Omega) \subset\subset L^2(\Omega)$. This domain clearly satisfies the Dirichlet boundary conditions $\psi|_{\partial\Omega} = 0$. Integrating by parts, it is simple matter to confirm that $\tilde{H}$ is symmetric. Furthermore, $\left\langle \psi, \tilde{H}\psi \right\rangle = \|\nabla\psi\|^2 \geq 0$, thus $\tilde{H}$ is accretive and, in particular, bounded from below. Using the Friedrichs extension (Subsection 1.1.6), we obtain a self-adjoint operator H that is associated to the closure of the sesquilinear form generated by $\tilde{H}$.

In accordance with Definition 1.50 and Definition 1.84, we have $\|\psi\|_{\tilde{h}} = \|\psi\|_1 = \sqrt{\|\psi\|^2 + \|\nabla\psi\|^2}$. Therefore, since $\operatorname{dom} h$ is the completion of $C_0^\infty(\Omega)$ under $\|\cdot\|_{\tilde{h}}$, we have $\operatorname{dom} h = H_0^1(\Omega)$. For sequence $\{\psi_n\}_{n=1}^\infty$ that is Cauchy in norm $\|\cdot\|_1$, with the limit function ψ, it holds that $h[\psi] = \lim \tilde{h}[\psi_n]$. Indeed, we can proceed analogously to the proof of Proposition 1.85:

Let $\omega \in (L^2(\Omega))^d$ denote the limit function of $\{\nabla\psi_n\}_{n=1}^\infty$. Then by continuity we have for any $\varphi \in (C_0^\infty(\Omega))^d$:

$$\langle \omega, \varphi \rangle = \lim \langle \nabla\psi_n, \varphi \rangle = -\lim \langle \psi_n, \nabla\varphi \rangle .$$

Hence, $\omega = \nabla\psi$ in the weak (Sobolev) sense and $h[\psi] = \|\nabla\psi\|^2$.

Additionally, (1.3) gives us

$$\operatorname{dom} H = \left\{ \psi \in H_0^1(\Omega) \mid (\exists \eta \in L^2(\Omega))(\forall \phi \in H_0^1(\Omega))(h(\phi, \psi) = \langle \phi, \eta \rangle) \right\}, \quad H\psi = \eta.$$

Consider any $\psi \in \operatorname{dom} H$. Then for all $\varphi \in C_0^\infty(\Omega) \subset H_0^1(\Omega)$, we have

$$\langle \varphi, \eta \rangle = h(\varphi, \psi) = \langle \nabla\varphi, \nabla\psi \rangle = -\langle \Delta\varphi, \psi \rangle .$$

Therefore, $\eta = -\Delta\psi$ in the weak sense. Consequently, the desired domain is

$$\operatorname{dom} H = \left\{ \psi \in H_0^1(\Omega) \mid \Delta\psi \in L^2(\Omega) \right\}. \tag{1.11}$$

Interestingly, there is another related norm we can define on $C_0^\infty(\Omega)$ and obtain a space of functions by completing it under this norm.

Definition 1.86. We define the following inner product on $C_0^\infty(\Omega)$:

$$\langle \phi, \psi \rangle_2 := \sum_{j=1}^{d} \left\langle \frac{\partial\phi}{\partial x_j}, \frac{\partial\psi}{\partial x_j} \right\rangle \equiv \langle \nabla\phi, \nabla\psi \rangle .$$

The space $\dot{H}_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|_2$ induced by this inner product.

Remark. In general, $H_0^1(\Omega) \subset \dot{H}_0^1(\Omega)$ and equality does not need to hold – for example a constant non-zero function on $\mathbb{R}$ is a member of the latter but not a member of the first. However, we will show further in the text that on bounded intervals in $\mathbb{R}$, the spaces are identical.

The one-dimensional case

In this paragraph, we will investigate the simple case $d = 1$, i.e. the Laplace operator on an interval. It turns out that in this setting, H^1 functions have a continuous representative. First, we will introduce a handy characterisation of $H^1(a, b)$, where (a, b) is a compact interval, that is due to [8], Lemma 7.1.1.

Lemma 1.87. *Let $-\infty < a < b < +\infty$. Then $\psi \in H^1(a,b)$ if and only if there exist $c \in \mathbb{C}$ and $\psi \in L^2(a,b)$ such that*

$$\psi(x) = c + \int_a^x \eta(y)\, dy \tag{1.12}$$

for all $x \in (a,b)$. Additionally, η is the weak derivative of ψ.

Remark. By continuity of an H^1 function we mean that there exists a continuous function in its equivalence class.

Proof. First suppose that ψ is of the form (1.12). Then for any $\varphi \in C_0^\infty(a,b)$ we have:

$$\begin{aligned}\langle \psi, \varphi' \rangle &= \overline{c} \int_a^b \varphi'(x)\, \mathrm{d}x + \int_a^b \int_a^x \overline{\eta}(y)\, \mathrm{d}y \varphi'(x)\, \mathrm{d}x = \int_a^b \int_y^b \overline{\eta}(y)\varphi'(x)\, \mathrm{d}x\, \mathrm{d}y \\ &= -\int_a^b \overline{\eta}(y)\varphi(y)\, \mathrm{d}y = -\langle \eta, \varphi \rangle .\end{aligned}$$

Therefore, $\psi \in H^1(a,b)$ with η as its weak derivative.

On the other hand, suppose $\psi \in H^1(a,b)$ and let $\eta := \psi'$ in the weak sense. Define $\omega(x) := \int_a^x \eta(y)\, \mathrm{d}y$. Similarly as above, we can calculate

$$\langle \omega, \varphi' \rangle = -\langle \eta, \varphi \rangle .$$

In other words, $\omega \in H^1(a,b)$ and $\omega' = \eta = \psi'$. It follows from the theory of distributions that $\psi - \omega = c \in \mathbb{C}$. Hence $\psi = \omega + c$ is of the form (1.87). □

Corollary 1.88. *(Newton's Formula for H^1). Let $\psi \in H^1(a,b)$, $x, y \in (a,b)$. Then*

$$\psi(x) - \psi(y) = \int_y^x \psi'(t)\, dt.$$

Theorem 1.89. *Let Ω be an open subset of $\mathbb{R}$, $\psi \in H^1(\Omega)$. Then ψ is continuous.*

Proof. It is obviously sufficient to consider Ω to be a bounded interval, let us denote it by $\Omega =: (a,b)$. For any $x, y \in (a,b)$, using Corollary 1.88, we have

$$|\psi(x) - \psi(y)| = \left| \int_y^x \psi'(t)\, \mathrm{d}t \right| \le \left\| \psi' \right\| |x-y|^{1/2}$$

thanks to the Cauchy-Schwarz inequality. □

Consider the bounded interval (a,b). We aim to show the equality of $H_0^1(a,b)$ and $\dot{H}_0^1(a,b)$. In doing so, we will make use of Theorem 1.39.

We will closely observe the free Laplace operator H with Dirichlet boundary conditions in the space $L^2(a,b)$. We recall that by (1.11), its domain is $\operatorname{dom} H = \left\{ \psi \in H_0^1(a,b) \mid \psi'' \in L^2(a,b) \right\} = H_0^1(a,b) \cap H^2(a,b)$. First, let us find its point spectrum.

Let $H\psi = \lambda\psi$ for $\psi \in \operatorname{dom} H = H_0^1(a,b) \cap H^2(a,b)$. That is $-\psi'' = \lambda\psi$. Since H is accretive, $\lambda \le 0$ only leads to trivial solution. For $\lambda > 0$, we have $\psi(x) = A \sin\left(\sqrt{\lambda}(x-a)\right)$ with $\sin\left(\sqrt{\lambda}(b-a)\right) = 0$. Therefore

$$\lambda_n = \frac{n^2\pi^2}{(b-a)^2}, \quad \psi_n(x) = \sqrt{\frac{2}{b-a}} \sin\left(\frac{n\pi}{b-a}(x-a)\right), \quad n \in \mathbb{N}, \tag{1.13}$$

after normalisation.

It is shown in the appendix (Section A.1) that $\{\psi_n\}_{n=1}^{\infty}$ is an ON basis of $L^2(a,b)$. By the Spectral Theorem, H has purely discrete spectrum. Furthermore, we already know that H is bounded from below. As a consequence, we can use Theorem 1.39 that, combined with (1.13), gives us

$$\frac{\pi^2}{(b-a)^2} = \lambda_1 = \inf_{\psi \in \operatorname{dom} H \setminus \{0\}} \frac{\langle \psi, H\psi \rangle}{\|\psi\|^2} = \inf_{\psi \in \operatorname{dom} H \setminus \{0\}} \frac{\|\psi'\|^2}{\|\psi\|^2}.$$

Since $C_0^\infty \subset \operatorname{dom} H$, we can simply conclude in the following proposition.

Proposition 1.90. *(Poincaré Inequality). Let* $-\infty < a < b < +\infty$, $\psi \in C_0^\infty(a,b)$. *Then*

$$\left\|\psi'\right\|^2 \geq \frac{\pi^2}{(b-a)^2} \|\psi\|^2 .$$

Finally, we are able to prove the desired equality.

Proposition 1.91. *Let* $-\infty < a < b < +\infty$. *Then* $H_0^1(a,b) = \dot{H}_0^1(a,b)$.

Proof. It suffices to show that norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $C_0^\infty(a,b)$ are equivalent. Let $\psi \in C_0^\infty(a,b)$. Clearly

$$\|\psi\|_2^2 = \left\|\psi'\right\|^2 \leq \|\psi\|^2 + \left\|\psi'\right\|^2 = \|\psi\|_1^2 .$$

On the other hand, by the Poincaré Inequality (Proposition 1.90), we have

$$\|\psi\|_2^2 = \frac{1}{2}\left\|\psi'\right\|^2 + \frac{1}{2}\left\|\psi'\right\|^2 \geq \frac{1}{2}\left\|\psi'\right\| + \frac{\pi^2}{2(b-a)^2}\|\psi\|^2 \geq \frac{1}{2}\min\left\{1, \frac{\pi^2}{(b-a)^2}\right\}\|\psi\|_1^2 . \qquad \square$$

Chapter 2

Spectral analysis of the wave equation

2.1 Wave equation

We consider the wave equation on the interval $[0, \pi]$ with frictional resistance $b : [0, \pi] \to \mathbb{C}$ in the form

$$u_{tt} - u_{xx} + b(x)u_t = 0, \tag{2.1}$$

where $u : [0, \pi] \times [0, \infty) \to \mathbb{C}$ is the wave evolution function. Here b can be understood in the sense of distributions. Consistently with our model, we suppose u satisfies Dirichlet boundary conditions with

$$u(0, t) = u(\pi, t) = 0, \quad t \in [0, \infty). \tag{2.2}$$

Furthermore, let the initial data $u(\cdot, 0) \in H_0^1(0, \pi)$ and $u_t(\cdot, 0) \in L^2(0, \pi)$. If we define $\psi := \begin{pmatrix} u \\ u_t \end{pmatrix}$, equation (2.1) with boundary conditions (2.2) can be transformed to

$$A\psi = \psi_t, \tag{2.3}$$

where

$$A := \begin{pmatrix} 0 & I \\ \partial_{xx} & -b \end{pmatrix}$$

$$\operatorname{dom} A = \left(H_0^1(0, \pi) \cap H^2(0, \pi)\right) \times H_0^1(0, \pi) \tag{2.4}$$

for b regular (non-distributional).

Theory of semigroups of linear operators gives us the solution

$$\psi(x, t) = \exp(tA)\, \psi(x, 0),$$

provided that the semigroup $\exp(tA)$ exists. Its existence and behaviour strongly depends on spectral properties of A – which is why they are the central point of interest of this thesis. Namely, we will be interested in the special damping $b(x) = b\delta(x - a)$ for some $b \in \mathbb{C}$ and $a \in (0, \pi)$.

2.2 Model and motivation

Our mathematical setting will be the Hilbert space $\mathcal{H} := H_0^1(0, \pi) \times L^2(0, \pi)$ which is reasonable considering (2.1) and (2.2). We endow $\mathcal{H}$ with the natural inner product $\langle \psi, \phi \rangle := \left\langle \psi_1', \phi_1' \right\rangle + \langle \psi_2, \phi_2 \rangle$.

Technically speaking, the first term is the inner product inherited from $\dot{H}_0^1(0,\pi)$, but we have shown in Proposition 1.91 that this space is identical with $H_0^1(0,\pi)$.

In [2], Bamberger, Rauch, and Taylor proposed a model for playing harmonics on stringed instruments. The motivation was to find the correct touch, i.e. the strength one has to apply to a string in order to damp the non-harmonic modes the fastest. They considered a sequence of highly localized, smooth, and compactly supported friction coefficients $b_n : [0,\pi] \to [0,\infty)$ such that

$$\lim_{n\to\infty} \int_0^\pi b_n(x)\varphi(x)\,\mathrm{d}x = b\varphi(a),$$

where $b \in [0,\infty)$ and φ is an arbitrary continuous function in $[0,\pi]$. They proceeded to show that if ψ_n are solutions of

$$A_n\psi_n = (\psi_n)_t, \quad A_n = \begin{pmatrix} 0 & I \\ \partial_{xx} & -b_n \end{pmatrix}, \quad \psi_n(x,0) = \psi_0(x),$$

then they converge in $\mathcal{H}$ to the solution ψ of

$$A(a,b)\psi = \psi_t, \quad A(a,b) = \begin{pmatrix} 0 & I \\ \partial_{xx} & 0 \end{pmatrix}, \quad \psi(x,0) = \psi_0(x),$$

$$\operatorname{dom} A(a,b) = \left\{\psi \in \left(H_0^1(0,\pi) \cap H^2(0,a) \cap H^2(a,\pi)\right) \times H_0^1(0,\pi) \mid \psi_1'(a+) - \psi_1'(a-) = b\psi_2(a)\right\}. \quad (2.5)$$

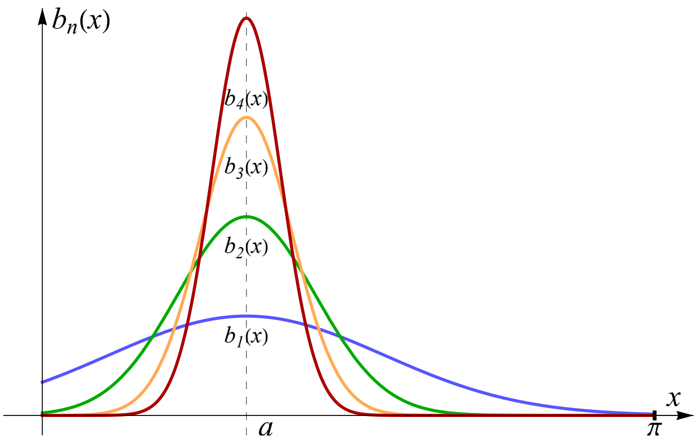


Figure 2.1: An illustration of the functions b_n approximating the δ function.

Furthermore, they showed that $\exp(tA_n)$ converges to $\exp(tA)$ in operator norm on any compact time interval and strongly for $t \geq 0$.

The semigroups $\exp(tA_n)$, $\exp(tA)$ can be constructed by the Lumer-Phillips Theorem ([10], Theorem 4.3) since both A_n and $A(a,b)$ with $b > 0$ are maximally dissipative. The result is due to [2].

Proposition 2.1. *[2]. Operator $A(a,b)$ defined above is maximally dissipative.*

Remark. It is a simple extension to show the following for a general $b \in \mathbb{C}$:

1. If $\operatorname{Re} b \geq 0$, operator $A(a,b)$ is maximally dissipative.
2. If $\operatorname{Re} b \leq 0$, operator $A(a,b)$ is maximally accretive.
3. In particular, if $\operatorname{Re} b = 0$, operator $A(a,b)$ is skew-adjoint as shown further.

Among other results stated in [2], Bamberger, Rauch, and Taylor observed that harmonic modes (i.e. eigenfunctions corresponding to imaginary eigenvalues of A) are only elicited for $\frac{a}{\pi}$ rational – a result that we will replicate and further expand on. In doing so, we will partially follow the footsteps of Cox and Henrot [3]. First of all, we shall investigate the simplest case – the undamped wave equation.

2.3 Spectrum of the undamped wave equation

Let us consider the simple case $b = 0$. We have $u_{tt} - u_{xx} = 0$ with Dirichlet boundary conditions $u(0,t) = u(\pi, t)$ for $t \geq 0$. We define the following linear operator in $\mathcal{H}$:

$$A := \begin{pmatrix} 0 & I \\ \partial_{xx} & 0 \end{pmatrix}, \quad \operatorname{dom} A := \left(H_0^1(0,\pi) \cap H^2(0,\pi)\right) \times H_0^1(0,\pi). \tag{2.6}$$

Solving the eigenvalue problem $A\begin{pmatrix} f \\ g \end{pmatrix} = \lambda \begin{pmatrix} f \\ g \end{pmatrix}$ gives us $g = \lambda f$ and $f'' = \lambda g = \lambda^2 f$. Therefore, we obtain the ODE

$$f'' - \lambda^2 f = 0,$$

with boundary conditions $f(0) = f(\pi) = 0$. Denoting $\lambda = \mathrm{i}\kappa$, we have $f(x) = A\cos(\kappa x) + B\sin(\kappa x)$. The boundary conditions force $A = 0$ and $\sin(\kappa\pi) = 0$, thus $\kappa =: n \in \mathbb{Z} \backslash \{0\}$. We have found the point spectrum of A with respective eigenfunctions to be

$$\lambda_n = \mathrm{i}n, \quad f_n = B_n \sin(nx), \quad g_n = \mathrm{i}n f_n, \quad n \in \mathbb{Z} \setminus \{0\}. \tag{2.7}$$

Normalising the eigenfunctions, we get

$$1 = \left\| \begin{pmatrix} f_n \\ g_n \end{pmatrix} \right\|^2 = \int_0^\pi \left|{f_n}'\right|^2 + |g_n|^2 = n^2 B_n^2 \int_0^\pi \cos^2(nx) + \sin^2(nx)\,\mathrm{d}x = \pi n^2 B_n^2 \implies B_n = \frac{1}{n\sqrt{\pi}}.$$

The eigenfunctions are of the form

$$\psi_n = \begin{pmatrix} f_n \\ g_n \end{pmatrix} = \frac{1}{n\sqrt{\pi}} \sin(nx) \begin{pmatrix} 1 \\ in \end{pmatrix}.$$

Proposition 2.2. *Operator A defined above is skew-adjoint.*

Proof.

(i) Showing that A is skew-symmetric is simple. Let $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \operatorname{dom} A$. Integrating by parts and exploiting the boundary conditions, we have:

$$\langle \psi, A\phi \rangle = \int_0^\pi \overline{\psi_1'}\phi_2' + \overline{\psi_2}\phi_1'' = -\int_0^\pi \overline{\psi_1''}\phi_2 + \overline{\psi_2'}\phi_1' = -\langle A\psi, \phi \rangle.$$

(ii) It remains to prove that $\operatorname{dom} A = \operatorname{dom} A^*$. Since A is skew-symmetric, clearly $\operatorname{dom} A \subset \operatorname{dom} A^*$. Consider $\phi = \left(\phi_1, \phi_2\right) \in \operatorname{dom} A^*$. There exists exactly one $\eta \in \mathcal{H}$ such that for all $\psi \in \operatorname{dom} A$ holds $\langle \phi, A\psi \rangle = \langle \eta, \psi \rangle$. That means

$$\left\langle \phi_1', \psi_2' \right\rangle + \left\langle \phi_2, \psi_1'' \right\rangle = \left\langle \eta_1', \psi_1' \right\rangle + \langle \eta_2, \psi_2 \rangle \tag{2.8}$$

for all $\psi_1 \in H_0^1(0,\pi) \cap H^2(0,\pi)$, $\psi_2 \in H_0^1$.

First, we set $\psi_1 = 0$. Since $\phi \in \mathcal{H}$, we already know that $\phi_1 \in H_0^1(0,\pi)$. (2.8) now reads

$$\left\langle \phi_1', \psi_2' \right\rangle = \langle \eta_2, \psi_2 \rangle$$

for all $\psi_2 \in H_0^1(0,\pi)$ – in particular for all $\psi_2 \in C_0^\infty(0,\pi)$. Consequently, $-\left\langle \phi_1, \psi_2'' \right\rangle = \langle \eta_2, \psi_2 \rangle$ which by definition means that $-\eta_2 \in L^2(0,\pi)$ is the weak second derivative of ϕ_1; therefore, $\phi_1 \in H^2(0,\pi)$.

Second, let $\psi_2 = 0$. Equation (2.8) transforms to

$$-\langle \phi_2, H\psi_1 \rangle = \left\langle \phi_2, \psi_1'' \right\rangle = \left\langle \eta_1', \psi_1' \right\rangle = \langle \eta_1, H\psi_1 \rangle$$

for all $\psi_1 \in H_0^1(0,\pi) \cap H^2(0,\pi) = \operatorname{dom} H$, where H is the Dirichlet Laplacian on $(0,\pi)$ defined in Section 1.3. Recall that by (1.13), $0 \notin \sigma(H)$, therefore H is invertible with $\operatorname{ran} H = L^2(0,\pi)$. As a simple conclusion, $\phi_2 = \eta_1 \in H_0^1(0,\pi)$. □

It is shown in the appendix (Section A.2) that $\{\psi_n\}_{n=1}^\infty$ form an ON basis in $\mathcal{H}$; it thus follows from the Spectral Theorem that we have found the whole spectrum of A.

2.4 The eigenvalue problem

Here we attack the general problem of analysing the spectrum of $A(a,b)$. Note that by [2], Theorem 2, $A(a,b) \equiv A$ has compact resolvent and thus discrete spectrum for all $a \in (0,\pi)$ and $b \in \mathbb{C}$. Therefore, we will only consider the eigenvalue problem:

$$A\psi = \lambda\psi. \tag{2.9}$$

Denoting $\psi = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{H}$ for clarity, this becomes

$$v = \lambda u, \quad u'' = \lambda v + \lambda b \delta_a u,$$

or eliminating v and expressing the δ function as a transmission condition:

$$u'' - \lambda^2 u = 0, \quad u(0) = u(\pi) = 0, \quad u(a) := u(a+) = u(a-), \quad u'(a+) - u'(a-) = \lambda b u(a), \tag{2.10}$$

where we made use of the domain of A. Note that by Theorem 1.89, u is continuous on $(0,\pi)$ and u' is continuous everywhere except for a. The equation is solvable using elementary methods.

Additionally, recall that by the remark following Proposition (2.1), if $\operatorname{Re} b > 0$, then A is dissipative and Proposition 1.46 yields $\sigma_p(A) \subset \{z \in \mathbb{C} \mid \operatorname{Re} z \le 0\}$. Throughout this section, we consider $b \in \mathbb{R}$.

The harmonic spectrum

First, we investigate the imaginary eigenvalues and the eigenfunctions corresponding to them. Let $\lambda := \mathrm{i}\kappa$ for some $\kappa \in \mathbb{R}$. From (2.10), we derive $u'' + \kappa^2 u = 0$. It is immediately obvious that $\kappa = 0$ does not yield a solution – the transmission condition reduces to continuous first derivative and the only linear function satisfying the Dirichlet boundary conditions is zero. Therefore, we can assume the solution to be of the form

$$u(x) = \begin{cases} A_1 \sin(\kappa x) + B_1 \cos(\kappa x), & \text{for } 0 < x < a, \\ A_2 \sin(\kappa(\pi - x)) + B_2 \cos(\kappa(\pi - x)), & \text{for } a < x < \pi. \end{cases}$$

The Dirichlet boundary conditions force $B_1 = B_2 = 0$. The transmission condition at a now reads

$$-\kappa A_2 \cos(\kappa(\pi - a)) - \kappa A_1 \cos(\kappa a) = \mathrm{i}\kappa b A_1 \sin(\kappa a).$$

Since the left-hand side is real, this gives $\sin(\kappa a) = 0$ and from continuity at a also $\sin(\kappa(\pi - a)) = 0$; therefore $\sin(\kappa\pi) = 0$. Using continuity again, the solution consequently becomes

$$u(x) = \sin(\kappa x)$$

up to a scalar multiple.

Now we must determine κ and find additional conditions for a. We know that

$$\sin(\kappa a) = \sin(\kappa(\pi - a)) = \sin(\kappa\pi) = 0. \tag{2.11}$$

Hence, $\kappa \in \mathbb{Z} \setminus \{0\}$ and $\kappa a \in \pi\mathbb{Z} \setminus \{0\}$. This means that *$a$ is a rational multiple of π*. Let us denote

$$a = \frac{p}{q}\pi \tag{2.12}$$

for $p, q \in \mathbb{N}$, $p \perp q$ (p and q are coprime integers). Finally, we have $\kappa_n = nq$, $u_n(x) = \sin(nqx) n \in \mathbb{Z} \setminus \{0\}$. We can summarise our findings concerning the harmonic spectrum.

Proposition 2.3. *Let $b \in \mathbb{R}$. $A(a, b)$ has imaginary eigenvalues if and only if $a = \frac{p}{q}\pi$, p, $q \in \mathbb{N}$. In such case, the eigenvalues and (normalised) eigenfunctions are*

$$\lambda_n = inq, \quad \psi_n = \frac{1}{nq\sqrt{\pi}} \sin(nqx) \begin{pmatrix} 1 \\ inq \end{pmatrix}, \quad n \in \mathbb{Z} \setminus \{0\}. \tag{2.13}$$

Remark. The normalisation is simple:

$$1 = \|\psi_n\|^2 = \left\| \begin{pmatrix} u_n \\ inqu_n \end{pmatrix} \right\|^2 = \int_0^\pi |u_n'|^2 + n^2q^2 |u_n|^2 = n^2q^2A_n^2 \int_0^\pi \cos^2(nqx) + \sin^2(nqx)\,\mathrm{d}x = \pi n^2 q^2 A_n^2,$$

so $A_n = \frac{1}{nq\sqrt{\pi}}$.

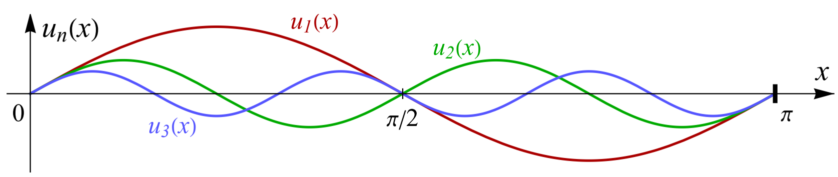


Figure 2.2: Functions $u_n(x) = \frac{1}{2n\sqrt{\pi}} \sin(2nx)$ generating the harmonic eigenfunctions of $A(\pi/2, b)$.

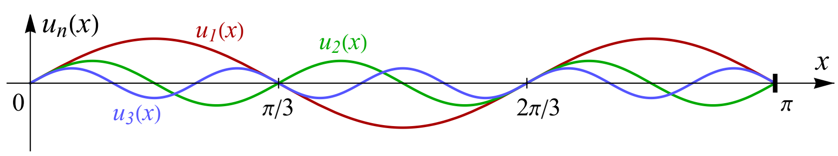


Figure 2.3: Functions $u_n(x) = \frac{1}{3n\sqrt{\pi}} \sin(3nx)$ generating the harmonic eigenfunctions of $A(\pi/3, b)$.

Remark. Note that the harmonic eigenfunctions of $A(p\pi/q, b)$ are zero at $a = p\pi/q$ as illustrated by Figure 2.2 and Figure 2.3.

The non-harmonic spectrum

We can assume a solution of (2.10) of the form

$$u(x) = \begin{cases} A_1 \sinh(\lambda x) + B_1 \cosh(\lambda x), & \text{for } 0 < x < a, \\ A_2 \sinh(\lambda(\pi - x)) + B_2 \cosh(\lambda(\pi - x)), & \text{for } a < x < \pi. \end{cases}$$

The Dirichlet boundary conditions force $B_1 = B_2 = 0$. Continuity at a then gives us $A_1 = \sinh(\lambda(\pi - a))$ and $A_2 = \sinh(\lambda a)$. Up to a scalar multiple we have

$$u(x) = \begin{cases} \sinh(\lambda(\pi - a)) \sinh(\lambda x), & \text{for } 0 < x < a, \\ \sinh(\lambda a) \sinh(\lambda(\pi - x)), & \text{for } a < x < \pi, \end{cases} \tag{2.14}$$

and $\psi = \begin{pmatrix} u \\ \lambda u \end{pmatrix}$. The transmisson condition at a tells us

$$-\lambda \sinh(\lambda a) \cosh(\lambda(\pi - a)) - \lambda \sinh(\lambda(\pi - a)) \cosh(\lambda a) = \lambda b \sinh(\lambda a) \sinh(\lambda(\pi - a)).$$

After algebraic manipulation, this is equivalent to

$$S(\lambda; a, b) := \sinh(\lambda\pi) + b \sinh(\lambda a) \sinh(\lambda(\pi - a)) = 0. \tag{2.15}$$

Eigenvalues λ are the roots of this transcendental equation except for 0 which is not an eigenvalue (the corresponding function is identically zero).

Remark. After arriving at this result in [3], Cox and Henrot noticed a handy expression for the function $S(\lambda; a, b)$ for $a = \frac{p}{q}\pi$:

$$\begin{aligned} S\left(\lambda; \frac{p\pi}{q}, b\right) &= \frac{e^{\lambda\pi} - e^{-\lambda\pi}}{2} + b \frac{e^{\lambda p\pi/q} - e^{-\lambda p\pi/q}}{2} \cdot \frac{e^{\lambda(q-p)\pi/q} - e^{-\lambda(q-p)\pi/q}}{2} \\ &= -\frac{1}{4} e^{\lambda\pi} \left[(2 - b) e^{-2\lambda\pi p} + b e^{-2\lambda\pi/q} + b e^{-2\lambda\pi(q-p)/q} - (2 + b) \right] = -\frac{1}{4} e^{\lambda\pi} P_b(e^{-2\lambda\pi/q}), \end{aligned} \tag{2.16}$$

where

$$P_b(z) := (2 - b) z^q + b z^p + b z^{q-p} - (2 + b) \tag{2.17}$$

is a polynomial of degree q (for $b \neq 2$). If we denote its roots by $\zeta_k = |\zeta_k| e^{i\theta_k}$, we arrive at the countable system of eigenvalues

$$\lambda_{1,n} = iqn, \tag{2.18}$$

$$\lambda_{k,n} = -\frac{q}{2\pi} \left(\ln |\zeta_k| + i(\theta_k + 2\pi n) \right), \quad k \in \{2, \ldots, q\}, \; n \in \mathbb{Z}. \tag{2.19}$$

where we choose the convention $\zeta_1 = 1$, $\lambda_{1,n} = iqn$. The following paragraph provides a brief analysis of properties of P_b and $S(\cdot; a, b)$.

The characteristic function

Throughout this paragraph, by the fact that a complex number λ is a double root of a function f holomorphic on some neighbourhood of λ we mean that it is a zero of both f and f' and it is not a zero of f''.

Proposition 2.4. *Let $b \in \mathbb{R}$. The functions $S(\cdot; a, b)$ and P_b satisfy:*

1. *1 is a simple root of P_b.*
2. *If $b > 0$, all other roots of P_b are outside of the unit circle. If $b = 0$, all other roots lie on the unit circle. If $b < 0$, all other roots are inside the unit circle.*
3. *All roots of $S(\cdot; a, b)$ (and thus also of P_b) are at most double.*
4. *All double roots of P_b are real.*
5. *Let $a = \frac{p\pi}{q}$.. If λ is a double root of $S(\cdot; a, b)$, then $\operatorname{Im} \lambda \equiv \frac{q}{2} \bmod q$.*
6. *$S\left(\overline{\lambda}; a, b\right) = \overline{S(\lambda; a, b)}$ for all $\lambda \in \mathbb{C}$.*
7. *If $a = \frac{p}{q}\pi$, then $S\left(\lambda + \mathrm{i}q; \frac{p\pi}{q}, b\right) = (-1)^q S\left(\lambda; \frac{p\pi}{q}, b\right)$ for all $\lambda \in \mathbb{C}$.*

Proof.

1. It is obvious that 1 is a root of P_b and $P_b'(1) = 2q \neq 0$.
2. $\lambda_{1,n} := inq$ are eigenvalues of $A(p\pi/q, b)$ corresponding to the root 1. We already know that these are the only imaginary eigenvalues and for all other lie in the strict left half-plane for $b > 0$. Therefore, from (2.19) we can see that $|\zeta_k| > 1$ for $k > 1$. Analogously, if $b < 0$, $A(a, b)$ is accretive and its eigenvalues lie within the right half-plane. For $b = 0$, the statement is evident.
3. We will follow the approach of [3]. For the purpose of differentiation, it is useful to note that

$$S(\lambda; a, b) = \sinh(\lambda\pi) + b\sinh(\lambda a)\sinh(\lambda(\pi - a)) = \sinh(\lambda\pi) + \frac{b}{2}\cosh(\lambda\pi) - \frac{b}{2}\cosh(\lambda(\pi - 2a)).$$

We have

$$S'(\lambda; a, b) = \pi\cosh(\lambda\pi) + \pi\frac{b}{2}\sinh\lambda\pi - (\pi - 2a)\frac{b}{2}\cosh(\lambda(\pi - 2a))$$

and for the second derivative

$$\begin{aligned} S''(\lambda; a, b) &= \pi^2\sinh(\lambda\pi) + \pi^2\frac{b}{2}\cosh\lambda\pi - (\pi - 2a)^2\frac{b}{2}\cosh\lambda(\pi - 2a) \\ &= \pi^2 S(\lambda; a, b) + 2ab(\pi - a)\cosh(\lambda(\pi - 2a)). \end{aligned} \tag{2.20}$$

Suppose $S(\lambda_0; a, b) = S'(\lambda_0; a, b) = S''(\lambda_0; a, b) = 0$ for some $\lambda_0 \in \mathbb{C}$. Then clearly

$$\cosh(\lambda_0(\pi - 2a)) = 0 \implies \lambda_(\pi - 2a) = \mathrm{i}\pi\left(n + \frac{1}{2}\right) \implies \lambda_0 = \frac{\mathrm{i}\pi\left(n + \frac{1}{2}\right)}{\pi - 2a}.$$

At the same time

$$\begin{aligned} 0 = S(\lambda_0; a, b) &= \sinh(\lambda_0\pi) + \frac{b}{2}\cosh(\lambda_0\pi) = \sinh\left(\frac{\mathrm{i}\pi^2\left(n + \frac{1}{2}\right)}{\pi - 2a}\right) + \frac{b}{2}\cosh\left(\frac{\mathrm{i}\pi^2\left(n + \frac{1}{2}\right)}{\pi - 2a}\right) \\ &= \mathrm{i}\sin\left(\frac{\pi^2\left(n + \frac{1}{2}\right)}{\pi - 2a}\right) + \frac{b}{2}\cos\left(\frac{\pi^2\left(n + \frac{1}{2}\right)}{\pi - 2a}\right) =: \mathrm{i}\sin\omega + \frac{b}{2}\cos\omega. \end{aligned}$$

Therefore, both $\sin\omega$ and $\cos\omega$ must be zero – a contradiction.

4. Note that
$$P_b'(z) = q(2-b)z^{q-1} + pbz^{p-1} + (q-p)bz^{q-p-1}.$$
Let $z_0 \in \mathbb{C}$ such that $P_b(z_0) = P_b'(z_0) = 0$. A simple calculation gives
$$0 = qP_b(z_0) - z_0P_b'(z_0) = (q-p)bz_0^p + pbz_0^{q-p} - q(2+b) \implies tz_0^{q-p} + (1-t)z_0^p = \frac{b+2}{b}, \quad (2.21)$$
where we denoted $t := p/q$. Additionally
$$0 = z_0^{1-q}P_b'(z_0) = q(2-b) + pbz_0^{p-q} + (q-p)bz_0^{-p} \implies \frac{b-2}{b} = \frac{t}{z_0^{q-p}} + \frac{1-t}{z_0^p}. \quad (2.22)$$
Now if we put $\alpha := z_0^{q-p}$, $\beta := z_0^p$, equations (2.21) and (2.22) yield
$$t\alpha + (1-t)\beta = \frac{b+2}{b}, \quad t\frac{\overline{\alpha}}{|\alpha|^2} + (1-t)\frac{\overline{\beta}}{|\beta|^2} = \frac{b-2}{2}.$$
Taking the imaginary part of both equations gives us
$$\begin{pmatrix} t & 1-t \\ \frac{t}{|\alpha|^2} & \frac{1-t}{|\beta|^2} \end{pmatrix} \begin{pmatrix} \operatorname{Im}\alpha \\ \operatorname{Im}\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$
For $|\alpha| \neq |\beta|$, this means $\alpha, \beta \in \mathbb{R}$. If $|\alpha| = |\beta|$, then $|z_0| = 1$, which is a contradiction with statement 2. of this proposition.
To finish the argument, since z_0^p and z_0^{q-p} are real and $p \perp q-p$, z_0 must also be real.

5. If $\lambda = \frac{p\pi}{q}$ is a double root of $S(\cdot; a, b)$, then using (2.16) and (2.19) it is of the form $\lambda = -\frac{q}{2\pi}\ln\zeta$, where ζ is a double root of P_b. (Note that $\lambda = 0$ is not a double root.) We know that ζ is real from the previous statement. By statement 1. and Descartes' rule of signs, P_b cannot have positive double roots, so
$$\lambda = -\frac{q}{2\pi}\ln|\zeta| - \mathrm{i}\frac{q}{2} - \mathrm{i}qn, \quad n \in \mathbb{Z}.$$

6. The statement is immediately evident from the definition of $S(\cdot; a, b)$.

7. A simple calculation from the definition. □

Remark. It follows directly from statement 6. of Proposition 2.4 that with $b \in \mathbb{R}$, $\sigma_p(A(a,b))$ is symmetric with respect to the real axis. For $a = \frac{p\pi}{q}$, the imaginary part of the spectrum is also q-periodic by statement 7. and combined this means that within a period the spectrum is symmetric with respect to $\frac{\mathrm{i}q}{2}$.

The following theorem clarifies the relationship between multiplicity of λ as a root of $S(\cdot; a, b)$ and its algebraic multiplicity as an eigenvalue of $A(a, b)$.

Theorem 2.5. *Let $\lambda \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{R}$. Then λ is an eigenvalue of $A(a,b)$ if and only if $S(\lambda; a, b) = 0$. Furthermore, the algebraic multiplicity of the eigenvalue λ is equal to its multiplicity as a root of $S(\cdot; a, b)$.*

Proof. We have already discussed the first part of the statement (imaginary eigenvalues in the case $a \in \pi\mathbb{Q}$ correspond to root 1 of P_b).

We will show that the existence of a generalised eigenvector (other then the original eigenvector) corresponding to λ is equivalent to the satisfaction of $S(\lambda; a, b) = S'(\lambda; a, b) = 0$. The generalised eigenfunction equation reads

$$(A - \lambda I)\tilde{\psi} = \psi, \tag{2.23}$$

where $A\psi = \lambda\psi$. Denoting $\psi = \begin{pmatrix} u \\ v \end{pmatrix}$, $\tilde{\psi} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}$, equation (2.23) transforms to

$$-\lambda\tilde{u} + \tilde{v} = u, \quad \tilde{u}'' - \lambda\tilde{v} = v.$$

Eliminating v and taking into account the domain of A, gives us the following ordinary differential equation:

$$\tilde{u}'' - \lambda^2\tilde{u} = 2\lambda u, \quad \tilde{u}(0) = \tilde{u}(\pi) = 0, \quad \tilde{u}(a) := \tilde{u}(a+) = \tilde{u}(a-), \quad \tilde{u}'(a+) - \tilde{u}'(a-) = b\tilde{u}(a) + \lambda b\tilde{u}(a), \tag{2.24}$$

with $\tilde{v} = \lambda\tilde{u} + u$.

First, we must show that imaginary eigenvalues are algebraically simple. Recall that $a = \frac{p\pi}{q}$ and by (2.13), we have

$$\lambda_n = inq, \quad \psi_n = \sin(nqx)\begin{pmatrix} 1 \\ inq \end{pmatrix}, \quad n \in \mathbb{Z} \setminus \{0\}.$$

We omit the normalisation for the sake of simplicity. It is useful to keep in mind that u is zero at 0, a, and π. (2.24) provides

$$\tilde{u}'' + n^2q^2\tilde{u} = 2inq\sin(nqx).$$

We can take advantage of the special right-hand side to arrive at

$$\tilde{u}(x) = -ix\cos(nqx) + \begin{cases} A_1\sin(nqx) + B_1\cos(nqx), & \text{for } 0 < x < a, \\ A_2\sin(nqx) + B_2\cos(nqx), & \text{for } a < x < \pi. \end{cases}$$

Dirichlet boundary conditions at 0 and π force $B_1 = 0$ and $B_2 = \mathrm{i}\pi$, respectively. However, continuity at a gives us $B_2 = 0$ – a contradiction.

Second, we consider λ with non-zero real part. From (2.14), we can similarly as above obtain

$$\tilde{u}'' - \lambda^2\tilde{u} = 2\lambda\begin{cases} \sinh(\lambda(\pi - a))\sinh(\lambda a), & \text{for } 0 < x < a, \\ \sinh(\lambda a)\sinh(\lambda(\pi - x)), & \text{for } a < x < \pi. \end{cases}$$

This is again a special right-hand side for both intervals, so using elementary tools we derive

$$\tilde{u}(x) = \begin{cases} x\sinh(\lambda(\pi - a))\cosh(\lambda x) + A_1\sinh(\lambda x) + B_1\cosh(\lambda x), & \text{for } 0 < x < a, \\ -x\cosh(\lambda a)\cosh(\lambda(\pi - x)) + A_2\sinh(\lambda(\pi - x)) + B_2\cosh(\lambda(\pi - x)), & \text{for } a < x < \pi. \end{cases}$$

Dirichlet boundary conditions at 0 and π force $B_1 = 0$ and $B_2 = \pi\sinh(\lambda a)$, respectively. Choosing $A_1 = (\pi - a)$ for convenience then yields $A_2 = a\cosh(\lambda a)$. Now for the transmisson condition at a: after algebraic manipulation, we can write

$$\begin{aligned} \tilde{u}'(a+) - \tilde{u}'(a-) &= -\sinh(\lambda\pi) - \lambda\pi\cosh(\lambda\pi) \\ b\lambda\tilde{u}(a) + bu(a) &= b\lambda\left[a\sinh(\lambda(\pi - a))\cosh(\lambda a) + (\pi - a)\cosh(\lambda(\pi - a))\sinh(\lambda a)\right] \\ &\quad + b\sinh(\lambda(\pi - a))\sinh(\lambda a). \end{aligned} \tag{2.25}$$

Also note that

$$
\begin{aligned}
S(\lambda; a, b) &= \sinh(\lambda\pi) + b\sinh(\lambda a)\sinh(\lambda(\pi - a)) \\
S'(\lambda; a, b) &= \pi\cosh(\lambda\pi) + ba\cosh(\lambda a)\sinh(\lambda(\pi - a)) + b(\pi - a)\sinh(\lambda a)\cosh(\lambda(\pi - a)).
\end{aligned} \tag{2.26}
$$

Now supposing $S(\lambda; a, b) = 0$ and substituting that into one of the equations of (2.25), satisfaction of the transmission equation becomes equivalent to $S'(\lambda; a, b) = 0$, which was to be proven.

The non-existence of a third generalised eigenvector can be shown in a similar manner. □

Corollary 2.6. *Let $b \in \mathbb{R}$. All eigenvalues of $A(a, b)$ are geometrically simple and of algebraic multiplicity at most 2.*

Suppose that $b > 0$. If $\lambda \in \sigma_p\left(A\left(\frac{p\pi}{q}, b\right)\right)$ and $\nu_a(\lambda) = 2$, then $\operatorname{Im}\lambda \equiv \frac{q}{2} \bmod q$. In such case, the generalised eigenvectors are of the form $\tilde{\psi} = \begin{pmatrix} \tilde{u} \\ u + \lambda\tilde{u} \end{pmatrix}$, where u is the function (2.14) and

$$
\tilde{u}(x) = \begin{cases} x\sinh(\lambda(\pi - a))\cosh(\lambda x) + (\pi - a)\sinh(\lambda x), & \text{for } 0 < x < a, \\ -x\cosh(\lambda a)\cosh(\lambda(\pi - x)) + a\cosh(\lambda a)\sinh(\lambda(\pi - x)), & \text{for } a < x < \pi. \end{cases} \tag{2.27}
$$

Proof. It is only left to prove that the geometric multiplicity of any eigenvalue is 1; rest of the statement is a consequence of Theorem 2.5 and Proposition 2.4. In doing so, we will once again follow the path of [2].

The general form of the eigenvalue problem $A\psi = \lambda\psi$ is (2.10). If both $\psi = \begin{pmatrix} u \\ v \end{pmatrix}$ and $\phi = \begin{pmatrix} w \\ z \end{pmatrix}$ are solutions of this equation for some $\lambda \in \mathbb{C}$, then it is possible to create a linear combination of them such that $\alpha u'(0) + \beta w'(0) = 0$. Let $\eta \equiv \begin{pmatrix} f \\ g \end{pmatrix} = \alpha\psi + \beta\phi$. The function f is a solution to the second-order ODE $f'' - \lambda^2 f = 0$ with initial conditions $f(0) = f'(0) = 0$, so $f = 0$ on $(0, a)$. The transmission condition $f'(a+) - f'(a-) = \lambda b f(a) = 0$ gives also $f(a) = f'(a) = 0$. Consequently, $f = 0$ on $(0, \pi)$ and $g = \lambda f = 0$ implying $\eta = 0$. Therefore, ψ and ϕ are linearly dependent. □

2.5 Disposition of the spectrum

In this section, we will briefly introduce some results of Cox and Henrot [3] concerning the localization of the spectrum of A. We remind that they only consider the case $b > 0$.

Proposition 2.7. *([3], Theorem 2.1). Let $\lambda = \alpha + \mathrm{i}\beta \in \mathbb{C}$ with $\alpha, \beta \in \mathbb{R}$. The following hold:*

1. *If $b < 2$, then $\sigma_p(A) \subset \mathbb{C} \setminus \mathbb{R}$. If $b > 2$, then there exists a unique $\alpha_0 = \alpha_0(b) \in \sigma_p(A) \cap \mathbb{R}$. The function $\alpha_0 : (2, +\infty) \to (-\infty, 0)$ is strictly increasing with $\lim_{b\to 2+} \alpha_0(b) = -\infty$, $\lim_{b\to+\infty} \alpha_0(b) = 0$.*

2. *Let $a = \frac{p\pi}{q}$, $q \in 2\mathbb{N}$, $b < 2$. Then $\alpha_1(b) + \frac{\mathrm{i}q}{2} \in \sigma_p(A)$ with α_1 being strictly decreasing and $\lim_{b\to 2-} = -\infty$.*

3. *Let $a = \frac{p\pi}{q}$, $q \in 2\mathbb{N} - 1$, $p \in 2\mathbb{N}$, $b > 2$. Then $\alpha_2(b) + \frac{\mathrm{i}q}{2} \in \sigma_p(A)$ with α_2 being strictly increasing and $\lim_{b\to 2+} \alpha_2(b) = -\infty$, $\lim_{bo+\infty} \alpha_2(b) = 0$.*

4. *Let $a = \frac{p\pi}{q}$, $q, p \in 2\mathbb{N}-1$. Then there exists a unique $b^* < 2$ and $\alpha(b^*)$ such that $\alpha(b^*) + \frac{\mathrm{i}q}{2} \in \sigma_p(A)$ is an algebraically double eigenvalue of A. For $b > b^*$, $\alpha_\pm(b) + \frac{\mathrm{i}q}{2} \in \sigma_p(A)$, where $\alpha_- < \alpha(b^*) < \alpha_+(b)$ and $\lim_{b\to 2-} \alpha_-(b) = -\infty$ and $\lim_{b\to+\infty} \alpha_+(b) = 0$.*

Proof. We will prove statements 1. and 4., statements 2. and 3. can be proven analogously.

1. For $\lambda \equiv \alpha \in \mathbb{R}$, equality $S(\alpha; a, b) = 0$ can be written as

$$b_0(\alpha) = -\frac{\sinh(\alpha\pi)}{\sinh(\alpha a)\sinh(\alpha(\pi - a))}.$$

We will show that b_0 is strictly increasing in α by showing this strict monotonicity for

$$f_0(\alpha; a) := -\frac{1}{b_0(\alpha)} = \frac{\sinh(\alpha a)\sinh(\alpha(\pi - a))}{\sinh(\alpha\pi)}.$$

Since f_0 is odd in α, we can suppose $\alpha > 0$ and from the symmetry $f_0(\alpha; a) = f_0(\alpha, \pi - a)$ it is enough to consider $a \in (0, \pi/2)$. Differentiating with respect to α, we obtain

$$f'(\alpha; a) = a\cosh(2\alpha a) - a\sinh(2\alpha a)\coth(\alpha\pi) + \pi\frac{\sinh^2(\alpha a)}{\sinh^2(\alpha\pi)}.$$

Multiplying by $\sinh^2(\alpha\pi)$, we get

$$\sinh^2(\alpha\pi)f'(\alpha; a) = a\sinh(\alpha\pi)\sinh(\alpha(\pi - 2a)) + \pi\sinh^2(\alpha a) > 0$$

for $a \in (0, \pi/2)$.

Calculating the limits, we have

$$\lim_{\alpha\to-\infty} b_0(\alpha) = \lim_{\alpha\to+\infty} 2\frac{e^{\alpha\pi} - e^{-\alpha\pi}}{(e^{\alpha a} - e^{-\alpha a})(e^{\alpha(\pi-a)} - e^{\alpha(\pi-a)})} = 2,$$
$$\lim_{\alpha\to 0-} b_0(\alpha) = \lim_{\alpha\to 0+} \frac{\alpha\pi + O(\alpha^3)}{\alpha^2 a(\pi - a) + O(\alpha^3)} = +\infty.$$

Therefore, $b_0 : (-\infty, 0) \to (2, +\infty) : \alpha \mapsto b_0(\alpha)$ is a strictly increasing bijection and, consequently, we can define $\alpha_0 : (2, +\infty) \to (-\infty, 0) : b \mapsto \alpha_0(b)$ which is also a strictly increasing bijection.

4. Let $p = 2m - 1$, $q = 2n + 1$. It is routine to calculate

$$S\left(\alpha + \frac{iq}{2}, a, b\right) = i(-1)^n\left[\cosh(\alpha\pi) + b\cosh(\alpha a)\sinh(\alpha(\pi - a))\right].$$

Analogously as above we can rewrite $S(\lambda; a, b) = 0$ as

$$b_3(\alpha) = -\frac{\cosh(\alpha\pi)}{\cosh(\alpha a)\sinh(\alpha(\pi - a))}.$$

Once again, we will investigate the function

$$f_3(\alpha; a) := -\frac{1}{b_3(\alpha)} = \frac{\cosh(\alpha a)\sinh(\alpha(\pi - a))}{\cosh(\alpha\pi)}.$$

Clearly $f_3(0; a) = 0$ and $\lim_{\alpha\to-\infty} f_3(\alpha; a) = -1/2$. Furthermore, simple estimates for $\alpha < 0$ lead us to

$$f_3(\alpha; a) = \cosh^2(\alpha a)\tanh(\alpha\pi) - \sinh(\alpha a)\cosh(\alpha a) > -\cosh(\alpha a)[\cosh(\alpha a) + \sinh(\alpha a)] = -\frac{e^{2\alpha a} + 1}{2} > -1.$$

Therefore, $-1 < f_3(\alpha; a) < 0$. Differentiating with respect to α, we obtain an expression of the form

$$f_3'(\alpha; a) = \frac{g_3(\alpha; a)}{\cosh^2(\alpha\pi)},$$

where g_3 satisfies

$$g_3(0; a) = \pi - a > 0, \quad \lim_{\alpha \to -\infty} g_3(\alpha; a) = -\infty.$$

Additionally, differentiating g_3 with respect to α, we have

$$g_3'(\alpha; a) = -2a(\pi - a)\cosh(\alpha\pi)\sinh(\alpha(\pi - 2a)) > 0$$

for $\alpha < 0$ and $a \in (0, \pi/2)$. Therefore, f_3' has a zero α_a that is the global minimizer for f_3. The corresponding point $b^* := -\frac{1}{f_3(\alpha_a; a)}$ is the desired number. For $b < b^*$, there is no solution α; for $b = b^*$, there is one double root; and for $b > b^*$, there are two simple roots. □

Remark. The proposition fully characterises algebraic multiplicities of eigenvalues for $a = p\pi/q$, $b > 0$ up to the determination of b^* and the case $b = 2$. For even q (and naturally odd p) as well as for odd q and even p, all eigenvalues are algebraically simple (note that by Corollary 2.6, we have analysed all possible double roots within a period). For q and p odd, there is a special choice of damping $b = b^* < 2$ which provides double eigenvalues λ_n^* with $\operatorname{Im}\lambda_n^* = q/2 + nq$. We will show the properties of the spectrum on two simple examples.

Example. Let $a = \pi/2$, i.e. $p = 1, q = 2$. Then $P_b(z) = (2-b)z^2 + 2bz - (2+b)$. We obtain the two roots:

$$\zeta_1 = 1, \quad \zeta_2 = \frac{b+2}{b-2}.$$

Using (2.19), the resulting eigenvalues are

$$\lambda_{1,n} = 2in, \quad \lambda_{2,n} = -\frac{1}{\pi}\ln\left|\frac{b+2}{b-2}\right| - i(\Theta(2-b) + 2n), \tag{2.28}$$

where Θ is the Heaviside step function.

In [3], Cox and Henrot show that the correct touch in the sense of Bamberger, Rauch, and Taylor [2] is the minimizer b^* of the spectral abscissa of the operator induced by $A(a, b)$ on the orthogonal complement of the closed linear span of the harmonic eigenfunctions:

$$\mu(a, b) := \max\{\operatorname{Re}\lambda \mid \lambda \in \sigma(A(a, b)),\ \lambda \notin i\mathbb{R}\}.$$

Using the form (2.28), we can clearly see that for $a = \pi/2$ the answer is $b^* = 2$ in accordance with [2].

Example. Let $a = \pi/3$, i.e. $p = 1, q = 3$. Then $P_b(z) = (2-b)z^3 + bz^2 + bz - (2+b)$. Dividing by $z - 1$ (1 is always a root of P_b), we obtain

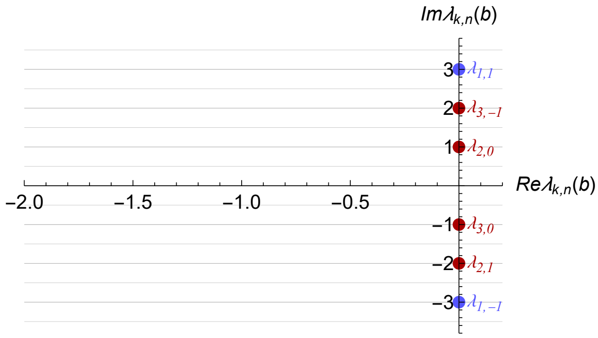
$$\frac{P_b(z)}{z-1} = (2-b)z^2 + 2z + b + 2 \implies \zeta = \frac{1 \pm \sqrt{b^2 - 3}}{b-2}.$$

So for $b^* = \sqrt{3}$ we obtain the double root $\zeta = -2 - \sqrt{3}$ corresponding to the double eigenvalues

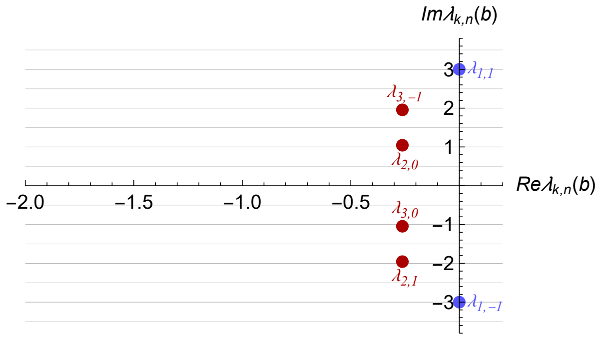
$$\lambda_n^* = -\frac{3}{2\pi}\ln\left(2 - \sqrt{3}\right) - i\left(\frac{3}{2} + 3n\right)$$

using (2.19). Note that this is indeed the correct touch in the sense of Bamberger, Rauch, and Taylor [2], as can be seen in Figure 2.4.

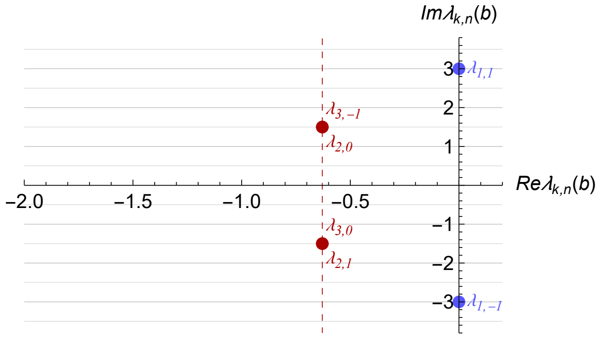
Figure 2.4 also shows that for $b > \sqrt{3} = b^*$, the part of the spectrum $\lambda_{2,n}$ exits to complex infinity along the lines $i(3/2 + 3n\pi)$, while the other part $\lambda_{3,n}$ begins its return to the imaginary axis along the same lines. For $b = 2$, we observe an abrupt change of spectral properties as the part approaching infinity disappears. This change will be quantitatively analysed in Chapter 3. For $b > 2$, the eigenvalues $\lambda_{2,n}$ return from infinity along different trajectories. Finally, as b approaches $+\infty$, we obtain a new Dirichlet boundary condition at $\pi/3$, effectively creating two separate strings of lengths $\pi/3$ and $2\pi/3$.



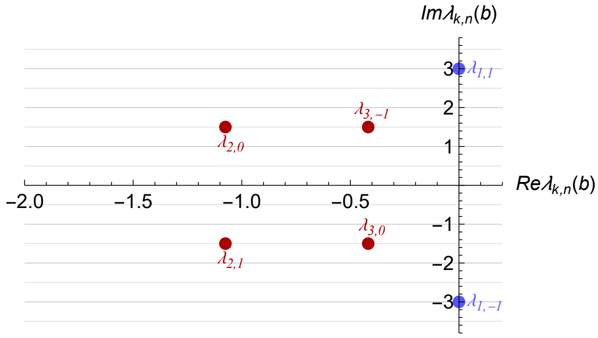
(a) $b = 0$ – the skew-adjoint case.



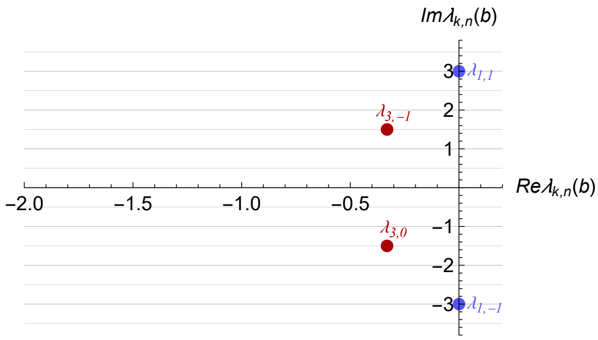
(b) $b = 1$.



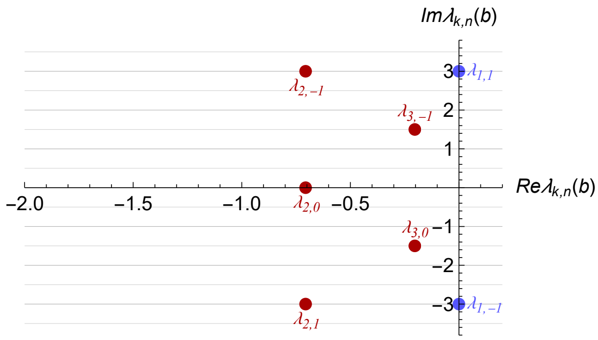
(c) $b = \sqrt{3}$ – the optimal damping.



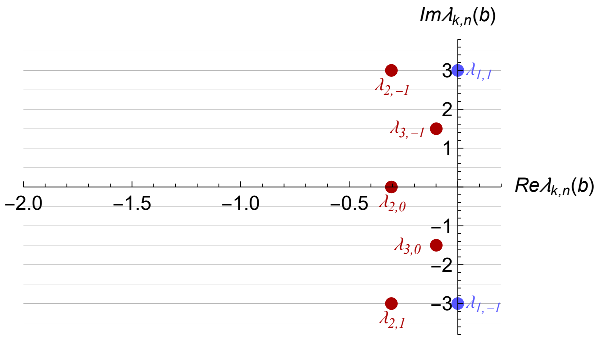
(d) $b = \sqrt{3} + 0.1$.



(e) $b = 2$ – the loss of a component of the spectrum.



(f) $b = 2.7$.



(g) $b = 5$.

Figure 2.4: Behaviour of the eigenvalues for $a = \pi/3$ with respect to the damping parameter b.

In [3], Cox and Henrot provide further description of the spectrum's disposition. However, they omit the discussion of the limiting case $b = 2$. Here we will complete the argument. It can be seen from the proof of statement 1. of Proposition 2.7 that for $b = 2$, there are no real eigenvalues – the function b_0 is a strictly increasing bijection $(-\infty, 0) \to (2, +\infty)$. Similar cases arise in statements 2. and 3. – if $a = p\pi/q$ with q even or q odd and p even, no eigenvalues of the form $\alpha + \mathrm{i}q/2$ exist. A different situation occurs with q and p being both odd. Since $b^* < 2$, there exists a simple root α for $b = 2$.

Corollary 2.8. *Let $a = p\pi/q$, $b = 2$. Then all eigenvalues of $A(a, b)$ are algebraically simple.*

Chapter 3

Root vectors of the wave operator

In this chapter, we will concern ourselves with the generalised eigenvectors (or *root vectors*) of the operator $A(a, b)$ defined in the previous chapter. We will be mainly studying if they are total and whether or not they form a Riesz basis in $\mathcal{H}$. We once again note that research on the topic has been done in [3]. Here, we will provide a detailed construction of the adjoint operator, show that harmonic and non-harmonic eigenfunctions are orthogonal, and finally, using the approach that led Cox and Henrot to show that the root vectors are total for a positive $b \neq 2$, we will disprove it for $b = \pm 2$.

Perhaps most importantly, Cox and Henrot only dealt with the case $a = p\pi/q$. We will provide the proof of the Riesz basis property (for $b \neq 2$) and its absence (for $b = 2$) for a general placement a of the damping. Additionally, instead of restricting ourselves to $b > 0$, we consider $b \in \mathbb{R}$ or even $b \in \mathbb{C}$ where possible.

First, recall that in (2.19), we introduced the notation $\lambda_{k,n}$ for the eigenvalues of $A(p\pi/q, b)$ expressed using roots ζ_k of the polynomial P_b. We consider $\zeta_1 = 1$, so eigenvalues $\lambda_{1,n} = inq$ correspond to eigenfunctions in (2.13):

$$\psi_{1,n} \equiv \begin{pmatrix} u_{1,n} \\ \lambda_{1,n} u_{1,n} \end{pmatrix} = \frac{1}{nq\sqrt{\pi}} \sin(nqx) \begin{pmatrix} 1 \\ inq \end{pmatrix}, \quad n \in \mathbb{Z} \setminus \{0\}.$$

For $k \geq 2$, we have the non-harmonic eigenfunctions from (2.14):

$$\psi_{k,n} = \begin{pmatrix} u_{k,n} \\ \lambda_{k,n} u_{k,n} \end{pmatrix}, \quad u_{k,n}(x) = \begin{cases} \sinh(\lambda_{k,n}(\pi - a)) \sinh(\lambda_{k,n} x), & \text{for } 0 < x < a, \\ \sinh(\lambda_{k,n} a) \sinh(\lambda_{k,n}(\pi - x)), & \text{for } a < x < \pi. \end{cases}$$

If $\lambda_{k+1,n} = \lambda_{k,n}$, we have the additional generalised eigenvector (2.27):

$$\psi_{k+1,n} \equiv \tilde{\psi}_{k,n} = \begin{pmatrix} \tilde{u}_{k,n} \\ u_{k,n} + \lambda_{k,n} \tilde{u}_{k,n} \end{pmatrix},$$

$$\tilde{u}_{k,n}(x) = \begin{cases} x \sinh(\lambda_{k,n}(\pi - a)) \cosh(\lambda_{k,n} x) + (\pi - a) \sinh(\lambda_{k,n} x), & \text{for } 0 < x < a, \\ -x \cosh(\lambda_{k,n} a) \cosh(\lambda_{k,n}(\pi - x)) + a \cosh(\lambda_{k,n} a) \sinh(\lambda_{k,n}(\pi - x)), & \text{for } a < x < \pi. \end{cases}$$

Proposition 3.1. *Let $a = p\pi/q$, $k \geq 2$. Then $\psi_{1,n} \perp \psi_{k,m}$ for all $n \in \mathbb{Z} \setminus \{0\}$ and $m \in \mathbb{Z}$.*

Proof.

(i) Suppose that $\psi_{k,m}$ is an eigenvector of A. We have

$$\begin{aligned} \langle \psi_{1,n}, \psi_{k,m} \rangle &= \left\langle u'_{1,n}, u'_{k,m} \right\rangle + \overline{\lambda_{1,n}} \lambda_{k,m} \langle u_{1,n}, u_{k,m} \rangle = -\left\langle u_{1,n}, u''_{k,m} \right\rangle + \overline{\lambda_{1,n}} \lambda_{k,m} \langle u_{1,n}, u_{k,m} \rangle \\ &= \lambda_{k,m} \left(\overline{\lambda_{1,n}} - \lambda_{k,m} \right) \langle u_{1,n}, u_{k,m} \rangle, \end{aligned}$$

where we used equation (2.10) to express $u''_{k,m}$ and made use of the fact that $u_{1,n}$ is zero at 0, a, and π.

Integrating by parts the other way round yields

$$\langle\psi_{1,n},\psi_{k,m}\rangle = -\left\langle u''_{1,n}, u_{k,m}\right\rangle + \overline{\lambda_{1,n}}\lambda_{k,m}\langle u_{1,n}, u_{k,m}\rangle = \overline{\lambda_{1,n}}\left(\lambda_{k,m}-\overline{\lambda_{1,n}}\right)\langle u_{1,n}, u_{k,m}\rangle .$$

Note that we used the fact that $u'_{1,n}(a+) - u'_{1,n}(a-) = u_{1,n}(a) = 0$. Combining these identities, we get

$$\left(\lambda_{k,m}-\overline{\lambda_{1,n}}\right)\left(\lambda_{k,m}+\overline{\lambda_{1,n}}\right)\langle u_{1,n}, u_{k,m}\rangle = 0 \implies \langle u_{1,n}, u_{k,m}\rangle = 0$$

and consequently $\langle\psi_{1,n},\psi_{k,m}\rangle = 0$.

(ii) Consider the generalised eigenvector $\tilde{\psi}_{k,m}$. Then

$$\begin{aligned}\left\langle\psi_{1,n},\tilde{\psi}_{k,m}\right\rangle &= \left\langle u'_{1,n},\tilde{u}'_{k,m}\right\rangle + \overline{\lambda_{1,n}}\langle u_{1,n}, \lambda_{k,m}\tilde{u}_{k,m}+u_{k,m}\rangle = -\left\langle u_{1,n}, u''_{k,m}\right\rangle + \overline{\lambda_{1,n}}\lambda_{k,m}\langle u_{1,n},\tilde{u}_{k,m}\rangle \\ &= \lambda_{k,m}\left(\overline{\lambda_{1,n}}-\lambda_{k,m}\right)\langle u_{1,n},\tilde{u}_{k,m}\rangle - 2\lambda\langle u_{1,n}, u_{k,m}\rangle = \lambda_{k,m}\left(\overline{\lambda_{1,n}}-\lambda_{k,m}\right)\langle u_{1,n},\tilde{u}_{k,m}\rangle ,\end{aligned}$$

where we used the generalised eigenvector equation (2.23) and the already proven case for an eigenvector. The rest of the proof is identical to the first part. □

3.1 Adjoint operator

In this section, we will construct the adjoint operator $A^*(a,b)$. Recall the definition (2.5) of $A(a,b)$:

$$A(a,b) = \begin{pmatrix} 0 & I \\ \partial_{xx} & 0\end{pmatrix},$$
$$\operatorname{dom} A(a,b) = \left\{\psi \in \left(H^1_0(0,\pi)\cap H^2(0,a)\cap H^2(a,\pi)\right)\times H^1_0(0,\pi) \mid \psi'_1(a+) - \psi_1(a-) = b\psi_2(a)\right\}.$$

We want to find the operator A^* in $\mathcal{H}$ such that

$$\operatorname{dom} A^* = \{\phi\in\mathcal{H} \mid (\exists\eta\in\mathcal{H})(\forall\psi\in\operatorname{dom}A)(\langle\phi,A\psi\rangle = \langle\eta,\psi\rangle)\},\quad A^*\phi=\eta.$$

Note that $\langle\phi,A\psi\rangle = \left\langle\phi'_1,\psi'_2\right\rangle + \left\langle\phi_2,\psi''_1\right\rangle$. The proof that $\phi_1 \in H^1_0(0,\pi)\cap H^2(0,a)\cap H^2(a,\pi)$ and $\phi_2 \in H^1_0(0,\pi)$ is analogous to the proof of Proposition 2.2. We will derive the matrix form of the operator and transmission condition at a. Integrating by parts, we obtain

$$\begin{aligned}\langle\phi,A\psi\rangle &= -\psi_2(a)\left(\overline{\phi'_1(a+)}-\overline{\phi'_1(a-)}\right) - \overline{\phi_2(a)}\left(\psi'_1(a+)-\psi'_1(a-)\right) - \left\langle\phi''_1,\psi_2\right\rangle - \left\langle\phi'_2,\psi'_1\right\rangle \\ &= -\psi_2(a)\left(\overline{\phi'_1(a+)}-\overline{\phi'_1(a-)}\right) - \overline{\phi_2(a)}\left(\psi'_1(a+)-\psi'_1(a-)\right) + \left\langle\begin{pmatrix}0 & -I\\ -\partial_{xx} & 0\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix},\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\right\rangle.\end{aligned}$$

We used the continuity of $\psi_2 \in H^1_0(0,\pi)$ provided by Theorem 1.89. Substituting $\psi'_1(a+) - \psi'_1(a-) = b\psi_2(a)$, we arrive at

$$\psi_2(a)\left(\overline{\phi'_1(a+)}-\overline{\phi'_1(a-)}+b\overline{\phi_2(a)}\right) = 0.$$

Therefore, since $\psi(a)$ can be chosen non-zero, $\phi'_1(a+) - \phi'_1(a-) = -\overline{b}\phi_2(a)$. In summary, we have found

$$A^*(a,b) = \begin{pmatrix} 0 & -I \\ -\partial_{xx} & 0\end{pmatrix},$$
$$\operatorname{dom} A^*(a,b) = \left\{\phi \in \left(H^1_0(0,\pi)\cap H^2(0,a)\cap H^2(a,\pi)\right)\times H^1_0(0,\pi) \mid \phi'_1(a+) - \phi_1(a-) = -\overline{b}\phi_2(a)\right\}. \quad (3.1)$$

Proposition 3.2. *The adjoint operator to $A(a,b)$ is $A^*(a,b) = -A\left(a, -\overline{b}\right)$.*

Remark. From here, it can be simply observed that for $\operatorname{Re} b = 0$, operator A is skew-adjoint.

Now we will discuss the spectrum and eigenvectors of A^*. Note that the eigenvalue problem is quite similar to (2.9); if we put $\phi = \begin{pmatrix} y \\ z \end{pmatrix} \in \operatorname{dom} A^*$, then

$$A^*\phi = \kappa\phi \implies z = -\kappa y, \quad y'' = -\kappa z - \kappa\overline{b}\delta_a y.$$

Once again, we can eliminate z to obtain

$$y'' - \kappa^2 y = 0, \quad y(0) = y(\pi) = 0, \quad y(a) := y(a+) = y(a-), \quad y'(a+) - y'(a-) = \kappa\overline{b}y(a).$$

Note that this equation is identical with (2.10) (the difference being that now $z = -\kappa y$ and we have the complex conjugate of b). Thus, we must get the solutions (2.13) and (2.14) for y. For $b \in \mathbb{R}$, knowing that the spectrum of A is symmetric about the real axis, we can write $\kappa \equiv \overline{\lambda}$ and thus $y = \overline{u}$ and $z = -\overline{\lambda u}$.

As for the algebraically double eigenvalues, the generalised eigenvector equation reads

$$(A^* - \overline{\lambda}I)\tilde{\phi} = \phi \implies \tilde{y}'' - \overline{\lambda}^2\tilde{y} = 2\overline{\lambda}y, \quad \tilde{y}'(a+) - \tilde{y}'(a-) = \overline{\lambda b}\tilde{y}(a) + \overline{b}y(a), \quad \tilde{z} = -\overline{\lambda}\tilde{y} - y.$$

Taking the complex conjugates of the respective equations for $b \in \mathbb{R}$ and using the fact that $y = \overline{u}$, we arrive at the same equation as in (2.23). Therefore, $\tilde{y} = \overline{\tilde{u}}$ and $\tilde{z} = -\overline{\lambda\tilde{u}} - \overline{u}$. The existence of such solution is determined by the equation $S\left(\overline{\lambda}; a, \overline{b}\right) = S'\left(\overline{\lambda}; a, \overline{b}\right) = 0$, which is equivalent to the existence of $\tilde{\psi}$ for A.

We will summarise our findings in a proposition.

Proposition 3.3. *The spectrum of A^* is $\sigma_p(A^*) = \overline{\sigma_p(A)}$ [1] ($= \sigma_p(A)$ for $b \in \mathbb{R}$). Eigenfunction ϕ corresponding to the eigenvalue $\overline{\lambda}$ is of the form*

$$\phi = \begin{pmatrix} \overline{u} \\ -\overline{\lambda u} \end{pmatrix}, \tag{3.2}$$

where u is given by (2.13) *for imaginary eigenvalues and by* (2.14) *otherwise.*

Additionally, $\overline{\lambda}$ is an algebraically double eigenvalue of A^* if and only if λ is an algebraically double eigenvalue of A. The generalised eigenvector $\tilde{\phi}$ of A^* corresponding to $\overline{\lambda}$ is of the form

$$\tilde{\phi} = \begin{pmatrix} \overline{\tilde{u}} \\ -\overline{\lambda\tilde{u}} - \overline{u}, \end{pmatrix} \tag{3.3}$$

where $\tilde{u}$ is given by (2.27).

For $a = p\pi/q$ and $b > 0$, let $\phi_{k,n}$ denote the root vector of A^* corresponding to $\overline{\lambda_{k,n}}$.

Proposition 3.4. *Let $a = p\pi/q$. The sequences $\{\psi_{k,n}\}_{k,n}$ and $\{\phi_{k,n}\}_{k,n}$ can be ordered to be biorthogonal (after normalisation).*

Proof. Recall that by [2], Theorem 2, $A(a,b)$ has compact resolvent $A^{-1}(a,b)$. Therefore, we may apply Proposition 1.76. □

[1] The bar stands for complex conjugate, not closure.

3.2 Basis of root vectors

3.2.1 Rational placement of the damping

First, let us consider the case $a = p\pi/q$. Recall that by Theorem 1.66, the eigenvectors $\{\psi_{k,n}\}_{k,n}$ form a Riesz basis in $\mathcal{H}$ if and only if they are total and Bessel and possess a biorthogonal sequence that is also total and Bessel. The biorthogonal sequence $\{\phi_{k,n}\}_{k,n}$ was constructed in the previous section as the root vectors of the adjoint operator.

We will start by showing the Bessel equality. We remind of Lemma 1.71 that allows us to only show the following.

Proposition 3.5. *Let $a = p\pi/q$, $b \in \mathbb{R}$. Then for all $\psi \in \mathcal{H}$ holds:*

$$\sum_{k,n} \left|\langle \psi, \psi_{k,n}/\lambda_{k,n}\rangle\right|^2 < \infty, \quad \sum_{k,n} \left|\left\langle \psi, \phi_{k,n}/\overline{\lambda_{k,n}}\right\rangle\right|^2 < \infty.$$

Proof. Let us start with the harmonic eigenfunctions $\psi_{1,n}$. Since they are a subset of the ON basis (13), they are automatically Bessel. Note that $\lambda_{k,n} = \lambda_{k,0} - inq$. For $k \geq 2$, we can choose appropriate normalisation and calculate

$$\langle \psi, \psi_{k,n}/\lambda_{k,n}\rangle = \frac{1}{\lambda_{k,n}} \left\langle \psi_1', u_{k,n}'\right\rangle + \langle \psi_2, u_{k,n}\rangle . \tag{3.4}$$

We have

$$\begin{aligned}\left\langle \psi_1', u_{k,n}'/\lambda_{k,n}\right\rangle &= \sinh(\lambda_{k,n}(\pi - a)) \int_0^a \overline{\psi_1'}(x) \cosh(\lambda_{k,n} x)\, \mathrm{d}x - \sinh(\lambda_{k,n} a) \int_a^\pi \overline{\psi_1'}(x) \cosh(\lambda_{k,n}(\pi - x))\, \mathrm{d}x \\ &= (-1)^{n(q-p)} \sinh(\lambda_{k,0}(\pi - a)) \int_0^a \overline{\psi_1'}(x) \cosh(\lambda_{k,0} x - \mathrm{i}qnx)\, \mathrm{d}x \\ &\quad - (-1)^{np} \sinh(\lambda_{k,0}(\pi - a)) \int_a^\pi \overline{\psi_1'}(x) \cosh(\lambda_{k,0}(\pi - x) - \mathrm{i}qn(\pi - x))\, \mathrm{d}x.\end{aligned}$$

Now it is sufficient to take a look at Section A.1, from which it follows that both $\{\sin(nx)_n\}_{n=1}^\infty$ and $\{\cos(nx)_n\}_{n=1}^\infty$ are Bessel in $L^2(0,\pi)$, which implies that the last expression is square summable in n. The same holds true for the second term at the right-hand side of (3.4). To finish the argument, note that the form of generalised eigenfunctions (2.27) also implies that they are Bessel. Finally, the sums are identical for the root vectors of the adjoint operator (see Proposition 3.3). □

Next, we must deal with the question whether the root vectors are total or not. Following the path of [3], we will use a trace criterion (whose proof can be found in the appendix).

Theorem 3.6. *(Livšic; [11], Theorem V. 2.1). Let T be a compact operator in $\mathcal{H}$. Suppose that* $\operatorname{Re} T := \frac{1}{2}(T + T^*)$ *is dissipative and* $|\operatorname{tr} \operatorname{Re} T| < \infty$. *Then*

$$\operatorname{tr}(\operatorname{Re} T) \leq \sum_{\lambda \in \sigma_p(T)} \operatorname{Re} \lambda,$$

with eigenvalues repeated according to their algebraic multiplicity. Equality holds if and only if the root vectors of T are total in $\mathcal{H}$.

Remark. It is straightforward to reformulate the criterion for $\operatorname{Re} T$ accretive; we obtain the opposite inequality.

In accordance with Proposition 1.38, we will apply the criterion to the compact resolvent at 0, i.e. $T := A^{-1}(a,b)$.

Proposition 3.7. $\operatorname{tr}(\operatorname{Re} A^{-1}(a,b)) = -\frac{\operatorname{Re} b(\pi-a)a}{\pi}$.

Proof. Consider the equation

$$\begin{pmatrix} u \\ v \end{pmatrix} = A^{-1}\begin{pmatrix} f \\ g \end{pmatrix}.$$

This means $v = f$ and $u'' = g$ with $u'(a+) - u'(a-) = bf(a)$ (or in other words $u'' = g + b\delta_a f$). The solution can be found using standard methods:

$$u(x) = \int_0^x (x-t)g(t)\,\mathrm{d}t - \frac{x}{\pi}\int_0^\pi (\pi - t)g(t)\,\mathrm{d}t - \frac{b}{\pi}f(a)\rho(x),$$

where

$$\rho(x) := \begin{cases} (\pi - a)x, & \text{for } 0 < x < a, \\ a(\pi - x), & \text{for } a < x < \pi. \end{cases}$$

Analogously if we take a look at

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = (A^{-1})^* \begin{pmatrix} \tilde{f} \\ \tilde{g} \end{pmatrix},$$

then we get to $\tilde{v} = -\tilde{f}$, $\tilde{u}'' = -\tilde{g}$ and $\tilde{u}'(a+) - \tilde{u}'(a-) = \overline{b}\tilde{f}(a)$. The solution is of the form

$$\tilde{u}(x) = -\int_0^x (x-t)g(t)\,\mathrm{d}t + \frac{x}{\pi}\int_0^\pi (\pi - t)g(t)\,\mathrm{d}t - \frac{\overline{b}}{\pi}f(a)\rho(x).$$

Together we have

$$\operatorname{Re} A^{-1}\begin{pmatrix} f \\ g \end{pmatrix} = \frac{1}{2}\left(A^{-1} + (A^{-1})^*\right)\begin{pmatrix} f \\ g \end{pmatrix} = -\frac{\operatorname{Re} b}{\pi} f(a)\rho\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Therefore, $\operatorname{ran} \operatorname{Re} A^{-1}$ is spanned by the single vector $\begin{pmatrix} \rho \\ 0 \end{pmatrix}$. Normalisation gives

$$1 = c^2 \left\|\rho'\right\|^2 = c^2 \int_0^a (\pi - a)^2\,\mathrm{d}x + c^2 \int_a^\pi a^2\,\mathrm{d}x = c^2(\pi - a)a\pi \implies c = \frac{1}{\sqrt{a\pi(\pi - a)}}.$$

We choose the one-member ON basis of the range to compute the trace:

$$\operatorname{tr}(\operatorname{Re} A^{-1}) = \frac{1}{a\pi(\pi - a)}\left\langle \begin{pmatrix} \rho \\ 0 \end{pmatrix}, \operatorname{Re} A^{-1}\begin{pmatrix} \rho \\ 0 \end{pmatrix}\right\rangle = -\frac{\operatorname{Re} b}{\pi^2}\left\|\rho'\right\|^2 = -\frac{\operatorname{Re} b(\pi - a)a}{\pi}. \qquad \square$$

In the following part, we will restrain ourselves to the case $b = \pm 2$, omitted to be analysed by Cox and Henrot in [3]. The approach can be modified in the case $b \neq \pm 2$ to replicate their result and extend it for $b < 0$ as follows:

Theorem 3.8. *Let $a = p\pi/q$ and $b \in \mathbb{R} \setminus \{-2, 2\}$. Then the root vectors of $A(a,b)$ are total in $\mathcal{H}$. In particular, it holds that*

$$\sum_{\lambda \in \sigma(A)} \operatorname{Re} \frac{1}{\lambda} = -\frac{b(\pi - a)a}{\pi}. \tag{3.5}$$

We will need to calculate the following series (which is done in the appendix – see Section A.3):

Lemma 3.9. *Let $\alpha \in \mathbb{R}$, $\beta > 0$. Then*

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n+\alpha)^2+\beta^2} = \frac{\pi}{2\beta}\frac{\sinh(2\pi\beta)}{\cosh^2(\pi\beta)-\cos^2(\pi\alpha)}. \tag{3.6}$$

Proposition 3.10. $S(\lambda; p\pi/q, 2) = \pi\lambda - \pi\lambda^2\left(\sum_{\lambda\in\sigma(A} \operatorname{Re}\frac{1}{\lambda} - \frac{\pi(q-r)}{q}\right) + O(\lambda^3)$*, where* $r := \max\{p, q-p\}$ *and* $A \equiv A(p\pi/q, 2)$.

Proof. Recall that $P_2(z) = 2(z^r + z^{q-r} - 2)$. Then according to (2.16) we can write

$$S(\lambda; p\pi/q, 2) \equiv S(\lambda) = -\frac{1}{4}e^{\lambda\pi}P_2\left(e^{-2\lambda\pi/q}\right) = -\frac{1}{2}e^{\lambda\pi}\prod_{k=1}^{r}(e^{-2\lambda\pi/q}-\zeta_k),$$

where ζ_k are the roots of P_2. Keep in mind that $\zeta_1 = 1$. Differentiating with respect to λ, we have

$$S'(\lambda) = -\frac{\pi}{2}e^{\lambda\pi}\prod_{k=1}^{r}(e^{-2\lambda\pi/q}-\zeta_k) + \frac{\pi}{q}e^{\lambda\pi(1-2/q)}\sum_{j=1}^{r}\prod_{k\neq j}(e^{-2\lambda\pi/q}-\zeta_k).$$

Consequently, for the first MacLaurin coefficient holds

$$S'(0) = \frac{\pi}{q}\prod_{k=2}^{r}(1-\zeta_k) = \frac{\pi}{2q}\lim_{z\to 1}\frac{P_2(z)}{z-1} = \pi.$$

Let us differentiate again:

$$\begin{aligned} S''(\lambda) = {} & \frac{\pi^2}{2}e^{\lambda\pi}\prod_{k=1}^{r}(e^{-2\lambda\pi/q}-\zeta_k) + \frac{2\pi^2}{q^2}(q-1)e^{\lambda\pi(1-2/q)}\sum_{j=1}^{r}\prod_{k\neq j}(e^{-2\lambda\pi/q}-\zeta_k) \\ & - \frac{2\pi^2}{q^2}e^{\lambda\pi(1-4/q)}\sum_{j=1}^{r}\sum_{i=1}^{r}\prod_{k\neq i,j}(e^{-2\lambda\pi/q}-\zeta_k). \end{aligned}$$

At $\lambda = 0$ this becomes

$$\begin{aligned} S''(0) &= \frac{2\pi^2(q-1)}{q^2}\prod_{k=2}^{r}(1-\zeta_k) - \frac{2\pi^2}{q^2}\sum_{j=2}^{r}\prod_{k\neq 1,j}(1-\zeta_k) = \frac{2\pi^2(q-1)}{q} - \frac{4\pi^2}{q^2}\sum_{j=2}^{r}\frac{\lim_{z\to 1}\frac{P_2(z)}{2(z-1)}}{1-\zeta_j} \\ &= \frac{2\pi^2(q-1)}{q} - \frac{4\pi^2}{q}\sum_{k=2}^{r}\frac{1}{1-\zeta_k} = \frac{2\pi^2}{q}\left[\sum_{k=2}^{r}\left(\frac{1-\zeta_k}{1-\zeta_k} - \frac{2}{1-\zeta_k}\right) + q - r\right] \\ &= \frac{2\pi^2}{q}\left[\sum_{k=2}^{r}\frac{\zeta_k+1}{\zeta_k-1} + q - r\right]. \end{aligned}$$

On the other hand, by (2.19) we know that

$$\lambda_{k,n} = -\frac{q}{2\pi}\left(\ln|\zeta_k| + i(\Theta_k + 2\pi n)\right) \implies \frac{1}{\lambda_{k,n}} = -\frac{2\pi}{q}\frac{\ln|\zeta_k| - i(\Theta_k + 2\pi n)}{\ln^2|\zeta_k| + (\Theta_k + 2\pi n)^2}.$$

The real part can be expressed as follows

$$\operatorname{Re}\frac{1}{\lambda_{k,n}} = -\frac{\ln|\zeta_k|}{2\pi q}\frac{1}{\frac{\ln^2|\zeta_k|}{4\pi^2} + \left(n + \frac{\Theta_k}{2\pi}\right)^2}.$$

Note that we do not need to worry about the harmonic spectrum, because we are interested in real parts – and so we take $n \in \mathbb{Z}$. Let $\alpha := \Theta_k/2\pi$ and $\beta := \ln|\zeta_k|/2\pi$ in (3.6). It follows that

$$\sum_{n\in\mathbb{Z}}\operatorname{Re}\frac{1}{\lambda_{k,n}} = -\frac{\ln|\zeta_k|}{2\pi q}\frac{\pi^2}{\ln|\zeta_k|}\frac{\sinh(\ln|\zeta_k|)}{\cosh^2\left(\frac{\ln|\zeta_k|}{2}\right) - \cos^2(\Theta_k/2)} = -\frac{\pi}{4q}\frac{|\zeta_k| - |\zeta_k|^{-1}}{\frac{1}{4}\left(|\zeta_k| + 2 + |\zeta_k|^{-1}\right) - \cos^2(\Theta_k/2)}$$
$$= \frac{\pi}{q}\frac{1 - |\zeta_k|^2}{|\zeta_k|^2 + 2|\zeta_k| + 1 - 4|\zeta_k|\cos^2(\Theta_k/2)} = \frac{\pi}{q}\frac{1 - |\zeta_k|^2}{|\zeta_k|^2 - 2\operatorname{Re}\zeta_k + 1}.$$

If $\zeta_k \in \mathbb{R}$, then

$$\frac{1 - |\zeta_k|^2}{|\zeta_k|^2 - 2\operatorname{Re}\zeta_k + 1} = \frac{\zeta_k + 1}{1 - \zeta_k}.$$

If $\zeta_k \in \mathbb{C} \setminus \mathbb{R}$, then, using the fact that P_b has real coefficients, also $\overline{\zeta_k}$ is a root of P_b. We compute

$$\frac{1 - |\zeta_k|^2}{|\zeta_k|^2 - 2\operatorname{Re}\zeta_k + 1} + \frac{1 - \left|\overline{\zeta_k}\right|^2}{\left|\overline{\zeta_k}\right|^2 - 2\operatorname{Re}\zeta_k + 1} = \frac{\zeta_k + 1}{1 - \zeta_k} + \frac{\overline{\zeta_k} + 1}{1 - \overline{\zeta_k}}.$$

Comparing this result with the second MacLaurin coefficient, we have

$$S''(0) = -2\pi\sum_{k=2}^{r}\sum_{n\in\mathbb{Z}}\operatorname{Re}\frac{1}{\lambda_{k,n}} + \frac{2\pi^2(q - r)}{q},$$

where we used the fact that $\operatorname{Re}\lambda_{1,n} = 0$. The proposition is proven. □

Proposition 3.11. $S(\lambda; a, b) = \pi\lambda + ab(\pi - a)\lambda^2 + O(\lambda^3)$.

Proof. This time, we will just differentiate $S(\lambda; a, b)$ from the definition

$$S(\lambda; a, b) = \sinh(\lambda\pi) + b\sinh(\lambda a)\sinh(\lambda(\pi - a)).$$

We have

$$S'(0; a, b) = \pi$$

and for the second derivative we have from (2.20)

$$S''(\lambda; a, b) = \pi^2 S(\lambda; a, b) + 2ab(\pi - a)\cosh(\lambda(\pi - 2a)).$$

Putting $\lambda = 0$ gets us the desired result

$$S''(0; a, b) = 2ab(\pi - a).$$ □

Comparing the two propositions, we derive the final conclusion.

Theorem 3.12. *Let $a = p\pi/q$, $b = 2$. Then*

$$\sum_{\lambda\in\sigma(A)}\operatorname{Re}\frac{1}{\lambda} = -\frac{b(\pi - a)a}{\pi} + \frac{\pi}{q}(q - r) = \begin{cases} -\frac{b(\pi-a)a}{\pi} + a, & \text{for } 0 < a \leq \frac{\pi}{2}, \\ -\frac{b(\pi-a)a}{\pi} + (\pi - a), & \text{for } \frac{\pi}{2} < a < \pi, \end{cases} \tag{3.7}$$

where $r := \max\{p, q - p\}$ and $A \equiv A(p\pi/q, 2)$

This result can be simply extended for the accretive operator with $b = -2$. Recall that by Proposition 3.2, it holds that $\operatorname{Re} A^{-1}(a,-2) = -\operatorname{Re} A^{-1}(a,2)$.

Proposition 3.13. *Let $a = p\pi/q$, $b = -2$. Then*

$$\sum_{\lambda\in\sigma(A)} \operatorname{Re}\frac{1}{\lambda} = -\frac{b(\pi-a)a}{\pi} - \frac{\pi}{q}(q-r) = \begin{cases} -\frac{b(\pi-a)a}{\pi} - a, & \text{for } 0 < a \le \frac{\pi}{2}, \\ -\frac{b(\pi-a)a}{\pi} - (\pi - a), & \text{for } \frac{\pi}{2} < a < \pi, \end{cases} \tag{3.8}$$

where $r := \max\{p,\ q-p\}$ and $A \equiv A(p\pi/q,-2)$.

Since Proposition 3.7 gives $\operatorname{tr}(\operatorname{Re} A^{-1}) = -\frac{b(\pi-a)a}{\pi}$, in the case $b = 2$ we have

$$\operatorname{tr}(\operatorname{Re} A^{-1}) < \sum_{\lambda\in\sigma(A)} \operatorname{Re}\frac{1}{\lambda}.$$

Analogously, in the case $b = -2$ holds

$$\operatorname{tr}(\operatorname{Re} A^{-1}) > \sum_{\lambda\in\sigma(A)} \operatorname{Re}\frac{1}{\lambda}.$$

Livšic criterion (Theorem 3.6) provides the following result.

Theorem 3.14. *Let $a = p\pi/q$, $b = \pm 2$. Then the root vectors of $A(a,b)$ are not total in $\mathcal{H}$ (and consequently do not form a Riesz basis).*

3.2.2 General placement of the damping

Suppose now that $a \in (0,\pi)$ is arbitrary and $b \in \mathbb{R}$. Let $\{\lambda_j\}_{j=1}^{\infty}$ denote the spectrum of $A(a,b)$ with eigenvalues repeated according to their algebraic multiplicity, i.e. the roots of (2.15) (except for 0 in the case $a = p\pi/q$). We will make use of the following recent result of Krejčiřík and Lipovský:

Theorem 3.15. *([14], Section 4). Let $b \in \mathbb{C}$ be arbitrary and $a \in (0,\pi/2)$. Let $\lambda_j^+(a)$ denote the j-th eigenvalue in the upper half-plane sorted in the non-decreasing order according to the imaginary part.*

$$\lambda_j^+(a) = \begin{cases} \mathrm{i}j + f_j(a), & \text{for } b \neq 2, \\ \frac{\mathrm{i}j\pi}{\pi-a}, & \text{for } b = 2, \end{cases} \tag{3.9}$$

where

1. *$f_j(a)$ are analytic in a with at most algebraic singularities. If for certain a_0 a finite number of $\lambda_j(a_0)$ have the same imaginary part, one may need to interchange their indices to get the analyticity.*
2. *The real parts of $f_j(a)$ satisfy $|\operatorname{Re} f_j(a)| < c_1$, where $c_1 > 0$ is independent of j and a.*
3. *The imaginary parts of $f_j(a)$ satisfy $|\operatorname{Im} f_j(a)| < c_2$, where $c_2 > 0$ is independent of j and a.*

Similar statement holds for the eigenvalues λ_j^- in the lower half-plane.

We will use this knowledge to prove the Riesz basis property by showing the Bessel property and performing the limit in the Livšic criterion. Let us start with the latter.

For any $a_0 \in (0,\pi/2)$, consider a sequence $\{a_n\}_{n=1}^{\infty} \subset \pi\mathbb{Q}$ such that $a_0 = \lim_{n\to\infty} a_n$. Recall that for any $n \in \mathbb{N}$ we can use formulas (3.5), (3.7), and (3.8) for $b \neq 2$, $b = 2$, and $b = -2$, respectively.

We have

$$\left|\operatorname{Re}\frac{1}{\lambda_j(a)}\right| \le \frac{c_1}{|\lambda_j(a)|^2},$$

where the term on the right-hand side is eventually dominated by αj^{-2} for some $\alpha > 0$ thanks to Theorem 3.15. The Lebesgue Dominated Convergence Theorem then for $b \neq \pm 2$ yields

$$\sum_{j=1}^{\infty} \operatorname{Re}\frac{1}{\lambda_j(a_0)} = \lim_{n\to\infty}\sum_{j=1}^{\infty} \operatorname{Re}\frac{1}{\lambda_j(a_n)} = \lim_{n\to\infty} -\frac{b(\pi - a_n)a_n}{\pi} = -\frac{b(\pi - a_0)a_0}{\pi} = \operatorname{tr}(\operatorname{Re} A^{-1}(a_0, b)).$$

In a similar manner, we can show that for $b = 2$ holds

$$\sum_{j=1}^{\infty} \operatorname{Re}\frac{1}{\lambda_j(a_0)} = -\frac{b(\pi - a_0)a_0}{\pi} + a_0 > \operatorname{tr}(\operatorname{Re} A^{-1}(a_0, b));$$

and for $b = -2$:

$$\sum_{j=1}^{\infty} \operatorname{Re}\frac{1}{\lambda_j(a_0)} = -\frac{b(\pi - a_0)a_0}{\pi} - a_0 < \operatorname{tr}(\operatorname{Re} A^{-1}(a_0, b)).$$

Once again, we have shown that the Livšic criterion is satisfied for $b \neq \pm 2$ and it is not fulfilled for $b = \pm 2$. The argumentation for general $a_0 \in (0, \pi)$ is finished if we realise that the right-hand sides of (3.5), (3.7), and (3.8), the trace of A^{-1}, and equation (2.15) are symmetric under the exchange of a and $\pi - a$.

Theorem 3.16. *Let $b \in \mathbb{R}$, $a \in (0, \pi)$. The root vectors of $A(a, b)$ are total in $\mathcal{H}$ if and only if $b \neq \pm 2$.*

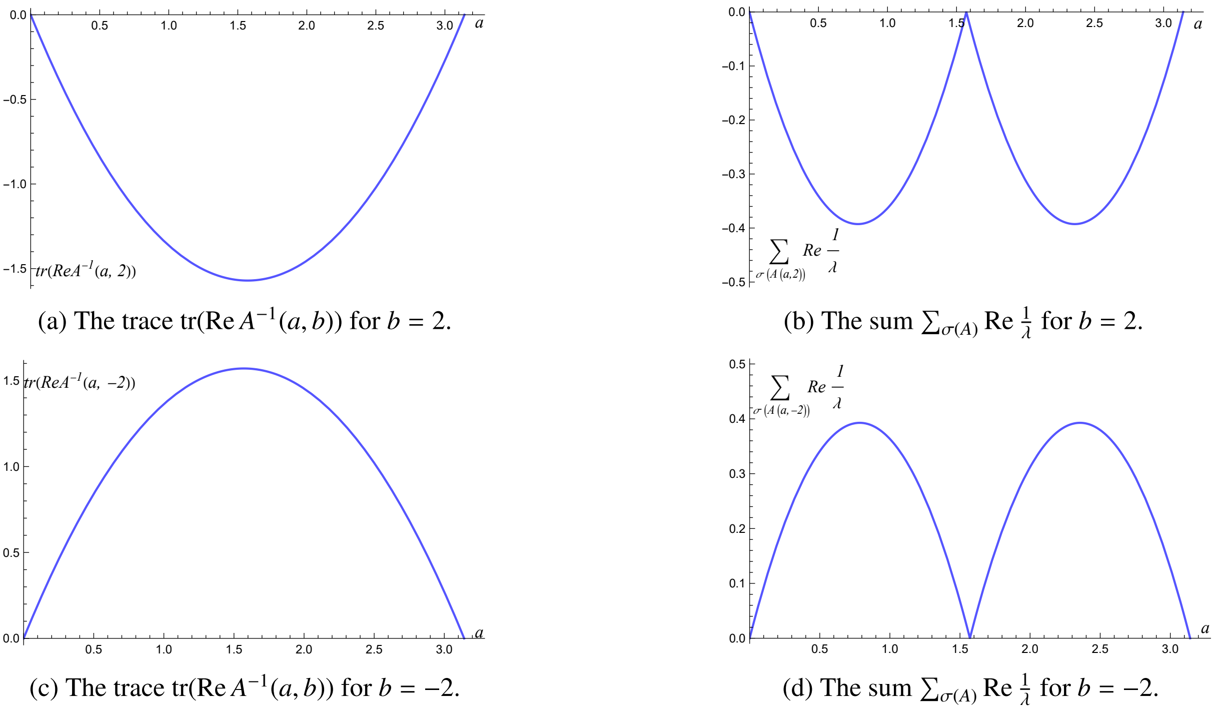


Figure 3.1: The left- and right-hand sides of the condition in the Livšic criterion (Theorem 3.6) as functions of a with $b = \pm 2$.

It remains to prove the Bessel property. The proof is quite similar to the proof of Proposition 3.5.

Proposition 3.17. *Let $\psi \in \mathcal{H}$ and denote $\psi_j := \begin{pmatrix} u_j \\ \lambda_j u_j \end{pmatrix}$ the root vector corresponding to λ_j in accordance with* (2.14) *and* (2.27). *Then*

$$\sum_{j=1}^{\infty} \left| \left\langle \psi, \psi_j/\lambda_j \right\rangle \right|^2 < \infty.$$

Proof. Similarly as in the proof of Proposition 3.5, we can calculate

$$\left\langle \psi, \psi_j/\lambda_j \right\rangle = \frac{1}{\lambda_j} \left\langle \psi_1', u_{j1}' \right\rangle + \left\langle \psi_2, u_{j2} \right\rangle . \tag{3.10}$$

Furthermore,

$$\frac{1}{\lambda_j} \left\langle \psi_1', u_{j1}' \right\rangle = \sinh(\lambda_j(\pi - a)) \int_0^a \overline{\psi_1'}(x) \cosh(\lambda_j x)\, \mathrm{d}x - \sinh(\lambda_j a) \int_a^\pi \overline{\psi_1'}(x) \cosh \lambda_j(\pi - x)\, \mathrm{d}x. \tag{3.11}$$

We remind of the useful identities

$$\sinh(\alpha + i\beta) = \sinh\alpha \cos\beta + \mathrm{i} \cosh\alpha \sin\beta, \quad \cosh(\alpha + i\beta) = \cosh\alpha\cos\beta + \mathrm{i}\sinh\alpha\sin\beta.$$

It follows from Theorem 3.15 and the identities above that the factors in front of the integrals are bounded uniformly in j. Denoting $f_j =: g_j + \mathrm{i}h_j$ with $g_j,\ h_j \in \mathbb{R}$, further computation gives

$$\begin{aligned}
\int_0^a \overline{\psi_1'}(x) \cosh(\lambda_j x)\, \mathrm{d}x &= \int_0^a \overline{\psi_1'}(x) \cosh(g_j x) \cos((j + h_j)x)\, \mathrm{d}x + \mathrm{i} \int_0^a \overline{\psi_1'}(x) \sinh(g_j x) \sin((j + h_j)x)\, \mathrm{d}x \\
&= \int_0^a \overline{\psi_1'}(x) \Big(\cosh(g_j x)\cos(h_j x) + \sinh(g_j x)\sin(h_j x)\Big) \cos(jx)\, \mathrm{d}x \\
&+ \int_0^a \overline{\psi_1'}(x) \Big(-\cosh(g_j x)\sin(h_j x) + \sinh(g_j x)\cos(h_j x)\Big) \sin(jx)\, \mathrm{d}x.
\end{aligned} \tag{3.12}$$

Let us denote

$$C := |\cosh(c_1\pi)|, \quad S := |\sinh(c_1\pi)|.$$

Using the uniform boundedness of g_j and h_j in j we can estimate

$$\left\| \psi_1'(x) \overline{\Big(\cosh(g_j x)\cos(h_j x) + \sinh(g_j x)\sin(h_j x)\Big)} \right\|^2 \le (C + S)^2 \left\| \psi_1' \right\|^2 .$$

Once again making use of the fact that both $\{\sin(jx)\}_{j=1}^\infty$ and $\{\cos(jx)\}_{j=1}^\infty$ are Bessel in $L^2(0,\pi)$, the first term on the right-hand side of (3.12) is square-summable. The same goes for the second term and thus for the right-hand side as a whole.

The exact same arguments can be used to show that the whole right-hand side of (3.11) is square-summable. Finally, the same can be done for the term $\left\langle \psi_2, u_{j2} \right\rangle$ in (3.10). Therefore, for eigenvectors, the Bessel property holds.

For generalised eigenvectors, the proof is analogous using the form (2.27). □

Clearly, the eigenvectors of the adjoint operator posses the same properties. Realising that for $b \in \mathbb{R}$ (so the eigenvalues are algebraically at most double), Proposition 1.76 may be used to conclude the general result:

Theorem 3.18. *If $b \in \mathbb{R} \setminus \{-2,\ 2\}$, then the root vectors of $A(a, b)$ form a Riesz basis in $\mathcal{H}$. If $b = \pm 2$, then the root vectors of $A(a, b)$ are not total in $\mathcal{H}$ (and thus they do not form a Riesz basis).*

Conclusion

In the thesis, we introduced the basics of the theory of unbounded linear operators and their spectra. We stated several theorems on Riesz bases in a Hilbert space. We also studied the Sobolev spaces and used them to define the Dirichlet Laplacian on an open connected subset of $\mathbb{R}^n$,

We provided a spectral analysis of the wave operator with the distributional Dirac damping. We derived the form and properties of a characteristic function that determines its spectrum. We then studied the spectrum with respect to the placement and the strength of the damping, arriving at solutions of the optimal damping problem for simple placements of the damping and acquiring a criterion for algebraically double eigenvalues. We extended several previous results from positive to real and occasionally even complex damping and completed arguments for the singular damping value $b = 2$.

Finally, we treated the root vectors of the wave operator, finding their explicit form in terms of the corresponding eigenvalues. Using the Livšic trace criterion, we proceeded to show that for rational placement of the damping of strength $b = 2$, the root vectors are not total and consequently do not form a Riesz basis. We then extended this result and the opposite result for $b \in (0, \infty) \setminus \{2\}$ first by symmetry for $b \in \mathbb{R}$ and then, perhaps most importantly, also for a general placement of the damping.

Several problems concerning the presented model remain open. First, it is challenging to further determine spectral properties of the operator $A(a, b)$ for a general $b \in \mathbb{C}$ since one loses several properties of the characteristic function $S(\lambda; a, b)$. A simpler case may arise with $b \in \mathrm{i}\mathbb{R}$. Second, in [4], the authors consider the wave equation with Dirac damping on a non-compact star graph. One may follow their footsteps with our model and extend the spectral analysis to the wave equation on a compact star graph with the view of clearing up the mystery of the singular values $b = \pm 2$. Finally, a much less straightforward generalisation appears to be moving to higher dimensions. Considering the wave equation on bounded domains in $\mathbb{R}^n$, one would have to deal with non-trivial geometric aspects as well as the proper introduction of the Dirac or a thin shell damping. For such extension, one may test the applicability of the unorthodox method presented in [16].

Appendix

A.1 Eigenfunctions of the Dirichlet Laplacian

In (1.13), we have shown the form of the eigenfunctions of the Laplace operator with Dirichlet boundary conditions on a compact interval. To finish the argument that the operator has purely discrete spectrum, we must show that the eigenfunctions form an ON basis in $L^2(a,b)$. The claim then follows from the Spectral Theorem. For simplicity, we will consider $(a,b) = (0,\pi)$, extension to an arbitrary interval is straightforward. We have

$$\psi_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx).$$

First, note that

$$\langle \psi_n, \psi_m \rangle = \frac{2}{\pi} \int_0^\pi \sin(nx)\sin(mx)\,\mathrm{d}x = \delta_{n,m}.$$

We will make use of the fact that $U := \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \sin(nx), \frac{1}{\sqrt{\pi}} \cos(nx) \right\}_{n=1}^\infty$ is an ON basis in $L^2(-\pi,\pi)$.

Take any $\psi \in L^2(0,\pi)$. Define

$$\tilde{\psi}(x) := \begin{cases} \psi(x), & \text{for } x \in (0,\pi), \\ 0, & \text{for } x = 0, \\ -\psi(-x), & \text{for } x \in (-\pi,0). \end{cases}$$

That is, $\tilde{\psi}$ is the odd extension of ψ to $(-\pi,\pi)$. Clearly $\left\|\tilde{\psi}\right\|^2 = 2\|\psi\|^2$, so $\tilde{\psi} \in L^2(-\pi,\pi)$. Since $\tilde{\psi}$ is odd, all its Fourier coefficients in U corresponding to even functions (constant and cosines) are zero. Consequently, Parseval equality yields

$$\|\psi\|^2 = \frac{1}{2}\left\|\tilde{\psi}\right\|^2 = \frac{1}{2}\sum_{n=1}^\infty \left|\left\langle \frac{1}{\sqrt{\pi}}\sin(nx), \tilde{\psi}\right\rangle\right|^2 = \frac{1}{4}\sum_{n=1}^\infty \left|\left\langle \sqrt{\frac{2}{\pi}}\sin(nx), \tilde{\psi}\right\rangle\right|^2 = \sum_{n=1}^\infty \left|\left\langle \sqrt{\frac{2}{\pi}}\sin(nx), \psi\right\rangle\right|^2,$$

where the last equality follows from the fact that $\sin(nx)\tilde{\psi}(x)$ is an even function. We have proven that Parseval equality holds for $\{\psi_n\}_{n=1}^\infty$ and by Theorem 1.59, it is an ON basis in $L^2(0,\pi)$. [2]

Remark. Using the very same approach (only with the even extension of a function), we can easily show that $\left\{ \sqrt{\frac{2}{\pi}} \cos(nx) \right\}_{n=1}^\infty$ is also an ON basis in $L^2(0,\pi)$. In fact it is the basis of eigenvectors of the Neumann Laplacian.

[2] We didn't differentiate in notation between norms and inner products on $L^2(0,\pi)$ and $L^2(-\pi,\pi)$. It is obvious from the context that expressions with ψ use the earlier and those with $\tilde{\psi}$ use the latter.

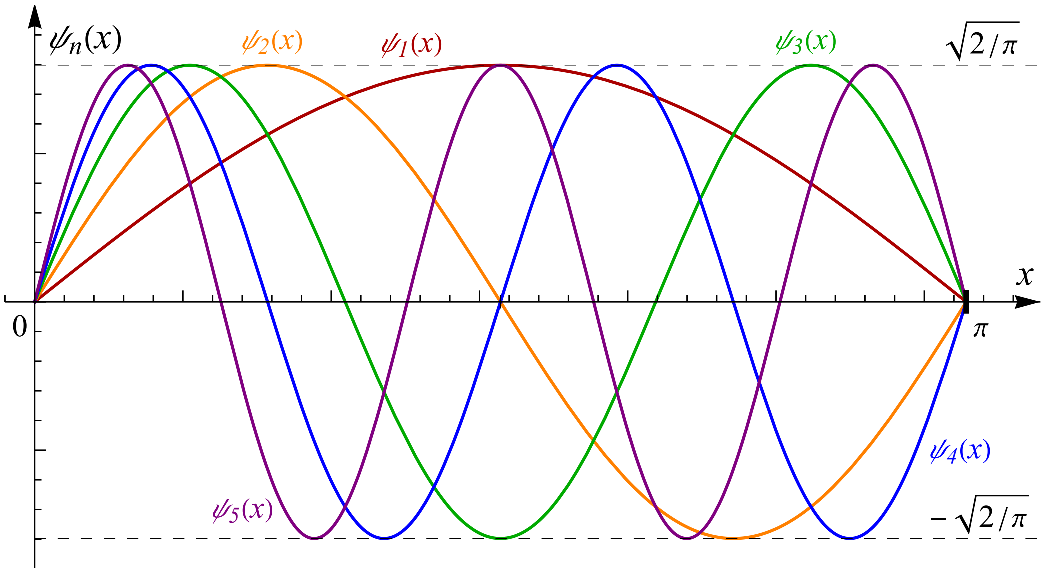


Figure 2: First 5 members of the ON basis $\{\psi_n\}_{n=1}^{\infty}$ in $L^2(0,\pi)$ formed by the eigenfunctions of the Dirichlet Laplacian.

A.2 Eigenfunctions of the undamped wave operator

We show in Section 2.3 that the eigenfunctions of the undamped wave operator A on $(0,\pi)$ are

$$\psi_n(x) = \frac{1}{n\sqrt{\pi}}\sin(nx)\begin{pmatrix}1\\ in\end{pmatrix}, \quad n\in\mathbb{Z}\setminus\{0\}. \tag{13}$$

Once again, we need to show that the operator has purely discrete spectrum by invoking the Spectral Theorem and showing that $\{\psi_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ is an ON basis in $\mathcal{H} = H_0^1(0,\pi)\times L^2(0,\pi)$. Take any $\psi = (\psi_1,\psi_2)\in\mathcal{H}$. We have

$$|\langle\psi_n,\psi\rangle|^2 = \frac{1}{n^2\pi}\left|n\left\langle\cos(nx),\psi_1'\right\rangle + in\left\langle\sin(nx),\psi_2\right\rangle\right|^2 = \frac{1}{\pi}\left(\left|\left\langle\cos(nx),\psi_1'\right\rangle\right|^2 + |\langle\sin(nx),\psi_2\rangle|^2\right).$$

Consequently, we can calculate

$$\begin{aligned}\sum_{n\in\mathbb{Z}\setminus\{0\}}|\langle\psi_n,\psi\rangle|^2 &= \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\left|\left\langle\cos(nx),\psi_1'\right\rangle\right|^2 + |\langle\sin(nx),\psi_2\rangle|^2\right)\\ &= \sum_{n=1}^{\infty}\left|\left\langle\sqrt{\frac{2}{\pi}}\cos(nx),\psi_1'\right\rangle\right|^2 + \sum_{n=1}^{\infty}\left|\left\langle\sqrt{\frac{2}{\pi}}\sin(nx),\psi_2\right\rangle\right|^2 = \left\|\psi_1'\right\|^2 + \|\psi_2\|^2 = \|\psi\|^2.\end{aligned}$$

We made use of the fact that both $\left\{\sqrt{\frac{2}{\pi}}\sin(nx)\right\}_{n=1}^{\infty}$ and $\left\{\sqrt{\frac{2}{\pi}}\cos(nx)\right\}_{n=1}^{\infty}$ are ON bases in $L^2(0,\pi)$. The Parseval equality is established and by Theorem 1.59, $\{\psi_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ is an ON basis in $\mathcal{H}$

A.3 A series

In Section 3.2 we made great use of the formula (3.6):

$$\sum_{n\in\mathbb{Z}}\frac{1}{(n+\alpha)^2+\beta^2} = \frac{\pi}{2\beta}\frac{\sinh(2\pi\beta)}{\cosh^2(\pi\beta)-\cos^2(\pi\alpha)}$$

with $\alpha \in \mathbb{R}, \beta > 0$. Here we will provide a short proof that uses the Poisson summation formula.

Theorem A.19. *(Poisson Summation Formula; [12], Theorem 2.4).*
Let $a > 0$ and suppose that $f : \mathbb{C} \to \mathbb{C}$ satisfies:

(i) f is holomorphic on $\{z \in \mathbb{C} \mid |\mathrm{Im}\, z| < a\}$.

(ii) There exists $A > 0$ such that for all $x, y \in \mathbb{R}$, $|y| < a$ holds $|f(x + \mathrm{i}y)| \leq \frac{A}{1+x^2}$.

Then

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n), \quad \textit{where} \quad \hat{f}(\xi) := \int_{\mathbb{R}} f(x)\mathrm{e}^{-2\pi\mathrm{i}\xi x}\, dx.$$

First, we will show that $f(z) := ((z + \alpha)^2 + \beta^2)^{-1}$ satisfies the two conditions. For the first one, we simply set $a := \beta/2$. For the second one, denoting $t := (x + \alpha)/\beta$, we have

$$\begin{aligned} |f(x + \mathrm{i}y)| &= \frac{1}{(x + \mathrm{i}y)^2 + \beta^2} = \frac{1}{\beta^2} \frac{1}{\left|1 + t^2 - y^2/\beta^2 + 2\mathrm{i}ty/\beta\right|} \leq \frac{1}{\beta^2} \frac{1}{\left|1 - y^2/\beta^2 + t^2\right|} \\ &\leq \frac{1}{\beta^2(1 - \frac{1}{4})} \frac{1}{1 + \frac{t^2}{1-1/4}} = \frac{\frac{4}{3}\beta^{-2}}{1 + \frac{3}{4}t^2}. \end{aligned}$$

Therefore, setting $A := \frac{4}{3}\beta^{-2}$ fulfills the second condition.

Now we can use Theorem A.19 to calculate the sum of the series. The Fourier transform gives

$$\hat{f}(\xi) = \int_{\mathbb{R}} \frac{\mathrm{e}^{-2\pi\mathrm{i}\xi x}}{(x + \alpha)^2 + \beta^2}\, \mathrm{d}x = \mathrm{e}^{2\pi\mathrm{i}\xi\alpha} \int_{\mathbb{R}} \frac{\mathrm{e}^{-2\pi\mathrm{i}\xi x}}{x^2 + \beta^2}\, \mathrm{d}x = \frac{\pi}{\beta}\mathrm{e}^{2\pi\mathrm{i}\xi\alpha - 2\pi|\xi|\beta)}$$

using residues. Finally, we obtain

$$\begin{aligned} \sum_{n\in\mathbb{Z}} f(n) &= \sum_{n\in\mathbb{Z}} \hat{f}(n) = \frac{\pi}{\beta} \sum_{n\in\mathbb{Z}} \mathrm{e}^{2\pi\mathrm{i}n\alpha - 2\pi\mathrm{i}|n|\beta} = \frac{\pi}{\beta}\left[\frac{1}{1 - \mathrm{e}^{2\pi\mathrm{i}\alpha - 2\pi\beta}} + \frac{1}{1 - \mathrm{e}^{-2\pi\mathrm{i}\alpha - 2\pi\beta}} - 1\right] \\ &= \frac{\pi}{\beta} \frac{1 - \mathrm{e}^{4\pi\beta}}{1 - \mathrm{e}^{2\pi\mathrm{i}\alpha - 2\pi\beta} - \mathrm{e}^{-2\pi\mathrm{i}\alpha - 2\pi\beta} + \mathrm{e}^{-4\pi\beta}} = \frac{\pi}{\beta} \frac{\sinh(2\pi\beta)}{\cosh(2\pi\beta) - \cos(2\pi\alpha)} = \frac{\pi}{2\beta} \frac{\sinh(2\pi\beta)}{\cosh^2(\pi\beta) - \cos^2(\pi\alpha)}. \end{aligned}$$

A.4 Proof of the Livšic criterion

In this section, we will provide a proof of Livšic criterion used in Section 3.2. First of all, we will need an auxiliary lemma that can be found in [11], Lemma I.4.1.

Lemma A.20. *(Schur). Let T be a compact operator in $\mathcal{H}$. Let $\mathcal{V}$ denote the closed linear span of its root vectors corresponding to non-zero eigenvalues and ν_a the sum of algebraic multiplicities of all non-zero eigenvalues of T. Then there exists an ON basis $\{u_n\}_{n=1}^{\nu_a}$ in $\mathcal{V}$ such that the matrix of $T|_{\mathcal{V}}$ has triangular form, i.e.*

$$Tu_n = \sum_{j=1}^{n} \alpha_{nj}u_j, \quad \alpha_{jj} = \lambda_j, \quad j \in \{1, 2, \ldots, \nu_a\},$$

where $\lambda_j \in \sigma_p(T) \setminus \{0\}$.

Proof. It is enough to choose a Jordan basis in every root subspace (they are finite-dimensional thanks to Lemma 1.42 and invariant with respect to T) and gather them into a set $\{x_n\}_{n=1}^{\nu_a}$. Then either $(T - \lambda_n I)x_n = 0$ or $(T - \lambda_n I)x_n = x_{n-1}$ (in other words x_n is either an eigenvector or a generalised eigenvector of A). The desired ON basis is obtained by successive orthonormalisation. □

Moreover, we will need to use the spectral theorem. To provide the proof in completeness, we will not reference it here and rather prove it for compact operators (also known as Hilbert-Schmidt Theorem). This formulation is sufficient for our needs.

Lemma A.21. *Let T be a compact self-adjoint operator in $\mathcal{H}$. Then one of $\|T\|$ and $-\|T\|$ is an eigenvalue of T. In particular, $\sigma_p(T) \neq \emptyset$.*

Proof. If $T = 0$, the statement is obvious. Suppose $T \neq 0$. Since

$$\|T\| = \sup_{\|x\|=1} |\langle x, Tx\rangle|,$$

we can choose $\{x_n\}_{n=1}^{\infty} \subset \mathcal{H}$ such that $\|x_n\| = 1$ and $|\langle x_n, Tx_n\rangle| \to \|T\|$. There exists a subsequence $\{x'_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \langle x'_n, Tx'_n\rangle =: \lambda \in \{\pm \|T\|\}.$$

Since $\{x'_n\}_{n=1}^{\infty}$ is bounded, its image under T is precompact, so there exists a subsequence $\{x''_n\}_{n=1}^{\infty}$ such that $Tx''_n \to y \in \mathcal{H}$. Note that $y \neq 0$ since otherwise we would have $\lambda = 0$ by continuity of the inner product and T. At the same time

$$\left\|Tx''_n - \lambda x''_n\right\|^2 = \left\|Tx''_n\right\|^2 + \lambda^2 \left\|x''_n\right\|^2 - 2\lambda \operatorname{Re}\langle x''_n, Tx''_n\rangle \to 0,$$

thus $\lambda x''_n \to y$. Finally, we have

$$Ty = T \lim_{n\to\infty} \lambda x''_n = \lambda \lim_{n\to\infty} Tx''_n = \lambda y. \qquad \square$$

Theorem A.22. *(Spectral Theorem for Compact Self-Adjoint Operators).*
Let T be a compact self-adjoint operator in $\mathcal{H}$. Then there exists an ON basis of $\mathcal{H}$ consisting of eigenvectors of T.

Proof. Note that all root subspaces are finite-dimensional and invariant with respect to T; therefore, T restricted to any of these is a finite-dimensional Hermitian operator. As a consequence, there always exists an ON basis of eigenvectors for the root subspace making them in fact eigenspaces. (In other words, all Jordan blocks from Lemma A.20 are diagonal.)

Orthogonality of eigenvectors corresponding to different eigenvalues is a simple well-known fact.

It remains to be shown that the eigenvectors form an ON basis for the whole $\mathcal{H}$ or, in the notation of Lemma A.20, that $\mathcal{V} = \mathcal{H}$. Let $\mathcal{V}_0$ denote the closed linear hull of some eigenspaces of T. Then $T|_{\mathcal{V}_0^{\perp}}$ is a self-adjoint operator. Indeed, if $x \in \mathcal{V}_0^{\perp}$, then for any $y \in \mathcal{V}_0$ holds

$$\langle y, Tx\rangle = \langle Ty, x\rangle = 0$$

since V_0 is invariant under T.

Therefore, we can always apply Lemma A.21 to $V_0^{\perp}$ to find an eigenvector of T. Invoking Zorn's lemma, we obtain a maximal ON subset comprised of eigenvectors of T. $\square$

Let us now restate the criterion.

Theorem A.23. *(Livšic; [11], Theorem V. 2.1). Let T be a compact operator in $\mathcal{H}$. Suppose that* $\operatorname{Re} T := \frac{1}{2}(T + T^*)$ *is dissipative and* $|\operatorname{tr}\operatorname{Re} T| < \infty$. *Then*

$$\operatorname{tr}\operatorname{Re} T \leq \sum_{\lambda\in\sigma_p(T)} \operatorname{Re}\lambda,$$

with eigenvalues repeated according to their algebraic multiplicity. Equality holds if and only if the root vectors of T are total in $\mathcal{H}$.

Proof. First, we will show the inequality. Consider the basis $\{u_n\}_{n=1}^{\nu_a}$ of $\mathcal{V} \subset\subset \mathcal{H}$ from Lemma A.20. Then we have

$$\langle u_n, Tu_n \rangle = \lambda_n, \quad \langle u_n, \operatorname{Re} Tu_n \rangle = \operatorname{Re} \lambda_n, \quad \sum_{n=1}^{\nu_a} \langle u_n, \operatorname{Re} Tu_n \rangle = \sum_{\lambda \in \sigma_p(T)} \operatorname{Re} \lambda.$$

We can choose an arbitrary ON basis $\{v_k\}_k$ of $\mathcal{V}^\perp$ and knowing that $\operatorname{Re} T$ is dissipative, we obtain

$$\operatorname{tr} \operatorname{Re} T = \sum_n \langle u_n, \operatorname{Re} Tu_n \rangle + \sum_k \langle v_k, \operatorname{Re} Tv_k \rangle \leq \sum_n \langle u_n, \operatorname{Re} Tu_n \rangle = \sum_{\lambda \in \sigma_p(T)} \operatorname{Re} \lambda. \tag{14}$$

Suppose now that equality holds. Then

$$\sum_{n=1}^{\nu_a} \langle u_n, \operatorname{Re} Tu_n \rangle = \operatorname{tr} \operatorname{Re} T.$$

In accordance with the previous part of the proof, namely (14), this means that $\operatorname{Re} T|_{\mathcal{V}^\perp} = 0$. Therefore, on $\mathcal{V}^\perp$, we have $T = -T^*$, i.e. T is skew-adjoint. Applying Theorem A.22 (to $\mathrm{i}T = (\mathrm{i}T)^*$), we obtain an ON basis of $\mathcal{V}^\perp$ comprised of eigenvectors of T. Consequently, the root vectors are total in $\mathcal{H} = \mathcal{V} \oplus \mathcal{V}^\perp$.

On the contrary, suppose that the root vectors of T are total in $\mathcal{H}$. Then $\{v_k\}_k$ can be formed by linear combinations of root vectors corresponding to 0 similarly to the proof of Lemma A.20, so $\langle v_k, Tv_k \rangle = 0$. Using (14) again directly yields

$$\operatorname{tr} \operatorname{Re} T = \sum_{\lambda \in \sigma_p(T)} \operatorname{Re} \lambda. \qquad \square$$

A.5 Equivalence of definitions of algebraic multiplicity

On finite-dimensional vector spaces, the algebraic multiplicity of an eigenvalue of a linear operator is typically defined as its multiplicity as a root of the characteristic polynomial. Without loss of generality, let us consider complex $n \times n$ matrices instead of operators in a vector space over an arbitrary algebraically closed field.

Definition A.24. Let $\mathbb{A} \in \mathbb{C}^{n,n}$. The *characteristic polynomial* of $\mathbb{A}$ is defined as $p(t) := \det(\mathbb{A} - t\mathbb{I})$.

Proposition A.25. *Let $\mathbb{A} \in \mathbb{C}^{n,n}$, $\lambda \in \mathbb{C}$. Then $\lambda \in \sigma(\mathbb{A})$ if and only if $p(\lambda) = 0$.*

Definition A.26. Let $\mathbb{A} \in \mathbb{C}^{n,n}$, $\lambda \in \sigma(A)$. The *algebraic multiplicity* $\nu_a(\lambda)$ of the eigenvalue λ is its multiplicity as a root of $p(\lambda)$.

However, when dealing with closed operators in infinite-dimensional vector spaces, one cannot (in general) construct the characteristic polynomial. Instead, the algebraic multiplicity is defined as in Definition 1.37:

Definition A.27. Let $A \in \mathcal{C}(\mathcal{X})$ and $\lambda \in \sigma_p(A)$. The *algebraic multiplicity* of λ is

$$\nu_a(\lambda) := \dim \cup_{n=1}^{\infty} \ker(A - \lambda I)^n.$$

The subspace $\cup_{n=1}^{\infty} \ker(A - \lambda I)^n$ is called the *generalised eigenspace* corresponding to λ. Its non-zero elements are called *generalised eigenvectors* or *root vectors* associated with λ.

The goal of this section is to provide a simple proof of the equivalence of the above definitions on finite-dimensional vector spaces. In other words, we strive to prove that if $A \in \mathbb{C}^{n,n}$ and $\lambda \in \sigma(A)$, then

$$\nu_a(\lambda) = \dim \cup_{k=1}^{\infty} \ker(A - \lambda I)^k$$

For simplicity, from now on A always denotes a matrix from $\mathbb{C}^{n,n}$ and the expression $A - \lambda$, where $\lambda \in \mathbb{C}$, means $A - \lambda\mathbb{I}$. We will not differentiate between A as a matrix and the corresponding linear mapping on $\mathbb{C}^n$.

Proposition A.28. *Let $m \in \mathbb{N}_0$. Then the following holds:*

1. $\ker A^m \subset \ker A^{m+1}$.
2. *If* $\ker A^m = \ker A^{m+1}$*, then for all* $j \in \mathbb{N}$ *also* $\ker A^{m+j} = \ker A^m$.
3. $\ker A^n = \ker A^{n+1}$.

Proof.

1. $x \in \ker A^m \iff A^m x = 0 \implies A^{m+1}x = A(A^m x) = A(0) = 0 \implies x \in \ker A^{m+1}$.
2. We only need to show the case $j = 2$, the rest follows simply by induction. We have $x \in \ker A^{m+2} \iff A^{m+2}x = 0 \iff A^{m+1}Ax = 0 \iff Ax \in \ker A^{m+1} = \ker A^m$. Therefore, $A^m Ax = A^{m+1}x = 0$, so $x \in \ker A^{m+1}$.
3. Suppose for contradiction that $\ker A^n \subsetneq \ker A^{n+1}$. Then by the second statement $\{0\} = \ker A^0 \subsetneq \ker A \subsetneq \cdots \subsetneq \ker A^n \subsetneq \ker A^{n+1}$. Since all of these are subspaces in $\mathbb{C}^n$, there must exist $n + 1$ linearly independent vectors in $\mathbb{C}^n$ - a contradiction. □

Definition A.29. $x \in \mathbb{C}^n$ is said to be a generalised eigenvector of A associated to $\lambda \in \sigma(A)$ if there exists an integer $k \geq 1$ such that $(A - \lambda)^k x = 0$. If k is the smallest integer with this property, we say that x is a generalised eigenvector of rank k.

Remark. It follows from Proposition A.28 that the rank of a generalised eigenvector is well-defined and never greater than n.

The following proposition is also a simple corollary of Proposition A.28.

Proposition A.30. *$x \in \mathbb{C}^n$ is a generalised eigenvector of A if and only if $(A - \lambda)^n x = 0$.*

Definition A.31. The space $\ker(A - \lambda)^n$ is called the generalised eigenspace associated to λ.

Proposition A.32. *The generalised eigenspace* $\ker(A - \lambda)^n$ *and* $\operatorname{ran}(A - \lambda)^n$ *are invariant under the mapping $A - \lambda$. The two subspaces are independent, i.e.* $\ker(A - \lambda)^n \cap \operatorname{ran}(A - \lambda)^n = \{0\}$.

Proof. The invariance is trivial for $\operatorname{ran}(A - \lambda)^n$ and it also holds for $\ker(A - \lambda)^n$ thanks to Proposition A.28.

We will show that $\ker(A - \lambda)^n \cap \operatorname{ran}(A - \lambda)^n = \{0\}$. If $y \in \operatorname{ran}(A - \lambda)^n$, i.e. $y = (A - \lambda)^n x$ for some $x \in \mathbb{C}^n$, then $(A - \lambda)^n y = 0$ implies $(A - \lambda)^{2n} x = 0$. Therefore, by Proposition A.28, $x \in \ker(A - \lambda)^n$ and thus $y = 0$. □

Proposition A.33. *Let $\lambda_1, \lambda_2 \in \sigma(A)$, $\lambda_1 \neq \lambda_2$. Then* $\ker(A - \lambda_1)^n \cap \ker(A - \lambda_2)^n = \{0\}$.

Proof. Denote $\mathcal{N}_1 := \ker(A-\lambda_1)^n$, $\mathcal{N}_2 := \ker(A-\lambda_2)^n$. Suppose for contradiction that $x \neq 0$ and $x \in \mathcal{N}_1 \cap \mathcal{N}_2$. Let $k \geq 1$ be the smallest integer such that $(A-\lambda_1)^k x = 0$. Then $y := (A-\lambda_1)^{k-1} x \neq 0$ is an eigenvector of A associated to λ_1. At the same time $y = (A-\lambda_2+(\lambda_2-\lambda_1))^k x$. Since $\mathcal{N}_2$ is invariant with respect to $A-\lambda_2$ (and thus with respect to $A-\lambda_2+(\lambda_2-\lambda_1)$), surely $y \in \mathcal{N}_2$ - a contradiction. □

Theorem A.34. *On a finite-dimensional vector space, Definition A.27 is equivalent to Definition A.26.*

Proof. By Proposition A.28 we have

$$M_\lambda = \bigcup_{k=1}^{\infty} \ker((A-\lambda I)^k) = \ker(A-\lambda)^n.$$

Notice that λ is the only eigenvalue of $A|_{M_\lambda}$ as follows from Proposition A.33. Therefore, its characteristic polynomial is

$$\tilde{p}(t) = (t-\lambda)^d, \tag{15}$$

where $d := \dim M_\lambda$. Also note that λ is not an eigenvalue of $A|_{\mathrm{ran}(A-\lambda)^n}$ thanks to Proposition A.32. Let X denote a regular matrix with d generalised eigenvectors corresponding to λ in the first d columns (and basis vectors of $\mathrm{ran}(A-\lambda)$ in the other $n-d$ columns). Then, since $X^{-1}(A-t)X$ is block-diagonal (thanks to Proposition A.32), we have $p(t) = \det(A-t) = \det\left(X^{-1}(A-t)X\right) = \tilde{p}(t) * q(t)$, where $q(t)$ is not divisible by $t-\lambda$ because it is the characteristic polynomial of $A|_{\mathrm{ran}(A-\lambda)^n}$. Finally, from (15) we derive $p(t) = (t-\lambda)^d q(t)$. Therefore, $\nu_a(\lambda) = d = \dim M_\lambda$. □

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